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Global existence, blow-up and stability for a stochastic transport equation with non-local velocity

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Abstract

In this paper we investigate a non-linear and non-local one dimensional transport equation under random perturbations on the real line. We first establish a local-in-time theory, i.e., existence, uniqueness and blow-up criterion for pathwise solutions in Sobolev spaces H^s with $s > 3$. Thereafter, we give a picture of the long time behavior of the solutions based on the type of noise we consider. On one hand, we identify a family of noises such that blow-up can be prevented with probability 1, guaranteeing the existence and uniqueness of global solutions almost surely. On the other hand, in the particular linear noise case, we show that singularities occur in finite time with positive probability, and we derive lower bounds of these probabilities. To conclude, we introduce the notion of stability of exiting times and show that one cannot improve the stability of the exiting time and simultaneously improve the continuity of the dependence on initial data.

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Keywords: Stochastic evolution equations; Pathwise solution; Blow-up criterion; Noise prevents blow-up; Weak instability

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1. Introduction and main results

Consider the following one dimensional non-local transport equation

$$u_t + (\mathcal{H}u)u_x = 0, \tag{1.1}$$

where $\mathcal{H}$ denotes the Hilbert transform. This equation first appears in the literature due to its analogy with the Birkhoff-Rott equations describing the evolution of vortex sheets with surface tension [4,45]. Moreover, (1.1) can be also viewed as a toy-model of the two-dimensional surface quasi-geostrophic equation (SQG) which describes the evolution of the potential temperature in a rapidly rotating stratified fluid with uniform potential vorticity [25,47]. A striking result showing the finite time blow-up of classical solutions to (1.1) for a generic class of smooth initial data was first obtained by Córdoba, Córdoba and Fontelos in [11] by means of complex analysis techniques. After that breakthrough, equation (1.1) is known as the CCF equation. Subsequent works have shown finite blow-up avoiding complex analysis approach (cf. [38,53]). In particular, in the latter Silvestre and Vicol provided four elegant and simple real analysis proofs of the blow-up phenomena.

In this paper, we are interested in stochastic variants of the CCF equation (1.1). Indeed, the introduction of stochasticity into ideal fluid dynamics has received special attention over the past two decades. The inclusion of stochastic noise can be a way of representing model uncertainty and turbulence. For example, in weather forecasting, phenomena including cloud formation are to this day poorly understood and the inclusion of stochastic noise has become an essential tool for gaining better understanding about it. Since the pioneering work of Holm in [31], where a variational approach to introducing noise in equations in a fashion that respects the geometry of the system is developed, the literature regarding the analysis of non-linear stochastic partial differential equations with transport type noise has increased substantially (cf. [1,2,9,10,20] and the references therein). To the best of the authors' knowledge, there are very few results regarding the CCF model under random perturbations. Only recently, by applying an abstract framework for singular stochastic partial differential equations (SPDEs) derived by two of the authors, cf. [3], the local existence, uniqueness and blow-up criterion of pathwise solutions to (1.1) with transport noise have been addressed in the periodic setting, i.e., for $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

To extend the result on stochastic CCF equation developed in [3] to the real line case, i.e., $x \in \mathbb{R}$, and to study the noise effect, in this paper we will consider the following Cauchy problem

$$\begin{cases} \mathrm{d}u + (\mathcal{H}u)u_x \mathrm{d}t = h(t,u)\mathrm{d}\mathcal{W}, & x \in \mathbb{R},\ t > 0, \\ u(\omega, 0, x) = u_0(\omega, x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $\mathcal{W}$ is a cylindrical Wiener process and h is a non-linear function. In this work, under some natural assumptions collected in Assumption **(A)**, we obtain the local existence, uniqueness and a blow-up criterion of pathwise solutions to (1.2). The detailed result is stated in Theorem 1.1. Here we notice that classical probabilistic compactness arguments cannot be directly applied to deal with the whole space $\mathbb{R}$ and new ideas are in order, see Remark 1.2 for more precise explanation.

It is well-known that noise effect is one of the crucial subjects in the study of SPDEs. Indeed, regularization effects due to noise have been observed for various models and different settings. For example, it is known that the well-posedness of linear stochastic transport equation with noise can be established under weaker hypotheses than its deterministic counterpart and restore uniqueness of solutions, cf. [15,18]. Regularization effects caused by the noise on flux for stochastic scalar conservation laws have been studied in [22]. Extensions of the previous works for the stochastic transport and continuity equations to k-forms have been recently addressed in [5]. In terms of numerical simulations, the regularization effects of noise can be found in [40]. Moreover, for different fluid models with linear multiplicative noise, we refer to [24,37,50,54], where noise provides a damping effect on the pathwise behavior of solutions.

Compared to the deterministic counterpart of (1.2), i.e., (1.1), in this article we focus on the following issues for the problem (1.2) regarding the noise effect:

- Noise versus finite time blow-up;
- Noise versus dependence on initial data.

1.1. Noise versus finite time blow-up

In this direction, we attempt to answer the following two important questions:

(Q-1) What kind of noise can prevent blow-up?
(Q-2) If blow-up may occur, what is the corresponding probability?

We remark here that most previous results in the literature on regularization by noise are restricted to linear equations or linear noises. For instance, we refer to [15,18,16,37,46] for linear transport equations, and to [19,24,50,54] for linear noise. Therefore, for non-linear SPDEs, it is very natural to analyze the validity of the regularization effects by non-linear noise. Indeed, searching for nonlinear noise such that blow-up can be prevented is important because it helps us to understand the regularization mechanisms of noise, and this is the main motivation to study question **(Q-1)**. Actually, even in the case of non-linear equations with linear noise, the noise effects are complicated because there are both examples in positive direction, i.e., noises can regularize singularities, and negative direction, i.e., noises cannot regularize singularities. For example, for the stochastic 2D Euler equations, coalescence of vortices disappears (see [19]) but noise cannot prevent the formation of shocks in the Burgers' equation (see [1,17]).

For simplicity, we set **(Q-1)** in the framework where $h(t,u)\,\mathrm{d}\mathcal{W} = \alpha(t,u)\,\mathrm{d}W$, with W a standard 1-D Brownian motion and α a non-linear function. We then focus on the system

$$\begin{cases} \mathrm{d}u + (\mathcal{H}u)u_x\,\mathrm{d}t = \alpha(t,u)\,\mathrm{d}W, & x \in \mathbb{R},\ t > 0, \\ u(\omega,0,x) = u_0(\omega,x), & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

We will show in Theorem 1.2 that if $\alpha(t,\cdot)$ grows fast enough, then global existence of pathwise solutions holds true with probability 1. This is strongly in contrast with its deterministic counterpart where the breakdown of classical solutions to (1.1) with generic smooth initial data occurs, cf. [11]. Hence we justify the idea that fast growing non-linear noise (*strong* noise) has regularization effects on the solutions in terms of preventing singularities.

By Theorem 1.2, we have identified a family of noises that can prevent blow-up, and this partially answers **(Q-1)**. Next, we will pay our attention to the case that blow-up may occur. Indeed, as a toy-model for the 2D surface quasi-geostrophic equation (SQG) (and hence for the 3D incompressible Euler equation), analyzing the possible blow-up of solutions is one of the central questions in the study of non-local transport type equations, cf. [11,14,38,43,53]. In this paper we are also interested in identifying the possible formation of singularities in finite time and estimating its probability, hence giving a partial answer to **(Q-2)**. Since Theorem 1.2 shows that fast enough growing noises can prevent blow-up, it is natural to ponder that singularities can only occur when the noise is somehow weak. Indeed, in contrast to the fast growing noise, we will show that in the case of linear noise (*weak* noise) given by equation

$$\begin{cases} \mathrm{d}u + (\mathcal{H}u)u_x \mathrm{d}t = b(t)u\,\mathrm{d}W, \quad x \in \mathbb{R},\ t > 0, \\ u(\omega, 0, x) = u_0(\omega, x), \quad x \in \mathbb{R}, \end{cases} \tag{1.4}$$

where b is some continuous function and W is a standard 1-D Brownian motion, finite time blow-up cannot be prevented and finite time singularities occur. The precise statement of this result is given in Theorem 1.3.

1.2. Noise versus dependence on initial data

Now we turn to the problem of noise effect on the initial-data dependence. There are very few results concerning the noise effect in the direction of dependence on initial data. In this work we will partially answer the following question:

(Q-3) Can noise affect the initial-data dependence?

The main motivation to consider question **(Q-3)** relies on the following observation. On the one hand, regularization provided by noise may look related to regularization effects induced by an additional dissipative term (a Laplacian). On the other hand, if one would add a real Laplacian to the governing equations, parabolic techniques may be used to improve the continuity of the initial-data dependence. For example, in the deterministic incompressible Euler equations, the solution map $u_0 \mapsto u$ cannot be better than continuous [29] but for the deterministic incompressible Navier-Stokes equations with sufficiently large viscosity, it is at least Lipschitz continuous in sufficiently high Sobolev spaces (see pp. 79–81 in [26]). In the deterministic setting, similar questions regarding the continuity map on the initial-data have been widely investigated for various non-linear dispersive and integrable equations of which we only mention a few related results. Koch and Tzvetkov [39] proved that the solution map of the Benjamin–Ono equation cannot be uniform continuous. For Camassa–Holm type equations, we refer to [27,28] for the non-uniform dependence on initial data in Sobolev spaces H^s with $s > 3/2$. Similar results in Besov spaces $B^s_{p,q}$ first appeared in [57,56], where the critical index s can be also covered. In the case of the SQG system, we refer the reader to [32]. In the stochastic setting, the interplay between regularization provided by noise and the dependence on initial conditions is first studied in [52,55].

In this article we consider question **(Q-3)** for (1.2). More precisely, we first recall the concept of stability of the exiting time as in [52,55]. Roughly speaking, this notion refers to the continuous changes of the point in time with respect to the initial condition, where such point is defined as the time when the solution leaves a certain range, see Definition 1.2 below. Later on, in Theorem 1.4, we show that when $h(t,u)$ satisfies certain conditions (see Assumption **(D)**), the multiplicative noise cannot improve the stability of the exiting time, and, at the same time, improve the continuity of the map $u_0 \mapsto u$ defined by (1.2).

1.3. Notations, definitions and hypotheses

We now introduce some notations. $L^2(\mathbb{R})$ is the usual space of square–integrable functions on $\mathbb{R}$. For $s \in \mathbb{R}$, $D^s = (1-\partial_{xx}^2)^{s/2}$ is defined by $\widehat{D^s f}(\xi) = (1+\xi^2)^{s/2}\widehat{f}(\xi)$, where $\widehat{g}$ is the Fourier transform of g. The Sobolev space $H^s(\mathbb{R})$ is defined as

$$H^s(\mathbb{R}) \triangleq \left\{ f \in L^2(\mathbb{R}) : \|f\|^2_{H^s(\mathbb{R})} = \int_{\mathbb{R}} (1+\xi^2)^s |\widehat{f}(\xi)|^2 \mathrm{d}\xi < +\infty \right\},$$

in which the inner product $(f,g)_{H^s}$ is given by

$$(f,g)_{H^s} \triangleq \int_{\mathbb{R}} (1+\xi^2)^s \widehat{f}(\xi) \cdot \overline{\widehat{g}}(\xi) \mathrm{d}\xi = (D^s f, D^s g)_{L^2}.$$

When the function space refers to $\mathbb{R}$, we will drop $\mathbb{R}$ if there is no ambiguity. $x \lesssim y$ ($x \gtrsim y$) means that $x \le cy$ ($x \ge cy$) holds for some universal *deterministic* constant c. Such constant may differ from line to line. For linear operators A and B, the commutator $[A,B]$ is defined by $[A,B] = AB - BA$.

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space, where $\mathbb{P}$ is a probability measure on Ω and $\mathcal{F}$ is a σ-algebra. $\mathbb{E}X$ is the mathematical expectation of X with respect to $\mathbb{P}$. Let $\mathcal{W}(t) = \mathcal{W}(\omega, t)$, $\omega \in \Omega$ be a cylindrical Wiener process. More precisely, we consider a separable Hilbert space $\mathcal{U}$ and let $\{e_k\}$ be a complete orthonormal basis of $\mathcal{U}$. Then we define

$$\mathcal{W} \triangleq \sum_{k=1}^{\infty} W_k e_k,$$

where $\{W_k\}_{k\ge1}$ is a sequence of mutually independent standard 1-D Brownian motions. We call $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge0}, \mathcal{W})$ a stochastic basis, where $\{\mathcal{F}_t\}_{t\ge0}$ is a right-continuous filtration endowed on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\{\mathcal{F}_0\}$ contains all the $\mathbb{P}$-negligible subsets.

$\mathcal{L}_2(\mathcal{U}; \mathcal{X})$ stands for the set of Hilbert-Schmidt operators from $\mathcal{U}$ to another Hilbert space $\mathcal{X}$. For a predictable process $Z \in \mathcal{L}_2(\mathcal{U}; \mathcal{X})$,

$$\int_0^t Z \mathrm{d}\mathcal{W} \triangleq \sum_{k=1}^{\infty} \int_0^t Z e_k \mathrm{d}W_k$$

is a well-defined $\mathcal{X}$-valued continuous square integrable martingale, see [12,48] for more details. In the sequel of the paper, when a stopping time is defined, we set $\inf \emptyset \triangleq \infty$ by convention.

We now give the precise notion of a pathwise solution to (1.2).

Definition 1.1 *(Pathwise solutions).* Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge0}, \mathcal{W})$ be a fixed stochastic basis. Let $s > 3/2$ and u_0 be an H^s-valued $\mathcal{F}_0$-measurable random variable.

1. A local pathwise solution to (1.2) is a pair (u, τ), where τ is a stopping time satisfying $\mathbb{P}\{\tau > 0\} = 1$ and $u : \Omega \times [0, \infty] \to H^s$ is an $\mathcal{F}_t$-predictable H^s-valued process satisfying

$$u(\cdot\wedge\tau)\in C([0,\infty);H^s)\ \ \mathbb{P}-a.s.,$$

and for all $t>0$,

$$u(t\wedge\tau)-u(0)+\int_0^{t\wedge\tau}(\mathcal{H}u)u_x\mathrm{d}t'=\int_0^{t\wedge\tau}h(t',u)\mathrm{d}\mathcal{W}\ \ \mathbb{P}-a.s.$$

2. The local pathwise solutions are said to be pathwise unique, if given any two pairs of local pathwise solutions (u_1,τ_1) and (u_2,τ_2) with $\mathbb{P}\{u_1(0)=u_2(0)\}=1$, we have

$$\mathbb{P}\{u_1(t,x)=u_2(t,x),\ (t,x)\in[0,\tau_1\wedge\tau_2]\times\mathbb{R}\}=1.$$

3. Additionally, (u,τ^*) is called a maximal pathwise solution to (1.2) if $\tau^*>0$ almost surely and if there is an increasing sequence $\tau_n\to\tau^*$ such that for any $n\in\mathbb{N}$, (u,τ_n) is a pathwise solution to (1.2) and on the set $\{\tau^*<\infty\}$,

$$\sup_{t\in[0,\tau_n]}\|u\|_{H^s}\geq n.$$

4. If (u,τ^*) is a maximal pathwise solution and $\tau^*=\infty$ almost surely, then we say that the pathwise solution exists globally.

Inspired by [52,55], we introduce the concept on stability of exiting time in Sobolev spaces. Exiting time, as the name suggests, describes the first time that the solution leaves a given range. More precisely, we introduce

Definition 1.2 *(Stability of exiting time).* Let $s>3/2$ and $\mathcal{S}=(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P},\mathcal{W})$ be a fixed stochastic basis. Let $u_0\in L^2(\Omega;H^s)$ and $\{u_{0,n}\}\subset L^2(\Omega;H^s)$ be $\mathcal{F}_0$-measurable. For each n, let u and u_n be the unique solutions to (1.2), as in Definition 1.1, with initial values u_0 and $u_{0,n}$, respectively. For any $R>0$, define the R-exiting times as

$$\tau_n^R\triangleq\inf\{t\geq0:\|u_n\|_{H^s}>R\},\quad \tau^R\triangleq\inf\{t\geq0:\|u\|_{H^s}>R\}. \tag{1.5}$$

Then we define the following properties on stability:

1. If $u_{0,n}\to u_0$ in H^s $\mathbb{P}-a.s.$ implies that

$$\lim_{n\to\infty}\tau_n^R=\tau^R\ \ \mathbb{P}-a.s., \tag{1.6}$$

then the R-exiting time of u is said to be stable.
2. If $u_{0,n}\to u_0$ in $H^{s'}$ for all $s'<s$ almost surely also implies that (1.6) holds true, the R-exiting time of u is said to be strongly stable.

To study the existence of pathwise solutions to (1.2), we need the following assumptions on the h:

Assumption (A). We assume that when $s > 3/2$, $h : [0, \infty) \times H^s \ni (t, u) \mapsto h(t, u) \in \mathcal{L}_2(\mathcal{U}; H^s)$ is continuous. Furthermore, we assume that there are two non-decreasing locally bounded functions $f, q : [0, +\infty) \to [0, +\infty)$ such that

- For any $t > 0$ and $u \in H^s$,

$$\|h(t, u)\|_{\mathcal{L}_2(\mathcal{U};H^s)} \leq f(\|u_x\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty})(1 + \|u\|_{H^s}). \tag{1.7}$$

- For any $t > 0$,

$$\sup_{\|u\|_{H^s},\|v\|_{H^s}\leq N} \left\{\mathbf{1}_{\{u\neq v\}} \frac{\|h(t, u) - h(t, v)\|_{\mathcal{L}_2(\mathcal{U};H^s)}}{\|u - v\|_{H^s}}\right\} \leq q(N), \quad N \geq 1. \tag{1.8}$$

Example. Now we give an example of noise coefficient satisfying Assumption **(A)**. For simplicity, we only consider the 1-D case, i.e., $\mathcal{W} = W$, and $h : (t, u) \mapsto h(t, u)$ is a map from $[0, \infty) \times H^s$ to H^s. The extension of this example to the general $\mathcal{L}_2(\mathcal{U}; H^s)$-valued h can be carried out by considering $h(e_k)$, where $\{e_k\}_{k\geq 1}$ is a complete orthonormal basis of $\mathcal{U}$ (as 1-D case with suitable coefficient) such that $\sum_{k\geq 1} \|h(e_k)\|^2_{H^s} < \infty$. To that purpose, let $G = \frac{1}{2}e^{-|x|}$. Then in 1-D case, $(1 - \partial^2_{xx})^{-1}$ can be understood as

$$\left[(1 - \partial^2_{xx})^{-1} f\right](x) = [G \star f](x) \text{ for } f \in L^2(\mathbb{R}), \tag{1.9}$$

where $\star$ stands for the convolution. Now we let

$$h(t, u) = q(t)(1 - \partial^2_{xx})^{-1}\partial_x[(u_x)^k + (\mathcal{H}u_x)^n], \quad k, n \geq 1.$$

If $q(\cdot)$ is smooth with both upper and lower bounds, then it is easy to see that h satisfies Assumption **(A)**.

To find global existence, we need some stronger conditions on the noise coefficient h and we make the following assumption:

Assumption (B). We assume that when $s > \frac{3}{2}$, $\alpha : [0, \infty) \times H^s \ni (t, u) \mapsto \alpha(t, u) \in H^s$ is continuous. Moreover, we assume the following properties hold true:

- $\alpha(\cdot, u)$ is bounded for all $u \in H^s$ and there is a non-decreasing locally bounded function $l(\cdot) : [0, \infty) \to [0, \infty)$ such that for any $t \geq 0$,

$$\sup_{\|u\|_{H^s},\|v\|_{H^s}\leq N} \left\{\mathbf{1}_{\{u\neq v\}} \frac{\|\alpha(t, u) - \alpha(t, v)\|_{H^s}}{\|u - v\|_{H^s}}\right\} \leq l(N), \quad N \geq 1, \ s > 3/2. \tag{1.10}$$

- Define

$$\mathfrak{G} = \left\{\mathcal{G} \in C^2([0, \infty); [0, \infty)) : \mathcal{G}(0) = 0, \ \mathcal{G}'(x) > 0, \ \mathcal{G}''(x) \leq 0 \text{ and } \lim_{x\to\infty} \mathcal{G}(x) = \infty\right\},$$

  and we assume that there is a function $\mathcal{G} \in \mathfrak{G}$ and constants $K_1, K_2 > 0$ such that for all $(t, u) \in [0, \infty) \times H^s$ with $s > 5/2$,

$$\mathcal{G}'(\|u\|^2_{H^{s-1}})\mathcal{M}(t)+2\mathcal{G}''(\|u\|^2_{H^{s-1}})\left|(\alpha(t,u),u)_{H^{s-1}}\right|^2 \le K_1-K_2\frac{\left\{\mathcal{G}'(\|u\|^2_{H^{s-1}})\left|(\alpha(t,u),u)_{H^{s-1}}\right|\right\}^2}{1+\mathcal{G}(\|u\|^2_{H^{s-1}})}, \tag{1.11}$$

where

$$\mathcal{M}(t)=2Q(\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty})\|u\|^2_{H^{s-1}}+\|\alpha(t,u)\|^2_{H^{s-1}}$$

and Q is the constant given in Lemma 2.3.

Example. Let $q(t)$ be a continuous function such that $q_*<q^2(t)<q^*$ for all t and let Q be the constant given in Lemma 2.3 below. Then it is easy to check that

$$\alpha(t,u)\triangleq q(t)\,(1+\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty})^\theta\,u \tag{1.12}$$

with

$$\text{either } \theta>\frac{1}{2},\ q^*>q_*>0 \text{ or } \theta=\frac{1}{2},\ q^*>q_*>2Q \tag{1.13}$$

satisfies Assumption **(B)** with $\mathcal{G}(x)=\log(1+x)\in\mathfrak{G}$. For simplicity we only verify (1.11). Indeed, we observe that

$$\begin{aligned}
&\mathcal{G}'(\|u\|^2_{H^{s-1}})\mathcal{M}(t)+2\mathcal{G}''(\|u\|^2_{H^{s-1}})\left|(\alpha(t,u),u)_{H^{s-1}}\right|^2\\
&+K_2\frac{\left\{\mathcal{G}'(\|u\|^2_{H^{s-1}})\left|(\alpha(t,u),u)_{H^{s-1}}\right|\right\}^2}{1+\mathcal{G}(\|u\|^2_{H^{s-1}})}\\
=&\frac{2Q(\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty})\|u\|^2_{H^{s-1}}+q^2(t)\,(1+\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty})^{2\theta}\,\|u\|^2_{H^{s-1}}}{1+\|u\|^2_{H^{s-1}}}\\
&-\frac{2q^2(t)\,(1+\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty})^{2\theta}\,\|u\|^4_{H^{s-1}}}{\left(1+\|u\|^2_{H^{s-1}}\right)^2}\\
&+K_2\frac{q^2(t)\,(1+\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty})^{2\theta}\,\|u\|^4_{H^{s-1}}}{\left(1+\|u\|^2_{H^{s-1}}\right)^2\left(1+\log(1+\|u\|^2_{H^{s-1}})\right)}\\
\le&\,2Q(\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty})+q^2(t)\,(1+\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty})^{2\theta}\\
&-\frac{2q^2(t)\,(1+\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty})^{2\theta}\,\|u\|^4_{H^{s-1}}}{\left(1+\|u\|^2_{H^{s-1}}\right)^2}+K_2\frac{q^2(t)\,(1+\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty})^{2\theta}}{\left(1+\log(1+\|u\|^2_{H^{s-1}})\right)}\\
:=&\,\mathfrak{J}
\end{aligned}$$

If $\|u_x\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty}$ is bounded, then $\mathfrak{J}$ is also bounded, and hence it can be controlled by some constant $K_1 > 0$. To prove (1.11), we only need to check that $\mathfrak{J}$ can be also controlled by K_1 when $\|u_x\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty} \to \infty$. Let $\mathcal{P} = q^2(t)\,(1 + \|u_x\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty})^{2\theta}$. Due to the embedding $H^{s-1} \hookrightarrow W^{1,\infty}$, when $\|u_x\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty} \to \infty$, $\|u\|_{H^{s-1}} \to \infty$. Hence, we have that

$$\limsup_{\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty}\to+\infty} \mathfrak{J} \le \limsup_{\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty}\to+\infty} \left\{ \frac{2Q(\|u_x\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty})}{\mathcal{P}} + 1 - \frac{2\|u\|^4_{H^{s-1}}}{\left(1+\|u\|^2_{H^{s-1}}\right)^2} + K_2 \frac{\|u\|^2_{H^{s-1}}}{\left(1+\|u\|^2_{H^{s-1}}\right)^2 \left(1+\log(1+\|u\|^2_{H^{s-1}})\right)} \right\} \mathcal{P}.$$

Moreover, if (1.13) is satisfied,

$$\limsup_{\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty}\to+\infty} \frac{2Q(\|u_x\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty})}{\mathcal{P}} < 1$$

and consequently $\limsup_{\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty}\to+\infty} \mathfrak{J} \le 0$. Therefore α given by (1.12) and $\mathcal{G}(\cdot) = \log(1+\cdot)$ satisfies (1.11).

In the linear noise case (1.4), we make the following assumption on b:

Assumption (C). We assume $b(t)$ in (1.4) satisfies that $b(t) \in C([0,\infty);[0,\infty))$ and there exists some $b^* > 0$ such that $b^2(t) < b^*$ for all $t \ge 0$.

Finally, we assume the following conditions to study the question **(Q-3)**:

Assumption (D). When considering (1.2) in Section 6, we assume that for $s > 3/2$, $h : [0,\infty) \times H^s \ni (t,u) \mapsto h(t,u) \in \mathcal{L}_2(\mathcal{U}; H^s)$ is continuous. Moreover, we assume the following:

- There exists a non-decreasing and locally bounded function $l(\cdot) : [0,+\infty) \to [0,+\infty)$ such that for any $t \ge 0$ and $u \in H^s$ with $s > 3$, we have that

$$\|h(t,u)\|_{\mathcal{L}_2(\mathcal{U};H^s)} \le l(\|u_x\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty})\|u\|_{H^s}.$$

- There exists a non-decreasing and locally bounded function $g(\cdot) : [0,+\infty) \to [0,+\infty)$ and a real number $\sigma_0 \in (3/2, 7/4)$ such that for all $N \ge 1$,

$$\sup_{t\ge 0,\|u\|_{H^s}\le N} \|h(t,u)\|_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})} \le g(N) \mathrm{e}^{-\frac{1}{\|u\|_{H^{\sigma_0}}}}.$$

- Property (1.8) holds true.

Example. Let us here give an example of noise structure satisfying Assumption **(D)**. As above, we only consider the case where $h(t,u)\,\mathrm{d}\mathcal{W} = \gamma(t,u)\,\mathrm{d}W$ with W a standard 1-D Brownian motion. Let $k \ge 1$ and $q(\cdot)$ be a continuous and bounded function, then for any $k, n \ge 1$,

$$\gamma(t,u)=q(t)\mathrm{e}^{-\frac{1}{\|u\|_{H^{\sigma_0}}}}(1-\partial_{xx}^2)^{-1}\partial_x[(u_x)^k+(\mathcal{H}u_x)^n]$$

satisfies Assumption **(D)**, where $(1-\partial_{xx}^2)^{-1}$ is defined in (1.9).

1.4. Main results and remarks

In this subsection, we present the precise statements of the different results shown in this article. The first result reads:

Theorem 1.1. *Let $s>3$ and let $h(t,u)$ satisfy Assumption (A). If u_0 is an H^s-valued $\mathcal{F}_0$-measurable random variable satisfying $\mathbb{E}\|u_0\|_{H^s}^2<\infty$, then there is a local unique pathwise solution (u,τ) to (1.2) in the sense of Definition 1.1 with*

$$u(\cdot\wedge\tau)\in L^2\left(\Omega;C\left([0,\infty);H^s\right)\right). \tag{1.14}$$

Moreover, (u,τ) can be extended to a unique maximal pathwise solution (u,τ^*) with the following blow-up criterion:

$$\mathbf{1}_{\left\{\limsup_{t\to\tau^*}\|u(t)\|_{H^s}=\infty\right\}}=\mathbf{1}_{\left\{\limsup_{t\to\tau^*}\|u_x(t)\|_{L^\infty}+\|\mathcal{H}u_x(t)\|_{L^\infty}=\infty\right\}}\ \mathbb{P}-a.s. \tag{1.15}$$

Remark 1.1. Before we explain the ideas and difficulties regarding the existence of local pathwise solutions, let us first stress some important remarks on the blow-up criterion (1.15).

- The blow-up criterion (1.15) implies that the $H^{s'}$ norm of the solution within the range $s'\in(\frac{3}{2},s]$ blows up at the same time τ^*. Indeed, for any fixed s and $s'\in(\frac{3}{2},s)$, since

  $$\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty}\lesssim\|u\|_{H^{s'}}\le\|u\|_{H^s},$$

  we can conclude that $\|u\|_{H^{s'}}$ blows up no later than the time $\|u_x(t)\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty}$ blows up but no earlier than the time $\|u\|_{H^s}$ blows up. Therefore, equality (1.15) shows that all of the $H^{s'}$ norms have the same blow-up time. This fact will be used to prove Theorem 1.2 (see (4.1)).
- Invoking the well-known logarithmic Sobolev inequality involving the Hilbert transform $\mathcal{H}$

  $$\|\mathcal{H}u_x\|_{L^\infty}\lesssim\left(1+\|u_x\|_{L^\infty}\log\left(\mathrm{e}+\|u_x\|_{H^1}\right)+\|u_x\|_{L^2}\right), \tag{1.16}$$

  one can show in deterministic case, by using (1.16), that the blow-up criterion (1.15) can be improved into (cf. [14])

  $$\limsup_{t\to\tau^*}\|u(t)\|_{H^s}=\infty\Longleftrightarrow\limsup_{t\to\tau^*}\|u_x(t)\|_{L^\infty}=\infty.$$

  However, it is still not clear how to achieve this in the stochastic setting. Technically, because $\mathbb{E}\left[\left(1+\|u_x\|_{L^\infty}\log\left(\mathrm{e}+\|u_x\|_{H^1}\right)+\|u_x\|_{L^2}\right)\|u\|_{H^s}\right]$ is involved, we have not been able to close the estimate for $\mathbb{E}\|u\|_{H^s}^2$.

Remark 1.2. Now we give a remark regarding the existence part in Theorem 1.1. Since we focus on a Cauchy problem defined on the whole space $\mathbb{R}$, the classical probabilistic compactness argument for non-linear SPDEs in bounded domain seems inapplicable in this work. Indeed, we have the following essential difficulties:

- For smooth u, in the *a priori* $L^2(\Omega; H^s)$ estimate for u, after using the Itô formula for $\|u\|^2_{H^s}$, we will have to deal with $\mathbb{E}\left(f^2(\|u_x\|_{L^\infty})(1+\|u\|^2_{H^s})\right)$ and $\mathbb{E}\left((\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty})\,\|u\|^2_{H^s}\right)$, coming from $h(t,u)$ and $(\mathcal{H}u)u_x$, respectively. To close the estimate, these two terms should be controlled in terms of $\mathbb{E}\left(1+\|u\|^2_{H^s}\right)$. Since we cannot split the mathematical expectation, we add a cut-off function to the original problem to cut the non-linear parts in terms of $\|\cdot\|_{H^r}$ (see (3.1) below) with suitable r such that $H^s \hookrightarrow H^r \hookrightarrow W^{1,\infty}$. This enables us to close the *a priori* estimate. In this work, $s>3$ and we let $r=s-3/2$.
- As the second step, we construct the approximation scheme and obtain certain uniform estimates. Our next aim is to pass to the limit to obtain the existence of a solution. If the target problem is defined on bounded domain, usually one can find $q\in(r,s)$ such that $H^s \hookrightarrow\hookrightarrow H^q \hookrightarrow H^r$ (here $\hookrightarrow\hookrightarrow$ means the embedding is compact). By the uniform estimates, one can follow the compactness argument (cf. Prokhorov's Theorem and Skorokhod's Theorem) to obtain the convergence in H^q, which enables us to pass to the limit to find a martingale solution to the *cut-off* version of the target SPDE. We can *a posteriori* introduce a stopping time to remove the cut-off. We refer to [6,8,13,24,30,54,55] for different examples. On the contrary, in unbounded domains, the compact embedding only holds true for the "local" space $H^s_{\text{loc}} \hookrightarrow\hookrightarrow H^q_{\text{loc}}$. Because the *cut-off* $\|\cdot\|_{H^r}$ appears in the problem itself, even though we obtain the convergence in H^q_{loc} (cf. Prokhorov's Theorem and Skorokhod's Theorem), it is still *not* enough to guarantee the convergence of $\|\cdot\|_{H^r}$ because $\|\cdot\|_{H^r}$ is a *global* object (for space variable), which cannot be controlled by a *local* condition. As mentioned above, because the *cut-off* is needed, we have to establish a convergence result in some topology no weaker than H^r.
- In this paper, we will show that there exists a sub-sequence of the approximate solutions converging in $C([0,T];H^r)$ almost surely, and the essential part of proving this is motivated by [42,44]. We also remark that our analysis is also available for the torus case, i.e., $x\in\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$, since all the estimates can be performed in the same way (up to some obvious modifications) for $x\in\mathbb{T}$. In the deterministic case where the *cut-off* is no longer needed, convergence in local Sobolev spaces is sufficient to take limit to find a solution.

Hereafter, we consider the noise effect versus finite time blow-up. Our second result gives a partial answer to **(Q-1)**:

Theorem 1.2 *(Noise preventing blow-up). Let $s>3$ and $u_0\in H^s$ be an H^s-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|u_0\|^2_{H^s}<\infty$. If Assumption **(B)** holds true, then the corresponding maximal pathwise solution (u,τ^*) to (1.3) satisfies*

$$\mathbb{P}\left\{\tau^*=\infty\right\}=1.$$

Remark 1.3. Let us first recall that in the deterministic counterpart of (1.3), the blow-up of regular enough solutions actually cannot be prevented, cf. [11,43,53]. Therefore, Theorem 1.2 justifies the idea that fast growing non-linear noise (*strong* noise) can regularize the solutions in terms of preventing singularities. Here we use the terminology "fast growing" condition, described by (1.11) in Assumption **(B)**, to cancel (notice that $\mathcal{G}'' < 0$ in (1.11)) the growth of the non-local transport term $(\mathcal{H}u)u$ such that $\mathbb{E}\mathcal{G}(\|u\|^2_{H^s})$ can be controlled with a Lyapunov type function $\mathcal{G}$. The idea of using a Lyapunov type function is motivated by the works [7,36,49,51].

Complementarily, in the case of linear noise, we can show that singularities occur in finite time with positive probability which yields a partial answer to **(Q-2)**. The precise statement reads:

Theorem 1.3 *(Blow-up for linear noise). Let $0 < K < 1$, $s > 3$ and $b(t)$ satisfy Assumption* **(C)**. *Assume that $u_0 = u_0(\omega, x)$ is an H^s valued $\mathcal{F}_0$-measurable random variable satisfying $\mathbb{E}\,\|u_0\|^2_{H^s} < \infty$ and attaining a global maximum at x_0. If*

$$\Lambda u_0(\omega, x_0) > \frac{b^*}{K} \ \ \mathbb{P} - a.s., \tag{1.17}$$

where b^* is given in Assumption **(C)** *and Λ is defined in (2.5), then the unique maximal pathwise solution (u, τ^*) to (1.4) satisfies that*

$$\mathbb{P}\{\tau^* < \infty\} \geq \mathbb{P}\{\limsup_{t\to\tau^*} \max_{x\in\mathbb{R}} \Lambda u(t) = +\infty\} \geq \mathbb{P}\{e^{\int_0^t b(t')\,\mathrm{d}\mathcal{W}_{t'}} > K \ \ \forall t > 0\} > 0. \tag{1.18}$$

Remark 1.4. We make the following remarks regarding Theorem 1.3:

- Motivated by previous works [24,50,51], to study (1.4), we make use of Girsanov type transform to obtain a PDE with random coefficient instead of a SPDE to study blow-up. In this linear noise case, the blow-up criterion (1.15) also holds true. By direct computation, we have $\mathcal{H}u_x = \Lambda u$ for the fractional Laplace operator $\Lambda = (-\Delta)^{\frac{1}{2}}$. Hence the blow-up criterion (1.15) becomes

$$\mathbf{1}_{\{\limsup_{t\to\tau^*} \|u(t)\|_{H^s} = \infty\}} = \mathbf{1}_{\{\limsup_{t\to\tau^*} \|u_x(t)\|_{L^\infty} + \|\Lambda u(t)\|_{L^\infty} = \infty\}} \ \mathbb{P} - a.s.$$

  This motivates us to consider blow-up for the particular case $\limsup_{t\to\tau^*} \|\Lambda u(t)\|_{L^\infty} = \infty$. Actually the case we obtain, as shown in (1.18), is $\limsup_{t\to\tau^*} \max_{x\in\mathbb{R}} \Lambda u(t) = +\infty$. So far it is still not clear if $\limsup_{t\to\tau^*} \min_{x\in\mathbb{R}} \Lambda u(t) = -\infty$ holds for blow-up.
- The key identity used in our analysis on blow-up is originally introduced in [53, Proposition 3.5], where the authors show that C^1 global solutions to (1.4) cannot exist. The precise statement of this identity in our stochastic context is given in (5.9). Although the idea towards the proof of (5.9) is similar to [53], we cannot assume without loss of generality (as in [53, Proposition 3.5]) that $\tilde{v}(z_0) = 0$, which simplifies the proof of (5.9). The main reason relies on the fact that in the stochastic setting z_0 depends on t and $\omega \in \Omega$. As a consequence, for different ω, the time t such that $\tilde{v}(z_0) = 0$ may be different, and therefore we cannot assume that $\tilde{v}(z_0) = 0$. On the other hand, if we fix t such that $\tilde{v}(z_0) = 0$, then $\tilde{v}(z_0) = 0$ may not hold almost surely, which brings further obstacles in proving (5.9).
- For trivial $b^* = 0$, which means $b(t) \equiv 0$, Theorem 1.3 recovers the deterministic result [53, Theorem 3.7]. Due to the presence of the noise term $b(t)u\mathrm{d}W$, the condition (1.17) is more restrictive compared to the deterministic case where $\Lambda u_0(x_0) > 0$ suffices to show the finite time blow-up, [53, Theorem 3.7]. Therefore, although finite time singularities can be shown

in the stochastic case, it is somehow restrictive and strongly depends on the choice of the coefficient $b(t)$.

Regarding question **(Q-3)**, we have the following partial (negative) answer:

Theorem 1.4 *(Weak instability). Let $s > 3$. If h satisfies Assumption (D), then at least one of the following properties holds true.*

1. *For any $R \gg 1$, the R-exiting time is not strongly stable for the zero solution to (1.2) in the sense of Definition 1.2;*
2. *There is a $T > 0$ such that solution map $u_0 \mapsto u$ defined by (1.2) is not uniformly continuous as a map from $L^\infty(\Omega, H^s)$ into $L^1(\Omega; C([0, T], H^s))$. More precisely, there exist two sequences of solutions $u^{1,n}(t, x)$ and $u^{2,n}(t, x)$, and two sequences of stopping time $\tau_{1,n}$ and $\tau_{2,n}$, such that*

   - *For $i = 1, 2$, $\mathbb{P}\{\tau_{i,n} > 0\} = 1$ for each $n > 1$. Besides,*

$$\lim_{n\to\infty} \tau_{1,n} = \lim_{n\to\infty} \tau_{2,n} = \infty \ \ \mathbb{P} - a.s. \tag{1.19}$$

   - *For $i = 1, 2$, $u^{i,n} \in C([0, \tau_{i,n}], H^s)$ $\mathbb{P} - a.s.$, and*

$$\mathbb{E}\left(\sup_{t\in[0,\tau_{1,n}]} \|u^{1,n}(t)\|_{H^s} + \sup_{t\in[0,\tau_{2,n}]} \|u^{2,n}(t)\|_{H^s}\right) \lesssim 1. \tag{1.20}$$

   - *At initial time $t = 0$,*

$$\lim_{n\to\infty} \sup_{\omega\in\Omega} \|u^{1,n}(0) - u^{2,n}(0)\|_{H^s} = 0. \tag{1.21}$$

   - *When $t > 0$,*

$$\liminf_{n\to\infty} \mathbb{E} \sup_{t\in[0,T\wedge\tau_{1,n}\wedge\tau_{2,n}]} \|u^{1,n}(t) - u^{2,n}(t)\|_{H^s} \gtrsim \sup_{t\in[0,T]} |\sin(t)|. \tag{1.22}$$

Remark 1.5. The idea behind Theorem 1.4 can be understood as follows: one cannot improve the continuity of the map $u_0 \mapsto u$, and simultaneously, the stability of the exiting time at $u \equiv 0$. Hereafter, we briefly outline the main difficulties encountered in the proof of Theorem 1.4 and the main strategies used to tackle them.

- For system (1.2), we do not know how to obtain any explicit expression of the solutions. Therefore, to establish (1.22), the idea relies on finding two sequences of approximate solutions $\{u_{m,n}\}$ $(m \in \{1, 2\})$ such that the difference between $u_{m,n}$ and the actual solution $u^{m,n}$ tends to zero as $n \to \infty$, from which one can prove (1.22) by using $\{u_{m,n}\}$ rather than $\{u^{m,n}\}$. In this way, the first difficulty lies in the construction of such approximate solutions $\{u_{m,n}\}$. In this work, by some delicate calculation, we are able to construct two sequences of approximate solutions $\{u_{m,n}\}$ such that the actual solutions $\{u^{m,n}\}$ satisfy

$$u^{m,n}(0)=u_{m,n}(0),\quad \lim_{n\to\infty}\mathbb{E}\sup_{[0,\tau_{m,n}]}\|u^{m,n}-u_{m,n}\|_{H^s}=0, \tag{1.23}$$

where $u^{m,n}$ exists at least on $[0,\tau_{m,n}]$. Technically, since the equation involves the non-local Hilbert transform $\mathcal{H}$ and the problem is defined on $\mathbb{R}$, the construction of approximate solutions $u_{m,n}$ is more elaborated than the constructions in [44,52,55].

- The second difficulty we have to surpass is that we need to guarantee that $\inf_n \tau_{m,n}>0$ almost surely when dealing with (1.23). This obstacle comes from the lack of lifespan estimates which we believe is quite a common issue in SPDEs. Indeed, in deterministic cases, one can easily obtain the lifespan estimate, from which it is not difficult to find a common interval $[0,T]$ such that all actual solutions exist on $[0,T]$ (see for example Lemma 6.1). In the stochastic setting, we are not able to show this precise estimate. As in [52,55], the key observation to surpass this difficulty is that the property $\inf_n \tau_{m,n}>0$ can be connected to the stability property of the exiting time (see Definition 1.2). The condition comprising the fact that the R_0-exiting time is strongly stable at the zero solution will be used to provide a common existence time $T>0$ such that for all n, $u^{m,n}$ exists up to T (see Lemma 6.4 below). Therefore, to prove Theorem 1.4, we will show that, if the R_0-exiting time is strongly stable at the zero solution for some $R_0\gg 1$, then the solution map $u_0\mapsto u$ defined by (1.2) cannot be uniformly continuous.

1.5. Plan of the paper

We outline the structure of the paper. In Section 2 we provide some relevant preliminaries and recall well-known estimates that will be employed throughout the paper. In Section 3 we show the existence and uniqueness of local pathwise solutions and derive a blow-up criterion proving Theorem 1.1. We will divide the proof into several subsections. First we use an approximation scheme and perform uniform bounds in Subsection 3.1. Next, in Subsection 3.2 we show the convergence of the approximated solutions and afterwards in Subsection 3.3 we prove the global pathwise solutions to the cur-off problem. We conclude the proof of Theorem 1.1 in Subsection 1.1. Section 4 is devoted to studying the effect of strong noise and proving Theorem 1.2. In Section 5 we show that finite time singularity occurs in the case of linear noise thus proving Theorem 1.3. To conclude the article, in Section 6, we prove Theorem 1.4. The proof is divided into Subsection 6.1 - Subsection 6.3.

2. Preliminaries

For any $\varepsilon\in(0,1)$, J_ε is the Friedrichs mollifier defined by $J_\varepsilon f(x)=j_\varepsilon\star f(x)$, where $\star$ stands for the convolution, $j_\varepsilon(x)=\frac{1}{\varepsilon}j(\frac{x}{\varepsilon})$ and $j(x)$ is a Schwartz function satisfying $0\le\widehat{j}(\xi)\le 1$ for all $\xi\in\mathbb{R}$ and $\widehat{j}(\xi)=1$ for any $|\xi|\le 1$, where $\widehat{f}$ denotes the Fourier transform of f. It is obvious that $\widehat{j_\varepsilon}(\xi)=\widehat{j}(\varepsilon\xi)$. Moreover, for any $u\in H^s$, we have, cf. [54,55],

$$\|I-J_\varepsilon\|_{\mathcal{L}(H^s;H^r)}\sim o(\varepsilon^{s-r}),\quad r\le s, \tag{2.1}$$

$$\|J_\varepsilon\|_{\mathcal{L}(H^s;H^r)}\lesssim \varepsilon^{s-r},\quad r>s, \tag{2.2}$$

$$(J_\varepsilon f,g)_{L^2}=(f,J_\varepsilon g)_{L^2}. \tag{2.3}$$

The Hilbert transform is defined as

$$\mathcal{H}f(x) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{y-x}\,\mathrm{d}y. \tag{2.4}$$

The fractional differential operator $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ is defined as the following singular integral operator

$$\Lambda^\alpha f(x) = c_\alpha \text{p.v.}\int_{\mathbb{R}} \frac{f(x)-f(y)}{|x-y|^{1+\alpha}}\,\mathrm{d}y, \quad \alpha \in (0,2), \tag{2.5}$$

where the constant $c_\alpha = \frac{4^{\frac{\alpha}{2}}\Gamma(\frac{1}{2}+\alpha/2)}{\sqrt{\pi}|\Gamma(-\alpha/2)|}$ is a normalization constant and Γ represents the classical gamma function. In particular, when $\alpha = 1$, $\mathcal{H}f_x = -\Lambda f$. Since J_ε, $\mathcal{H}$ and D^s can be characterized by their Fourier multipliers, it is easy to see

$$[D^s, \mathcal{H}] = [D^s, J_\varepsilon] = [\partial_x, \mathcal{H}] = [\partial_x, J_\varepsilon] = [J_\varepsilon, \mathcal{H}] = 0, \tag{2.6}$$

and for any $s \ge 0$,

$$\|\mathcal{H}u\|_{H^s} \le \|u\|_{H^s}, \ \|J_\varepsilon u\|_{H^s} \le \|u\|_{H^s}. \tag{2.7}$$

Lemma 2.1 *(Page 3 in [58]). Let J_ε be defined as in the above. Assume $g \in W^{1,\infty}$ and $f \in L^2$. Then for some $C > 0$,*

$$\|[J_\varepsilon, g]\partial_x f\|_{L^2} \le C\|\partial_x g\|_{L^\infty}\|f\|_{L^2}.$$

We also recall the following well-known estimates.

Lemma 2.2 *([34,35]). If $f, g \in H^s \bigcap W^{1,\infty}$ with $s > 0$, then for $p, p_i \in (1,\infty)$ with $i = 2, 3$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, we have*

$$\left\|\left[D^s, f\right]g\right\|_{L^p} \le C_{s,p}(\|\nabla f\|_{L^{p_1}}\|D^{s-1}g\|_{L^{p_2}} + \|D^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}),$$

and

$$\|D^s(fg)\|_{L^p} \le C_{s,p}(\|f\|_{L^{p_1}}\|D^s g\|_{L^{p_2}} + \|D^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}).$$

Lemma 2.3. *Let $s > 3$. Let J_ε be the Friedrichs mollifier defined before. There is a constant $Q = Q(s) > 0$ such that*

$$\left|((\mathcal{H}u)u_x, u)_{H^{s-1}}\right| \le Q(\|u_x\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty})\|u\|^2_{H^{s-1}}, \tag{2.8}$$

and for all $\varepsilon > 0$,

$$|(J_\varepsilon[(\mathcal{H}u)u_x], J_\varepsilon u)_{H^s}| \le Q(\|u_x\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty})\|u\|^2_{H^s}. \tag{2.9}$$

Proof. We first prove (2.9). Due to (2.6) and (2.3), we commute the operator to derive

$$\begin{aligned}&\left(D^s J_\varepsilon\left[(\mathcal{H}u)u_x\right], D^s J_\varepsilon u\right)_{L^2}\\ &=\left(\left[D^s, \mathcal{H}u\right]u_x, D^s J_\varepsilon^2 u\right)_{L^2}+\left([J_\varepsilon, \mathcal{H}u]D^s u_x, D^s J_\varepsilon u\right)_{L^2}+\left((\mathcal{H}u)D^s J_\varepsilon u_x, D^s J_\varepsilon u\right)_{L^2}.\end{aligned}$$

Then it follows from Lemmas 2.1 and 2.2, integration by parts, (2.7) and $H^s \hookrightarrow W^{1,\infty}$ that

$$\left|\left(\left[D^s, \mathcal{H}u\right]u_x, D^s J_\varepsilon^2 u\right)_{L^2}\right| \lesssim \left(\|u_x\|_{L^\infty}+\|\mathcal{H}u_x\|_{L^\infty}\right)\|u\|_{H^s}^2,$$

$$\left|\left([J_\varepsilon, \mathcal{H}u]D^s u_x, D^s J_\varepsilon u\right)_{L^2}\right| \lesssim \|\mathcal{H}u_x\|_{L^\infty}\|u\|_{H^s}^2,$$

and

$$\left|\left((\mathcal{H}u)D^s J_\varepsilon u_x, D^s J_\varepsilon u\right)_{L^2}\right| \lesssim \|\mathcal{H}u_x\|_{L^\infty}\|u\|_{H^s}^2.$$

Combining the above inequalities gives rise to (2.9). For (2.8), since $((\mathcal{H}u)u_x, u)_{H^{s-1}}$ is well-defined in this case, by repeating the above analysis, one can easily obtain the (2.8) and the proof is therefore completed. □

Remark 2.1. We remark that (2.9) will be used in the proof of blow-up criterion (1.15) (see (3.26) below) and (2.8) will be used in the proof of Theorem 1.2 (see (4.3) below). We refer to Remark 4.1 for more details.

Let us collect some identities and formulas regarding the Hilbert transform and the fractional Laplacian operator, cf. [11,53]. The first one is the so-called Cotlar's identity

$$2\mathcal{H}(f(\mathcal{H}f)) = (\mathcal{H}f)^2 - f^2. \tag{2.10}$$

The second one is the following equality:

Lemma 2.4 *(Corollary 3.3, [53]). For any $f \in L^1(\mathbb{R};\mathbb{R})$, we have that for $g(x) = xf(x)$,*

$$\Lambda g(x) = x\Lambda f(x) - \mathcal{H}f(x)$$

Finally, we recall the following estimate on the product of a Schwartz function and a trigonometric function.

Lemma 2.5 *([27]). Let $\delta > 0$ and $\alpha \in \mathbb{R}$. Then for any $r \geq 0$ and any Schwartz function ψ, the following equation holds true:*

$$\lim_{n\to\infty} n^{-\frac{\delta}{2}-r}\left\|\psi\left(\frac{x}{n^\delta}\right)\cos(nx-\alpha)\right\|_{H^r} = \frac{1}{\sqrt{2}}\|\psi\|_{L^2}. \tag{2.11}$$

Moreover, (2.11) also holds true when cos *is replaced by* sin.

3. Proof of Theorem 1.1

For the sake of clarity, the proof is divided into several subsections.

3.1. Approximation scheme and uniform estimates

The first step is to construct a suitable approximation scheme. For any $R > 1$, we let $\chi_R(x) : [0, \infty) \to [0, 1]$ be a C^∞ function such that $\chi_R(x) = 1$ for $x \in [0, R]$ and $\chi_R(x) = 0$ for $x > 2R$. Then we consider the following cut-off problem on $\mathbb{R}$,

$$\begin{cases} \mathrm{d}u + \chi_R(\|u\|_{H^{s-3/2}})(\mathcal{H}u)u_x \mathrm{d}t = \chi_R(\|u\|_{H^{s-3/2}})h(t,u)\mathrm{d}\mathcal{W}, \\ u(\omega, 0, x) = u_0(\omega, x) \in H^s, \quad s > 3. \end{cases} \tag{3.1}$$

To apply the theory of SDEs in Hilbert space to (3.1), we will have to mollify the transport term $(\mathcal{H}u)u_x$ since the product $(\mathcal{H}u)u_x$ loses regularity. To this end, we consider the following approximation scheme:

$$\begin{cases} \mathrm{d}u + G_{1,\varepsilon}(u)\mathrm{d}t = G_2(t,u)\mathrm{d}\mathcal{W}, \\ \qquad G_{1,\varepsilon}(u) = \chi_R(\|u\|_{H^{s-3/2}})J_\varepsilon[(\mathcal{H}J_\varepsilon u)\,\partial_x J_\varepsilon u], \\ \qquad G_2(t,u) = \chi_R(\|u\|_{H^{s-3/2}})h(t,u), \\ \qquad u(0,x) = u_0(x) \in H^s, \end{cases} \tag{3.2}$$

where J_ε is the Friedrichs mollifier defined in Section 2.

After mollifying the non-local transport term $(\mathcal{H}u)u_x$, we see that $G_{1,\varepsilon}(\cdot)$ and $G_2(t,\cdot)$ are locally Lipschitz continuous in H^s. Moreover, the cut-off function $\chi_R(\|\cdot\|_{H^{s-3/2}})$ gives the linear growth condition (cf, Lemma 2.3 and (1.7)), i.e., there are constants $l_1 = l_1(\varepsilon, R) > 0$ and $l_2 = l_2(R) > 0$ such that for all $t \ge 0$ and $s > 3$,

$$\|G_{1,\varepsilon}(u)\|_{H^s} \le l_1(1 + \|u\|_{H^s}), \quad \|G_2(t,u)\|_{\mathcal{L}_2(\mathcal{U};H^s)} \le l_2(1 + \|u\|_{H^s}), \quad t \in [0,T]. \tag{3.3}$$

Therefore, for a fixed stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge0}, \mathcal{W})$ and for $u_0 \in L^2(\Omega; H^s)$ with $s > 3$, the existence theory of SDEs in Hilbert space (see for example [48, Theorem 4.2.4 with Example 4.1.3] and [33]) shows that for any $R > 0$, (3.2) admits a unique solution $u_\varepsilon \in C([0,\infty), H^s)$ $\mathbb{P} - a.s.$

We have the following uniform-in-ε estimate:

Proposition 3.1. *Let $s > 3$, $R > 1$ and $\varepsilon \in (0,1)$. Assume h satisfies Assumption (A) and $u_0 \in L^2(\Omega; H^s)$ is an $\mathcal{F}_0$-measurable random variable. Let $u_\varepsilon \in C([0,\infty); H^s)$ solve (3.2) $\mathbb{P} - a.s.$, then for any $T > 0$, there is $C = C(R, T, u_0) > 0$ such that*

$$\sup_{\varepsilon>0} \mathbb{E} \sup_{t\in[0,T]} \|u_\varepsilon(t)\|^2_{H^s} \le C. \tag{3.4}$$

Proof. Using the Itô formula for $\|u_\varepsilon\|_{H^s}^2$, we have that for any $t>0$,

$$\begin{aligned} \mathrm{d}\|u_\varepsilon(t)\|_{H^s}^2 =& 2\chi_R(\|u_\varepsilon\|_{H^{s-3/2}})\,(h(t,u_\varepsilon)\mathrm{d}\mathcal{W},u_\varepsilon)_{H^s}\\ &-2\chi_R(\|u_\varepsilon\|_{H^{s-3/2}})\left(D^sJ_\varepsilon\left[J_\varepsilon(\mathcal{H}u_\varepsilon)\partial_xJ_\varepsilon u_\varepsilon\right],D^su_\varepsilon\right)_{L^2}\mathrm{d}t\\ &+\chi_R^2(\|u_\varepsilon\|_{H^{s-3/2}})\|h(t,u_\varepsilon)\|_{\mathcal{L}_2(\mathcal{U};H^s)}^2\mathrm{d}t. \end{aligned}$$

Then, by means of the BDG inequality, (1.7), $H^{s-3/2}\hookrightarrow W^{1,\infty}$ and (2.7), we find that for some constant $C=C(R)>0$,

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^2-\mathbb{E}\|u_0\|_{H^s}^2\\ \lesssim&\mathbb{E}\left(\int_0^T\chi_R^2(\|u_\varepsilon\|_{H^{s-3/2}})f^2(2\|u_\varepsilon\|_{H^{s-3/2}})(1+\|u_\varepsilon\|_{H^s}^2)\|u_\varepsilon\|_{H^s}^2\mathrm{d}t\right)^{\frac12}\\ &+2\mathbb{E}\int_0^T\chi_R(\|u_\varepsilon\|_{H^{s-3/2}})\left|\left(D^sJ_\varepsilon\left[J_\varepsilon(\mathcal{H}u_\varepsilon)\partial_xJ_\varepsilon u_\varepsilon\right],D^su_\varepsilon\right)_{L^2}\right|\mathrm{d}t\\ &+\mathbb{E}\int_0^T\chi_R^2(\|u_\varepsilon\|_{H^{s-3/2}})f^2(2\|u_\varepsilon\|_{H^{s-3/2}})(1+\|u_\varepsilon\|_{H^s}^2)\mathrm{d}t\\ \le&\frac12\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon\|_{H^s}^2+C(R)\mathbb{E}\int_0^T\left(1+\|u_\varepsilon\|_{H^s}^2\right)\mathrm{d}t\\ &+2\mathbb{E}\int_0^T\chi_R(\|u_\varepsilon\|_{H^{s-3/2}})\left|\left(D^sJ_\varepsilon\left[J_\varepsilon(\mathcal{H}u_\varepsilon)\partial_xJ_\varepsilon u_\varepsilon\right],D^su_\varepsilon\right)_{L^2}\right|\mathrm{d}t. \end{aligned}$$

Let $J_\varepsilon u_\varepsilon=v$. It follows from (2.3), Lemma 2.2, integration by parts and (2.7) that

$$\begin{aligned} \left|\left(D^sJ_\varepsilon\left[J_\varepsilon(\mathcal{H}u_\varepsilon)\partial_xJ_\varepsilon u_\varepsilon\right],D^su_\varepsilon\right)_{L^2}\right|\le&\left|\left(\left[D^s,\mathcal{H}v\right]v_x,D^sv\right)_{L^2}\right|+\left|\left((\mathcal{H}v)D^sv_x,D^sv\right)_{L^2}\right|\\ \le&C\left(\|\mathcal{H}v\|_{H^s}\|v_x\|_{L^\infty}+\|\mathcal{H}v_x\|_{L^\infty}\|v\|_{H^s}\right)\|v\|_{H^s}\\ \le&C\|v\|_{H^{s-3/2}}\|v\|_{H^s}^2, \end{aligned}$$

which implies

$$\mathbb{E}\int_0^T\chi_R(\|u_\varepsilon\|_{H^{s-3/2}})\left|\left(D^sJ_\varepsilon\left[J_\varepsilon(\mathcal{H}u_\varepsilon)\partial_xJ_\varepsilon u_\varepsilon\right],D^su_\varepsilon\right)_{L^2}\right|\mathrm{d}t\le C(R)\mathbb{E}\int_0^T\|u_\varepsilon\|_{H^s}^2\mathrm{d}t.$$

Therefore we obtain

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^2 \le 2\mathbb{E}\|u_0\|_{H^s}^2 + C(R)\int_0^T \left(1+\mathbb{E}\sup_{t'\in[0,t]}\|u(t')\|_{H^s}^2\right)\mathrm{d}t.$$

Using Grönwall's inequality to the above estimate implies that for some $C = C(R, T, u_0) > 0$,

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^2 \le C(R, T, u_0),$$

which is (3.4). □

3.2. *Convergence of approximate solutions*

Now we are going to show that, there is a subsequence of u_ε converging in $C([0, T], H^{s-3/2})$ almost surely. To this end, for the solutions u_ε and u_η to (3.2), we consider the following problem for $v_{\varepsilon,\eta} = u_\varepsilon - u_\eta$,

$$\mathrm{d}v_{\varepsilon,\eta} + \left[G_{1,\varepsilon}(u_\varepsilon) - G_{1,\eta}(u_\eta)\right]\mathrm{d}t = \left[G_2(t, u_\varepsilon) - G_2(t, u_\eta)\right]\mathrm{d}\mathcal{W}, \quad v_{\varepsilon,\eta}(0) = 0. \tag{3.5}$$

We notice that

$$\begin{aligned}
&G_{1,\varepsilon}(u_\varepsilon) - G_{1,\eta}(u_\eta)\\
=&\chi_R(\|u_\varepsilon\|_{H^{s-3/2}})\left[J_\varepsilon\left(J_\varepsilon(\mathcal{H}u_\varepsilon)\partial_x J_\varepsilon u_\varepsilon\right)\right] - \chi_R(\|u_\eta\|_{H^{s-3/2}})\left[J_\eta\left(J_\eta(\mathcal{H}u_\eta)\partial_x J_\eta u_\eta\right)\right]\\
=&\left[\chi_R(\|u_\varepsilon\|_{H^{s-3/2}}) - \chi_R(\|u_\eta\|_{H^{s-3/2}})\right] J_\varepsilon[J_\varepsilon(\mathcal{H}u_\varepsilon)\partial_x J_\varepsilon u_\varepsilon]\\
&+ \chi_R(\|u_\eta\|_{H^{s-3/2}})(J_\varepsilon - J_\eta)[J_\varepsilon(\mathcal{H}u_\varepsilon)\partial_x J_\varepsilon u_\varepsilon]\\
&+ \chi_R(\|u_\eta\|_{H^{s-3/2}})J_\eta[(J_\varepsilon - J_\eta)(\mathcal{H}u_\varepsilon)\partial_x J_\varepsilon u_\varepsilon] + \chi_R(\|u_\eta\|_{H^{s-3/2}})J_\eta[J_\eta(\mathcal{H}u_\varepsilon - \mathcal{H}u_\eta)\partial_x J_\varepsilon u_\varepsilon]\\
&+ \chi_R(\|u_\eta\|_{H^{s-3/2}})J_\eta[J_\eta(\mathcal{H}u_\eta)\partial_x(J_\varepsilon - J_\eta)u_\varepsilon] + \chi_R(\|u_\eta\|_{H^{s-3/2}})J_\eta[J_\eta(\mathcal{H}u_\eta)\partial_x J_\eta(u_\varepsilon - u_\eta)]\\
\triangleq&\sum_{i=1}^{6} q_i
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&G_2(t, u_\varepsilon) - G_2(t, u_\eta)\\
=&\chi_R(\|u_\varepsilon\|_{H^{s-3/2}})h(t, u_\varepsilon) - \chi_R(\|u_\eta\|_{H^{s-3/2}})h(t, u_\eta)\\
=&\left[\chi_R(\|u_\varepsilon\|_{H^{s-3/2}}) - \chi_R(\|u_\eta\|_{H^{s-3/2}})\right] h(t, u_\varepsilon) + \chi_R(\|u_\eta\|_{H^{s-3/2}})[h(t, u_\varepsilon) - h(t, u_\eta)]\\
\triangleq&\sum_{i=7}^{8} q_i.
\end{aligned} \tag{3.7}$$

Invoking Itô's formula in (3.5) and recalling (3.6) and (3.7), we find that for any $t > 0$,

$$\|v_{\varepsilon,\eta}(t)\|^2_{H^{s-3/2}} = \mathcal{Q}_1 - \int_0^t \mathcal{Q}_2 \mathrm{d}t' + \int_0^t \mathcal{Q}_3 \mathrm{d}t', \tag{3.8}$$

where

$$\mathcal{Q}_1 = 2\int_0^t \left(\sum_{i=7}^{8} q_i \mathrm{d}\mathcal{W}, v_{\varepsilon,\eta}\right)_{H^{s-3/2}}, \quad \mathcal{Q}_2 = 2\sum_{i=1}^{6} \left(q_i, v_{\varepsilon,\eta}\right)_{H^{s-3/2}}, \quad \mathcal{Q}_3 = \left\|\sum_{i=7}^{8} q_i\right\|^2_{\mathcal{L}_2(U;H^{s-3/2})}. \tag{3.9}$$

Lemma 3.1. *Let $s > 3$. For any $\varepsilon, \eta \in (0,1)$, there is a constant $C > 0$ such that $\mathcal{Q}_2$ given by (3.9) satisfies*

$$|\mathcal{Q}_2| \le C(1 + \|u_\varepsilon\|^2_{H^s} + \|u_\eta\|^2_{H^s})\|v_{\varepsilon,\eta}\|^2_{H^{s-3/2}} + C(\|u_\varepsilon\|^4_{H^s} + \|u_\eta\|^4_{H^s}) \max\{\varepsilon, \eta\}.$$

Proof. Using the mean value theorem for $\chi_R(\cdot)$ and (2.7), we have

$$\|q_1\|_{H^{s-3/2}} \lesssim \|v_{\varepsilon,\eta}\|_{H^{s-3/2}} \|u_\varepsilon\|^2_{H^s}.$$

Using (2.1) and (2.7), we see that

$$\begin{aligned}
\|q_i\|_{H^{s-3/2}} &\lesssim \max\{\varepsilon^{1/2}, \eta^{1/2}\}\|u_\varepsilon\|^2_{H^s}, \ i = 2, 3, \\
\|q_4\|_{H^{s-3/2}} &\lesssim \|v_{\varepsilon,\eta}\|_{H^{s-3/2}} \|u_\varepsilon\|_{H^s}, \\
\|q_5\|_{H^{s-3/2}} &\lesssim \max\{\varepsilon^{1/2}, \eta^{1/2}\}\|u_\varepsilon\|_{H^s} \|u_\eta\|_{H^s}.
\end{aligned}$$

For q_6, using (2.3), (2.6) and then integrating by parts, we have

$$\begin{aligned}
\left(q_6, v_{\varepsilon,\eta}\right)_{H^{s-3/2}} =& \chi_R(\|u_\eta\|_{H^{s-3/2}}) \int_{\mathbb{R}} D^{s-3/2}[J_\eta(\mathcal{H}u_\eta)\partial_x J_\eta v_{\varepsilon,\eta}] \cdot D^{s-3/2} J_\eta v_{\varepsilon,\eta} \mathrm{d}x \\
=& \chi_R(\|u_\eta\|_{H^{s-3/2}}) \int_{\mathbb{R}} \left[D^{s-3/2}, J_\eta(\mathcal{H}u_\eta)\right] \partial_x J_\eta v_{\varepsilon,\eta} \cdot D^{s-3/2} J_\eta v_{\varepsilon,\eta} \mathrm{d}x \\
&+ \chi_R(\|u_\eta\|_{H^{s-3/2}}) \int_{\mathbb{R}} J_\eta(\mathcal{H}u_\eta)\partial_x D^{s-3/2} J_\eta v_{\varepsilon,\eta} \cdot D^{s-3/2} J_\eta v_{\varepsilon,\eta} \mathrm{d}x \\
=& \chi_R(\|u_\eta\|_{H^{s-3/2}}) \int_{\mathbb{R}} \left[D^{s-3/2}, J_\eta(\mathcal{H}u_\eta)\right] \partial_x J_\eta v_{\varepsilon,\eta} \cdot D^{s-3/2} J_\eta v_{\varepsilon,\eta} \mathrm{d}x \\
&- \frac{1}{2}\chi_R(\|u_\eta\|_{H^{s-3/2}}) \int_{\mathbb{R}} \partial_x J_\eta(\mathcal{H}u_\eta) \cdot (D^{s-3/2} J_\eta v_{\varepsilon,\eta})^2 \mathrm{d}x.
\end{aligned}$$

Using $\chi_R(\cdot) \le 1$, Lemma 2.2, (2.7) and the embedding $H^{s-3/2} \hookrightarrow W^{1,\infty}$, we have

$$\begin{aligned}\big(q_6, v_{\varepsilon,\eta}\big)_{H^{s-3/2}} \lesssim& \|u_\eta\|_{H^{s-3/2}}\|\partial_x J_\eta v_{\varepsilon,\eta}\|_{L^\infty}\|v_{\varepsilon,\eta}\|_{H^{s-3/2}} + \|\partial_x J_\eta(\mathcal{H}u_\eta)\|_{L^\infty}\|v_{\varepsilon,\eta}\|^2_{H^{s-3/2}}\\ \lesssim& \|u_\eta\|_{H^s}\|v_{\varepsilon,\eta}\|^2_{H^{s-3/2}}.\end{aligned}$$

Therefore we can put these all together to see that there is a constant $C > 0$ such that

$$|\mathcal{Q}_2| \le C(1 + \|u_\varepsilon\|^2_{H^s} + \|u_\eta\|^2_{H^s})\|v_{\varepsilon,\eta}\|^2_{H^{s-3/2}} + C(\|u_\varepsilon\|^4_{H^s} + \|u_\eta\|^4_{H^s})\max\{\varepsilon, \eta\},$$

which is the desired estimate. □

Lemma 3.2. *Let $s > 3$, $R > 1$ and $\varepsilon \in (0, 1)$. Let $u_\varepsilon \in C([0, \infty); H^s)$ solve (3.2) $\mathbb{P} - a.s.$ For any $T > 0$ and $K > 1$, we define*

$$\tau^T_{\varepsilon,K} = \inf\{t \ge 0 : \|u_\varepsilon(t)\|_{H^s} \ge K\} \wedge T, \tag{3.10}$$

and

$$\tau^T_{\varepsilon,\eta,K} = \tau^T_{\varepsilon,K} \wedge \tau^T_{\eta,K}. \tag{3.11}$$

Then we have

$$\lim_{\varepsilon\to 0}\sup_{\eta\le\varepsilon}\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|u_\varepsilon - u_\eta\|_{H^{s-3/2}} = 0, \quad K > 1. \tag{3.12}$$

Proof. Recalling (3.8) and (3.9), we have

$$\|v_{\varepsilon,\eta}(t)\|^2_{H^{s-3/2}} \le |\mathcal{Q}_1| + \int_0^t |\mathcal{Q}_2|\mathrm{d}t' + \int_0^t |\mathcal{Q}_3|\mathrm{d}t'. \tag{3.13}$$

The mean value theorem for $\chi_R(\cdot)$ and Assumption **(A)** yield that

$$\|q_7\|_{\mathcal{L}_2(U;H^{s-3/2})} \le C\|v_{\varepsilon,\eta}\|_{H^{s-3/2}} f(\|u_\varepsilon\|_{H^s})(1 + \|u_\varepsilon\|_{H^s}).$$

By (3.11) and Assumption **(A)**, we see that

$$\|q_8\|_{\mathcal{L}_2(U;H^{s-3/2})} \le C\|v_{\varepsilon,\eta}\|_{H^{s-3/2}} q(K), \quad t \in [0, \tau^T_{\varepsilon,\eta,K}] \ \mathbb{P} - a.s.,$$

where $q(\cdot)$ is given in Assumption **(A)**. Therefore we find a constant $C = C(K) > 0$ such that

$$\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}} |\mathcal{Q}_3|\mathrm{d}t \le C(K)\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}} \|v_{\varepsilon,\eta}\|^2_{H^{s-3/2}}\mathrm{d}t \le C(K)\int_0^T \mathbb{E}\sup_{t'\in[0,\tau^t_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t')\|^2_{H^{s-3/2}}\mathrm{d}t. \tag{3.14}$$

Then we employ the BDG inequality to (3.8) to find

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t)\|^2_{H^{s-3/2}}\\
\le& C(K)\mathbb{E}\left(\int_0^{\tau^T_{\varepsilon,\eta,K}}\|v_{\varepsilon,\eta}\|^4_{H^{s-3/2}}\mathrm{d}t\right)^{\frac{1}{2}}+\sum_{i=2}^3\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}|\mathcal{Q}_i|\mathrm{d}t\\
\le&\frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}\|^2_{H^{s-3/2}}+C(K)\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}\|v_{\varepsilon,\eta}\|^2_{H^{s-3/2}}\mathrm{d}t+\sum_{i=2}^3\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}|\mathcal{Q}_i|\mathrm{d}t\\
\le&\frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}\|^2_{H^{s-3/2}}+C(K)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t')\|^2_{H^{s-3/2}}\mathrm{d}t+\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}|\mathcal{Q}_2|\mathrm{d}t.
\end{aligned}
$$

On account of Lemma 3.1, we arrive at

$$
\begin{aligned}
\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}|\mathcal{Q}_2|\mathrm{d}t \le& C(K)\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}\|v_{\varepsilon,\eta}\|^2_{H^{s-3/2}}\mathrm{d}t+C(K)T\max\{\varepsilon,\eta\}\\
\le& C(K)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t')\|^2_{H^{s-3/2}}\mathrm{d}t+C(K)T\max\{\varepsilon,\eta\}. \qquad (3.15)
\end{aligned}
$$

Now we can put these all together to obtain

$$
\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t)\|^2_{H^{s-3/2}}\le C(K)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t')\|^2_{H^{s-3/2}}\mathrm{d}t+C(K)T\max\{\varepsilon,\eta\}, \qquad (3.16)
$$

which means that

$$
\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t)\|^2_{H^{s-3/2}}\le C(K,T)\max\{\varepsilon,\eta\}, \qquad (3.17)
$$

and hence (3.12) holds true. □

Lemma 3.3. *For any fixed $s>3$ and $T>0$, there is an $\{\mathcal{F}_t\}_{t\ge0}$ progressive measurable H^s-valued process*

$$
u\in L^2\left(\Omega;L^\infty\left(0,T;H^s\right)\right) \qquad (3.18)
$$

and a countable subsequence of $\{u_\varepsilon\}$ (still denoted as $\{u_\varepsilon\}$) such that

$$u_\varepsilon \xrightarrow{\varepsilon\to 0} u \text{ in } C([0,T];H^{s-3/2}) \ \ \mathbb{P}-a.s. \tag{3.19}$$

Proof. We notice that for each $\varepsilon \in (0,1)$, (3.2) has solution u_ε almost surely. Now we first take ε to be discrete such that for all ε, u_ε can be defined on the same set $\widetilde{\Omega}$ with $\mathbb{P}\{\widetilde{\Omega}\}=1$ (actually, one can pick discrete $\varepsilon=\frac{1}{n}$ from the beginning in (3.2)). Recall (3.10) and (3.11). For any $\epsilon>0$, by using Proposition 3.1 and Chebyshev's inequality, we see that

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[0,T]}\|u_\varepsilon-u_\eta\|_{H^{s-\frac{3}{2}}}>\epsilon\right\}\\
=&\mathbb{P}\left\{\left(\left\{\tau^T_{\varepsilon,\eta,K}<T\right\}\cup\left\{\tau^T_{\varepsilon,\eta,K}=T\right\}\right)\cap\left\{\sup_{t\in[0,T]}\|u_\varepsilon-u_\eta\|_{H^{s-\frac{3}{2}}}>\epsilon\right\}\right\}\\
\le&\mathbb{P}\left\{\tau^T_{\varepsilon,K}<T\right\}+\mathbb{P}\left\{\tau^T_{\eta,K}<T\right\}+\mathbb{P}\left\{\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|u_\varepsilon-u_\eta\|_{H^{s-\frac{3}{2}}}>\epsilon\right\}\\
\le&\frac{2C(R,T,u_0)}{K^2}+\mathbb{P}\left\{\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|u_\varepsilon-u_\eta\|_{H^{s-\frac{3}{2}}}>\epsilon\right\}.
\end{aligned}$$

Now (3.12) clearly forces

$$\limsup_{\varepsilon\to 0}\sup_{\eta\le\varepsilon}\mathbb{P}\left\{\sup_{t\in[0,T]}\|u_\varepsilon-u_\eta\|_{H^{s-3/2}}>\epsilon\right\}\le\frac{2C(R,T,u_0)}{K^2},\quad K>1.$$

Letting $K\to\infty$, we see that u_ε converges in probability in $C([0,T];H^{s-3/2})$. Therefore, up to a further subsequence, (3.19) holds true.

Now we prove (3.18). Indeed, since $H^s\hookrightarrow H^{s-3/2}$ is continuous, there exist continuous maps $\pi_m: H^{s-3/2}\to H^s,\ m\ge 1$ such that

$$\|\pi_m u\|_{H^s}\le\|u\|_{H^s},\quad \lim_{m\to\infty}\|\pi_m u\|_{H^s}=\|u\|_{H^s},\quad u\in H^{s-3/2},$$

where $\|u\|_{H^s}\triangleq\infty$ if $u\notin H^s$. For example, one may take π_m as the standard mollifier. Then it follows from Proposition 3.1 and Fatou's lemma that

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|^2_{H^s}\le&\liminf_{m\to\infty}\mathbb{E}\sup_{t\in[0,T]}\|\pi_m u(t)\|^2_{H^s}\\
\le&\liminf_{m\to\infty}\liminf_{\varepsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}\|\pi_m u_\varepsilon(t)\|^2_{H^s}\\
\le&\liminf_{m\to\infty}\liminf_{\varepsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|^2_{H^s}<C(R,u_0,T).
\end{aligned}$$

Hence (3.18) holds true. □

3.3. Global pathwise solution to the cut-off problem

Proposition 3.2. *Let $s>3$, $R>1$ and $\varepsilon\in(0,1)$. Assume Assumption* **(A)** *is satisfied. Let $u_0\in L^2(\Omega;H^s)$ be an $\mathcal{F}_0$-measurable random variable. Then for any $T>0$, (3.1) has a solution $u\in L^2(\Omega;C([0,T];H^s))$. Moreover, there is a constant $C(R,T,u_0)>0$ such that*

$$\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^s}^2\le C(R,T,u_0).$$

Proof. Since for each $\varepsilon\in(0,1)$, u_ε is $\{\mathcal{F}_t\}_{t\ge0}$ progressive measurable, so is u. By Lemma 3.3 and the embedding $H^{s-3/2}\hookrightarrow W^{1,\infty}$, we can send $\varepsilon\to0$ in (3.2) to conclude that u solves (3.1). Now we only need to prove (1.14). Due to Lemma 3.3, $u\in C([0,T];H^{s-3/2})\cap L^\infty(0,T;H^s)$ almost surely. Since H^s is dense in $H^{s-3/2}$, we see that ([59, page 263, Lemma 1.4]) $u\in C_w([0,T];H^s)$, where $C_w([0,T];H^s)$ is the space of weakly continuous functions with values in H^s. Therefore to prove (1.14), we only need to prove the continuity of $[0,T]\ni t\mapsto\|u(t)\|_{H^s}$.

However, we cannot directly apply the Itô formula for $\|u\|_{H^s}^2$ to get control of $\mathbb{E}\|u(t)\|_{H^s}^2$ because we only have $u\in H^s$ and $(\mathcal{H}u)u_x\in H^{s-1}$. Indeed, the Itô formula in a Hilbert space ([12, Theorem 4.32] or [21, Theorem 2.10]) requires $((\mathcal{H}u)u_x,u)_{H^s}$ to be well-defined and the Itô formula under a Gelfand triplet ([41, Theorem I.3.1] or [48, Theorem 4.2.5]) requires the dual product ${}_{H^{s-1}}\langle(\mathcal{H}u)u_x,u\rangle_{H^{s+1}}$ to be well-defined. In our case neither of them is satisfied. To this end, we recall the mollifier J_ε defined in Section 2 and apply the Itô formula to $\|J_\varepsilon u\|_{H^s}^2$ to obtain

$$\begin{aligned}\mathrm{d}\|J_\varepsilon u(t)\|_{H^s}^2=&2\chi_R(\|u\|_{W^{1,\infty}})\,(J_\varepsilon h(t,u)\mathrm{d}\mathcal{W},J_\varepsilon u)_{H^s}\\&-2\chi_R(\|u\|_{W^{1,\infty}})\,(J_\varepsilon[(\mathcal{H}u)u_x],J_\varepsilon u)_{H^s}\,\mathrm{d}t\\&+\chi_R^2(\|u\|_{W^{1,\infty}})\|J_\varepsilon h(t,u)\|_{\mathcal{L}_2(\mathcal{U};H^s)}^2\mathrm{d}t.\end{aligned}\tag{3.20}$$

By (3.18),

$$\tau_N=\inf\{t\ge0:\|u(t)\|_{H^s}>N\}\to\infty\text{ as }N\to\infty\ \ \mathbb{P}-a.s.\tag{3.21}$$

Then we only need to prove the continuity up to time $\tau_N\wedge T$ for each $N\ge1$. We first notice that J_ε satisfies (2.3), (2.6) and (2.7). Therefore for any $[t_2,t_1]\subset[0,T]$ with $t_1-t_2<1$, we use Lemma 2.3, the BDG inequality, Assumption **(A)** and (3.21) to find

$$\mathbb{E}\left[\left(\|J_\varepsilon u(t_1\wedge\tau_N)\|_{H^s}^2-\|J_\varepsilon u(t_2\wedge\tau_N)\|_{H^s}^2\right)^4\right]\le C(N,T)|t_1-t_2|^2.$$

Using Fatou's lemma, we arrive at

$$\mathbb{E}\left[\left(\|u(t_1\wedge\tau_N)\|_{H^s}^2-\|u(t_2\wedge\tau_N)\|_{H^s}^2\right)^4\right]\le C(N,T)|t_1-t_2|^2.$$

This and Kolmogorov's continuity theorem ensure the continuity of $t\mapsto\|u(t\wedge\tau_N)\|_{H^s}$, completing the proof. □

3.4. Concluding the proof of Theorem 1.1

Finally, we are in the position to finish the proof of Theorem 1.1. For the sake of clarity, we split the proof into three steps.

Step 1: Existence. For $u_0(\omega, x) \in L^2(\Omega; H^s)$, we let

$$\Omega_k = \{k-1 \le \|u_0\|_{H^s} < k\},\ k \in \mathbb{N},\ k \ge 1.$$

Since $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$, we have

$$u_0(\omega, x) = \sum_{k\ge 1} u_{0,k}(\omega, x) = \sum_{k\ge 1} u_0(\omega, x)\mathbf{1}_{\{k-1\le \|u_0\|_{H^s}<k\}}\ \ \mathbb{P}-a.s.$$

On account of Proposition 3.2, we let $u_{k,R}$ be the pathwise global solution to the cut-off problem (3.1) with initial value $u_{0,k}$ and cut-off function $\chi_R(\cdot)$. Define

$$\tau_{k,R} = \inf\left\{t>0: \sup_{t'\in[0,t]} \|u_{k,R}(t')\|_{H^s}^2 > \|u_{0,k}\|_{H^s}^2 + 2\right\}. \tag{3.22}$$

Then for any $R > 0$ and $k \in \mathbb{N}$, we have $\mathbb{P}\{\tau_{k,R} > 0\} = 1$. The difficulty here is that we have to take R to be deterministic. Otherwise Proposition 3.1 will fail. To overcome this difficulty, we let $R = R_k$ be discrete and then denote $(u_k, \tau_k) = (u_{k,R_k}, \tau_{k,R_k})$. It is clear that $\mathbb{P}\{\tau_k > 0,\ k \ge 1\} = 1$. Let $E > 0$ be the embedding constant such that $\|\cdot\|_{W^{1,\infty}} \le E\|\cdot\|_{H^s}$ for $s > 3$. Particularly, we take $R_k^2 > E^2\|u_{0,k}\|_{H^s}^2 + 2E^2$, and then we have

$$\mathbb{P}\left\{\|u_k\|_{W^{1,\infty}}^2 \le E^2\|u_k\|_{H^s}^2 \le E^2\|u_{0,k}\|_{H^s}^2 + 2E^2 < R_k^2,\ t\in[0,\tau_k],\ k\ge 1\right\} = 1,$$

which means

$$\mathbb{P}\left\{\chi_{R_k}(\|u_k\|_{W^{1,\infty}}) = 1,\ t\in[0,\tau_k],\ k\ge 1\right\} = 1.$$

Therefore (u_k, τ_k) is the pathwise solution to (1.2) with initial value $u_{0,k}$. Notice that

$$\mathbf{1}_{\Omega_k} u_k(t\wedge\tau_k) - \mathbf{1}_{\Omega_k} u_{0,k} = -\int_0^{t\wedge \mathbf{1}_{\Omega_k}\tau_k} \mathbf{1}_{\Omega_k}[(\mathcal{H}u_k)\partial_x u_k]\,\mathrm{d}t' + \int_0^{t\wedge \mathbf{1}_{\Omega_k}\tau_k} \mathbf{1}_{\Omega_k} h(t, u_k)\mathrm{d}\mathcal{W},$$

$$\mathbf{1}_{\Omega_k} h(t, u_k) = h(t, \mathbf{1}_{\Omega_k} u_k) - \mathbf{1}_{\Omega_k^C} h(t, 0)$$

and

$$\mathbf{1}_{\Omega_k}[(\mathcal{H}u_k)\partial_x u_k] = (\mathcal{H}\mathbf{1}_{\Omega_k} u_k)\partial_x \mathbf{1}_{\Omega_k} u_k.$$

By Assumption (A), we have $\|h(t,\mathbf{0})\|_{\mathcal{L}_2(\mathcal{U};H^s)}<\infty$. Then we have

$$\begin{aligned}&\mathbf{1}_{\Omega_k}u_k(t\wedge\tau_k)-\mathbf{1}_{\Omega_k}u_{0,k}\\ =&\mathbf{1}_{\Omega_k}u_k(t\wedge\mathbf{1}_{\Omega_k}\tau_k)-u_{0,k}\\ =&-\int_0^{t\wedge\mathbf{1}_{\Omega_k}\tau_k}(\mathcal{H}\mathbf{1}_{\Omega_k}u_k)\partial_x(\mathbf{1}_{\Omega_k}u_k)\,\mathrm{d}t'+\int_0^{t\wedge\mathbf{1}_{\Omega_k}\tau_k}h(t,\mathbf{1}_{\Omega_k}u_k)\mathrm{d}\mathcal{W}.\end{aligned}$$

Therefore $(\mathbf{1}_{\Omega_k}u_k,\mathbf{1}_{\Omega_k}\tau_k)$ is a solution to (1.2) with initial data $u_{0,k}$. Since $\Omega_k\bigcap\Omega_{k'}=\emptyset$ for $k\neq k'$ and $\bigcup_{k\geq1}\Omega_k$ is a set of full measure, we see that

$$\left(u=\sum_{k\geq1}\mathbf{1}_{\{k-1\leq\|u_0\|_{H^s}<k\}}u_k,\ \ \tau=\sum_{k\geq1}\mathbf{1}_{\{k-1\leq\|u_0\|_{H^s}<k\}}\tau_k\right)$$

is a pathwise solution to (1.2) corresponding to the initial condition u_0. Besides, using (3.22), we have

$$\begin{aligned}\sup_{t\in[0,\tau]}\|u\|^2_{H^s}=&\sum_{k\geq1}\mathbf{1}_{\{k-1\leq\|u_0\|_{H^s}<k\}}\sup_{t\in[0,\tau_k]}\|u_k\|^2_{H^s}\\ \leq&\sum_{k\geq1}\mathbf{1}_{\{k-1\leq\|u_0\|_{H^s}<k\}}\left(\|u_{0,k}\|^2_{H^s}+2\right)\leq2\|u_0\|^2_{H^s}+4.\end{aligned}$$

Taking expectation gives rise to (1.14).

Step 2: Uniqueness and maximal pathwise solution. With (u,τ) in hand, we can extend (u,τ) to a maximal pathwise solution in the sense of Definition 1.1 by following the techniques as in [9,23,24,50]. For uniqueness, we let (u_1,τ_1) and (u_2,τ_2) be two solutions to (1.2) such that $u_j(0)=u_0$ almost surely and $u_j(\cdot\wedge\tau_j)\in L^2(\Omega;C([0,\infty);H^s))$ with $s>3$ for $j=1,2$. Let $\frac{1}{2}<\delta<s-1$ and define

$$\tau_K^T=\inf\{t\geq0:\|u_1(t)\|_{H^s}+\|u_2(t)\|_{H^s}\geq K\}\wedge T,\quad K\in\mathbb{N},\ T>0.$$

Using (1.8) and the definition of τ_K^T, then the estimate of $\mathbb{E}\sup_{t\in[0,\tau_K^T]}\|u_1(t)-u_2(t)\|^2_{H^\delta}$ is essential as in the derivation of (3.12) and we have

$$\mathbb{E}\sup_{t\in[0,\tau_K^T]}\|u_1(t)-u_2(t)\|^2_{H^\delta}=0.$$

If necessary, to guarantee $\tau_K^T>0$ almost surely, we can first assume $u_0\in L^\infty(\Omega;H^s)$ and then remove this restriction by using the techniques as in *Step 1*. Hence we obtain uniqueness and the details are omitted here for brevity.

Step 3: Blow-up criterion. We first define

$$\tau_{1,m}=\inf\{t\geq0:\|u(t)\|_{H^s}\geq m\},\quad\tau_{2,n}=\inf\{t\geq0:\|u_x(t)\|_{L^\infty}+\|\mathcal{H}u_x(t)\|_{L^\infty}\geq n\},$$

and then let $\tau_1 = \lim_{m\to\infty} \tau_{1,m}$ and $\tau_2 = \lim_{n\to\infty} \tau_{2,n}$. We notice that for fixed $m, n > 0$, even if $\mathbb{P}\{\tau_{1,m} = 0\}$ or $\mathbb{P}\{\tau_{2,n} = 0\}$ is larger than 0, for a.e. $\omega \in \Omega$, there is $m > 0$ or $n > 0$ such that $\tau_{1,m}, \tau_{2,n} > 0$. By continuity of $\|u(t)\|_{H^s}$ and the uniqueness of u, it is easy to check that $\tau_1 = \tau_2$ is actually the maximal existence time τ^* of u in the sense of Definition 1.1. Therefore to prove (1.15), we only need to verify that

$$\tau_1 = \tau_2 \ \ \mathbb{P} - a.s. \tag{3.23}$$

The approach here is motivated by [3,9]. Since $H^s \hookrightarrow W^{1,\infty}$ and $\mathcal{H}$ is continuous in H^s (cf. (2.7)), there exists a constant $M > 0$ such that,

$$\sup_{t\in[0,\tau_{1,m}]} (\|u_x(t)\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty}) \le M \sup_{t\in[0,\tau_{1,m}]} \|u(t)\|_{H^s} \le ([M]+1)m,$$

where $[M]$ denotes the integer part of M. Therefore we have $\tau_{1,m} \le \tau_{2,([M]+1)m} \le \tau_2 \ \mathbb{P} - a.s.$, which means that $\tau_1 \le \tau_2 \ \mathbb{P} - a.s.$ Now we only need to prove $\tau_2 \le \tau_1 \ \mathbb{P} - a.s.$ We do the following claim:

Claim:

$$\mathbb{P}\left\{\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]} \|u(t)\|_{H^s} < \infty\right\} = 1 \ \ \forall\, n_1, n_2 \in \mathbb{N}. \tag{3.24}$$

As is explained before, we cannot directly apply the Itô formula for $\|u\|^2_{H^s}$ to get control of $\mathbb{E}\|u(t)\|^2_{H^s}$. Similar to (3.20), by applying J_ε to (1.2) and using the Itô formula for $\|J_\varepsilon u\|^2_{H^s}$, we have that for any $t > 0$,

$$\begin{aligned} \mathrm{d}\|J_\varepsilon u(t)\|^2_{H^s} = & (J_\varepsilon h(t,u)\mathrm{d}\mathcal{W}, J_\varepsilon u)_{H^s} - 2\left(D^s J_\varepsilon[(\mathcal{H}u)u_x], D^s J_\varepsilon u\right)_{L^2}\mathrm{d}t \\ & + \|J_\varepsilon h(t,u)\|^2_{\mathcal{L}_2(\mathcal{U};H^s)}\mathrm{d}t. \end{aligned} \tag{3.25}$$

By the BDG inequality, we have

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]} \|J_\varepsilon u(t)\|^2_{H^s} \\ \le & \mathbb{E}\|J_\varepsilon u_0\|^2_{H^s} + C\mathbb{E}\left(\int_0^{\tau_{2,n_1}\wedge n_2} \|J_\varepsilon h(t,u)\|^2_{\mathcal{L}_2(\mathcal{U};H^s)}\|J_\varepsilon u\|^2_{H^s}\mathrm{d}t\right)^{\frac{1}{2}} \\ & + 2\mathbb{E}\int_0^{\tau_{2,n_1}\wedge n_2} |\left(D^s J_\varepsilon[(\mathcal{H}u)u_x], D^s J_\varepsilon u\right)_{L^2}|\mathrm{d}t + \mathbb{E}\int_0^{\tau_{2,n_1}\wedge n_2} \|J_\varepsilon h(t,u)\|^2_{\mathcal{L}_2(\mathcal{U};H^s)}\mathrm{d}t. \end{aligned}$$

Then (1.7) and (2.7) lead to

$$C\mathbb{E}\left(\int_0^{\tau_{2,n_1}\wedge n_2} \|J_\varepsilon h(t,u)\|^2_{\mathcal{L}_2(\mathcal{U};H^s)}\|J_\varepsilon u\|^2_{H^s}\mathrm{d}t\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]}\|J_\varepsilon u\|^2_{H^s} + Cf^2(2n_1)\int_0^{n_2}\left(1+\mathbb{E}\|u\|^2_{H^s}\right)\mathrm{d}t.$$

By Lemma 2.3, we find

$$2\mathbb{E}\int_0^{\tau_{2,n_1}\wedge n_2} |\left(D^s J_\varepsilon\left[(\mathcal{H}u)u_x\right], D^s J_\varepsilon u\right)_{L^2}|\mathrm{d}t \leq Cn_1\int_0^{n_2}\left(1+\mathbb{E}\|u\|^2_{H^s}\right)\mathrm{d}t. \tag{3.26}$$

It follows from (1.7) that for some constant $C>0$,

$$\mathbb{E}\int_0^{\tau_{2,n_1}\wedge n_2} \|J_\varepsilon h(t,u)\|^2_{\mathcal{L}_2(\mathcal{U};H^s)}\,\mathrm{d}t \leq Cf^2(2n_1)\int_0^{n_2}\left(1+\mathbb{E}\|u\|^2_{H^s}\right)\mathrm{d}t.$$

Therefore we combine the above estimates, use (2.7), and then send $\varepsilon\to 0$ in the resulting inequality to obtain

$$\mathbb{E}\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]}\|u(t)\|^2_{H^s} \leq C\mathbb{E}\|u_0\|^2_{H^s} + C\int_0^{n_2}\left(1+\mathbb{E}\sup_{t'\in[0,t\wedge\tau_{2,n_1}]}\|u(t')\|^2_{H^s}\right)\mathrm{d}t. \tag{3.27}$$

Then Grönwall's inequality shows that for each $n_1,n_2\in\mathbb{N}$, there is a constant $C=C(n_1,n_2,u_0)>0$ such that

$$\mathbb{E}\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]}\|u(t)\|^2_{H^s} < C(n_1,n_2,u_0),$$

which gives (3.24) and concludes the claim.

Hence (3.24) implies that for all $n_1,n_2\in\mathbb{N}$, $\mathbb{P}\left\{\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]}\|u(t)\|_{H^s}<\infty\right\}=1$. On the other hand, it is easy to see that for all $n_1,n_2\in\mathbb{N}$,

$$\left\{\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]}\|u(t)\|_{H^s}<\infty\right\} \subset \bigcup_{m\in\mathbb{N}}\left\{\tau_{2,n_1}\wedge n_2\leq\tau_{1,m}\right\} \subset \left\{\tau_{2,n_1}\wedge n_2\leq\tau_1\right\}.$$

Consequently,

$$\mathbb{P}\left\{\tau_2\leq\tau_1\right\} = \mathbb{P}\left\{\bigcap_{n_1\in\mathbb{N}}\left\{\tau_{2,n_1}\leq\tau_1\right\}\right\} = \mathbb{P}\left\{\bigcap_{n_1,n_2\in\mathbb{N}}\left\{\tau_{2,n_1}\wedge n_2\leq\tau_1\right\}\right\} = 1. \tag{3.28}$$

Combining the above three steps, we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2

To begin with, we can follow the steps as in the proof of Theorem 1.1 to obtain that, if u_0 is an H^s-valued $\mathcal{F}_0$-measurable random variable satisfying $\mathbb{E}\|u_0\|^2_{H^s} < \infty$ with $s > 3$, then (1.3) has a unique pathwise solution $u \in H^s$ with maximal existence time τ^*. Now the target is to show that $\mathbb{P}\{\tau^* = \infty\} = 1$. To this end, we define

$$\widehat{\tau_k} = \inf\left\{t \geq 0 : \|u(t)\|_{H^{s-1}} \geq k\right\},\quad k \geq 1 \ \text{ and } \ \widehat{\tau^*} = \lim_{k\to\infty} \widehat{\tau_k}.$$

Recalling Remark 1.1, we have

$$\widehat{\tau^*} = \tau^* \ \ \mathbb{P} - a.s. \tag{4.1}$$

Therefore we only need to show $\mathbb{P}\{\widehat{\tau^*} = \infty\} = 1$. Applying the Itô formula to $\|u(t)\|^2_{H^{s-1}}$ gives

$$\mathrm{d}\|u\|^2_{H^{s-1}} = 2\,(\alpha(t,u),u)_{H^{s-1}}\,\mathrm{d}W - 2\,((\mathcal{H}u)u_x,u)_{H^{s-1}}\,\mathrm{d}t + \|\alpha(t,u)\|^2_{H^{s-1}}\,\mathrm{d}t. \tag{4.2}$$

Let $\mathcal{G} \in \mathfrak{G}$. On account of the Itô formula, we derive

$$\begin{aligned}\mathrm{d}\mathcal{G}(\|u\|^2_{H^{s-1}}) = {}& 2\mathcal{G}'(\|u\|^2_{H^{s-1}})\,(\alpha(t,u),u)_{H^{s-1}}\,\mathrm{d}W\\ &+ \mathcal{G}'(\|u\|^2_{H^{s-1}})\left\{-2\,((\mathcal{H}u)u_x,u)_{H^{s-1}} + \|\alpha(t,u)\|^2_{H^{s-1}}\right\}\mathrm{d}t\\ &+ 2\mathcal{G}''(\|u\|^2_{H^{s-1}})\left|(\alpha(t,u),u)_{H^{s-1}}\right|^2\mathrm{d}t.\end{aligned}$$

Recall that in Assumption **(B)**,

$$\mathcal{M}(t) = 2Q(\|u_x(t)\|_{L^\infty} + \|\mathcal{H}u_x(t)\|_{L^\infty})\|u(t)\|^2_{H^{s-1}} + \|\alpha(t,u)\|^2_{H^{s-1}}.$$

Hence taking expectation, using inequality (2.7), Lemma 2.3 and Assumption **(B)** we find that for any $t > 0$,

$$\begin{aligned}&\mathbb{E}\mathcal{G}(\|u(t)\|^2_{H^{s-1}})\\ &= \mathbb{E}\mathcal{G}(\|u_0\|^2_{H^{s-1}}) + \mathbb{E}\int_0^t \mathcal{G}'(\|u\|^2_{H^{s-1}})\left\{-2\,((\mathcal{H}u)u_x,u)_{H^{s-1}} + \|\alpha(t',u)\|^2_{H^{s-1}}\right\}\mathrm{d}t'\\ &\quad + \mathbb{E}\int_0^t 2\mathcal{G}''(\|u\|^2_{H^{s-1}})\left|\left(\alpha(t',u),u\right)_{H^{s-1}}\right|^2\mathrm{d}t'\\ &\leq \mathbb{E}\left\{\mathcal{G}(\|u_0\|^2_{H^{s-1}}) + \mathbb{E}\int_0^t \mathcal{G}'(\|u\|^2_{H^{s-1}})\mathcal{M}(t') + 2\mathcal{G}''(\|u\|^2_{H^{s-1}})\left|\left(\alpha(t',u),u\right)_{H^{s-1}}\right|^2\right\}\mathrm{d}t'\end{aligned}$$

$$\leq \mathbb{E}\mathcal{G}(\|u_0\|^2_{H^{s-1}}) + K_1 t - \mathbb{E}\int_0^t K_2 \frac{\left\{\mathcal{G}'(\|u\|^2_{H^{s-1}})\left|\left(\alpha(t',u),u\right)_{H^{s-1}}\right|\right\}^2}{1+\mathcal{G}(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t', \tag{4.3}$$

which shows that there exists a constant $C = C(u_0, K_1, K_2, t) > 0$ such that

$$\mathbb{E}\int_0^t \frac{\left\{\mathcal{G}'(\|u\|^2_{H^{s-1}})\left|\left(\alpha(t',u),u\right)_{H^{s-1}}\right|\right\}^2}{1+\mathcal{G}(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t' \leq C(u_0, K_1, K_2, t). \tag{4.4}$$

Moreover, for any $T > 0$, it follows from Assumption **(B)** and the BDG inequality that

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]} \mathcal{G}(\|u\|^2_{H^{s-1}}) \\
&\leq \mathbb{E}\mathcal{G}(\|u_0\|^2_{H^{s-1}}) + C\mathbb{E}\left(\int_0^T \left\{\mathcal{G}'(\|u\|^2_{H^{s-1}})\left|(\alpha(t,u),u)_{H^{s-1}}\right|\right\}^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
&\quad + K_1 T + K_2\mathbb{E}\int_0^T \frac{\left\{\mathcal{G}'(\|u\|^2_{H^{s-1}})\left|(\alpha(t,u),u)_{H^{s-1}}\right|\right\}^2}{1+\mathcal{G}(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t \\
&\leq \mathbb{E}\mathcal{G}(\|u_0\|^2_{H^{s}}) + \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\left(1+\mathcal{G}(\|u\|^2_{H^{s-1}})\right) + C\mathbb{E}\int_0^T \frac{\left\{\mathcal{G}'(\|u\|^2_{H^{s-1}})\left|(\alpha(t,u),u)_{H^{s-1}}\right|\right\}^2}{1+\mathcal{G}(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t \\
&\quad + K_1 T + K_2\mathbb{E}\int_0^T \frac{\left\{\mathcal{G}'(\|u\|^2_{H^{s-1}})\left|(\alpha(t,u),u)_{H^{s-1}}\right|\right\}^2}{1+\mathcal{G}(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t.
\end{aligned}$$

Thus, using (4.4) we obtain

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]} \mathcal{G}(\|u\|^2_{H^{s-1}}) &\leq C(u_0, K_1, T) + C(K_2)\mathbb{E}\int_0^T \frac{\left\{\mathcal{G}'(\|u\|^2_{H^{s-1}})\left|(\alpha(t,u),u)_{H^{s-1}}\right|\right\}^2}{1+\mathcal{G}(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t \\
&\leq C(u_0, K_1, K_2, T).
\end{aligned}$$

We can infer from the above estimate that

$$\mathbb{P}\{\widehat{\tau^*} < T\} \leq \mathbb{P}\{\widehat{\tau}_k < T\} \leq \mathbb{P}\left\{\sup_{t\in[0,T]} \mathcal{G}(\|u\|^2_{H^{s-1}}) \geq \mathcal{G}(k^2)\right\} \leq \frac{C(u_0, K_1, K_2, T)}{\mathcal{G}(k^2)}.$$

Therefore, since $\lim_{x\to\infty}\mathcal{G}(x) = \infty$, one can send $k \to \infty$ to identify that $\mathbb{P}\{\widehat{\tau^*} < T\} = 0$. Since $T > 0$ is arbitrary, we have that $\mathbb{P}\{\widehat{\tau^*} = \infty\} = 1$, which shows the desired assertion.

Remark 4.1. We remark that the using of Lyapunov is motivated by the non-explosion test [36], see also [7,49,51]. In the above proof, (2.8) is used to obtain (4.3), but we remark that (2.9) is also used implicitly. Indeed, as in the proof of (3.23), (2.9) is used to obtain (3.26), and here (4.1) also requires (2.9) because it is a consequence of (3.23). In this work we estimate H^{s-1} norm (i.e., $\mathbb{E}\sup_{t\in[0,T]}\mathcal{G}(\|u\|^2_{H^{s-1}})$) and use the fact (4.1) to prove global existence. Let us stress that this extends the recent work [51], where the authors estimate H^s norm of the solution to show global existence.

5. Proof of Theorem 1.3

Due to the linear nature of the noise, we use the Girsanov type transformation

$$v=\frac{1}{\beta(\omega,t)}u,\quad \beta(\omega,t)=\mathrm{e}^{\int_0^t b(t')\mathrm{d}W_{t'}-\int_0^t \frac{b^2(t')}{2}\mathrm{d}t'}. \tag{5.1}$$

In the following lemma we show that v is the solution to a random PDE enjoying desired regularity properties.

Lemma 5.1. *Let $s>3$, $b(t)$ satisfies Assumption **(C)** and fix $\mathcal{S}=\big(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq 0},W\big)$. Assume that $u_0(\omega,x)$ is an H^s-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|u_0\|^2_{H^s}<\infty$ and (u,τ^*) is the corresponding unique maximal pathwise solution to (1.4). Then for $t\in[0,\tau^*)$, the process v as in (5.1) is a solution $\mathbb{P}-a.s.$ to*

$$\begin{cases} v_t+\beta(\mathcal{H}v)v_x=0, \quad \beta(\omega,t)=\mathrm{e}^{\int_0^t b(t')\mathrm{d}W_{t'}-\int_0^t \frac{b^2(t')}{2}\mathrm{d}t'},\\ v(\omega,0,x)=u_0(\omega,x), \quad x\in\mathbb{R}\end{cases} \tag{5.2}$$

with $v\in C\left([0,\tau^*);H^s\right)\bigcap C^1([0,\tau^*);H^{s-1})$ $\mathbb{P}-a.s.$

Proof. Applying Theorem 1.1 for the particular case $h(t,u)=b(t)u$ and noticing that $b(t)$ satisfies Assumption **(C)** (and therefore $h(t,u)=b(t)u$ satisfies Assumption **(A)**), we infer that equation (1.4) has a unique maximal pathwise solution (u,τ^*). Itô's formula yields

$$\mathrm{d}\frac{1}{\beta}=-b(t)\frac{1}{\beta}\mathrm{d}W+b^2(t)\frac{1}{\beta}\mathrm{d}t,$$

and hence straightforward computation shows that

$$\mathrm{d}v=-\beta(\mathcal{H}v)v_x\mathrm{d}t, \tag{5.3}$$

yielding the first equation in (5.2). At time $t=0$, $v(\omega,0,x)=u_0(\omega,x)$ since $\beta(\omega,0)=1$ almost surely, so v satisfies (5.2) almost surely. Furthermore, Theorem 1.1 shows that $u\in C\left([0,\tau^*);H^s\right)$ $\mathbb{P}-a.s.$, hence $v\in C\left([0,\tau^*);H^s\right)$ and $v\in C^1\left([0,\tau^*);H^{s-1}\right)$ $\mathbb{P}-a.s.$ □

Invoking Lemma 5.1, we have that for a.e. $\omega\in\Omega$ the process $v(\omega,t,x)$ solves (5.2) on $[0,\tau^*)$ and $v\in C\left([0,\tau^*);H^s\right)\bigcap C^1([0,\tau^*);H^{s-1})$ for $s>3$. In particular, by the Sobolev embedding,

$v \in C\left([0,\tau^*); C^1\right)$, therefore for a.e. $\omega \in \Omega$, the particle trajectory mapping related to the process v given by

$$\begin{cases} \dfrac{\mathrm{d}\phi(\omega,t,x)}{\mathrm{d}t} = \beta(\omega,t)\mathcal{H}v(\omega,t,\phi(\omega,t,x)), & t \in [0,\tau^*), \\ \phi(\omega,0,x) = x, \quad x \in \mathbb{R}, \end{cases} \tag{5.4}$$

has a unique solution $\phi(\omega,t,x) \in C^1([0,\tau^*) \times \mathbb{R})$. Now for a.e. $\omega \in \Omega$, we let $x_0 = x_0(\omega) \in \mathbb{R}$ be the point that u_0 attains its global maximum, i.e.,

$$v(\omega,0,x_0(\omega)) = u_0(\omega,x_0(\omega)) = \max_{x\in\mathbb{R}} u_0(\omega,x), \text{ for a.e. } \omega \in \Omega.$$

Then we focus on the particle trajectory mapping from x_0 in (5.4), i.e.,

$$\begin{cases} \dfrac{\mathrm{d}\phi(\omega,t,x_0)}{\mathrm{d}t} = \beta(\omega,t)\mathcal{H}v(\omega,t,\phi(\omega,t,x_0)), & t \in [0,\tau^*), \\ \phi(\omega,0,x_0) = x_0. \end{cases} \tag{5.5}$$

On the other hand, by the transport nature of equation (5.2), the value of v is constant along characteristics and since $v(\omega,0,x_0)$ attains a global maximum, we have that

$$\partial_x v(\omega,t,\phi(\omega,t,x_0)) = 0, \quad t \in [0,\tau^*) \ \ \mathbb{P}-a.s. \tag{5.6}$$

Computing the quantity $\Lambda v(\omega,t,\phi(\omega,t,x_0))$, i.e., the evolution of the Λ operator of v along the trajectory, we see that for a.e. $\omega \in \Omega$ and $t \in [0,\tau^*)$, there holds

$$\begin{aligned} &\frac{\mathrm{d}\Lambda v(\omega,t,\phi(\omega,t,x_0))}{\mathrm{d}t} \\ &= \Lambda v_t(\omega,t,\phi(\omega,t,x_0)) + \frac{\mathrm{d}\Lambda v(\omega,t,\phi(\omega,t,x_0))}{\mathrm{d}\phi(\omega,t,x_0))}\frac{\mathrm{d}\phi(\omega,t,x_0)}{\mathrm{d}t} \\ &= -\beta(\omega,t)\left\{\Lambda[(\mathcal{H}v)v_x](\omega,t,\phi(\omega,t,x_0)) - (\mathcal{H}v)(\omega,t,\phi(\omega,t,x_0))\Lambda v_x(\omega,t,\phi(\omega,t,x_0))\right\}, \end{aligned} \tag{5.7}$$

where chain rule is used in the former equality and the fact that v solves (5.2) and the particle trajectory equation (5.5) in the latter. Let $\tilde{v} = \mathcal{H}v$. Then we have $v_x = \Lambda\tilde{v}$ and $\Lambda v_x = -\tilde{v}_{xx}$, and then we can rewrite equation (5.7) as

$$\begin{aligned} &\frac{\mathrm{d}\Lambda v(\omega,t,\phi(\omega,t,x_0))}{\mathrm{d}t} \\ &= -\beta(\omega,t)\left[\Lambda(\tilde{v}\Lambda\tilde{v})(\omega,t,\phi(\omega,t,x_0)) + \tilde{v}(\omega,t,\phi(\omega,t,x_0))\tilde{v}_{xx}(\omega,t,\phi(\omega,t,x_0))\right]. \end{aligned} \tag{5.8}$$

Now we denote $z_0 \triangleq \phi(\omega,t,x_0)$ and omit the dependence of ω and t in (5.8) for simplicity if there is no ambiguity.

5.1. An identity for the fractional Laplacian Λ

Lemma 5.2. *For a.e. $\omega \in \Omega$ and $t \in [0, \tau^*)$, there holds the following equation*

$$\Lambda(\tilde{v}\Lambda\tilde{v})(z_0) + \tilde{v}(z_0)\tilde{v}_{xx}(z_0) = -\frac{1}{2}(\Lambda v(z_0))^2 - \frac{1}{\pi}\left\| \frac{\mathcal{H}v(z_0) - \mathcal{H}v(\cdot)}{z_0 - \cdot} \right\|^2_{\dot{H}^{\frac{1}{2}}}. \tag{5.9}$$

Proof. We remark that (5.9) has been obtained in [53, Proposition 3.5] in the deterministic case. However, notice that one cannot assume without loss of generality (as in the deterministic case) that $\tilde{v}(z_0) = 0$, which simplifies the proof of (5.9) (see Remark 1.4), and hence we present also the complete proof here.

Recalling (5.6) and $v_x = \Lambda\tilde{v}$, we have $\Lambda\tilde{v}(z_0) = 0$ for $t \in [0, \tau^*)$ almost surely. Then, invoking the integral representation (2.5) for $\alpha = 1$, we arrive at

$$\begin{aligned}
\Lambda(\tilde{v}\Lambda\tilde{v})(z_0) =& \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{\tilde{v}(z_0)\Lambda\tilde{v}(z_0) - \tilde{v}(y)\Lambda\tilde{v}(y)}{|z_0 - y|^2}\,\mathrm{d}y \\
=& \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{-\tilde{v}(y)\Lambda\tilde{v}(y)}{|z_0 - y|^2}\,\mathrm{d}y, \quad t \in [0, \tau^*) \ \ \mathbb{P} - a.s.
\end{aligned} \tag{5.10}$$

On the other hand, we have that

$$\tilde{v}(z_0)\tilde{v}_{xx}(z_0) = -\tilde{v}(z_0)\Lambda(\Lambda\tilde{v})(z_0) = \tilde{v}(z_0)\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{\Lambda\tilde{v}(y)}{|z_0 - y|^2}\,\mathrm{d}y, \quad t \in [0, \tau^*) \ \ \mathbb{P} - a.s.,$$

where we have used in the first equality the fact that $\partial^2_{xx} = -\Lambda^2$ and the semigroup property of the fractional Laplace operator $\Lambda^2 = \Lambda\Lambda$. Therefore, we have that

$$\Lambda(\tilde{v}\Lambda\tilde{v})(z_0) + \tilde{v}(z_0)\tilde{v}_{xx}(z_0) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{(\tilde{v}(z_0) - \tilde{v}(y))\,\Lambda\tilde{v}(y)}{|z_0 - y|^2}\,\mathrm{d}y, \quad t \in [0, \tau^*) \ \ \mathbb{P} - a.s. \tag{5.11}$$

Recalling the notation $\tilde{v}(z_0) = \tilde{v}(\omega, t, z_0)$, then for a.e. $\omega \in \Omega$ and $t \in [0, \tau^*)$ we define

$$\overline{v}(\omega, t, y) \triangleq \tilde{v}(\omega, t, z_0) - \tilde{v}(\omega, t, y). \tag{5.12}$$

Since $\overline{v}(\omega, t, z_0) = 0$, factorizing the root implies that there exists a process $\eta(\omega, t, y)$ such that

$$\overline{v}(\omega, t, y) = (z_0 - y)\eta(\omega, t, y), \quad t \in [0, \tau^*) \ \ \mathbb{P} - a.s. \tag{5.13}$$

Therefore we can observe that

$$\eta(\omega, t, z_0) = -\overline{v}_x(\omega, t, z_0), \ \ \mathcal{H}\eta(\omega, t, z_0) = \Lambda\overline{v}(\omega, t, z_0), \quad t \in [0, \tau^*) \ \ \mathbb{P} - a.s. \tag{5.14}$$

Again, we drop ω and t if there is no ambiguity. Then (5.11) reduces to

$$\Lambda(\tilde{v}\Lambda\tilde{v})(z_0)+\tilde{v}(z_0)\tilde{v}_{xx}(z_0)=-\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}}\frac{\overline{v}(y)\Lambda\overline{v}(y)}{|z_0-y|^2}\,\mathrm{d}y$$

$$=-\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}}\frac{(z_0-y)\eta(y)[\Lambda((z_0-\cdot)\eta(\cdot))](y)}{|z_0-y|^2}\,\mathrm{d}y,\quad t\in[0,\tau^*)\ \ \mathbb{P}-a.s.$$

Using the linearity of the fractional Laplacian operator and Lemma 2.4, we have

$$[\Lambda((z_0-\cdot)\eta(\cdot))](y)=z_0\Lambda\eta(y)-y\Lambda\eta(y)+\mathcal{H}\eta(y)=(z_0-y)\Lambda\eta(y)+\mathcal{H}\eta(y)$$

and hence

$$\Lambda(\tilde{v}\Lambda\tilde{v})(z_0)+\tilde{v}(z_0)\tilde{v}_{xx}(z_0)=\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}}\frac{-(z_0-y)\eta(y)\left((z_0-y)\Lambda\eta(y)+\mathcal{H}\eta(y)\right)}{|z_0-y|^2}\,\mathrm{d}y$$

$$=\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}}\frac{-(z_0-y)^2\eta(y)\Lambda\eta(y)\,\mathrm{d}y}{|z_0-y|^2}-\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}}\frac{(z_0-y)\eta(y)\mathcal{H}\eta(y)}{|z_0-y|^2}\,\mathrm{d}y$$

$$=-\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}}\eta(y)\Lambda\eta(y)\,\mathrm{d}y+\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}}\frac{\eta(y)\mathcal{H}\eta(y)}{y-z_0}\,\mathrm{d}y$$

$$=-\frac{1}{\pi}\left\|\Lambda^{1/2}\eta\right\|_{L^2}^2+[\mathcal{H}(\eta\mathcal{H}\eta)](z_0),\quad t\in[0,\tau^*)\ \ \mathbb{P}-a.s.$$

Therefore, applying the identity (2.10), we can rewrite the above equation as

$$\Lambda(\tilde{v}\Lambda\tilde{v})(z_0)+\tilde{v}(z_0)\tilde{v}_{xx}(z_0)=-\frac{1}{2}(\eta(z_0))^2+\frac{1}{2}(\mathcal{H}\eta(z_0))^2-\frac{1}{\pi}\left\|\Lambda^{1/2}\eta\right\|_{L^2}^2,\quad t\in[0,\tau^*)\ \ \mathbb{P}-a.s.$$

Using (5.12), (5.13), (5.14) and noticing $\Lambda\tilde{v}(z_0)=0$, we arrive at

$$\Lambda(\tilde{v}\Lambda\tilde{v})(z_0)+\tilde{v}(z_0)\tilde{v}_{xx}(z_0)=-\frac{1}{2}(\overline{v}_x(z_0))^2+\frac{1}{2}(\Lambda\overline{v}(z_0))^2-\frac{1}{\pi}\left\|\frac{\overline{v}(\cdot)}{z_0-\cdot}\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

$$=-\frac{1}{2}\left(\tilde{v}_x(z_0)\right)^2-\frac{1}{\pi}\left\|\frac{\tilde{v}(z_0)-\tilde{v}(\cdot)}{z_0-\cdot}\right\|_{\dot{H}^{\frac{1}{2}}}^2,\quad t\in[0,\tau^*)\ \ \mathbb{P}-a.s.$$

Then (5.9) is a direct consequence of the above equation and the fact $\tilde{v}=\mathcal{H}v$. □

5.2. *Proof of Theorem 1.3*

Now we are in the position to prove Theorem 1.3. To begin with, we can infer from Lemma 5.2 and (5.8) that

$$\frac{\mathrm{d}\Lambda v(z_0)}{\mathrm{d}t}=\beta\left[\frac{1}{2}(\Lambda v(z_0))^2+\frac{1}{\pi}\left\|\frac{\mathcal{H}v(z_0)-\mathcal{H}v(\cdot)}{z_0-\cdot}\right\|_{\dot{H}^{\frac{1}{2}}}^2\right]\geq\beta\frac{1}{2}(\Lambda v(z_0))^2,\quad t\in[0,\tau^*)\ \ \mathbb{P}-a.s.$$

Let $F(\omega,t) \triangleq \Lambda v(\omega,t,z_0)$. Then the above estimate becomes

$$\frac{\mathrm{d}F(\omega,t)}{\mathrm{d}t} \geq \frac{1}{2}\beta(\omega,t)F^2(\omega,t), \quad t\in[0,\tau^*) \;\; \mathbb{P}-a.s. \tag{5.15}$$

Let $0<K<1$ and define $\Omega^* \triangleq \{\omega : \beta(t) \geq Ke^{-\frac{b^*}{2}t} \text{ for all } t\}$. If $F(\omega,0) > \frac{b^*}{K}$ almost surely, then $\tau^* < \infty$ for a.e. $\omega \in \Omega^*$. Indeed, integrating (5.15) leads to

$$-\frac{1}{F(\omega,t)} + \frac{1}{F(\omega,0)} \geq \frac{1}{2}\int_0^t \beta(\omega,t')\,\mathrm{d}t', \quad t\in[0,\tau^*) \;\; \mathbb{P}-a.s.$$

Since $F(\omega,0) > \frac{b^*}{K}$ almost surely, (5.15) means that F is increasing almost surely. Therefore we restrict the above inequality to $\omega\in\Omega^*$ and we arrive at

$$\frac{1}{F(\omega,0)} \geq \frac{1}{2}K\int_0^{\tau^*} e^{-\frac{b^*}{2}t'}dt' = \frac{K}{b^*}\left(1-e^{-\frac{b^*}{2}\tau^*}\right)$$

and hence

$$\frac{1}{F(\omega,0)} - \frac{K}{b^*}\left(1-e^{-\frac{b^*}{2}\tau^*}\right) \geq 0.$$

By the assumption $F(\omega,0) > \frac{b^*}{K}$ almost surely, we arrive at

$$\frac{K}{b^*}e^{-\frac{b^*}{2}\tau^*} \geq \frac{K}{b^*} - \frac{1}{F(0)} > 0, \qquad \text{a.e } \omega\in\Omega^*.$$

Thus $\tau^* < \infty$ a.e. on Ω^* as desired. Recalling that $\beta(\omega,t) = \mathrm{e}^{\int_0^t b(t')\mathrm{d}W_{t'} - \int_0^t \frac{b^2(t')}{2}\mathrm{d}t'}$, we have shown that

$$\mathbb{P}\{\tau^* < \infty\} \geq \mathbb{P}\{\beta(t) \geq Ke^{-\frac{b^*}{2}t} \text{ for all } t\},$$

which together with $b^2(t) < b^*$ $(t \geq 0)$ implies that

$$\mathbb{P}\{\tau^* < \infty\} \geq \mathbb{P}\{\mathrm{e}^{\int_0^t b(t')\mathrm{d}W_{t'}} > K \text{ for all } t\}.$$

The proof is now completed.

6. Proof of Theorem 1.4

In this section, we provide the proof of Theorem 1.4. As is mentioned in Remark 1.5, since we cannot get an explicit expression of the solution to (1.2), we start with constructing some approximate solutions from which (1.22) can be established. Similarly as before we divide the proof into several subsections.

6.1. Approximate solutions and actual solutions

Following [44], we construct the approximate solution as follows. First, we fix two functions $\phi, \tilde{\phi} \in C_c^\infty$ such that

$$\phi(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 2, \end{cases} \quad \text{and} \quad \tilde{\phi}(x) = 1 \text{ if } x \in \operatorname{supp} \phi. \tag{6.1}$$

Next, we construct the following sequence of approximate solutions

$$u_{m,n} = u_h + u_l, \quad m \in \{-1, 1\}, \tag{6.2}$$

where

- $u_h = u_{h,m,n}$ is the high-frequency part defined by

$$u_h = u_{h,m,n}(t,x) = n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx - mt), \quad n \in \mathbb{N}. \tag{6.3}$$

- $u_l = u_{l,m,n}$ is the low-frequency part defined as the solution to the following problem:

$$\begin{cases} \partial_t u_l + (\mathcal{H}u_l)\partial_x u_l = 0, & x \in \mathbb{R},\ t > 0, \\ u_l(0,x) = -\mathcal{H}\left(mn^{-1}\tilde{\phi}\left(\frac{x}{n^\delta}\right)\right), & x \in \mathbb{R}. \end{cases} \tag{6.4}$$

In (6.3)-(6.4), $\delta > 0$ is a parameter that will be determined later in the proof.

Let us consider the problem (1.2) with the initial data $u_{m,n}(0,x)$, i.e.,

$$\begin{cases} \mathrm{d}u + (\mathcal{H}u)u_x \mathrm{d}t = h(t,u)\mathrm{d}\mathcal{W}, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = -\mathcal{H}\left(mn^{-1}\tilde{\phi}\left(\frac{x}{n^\delta}\right)\right) + n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx), & x \in \mathbb{R}. \end{cases} \tag{6.5}$$

Since Assumption **(D)** implies Assumption **(A)**, Theorem 1.1 immediately yields that for each fixed $n \in \mathbb{N}$, (6.5) has a unique pathwise solution $(u^{m,n}, \tau^{m,n})$ such that $u^{m,n} \in C\left([0, \tau^{m,n}]; H^s\right)$ $\mathbb{P} - a.s.$ with $s > 3$.

6.2. Estimates on the errors

Substituting (6.2) into (1.2), we define the error $\mathcal{E}(\omega, t, x)$ as

$$\mathcal{E}(\omega,t,x) = u_{m,n}(t,x) - u_{m,n}(0,x) + \int_0^t (\mathcal{H}u_{m,n})\partial_x u_{m,n}\,\mathrm{d}t' - \int_0^t h(t', u_{m,n})\,\mathrm{d}\mathcal{W} \ \ \mathbb{P} - a.s.$$

By using (6.2) and (6.4), we reformulate $\mathcal{E}(\omega, t, x)$ as

$$
\begin{aligned}
&\mathcal{E}(\omega,t,x)\\
=&u_l(t,x)-u_l(0,x)+\int_0^t(\mathcal{H}u_l)\partial_x u_l\,\mathrm{d}t'+u_h(t,x)-u_h(0,x)\\
&+\int_0^t(\mathcal{H}u_l)\partial_x u_h+(\mathcal{H}u_h)(\partial_x u_l+\partial_x u_h)\,\mathrm{d}t'-\int_0^t h(t',u_{m,n})\,\mathrm{d}\mathcal{W}\\
=&u_h(t,x)-u_h(0,x)\\
&+\int_0^t(\mathcal{H}u_l)\partial_x u_h+(\mathcal{H}u_h)(\partial_x u_l+\partial_x u_h)\,\mathrm{d}t'-\int_0^t h(t',u_{m,n})\,\mathrm{d}\mathcal{W}\ \ \mathbb{P}-a.s.
\end{aligned}
\tag{6.6}
$$

The following lemma shows the decay estimate for the low-frequency part of $u_{m,n}$.

Lemma 6.1. *Let $|m|=1$, $s>3$, $\delta\in(0,2)$ and $n\gg 1$. Then there exists a $T_l>0$, independent of n, such that the initial value problem (6.4) has a unique smooth solution $u_l=u_{l,m,n}\in C([0,T_l];H^s)$ for all $n\gg 1$. Besides, for any $r>0$, there exists a constant $C=C_{r,\tilde{\phi},T_l}>0$ such that u_l satisfies*

$$
\|u_l(t)\|_{H^r}\le C|m|n^{\frac{\delta}{2}-1},\quad t\in[0,T_l]. \tag{6.7}
$$

Proof. For $|m|=1$ and any fixed $n\ge 1$, since $u_l(0,x)\in H^\infty$, by applying Theorem 1.1 with $h=0$ and deterministic initial data, we see that for any $s>3$, (6.4) has a unique (deterministic) solution $u_l=u_{l,m,n}\in C([0,T_l];H^s)$. We will show that there exists a lower bound of the existence time, i.e., there is a $T_l>0$ such that for all $n\gg 1$, $u_l=u_{l,m,n}$ exists on $[0,T_l]$ and satisfies (6.7). The proof of Lemma 6.1 consists of three main steps.

Step 1: Estimate $\|u_l(0,x)\|_{H^r}$. Let $g(x)=mn^{-1}\tilde{\phi}\left(\frac{x}{n^\delta}\right)$. For $n\gg 1$, by using (2.7), we have that

$$
\begin{aligned}
\|u_l(0,x)\|_{H^r}^2\le&\left\|mn^{-1}\tilde{\phi}\left(\frac{x}{n^\delta}\right)\right\|_{H^r}^2\\
=&\int_{\mathbb{R}}(1+|\xi|^2)^r|\widehat{g}(\xi)|^2\,\mathrm{d}\xi\\
=&m^2n^{2\delta-2}\int_{\mathbb{R}}(1+|\xi|^2)^r\left|\widehat{\tilde{\phi}}(n^\delta\xi)\right|^2\,\mathrm{d}\xi\\
=&m^2n^{\delta-2}\int_{\mathbb{R}}\left(1+\left|\frac{z}{n^\delta}\right|^2\right)^r\left|\widehat{\tilde{\phi}}(z)\right|^2\,\mathrm{d}z\\
\le&m^2n^{\delta-2}\int_{\mathbb{R}}\left(1+|z|^2\right)^r\left|\widehat{\tilde{\phi}}(z)\right|^2\,\mathrm{d}z\le Cm^2n^{\delta-2},
\end{aligned}
$$

for some constant $C = C_{r,\tilde{\phi}} > 0$. Therefore we find that

$$\|u_l(0,x)\|_{H^r} \le C|m|n^{\frac{\delta}{2}-1}.$$

Step 2: Proof of (6.7) *for* $r > 3/2$. In this case, we apply Lemma 2.2, (2.7), $H^r \hookrightarrow W^{1,\infty}$ and integration by parts to find

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_l\|_{H^r}^2 \\
&= -\int_{\mathbb{R}} D^r u_l D^r\left((\mathcal{H}u_l)\partial_x u_l\right) \mathrm{d}x \\
&\le \left|\left(D^r u_l, D^r\left((\mathcal{H}u_l)\partial_x u_l\right)\right)_{L^2}\right| \\
&\le \left|\left([D^r, \mathcal{H}u_l]\partial_x u_l, D^r u_l\right)_{L^2}\right| + \left|\left((\mathcal{H}u_l)D^r\partial_x u_l, D^r u_l\right)_{L^2}\right| \\
&\lesssim \left(\|D^r(\mathcal{H}u_l)\|_{L^2}\|\partial_x u_l\|_{L^\infty} + \|\partial_x(\mathcal{H}u_l)\|_{L^\infty}\|D^{r-1}\partial_x u_l\|_{L^2}\right)\|u_l\|_{H^r} \\
&\quad + \left|\frac{1}{2}\int_{\mathbb{R}}(\mathcal{H}u_l)\,\partial_x(D^r u_l)^2\,\mathrm{d}x\right| \\
&\lesssim \|\partial_x u_l\|_{L^\infty}\|u_l\|_{H^r}^2 + \|\partial_x(\mathcal{H}u_l)\|_{L^\infty}\|u_l\|_{H^r}^2 \\
&\lesssim \|u_l\|_{W^{1,\infty}}\|u_l\|_{H^r}^2 + \|\mathcal{H}u_l\|_{W^{1,\infty}}\|u_l\|_{H^r}^2 \\
&\le C\|u_l\|_{H^r}^3, \quad C = C_r > 0.
\end{aligned}$$

Solving the above inequality gives

$$\|u_l\|_{H^r} \le \frac{\|u_l(0)\|_{H^r}}{1 - Ct\|u_l(0)\|_{H^r}}, \quad 0 \le t < \frac{1}{C\|u_l(0)\|_{H^r}}.$$

Remember that $u_l = u_{l,m,n}$. Then we define the time interval $[0, T_{l,m,n}]$ such that

$$\|u_l\|_{H^r} \le 2\|u_l(0)\|_{H^r}, \quad t \in [0, T_{l,m,n}], \quad T_{l,m,n} = \frac{1}{2C\|u_l(0)\|_{H^r}}. \tag{6.8}$$

By *Step 1*, we have that for $|m| = 1$, $T_{l,m,n} \gtrsim \frac{1}{2Cn^{\frac{\delta}{2}-1}} \to \infty$, as $n \to \infty$. Therefore we can find a common time interval $[0, T_l]$ such that

$$\|u_l\|_{H^r} \le 2\|u_l(0)\|_{H^r} \le C|m|n^{\frac{\delta}{2}-1}, \quad t \in [0, T_l], \quad C = C_{r,\tilde{\phi}} > 0, \tag{6.9}$$

which is (6.7).

Step 3: Proof of (6.7) *for* $0 < r \le 3/2$. Applying Lemma 2.2, (2.7), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_l\|_{H^r}^2 \\
&= -\int_{\mathbb{R}} D^r u_l D^r\left((\mathcal{H}u_l)\partial_x u_l\right)\mathrm{d}x \\
&\le \left|\left(D^r u_l, D^r\left((\mathcal{H}u_l)\partial_x u_l\right)\right)_{L^2}\right| \\
&\le \left|\left([D^r, \mathcal{H}u_l]\partial_x u_l, D^r u_l\right)_{L^2}\right| + \left|\left((\mathcal{H}u_l)D^r\partial_x u_l, D^r u_l\right)_{L^2}\right| \\
&\lesssim \left(\|D^r(\mathcal{H}u_l)\|_{L^2}\|\partial_x u_l\|_{L^\infty} + \|\partial_x(\mathcal{H}u_l)\|_{L^\infty}\|D^{r-1}\partial_x u_l\|_{L^2}\right)\|u_l\|_{H^r} \\
&\quad + \left|\frac{1}{2}\int_{\mathbb{R}}(\mathcal{H}u_l)\,\partial_x(D^r u_l)^2\,\mathrm{d}x\right| \\
&\lesssim \|\partial_x u_l\|_{L^\infty}\|u_l\|_{H^r}^2 + \|\partial_x(\mathcal{H}u_l)\|_{L^\infty}\|u_l\|_{H^r}^2.
\end{aligned}
$$

It follows from the embedding $H^{r+\frac{3}{2}} \hookrightarrow W^{1,\infty}$ that

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_l\|_{H^r}^2 &\lesssim \|\partial_x u_l\|_{L^\infty}\|u_l\|_{H^r}^2 + \|\partial_x(\mathcal{H}u_l)\|_{L^\infty}\|u_l\|_{H^r}^2 \\
&\lesssim \|u_l\|_{W^{1,\infty}}\|u_l\|_{H^r}^2 + \|\mathcal{H}u_l\|_{W^{1,\infty}}\|u_l\|_{H^r}^2 \\
&\lesssim \|u_l\|_{H^{r+\frac{3}{2}}}\|u_l\|_{H^r}^2 + \|\mathcal{H}u_l\|_{H^{r+\frac{3}{2}}}\|u_l\|_{H^r}^2 \\
&\lesssim \|u_l\|_{H^{r+\frac{3}{2}}}\|u_l\|_{H^r}^2.
\end{aligned}
$$

Using the conclusion of *Step 2* for $r + \frac{3}{2} > \frac{3}{2}$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|u_l\|_{H^r} \lesssim \|u_l\|_{H^r}\|u_l(0)\|_{H^{r+\frac{3}{2}}}, \quad t \in [0, T_l],
$$

and hence

$$
\|u_l(t)\|_{H^r} \lesssim \|u_l(0)\|_{H^r} + \int_0^t \|u_l\|_{H^r}\|u_l(0)\|_{H^{r+\frac{3}{2}}}\,\mathrm{d}t, \quad t \in [0, T_l].
$$

Applying Grönwall's inequality to the above estimate, we have that

$$
\|u_l\|_{H^r} \lesssim \|u_l(0)\|_{H^r}\exp\left\{\|u_l(0)\|_{H^{r+\frac{3}{2}}}T_l\right\}, \quad t \in [0, T_l].
$$

Since $\delta \in (0, 2)$, we can infer from *Step 1* that $\exp\left\{\|u_l(0)\|_{H^{r+\frac{3}{2}}}T_l\right\} < C(r, \tilde{\phi}, T_l)$ for some constant $C(r, \tilde{\phi}, T_l) > 0$. Therefore we see that there exists a constant $C = C_{r,\tilde{\phi},T_l} > 0$ such that

$$\|u_l\|_{H^r} \le C|m|n^{\frac{\delta}{2}-1}, \quad t \in [0, T_l],$$

concluding the desired estimate (6.7). □

The above result implies that the H^s-norm of u_l, the low-frequency part of the approximate solution defined by (6.2), is decaying. For the high-frequency part u_h, due to Lemma 2.5, its H^s-norm is bounded. To sum up, let T_l be given in Lemma 6.1, for any $s > 0$, there is a constant $M = M_{s,\tilde{\phi},\phi,T_l} > 0$ such that

$$\|u_{m,n}(t)\|_{H^s} \lesssim M, \quad t \in [0, T_l]. \tag{6.10}$$

Moreover, although not strictly necessary, we can infer from (6.9) that M can be independent of T_l when $s > 3/2$.

6.2.1. Estimating the error $\mathcal{E}$

Recall (6.6). By using (6.1), we have that $\phi = \tilde{\phi}\phi$. Then by (6.3) and $u_l(0, x)$ in (6.4), we see that for $m \in \{-1, 1\}$,

$$\begin{aligned}
u_h(t, x) - u_h(0, x)
&= n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx - mt) - n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx)\\
&= m^{-1}m\tilde{\phi}\left(\frac{x}{n^\delta}\right)n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx - mt)\\
&\quad - m^{-1}m\tilde{\phi}\left(\frac{x}{n^\delta}\right)n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx)\\
&= m^{-1}\left(\mathcal{H}u_l(0, x)\right)n^{1-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx - mt)\\
&\quad - m^{-1}\left(\mathcal{H}u_l(0, x)\right)n^{1-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx)\\
&= \int_0^t \left(\mathcal{H}u_l(0, x)\right)n^{1-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\sin(nx - mt')\,\mathrm{d}t'.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\int_0^t (\mathcal{H}u_l)\partial_x u_h \,\mathrm{d}t' &= -\int_0^t (\mathcal{H}u_l)(t')n^{1-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\sin(nx - mt')\,\mathrm{d}t'\\
&\quad + \int_0^t (\mathcal{H}u_l)(t')n^{-\frac{3\delta}{2}-s}\partial_x\phi\left(\frac{x}{n^\delta}\right)\cos(nx - mt')\,\mathrm{d}t'.
\end{aligned} \tag{6.11}$$

Thus, (6.6) becomes

$$\begin{aligned}\mathcal{E}(\omega,t,x)=&\int_0^t[(\mathcal{H}u_l)(0)-(\mathcal{H}u_l)(t')]n^{1-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\sin(nx-mt')\,\mathrm{d}t'\\&+\int_0^t(\mathcal{H}u_l)n^{-\frac{3\delta}{2}-s}\partial_x\phi\left(\frac{x}{n^\delta}\right)\cos(nx-mt')\,\mathrm{d}t'\\&+\int_0^t(\mathcal{H}u_h)(\partial_xu_l+\partial_xu_h)\,\mathrm{d}t'-\int_0^t h(t',u_{m,n})\,\mathrm{d}\mathcal{W}\\=&\int_0^t E\,\mathrm{d}t'-\int_0^t h(t',u_{m,n})\,\mathrm{d}\mathcal{W}\ \ \mathbb{P}-a.s.,\end{aligned}\tag{6.12}$$

where

$$\begin{aligned}E=E(\omega,t,x)=&[(\mathcal{H}u_l)(0)-(\mathcal{H}u_l)(t)]n^{1-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\sin(nx-mt)\\&+(\mathcal{H}u_l)n^{-\frac{3\delta}{2}-s}\partial_x\phi\left(\frac{x}{n^\delta}\right)\cos(nx-mt)+(\mathcal{H}u_h)(\partial_xu_l+\partial_xu_h).\end{aligned}\tag{6.13}$$

Now we shall estimate the H^{σ_0}-norm of the error $\mathcal{E}$, where σ_0 is given in Assumption **(D)**. Actually, we will show that the H^{σ_0}-norm of $\mathcal{E}$ is decaying.

Lemma 6.2. *Let $n \gg 1$, $s > 3$, $\frac{3}{4} < \delta < 1$. Let T_l be given in Lemma 6.1, and σ_0 be given in Assumption **(D)**. Let*

$$r_s=-s-1+\sigma_0+\delta.\tag{6.14}$$

Then $r_s<0$ and the error $\mathcal{E}$ given by (6.12) satisfies

$$\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|^2_{H^{\sigma_0}}\le Cn^{2r_s},$$

where $C=C(\sigma_0,\tilde{\phi},\phi,T_l)>0$ is a constant independent of n.

Proof. It is obvious by construction that r_s given by (6.14) is negative. Combining (6.13), the embedding $H^{\sigma_0}\hookrightarrow L^\infty$ and Lemmas 6.1 and 2.5, we find that for $t\in[0,T_l]$,

$$\begin{aligned}\|E\|_{H^{\sigma_0}}\le&\left\|[(\mathcal{H}u_l)(0)-(\mathcal{H}u_l)(t)]n^{1-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\sin(nx-mt)\right\|_{H^{\sigma_0}}\\&+\left\|(\mathcal{H}u_l)n^{-\frac{3\delta}{2}-s}\partial_x\phi\left(\frac{x}{n^\delta}\right)\cos(nx-mt)\right\|_{H^{\sigma_0}}\\&+\|(\mathcal{H}u_h)\partial_xu_l\|_{H^{\sigma_0}}+\|(\mathcal{H}u_h)\partial_xu_h\|_{H^{\sigma_0}}\\\lesssim&\, n^{1-\frac{\delta}{2}-s}\|(\mathcal{H}u_l)(0)-(\mathcal{H}u_l)(t)\|_{H^{\sigma_0}}\left\|\phi\left(\frac{x}{n^\delta}\right)\sin(nx-mt)\right\|_{H^{\sigma_0}}\end{aligned}$$

$$
\begin{aligned}
&+ n^{-\frac{3\delta}{2}-s}\,\|\mathcal{H}u_l\|_{H^{\sigma_0}}\left\|\partial_x\phi\left(\frac{x}{n^\delta}\right)\cos(nx-mt')\right\|_{H^{\sigma_0}}\\
&+\|(\mathcal{H}u_h)\partial_x u_l\|_{H^{\sigma_0}}+\|(\mathcal{H}u_h)\partial_x u_h\|_{H^{\sigma_0}}\\
\lesssim\,& n^{1-s+\sigma_0}\,\|(\mathcal{H}u_l)(0)-(\mathcal{H}u_l)(t)\|_{H^{\sigma_0}}+n^{-s-1+\sigma_0-\frac{1}{2}\delta}\\
&+\|(\mathcal{H}u_h)\partial_x u_l\|_{H^{\sigma_0}}+\|(\mathcal{H}u_h)\partial_x u_h\|_{H^{\sigma_0}}\\
\lesssim\,& n^{1-s+\sigma_0}\,\|(\mathcal{H}u_l)(0)-(\mathcal{H}u_l)(t)\|_{H^{\sigma_0}}+n^{r_s}\\
&+\|(\mathcal{H}u_h)\partial_x u_l\|_{H^{\sigma_0}}+\|(\mathcal{H}u_h)\partial_x u_h\|_{H^{\sigma_0}}\,.
\end{aligned}
\tag{6.15}
$$

For $\|(\mathcal{H}u_l)(0)-(\mathcal{H}u_l)(t)\|_{H^{\sigma_0}}$, it follows from the fundamental theorem of calculus and $H^{\sigma_0}\hookrightarrow L^\infty$ that for $t\in[0,T_l]$,

$$
\begin{aligned}
\|(\mathcal{H}u_l)(0)-(\mathcal{H}u_l)(t)\|_{H^{\sigma_0}} &\lesssim \|u_l(0)-u_l(t)\|_{H^{\sigma_0}}\\
&=\left\|\int_0^t \partial_t u_l(t')\,\mathrm{d}t'\right\|_{H^{\sigma_0}}\\
&\lesssim \int_0^t \|(\mathcal{H}u_l)\partial_x u_l\|_{H^{\sigma_0}}\,\mathrm{d}t'\\
&\lesssim \int_0^t \|u_l\|^2_{H^{\sigma_0+1}}\,\mathrm{d}t' \lesssim n^{\delta-2}T_l,
\end{aligned}
$$

where we used (6.4) with $t\in[0,T_l]$, Lemma 6.1 and the embedding $H^{\sigma_0+1}\hookrightarrow W^{1,\infty}$. Therefore,

$$
n^{1-s+\sigma_0}\,\|(\mathcal{H}u_l)(0)-(\mathcal{H}u_l)(t)\|_{H^{\sigma_0}}\lesssim n^{1-s+\sigma_0+\delta-2}T_l=n^{r_s}T_l,\quad t\in[0,T_l]. \tag{6.16}
$$

Next, applying Lemma 6.1 and 2.5, we have for $t\in[0,T_l]$,

$$
\begin{aligned}
\|(\mathcal{H}u_h)\partial_x u_l\|_{H^{\sigma_0}} &\lesssim \|u_h\|_{H^{\sigma_0}}\|u_l\|_{H^{\sigma_0+1}}\\
&\lesssim n^{-s+\sigma_0}n^{\frac{\delta}{2}-1}=n^{-s+\sigma_0+\frac{\delta}{2}-1}\lesssim n^{r_s},
\end{aligned}
\tag{6.17}
$$

$$
\begin{aligned}
\|(\mathcal{H}u_h)\partial_x u_h\|_{H^{\sigma_0}} &\lesssim \|u_h\|_{H^{\sigma_0}}\|u_h\|_{H^{\sigma_0+1}}\\
&\lesssim n^{-s+\sigma_0}n^{-s+\sigma_0+1}=n^{-2s+2\sigma_0+1}\lesssim n^{r_s}.
\end{aligned}
\tag{6.18}
$$

Here in (6.18) we used the assumption $\sigma_0\in(3/2,7/4)$ to guarantee $n^{-2s+2\sigma_0+1}\lesssim n^{r_s}$. Inserting (6.16), (6.17) and (6.18) into (6.15), we finally obtain

$$
\|E\|_{H^{\sigma_0}}\lesssim n^{r_s},\quad t\in[0,T_l]. \tag{6.19}
$$

With (6.19) at hand, we are in the position to estimate $\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|^2_{H^{\sigma_0}}$. Invoking Itô formula in (6.12) leads to

$$\|\mathcal{E}(t,x)\|_{H^{\sigma_0}}^2 \leq \left| -2\int_0^t (h(t',u_{m,n})\,\mathrm{d}\mathcal{W},\mathcal{E})_{H^{\sigma_0}} \right| + 2\int_0^t |(E,\mathcal{E})_{H^{\sigma_0}}|\,\mathrm{d}t' + \int_0^t \|h(t',u_{m,n})\|_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})}^2\,\mathrm{d}t'.$$

Taking supremum with respect to $t \in [0, T_l]$ and using the BDG inequality, we can find some $\overline{C} > 0$ such that

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|_{H^{\sigma_0}}^2\\ &\leq \frac{1}{2}\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|_{H^{\sigma_0}}^2 + \overline{C}\mathbb{E}\int_0^{T_l}\|h(t,u_{m,n})\|_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})}^2\,\mathrm{d}t + \overline{C}\int_0^{T_l}\left[\mathbb{E}\|E\|_{H^{\sigma_0}}^2 + \mathbb{E}\|\mathcal{E}(t)\|_{H^{\sigma_0}}^2\right]\mathrm{d}t.\end{aligned}$$

By virtue of (6.19), we arrive at

$$\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|_{H^{\sigma_0}}^2 \lesssim T_l n^{2r_s} + \mathbb{E}\int_0^{T_l}\|h(t,u_{m,n})\|_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})}^2\,\mathrm{d}t + \int_0^{T_l}\mathbb{E}\sup_{t'\in[0,t]}\|\mathcal{E}(t')\|_{H^{\sigma_0}}^2\,\mathrm{d}t.$$

Now, we estimate $\|h(t,u_{m,n})\|_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})}$. For any fixed $s > 3$, on account of Assumption **(D)**, Lemmas 2.5 and 6.1, we can pick $\kappa > 2(1 + \frac{s-\sigma_0-1}{2-\delta})$ such that

$$\|h(t,u_{m,n})\|_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})}^2 \lesssim \left(\mathrm{e}^{\frac{-1}{\|u_{m,n}\|_{H^{\sigma_0}}}}\right)^2 \lesssim \left(n^{-s+\sigma_0} + n^{\frac{\delta}{2}-1}\right)^{2\kappa} \lesssim n^{2r_s},$$

which gives

$$\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|_{H^{\sigma_0}}^2 \lesssim T_l n^{2r_s} + \int_0^{T_l}\mathbb{E}\sup_{t'\in[0,t]}\|\mathcal{E}(t')\|_{H^{\sigma_0}}^2\,\mathrm{d}t.$$

Obviously, for each $n \geq 1$, $\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|_{H^{\sigma_0}}^2$ is finite and $T_l > 0$ is fixed. Then by the Grönwall inequality, we have

$$\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|_{H^{\sigma_0}}^2 \leq Cn^{2r_s}.$$

The proof is completed. □

6.2.2. Estimating $u_{m,n} - u^{m,n}$

Recall the approximate solutions $u_{m,n}$ given by (6.2). Then we have the following estimates on the difference between the actual solutions and the approximate solutions.

Lemma 6.3. *Let $s > 3$, $\frac{3}{4} < \delta < 1$, σ_0 be given in Assumption **(D)** and $r_s < 0$ be given in (6.14). For any $R > 1$, we define*

$$\tau_R^{m,n} := \inf\{t > 0 : \|u^{m,n}\|_{H^s} > R\}. \tag{6.20}$$

Then for $n \gg 1$ and $T_l > 0$ given in Lemma 6.1,

$$\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|u_{m,n} - u^{m,n}\|^2_{H^{\sigma_0}} \le Cn^{2r_s}, \tag{6.21}$$

$$\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|u_{m,n} - u^{m,n}\|^2_{H^{2s-\sigma_0}} \le Cn^{2s-2\sigma_0}, \tag{6.22}$$

where $C = C(s, \sigma_0, \tilde{\phi}, \phi, T_l, R) > 0$ is a constant independent of n.

Proof. Let $v = v_{m,n} = u_{m,n} - u^{m,n}$. Then v satisfies $v(0) = 0$ and

$$v(t) + \int_0^t \left((\mathcal{H}v)\,\partial_x u_{m,n} + \left(\mathcal{H}u^{m,n}\right)\partial_x v\right) \mathrm{d}t' = -\int_0^t h(t', u^{m,n})\,\mathrm{d}\mathcal{W} + \int_0^t E\,\mathrm{d}t',$$

where (6.12) is used. For $T_l > 0$, we use the Itô formula to find

$$\begin{aligned}\|v(t)\|^2_{H^{\sigma_0}} = &-2\int_0^t \left(h(t', u^{m,n})\,\mathrm{d}\mathcal{W}, v\right)_{H^{\sigma_0}} + 2\int_0^t (E, v)_{H^{\sigma_0}}\,\mathrm{d}t' - 2\int_0^t \left((\mathcal{H}v)\,\partial_x u_{m,n}, v\right)_{H^{\sigma_0}}\,\mathrm{d}t' \\ &- 2\int_0^t \left(\left(\mathcal{H}u^{m,n}\right)\partial_x v, v\right)_{H^{\sigma_0}}\,\mathrm{d}t' + \int_0^t \|h(t', u^{m,n})\|^2_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})}\,\mathrm{d}t'.\end{aligned}$$

Taking supremum with respect to $t \in [0, T_l \wedge \tau_R^{m,n}]$, and then using the BDG inequality yields that for some $C, \overline{C} > 0$,

$$\begin{aligned}&\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|v(t)\|^2_{H^{\sigma_0}} \\ &\le C\mathbb{E}\left(\int_0^{T_l\wedge\tau_R^{m,n}} \|v\|^2_{H^{\sigma_0}}\|h(t, u^{m,n})\|^2_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})}\,\mathrm{d}t\right)^{1/2} + 2\mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} |(E, v)_{H^{\sigma_0}}|\,\mathrm{d}t \\ &+ 2\mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} \left|\left((\mathcal{H}v)\,\partial_x u_{m,n}, v\right)_{H^{\sigma_0}}\right|\mathrm{d}t + 2\mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} \left|\left(\left(\mathcal{H}u^{m,n}\right)\partial_x v, v\right)_{H^{\sigma_0}}\right|\mathrm{d}t\end{aligned}$$

$$+\mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} \|h(t,u^{m,n})\|^2_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})}\,\mathrm{d}t$$

$$\le \frac{1}{2}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_R^{m,n}]}\|v(t)\|^2_{H^{\sigma_0}} + \overline{C}\mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} |(E,v)_{H^{\sigma_0}}|\,\mathrm{d}t$$

$$+\overline{C}\mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} \left|\left((\mathcal{H}v)\,\partial_x u_{m,n}, v\right)_{H^{\sigma_0}}\right|\mathrm{d}t + \overline{C}\mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} \left|\left(\left(\mathcal{H}u^{m,n}\right)\partial_x v, v\right)_{H^{\sigma_0}}\right|\mathrm{d}t$$

$$+\overline{C}\mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} \|h(t,u^{m,n})\|^2_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})}\,\mathrm{d}t.$$

Recall that (6.10) gives $\|u_{m,n}\|_{H^s}\lesssim M_{s,\tilde{\phi},\phi,T_l}$ on $t\in[0,T_l\wedge\tau_R^{m,n}]$. Hence we can infer from Assumption **(D)** that for some $\overline{C}>0$

$$\begin{aligned}\|h(t,u^{m,n})\|^2_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})} &\lesssim \|h(t,u_{m,n})\|^2_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})} + \|h(t,u_{m,n})-h(t,u^{m,n})\|^2_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})}\\ &\le \overline{C}\left(\mathrm{e}^{\frac{-1}{\|u_{m,n}\|_{H^{\sigma_0}}}}\right)^2 + q(\overline{C})\|v\|^2_{H^{\sigma_0}},\ \ t\in[0,T_l\wedge\tau_R^{m,n}]\ \ \mathbb{P}-a.s.,\end{aligned}$$

where $q(\cdot)$ is given in (1.8). As a result, for any fixed $s>3$, by applying Lemmas 2.5 and 6.1 again, we can pick $\lambda>2(1+\frac{s-\sigma_0-1}{2-\delta})$ to derive

$$\begin{aligned}\|h(t,u^{m,n})\|^2_{\mathcal{L}_2(\mathcal{U};H^{\sigma_0})} &\lesssim \|u_{m,n}\|^{2\lambda}_{H^{\sigma_0}} + \|v\|^2_{H^{\sigma_0}}\\ &\lesssim \left(n^{-s+\sigma_0}+n^{\frac{\delta}{2}-1}\right)^{2\lambda} + \|v\|^2_{H^{\sigma_0}}\\ &\lesssim n^{2r_s} + \|v\|^2_{H^{\sigma_0}},\ \ t\in[0,T_l\wedge\tau_R^{m,n}]\ \ \mathbb{P}-a.s.\end{aligned}$$

Via (6.19), we have

$$2|(E,v)_{H^{\sigma_0}}| \le 2\|E\|_{H^{\sigma_0}}\|v\|_{H^{\sigma_0}} \lesssim \|E\|^2_{H^{\sigma_0}} + \|v\|^2_{H^{\sigma_0}} \lesssim n^{2r_s} + \|v\|^2_{H^{\sigma_0}}.$$

Using Lemma 2.2, (6.10), (2.7), integration by parts, and the embedding $H^s\hookrightarrow H^{\sigma_0}\hookrightarrow W^{1,\infty}$, we obtain that almost surely

$$\begin{aligned}\left|\left((\mathcal{H}v)\,\partial_x u_{m,n}, v\right)_{H^{\sigma_0}}\right| &\lesssim \|\mathcal{H}v\|_{H^{\sigma_0}}\|u_{m,n}\|_{H^{\sigma_0+1}}\|v\|_{H^{\sigma_0}} \lesssim \|u_{m,n}\|_{H^s}\|v\|^2_{H^{\sigma_0}}\\ &\lesssim \|v\|^2_{H^{\sigma_0}},\ \ t\in[0,T_l\wedge\tau_R^{m,n}],\end{aligned}$$

and

$$\begin{aligned}
&\left|\left(\left(\mathcal{H}u^{m,n}\right)\partial_x v, v\right)_{H^{\sigma_0}}\right| \\
&= \left|([D^{\sigma_0}, \mathcal{H}u^{m,n}]\partial_x v, D^{\sigma_0}v)_{L^2} + (\mathcal{H}u^{m,n}D^{\sigma_0}\partial_x v, D^{\sigma_0}v)_{L^2}\right| \\
&\lesssim \left(\|D^{\sigma_0}\mathcal{H}u^{m,n}\|_{L^2}\|\partial_x v\|_{L^\infty} + \|\partial_x\mathcal{H}u^{m,n}\|_{L^\infty}\|D^{\sigma_0-1}\partial_x v\|_{L^2}\right)\|v\|_{H^{\sigma_0}} + \|\partial_x\mathcal{H}u^{m,n}\|_{L^\infty}\|v\|^2_{H^{\sigma_0}} \\
&\lesssim \|\mathcal{H}u^{m,n}\|_{H^s}\|v\|^2_{H^{\sigma_0}} \lesssim \|u^{m,n}\|_{H^s}\|v\|^2_{H^{\sigma_0}} \lesssim \|v\|^2_{H^{\sigma_0}}, \ t \in [0, T_l \wedge \tau_R^{m,n}].
\end{aligned}$$

To sum up, we obtain that

$$\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_R^{m,n}]}\|v(t)\|^2_{H^{\sigma_0}} \lesssim T_l n^{2r_s} + \int_0^{T_l}\mathbb{E}\sup_{t'\in[0,t\wedge\tau_R^{m,n}]}\|v(t')\|^2_{H^{\sigma_0}}\,\mathrm{d}t.$$

Using the Grönwall inequality, we obtain (6.21).

Now we prove (6.22). To this end, we first notice that $2s-\sigma_0 > 3$ and $u^{m,n}$ is the unique solution to (6.5). Then, similar to (3.27), we can use (6.20) and Assumption **(D)** to find for each fixed $n \in \mathbb{N}$ that

$$\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_R^{m,n}]}\|u^{m,n}\|^2_{H^{2s-\sigma_0}} \lesssim \mathbb{E}\|u_{m,n}(0)\|^2_{H^{2s-\sigma_0}} + \int_0^{T_l}\mathbb{E}\sup_{t'\in[0,t\wedge\tau_R^{m,n}]}\|u^{m,n}\|^2_{H^{2s-\sigma_0}}\,\mathrm{d}t.$$

Using the Grönwall inequality and Lemmas 6.1 and 2.5, we have

$$\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_R^{m,n}]}\|u^{m,n}\|^2_{H^{2s-\sigma_0}} \lesssim \mathbb{E}\|u_{m,n}(0)\|^2_{H^{2s-\sigma_0}} \lesssim (n^{\frac{\delta}{2}-1} + n^{s-\sigma_0})^2 \lesssim n^{2s-2\sigma_0}, \quad n \geq 1.$$

Hence, by Lemmas 6.1 and 2.5 again, we arrive at

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_R^{m,n}]}\|u^{m,n} - u_{m,n}\|^2_{H^{2s-\sigma_0}} \\
&\leq 2\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_R^{m,n}]}\|u^{m,n}\|^2_{H^{2s-\sigma_0}} + 2\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_R^{m,n}]}\|u_{m,n}\|^2_{H^{2s-\sigma_0}} \leq Cn^{2s-2\sigma_0}, \quad n \geq 1.
\end{aligned}$$

Therefore, we complete the proof. □

6.3. Concluding the proof of Theorem 1.4

To begin with, we have the following property:

Lemma 6.4. *Let Assumption **(D)** hold true. Suppose that for some $R_0 \gg 1$, the R_0-exiting time of the zero solution to (1.2) is strongly stable. Then we have*

$$\lim_{n\to\infty}\tau_{R_0}^{m,n} = \infty\ \mathbb{P} - a.s. \tag{6.23}$$

Proof. By Assumption **(D)**, the unique solution with zero initial data to (1.2) is zero. Now we notice that for all $s' < s$, $\lim_{n\to\infty} \|u_{m,n}(0)\|_{H^{s'}} = \lim_{n\to\infty} \|u_{m,n}(0) - 0\|_{H^{s'}} = 0$ and the R_0-exiting time of the zero solution is ∞. Then the assumption that R_0-exiting time of the zero solution to (1.2) is strongly stable immediately implies (6.23). □

Proof of Theorem 1.4. We only need to show that if the R_0-exiting time is strongly stable at the zero solution for some $R_0 \gg 1$, then $\{u^{-1,n}\}$ and $\{u^{1,n}\}$ are two sequences of pathwise solutions such that (1.19), (1.20), (1.21) and (1.22) are satisfied.

For each $n > 1$ and for fixed $R_0 \gg 1$, Lemmas 6.1, 2.5 and (6.20) give $\mathbb{P}\{\tau_{R_0}^{m,n} > 0\} = 1$, and Lemma 6.4 implies (1.19). Then, it follows from Theorem 1.1 and (6.20) that $u^{m,n} \in C([0, \tau_{R_0}^{m,n}]; H^s)$ $\mathbb{P} - a.s.$ and (1.20) holds true. Next, we check (1.21). By interpolation, we have

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]} \|u_{m,n} - u^{m,n}\|_{H^s} \\
&\lesssim \left(\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]} \|u_{m,n} - u^{m,n}\|_{H^{\sigma_0}}\right)^{\frac{1}{2}} \left(\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]} \|u_{m,n} - u^{m,n}\|_{H^{2s-\sigma_0}}\right)^{\frac{1}{2}} \\
&\lesssim \left(\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]} \|u_{m,n} - u^{m,n}\|^2_{H^{\sigma_0}}\right)^{\frac{1}{4}} \left(\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]} \|u_{m,n} - u^{m,n}\|^2_{H^{2s-\sigma_0}}\right)^{\frac{1}{4}}.
\end{aligned}$$

Combining Lemma 6.3 and the above estimate yields

$$\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]} \|u_{m,n} - u^{m,n}\|_{H^s} \lesssim n^{\frac{1}{4}\cdot 2r_s} \cdot n^{\frac{1}{4}\cdot(2s-2\sigma_0)} = n^{r_s'}, \tag{6.24}$$

where r_s is defined by (6.14) and

$$0 > r_s' = r_s \cdot \frac{1}{2} + (s - \sigma_0) \cdot \frac{1}{2} = \frac{\delta - 1}{2}.$$

Since $r_s' < 0$, we can deduce that

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]} \|u_{m,n} - u^{m,n}\|_{H^s} = 0. \tag{6.25}$$

Since $\delta < 1$, we have

$$\begin{aligned}
\|u^{-1,n}(0) - u^{1,n}(0)\|_{H^s} &= \|u_{-1,n}(0) - u_{1,n}(0)\|_{H^s} \\
&= \left\|2\mathcal{H}\left(n^{-1}\tilde{\phi}\left(\frac{x}{n^\delta}\right)\right)\right\|_{H^s} \lesssim \left\|n^{-1}\tilde{\phi}\left(\frac{x}{n^\delta}\right)\right\|_{H^s} \\
&\lesssim n^{\frac{\delta}{2}-1} \left\|\tilde{\phi}\right\|_{H^s} \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Since $u^{-1,n}(0)$ and $u^{1,n}(0)$ are deterministic, the above estimate implies that (1.21) holds true.

Now we prove (1.22). Let $T_l > 0$ be given in Lemma 6.1. We can infer from (6.25) that

$$\begin{aligned}
&\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\|u^{-1,n}(t)-u^{1,n}(t)\|_{H^s}\\
\gtrsim&\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\|u_{-1,n}(t)-u_{1,n}(t)\|_{H^s}\\
&-\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\|u_{-1,n}(t)-u^{-1,n}(t)\|_{H^s}\\
&-\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\|u_{1,n}(t)-u^{1,n}(t)\|_{H^s}\\
\gtrsim&\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\|u_{-1,n}(t)-u_{1,n}(t)\|_{H^s}.
\end{aligned}$$

Then it follows from the construction of $u_{m,n}$, Lemmas 6.1, 2.5 and 6.4, and Fatou's lemma that

$$\begin{aligned}
&\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\|u_{-1,n}(t)-u_{1,n}(t)\|_{H^s}\\
=&\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\left\|-2n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\sin(nx)\sin(t)+[u_{l,-1,n}(t)-u_{l,1,n}(t)]\right\|_{H^s}\\
\gtrsim&\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}n^{-\frac{\delta}{2}-s}\left\|\phi\left(\frac{x}{n^\delta}\right)\sin(nx)\right\|_{H^s}|\sin t|-\liminf_{n\to\infty}n^{\frac{\delta}{2}-1}\\
\gtrsim&\sup_{t\in[0,T_l]}|\sin t|,
\end{aligned}$$

which is (1.22). The proof is therefore completed. □

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