KMS STATES OF QUASI-FREE DYNAMICS ON C^*-ALGEBRAS OF PRODUCT SYSTEMS OVER RIGHT LCM MONOIDS

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ABSTRACT. We generalise recent results of Afsar, Larsen and Neshveyev for product systems over quasi-lattice orders by showing that the equilibrium states of quasi-free dynamics on the Nica-Toeplitz C^*-algebras of product systems over right LCM monoids must satisfy a positivity condition encoded in a system of inequalities satisfied by their restrictions to the coefficient algebra. We prove that the reduction of this positivity condition to a finite subset of inequalities is valid for a wider class of monoids that properly includes finite-type Artin monoids, answering a question left open in their work. Our main technical tool is a combinatorially generated tree modelled on a recent construction developed by Boyu Li for dilations of contractive representations. We also obtain a reduction of the positivity condition to inequalities arising from a certain minimal subset that may not be finite but has the advantage of holding for all Noetherian right LCM monoids, and we present an example, arising from a finite-type Artin monoid, that exhibits a gap in its inverse temperature space.

1. INTRODUCTION

A very rich context for the study of KMS states is provided by the C^*-algebras $\mathcal{T}(X)$ and $\mathcal{O}(X)$ associated to a C^*-correspondence X by Pimsner [26]. These algebras can be endowed with various quasi-free dynamics, and one of the key features of the resulting C^*-dynamical systems is the relation between their KMS states and the traces of the coefficient algebra [17]. Since the C^*-algebras associated to product systems of C^*-correspondences are natural generalisations of Pimsner algebras [12, 28], it is natural to wonder about a similar study for product systems.

Along these lines, KMS states of Nica-Toeplitz C^*-algebras and their Cuntz-Pimsner quotients for special cases of product systems have been studied in [13, 1, 6]. More recently, partly motivated by the connection between combinatorial and analytic properties of quasi-lattice ordered monoids initially explored in [6], Afsar, Larsen and Neshveyev have completely characterised the traces of the coefficient algebra of a product system X that extend to KMS states of $\mathcal{NT}(X)$ for all compactly aligned product systems over quasi-lattice ordered semigroups [3]. Their characterisation provides a unified perspective of several special cases previously studied and consists of a system of inequalities that amount to a positivity condition for induced traces. In principle such a system involves infinitely many inequalities. But for product systems over right-angled Artin monoids A_M^+, the positivity condition reduces to a more tractable system of inequalities that depends only on the canonical generating set of A_M^+ and is finite whenever A_M^+ is finitely generated [3, Section 9]. This immediately raises the concrete question of whether a similar result holds for other Artin monoids or indeed other monoids P.

In this article we show that this 'reduction of positivity to generators' is valid for finite-type Artin monoids, as well as other Artin monoids that are neither right-angled nor finite-type. Along the way, we also obtain a simplified characterisation, valid when P is a Noetherian right LCM monoid, in which the inequalities arise from a certain minimal subset P_{inf} of P that was recently introduced in [20] by B. Li in the context of regular dilations of contractive representations of monoids.

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In order to describe our results in more precise terms, it is helpful to recall the abstract characterisation of KMS states in terms of the systems of inequalities from [3]. We refer to Section 2 below for the necessary definitions.

Suppose that (G, P) is a quasi-lattice ordered group, that $X = \{X_p\}_{p \in P}$ is a compactly aligned product system of C^*-correspondences over P with $X_e = A$, that the C^*-algebra $\mathcal{NT}(X)$ is endowed with a time evolution defined by a homomorphism $N : P \to (0, \infty)$ and that $\beta \in \mathbb{R}$. For a finite subset K of P we denote by $\vee K \in P \cup \{\infty\}$ the smallest upper bound of the elements in K. If τ is a tracial state on A and $p \in P$, then Tr_τ^p denotes the trace on A induced from τ through the bimodule X_p as in [17, Theorem 1.1]. This is the set-up of [3, Theorem 2.1], which shows that the inequality

$$\tau(a) + \sum_{\emptyset \neq K \subset J} (-1)^{|K|} N(\vee K)^{-\beta} \,\mathrm{Tr}_\tau^{\vee K}(a) \geq 0, \qquad a \in A_+, \quad J \subset P \backslash \{e\}, \tag{1.1}$$

must hold for the restriction $\tau = \phi|_A$ of a KMS_β state ϕ on $\mathcal{NT}(X)$ for every finite J. By convention, the summands corresponding to $\vee K = \infty$ are zero. Moreover every tracial state of A satisfying (1.1) is the restriction of a KMS_β-state ϕ on $\mathcal{NT}(X)$. In general, the restriction map may not be one to one, but [3, Theorem 5.1] shows that the gauge invariant KMS_β-states are in affine bijection with the tracial states of A satisfying the positivity condition (1.1).

Inequalities of the form (1.1) are often referred to as subinvariance conditions and have appeared before in the study of KMS states: the case of a single correspondence X requires one inequality [17, Theorem 2.1]; the case of a product system arising from finitely many $*$-commuting local homeomorphisms of a compact space involves finitely many inequalities [1, Proposition 4.1].

In principle, by [3, Theorem 5.1], the gauge-invariant KMS_β-states of the dynamics on $\mathcal{NT}(X)$ arising from a homomorphism N can be determined by finding all the traces on A that satisfy the positivity condition. Needless to say, this procedure may be unmanageable in practice, so it becomes important to identify cases in which a smaller system of inequalities suffices.

Such a simplification was obtained in [3, Theorem 9.1] for the case in which P is a right-angled Artin monoid A_M^+. Specifically, denoting the set of canonical generators of A_M^+ by S, in this case the positivity condition (1.1) is equivalent to

$$\tau(a) + \sum_{\substack{\emptyset \neq K \subset J \\ K \in cl(S)}} (-1)^{|K|} N(\vee K)^{-\beta} \,\mathrm{Tr}_\tau^{\vee K}(a) \geq 0 \text{ for all finite } J \subset S \text{ and } a \in A_+, \tag{1.2}$$

where the sum is over the collection $cl(S)$ of cliques of S, namely the collection of subsets of S that have an upper bound in P. We refer to such a result as reduction of the positivity condition (1.1) to atoms. Our main results are various reductions of the positivity condition to smaller systems of inequalities valid for several classes of monoids P.

As for our context, we begin by considering general right LCM monoids, which are the left cancellative monoids that have least common upper bounds and may contain nontrivial invertible elements. For some results we focus on the more restrictive class of group embeddable right LCM monoids that have no invertible elements other than the identity; these are known as *weak quasi-lattice ordered monoids* [2, Definition 2.1]. In most situations we will also assume P to be Noetherian; this is a suitable assumption that ensures the existence of a set of atoms which, together with the invertible elements, generate the monoid. It is known, for example, that the existence of a length function implies Noetherianity. Our main applications and most definite results are for finite-type Artin monoids, for which we answer in the positive, the question of reduction of positivity to generators raised in [3].

The paper is organised as follows. In Section 2 we review briefly the necessary background on monoids. The main results of Section 3 are the generalisations of Theorem 2.1 and Theorem 5.1 of

[3] to the Nica-Toeplitz algebra $\mathcal{NT}(X)$ of a compactly aligned product system $X = (X_p)_{p\in P}$ over a right LCM monoid P. This is fairly straightforward, except for the need to upgrade several results about inducing in stages for product systems over Noetherian monoids P that have nontrivial invertible elements. This widens the range of applications and may be of independent interest. Since we do not assume the existence of an ambient group, we also propose a natural notion of gauge-invariance for KMS states on $\mathcal{NT}(X)$. The key is to use certain conditional expectations that were implicit in [15] and [16] to replace the conditional expectations of the dual coactions that exist only when P embeds in a group.

In Section 4, motivated by [20], we construct a tree by an induction procedure that provides a recursion formula to keep track of how the positivity condition for a given level depends on the next level, Proposition 4.7. Our first main result is Theorem 4.8 where we show that reduction of positivity holds whenever the tree is finite. As applications we prove reduction results for finite-type Artin monoids, Corollary 4.11, and we also recover in Proposition 4.12 the reduction result for right-angled Artin monoids, which was proved by different means in [3, Theorem 9.1]. Further, Corollary 4.9 spells out the main practical method to find KMS states by verifying positivity on suitable finite subsets of P.

In Section 5 we take a different approach to reduction of the positivity condition. More precisely, we combine the technique from [3] for proving reduction for right-angled Artin monoids with a minimal set introduced by Boyu Li in [20]. This minimal set is the smallest subset of P that contains the atoms and is closed under taking left divisors of least upper bounds. The key step, Lemma 5.4, is to show that a certain algebra of subsets of P can be constructed from ideals $pP \subset P$ with p in the minimal set. Our second main result is Theorem 5.5, which shows that for all Noetherian weak quasi-lattice orders there is a reduction of positivity to a subsystem of inequalities corresponding to the minimal subset.

In Section 6 we specialise to product systems with one-dimensional fibres, that is, $X_p = \mathbb{C}$ for all $p \in P$, and we illustrate, for the C^*-algebras of Artin braid monoids B_n^+ for $n \geq 3$, the technique of finding KMS states by solving the system of equations (explicitly for $n = 4$).

Finally, in Section 7 we refine the search of a suitable algebra of subsets of P for which reduction of positivity holds by going all the way down to the set of atoms inside the minimal subset of P. Theorem 7.1 formalises the result, and its main application, Corollary 7.10, provides a reduction of the positivity condition to atoms for a class of Artin monoids, obtained through free and direct products, which properly contains the finite-type and the right-angled ones. Explicit examples are given in Proposition 7.7. It remains an open question whether reduction of positivity to atoms holds for all Artin monoids.

2. Background

2.1. **Semigroups, conditional right LCMs, and noetherianity.** All semigroups in this paper have an identity, denoted e, and thus are monoids. We concentrate our attention on left cancellative monoids that admit a partial order under which conditional right least common multiples exist, also referred to as *right LCM monoids*. These objects fall under a broader class of monoids, and we refer to the monograph [11] as reference.

For p, q in a monoid P we say that q is a *right multiple* of p, written $p \leq q$, provided that there is $q' \in P$ such that $pq' = q$. Equivalently, $q \in pP$. A further equivalent formulation is that p is a *left divisor* of q. The set of left-divisors of q will be denoted $\mathrm{Div}_l(q)$. The relation $\leq$ is an order when P is left cancellative with no non-trivial invertible elements. There are obvious definitions of the symmetric notions of left multiple (right divisor). The invertible elements in P form a group P^*, possibly the trivial group $\{e\}$.

Definition 2.1. [11, Definition 2.9] Let P be a left-cancellative monoid and let U be a subset of P. The element r of P is a least common right multiple, or *right LCM*, of U if r is a right multiple of each element of U and every element r' of P that is a right multiple of each element of U is a right multiple of r.

A subset U of P that admits a right LCM $r \in P$ is called a *clique* in P. We denote a right LCM of U by $\vee U$ and note that $(\vee U)u$ is a right LCM of U for every $u \in P^*$. If U is a clique consisting of two elements p and q, we write $p \vee q$ for a right LCM of $\{p, q\}$. We write $\vee U = \infty$ when U has no right LCM. By convention, the empty set is a clique. Obviously, every one-element subset of P is a clique. For a subset $J \subset P$ we denote the collection of cliques U of J by $cl(J)$.

Note that the operation $\vee$ is associative on P (since P is left-cancellative), a fact which will be used tacitly throughout.

We recall now an important class of monoids: in [11], their defining property is called existence of conditional right LCM's. Right-cancellative monoids that admit conditional lcm's with respect to left multiples have been studied in e.g. [19].

Definition 2.2. A left-cancellative monoid P with the property that every two elements p and q that admit a common right multiple admit a right LCM will be called a right LCM monoid.

A monoid P is *left-Noetherian* (respectively *right-Noetherian*, respectively *Noetherian*) provided that there exists no infinite descending sequence in P with respect to proper left-divisibility (respectively proper right-divisibility, respectively proper factor relation). By [11, Proposition II.2.29], a left-cancellative monoid is Noetherian if and only if it is both right- and left-Noetherian.

If P is a left-cancellative monoid that is right-Noetherian, and if $q \in P$, then every sequence in $\mathrm{Div}_l(q)$ of the form $p_1, p_1p_2, p_1p_2p_3, \dots$ with p_n non-invertible is necessarily finite, cf. [11, Proposition II.2.28] (i.e. The divisors of q have no infinite increasing sequences.) An analogous argument shows that if P is also right-cancellative and left-Noetherian, then for each $q \in P$, any sequence of the form $r_1, r_2r_1, r_3r_2r_1, \dots$ with r_n non-invertible (i.e. an increasing sequence with respect to right-divisibility) in the set $\mathrm{Div}_r(q)$ of right-divisors of q is finite.

An *atom* in a monoid P is an element $s \in P \setminus P^*$ such that every decomposition of s as a finite product of elements in P contains at most one element in $P \setminus P^*$, see [11, Ch. II, Definition 2.52]. We denote the set of atoms in P by P_a.

It is proved in [11, Corollary II.2.59] that in a Noetherian left-cancellative monoid P that contains no non-trivial invertible elements, the set of atoms P_a is the smallest subset generating P. In an arbitrary left-cancellative Noetherian monoid, a generating set P_{gen} is any subfamily that generates P^* and contains at least one element in each equivalence class of atoms for the relation $\sim$ with $s \sim s'$ if and only if $s' = su$ for $u \in P^*$, [11, Proposition 2.58].

We recall from [22], see also [9] that a discrete group G and a subsemigroup P so that $P \cap P^{-1} = \{e\}$ form a quasi-lattice ordered pair for the partial order $x \leq y \Leftrightarrow x^{-1}y \in P$, with $x, y \in G$, if the following two conditions hold:

(QL1) For all $p, q \in P$ such that $(pq^{-1})P \cap P \neq \emptyset$ there is (a necessarily unique) $r \in P$ so that $(pq^{-1})P \cap P = rP$, and

(QL2) Given any pair $p, q \in P$ so that $pP \cap qP \neq \emptyset$ there is (a necessarily unique) element $p \vee q$ in P, their *least common upper bound*, so that $pP \cap qP = (p \vee q)P$.

A discrete group G and a subsemigroup P so that $P \cap P^{-1} = \{e\}$ form a weak quasi-lattice ordered pair if condition (QL2) alone is satisfied.

2.2. **Artin monoids.** We now recall some basic facts about Artin monoids. A *Coxeter matrix* over a set S is a matrix $M = (m_{st})_{s,t \in S}$ such that $m_{st} = m_{ts} \in \{2, \dots, \infty\}$ for all $s \neq t$ and $m_{ss} = 1$ for all $s, t \in S$, with the convention that M may be an infinite matrix. The *Artin group* A_M associated to M is the group with generating set S and presentation

$$\{S \mid \langle st\rangle^{m_{st}} = \langle ts\rangle^{m_{ts}} \text{ for all } s,t\in S\},$$

where $\langle st\rangle^{m_{st}} = sts\ldots$ is the alternating product of length m_{st}; notice that the last letter of $\langle st\rangle^{m_{st}}$ is t when m_{st} is even and s when m_{st} is odd. The *Artin monoid* A_M^+ is the monoid with the same presentation. As a consequence of [4, Verkürzungslemma], every Artin monoid admits conditional right LCM's and the explicit formula for the least common upper bound of two generators is given by the obvious guess.

Lemma 2.3. *Every Artin monoid A_M^+ embeds in the corresponding Artin group A_M, and the pair (A_M, A_M^+) is weak quasi-lattice ordered. If s and t are generators, then $s\vee t = \langle st\rangle^{m_{s,t}} = \langle ts\rangle^{m_{t,s}}$ for all $s,t\in S$ with $m_{s,t} = m_{t,s} < \infty$.*

Proof. The general embedding result is from [24], although various special cases had been known previously. That the pair is weak quasi-lattice ordered is a consequence of [4, Verkürzungslemma]. For the last assertion, it is clear that $\langle st\rangle^{m_{s,t}} = \langle ts\rangle^{m_{t,s}}$ is an upper bound for both s and t in S. Suppose $x\in P$ is an arbitrary upper bound for s and t, that is, there exist $p,q\in P$ such that $x = sp = tq$. By [4, Verkürzungslemma], there exists a positive word w such that $p = \langle ts\rangle^{m_{s,t}-1}w$ and $q = \langle st\rangle^{m_{t,s}-1}w$. Hence $x = sp = \langle st\rangle^{m_{s,t}}w$, and hence $\langle st\rangle^{m_{s,t}} \le x$. Since x is arbitrary we can conclude that $s\vee t = \langle st\rangle^{m_{s,t}} = \langle ts\rangle^{m_{ts}}$. □

There are two particularly well understood classes of Artin groups and monoids: *Right-angled Artin groups*, where $m_{ij}\in\{2,\infty\}$ for all $i\ne j$ in Λ, and *finite-type Artin groups*, which are those for which the corresponding *Coxeter group*, which is the group obtained by adding the relations $s_i^2 = e$ to those of A_M, is finite, from this it follows that their Coxeter matrix is finite and that it contains no infinities. Both form quasi-lattice ordered pairs, cf. [4] and [9].

Remark 2.4. Since Coxeter matrices are symmetric, the relations defining Artin monoids are homogeneous, in the sense that they preserve the total number of generators in a word. This implies that the usual length function on the free monoid factors through Artin monoids. As a consequence, all Artin monoids are Noetherian.

2.3. **An algebra of subsets of P.** Suppose that P is a right LCM monoid. For each $p\in P$, let 1_{pP} denote the characteristic function of the set $\{q\in P : p\le q\}$. Then $B_P = \overline{\operatorname{span}}\{1_{pP} : p\in P\}$ is a commutative C^*-subalgebra of $l^\infty(P)$, see [16, Section 2] for further properties.

Following [3, Section 2] we denote by $\mathcal{B}_P$ the algebra of subsets of P generated by the sets pP for $p\in P$; by an algebra we mean a nonempty collection of subsets of P that is closed under taking complements and finite unions. Then the set of projections in B_P is precisely the set of characteristic functions of the elements of $\mathcal{B}_P$. For easy reference we state a version of [3, Lemma 2.4] valid for right LCM monoids.

Lemma 2.5. (cf. [3, Lemma 2.4]) *For each finite subset J of P let*

$$\Omega_J := \bigcap_{q\in J}(P\setminus qP), \tag{2.1}$$

where by convention $\Omega_\emptyset = P$ and $\Omega_J = \emptyset$ whenever $J\cap P^*\ne\emptyset$. Then every set $\mathcal{B}_P$ is a finite disjoint union of sets of the form $p\Omega_J$, where $p\in P$ and $J\subset P$ is a finite set.

In the reduction of the positivity condition to generators proved in [3, Theorem 9.1] in the case of right-angled Artin monoids, the key ingredient is that to generate $\mathcal{B}_P$ it suffices to take subsets $J\subset S$ of generators, see [3, Lemma 9.3].

3. Product systems over right LCM monoids

Suppose that P is a left cancellative monoid. The notion of a product system X over P of C^*-correspondences was introduced in [12]. In particular, [12] identified the important class of Nica-Toeplitz covariant representations of a compactly aligned product system X over a monoid P in a quasi-lattice ordered pair (G, P).

The analysis of KMS states for time evolutions on Nica-Toeplitz C^*-algebras of compactly aligned product systems over positive cones of quasi-lattice ordered groups was carried out in [3]. One of our aims here is to extend this analysis to product systems over the positive cones of weak quasi-lattice ordered groups, and, more generally, over right LCM monoids. Another more specific aim is to show that the reduction of the characterisation of KMS states to generators holds in other cases as well, notably for finite-type Artin monoids. We begin by briefly recalling some necessary background. For a comprehensive discussion of product systems of C^*-correspondences see [12, 28, 7] in the quasi-lattice ordered case and the generalisation [5, 16] to the right LCM case.

Suppose that P is a (countable) cancellative semigroup with identity e and A a C^*-algebra. Let X be a right C^*-Hilbert A-module. We denote the adjointable operators on X by $\mathcal{L}(X)$ with identity I_X. A C^*-correspondence X over A is a right C^*-Hilbert A-module equipped with a left action given by a $*$-homomorphism $\varphi : A \to \mathcal{L}(X)$. The correspondence is *essential* if $\overline{\varphi(A)(X)} = X$. A collection $(X_p)_{p\in P}$ of C^*-correspondences over A is a *product system*, cf. [12], if the following properties are satisfied:

- *(i)* $\bigcup_{p\in P} X_p$ has a monoid structure such that for any $\xi \in X_p$ and $\zeta \in X_q$ we have $\xi\zeta \in X_{pq}$ and the map $\xi \otimes \zeta \mapsto \xi\zeta$ extends to an isometric isomorphism $X_p \otimes_A X_q \simeq X_{pq}$ for all $p, q \in P$ with $p \neq e$.
- *(ii)* X_e is the canonical correspondence ${}_AA_A$ and the product maps $X_e \times X_p \to X_p$ and $X_p \times X_e \to X_p$ coincide with the structure maps of the A-bimodules X_p for all $p \in P$.

3.1. The Nica-Toeplitz C^*-algebra of X.

A universal C^*-algebra $\mathcal{T}(X)$ is associated to a product system X via representations ψ of X. We refer to [12] for the definition of an arbitrary representation $\psi : X \to B$ with B a C^*-algebra. Next we recall in some detail the notion of Nica covariant representation of X, under further assumptions on P and X.

Suppose that P is a right LCM monoid and X is a product system over P of C^*-correspondences over a C^*-algebra A such that X_p is essential for every $p \in P$. Note that if $P^* \neq \{e\}$ then X will automatically have essential fibres, cf. [16, Remark 1.3]. The requirement of essential fibres is not needed to define Nica covariant representations and the universal C^*-algebra $\mathcal{NT}(X)$, but is useful when we also bring into the discussion the reduced Nica-Toeplitz C^*-algebra $\mathcal{NT}^r(X)$, cf. also [3].

Let I_p denote the identity operator in $\mathcal{L}(X_p)$ for every $p \in P$. If $p, r \in P$ with $p \leq r$, then there is a natural $*$-homomorphism $\iota_p^r \colon \mathcal{L}(X_p) \to \mathcal{L}(X_r)$ obtained by identifying X_r with $X_p \otimes_A X_{p^{-1}r}$ and mapping $S \in \mathcal{L}(X_p)$ into $S \otimes I_{p^{-1}r}$. Let $\mathcal{K}(X_p)$ denote the ideal of generalised compact operators in $\mathcal{L}(X_p)$ for each p. We say that X is *compactly aligned* if for all $p, q, r \in P$ with $pP \cap qP = rP$ we have that

$$\iota_p^r(\mathcal{K}(X_p))\iota_q^r(\mathcal{K}(X_q)) \subset \mathcal{K}(X_r);$$

no condition is imposed when $pP \cap qP = \emptyset$. We define elements $\theta_{\xi,\zeta} \in \mathcal{K}(X_p)$ by $\theta_{\xi,\zeta}\eta = \xi\langle\zeta, \eta\rangle$, for $\xi, \zeta, \eta \in X_p$. Given a representation ψ of X into B, there are $*$-homomorphisms $\psi^{(p)} \colon \mathcal{K}(X_p) \to B$ given by $\psi^{(p)}(\theta_{\xi,\zeta}) = \psi_p(\xi)\psi_p(\zeta)^*$ for each $p \in P$. A representation ψ is Nica covariant if

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(r)}\big(\iota_p^r(S)\iota_q^r(T)\big) & \text{if } pP \cap qP = rP \\ 0 & \text{otherwise} \end{cases}$$

for all $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$. We refer to [5, Section 6] and [16, Lemma 2.4] for further details. By definition, the *Nica-Toeplitz algebra* of X is the C^*-algebra $\mathcal{NT}(X)$ generated by a universal Nica covariant representation $i_X \colon X \to \mathcal{NT}(X)$.

Following [3], for each multiplicative semigroup homomorphism $N : P \to (0, \infty)$ we will consider the time evolution σ on $\mathcal{NT}(X)$ such that $\sigma_t(i_X(\xi_p)) = N(p)^{it} i_X(\xi_p)$ for $\xi_p \in X_p$ and $t \in \mathbb{R}$. We will need to assume throughout that N is trivial on P^*, that is, $N_u = 1$ for every $u \in P^*$; this is automatic if the range of N is contained in $[1, \infty)$. Recall that a state ϕ of $\mathcal{NT}(X)$ is a KMS state at inverse temperature β (a KMS$_\beta$ state) provided that

$$\phi(xy) = \phi(y\sigma_{i\beta}(x))$$

for all x, y with x analytic with respect to σ, that is to say, $t \mapsto \sigma_t(x)$ extends to a C*-algebra-valued entire function. Since in the present case it suffices to consider the σ-invariant set of analytic elements $i_X(\xi_p)i_X(\eta_q)^*$ with $\xi_p \in X_p$ and $\eta_q \in X_q$ because it has a dense linear span (it suffices to know that $i_X(\xi)$ generate $\mathcal{NT}(X)$ as a C^*-algebra, see [3, Lemma 1.9]), we will see below that KMS$_\beta$ states of $(\mathcal{NT}(X), \sigma)$ are characterized by the condition

$$\phi(i_X(\xi_p)i_X(\eta_q)^*) = \delta_{p,q} N^{-\beta} \operatorname{Tr}_\tau^p(\theta_{\eta_q,\xi_p}), \qquad \xi_p \in X_p, \quad \eta_q \in X_q,$$

involving the induced trace of the tracial state $\tau := \phi|_A$.

3.2. **Inducing traces via C^*-correspondences.** We recall from [25, Sections 5.1 and 5.2] that a weight ψ on a C^*-algebra A is a map $\psi : A_+ \to [0, \infty]$ that satisfies positive homogeneity: $\psi(\alpha a) = \alpha\psi(a)$ for all $\alpha \in \mathbb{R}_+$ and $a \in A_+$, and additivity: $\psi(a + b) = \psi(a) + \psi(b)$ for all $a, b \in A_+$. The weight ψ is lower semicontinous if $\{a \mid \psi(a) \le c\}$ is closed for all $c \in \mathbb{R}_+$. With $A_+^\psi := \{a \in A_+ \mid \psi(a) < \infty\}$, the space $A^\psi = \operatorname{span} A_+^\psi$ is a hereditary C^*-algebra of A on which ψ admits a unique extension as a positive linear functional, still denoted ψ. If the weight ψ is invariant under conjugation by unitary elements in the unitisation $\tilde{A}$, we say that it is a trace, in which case A^ψ is an ideal of A. Moreover, if ψ is a finite trace, meaning that $A_+^\psi = A_+$, then $\psi(ab) = \psi(ba)$ for all $a, b \in A$.

Suppose that X is a C^*-correspondence over A and τ a tracial positive linear functional on A. We recall from [17, Theorem 1.1] that the *induced trace* Tr_τ^X is defined for $T \in \mathcal{L}(X)$, $T \ge 0$ by

$$\operatorname{Tr}_\tau^X(T) = \sup_{H \subset X} \sum_{\xi \in H} \tau(\langle \xi, T\xi \rangle), \tag{3.1}$$

where the supremum is taken over all finite subsets $H \subset X$ such that $\sum_{\xi \in H} \theta_{\xi,\xi} \le 1$. Then Tr_τ^X is strictly lower semicontinuous and satisfies

$$\operatorname{Tr}_\tau^X(T) = \lim_k \sum_{\xi \in H_k} \tau(\langle \xi, T\xi \rangle) \quad \text{for all } T \in \mathcal{L}(X)_+.$$

whenever H_k is a generalized sequence of finite sets in X such that $\{\sum_{\xi \in H_k} \theta_{\xi,\xi}\}_k$ is an approximate unit in $\mathcal{K}(X)$, moreover, Tr_τ^X extends to a positive semifinite trace on $\mathcal{L}(X)$.

We are interested in the restriction of Tr_τ^X to the image of A inside $\mathcal{L}(X)$ given by the left action. Recall from [17] that for each C^*-correspondence X over A there is an operator F_X mapping a tracial positive linear functional τ on A to a possibly infinite positive trace $F_X\tau$ on A defined by

$$(F_X\tau)(a) = \operatorname{Tr}_\tau^X(a)$$

for $a \in A_+$. Given a product system $\{X_p\}_{p \in P}$ of essential C^*-correspondences over P, we follow [3, Section 1.4] and write F_p instead of F_{X_p} and Tr_τ^p instead of $\operatorname{Tr}_\tau^{X_p}$ for all $p \in P$. Let φ_p be the left action on X_p and recall that if $(\operatorname{Tr}_\tau^q \circ \varphi_p)|_A$ is a finite positive trace on A for some $q \in P$, then we can use induction in stages [17, Proposition 1.2] and factor $F_{pq}(\tau)$ as $F_p(F_q(\tau))$ for all $p \in P$. In other words, for $r = pq$ so that $F_q(\tau) = \mathrm{Tr}^q_\tau|_{A_+}$ is finite we have

$$\mathrm{Tr}^r_\tau(a) = \mathrm{Tr}^p_{F_q(\tau)}(a) \text{ for } a \in A_+. \tag{3.2}$$

In general, the induced trace Tr^q_τ, or rather $\mathrm{Tr}^q_\tau \circ \varphi_p$, need not be a finite trace on A. However, we recall from [3] that if τ satisfies condition (1.1) with $J = \{q\}$, then

$$\mathrm{Tr}^q_\tau(a) \le \tau(a) N(q)^\beta < \infty \quad \text{for all } a \in A_+. \tag{3.3}$$

In particular, Tr^q_τ is a finite trace on A whenever τ is the restriction of a KMS_β-state.

3.3. **A characterisation of KMS states on $\mathcal{NT}(X)$.** We will now extend [3, Theorem 2.1] to the case of Nica-Toeplitz algebras $\mathcal{NT}(X)$ where X is a product system over a right LCM monoid. Before we can state the generalisation we need some preparation about induced traces for product systems over monoids that have nontrivial invertible elements.

Suppose that X is a product system over a right LCM monoid P with $X_e = A$. Recall that φ_p is the left action in X_p and I_p the identity operator in $\mathcal{L}(X_p)$, for all $p \in P$. As observed in [5], for each $u \in P^*$ the map $T \mapsto T \otimes I_{u^{-1}}$ is an adjunction from X_u to $X_{u^{-1}}$ in the sense of [8, Definition 2.17], meaning that it induces a natural isomorphism

$$\mathcal{L}(Z \otimes_A X_u, Y) \to \mathcal{L}(Z, Y \otimes_A X_{u^{-1}})$$

for any Hilbert A-modules Y, Z. One consequence is that by [14, Theorems 4.4(2) and 4.13], see also [8, Theorem 2.24], it necessarily must hold that $\varphi_u(a) \subseteq \mathcal{K}(X_u)$ for $u \in P^*$ and $a \in A_+$. A second consequence of the existence of this adjunction is that its restriction to the generalised compact operators is an isomorphism

$$\mathcal{K}(Z \otimes_A X_u, Y) \to \mathcal{K}(Z, Y \otimes_A X_{u^{-1}}),$$

cf. [8, Corollary 3.9], in other words it yields a local adjunction, for all $u \in P^*$. As in [8], we denote elements in $X_{u^{-1}}$ by ξ^*, where ξ runs over X_u (in the terminology of [8], the C^*-correspondence $X_{u^{-1}}$ is the conjugate of X_u). It follows from [8, Corollary 3.31] that A admits a direct sum decomposition

$$A = A_{u^{-1}} \oplus A^\perp_{u^{-1}}$$

into two-sided ideals $A_{u^{-1}} = \overline{\mathrm{span}}\{\langle \xi_1^*, \xi_2^*\rangle \mid \xi_1^*, \xi_2^* \in X_{u^{-1}}\}$ and $A^\perp_{u^{-1}} = \ker(\varphi_u)$. In particular, we have that $\varphi_u(A) = \varphi_u(A_{u^{-1}})$.

We now let $F_{p,q} : X_p \otimes_A X_q \to X_{pq}$ denote the subordinate isomorphisms of C^*-correspondences for $p, q \in P$. Note that for each $u \in P^*$, the map

$$F^{-1}_{u,u^{-1}} : X_e \to X_u \otimes_A X_{u^{-1}}$$

is adjointable with adjoint $F_{u,u^{-1}}$, and satisfies the properties of being a unit for the adjunction corresponding to tensoring by X_u and $X_{u^{-1}}$. By uniqueness of the unit, the map $F^{-1}_{u,u^{-1}}$ is the adjoint of the map δ from [8, Lemma 3.18], hence it corresponds to the left action φ_u through a canonical isomorphism of $X_u \otimes_A X_{u^{-1}}$ onto $\mathcal{K}(X_u)$ identifying $\xi_1 \otimes \xi_2^*$ with θ_{ξ_1,ξ_2}, where $\xi_1, \xi_2 \in X_u$, [8, Proposition 3.29]. Moreover, $F_{u,u^{-1}}$ can be identified with the map δ, so

$$F_{u,u^{-1}}(\xi_1 \otimes \xi_2^*) = \langle \xi_1^*, \xi_2^*\rangle \text{ for } \xi_1^*, \xi_2^* \in X_{u^{-1}}.$$

For the same reason, the map $F^{-1}_{u^{-1},u} : X_e \to X_{u^{-1}} \otimes_A X_u$ must coincide with ε from [8, Lemma 3.18], so it corresponds to the left action $\varphi_{u^{-1}}$ upon canonical identification of $\xi_1^* \otimes \xi_2$ with $\theta_{\xi_1^*,\xi_2^*} \in \mathcal{K}(X_{u^{-1}})$, [8, Proposition 3.32]. Further,

$$F_{u^{-1},u}(\xi_1^* \otimes \xi_2) = \langle \xi_1, \xi_2\rangle \text{ for } \xi_1, \xi_2 \in X_u.$$

Since $F^{-1}_{u^{-1},u} \circ F_{u,u^{-1}} : X_u \otimes_A X_{u^{-1}} \to X_{u^{-1}} \otimes_A X_u$ maps $\xi \otimes \xi^*$ to $\xi^* \otimes \xi$, where $\xi \in X_u$, it follows that

$$\langle \xi^*, \xi^* \rangle = \langle \xi, \xi \rangle \text{ for all } \xi \in X_u. \tag{3.4}$$

Lemma 3.1. *Suppose that $X = \{X_p\}_{p\in P}$ is a product system of C^*-correspondences over a right LCM monoid P with $X_e = A$. Suppose that τ is a tracial state on A. Then the trace on A defined as $\mathrm{Tr}_\tau^{X_u}(\varphi_u(a))$ for $u \in P^*, a \in A_+$ coincides with τ. In particular, for every $p \in P$ and all $a \in A_+$ we have that*

$$\mathrm{Tr}_\tau^{X_{pu}}(\varphi_{pu}(a)) = \mathrm{Tr}_\tau^{X_p}(\varphi_p(a)) = \mathrm{Tr}_\tau^{X_{up}}(\varphi_{up}(a)).$$

Proof. We recall that $\mathrm{Tr}_\tau^Y(\varphi(a))$ is defined in [17, Proposition 1.2] for an arbitrary C^*-correspondence Y over A. Given $u \in P^*$, let $(e_J)_J$ with $e_J \leq I_u$ be an approximate unit for $\mathcal{K}(X_u)$ where $e_J = \sum_{\eta\in J} \theta_{\eta,\eta}$. Let $a = \langle \xi^*, \xi^* \rangle \in A_+$ for $\xi \in X_u$. We have that

$$\begin{aligned}
\mathrm{Tr}_\tau^{X_u}(\varphi_u(a)) &= \sup\nolimits_J \sum_{\eta\in J} \tau(\langle \eta, \varphi_u(a)\eta\rangle) \\
&= \sup\nolimits_J \sum_{\eta\in J} \tau(\langle \eta, \langle \xi^*, \xi^*\rangle\eta\rangle) \\
&= \sup\nolimits_J \sum_{\eta\in J} \tau(\langle \eta, \theta_{\xi,\xi}\eta\rangle) \\
&= \sup\nolimits_J \sum_{\eta\in J} \tau(\langle \eta, \xi\rangle\langle \xi, \eta\rangle) \\
&= \sup\nolimits_J \sum_{\eta\in J} \tau(\langle \xi, \eta\rangle\langle \eta, \xi\rangle) \text{ since } \tau \text{ is a trace} \\
&= \sup\nolimits_J \sum_{\eta\in J} \tau(\langle \xi, \theta_{\eta,\eta}\xi\rangle) \\
&= \tau(\langle \xi, \xi\rangle) \text{ since } e_J \text{ is an approximate unit} \\
&= \tau(a)
\end{aligned}$$

by the assumption on (e_J) and (3.4). By linearity and continuity we have that $\mathrm{Tr}_\tau^{X_u}(\varphi_u(a)) = \tau(a)$ for all $a \in A_+$. The displayed equalities follow by two applications of [17, Proposition 1.2], namely by letting $q \in P^*$ and, respectively, $p \in P^*$ in equation (3.2). $\square$

Lemma 3.2. *Let P be a Noetherian right LCM monoid, $\{X_p\}_{p\in P}$ a compactly aligned product system of essential C^*-correspondences over P with $X_e = A$, $N : P \to (0,\infty)$ a multiplicative homomorphism, τ a finite positive trace on A and $\beta \in \mathbb{R}$.*

- (i) *Suppose that $\mathrm{Tr}_\tau^s(a) \leq N(s)^\beta\tau(a)$ for all $s \in P_a,\ a \in A_+$. Then $\mathrm{Tr}_\tau^p(a) \leq N(p)^\beta\tau(a)$ for all $p \in P$ and $a \in A_+$. In particular $\mathrm{Tr}_\tau^p|_A$ is a tracial linear functional on A for every p.*
- (ii) *Suppose that $\mathrm{Tr}_\tau^p|_A$ is finite for every $p \in P$ and for each finite subset U of P with $\vee U < \infty$ set $\tau_U = \mathrm{Tr}_\tau^{\vee U}|_A$. Then $\mathrm{Tr}_{\tau_U}^p(a) < \infty$ for all $p \in P$ and $a \in A_+$.*

Proof. Let $N_0(p) = N(p)^\beta$ for all $\in P$. Assume the hypothesis of (i) and let $p \in P$. If $p \in P^*$, then $\mathrm{Tr}_\tau^p(a) = \tau(a) = N_0(p)\tau(a)$ for all $a \in A_+$ by Lemma 3.1. If $p \in P \setminus P^*$, then the element p can be expressed as a product of the form $p = gs_1g_1s_2g_2\dots s_ng_n$, for some $n \geq 1$, with $s_j \in P_a$ and g, g_j in a generating set for P^* (which we assume contains the identity) for each $j = 1, \dots, n$, see Section 2. We will prove the claim by induction on the number of atoms in the expression of p.

Let first $p = gs_1g_1$ where $s_1 \in P_a$ and $g, g_1 \in P^*$. Then for $a \in A_+$ we have

$$\mathrm{Tr}_\tau^p(a) = \mathrm{Tr}_\tau^{s_1}(a) \leq N_0(s_1)\tau(a) = N_0(p)\tau(a)$$

by Lemma 3.1 and our hypothesis

Assume now that $\mathrm{Tr}_\tau^p|_A \leq N_0(p)\tau$ for all $p \in P$ that can be expressed as a product of generators containing n atoms, for $n \geq 2$. Let $q \in P$ be of the form $q = gs_1g_1s_2g_2 \dots s_ng_ns_{n+1}g_{n+1}$, with $s_j \in P_a$ and g, g_j in a generating set for P^* for $j = 1, \dots, n+1$. Set $p = g_1s_2g_2 \dots s_ng_ns_{n+1}g_{n+1}$, so that $q = gs_1p$.

By the inductive hypothesis applied to p, $\tau_0 = \mathrm{Tr}_\tau^p|_A$ is a finite trace on A. Then, applying Lemma 3.1, equation (3.2), the inductive hypothesis on p and our hypothesis on s_1, we get that

$$\begin{aligned}\mathrm{Tr}_\tau^q(a) = \mathrm{Tr}_\tau^{gs_1p}(a) = \mathrm{Tr}_{\tau_0}^{s_1}(a) &= \sup_F \sum_{\xi \in F} \tau_0(\langle \xi, a\xi \rangle) \\ &\leq \sup_F \sum_{\xi \in F} N_0(p)\tau(\langle \xi, a\xi \rangle) \\ &= N_0(p)\,\mathrm{Tr}_\tau^{s_1}(a) \leq N_0(p)N_0(s_1)\tau(a) \\ &= N_0(q)(a)\end{aligned}$$

for all $a \in A_+$, where F are finite subsets of X_{s_1}. We can conclude that $\mathrm{Tr}_\tau^q(a) \leq N_0(q)\tau(a) < \infty$ for all $a \in A_+$ and all $q \in P$. This shows (i).

For (ii), fix a finite subset $U \subset P$ with $\vee U < \infty$. We have that $\tau_U = \mathrm{Tr}_\tau^{\vee U}$ is a finite trace on A by assumption. Then

$$\mathrm{Tr}_{\tau_U}^p(a) = \mathrm{Tr}_\tau^{p(\vee U)}(a) < \infty$$

for all $a \in A_+$ and $p \in P$ follows by [17, Proposition 1.2], see equation (3.2). □

We shall need generalisations of [3, Theorem 2.1] and [3, Theorem 5.1] valid for right LCM monoids.

Theorem 3.3. (cf. [3, Theorem 2.1]) *Assume that P is a right LCM monoid and $X = \{X_p\}_{p\in P}$ is a compactly aligned product system of essential C^*-correspondences over P with $X_e = A$. Consider the time evolution on $\mathcal{NT}(X)$ defined by a homomorphism $N : P \to (0, \infty)$, and assume ϕ is a* KMS_β*-state on $\mathcal{NT}(X)$ for some $\beta \in \mathbb{R}$. Then $\tau = \phi|_A$ is a tracial state on A satisfying condition* (1.1) *for all finite $J \subset P\backslash\{e\}$ and $a \in A_+$.*

Proof. The argument follows the one from [3]. Let (π, H, ξ) be the Hilbert space of the GNS representation of $\mathcal{NT}(X)$ from ϕ. There are two main points where we need an extension to the right LCM case. First, we need to know that

$$N(pu)^{-\beta}\,\mathrm{Tr}_\tau^{X_{pu}}(\varphi_{pu}(a)) = N(p)^{-\beta}\,\mathrm{Tr}_\tau^{X_p}(\varphi_p(a))$$

for all $p \in P, u \in P^*$ and $a \in A$. Since $N(u) = 1$ because N is a homomorphism, this claim follows directly from Lemma 3.1.

Second, we need an analogue of the projections f_p, $p \in P$ from [3]. For each $p \in P$, let 1_{pP} denote the characteristic function of the set $\{q \in P : p \leq q\}$. Then $B_P = \overline{\mathrm{span}}\{1_{pP} : p \in P\}$ is a commutative C^*-subalgebra of $l^\infty(P)$. Furthermore, just as in the case of quasi-lattice ordered pairs cf [3, Section 2], the set $\{1_{pP} \mid p \in P\}$ is linearly independent.

Since the representation $\pi : \mathcal{NT}(X) \to B(H)$ corresponds to a Nica covariant representation $\psi : X \to B(H)$ through $\psi = \pi \circ i_X$, we let $\alpha_p^\psi(1)$ be the projection in $\psi_e(A)' \subset B(H)$ from [12, Proposition 4.1] (which is the case of quasi-lattice orders, see [16, Lemma 2.28] for the right LCM case). We have that $\alpha_p^\psi(1)\alpha_q^\psi(1) = \alpha_r^\psi(1)$ whenever $pP \cap qP = rP$ and $\alpha_p^\psi(1)\alpha_q^\psi(1) = 0$ when p and q do not admit a common right multiple, see [15, Proposition 9.5 and Lemma 9.3] in connection with [16, Lemma 2.4]. By [16, Proposition 2.30], there is a representation L_π of B_P on H determined by $L_\pi(1_{pP}) = \alpha_p^\psi(1)$. In particular, $\alpha_p^\psi(1)$ is the strong limit of $\pi(i_X^{(p)}(u_j))$ for every approximate unit

$(u_j)_j$ in $\mathcal{K}(X_p)$. Thus with projections $\alpha_p^{\psi}(1)$ instead of f_p, for $p \in P$, the proof of [1, Theorem 2.1] can be adapted verbatim to the case when P is right LCM. □

3.4. **Gauge-invariant KMS states.** It is natural to ask if the necessary condition (1.1) describing a KMS state is also sufficient for its existence. A positive answer is given in [3, Theorem 5.1], as we now briefly recall. Suppose that X is a compactly aligned product system X over a monoid P in a quasi-lattice ordered pair (G, P). Then there exists a canonical coaction δ^G of G on $\mathcal{NT}(X)$ that yields a grading into a family of subspaces $\mathcal{NT}(X)_g$ for $g \in G$, cf. [7, Proposition 3.5]. A state ϕ on $\mathcal{NT}(X)$ is gauge-invariant if it vanishes on all $\mathcal{NT}(X)_g$ where $g \neq e$, in other words if ϕ factors through the conditional expectation arising from δ^G, which maps $\mathcal{NT}(X)$ onto its core subalgebra. Then the map $\phi \mapsto \phi|_A$ defines a one-to-one correspondence between the gauge-invariant KMS$_\beta$-states on $\mathcal{NT}(X)$ and the tracial states on A satisfying (1.1), cf. [3, Theorem 5.1].

In order to extend this result to the case of product systems over right LCM monoids, the immediate question arises as to what should gauge-invariance mean in the absence of an ambient group containing the given monoid. The answer we suggest is that the state should factor through a canonical conditional expectation E of $\mathcal{NT}(X)$, similar to the case of a quasi-lattice ordered pair (G, P). The reason for this is that the Fock module $\bigoplus_{p \in P} X_p$ induces a grading of P on the reduced Nica-Toeplitz algebra $\mathcal{NT}^r(X)$, and this grading alone suffices to give rise to the desired conditional expectation on $\mathcal{NT}(X)$. We next recall this construction, see [15, 16].

Given a compactly aligned product system of (essential) C^*-correspondences over a right LCM monoid P, there is a natural Nica covariant representation l of X on $\bigoplus_{p \in P} X_p$, the Fock representation. For the sake of clarity we use the symbol l for this representation, just as in [12] (but unlike e.g. [16, 3]), and reserve ℓ for a length function on a monoid. The C^*-algebra generated by l is the reduced Nica-Toeplitz algebra $\mathcal{NT}^r(X)$, and there is a canonical surjective $*$-homomorphism $\Lambda : \mathcal{NT}(X) \to \mathcal{NT}^r(X)$, see [16, Section 2].

The next result is implicit in [15], where it is formulated using the C^*-precategory picture of $\mathcal{NT}^r(X)$ and $\mathcal{NT}(X)$, see [15, Proposition 5.4], respectively [15, Corollary 6.5]. For the sake of completeness and easy reference we state it in a form suitable to the more familiar picture of $\mathcal{NT}^r(X)$ viewed as the closure of the subalgebra spanned by monomials $l(\xi)l(\eta)^*$ for $\xi \in X_p, \eta \in X_q, p, q \in P$ and similarly with $\mathcal{NT}(X)$ as the closure of the subalgebra spanned by monomials $i_X(\xi)i_X(\eta)^*$. Denote by $d : X \to P$ the degree homomorphism of monoids so that the fibre over $p \in P$ is $X_p = d^{-1}(\{p\})$. By definition, the *core subalgebra* of $\mathcal{NT}(X)$ is the fibre over the identity, namely

$$B_e^{i_X} = \overline{\operatorname{span}}\{i_X(\xi)i_X(\eta)^* \mid \xi, \eta \in X, d(\xi) = d(\eta)\}$$

and B_e^l of $\mathcal{NT}^r(X)$ as

$$B_e^l = \overline{\operatorname{span}}\{l(\xi)l(\eta)^* \mid \xi, \eta \in X, d(\xi) = d(\eta)\}.$$

Proposition 3.4. *Let X be a compactly aligned product system of essential C^*-correspondences over a cancellative right LCM monoid P, where $X_e = A$. Then there is a faithful conditional expectation $E^r : \mathcal{NT}^r(X) \to B_e^l$ such that*

$$E^r(\sum_{\xi,\eta \in F} l(\xi)l(\eta)^*) = \sum_{\{\xi \in F \mid d(\xi) = d(\eta)\}} l(\xi)l(\eta)^*, \tag{3.5}$$

where F is a finite subset of $X = \bigcup_p X_p$.

Further, there is a conditional expectation $E : \mathcal{NT}(X) \to B_e^{i_X}$ such that

$$E(\sum_{\xi,\eta \in F} i_X(\xi)i_X(\eta)^*) = \sum_{\{\xi \in F \mid d(\xi) = d(\eta)\}} i_X(\xi)i_X(\eta)^*, \tag{3.6}$$

where F is a finite subset of $X = \bigcup_p X_p$.

Proof. A routine calculation shows that the Fock representation l of the product system X and the Fock representation $\mathbb{L}$ of the C^*-precategory associated to X are related in the form

$$\mathbb{L}(\Theta_{\xi,\eta}) = l(\xi)l(\eta)^*, \quad \xi \in X_p, \quad \eta \in X_q, \quad p, q \in P,$$

where $\Theta_{\xi,\eta} \in \mathcal{K}(X_q, X_p)$, as prescribed by [16, Lemma 2.6]. Inserting this in [15, Equation (5.4)] gives the claim about E^r. Using the formula for E^r in the proof of [15, Corollary 6.5] gives the claim about E. □

Theorem 3.5. (cf. [3, Theorem 5.1]) *Assume that $X = \{X_p\}_{p\in P}$ is a compactly aligned product system of essential C^*-correspondences over a cancellative right LCM monoid P, with $X_e = A$, and consider the time evolution on $\mathcal{NT}(X)$ defined by a homomorphism $N : P \to (0, \infty)$. For every $\beta \in \mathbb{R}$, the map $\phi \mapsto \phi|_A$ defines a one-to-one correspondence between KMS_β-states on $\mathcal{NT}(X)$ that factor through the conditional expectation E and the tracial states on A satisfying* (1.1).

The proof follows the strategy laid down in [3, Section 5], which in itself follows by now classical ideas, see for example [23].

Lemma 3.6. *Assume the hypotheses of Theorem 3.5. Let τ be a tracial state on A satisfying* (1.1). *Then there is a unique state ϕ_0 on $B_e^{i_X}$ such that*

$$\phi_0(i_X(\xi)i_X(\eta)^*) = N(p)^{-\beta}\tau(\langle\eta, \xi\rangle) \text{ for } \xi, \eta \in X_p, p \in P. \tag{3.7}$$

Proof. To get hold of ϕ_0 we exploit the structure of $B_e^{i_X}$ as the norm closure of a nested family of subalgebras B_J similar to [3, Section 4], see also [7, Lemma 3.6]. More precisely, we recall that a quasi-lattice I is a partially ordered set such that for all $p, q \in I$, either p, q have a unique least common upper bound, denoted $p \vee q$, or have no common upper bound. The example from [3, Section 4] is that of a monoid in a quasi-lattice ordered pair. The important observation in [3, Lemma 4.2] is that the order structure alone, regardless of the monoid structure, allows to express the core C^*-subalgebra as $\overline{\bigcup_F B_F}$, where F run over the finite $\vee$-closed subsets of I and $B_F = \oplus_{p\in F} B_p$, with $B_p = i_X^{(p)}(\mathcal{K}(X_p))$.

In our situation, for a right LCM monoid P we define an equivalence relation by $x \sim y$ if $y = xu$ for some invertible $u \in P^*$ and we denote the equivalence class of p by $[p] = pP^*$. The crux of the matter is that the quasi-lattice graded structure of the core and its dense subalgebra $\bigcup_F B_F$ exploited in [3, Lemma 4.2] is inherited by the set $P/_\sim := \{[p] \mid p \in P\}$ of equivalence classes. Indeed, as noticed in [15, Section 6], the set $P/_\sim$ is a *quasi-lattice* in the sense of [3, Section 4] for any right LCM monoid P. For each finite $\vee$-closed subset $\tilde{F}$ of $P/_\sim$ we obtain a C^*-subalgebra

$$B_{\tilde{F}} = \overline{\mathrm{span}}\{i_X^{(p)}(\mathcal{K}(X_p)) \mid [p] \in \tilde{F}\}$$

of $\mathcal{NT}(X)$, see [15, Lemma 6.6]. The collection of finite $\vee$-closed[1] subsets $\tilde{F}$ of $P/_\sim$ forms a directed set, which we denote $\mathcal{F}$, and we have that

$$B_e^{i_X} = \overline{\bigcup_{\tilde{F}\in\mathcal{F}} B_{\tilde{F}}}$$

by combining observations in the proof of Corollary 6.3 and of Theorem 6.1(b) from [15].

Thus we may carry out the construction of a positive linear functional ϕ_0 given by $\oplus_{[p]\in P/_\sim}\phi_{[p]}$ as in [3, Proposition 4.4 and section 5]. □

[1]Note that the definition of $\vee$-closed subset $F \subset P/_\sim$ in [15, Section 6.6] ought to be that $[p] \vee [q] \in F$ whenever $[p], [q] \in F$ admit a common upper bound.

Proof. (Proof of Theorem 3.5). Let ϕ_0 be the state on $B_e^{i_X}$ constructed in Lemma 3.6 from a tracial state τ on A satisfying condition (1.1). We claim that $\phi := \phi_0 \circ E$ satisfies the KMS condition, in which case we obtain surjectivity of the mapping in the statement of the theorem. To show the KMS_β condition, we claim that the calculations in the proof of [3, Theorem 5.1] are valid using the formula for the expectation E on $\mathcal{NT}(X)$ that comes from the Fock space grading, without reference to an ambient group. Thus, with $a = i_X(\xi)$, $\xi \in X_p$ and $b = i_X(\zeta)i_X(\eta)^*$, for $\zeta \in X_q, \eta \in X_r$, $p, q, r \in P$, we must show that $\phi(ab) = N(p)^{-\beta}\phi(ba)$. The case $pq = r$ is verbatim as in [3], but the case $pq \neq r$ requires some attention as we cannot form elements $pq^{-1}r$ and $qr^{-1}p$ in the absence of a group.

Assume that $pq \neq r$. Since $ab = i_X(\xi\zeta)i_X(\eta)^*$, it follows from (3.6) that $\phi(ab) = 0$. Turning to ba, we rewrite this following the idea of the proof of [15, Proposition 2.10], thus we let $\eta = R\eta'$ and $\xi = P\xi'$ for $R \in \mathcal{K}(X_r)$, $P \in \mathcal{K}(X_p)$, $\eta' \in X_r, \xi' \in X_p$. Then

$$ba = i_X(\zeta)i_X(\eta')^* i_X^{(r)}(R^*) i_X^{(p)}(P) i_X(\xi').$$

By Nica covariance of i_X we have that $ba = 0$ if $rP \cap pP = \emptyset$. Assume therefore that $rP \cap pP \neq \emptyset$. We have $rP \cap pP = sP$ with $s \in P$, and we let $pp' = rr' = s$ for unique p', r' in P. Therefore $ba \in i_X(X_{qr'})i_X(X_{p'})^*$, and we must prove that $qr' \neq p'$, in which case we can conclude that $\phi(ba) = 0$ by the definition of E. If $qr' = p'$, then $pqr' = pp' = rr'$, and right cancellation in P would give $pq = r$, a contradiction.

Injectivity of the mapping $\phi \mapsto \phi|_A$ follows because the restriction to $B_e^{i_X}$ of a KMS state is determined uniquely by the requirement (3.7). □

4. Reduction of the positivity condition to atoms

Throughout this section we assume that P is a Noetherian left cancellative right LCM monoid, that $\{X_p\}_{p\in P}$ is a compactly aligned product system of C^*-correspondences over P with $X_e = A$, that $N : P \to (0, \infty)$ is a multiplicative homomorphism and that $\beta \in \mathbb{R}$. We also assume that τ is a tracial state of A satisfying the positivity condition for finite subsets of atoms:

$$\tau(a) + \sum_{\emptyset \neq U \subset J} (-1)^{|U|} N(\vee U)^{-\beta}\, \mathrm{Tr}_\tau^{\vee U}(a) \geq 0 \qquad \text{for } a \in A_+ \quad J \subset P_a. \tag{4.1}$$

The goal of the section is to show that, under an extra hypothesis on P, the positivity also holds for all finite subsets of $P\backslash\{e\}$, as in (1.1). This gives a version of Theorem 3.5 that is stronger because it only requires verification of the subcollection of inequalities arising from subsets of atoms, which is finite if P is finitely generated. We are motivated by the simplification obtained for right-angled Artin monoids in [3, Theorem 9.1].

The key property we require for the reduction to atoms is the finiteness of a tree that we construct, largely following the proof of [20, Lemma 4.2]. In the formal arguments we replace finite subsets J of P by finite *lists* of elements of P, in which repetitions are allowed. The reason is that even if the initial list has no repetitions, repetitions may appear already at the second step.

4.1. **Positivity for finite lists in $P \cup \{\infty\}$.** We will use sums over lists of elements in P instead of subsets of P., and it is also convenient to allow ∞ in our lists. So for each $n \in \mathbb{N}$ we consider the index set $I_n := \{1, 2, \dots, n\}$ and define an n-list to be a function of the form $\lambda : I_n \to P \cup \{\infty\}$.

Lemma 4.1. *Given a tracial state τ on A that satisfies* (4.1)*, a list λ in $P \cup \{\infty\}$, and $\beta > 0$, there is a tracial linear functional $a \mapsto Z(\lambda, \tau, a)$ defined for each $a \in A$ such that*

$$Z(\lambda, \tau, a) := \sum_{U \subset I_n} (-1)^{|U|} N(\vee\lambda(U))^{-\beta}\, \mathrm{Tr}_\tau^{\vee\lambda(U)}(a), \quad a \in A, \tag{4.2}$$

where we follow the convention that if $\lambda(U)$ is not a clique, then $\vee\lambda(U) = \infty$ and $N(\vee\lambda(U))^{-\beta} = 0$.

Proof. When the subset J consists of a single atom, (4.1) is the assumption of Lemma 3.2(i), and then Lemma 3.2(ii) implies that $\mathrm{Tr}_\tau^{\vee\lambda(U)}$ is a tracial linear functional defined on all of A whenever $\vee\lambda(U) \in P$. When $\vee\lambda(U) = \infty$ it is irrelevant that $\mathrm{Tr}_\tau^{\vee\lambda(U)}(a)$ is not defined because then the corresponding coefficient vanishes. Hence $Z(\lambda, \tau, a)$ is well-defined and the function $a \mapsto Z(\lambda, \tau, a)$ is a tracial bounded linear functional on A. □

For each list λ, we let $\mathrm{Cl}_\lambda := \{U \subset I_n : \lambda(U) \in \mathrm{Cl}(P)\}$ be the collection of cliques in λ, explicitly, this is the collection of subsets of I_n on which λ has an upper bound in P. Since the terms associated to non-cliques vanish, we have

$$Z(\lambda, \tau, a) = \sum_{U \in \mathrm{Cl}_\lambda} (-1)^{|U|} N(\vee\lambda(U))^{-\beta} \,\mathrm{Tr}_\tau^{\vee\lambda(U)}(a),$$

where the sums are over over cliques or over lists, indistinctly.

If $J \subset P$ is a set with n elements, we replace it by a list $\lambda : I_n \to P$ with $\lambda(I_n) = J$ and write $Z(\lambda, \tau, a)$ as a sum over subsets of P. Again, the sum only depends on the part of the range of the list that lies in P, and terms corresponding to non-cliques vanish, that is,

$$Z(\lambda, \tau, a) = \sum_{K \subset \lambda(I) \cap P} (-1)^{|K|} N(\vee K)^{-\beta} \,\mathrm{Tr}_\tau^{\vee K}(a) = \sum_{K \in \mathrm{Cl}(\lambda(I))} (-1)^{|K|} N(\vee K)^{-\beta} \,\mathrm{Tr}_\tau^{\vee K}(a).$$

We see next that multiples, repetitions, and invertible elements in a list can be ignored as far as $Z(\lambda, \tau, a)$ is concerned.

Lemma 4.2. *Suppose $\lambda : I_n \to P$ is a list and assume $\lambda(i) \leq \lambda(j)$ for some i and j with $i \neq j$. Denote by $\hat{\lambda}$ the list obtained by restricting λ to $I_n \setminus \{j\}$. Then $Z(\lambda, \tau, a) = Z(\hat{\lambda}, \tau, a)$. In particular if $\lambda(I_n) \cap P^* \neq \emptyset$, then $Z(\lambda, \tau, a) = 0$ for all a.*

Proof. Suppose $\lambda(i) \leq \lambda(j)$ for $i, j \in I_n$ with $i \neq j$. The λ-cliques $U \subset I_n$ that contain j fall into two classes, namely those that contain i and those that do not. The operation of 'removing i' establishes a bijection of the first class onto the second one, reducing the cardinality of the clique by one but leaving the least upper bound unchanged. This changes the sign of the corresponding summand of $Z(\lambda, \tau, a)$, but not its absolute value, so summands paired this way cancel each other out.

If we assume now $\lambda(i) \in P^*$, then $\lambda(i) \leq \lambda(j)$ for every $j \in I_n$, and we may remove successively each $\lambda(j)$ for $j \neq i$ from λ, finally arriving at $Z(\lambda, \tau, a) = Z(\{i\}, \tau, a)$ which is easily seen to be zero because the only two cliques are $\emptyset$ and $\{i\}$, which have opposite signs and the same absolute value $\tau(a)$. □

Definition 4.3. A list $\lambda : I_n \to P \cup \{\infty\}$ will be called a *leaf* if either $\lambda(I_n) \subset P_a \cup \{\infty\}$ or $\lambda(I_n) \cap P^* \neq \emptyset$.

Next we define an iteration step. Let λ be a list in $P \cup \{\infty\}$. When λ is not a leaf, let i be the smallest number in $\{1, 2, \dots, n\}$ for which $\lambda(i) = pq$ with $p, q \in P \setminus P^*$ (here we may assume without loss of generality that p is a generator) and define two new lists, λ_1 and λ_2, in $P \cup \{\infty\}$ as follows.

$$\text{(4.3)} \qquad \lambda_1(j) := \begin{cases} \lambda(j) & \text{if } j \neq i \\ p & \text{if } j = i; \end{cases} \qquad \lambda_2(j) := \begin{cases} p^{-1}(p \vee \lambda(j)) & \text{if } p \vee \lambda(j) < \infty \\ \infty & \text{if } p \vee \lambda(j) = \infty. \end{cases}$$

In English: the list λ_1 is obtained from λ by simply replacing the i^{th} term pq by p and leaving the rest of the terms unchanged, and the list λ_2 is obtained by replacing every term in λ by its image under the transformation $x \mapsto p^{-1}(x \vee p)$ (here we let $p \vee x$ be any choice of least common upper bound of p and x if there is one and ∞ if there is none). Notice that, in particular, we may choose $\lambda_2(i) = p^{-1}(p \vee pq) = q$.

Clearly $\lambda_1(j) = \infty$ if and only if $\lambda(j) = \infty$, and $\lambda_2(j) = \infty$ if $\lambda(j) = \infty$, but it is certainly possible to have $\lambda_2(j) = \infty$ with $\lambda(j) < \infty$.

Lemma 4.4. *Let P be a right LCM monoid. Let $\lambda : I_n \to P \cup \{\infty\}$ be a list that is not a leaf and let $i \in I$ be the first index such that $\lambda(i) = pq$ for $p \in P_a$ and $q \in P \setminus P^*$. For each $U \subset I_n$ we have*

(1) *if $\lambda(U)$ is a clique, then so is $\lambda_1(U)$, and*

$$\bigvee \lambda_1(U) \leq \bigvee \lambda(U);$$

(2) *$\lambda(U) \cup \{p\}$ is a clique if and only if $\lambda_2(U)$ is a clique, and*

$$\bigvee \lambda(U) \vee p = p \cdot \bigvee \lambda_2(U);$$

(3) *when $i \in U$ we have that $\lambda_1(U)$ is a clique if and only if $\lambda_2(U \setminus \{i\})$ is a clique, and*

$$\bigvee \lambda_1(U) = p \cdot \bigvee \lambda_2(U \setminus \{i\}).$$

Proof. Suppose λ is not a leaf and $\lambda(i) = pq$ as in the statement. Part (1) can be split into two cases. First assume $i \notin U$; then obviously $\lambda(U) = \lambda_1(U)$, and the claim follows. Next assume $i \in U$; then clearly $\lambda(i)P = pqP \subset pP = \lambda_1(i)P$, and thus $\bigcap_{j \in U} \lambda(j)P \subset \bigcap_{j \in U} \lambda_1(j)P$. This gives the inequality in part (1). Cliques are preserved because $\bigvee \lambda(U) < \infty$ implies $\bigcap_j \lambda_1(j)P \neq \emptyset$.

For part (2), notice that

$$p \bigcap_{j \in U} \lambda_2(j)P = \bigcap_{j \in U} p\lambda_2(j)P = \bigcap_{j \in U} (p \vee \lambda(j))P = \bigcap_{j \in U} (pP \cap \lambda(j)P), \tag{4.4}$$

Interpreting this for sets of right LCMs, we see that

$$p \cdot \bigvee \lambda_2(U) = \bigvee p\lambda_2(U) = p \vee \bigvee \lambda(U) = \bigvee (\lambda(U) \cup \{p\}),$$

which proves the second assertion.

For part (3), we notice that since $\lambda_1(j) = \lambda(j)$ for $j \in U \setminus \{i\}$, the intersection over $U \setminus \{i\}$ can be transformed into one over U; specifically,

$$p \bigcap_{j \in U \setminus \{i\}} \lambda_2(j)P = \Big(\bigcap_{j \in U \setminus \{i\}} p\lambda_2(j)P\Big) \cap pP = \bigcap_{j \in U \setminus \{i\}} (pP \cap \lambda(j)P) \cap \lambda_1(i)P = \bigcap_{j \in U} \lambda_1(j)P, \tag{4.5}$$

so the sets of right LCMs satisfy

$$p \cdot \bigvee \lambda_2(U \setminus \{i\}) = \bigvee \lambda_1(U).$$

That cliques are preserved both ways in cases (2) and (3) is easy to see from (4.4) and (4.5). □

Remark 4.5. In some cases the map $\lambda \mapsto \lambda_1$ can send an unbounded list to a clique. For an easy example consider $P = \mathbb{F}_2^+$, the free monoid on two generators a and b. Then the list $\lambda = (ab, a^2)$ is unbounded, but $\lambda_1 = (a, a^2)$ is a clique.

The map $\lambda \mapsto \lambda_1$ does not introduce new invertibles or new infinites. On the other hand, $\lambda \mapsto \lambda_2$ has the capacity to generate new invertibles and new infinites.

Lemma 4.6. *Let X be an essential C^*-correspondence over A. Suppose $\tau_1, \ldots, \tau_n$ are positive finite traces on A such that $\mathrm{Tr}^X_{\tau_i}$ is finite for each $i = 1, \ldots, n$, and let $c_1, \ldots, c_n \in \mathbb{R}$. If $\sum_{i=1}^n c_i \tau_i$ is positive on A, then so is $\sum_{i=1}^n c_i \mathrm{Tr}^X_{\tau_i}(\cdot)$.*

Proof. By the linearity properties of the induced trace we have

$$\sum_{i=1}^{n} c_i \operatorname{Tr}_{\tau_i}^{X}(a) = \operatorname{Tr}^{X}_{\sum_{i=1}^{n} c_i\tau_i}(a) \quad \text{for all } a \in A_+.$$

The result follows because inducing preserves positivity. ☐

Proposition 4.7. *Suppose P is a Noetherian left cancellative right LCM monoid, that $\{X_p\}_{p\in P}$ is a compactly aligned product system of C^*-correspondences over P with $X_e = A$, that $N : P \to (0,\infty)$ is a multiplicative homomorphism and that $\beta \in \mathbb{R}$. Assume that τ is a tracial state of A satisfying the positivity condition* (4.1) *for finite subsets of atoms.*

If λ, λ_1, and λ_2 are lists, with $\lambda(i) = pq$ as in (4.3)*, then*

$$Z(\lambda, \tau, a) = Z(\lambda_1, \tau, a) + \frac{1}{N(p)^\beta} \sum_{U\subset I_n} (-1)^{|U|} N(\vee\lambda_2(U))^{-\beta} \operatorname{Tr}^{p}_{F_{\vee\lambda_2(U)}(\tau)}(a) \quad \text{for all } a \in A_+.$$

Proof. Let $i \in I_n$ be given as in (4.3). We compute the difference $Z(\lambda, \tau, a) - Z(\lambda_1, \tau, a)$ for $a \in A_+$. Notice that the terms corresponding to each clique $U \subset I_n$ with $i \notin U$ cancel, because then $\lambda(U) = \lambda_1(U)$. So we only need to write terms with $i \in U \subset I_n$. Therefore

$$Z(\lambda, \tau, a) - Z(\lambda_1, \tau, a) = \sum_{i\in U\subset I_n} (-1)^{|U|} \Big(N(\vee\lambda(U))^{-\beta} \operatorname{Tr}_\tau^{\vee\lambda(U)}(a) - N(\vee\lambda_1(U))^{-\beta} \operatorname{Tr}_\tau^{\vee\lambda_1(U)}(a) \Big)$$

Since $i \in U$, so that $\lambda(U) \cup \{p\} = \lambda(U)$, Lemma 4.4(2) gives $N(\vee\lambda(U)) = N(p)N(\vee\lambda_2(U))$ and Lemma 4.4(3) gives $N(\vee\lambda_1(U)) = N(p)N(\vee\lambda_2(U \setminus \{i\}))$, so

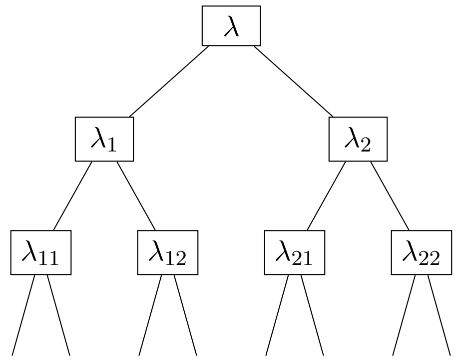
$$Z(\lambda, \tau, a) - Z(\lambda_1, \tau, a) = \sum_{i\in U\subset I_n} \frac{(-1)^{|U|}}{N(p)^\beta} \Big(N(\vee\lambda_2(U))^{-\beta} \operatorname{Tr}_\tau^{\vee\lambda(U)}(a) - N(\vee\lambda_2(U\setminus\{i\}))^{-\beta} \operatorname{Tr}_\tau^{\vee\lambda_1(U)}(a) \Big).$$

When we replace $U \setminus \{i\}$ by V, which is a set that ranges over the subsets of $I_n \setminus \{i\}$, and we adjust the sign to take into account that $|V| = |U| - 1$, we get

$$\begin{aligned} Z(\lambda,\tau, a) - Z(\lambda_1, \tau, a) = \frac{1}{N(p)^\beta} \Big(&\sum_{i\in U\subset I_n} (-1)^{|U|} N(\vee\lambda_2(U))^{-\beta} \operatorname{Tr}_\tau^{\bigvee \lambda(U)}(a) \\ &+ \sum_{i\notin V\subset I_n} (-1)^{|V|} N(\vee\lambda_2(V))^{-\beta} \operatorname{Tr}_\tau^{\lambda_1(i)\vee\bigvee \lambda_1(V)}(a) \Big). \end{aligned} \tag{4.6}$$

Recall now that by (3.2) we have $\operatorname{Tr}_\tau^{\bigvee \lambda(U)} = \operatorname{Tr}^{p}_{F_{\vee\lambda_2(U)}(\tau)}$ for each U containing i and $\operatorname{Tr}_\tau^{\lambda_1(i)\vee\bigvee \lambda(V)} = \operatorname{Tr}^{p}_{F_{\vee\lambda_2(V)}(\tau)}$ for each V that does not contain i. Thus we may collect the two sums from (4.6) into the single sum, completing the proof. ☐

By recursion, the maps $\lambda \mapsto \lambda_1$ and $\lambda \mapsto \lambda_2$ generate a binary tree rooted at λ. This is similar to the proof of [20, Theorem 4.8], but here we use the following tree with nodes indexed by lists,



At level k we have nodes indexed by λ_ω, where ω is a word of length k in the free monoid $\mathbb{F}^+\{1,2\}$ generated by the symbols 1 and 2. The level $k+1$ is generated by splitting each node λ_ω into two new nodes $\lambda_{\omega 1} := (\lambda_\omega)_1$ and $\lambda_{\omega 2} := (\lambda_\omega)_2$ defined according to the recursive step (4.3). As a matter of notation, whenever ω is a finite or infinite word with entries in $\{1,2\}$, we denote the initial subword of length k by $\omega[1,k]$.

According to Definition 4.3 a node λ_ω is a leaf if $\lambda_\omega(j) \in P^*$ for some j, or if for every $j \in I_n$ there is no notrivial factorisation of $\lambda_\omega(j)$. We are interested in a branching process that stops when $Z(\lambda_\omega, \tau, a) \geq 0$, and this will be ensured either by the positivity assumption for lists consisting of atoms and infinities, or by Lemma 4.2 for lists that have an invertible element, because then $Z(\lambda, \tau, a) = 0$. Thus we will say that a *branch* is a finite or infinite word ω on the symbols $\{1,2\}$ that starts at the root λ and either ends at the first node such that $\lambda_{\omega[1,k]}$ is a leaf, or else does not end at all, should such a node not exist.

Theorem 4.8. *Under the assumptions of Proposition 4.7, suppose that the tree (λ_ω) constructed above is finite for every list λ. Then reduction of positivity to atoms holds for P. Specifically, a tracial state τ on A satisfies* (1.1) *if and only if it satisfies the weaker condition* (4.1).

Proof. Suppose that τ is a tracial state of A that satisfies the positivity condition (4.1) for subsets of atoms. Let J be a subset of P and take a list λ with range J. The left hand side of (1.1) then equals $Z(\lambda, \tau, a)$ so it suffices to show that $Z(\lambda, \tau, a) \geq 0$ for every $a \in A^+$.

Assume that λ is a list in P such that the positivity condition (1.1) holds for both λ_1 and λ_2 for every $a \in A$. Notice that

$$Z(\lambda_2, \tau, \cdot) = \sum_{U \subset I_n} (-1)^{|U|} N(\vee\lambda_2(U))^{-\beta} \operatorname{Tr}_\tau^{\vee\lambda_2(U)} = \sum_{U \subset I_n} (-1)^{|U|} N(\vee\lambda_2(U))^{-\beta} (F_{\vee\lambda_2(U)}\tau)$$

is a linear combination of the finite positive traces $F_{\vee\lambda_2(U)}\tau$. Since $Z(\lambda_2, \tau, \cdot) \geq 0$ by assumption, Lemma 4.6 implies that $\frac{1}{N(p)^\beta} \sum_{U \subset I_n} (-1)^{|U|} N(\vee\lambda_2(U))^{-\beta} \operatorname{Tr}^p_{F_{\vee\lambda_2(U)}(\tau)}(a) \geq 0$, and then Proposition 4.7 shows that positivity holds for λ.

Hence, when positivity fails for a list λ, then it must also fail for either λ_1 or λ_2, and iteration of this process, say, choosing the first edge whenever positivity fails for both, generates a branch of the tree of λ such that positivity fails at every node. Since the tree of λ is finite by assumption, this branch must end at a leaf, on which positivity also fails. But this is a contradiction, because infinities in a list do not affect the sum, invertible elements cause the function Z to vanish, and if the list consists only of atoms, then the value is nonnegative by assumption (4.1). □

As an immediate consequence we obtain a stronger version of Theorem 3.5.

Corollary 4.9. *If in addition to the assumptions of Theorem 3.5 the tree of every list is finite, then $\phi \mapsto \phi|_A$ is a one-to-one correspondence between KMS_β-states on $\mathcal{NT}(X)$ that factor through the conditional expectation E and the tracial states on A satisfying* (4.1).

4.2. **The directed case.** If the right LCM monoid P is directed (hence a semi-lattice), in the sense that $pP \cap qP \neq \emptyset$ for every $p, q \in P$, we can actually prove that the tree (λ_ω) is finite.

Proposition 4.10. *Suppose P is a Noetherian, right LCM monoid such that $xP \cap yP \neq \emptyset$ for every pair $x, y \in P$, and let $\lambda : I_n \to P$ be a list of elements of P. Then the tree (λ_ω) is finite, and reduction of positivity to atoms holds for P.*

Proof. By König's lemma, the tree is infinite if and only if it has an infinite branch. By definition, an infinite branch consists of an infinite sequence $\lambda_\omega = (\lambda_{\omega[1,k]})_{k\in\mathbb{N}}$ of nodes associated to an infinite word ω over $\{1, 2\}$ such that for every level k the set $\lambda_{\omega[1,k]}(I_n)$ contains no invertible elements and has at least one element with a nontrivial factorisation. We will show that there are no such infinite branches.

We claim first that a branch ω can only have finitely many 2's. Aiming for a contradiction, we assume ω has infinitely many 2's, occurring precisely as ω_{k_j} for an infinite increasing sequence $(k_j)_{j\in\mathbb{N}}$, while the rest of the ω_i are equal to 1. Thus, $\omega[1, k_1 - 1]$ is a string consisting of $k_1 - 1 \geq 0$ symbols equal to 1, followed by $\omega_{k_1} = 2$ (the first 2 in ω) followed by the string $\omega[k_1 + 1, k_2 - 1]$ of $k_2 - k_1 - 1 \geq 0$ symbols equal to 1, followed ω_{k_2} (the second 2 in ω), and so on. Since $\omega(j) = 1$ for $j \leq k_1 - 1$. Lemma 4.4(1) applied $k_1 - 1$ times gives

$$\vee\lambda = \vee\lambda_{\omega[1]} g_1' = (\vee\lambda_{\omega[1,2]} g_2') g_1' = \cdots = (\vee\lambda_{\omega[1,k_1-1]} g_{k_1-1}') g_{k_1-2}' \cdots g_2' g_1',$$

and we set $g_{k_1} = g_{k_1-1}' \cdots g_2' g_1'$, where $g_{k_1} = e$ in case $k_1 = 1$. Since $\omega_{k_1} = 2$, Lemma 4.4(2) applied to the list $\lambda_{\omega[1,k_1-1]}$ with $U = I_n$ gives

$$\vee\lambda_{\omega[1,k_1-1]} = p_{k_1}(\vee(\lambda_{\omega[1,k_1-1]})_2) = p_{k_1}(\vee\lambda_{\omega[1,k_1]})$$

for some generator $p_{k_1} \in P_a$. Hence, writing $d_k := \vee\lambda_{\omega[1,k]}$ for $k = 1, 2, \ldots$, we have shown that

$$d_0 = \vee\lambda = d_{k_1-1} g_{k_1} = p_{k_1} d_{k_1} g_{k_1}. \tag{4.7}$$

Now, let $k_2 > k_1$ be the second occurrence of a 2 in ω. If $k_2 = k_1 + 1$, set $g_{k_2} = e$. Otherwise, by Lemma 4.4(1) applied $k_2 - k_1 - 1$ times and by Lemma 4.4 applied at the k_2 step we find $g_j' \in P$ for $j = k_1 + 1, \ldots, k_2 - 1$ and a generator $p_{k_2} \in P_a$ such that

$$\vee\lambda_{\omega[1,k_2-1]} = p_{k_2}(\vee\lambda_{\omega[1,k_2]})$$

and

$$\vee\lambda_{\omega[1,k_1]} = \vee\lambda_{\omega[1,k_2-1]} g_{k_2-1}' \cdots g_{k_1+1}'.$$

Putting $g_{k_2} = g_{k_2-1}' \cdots g_{k_1+1}'$, gives

$$d_{k_1} = d_{k_2-1} g_{k_2} = p_{k_2} d_{k_2} g_{k_2}.$$

Inserting this in (4.7) we get

$$d_0 = p_{k_1}(p_{k_2} d_{k_2} g_{k_2}) g_{k_1}.$$

Continuing this way we get two sequences $\{p_{k_m}\}$ in $P \setminus P^*$ and $\{g_{k_m}\}$ in P such that

$$d_0 = p_{k_1} \cdots p_{k_m} d_{k_m+1} g_{k_m} \cdots g_{k_1}.$$

Thus $(p_{k_1}, p_{k_1}p_{k_2}, p_{k_1}p_{k_2}p_{k_3}, \ldots)$ is a strictly increasing sequence of left divisors of d_0, which contradicts the assumption that P is Noetherian and completes the proof of the claim.

It follows that there exists $\bar{k}$ such that $\omega[k] = 1$ for all $k \geq \bar{k}$. We will show that this leads to a contradiction. Notice that, by definition, the process defining λ_1 will exhaust the factorisation of $\lambda_{\omega_{[1,\bar{k}]}}(1)$ before moving onto that of $\lambda_{\omega_{[1,\bar{k}]}}(2)$, and so on. Continuing, we see that the j-th term $\lambda_{\omega_{[1,\bar{k}]}}(j)$ of the list remains constant down the branch until it is changed for the first time after the initial $j-1$ terms have been replaced by generators, admitting no further nontrivial factorisations. Each step in the λ_1-process requires a nontrivial factorisation of exactly one of the terms in the list at that level, that is, at the $(\bar{k} + y)$-th step, there exist exactly one $j \in I_n$, $p_y \in P_a$ and $q_y \in P \setminus P^*$

such that $\lambda_{\omega_{[1,\bar{k}+y]}}(j) = p_y q_y$ and $\lambda_{\omega_{[1,\bar{k}+y+1]}} = p_y$. The other terms in the list remain unchanged. Since the list is finite, the λ_1-process must stop after finitely many steps. This contradicts the assumption that ω is an infinite branch, proving that the tree has to be finite. The last assertion now follows by Theorem 4.8 □

The question of which monoids have reduction of positivity to subsets of atoms was raised in the introduction of [3]. As a corollary we can add Artin monoids of finite type to that class.

Corollary 4.11. *Reduction of positivity to generators holds for Artin monoids of finite type.*

Proof. The result follows from Proposition 4.10 because Artin monoids of finite type are Noetherian, directed, right LCM monoids. □

4.3. **Right-angled Artin monoids.** Our methods also recover the fact that reduction of positivity holds for right-angled Artin monoids, cf. [3, Theorem 9.1].

Proposition 4.12. *Suppose P is a right-angled Artin monoid, and let $\lambda : I_n \to P$ be a list of elements of P. Then the tree (λ_ω) is finite, and reduction of positivity to generators holds for P.*

Proof. Let $\omega \in \mathbb{F}^+\{1,2\}$, and suppose that $\lambda_{\omega[1,k]}(I_n)$ is not a leaf for some $k \in \mathbb{N}$. Recall from the definition of λ_1 and λ_2 in (4.3) that when we factor the element $\lambda(i) = sq$ in $\lambda_\omega(I_n)$, we choose s to be a generator, that is $s \in P_a$.

Let $\ell : P \to \mathbb{N}$ be the normalized length function on P (see Remark 2.4). We claim that $\lambda_{\omega[1,k+1]}(j)$ either satisfies $\lambda_{\omega[1,k+1]}(j) = \infty$ or else $\ell(\lambda_{\omega[1,k+1]}(j)) \leq \ell(\lambda_{\omega[1,k]}(j))$ for all $j = 1, \dots, n$, and that the inequality is strict at least for the subindex i where the factorisation occurs. Indeed, the length reduction clearly holds in λ_1-step, that is, when $\omega_{k+1} = 1$, because the length of the element $\lambda_{\omega[1,k]}(i)$ being factorised goes down to 1, while the lengths of all the other elements remain the same. Next we show that the length reduction also holds in the case of a λ_2-step, that is, when $\omega_{k+1} = 2$. From [3, Lemma 9.2], we know that $s \vee p \in \{p, sp, s, \infty\}$ for all $p \in P$ and $s \in P_a$. Thus $s^{-1}(s \vee p) \in \{s^{-1}p, p, e, \infty\}$, where the case $s^{-1}p$ only occurs if s is an initial letter in some expression of p, in which case we have that $s^{-1}p \in P$ with $\ell(s^{-1}p) < \ell(p)$. From this it becomes clear that either $\lambda_{\omega[1,k+1]}(j) = \infty$ or

$$\lambda_{\omega[1,k+1]}(i) = \ell(s^{-1}(p \vee s)) \leq \ell(p) = \ell(\lambda_{\omega[1,k]}(i)),$$

for all $i = 1, \dots, n$. In this case the term $\lambda_{\omega[1,k]}(i)$ being factorised goes down in length by 1. This finishes the proof of our claim.

If we now account for the total length $\sum_j \ell(\lambda(j))$ using the convention that $\ell(e) = \ell(\infty) = 0$, we see that $k \mapsto \sum_j \ell(\lambda_{\omega[1,k]}(j))$ is a strictly decreasing sequence with values in the positive integers. This excludes the possibility of an infinite branch, and thus the tree (λ_ω) is finite by König's lemma. □

5. Reduction to minimal elements

In this section we show that for all weak quasi-lattice ordered groups (G, P) with P Noetherian there is a reduction of the positivity criterion (1.1) to a system of inequalities associated to subsets of a set P_{inf}, see the definition below, that contains the atoms but may have other elements, although in some special cases we can verify that $P_{\text{inf}} = P_a$. The definition of the set P_{inf} is motivated by [20, Definition 4.9].

Definition 5.1. Suppose (G, P) is a weak quasi-lattice ordered group with set of atoms P_a. Set $P_1 := P_a$ and, recursively, $P_n := P_a^{-1} P_{n-1} \cap P = \{x^{-1}p \mid x \in P_a, p \in P_n \cap xP\}$ for each $n \geq 2$; then define the minimal set

$$P_{\text{inf}} := \bigcup_{n \in \mathbb{N}} P_n.$$

Lemma 5.2. *The minimal set $P_{\rm inf}$ is the smallest subset of P that contains P_a and is closed under the operations $p \mapsto x^{-1}(x \vee p)$ for $x \in P_a$.*

Proof. Clearly the set $P_{\rm inf}$ contains P_a and is closed under $p \mapsto x^{-1}(x \vee p)$, and if $Q \subset P$ contains P_a and is closed under $p \mapsto x^{-1}(x \vee p)$, then it obviously contains each of the P_n, and hence $P_{\rm inf}$. ☐

Recall that Ω_K denotes the intersection $\bigcap_{k\in K}(P\backslash kP)$ for each finite subset $K \subset P$; and set $\Omega_\emptyset = P$. Given a nonempty subset R of P, we define $\mathcal{A}_R$ to be the collection of all *finite (or empty) disjoint unions* of sets $p\Omega_K$:

$$\mathcal{A}_R := \{\bigsqcup_{j=1}^{n} p_j\Omega_{K_j} : p_j \in P \text{ and } K_j \subset R \text{ finite or empty}\}. \tag{5.1}$$

By convention $\emptyset \in \mathcal{A}_R$, arising from the empty union. It is apparent that if $R \subset T \subset P$, then $\mathcal{A}_R \subset \mathcal{A}_T$. Our strategy for proving reductions of (1.1) to a smaller set of inequalities based on $P_{\rm inf}$ is to show that when P is Noetherian, $\mathcal{A}_{P_{\rm inf}}$ is itself an algebra of subsets of P, namely the algebra $\mathcal{B}_P$ from Lemma 2.5.

Lemma 5.3. *The collection $\mathcal{A}_{P_a} = \{\sqcup_{j=1}^{n} p_j\Omega_{K_j} : p_j \in P$ and $K_j \subset P_a$ finite or empty$\}$ contains the set pP and its complement $P \setminus pP$ for every $p \in P$.*

Proof. Let $p \in P$. Setting $K = \emptyset$ in the definition shows that $pP = p\Omega_\emptyset \in \mathcal{A}_{P_a}$. Suppose now that $p = s_{i_1} \dots s_{i_k}$ is an expression for p in terms of atoms.

Then we have

$$\begin{aligned} P \setminus pP &= (P\backslash s_{i_1}P) \sqcup (s_{i_1}P\backslash s_{i_1}s_{i_2}P) \sqcup \dots \sqcup (s_{i_1} \dots s_{i_{k-1}}P\backslash s_{i_1} \dots s_{i_k}P) \\ &= (P\backslash s_{i_1}P) \sqcup s_{i_1}(P\backslash s_{i_2}P) \sqcup \dots \sqcup s_{i_1} \dots s_{i_{k-1}}(P\backslash s_{i_k}P), \end{aligned}$$

which is in $\mathcal{A}_{P_a}$ by definition. ☐

Lemma 5.4. *Let (G, P) be a weak quasi-lattice ordered group with P Noetherian. Then $\mathcal{A}_{P_{\rm inf}}$ is the algebra $\mathcal{B}_P$.*

Proof. By [3, Lemma 2.4] it suffices to prove that $\Omega_K \in \mathcal{A}_{P_{\rm inf}}$ for every finite subset $K = \{q_1, \dots, q_n\}$ of P. We do this by a double induction argument: for an arbitrary n, we first apply induction on the number of atoms in q_i, and then on k, the cardinality of $K \cap P$.

Suppose first $K = \{q_1, \dots, q_n\}$ with $q_1 \in P$ and $q_2, \dots, q_n \in P_{\rm inf}$. We use induction on the number of atoms in a factorisation of q_1. If q_1 is an atom then $q_1 \in P_{\rm inf}$, then Ω_K is in $\mathcal{A}_{P_{\rm inf}}$ by definition. Assume now that $\Omega_K \in \mathcal{A}_{P_{\rm inf}}$ for all K as above in which $q_1 \in P$ is a product of at most N atoms, and let $q \in P$ be a product of $N+1$ atoms, so that $q = s_1q'$ for $s_1 \in P_a$ and q' a product of N atoms. Since $\Omega_q = (P\backslash s_1P) \sqcup s_1(P\backslash q'P)$, we have

$$\Omega_{\{q,q_2,\dots,q_n\}} = [(P\backslash s_1P) \sqcup s_1(P\backslash q'P)] \cap \Omega_{\{q_2,\dots,q_n\}} = \Omega_{\{s_1,q_2,\dots,q_n\}} \sqcup s_1(P\backslash q'P) \cap \Omega_{\{q_2,\dots,q_n\}}. \tag{5.2}$$

By definition $\Omega_{\{s_1,q_2,\dots,q_n\}} \in \mathcal{A}_{P_{\rm inf}}$, so we only need to show that $s_1(P\backslash q'P) \cap \Omega_{\{q_2,\dots,q_n\}} \in \mathcal{A}_{P_{\rm inf}}$. We rewrite this intersection as

$$\begin{aligned} s_1(P\backslash q'P) \cap \Omega_{\{q_2,\dots,q_n\}} &= s_1(P\backslash q'P) \cap (s_1P\backslash q_2P) \cap \dots \cap (s_1P\backslash q_nP) \\ &= s_1(P\backslash q'P) \cap (s_1P\backslash (s_1 \vee q_2)P) \cap \dots \cap (s_1P\backslash (s_1 \vee q_n)P) \\ &= s_1[(P\backslash q'P) \cap (P\backslash s_1^{-1}(s_1 \vee q_2)P) \cap \dots \cap (P\backslash s_1^{-1}(s_1 \vee q_n)P)]. \end{aligned} \tag{5.3}$$

By the definition of P_{inf} we have that $s_1^{-1}(s_1 \vee q_i) \in P_{inf}$, since $s_1, q_i \in P_{inf}$ for $i = 1, \dots, n$. Moreover, q' is a product of n atoms, thus, by the induction hypothesis, the intersection in the last line above is contained in $\mathcal{A}_{P_{inf}}$. From this we can conclude that $\mathcal{A}_{P_{\mathrm{inf}}}$ contains the set $\Omega_{\{q,q_2,\dots,q_n\}} \in \mathcal{A}_{P_{\mathrm{inf}}}$ whenever $q \in P$ and $q_j \in P_{\mathrm{inf}}$, for $j = 2, \dots, n$.

If $K = \{q_1, q_2, \dots, q_n\}$ with $q_1 \in P$ and $q_2, \dots, q_n \in P_{\mathrm{inf}}$, then $\Omega_K \in \mathcal{A}_{P_{\mathrm{inf}}}$ by the first part of the proof. Assume now that $\Omega_K \in \mathcal{A}_{P_{\mathrm{inf}}}$ for all $K = \{q_1, q_2, \dots, q_n\}$ such that $q_1, \dots, q_k \in P$ and $q_{k+1}, \dots, q_n \in P_{\mathrm{inf}}$. We aim to show that

$$(5.4) \qquad \Omega_K \in \mathcal{A}_{P_{\mathrm{inf}}} \qquad \text{when } q_1, \dots, q_{k+1} \in P \text{ and } q_{k+2}, \dots, q_n \in P_{\mathrm{inf}}.$$

As in the first part we use induction on the number of atoms in q_{k+1}. If q_{k+1} is an atom itself, then $q_{k+1} \in P_a \subset P_{\mathrm{inf}}$, thus $\Omega_{\{q_1,\dots,q_{k+1},\dots,q_n\}} \in \mathcal{A}_{P_{\mathrm{inf}}}$ by the induction hypothesis. Assume now that (5.4) holds true when the $(k+1)$th element is a product of at most M atoms. Suppose now the $(k+1)$th element is a product of $M+1$ atoms, and write $q_{k+1} = sq' \in P$ with $s \in P_a$ and $q' \in P$ a product of M atoms. As before,

$$(P \backslash q'_{k+1}P) = (P \backslash sP) \sqcup s(P \backslash q'P).$$

This splits $\Omega_{\{q_1,\dots,q_{k+1},\dots,q_n\}}$ into a disjoint union of two intersections. The first one has an atom s in the $(k+1)$th term so it is in $\mathcal{A}_{P_{\mathrm{inf}}}$ by the induction hypothesis, so proving (5.4) reduces to showing that the second one, namely $(P \backslash q_1P) \cap \dots \cap s(P \backslash q'P) \cap \dots \cap (P \backslash q_nP)$ is in $\mathcal{A}_{P_{\mathrm{inf}}}$. This holds by the second induction hypothesis because

$$(P \backslash q_1P) \cap \dots \cap s(P \backslash q'P) \cap \dots \cap (P \backslash q_nP) = \\ s[(P \backslash s^{-1}(s \vee q_1)P) \cap \dots \cap (P \backslash q'P) \cap \dots \cap (P \backslash s^{-1}(s \vee q_n)P))]$$

and q' is the product of M atoms. This concludes the proof. $\square$

We are now ready to prove the main result of this section.

Theorem 5.5. *Let (G, P) be a weak quasi-lattice ordered group with P Noetherian. Assume that we are given a compactly aligned product system $\{X_p\}_{p \in P}$ of essential C^*-correspondences over P with $X_e = A$, a homomorphism $N : P \to (0, \infty)$ and $\beta \in \mathbb{R}$. Then a tracial state τ on A satisfies the condition of* (1.1) *if and only if*

$$(5.5) \qquad \tau(a) + \sum_{\emptyset \neq K \subset J} (-1)^{|K|} N(\vee K)^{-\beta} \operatorname{Tr}_\tau^{\vee K}(a) \geq 0 \qquad \text{for all finite } J \subset P_{\mathrm{inf}} \text{ and } a \in A_+,$$

where the terms corresponding to $\vee K = \infty$ are set to zero.

Proof. Obviously (1.1) implies (5.5). The proof that (5.5) implies (1.1) goes along the lines of the proof of [3, Theorem 9.1], which is the particular case of right-angled Artin monoids.

Suppose τ is a tracial state of A satisfying (5.5). By taking $J = \{s\}$ for $s \in P_a$ we see that $F_s(\tau)(a) \leq N(s)^\beta \tau(a) < \infty$ for all $s \in P_a$ and $a \in A_+$. By Lemma 3.2, we have that $\operatorname{Tr}_\tau^p(a)$ is finite for all $p \in P$ and $a \in A_+$.

Next recall that $\mathcal{B}_P$ is the algebra generated by the collection $\{pP : p \in P\}$, and for each $p \in P$ define

$$\mu(pP) = N(p)^{-\beta} F_p(\tau) \text{ for } p \in P.$$

Then μ extends to a finitely additive measure, which we also denote by μ, defined on $(P, \mathcal{B}_P)$ with values in the finite traces of A. Evaluating μ at the sets $\Omega_J = \bigcap_{p \in J} (P \backslash pP)$ for finite nonempty subsets $J \subset P \backslash \{e\}$ yields

$$\mu(\Omega_J) = \mu(P) - \mu(\bigcup_{p \in J} pP) = F_e(\tau) + \sum_{\emptyset \neq K \subset J} (-1)^{|K|} N(\vee K)^{-\beta} F_{\vee K}(\tau).$$

Thus our assumption (5.5) guarantees that $\mu(\Omega_J) \geq 0$ for all finite subsets $J \subset P_{\mathrm{inf}}$ and we must prove that this positivity holds for all finite subsets $J \subset P \setminus \{e\}$. By Lemma 5.4 it suffices to show that μ is positive on all sets in the family $\mathcal{A}_{P_{\inf}}$. Fix a finite subset K of $P_{\inf}$. For $p \in P$ we have that

$$\begin{aligned}\mu(p\Omega_K) &= N(p)^{-\beta}F_p(\tau) + \sum_{\emptyset\neq H\subset K}(-1)^{|H|}N(p(\vee H))^{-\beta}F_{p(\vee H)}(\tau)\\ &= N(p)^{-\beta}F_p(\mu(\Omega_K)),\end{aligned}$$

and the right hand side is positive because $\mu(\Omega_K)$ is positive by assumption. By finite additivity, μ is positive on every set in $\mathcal{A}_{P_{\inf}}$, which completes the proof. □

Example 5.6. By exhibiting classes of examples, we point out that the set $P_{\inf}$ can be finite, in which case Theorem 5.5 provides a significant reduction, and that the inclusion $P_a \subset P_{\inf}$ can be strict for some monoids and an equality for others.

(i) When $P = A_M^+$ is an Artin monoid with finite generating set, the set $P_{\inf}$ finite because it is contained in the finite Garside family shown to exist in [10, Theorem 1.1].

(ii) If P is a finite-type Artin monoid that is not abelian, then there exist canonical generators s, t such that $m_{s,t} > 2$, so that $s^{-1}(s \vee t) = \langle ts\rangle^{m_{s,t}-1} \notin S$. Thus $P_{\inf} \neq S$. In this case the arguments of Section 4 yield the reduction of positivity to generators, which is stronger than Theorem 5.5.

(iii) In contrast, when P is a right-angled Artin monoid it is easy to see from [3, Lemma 9.2] that $P_{\inf}$ is equal to the set of canonical generators. Formally, this recovers [3, Theorem 9.1] as a corollary of Theorem 5.5, but unlike Proposition 4.12, this proof is not really different since the proof of Theorem 5.5 is modelled on that of [3, Theorem 9.1].

6. KMS-gaps

We assume in this section that (G, P) has no nontrivial invertible elements, that is, $P \cap P^{-1} = \{e\}$. It is known that the associated (full) semigroup C^*-algebra $C^*(P)$ may be viewed as the Nica-Toeplitz algebra $\mathcal{NT}(X)$ constructed from the product system $X = \{X_p\}_{p\in P}$ with one-dimensional fibres $X_p = \mathbb{C}$, where the left action is given by complex multiplication for all $p \in P$, see [28, Proposition 5.6]. Suppose that $N : P \to (0, \infty)$, let $\sigma_t(v_p) = N(p)^{it}v_p$ for $p \in P$ and $t \in \mathbb{R}$ and let us analyse (1.1) for $(C^*(P), \sigma)$.

Since $A = X_e \cong \mathbb{C}$, a tracial state τ on A is simply $\tau = id$. Moreover, for each finite non-empty $K \subset P$ such that $\vee K < \infty$ we have $\mathrm{Tr}_\tau^{\vee K} = \tau = id$. Thus for given $\beta \in \mathbb{R}$, (1.1) rewrites as

$$1 + \sum_{\emptyset\neq K\subset J}(-1)^{|K|}N(\vee K)^{-\beta} \geq 0, \text{ for all finite nonempty } J \subset P \setminus \{e\}. \tag{6.1}$$

This condition implies the existence of a KMS$_\beta$-state for $(C^*(P), \sigma)$ by Theorem 3.5. This condition also implies that the finitely additive measure on $\mathcal{B}_P$ given by

$$\mu(pP) = N(p)^{-\beta} \text{ for } pP \in \mathcal{B}_P \tag{6.2}$$

extends to a genuine measure on the σ-algebra generated by $\{pP : p \in P\}$.

A refinement of this observation was implicit already in [3]. More precisely, it was shown in [3, Lemma 7.2] that every finitely additive measure μ on $(P, \mathcal{B}_P)$ such that $\sum_{p\in P}\mu(pP) < \infty$ extends to a genuine measure on the σ-algebra of all subsets of P. When μ is as in (6.2), we are asking that $\sum_{p\in P} N(p)^{-\beta} < \infty$. Recall from [3, Definition 7.8] that the infimum of all such real β is the *critical inverse temperature* of the system $(C^*(P), \sigma)$, where we assume that $N(p) \geq 1$ for all $p \in P$. When β_c is finite, there is a unique KMS$_\beta$-state for every $\beta > \beta_c$, and it is of finite type in the sense of [3, Definition 6.4], see also [3, Example 9.6]. Assuming further that P is Noetherian, that P has a finite set of atoms and that $N(p) > 1$ for all $p \in P \setminus \{e\}$, any possible KMS$_\beta$-state of infinite type will arise at values β where

$$1+\sum_{\emptyset\neq K\subset P_a}(-1)^{|K|}N(q_K)^{-\beta}=0,$$

cf. [6] and [3]. Note that this condition means that (6.1) reduces to having equality at the single subset $J=P_a$ of $P\setminus\{e\}$.

We note at this point that in case (G,P) is lattice ordered, then for any homomorphism $N: P\to(0,\infty)$, condition (6.1) is trivially satisfied at $\beta=0$ for every J, because it reduces to $\sum_{K\subset J}(-1)^{|K|}=0$, which follows by the binomial formula. Thus $(C^*(P),\sigma)$ admits a gauge-invariant KMS_0-state. We note that this was first proved in [6, Proposition 3.7], cf. the equivalence of (2) and (4), which does not require the additional assumption that $N(p)=1$ only if $p=e$. In this case any KMS_0-state is of infinite type [3, Corollary 6.10(i)].

6.1. **Artin monoids of finite type and** KMS**-gaps.** Next we wish to apply the results of the previous sections to illustrate a new phenomenon in the context of C^*-algebras of Artin monoids, namely the appearance of gaps in the subset of inverse temperatures that support KMS-states. The next definition makes this precise.

Definition 6.1. Let (G,P) be quasi-lattice ordered. Assume that we are given a compactly aligned product system $\{X_p\}_{p\in P}$ of C^*-correspondences over P with $X_e=A$ and a homomorphism $N:P\to(0,\infty)$. We say that $(\mathcal{NT}(X),\sigma)$ has a KMS-gap if

$$\{\beta\in\mathbb{R}\mid \text{ there exists a gauge-invariant }\ \mathrm{KMS}_\beta\text{-state}\}\text{ is disconnected.}$$

A large class of monoids where there are no KMS-gaps is provided by non-abelian right-angled Artin monoids, cf. the last paragraph of Example 9.6 of [3]. More precisely, let P be a non-abelian right-angled Artin monoid with finite generating set $S\subset P\setminus\{e\}$. Let $N:P\to(0,\infty)$ be a homomorphism such that $N(p)=1$ only for $p=e$. Then P is not lattice ordered, $\beta_c>0$ by [6, Proposition 3.7], and the possible behavior of KMS_β-states is as follows: there are none for $\beta<\beta_c$, there is a unique KMS_{β_c}-state, which is of infinite type, and for each $\beta>\beta_c$ there is a unique KMS_β-state, which is of finite type. In particular, there are no KMS-gaps.

Next we see that the Artin braid monoids B_n^+ for $n\geq 3$ offer a different picture.

Proposition 6.2. *Suppose $n\geq 3$ and let G be the braid group B_n with generating set $\{s_1,s_2,\dots,s_{n-1}\}$ and relations*

$$\begin{aligned} s_is_js_i&=s_js_is_j &&\text{when } |i-j|=1\\ s_is_j&=s_js_i &&\text{when } |i-j|\geq 2. \end{aligned}$$

Let B_n^+ be the associated braid monoid and let $N:B_n^+\to[1,\infty)$ be the homomorphism given by $N(p)=\exp(\ell(p))$ for $p\in B_n^+$, where ℓ is the normalised length function on B_n^+. Then $(C^*(B_n^+),\sigma)$ has a gauge-invariant KMS_0-state and no KMS_β-states in the interval $(0,a)$, where $\exp(-a)=\sqrt{5}/2-1/2\approx 0.61803$ is (the reciprocal of) the golden ratio.

Proof. Since $P=B_n^+$ is lattice ordered, there is a gauge-invariant KMS_0-state, as we already observed. In order to show there is a KMS-gap that extends from 0 to at least a we show that (6.1) fails for the subset $J=\{s_1,s_2\}$, that is

$$g_J(\beta)=1+\sum_{\emptyset\neq K\subset J}(-1)^{|K|}N(\vee K)^{-\beta}$$

is strictly negative in the interval $(0,a)$. Since $\ell(s_1\vee s_2)=\ell(s_1s_2s_1)=3$, is is clear that

$$g_{\{s_1,s_2\}}(\beta)=1-2e^{-\beta}+(e^{-\beta})^3.$$

The polynomial $1-2t+t^3$ is negative in the interval $(\frac{\sqrt{5}}{2}-\frac{1}{2},1)$ determined by its two positive roots, so if we set $t=e^{-\beta}$ we see that $g_{\{s_1,s_2\}}(\beta)<0$ for $\beta\in(0,a)$. Hence (6.1) fails for $J=\{s_1,s_2\}$ and all β in $(0,a)$, so there are no KMS$_\beta$-states in $(0,a)$. □

Corollary 6.3. (1) *Let $t_1=\sqrt{5}/2-1/2\approx 0.618$ be the smallest positive root of the clique polynomial $1-2t+t^3$ of B_3^+, and define $a=-\log t_1$. Then the inverse temperature space of $(C^*(B_3^+),\sigma)$ is $\{0\}\cup[a,\infty]$*

(2) *Let r_1 be the smallest positive root of $1-2t-t^2+t^3+t^4+t^5$, and $b=-\log r_1$. Then the inverse temperature space of $(C^*(B_4^+),\sigma)$ is $\{0\}\cup[b,\infty]$.*

Proof. The clique polynomial is the reciprocal of the growth series, see [27], so a and b are the critical temperatures for B_3^+ and B_4^+ respectively and we know from [6, Theorem 3.5] and [6, Proposition 3.7] that there is a KMS$_\beta$-state for each $\beta\in\{0\}\cup[\beta_c,\infty]$, which is unique for $\beta\neq 0$. Recall also from [6, Proposition 4.5] that if there exists a KMS$_\beta$-state for some $\beta\in(0,\beta_c)$, then $e^{-\beta}$ has to be a root of the clique polynomial in the interval $(e^{-\beta_c},1)$; however, [6] does not decide whether there are any KMS$_\beta$-states corresponding to any intermediate roots. Our positivity criterion allows us to retrieve the known part of the temperature space and show that there are no KMS states in $(0,\beta_c)$.

We deal with B_3^+ first. Since here is only one relation, namely $s_1s_2s_1=s_2s_1s_2$, and all subsets are cliques, the clique polynomial is $1-2t+t^3$, which has roots $-\sqrt{5}/2-1/2<0$, $t_1=\sqrt{5}/2-1/2$, and 1 and is strictly positive on $(0,t_1)$. The other cliques have polynomials 1 and $1-t$, which are positive on all of $(0,1)$. Hence (6.1) holds for $\beta\geq\beta_c$, proving the assertion about B_3^+.

The relations in B_4^+ are

$$s_1s_2s_1=s_2s_1s_2,\ s_2s_3s_2=s_3s_2s_3 \text{ and } s_1s_3=s_3s_1,$$

and since B_4 is finite type, all the subsets of S are cliques:

$$\emptyset,\ \{s_1\},\ \{s_2\},\ \{s_3\},\ \{s_1,s_2\},\ \{s_2,s_3\},\ \{s_1,s_3\},\ \{s_1,s_2,s_3\}$$

The common upper bounds of all but the full clique have been computed above, and $\vee\{s_1,s_2,s_3\}=s_3s_2s_1s_3s_2s_3$, which has length 6. Thus, the clique polynomial for B_4^+ written with $t=e^{-\beta}$, is $h(t)=1-3t+t^2+2t^3-t^6=(1-t)(1-2t-t^2+t^3+t^4+t^5)$.

We may now use the criterion of (6.1) for cliques J in S to check whether there is a gauge-invariant KMS$_\beta$-state at $\beta=-\log r_1$. From the proof of Proposition 6.2 we know that the condition is trivially satisfied for the choices $J=\{s_1,s_2\}$ and $J=\{s_2,s_3\}$. For the choice $J=\{s_1,s_3\}$, the corresponding polynomial $1-2t+t^2$ is nonnegative everywhere. Finally, the one-element choices of J yield the polynomial $1-t$, which is positive at t_1. Hence there is a gauge-invariant KMS$_\beta$-state at $\beta=-\log r_1$ in view of Corollary 4.11 and [3, Theorem 5.1]. In the interval $(0,1)$ the polynomial $1-2t-t^2+t^3+t^4+t^5$ has two other roots, $r_2\approx 0.659$ and $r_3\approx 0.874$. But since the positivity condition fails at both these roots for the clique polynomial $1-2t+t^3$ of $J=\{s_1,s_2\}$, there are no KMS$_\beta$-state for $\beta\in(0,b)$. This completes the proof for B_4^+. □

7. Characterisation of reduction to generators

In this section we investigate more closely conditions under which the positivity condition (5.5) for the minimal set P_{inf} can be reduced to a smaller subset of inequalities, namely those involving only finite subsets J of the set P_a of atoms. We assume throughout that (G,P) is a weak quasi-lattice ordered group. Recall that the key ingredient in the proof of Theorem 5.5 is Lemma 5.4 showing that the collection $\mathcal{A}_{P_{\text{inf}}}$ defined via (5.1) contains all sets Ω_J for $J\subset P\backslash\{e\}$ and hence is the algebra $\mathcal{B}_P$ from [3, Section 2].

Theorem 7.1. *If the collection*

$$\mathcal{A}_{P_a}=\{\sqcup_{j=1}^n p_j\Omega_{K_j} : p_j\in P \text{ and } K_j\subset P_a \text{ finite or empty}\}$$

is an algebra, then $\mathcal{A}_{P_a} = \mathcal{B}_P$ and reduction of the positivity condition to generators holds for P.

Proof. By Lemma 5.3, $pP \in \mathcal{A}_{P_a}$ for all $p \in P$. Now, $\mathcal{B}_P$ is the algebra (closed under finite unions, intersections and complements) generated by the sets pP, so if $\mathcal{A}_{P_a}$ is an algebra we get that $\mathcal{B}_P \subseteq \mathcal{A}_{P_a}$, which implies that $\mathcal{A}_{P_a} = \mathcal{A}_{P_{\inf}}$. Following the proof Theorem 5.5 we see that condition (1.1) can be reduced to subsets of P_a. $\square$

In view of this we aim to find conditions ensuring that $\mathcal{A}_{P_a}$ is an algebra. By Lemma 5.3 the collection $\mathcal{A}_{P_a}$ contains the sets pP and their complements $P \setminus pP$, and obviously $\mathcal{A}_{P_a}$ is also closed under finite disjoint unions.

Lemma 7.2. *Let (G, P) be a weak quasi-lattice ordered group with P Noetherian. If $\mathcal{A}_{P_a}$ is closed under finite intersections, then it contains all arbitrary finite unions of sets of the form pP with $p \in P$.*

Proof. The proof is by induction on the cardinality of the union. If the cardinality is 1 the union is a set on the form pP, which is in $\mathcal{A}_{P_a}$ by Lemma 5.3. Let $n \geq 1$ and assume that $\mathcal{A}_{P_a}$ is closed under unions of at most n sets of the form pP with $p \in P$. Let $p_1, p_2, \ldots, p_{n+1} \in P$. We write

$$\begin{aligned}\bigcup_{i=1}^{n+1} p_iP &= \left(\bigcup_{i=1}^{n} p_iP\right) \bigcup\, p_{n+1}P \\ &= \left[\left(\bigcup_{i=1}^{n} p_iP\right)^c \bigcap p_{n+1}P\right] \bigsqcup \left[\bigcup_{i=1}^{n} p_iP \bigcap\, p_{n+1}P\right] \bigsqcup \left[\bigcup_{i=1}^{n} p_iP \bigcap (p_{n+1}P)^c\right] \\ &= B_1 \bigsqcup B_2 \bigsqcup B_3.\end{aligned}$$

We show next that the sets B_1, B_2 and B_3 are in $\mathcal{A}_{P_a}$. We have $B_1 = \bigcap_{i=1}^n (p_iP)^c \bigcap\, p_{n+1}P$, which is in $\mathcal{A}_{P_a}$ by Lemma 5.3 and our hypothesis that $\mathcal{A}_{P_a}$ is closed under finite intersections. The set B_2 takes the form $\bigcup_{i=1}^n (p_iP \bigcap\, p_{n+1}P) = \bigcup_{i=1}^n (p_i \vee p_{n+1})P$, where $(p_i \vee p_{n+1})P = \emptyset$ if $p_i \vee p_{n+1} = \infty$. Since this union has at most n terms, $B_2 \in \mathcal{A}_{P_a}$ by the induction hypothesis. Similarly, $\bigcup_{i=1}^n p_iP \in \mathcal{A}_{P_a}$ by the induction hypothesis, and $(p_{n+1}P)^c \in \mathcal{A}_{P_a}$ by Lemma 5.3, so $B_3 \in \mathcal{A}_{P_a}$ because $\mathcal{A}_{P_a}$ is closed under intersections. Since $\mathcal{A}_{P_a}$ is closed under disjoint finite unions, the proof is complete. $\square$

Proposition 7.3. *Let (G, P) be a weak quasi-lattice ordered group with P Noetherian. If $\mathcal{A}_{P_a}$ is closed under finite intersections, then $\mathcal{A}_{P_a} = \mathcal{B}_P$.*

Proof. We start by proving that $\mathcal{A}_{P_a}$ is closed under complements. Fix a set $A \in \mathcal{A}_{P_a}$ of the form $A = \bigsqcup_{i \in I} p_i\Omega_{K_i}$, where I is finite, $p_i \in P$ and $K_i \subset P_a$ is finite, for every $i \in I$. We have that

$$A^c = \bigcap_{i \in I} (p_i\Omega_{K_i})^c = \bigcap_{i \in I} \left[\left(p_i \bigcup_{k_i \in K_i} k_iP\right) \bigsqcup (p_iP)^c\right].$$

Now $(p_iP)^c \in \mathcal{A}_{P_a}$ by Lemma 5.3 and $\bigcup_{k_i \in K_i} k_iP \in \mathcal{A}_{P_a}$ by Lemma 7.2, and it follows that each set in the intersection over $i \in I$ is in $\mathcal{A}_{P_a}$. Since $\mathcal{A}_{P_a}$ is closed under finite intersections, we obtain that $A^c \in \mathcal{A}_{P_a}$, as claimed. It remains to show that $\mathcal{A}_{P_a}$ is closed under finite unions. Given $A_1, A_2 \in \mathcal{A}_{P_a}$, we write

$$A_1 \cup A_2 = (A_1 \cap A_2^c) \sqcup (A_1 \cap A_2) \sqcup (A_2 \cap A_1^c),$$

and use the first part of the proof in conjunction with the hypothesis to obtain that $A_1 \cup A_2 \in \mathcal{A}_{P_a}$. The general case follows by induction. This proves that $\mathcal{A}_{P_a}$ is an algebra, which is clearly contained in $\mathcal{B}_P$ and contains pP for every $p \in P$, so $\mathcal{A}_{P_a} = \mathcal{B}_P$. $\square$

Lemma 7.4. *Let (G, P) be a weak quasi-lattice ordered group with P Noetherian. Suppose that $(P\backslash sP) \cap q\Omega_K \in \mathcal{A}_{P_a}$ for all $s \in P_a$ and all finite $K \subset \mathcal{A}_{P_a}$. Then $\mathcal{A}_{P_a}$ is closed under finite intersections.*

Proof. Since every set in $\mathcal{A}_{P_a}$ is a disjoint union of sets of the form $q\Omega_K$ with $K \subset P_a$, it suffices to prove that $(p\Omega_L) \cap (q\Omega_K) \in \mathcal{A}_{P_a}$ for $p, q \in P$ and $K, L \subset P_a$ finite.

We proceed by induction on $|L|$. Assume first $L = \{s\}$ for $s \in P_a$. If $p \vee q = \infty$ we have $pP \cap qP = \emptyset$ and $p(P\backslash sP) \bigcap q\Omega_K = \emptyset \in \mathcal{A}_{P_a}$. If $p \vee q < \infty$, we have

$$\begin{aligned} p(P\backslash sP) \bigcap q\Omega_K &= p(P\backslash sP) \bigcap \left((p \vee q) \bigcap_{k \in K} (P\backslash kP) \right) \\ &= p \left[(P\backslash sP) \bigcap \left(p^{-1}(p \vee q) \bigcap_{k \in K} (P\backslash kP) \right) \right]. \end{aligned}$$

Since the intersection is in $\mathcal{A}_{P_a}$ by hypothesis and $\mathcal{A}_{P_a}$ is closed under left multiplication by p, this concludes the proof of the case $|L| = 1$.

Fix $p, q \in P$ and a finite $K \subset P_a$. Suppose $(p\Omega_{L'}) \cap (q\Omega_K) \in \mathcal{A}_{P_a}$ for all $L' \subset P_a$ with $|L'| = N$ and let $L \subset P_a$ be such that $L = L' \cup \{t\}$ for $t \in P_a$, so that $|L| = N + 1$. Then

$$\begin{aligned} \left[p \bigcap_{h \in L} (P\backslash hP) \right] \bigcap q\Omega_K &= p(P\backslash tP) \bigcap \left[p \bigcap_{h \in L'} (P\backslash hP) \right] \bigcap q\Omega_K \\ &= p(P\backslash tP) \bigcap A' \in \mathcal{A}_{P_a}. \end{aligned}$$

By the induction hypothesis A' is in $\mathcal{A}_{P_a}$, so it is a disjoint union of sets of the form $q_j\Omega_{K_j}$, to which we may apply the case $|L| = 1$ to complete the proof. $\square$

Next we illustrate the above results with an application to the braid group on two generators.

Lemma 7.5. *Let (B_3, B_3^+) the finite-type Artin group-monoid pair with generating set $S_2 = \{s_1, s_2\}$ and presentation*

$$\langle s_1, s_2 \, | s_1 s_2 s_1 = s_2 s_1 s_2 \rangle.$$

If $p \in B_3^+$ can be written as $p = p_1(s_1 s_2 s_1)p_2$ for $p_1, p_2 \in P$, then $s_1 \le p$ and $s_2 \le p$.

Proof. We only prove that $s_1 \le p$, as the other claim is analogous. Assume for a contradiction that $s_1 \not\le p$. Then we can write p in the form $p = p_1 s_1 s_2 s_1 p_2$ where $p_1 \neq e$ has the smallest possible length. Now p_1 must end with either s_1 or s_2. In case $p_1 = r_1 s_1$, we have $p = r_1(s_1 s_1 s_2 s_1 p_2) = r_1(s_1 s_2 s_1 s_2 p_2)$, contradicting the choice of p_1. The case $p_1 = r_1 s_2$ leads to a similar contradiction. Hence p admits an expression in the generators that starts with s_1, which precisely means that $s_1 \le p$. $\square$

Lemma 7.6. *For all $p \in B_3^+$ and $i = 1, 2$ we have that*

$$p \vee s_i \in \{s_1,\, s_2,\, p,\, ps_1,\, ps_2,\, ps_1 s_2,\, ps_2 s_1\}.$$

Proof. Assume $i = 1$ first. The case $p = e$ is trivially satisfied, and the case $p = s_2$ is easy to verify because $s_1 \vee s_2 = s_1 s_2 s_1 = s_2 s_1 s_2 = ps_1 s_2$. The case $s_1 \le p$ obviously yields $p \vee s_1 = p$. So we may assume that p has at least two letters, and also that $s_1 \not\le p$, so that p does not have a factor $s_1 s_2 s_1$ by Lemma 7.5. In this case $p \vee s_1 = pb$ for some $b \in B_3^+ \setminus \{e\}$. Factoring out the last two letters of p, we write $p = p_1 x$ where x is one of $s_1 s_2$, $s_2 s_1$, s_1^2, or s_2^2, and it is easy to see that b is one of s_1, s_2, $s_2 s_1$, or $s_1 s_2$, respectively. $\square$

Proposition 7.7. *Let (B_3, B_3^+) be the braid group with generating set $S_2 = \{s_1, s_2\}$. Then $\mathcal{A}_{S_2}$ is an algebra.*

Proof. By Proposition 7.3 and Lemma 7.4, we only need to show that the set

$$B := (P\backslash s_i P) \bigcap \left[q \bigcap_{k\in K} (P\backslash kP) \right]$$

is in $\mathcal{A}_{S_2}$ for $i = 1, 2$, finite $K \subseteq S_2$ and $q \in P$. We rewrite

$$B = (P\backslash s_i P) \bigcap \left[q \bigcap_{k\in K} (P\backslash kP) \right] = q \left[(P\backslash q^{-1}(q \vee s_i)P) \bigcap \left(\bigcap_{k\in K} (P\backslash kP) \right) \right].$$

By Lemma 7.6, we have $q \vee s_i \in \{s_i, q, qs_1, qs_2, qs_1s_2, qs_2s_1\}$ for $i = 1, 2$. This yields

$$(P\backslash q^{-1}(q \vee s_i)P) \in \{\emptyset, (P\backslash s_1P), (P\backslash s_2P), (P\backslash s_1s_2P), (P\backslash s_2s_1P)\}$$

for $i = 1, 2$. It is clear that if $(P\backslash q^{-1}(q \vee s_i)P)$ is one of $\emptyset, (P\backslash s_1P), (P\backslash s_2P)$, then B is contained in $\mathcal{A}_{S_2}$.

Next we consider the case that $(P\backslash q^{-1}(q \vee s_i)P) = (P\backslash s_1s_2P)$, and examine its possible intersection with Ω_K for given $K \subset S_2$. If $K = \{s_1\}$, then $(P\backslash s_1s_2P) \cap (P\backslash s_1P) = (P\backslash s_1P) \in \mathcal{A}_{S_2}$. If $K = \{s_2\}$, then $(P\backslash s_1s_2P) \cap (P\backslash s_2P) = [s_1(P\backslash s_2P) \sqcup (P\backslash s_1P)] \cap (P\backslash s_2P)$. We need to show that $s_1(P\backslash s_2P) \cap (P\backslash s_2P) \in \mathcal{A}_{S_2}$. This follows from

$$\begin{aligned} s_1(P\backslash s_2P) \cap (P\backslash s_2P) &= s_1 \left[(P\backslash s_2P) \cap (P\backslash s_1^{-1}(s_1 \vee s_2)P) \right] \\ &= s_1 \left[(P\backslash s_2P) \cap (P\backslash s_2s_1P) \right] = s_1(P\backslash s_2P). \end{aligned}$$

In case $K = \{s_1, s_2\}$ we obtain $(P\backslash s_1s_2P) \cap (P\backslash s_1P) \cap (P\backslash s_2P) = (P\backslash s_1P) \cap (P\backslash s_2P)$, which is in $\mathcal{A}_{S_2}$. Checking the remaining case $(P\backslash q^{-1}(q \vee s_i)P) = (P\backslash s_2s_1P)$ can be done in the same way. Hence $B \in \mathcal{A}_{S_2}$ and the proposition follows. □

Remark 7.8. It follows from Proposition 7.7 that reduction of positivity to generators holds for B_3^+, a fact that already follows from Corollary 4.11 because B_3 is of finite-type.

New examples of weak quasi-lattice ordered groups satisfying $\mathcal{A}_{P_a} = \mathcal{B}_P$ can be obtained by taking free and direct products. Suppose, for instance that G_1 and G_2 are two Artin groups with canonical generating sets P_a^1 and P_a^2 and Coxeter matrices M_1 and M_2, respectively. Then their free and direct products are also Artin groups with generating set $P_a = P_a^1 \cup P_a^2$, whose Coxeter matrices have diagonal blocks M_1 and M_2 and remaining entries equal to ∞, in the case of the free product, or 2, in the case of the direct product. And similarly for the corresponding Artin monoids.

Proposition 7.9. *Let (G_1, P_1) and (G_2, P_2) be two Artin group-monoid pairs, with canonical generating sets P_a^1 and P_a^2, respectively, and suppose that $\mathcal{A}_{P_a^1}$ and $\mathcal{A}_{P_a^2}$ are algebras of subsets of P_1 and P_2 respectively. Let $P_a = P_a^1 \cup P_a^2$.*

1. *$(G_1 * G_2, P_1 * P_2)$ is an Artin group-monoid pair with generating set P_a subject to the relations imposed in the presentations of G_1 and G_2, and the collection $\mathcal{A}_{P_a}$ is an algebra of subsets of $P_1 * P_2$.*
2. *$(G_1 \times G_2, P_1 \times P_2)$ is an Artin group-monoid pair with generating set P_a satisfying the relations $s_1s_2 = s_2s_1$ for all pairs of generators $s_1 \in P_a^1$ and $s_2 \in P_a^2$ in addition to the relations already imposed in the presentations of G_1 and G_2, the collection $\mathcal{A}_{P_a}$ is an algebra of subsets of $P_1 \times P_2$.*

Proof. We prove (1) first. Let (G, P) temporarily denote the free product to simplify the notation. By Lemma 7.4 and Proposition 7.3, it suffices to show that

$$(P\backslash tP) \cap q\Omega_K \in \mathcal{A}_{P_a} \qquad \text{for all } t \in P_a,\ q \in P,\ K \subset P_a. \tag{7.1}$$

We start by rewriting $B := (P\backslash tP) \cap q\Omega_K$ as

$$B = (P\backslash tP) \bigcap \left[q \bigcap_{k\in K} (P\backslash kP) \right] = q \left[(P\backslash q^{-1}(q\vee t)P) \bigcap \left(\bigcap_{k\in K} (P\backslash kP) \right) \right].$$

Assume first that $t \in P_a^1$. If $q \in P_1$, then $B \in \mathcal{A}_{P_a^1} \subseteq \mathcal{A}_{P_a}$ by the assumption that $\mathcal{A}_{P_a^1}$ forms an algebra of sets of P_1. Otherwise there exists at least one atom $s \in P_a^2$ that is a factor in q. Let s_1 be the first instance (from the left) of such an atom in q. Then $q = as_1b$ where $a \in P_1$ and $b \in P$. We then observe that $t \vee q = q$ if and only if t is a factor in a and t can be shuffled to the front of a, and that otherwise $t \vee q = \infty$, because t can not be shuffled past s_1. This implies that $q^{-1}(q \vee t) \in \{e, \infty\}$, and so $(P\backslash q^{-1}(q\vee t)P) \in \{P, \emptyset\}$, leading to $B \in \mathcal{A}_{P_a}$. The argument is analogous for the case where $t \in P_a^2$. This concludes the proof of (1).

In order to prove (2), let now (G, P) denote the direct product. As with (1) it suffices to verify (7.1). Writing $K = K_1 \sqcup K_2$ with $K_1 \subseteq P_a^1$ and $K_2 \subseteq P_a^2$, we see that

$$B = q \left[(P\backslash q^{-1}(q\vee t)P) \bigcap \left(\bigcap_{k_1\in K_1} (P\backslash k_1P) \right) \bigcap \left(\bigcap_{k_2\in K_2} (P\backslash k_2P) \right) \right].$$

Assume first that $t \in P_a^1$. If $q \in P_1$, then $q(P\backslash q^{-1}(q\vee t)P) \bigcap \left(\bigcap_{k_1\in K_1}(P\backslash k_1P)\right) \in \mathcal{A}_{P_a^1}$ by the assumption that $\mathcal{A}_{P_a^1}$ forms an algebra of sets of P_1. Otherwise there exists at least one atom $s \in P_a^2$ that is a factor in q. Since P_a^2 commutes with P_a^1 in P, we can write $q = q_1s_1\cdots s_n$, where $s_1, \ldots, s_n \in P_a^2$ and $q_1 \in P_1$. By assumption t is in P_a^1, so it commutes with all the elements $s_1, \ldots, s_n$. Thus, $q \vee t = \infty$ when $q_1 \vee t = \infty$, and $q \vee t = (q_1 \vee t)s_1 \cdots s_n = s_1 \cdots s_n(q_1 \vee t)$ when $q_1 \vee t < \infty$, in which case

$$q^{-1}(q \vee t) = q_1^{-1}s_n^{-1}\cdots s_1^{-1}s_1\cdots s_n(q_1\vee t) = q_1^{-1}(q_1 \vee t) \in P_1.$$

Hence, we either have that $(P\backslash q^{-1}(q\vee t)P) = P$ (in the case where $q_1 \vee t = \infty$), or that

$$(P\backslash q^{-1}(q\vee t)P) \bigcap \left(\bigcap_{k_1\in K_1} (P\backslash k_1P) \right) = (P\backslash q_1^{-1}(q_1\vee t)P) \bigcap \left(\bigcap_{k_1\in K_1} (P\backslash k_1P) \right) \in \mathcal{A}_{P_a^1}$$

(in the case where $q_1 \vee t < \infty$), by the algebra assumption on $\mathcal{A}_{P_a^1}$.

It remains to prove that $p_1\Omega_{K_1} \cap \Omega_{K_2} \in \mathcal{A}_{P_a}$ for all $p_1 \in P_1$, $K_1 \subseteq P_a^1$ and $K_2 \subseteq P_a^2$. We observe that

$$\begin{aligned} p_1\Omega_{K_1} \cap \Omega_{K_2} &= p_1 \left[\bigcap_{k_1\in K_1} (P\backslash k_1P) \bigcap \bigcap_{k_2\in K_2} (P\backslash p_1^{-1}(p_1\vee k_2)P) \right] \\ &= p_1 \left[\bigcap_{k_1\in K_1} (P\backslash k_1P) \bigcap \bigcap_{k_2\in K_2} (P\backslash p_1^{-1}p_1k_2P) \right] \\ &= p_1 \left[\bigcap_{k_1\in K_1} (P\backslash k_1P) \bigcap \bigcap_{k_2\in K_2} (P\backslash k_2P) \right] \end{aligned}$$

where the second equality follows by the fact that $p_1 \vee k_2 = p_1k_2 = k_2p_1$, since $k_2 \in P_a^2 \subseteq P_2$ and $p_1 \in P_1$ and all elements from P_1 commute with all elements from P_2. Since the set in the last line of the above calculation is clearly in $\mathcal{A}_{P_a}$, we conclude $p_1\Omega_{K_1} \cap \Omega_{K_2} \in \mathcal{A}_{P_a}$. Interchanging the roles of P_a^1 and P_a^2 proves the case $t \in P_a^2$ and finishes the proof. □

It is easy to see that by taking free and direct products of right-angled or finite-type Artin monoids we obtain other Artin monoids many of which are neither right-angled nor finite-type. As a result we obtain new monoids that satisfy reduction of positivity to generators.

Corollary 7.10. *Under the assumptions of Proposition 7.9, let (G, P) denote either the free or the direct product of (G_1, P_1) and (G_2, P_2). Assume moreover that we are given a compactly aligned product system $\{X_p\}_{p\in P}$ of C^*-correspondences over P with $X_e = A$, a homomorphism $N : P \to (0, \infty)$ and $\beta \in \mathbb{R}$. Then reduction of the positivity condition to generators holds for P.*

Proof. Combine Theorem 7.1 with Proposition 7.9. □

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