REAL PHASE STRUCTURES ON MATROID FANS AND MATROID ORIENTATIONS

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ABSTRACT. We introduce the notion of real phase structure on rational polyhedral fans in Euclidean space. Such a structure consists of an assignment of affine spaces over $\mathbb{Z}/2\mathbb{Z}$ to each top dimensional face of the fan subject to two conditions.

Given an oriented matroid we can construct a real phase structure on the fan of the underlying matroid. Conversely, we show that from a real phase structure on a matroid fan we can produce an orientation of the underlying matroid. Thus real phase structures are cryptomorphic to matroid orientations.

The topes of the orientated matroid are recovered immediately from the real phase structure. We also provide a direct way to recover the signed circuits of the oriented matroid from the real phase structure.

CONTENTS

1. INTRODUCTION

We propose a definition of real phase structures on rational polyhedral fans, with specific attention to matroid fans, also known as Bergman fans of matroids. A real phase structure on a fan in $\mathbb{R}^n$ is the specification of an affine subspace of $(\mathbb{Z}/2\mathbb{Z})^n$ for each top-dimensional cone of the fan and subject to two conditions, see Definition 2.2. Given an oriented matroid $\mathcal{M}$, its underlying non-oriented matroid is denoted by $\underline{\mathcal{M}}$. For a fixed matroid M, an oriented matroid $\mathcal{M}$ such that $M = \underline{\mathcal{M}}$ is called an orientation of

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M. Our main theorem states that a real phase structure on the associated matroid fan Σ_M is equivalent to an orientation of the underlying matroid M.

Theorem 1.1. *Given a fixed matroid M, there is a natural bijection between orientations $\mathcal{M}$ of M and real phase structure on the matroid fan Σ_M. In other words, oriented matroids and real phase structures on matroid fans are cryptomorphic concepts.*

Real phase structures have previously been defined on tropical curves [Mik06, Section 7.2], [Ber], [BBR17], and on non-singular tropical hypersurfaces [Ren17] and [RS18]. These two special cases of real phase structures on polyhedral complexes have been used in the study of real enumerative geometry, in the form of Welschinger invariants, and the topology of real algebraic varieties. Studying the fans of matroids has led to major breakthroughs in understanding the behaviour of many matroid invariants [AHK18], [ADH20]. Our hope is that matroid fans equipped with real phase structures will find similar applications in the study of oriented matroids.

The initial goal of our investigation into real phase structures on matroid fans was to generalise the spectral sequence from [RS18] and subsequent bounds obtained on Betti numbers of real algebraic hypersurfaces arising from Viro's patchworking procedure [Vir84], [Vir08] to more general spaces. This is the subject of a forthcoming paper, in which we define the real part of a tropical variety equipped with a real phase structure. This is similar to the patchworking of tropical linear spaces of Celaya, Loho and Yuen [CLY20], yet we do not require the underlying matroid of the oriented matroid to be uniform.

We construct the bijection from Theorem 1.1 explicitly by assigning to each oriented matroid $\mathcal{M}$ with $\underline{\mathcal{M}} = M$ a real phase structure on Σ_M. The real part of this real phase structure produces a topological representation of the oriented matroid in the sense of the famous theorem of Folkman and Lawrence [FL78]. This recovers similar constructions by Ardila, Klivans, and Williams [AKW06] and Celaya [Cel19] for the positive real part and real part of a matroid fan. In the forthcoming paper, we combine this fact with Theorem 1.1 to prove that the real part of a non-singular tropical variety is a PL-manifold.

The paper is organised as follows. In Section 2.1, we recall the definition of rational polyhedral fans and introduce the notion of real phase structures on them. Then we specialise to the case of fans arising from matroids. In Subsection 2.2, we recall the construction of Ardila and Klivans of a matroid fan from its lattice of flats. In Subsection 2.3, we introduce the notion of necklace line arrangements and translate one of the conditions for real phase structures into this language in the case of matroid fans. We show that real phase structures on matroid fans behave well under the operations of deletion and contraction of the underlying matroid in Subsection 2.4.

Section 3 contains our main results. First, in subsection 3.1 we show how to obtain a real phase structure from an oriented matroid. Then in Subsections 3.2 and 3.3, we prove that every real phase structure on a matroid fan is obtained in this fashion by considering oriented matroid quotients. Finally, for convenience, in Subsection 3.4 we give an explicit description of how to recover the signed circuits of the oriented matroid from the real phase structure on a matroid fan.

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2. Real phase structures

2.1. **Fans and real phase structures.** A polyhedral fan Σ in $\mathbb{R}^n$ is a collection of convex polyhedral cones such that every face of a cone in Σ is also a cone in Σ, and the intersection of two cones is a face of both of them. A fan is rational if all of its cones are generated over $\mathbb{Z}$. The faces of Σ which are maximal with respect to inclusion are called the facets of Σ. We denote the set of facets of Σ by $\text{Facets}(\Sigma)$. A rational polyhedral fan is pure dimensional if its facets are all of equal dimension. Here we only consider fans of pure dimension.

Throughout V will be a vector space over $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. We let $\text{Aff}_d(V)$ denote the set of all affine subspaces of dimension d in V. Given $A \in \text{Aff}_d(V)$, we denote by $T(A)$ the *tangent space* of A. In particular, the space $T(A)$ is the d-dimensional linear subspace in V generated by the vectors $x - y$ for $x, y \in A$. If σ is a rational polyhedral cone in $\mathbb{R}^n$, we denote its tangent space by $T(\sigma)$. This is equivalent to the linear span over $\mathbb{R}$ of the vectors generating the cone. The set of integer points in its tangent space of σ is denoted by $T_{\mathbb{Z}}(\sigma)$, and the reduction mod 2 of these points is denoted by $T_{\mathbb{Z}_2}(\sigma)$.

Definition 2.1. *A collection of subsets of a set such that every element in the union is contained in an even number of the subsets is called an* even covering.

Definition 2.2. *Let Σ be a rational polyhedral fan of pure dimension d in $\mathbb{R}^n$. A* real phase structure *$\mathcal{E}$ on Σ is a map*

$$\mathcal{E}\colon \mathit{Facets}(\Sigma) \to \mathit{Aff}_d(\mathbb{Z}_2^n)$$

such that

(1) *for every facet σ of Σ, the set $\mathcal{E}(\sigma)$ is an affine subspace of $\mathbb{Z}_2^n$ parallel to σ, in formulas, $T(\mathcal{E}(\sigma)) = T_{\mathbb{Z}_2}(\sigma)$;*
(2) *for every codimension one face τ of Σ with facets $\sigma_1, \dots, \sigma_k$ adjacent to it, the sets $\mathcal{E}(\sigma_1), \dots, \mathcal{E}(\sigma_k)$ are an even covering.*

Definition 2.3. *Let $\mathcal{E}$ be a real phase structure on Σ. A* reorientation of $\mathcal{E}$ *is a real phase structure $\mathcal{E}'$ obtained by translating all affine subspaces in a real phase structure $\mathcal{E}$ by a fixed vector $\varepsilon \in \mathbb{Z}_2^n$. In other words $\mathcal{E}'(\sigma) = \mathcal{E}(\sigma) + \varepsilon$ for all $\sigma \in \Sigma$.*

2.2. **Matroid fans.** A matroid M is a finite set E together with a function $\mathrm{r} : 2^E \to \mathbb{N}_{\geq 0}$, where 2^E denotes the power set of E. The set E is called the ground set of M and r the rank function. The rank function is subject to the axioms:

(1) $0 \leq \mathrm{r}(A) \leq |A|$ for all $A \subseteq E$
(2) if $A \subseteq B \subseteq E$, then $\mathrm{r}(A) \leq \mathrm{r}(B)$
(3) if $A, B \subseteq E$, then $\mathrm{r}(A \cup B) + \mathrm{r}(A \cap B) \leq \mathrm{r}(A) + \mathrm{r}(B)$.

The rank function defines a closure operator on subsets by

$$\mathrm{cl}(A) = \{i \in E \mid \mathrm{r}(A) = \mathrm{r}(A \cup i)\} \supseteq A.$$

A subset $F \subseteq E$ is a flat of M if it is closed with respect to this operator, namely $\mathrm{cl}(F) = F$. The flats of a matroid M ordered by inclusion form a lattice, known as the lattice of flats, which we denote by $\mathcal{L}$.

A loop is an element of the ground set for which $\mathrm{r}(i) = 0$. Parallel elements are pairs of non-loop elements for which $\mathrm{r}(ij) = 1$. A matroid is simple if it contains no loops or parallel elements. A circuit is any set $A \subset E$ such that $|A| = \mathrm{r}(A) + 1$ and $|A| = \mathrm{r}(A \setminus i)$ for any $i \in A$. A coloop is an element that does not belong to any circuit.

Given a loopfree matroid M on the base set E, we denote by Σ_M the *affine* matroid fan in $\mathbb{R}^E$ and by $\mathbb{P}\Sigma_M = \Sigma_M / \langle(1, \dots, 1)\rangle$ the *projective* matroid fan in $\mathbb{R}^E / \langle(1, \dots, 1)\rangle$. We now describe how to construct both of these fans following Ardila and Klivans [AK06]. Fix the vectors $v_i = -e_i$ where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^E$ for $E = \{1, \dots, n\}$ and set $v_I = \sum_{i \in I} v_i$ for any subset $I \subset E$. For a chain of flats

$$\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \subsetneq E\}$$

in the lattice of flats $\mathcal{L}$, define the $k+1$-dimensional cone

$$\sigma_{\mathcal{F}} = \langle v_{F_1}, \dots, v_{F_k}, \pm v_E \rangle_{\geq 0}.$$

The affine matroid fan Σ_M is the collection of all such cones ranging over the chains in $\mathcal{L}$. In particular, the top dimensional faces of Σ_M are in one to one correspondence with the maximal chains in the lattice of flats of M. The projective matroid fan $\mathbb{P}\Sigma_M$ is the image of Σ_M in the quotient $\mathbb{R}^E / (1, \dots, 1)$.

If a matroid M has loops $L = \mathrm{cl}(\emptyset)$, then we set $\Sigma_M := \Sigma_{M/L} \subset \mathbb{R}^{E \setminus L}$ and $\mathbb{P}\Sigma_M := \mathbb{P}\Sigma_{M/L} \subset \mathbb{R}^{E \setminus L} / (1, \dots, 1)$. This is a practical definition, the more coherent point of view is to regard the affine and projective matroid fans as subsets of boundary strata of tropical affine space and tropical projective space, respectively. See [Sha13] or [MR] for more details.

Example 2.4. The uniform matroid of rank $k+1$ on n elements will be denoted $U_{k+1,n}$. The flats of the matroid $M = U_{k+1,n}$ are all subsets of $\{1, \dots, n\}$ of size less than or equal to k and $\{1, \dots, n\}$. Therefore, the faces of top dimension of Σ_M are in bijection with ordered subsets of size k. For example, the ordered set $\{i_1 < i_2 < \dots < i_k\}$ corresponds to a chain of flats

$$\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq \dots \subsetneq F_k \subsetneq E\}$$

where $F_1 = \{i_1\}$ and $F_{i+1} = F_i \cup i_{i+1}$, which in turn corresponds to the cone

$$\sigma_{\mathcal{F}} = \langle v_{F_1}, \dots, v_{F_k}, \pm v_E \rangle_{\geq 0}.$$

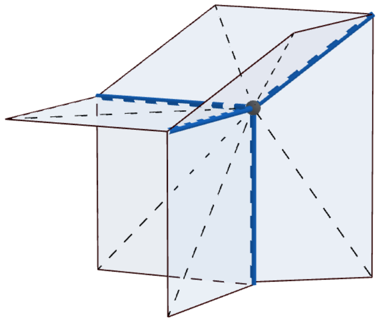

FIGURE 1. The projective fan of the matroid $U_{3,4}$ drawn in $\mathbb{R}^4/(1,\dots,1)$ as described in Example 2.4.

The projective fan for $U_{3,4}$ is shown in Figure 1. The dotted rays in this fan correspond to the 6 rank 2 flats.

There are coarser fan structures on the support of the fan Σ_M. For example, the fan with maximal cones $\sigma_I = \langle v_{i_1}, \dots, v_{i_k}, \pm v_E\rangle$ where $I = \{i_1, \dots, i_k\}$ ranges over all subsets I of E of size k, has the same support as Σ_M. This in known as the coarse matroid fan of M [AK06].

2.3. **Necklace arrangements.** We now give an equivalent reformulation of Condition (2) in Definition 2.2 in the case of matroid fans. Notice first that there is a one to one correspondence between real phase structures on Σ_M and $\mathbb{P}\Sigma_M$ induced by the projection $\mathbb{Z}_2^E \to \mathbb{Z}_2^E/(1,\dots,1)$.

Suppose $\mathcal{E}$ is a real phase structure on a d-dimensional rational polyhedral fan Σ and let τ be a codimension 1 face of Σ. Then the affine subspaces $\mathcal{E}(\sigma_i)$ where σ_i are the facets adjacent to τ all contain the direction of the $(d-1)$-dimensional linear space $T_{\mathbb{Z}_2}(\tau)$. The even covering property at the codimension one face τ can be equivalently checked on the lines in $\mathbb{Z}_2^n/T_{\mathbb{Z}_2}(\tau)$ obtained as projections of the $\mathcal{E}(\sigma_i)$'s.

Given an arrangement of lines $L_1, \dots, L_k \in \mathrm{Aff}_1(V)$, its *intersection complex* is the simplicial complex that consists of a vertex for every line and a simplex on the vertices $i_1, \dots, i_q$ for every point in $L_{i_1} \cap \dots \cap L_{i_q}$.

Definition 2.5. *An arrangement of lines $L_1, \dots, L_k \in \mathit{Aff}_1(V)$ is a* necklace of lines *if its intersection complex is a cycle graph. An arrangement of subspaces $E_1, \dots, E_k \in \mathit{Aff}_d(V)$ whose tangent spaces share a $d-1$-dimensional linear space W is called a* necklace arrangement *if the projection to V/W yields lines $L_1, \dots, L_k$ forming a necklace of lines.*

Remark 2.6. *We will use this definition exclusively for vector spaces over $\mathbb{Z}_2$. Under this assumption, two lines L_1, L_2 form a necklace if and only if $L_1 = L_2$. If a necklace arrangement consists of more than two lines, then these lines must be pairwise distinct.*

For subspace arrangements of higher dimension, note that the definition of necklace arrangement is independent of the choice of W. Indeed, if this choice is not unique, then the affine spaces are all parallel and hence form a necklace if and only if $k = 2$ and $E_1 = E_2$.

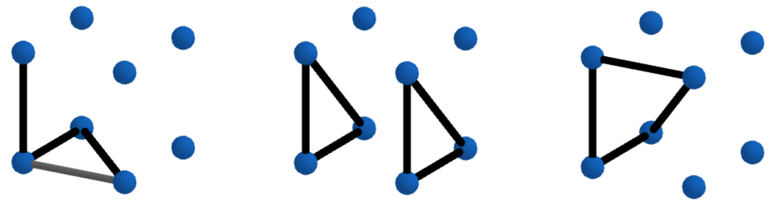

FIGURE 2. Three line arrangements in $\mathbb{Z}_2^3$ from Example 2.7.

Example 2.7. Figure 2 shows three different line arrangements in $\mathbb{Z}_2^3$. The points in $\mathbb{Z}_2^3$ are represented as vertices of a cube. Lines in $\mathbb{Z}_2^n$ are in correspondence with pairs of points. So in the figure a line is represented by an edge joining two points. The first line arrangement consists of 4 lines, and the 4 lines considered as subsets of $\mathbb{Z}_2^3$ do not form an even cover in the sense of Definition 2.1. In the second example, there are 6 lines, which form an even cover but do not form a necklace arrangement. The third example is collection of 4 lines forming a necklace arrangement.

We can establish the following alternative for condition (2) in Definition 2.2 in the case of matroid fans.

(2') For every codimension one face τ of Σ with facets $\sigma_1, \dots, \sigma_k$ adjacent to it, the subspaces $\mathcal{E}(\sigma_1), \dots, \mathcal{E}(\sigma_k)$ form a necklace arrangement.

Lemma 2.8. *Let $\mathcal{E}\colon \text{Facets}(\Sigma) \to \text{Aff}_d(\mathbb{Z}_2^n)$ be a map satisfying condition (1) from Definition 2.2. If $\mathcal{E}$ satisfies condition (2'), then it also satisfies condition (2). Moreover, if $\Sigma = \Sigma_M$ (or $\Sigma = \mathbb{P}\Sigma_M$) is an affine (or projective) matroid fan, then the two conditions are equivalent.*

Proof. Suppose Condition (2') is satisfied, then at each codimension one face τ of Σ the intersection complex of the subspaces $\mathcal{E}(\sigma_1), \dots, \mathcal{E}(\sigma_k)$ is a necklace arrangement. In particular, every element in $\mathcal{E}(\sigma_1) \cup \dots \cup \mathcal{E}(\sigma_k)$ is contained in exactly two of the affine spaces. This implies the first statement.

Let us now assume $\Sigma = \Sigma_M$ for some matroid M. The statement for projective fans is equivalent. Assume that $\mathcal{E}$ satisfies condition (2). For a face τ of codimension 1 of Σ_M and a facet σ_i adjacent to τ, let $v_i \in \mathbb{Z}^n / T_{\mathbb{Z}}(\tau)$ be the non-zero integer vector representing the image of $T_{\mathbb{Z}}(\sigma_i)$ under the quotient by $T_{\mathbb{Z}}(\tau)$. Let $\overline{v}_i \in \mathbb{Z}_2^n / T_{\mathbb{Z}_2}(\tau)$ denote the mod 2 reduction of v_i. Then the directions of the lines in the necklace line arrangement in $\mathbb{Z}_2^n / T_{\mathbb{Z}_2}(\tau)$ are $\overline{v}_i$. Any cycle in the intersection complex corresponds to a non-trivial linear relation among the vectors $\overline{v}_i$ in $\mathbb{Z}_2^n / T_{\mathbb{Z}_2}(\tau)$. In the case of matroid fans, there is a unique, up to scalar, linear relation among the $\overline{v}_i$'s, namely their sum is zero. Indeed, the chain of flats $\mathcal{F}$ associated to τ contains exactly one gap, namely flats $F \subsetneq G$ with $\text{r}(G) = \text{r}(F) + 2$. Then the statement follows from [Rau20, Remark 4.4 b)] and the fact that the sets $H_i \setminus F$, running through all in-between flats $F \subsetneq H_i \subsetneq G$, form a partition of $G \setminus F$. $\square$

Definition 2.9. *Given a finite set S, a* necklace ordering *of S is an equivalence class of two cyclic orderings of S, which are related by reversing the order. For example, a cycle graph defines a necklace ordering of its vertices.*

Remark 2.10. *A real phase structure on a matroid fan Σ_M determines at every codimension one face of Σ_M a necklace ordering of the facets of Σ_M adjacent to the codimension one face. The necklace ordering is defined by the cycle graph of the necklace line arrangement at each codimension one face. A reorientation of a real phase structure in the sense of Definition 2.3 induces the same necklace ordering of facets adjacent to codimension one faces as the original real phase structure.*

Question 2.11. *Is a real phase structure on a matroid fan determined, up to reorientation, by the induced necklace ordering of the facets at each codimension one face of the fan?*

In general, we may ask for a description of the set of real phase structures which produce a fixed collection of necklace orderings at codimension one faces. Using the correspondence between real phase structures on matroid fans and orientations of matroids which we will prove here, we can translate the question to one about oriented matroids: Up to reorientations, is an oriented matroid determined by its rank 2 minors?

Example 2.12. Consider the projective fan of the uniform matroid $M = U_{2,n}$. The fan $\mathbb{P}\Sigma_M \subset \mathbb{R}^{n-1}$ has n edges generated by the images of the vectors $v_1 = -e_1, \dots, v_n = -e_n$ in $\mathbb{R}^n/\langle(1, \cdots, 1)\rangle$. Denote by ρ_i the image of the vector v_i. Note that $\sum v_i = 0$. Choosing a real phase structure on $\mathbb{P}\Sigma_M$ amounts to choosing the following ingredients:

1. A necklace ordering of the n edges corresponding to $\rho_{i_1}, \dots, \rho_{i_n}$;
2. A point $p \in \mathbb{Z}_2^{n-1} \simeq \mathbb{Z}_2^n/\langle(1, \cdots, 1)\rangle$ that serves as the intersection point of $\mathcal{E}(\rho_{i_1})$ and $\mathcal{E}(\rho_{i_2})$.

From this information, a collection of affine lines $\mathcal{E}(\rho_i)$ satisfying the conditions of Definition 2.2 can be uniquely recovered. For example, the choice of point p determines both $\mathcal{E}(\rho_{i_1})$ and $\mathcal{E}(\rho_{i_2})$, since their tangent spaces are fixed. By the necklace arrangement property, the point $p + \sum_{k=2}^{j-1} v_{i_k}$ is in the affine line $\mathcal{E}(\rho_{i_j})$ for $j \geq 3$, where the vector sum is considered mod 2. This determines all of the affine lines $\mathcal{E}(\rho_{i_j})$.

Figure 3 shows the fan $\mathbb{P}\Sigma_M \subset \mathbb{R}^4/\langle(1, 1, 1, 1)\rangle \cong \mathbb{R}^3$ for $M = U_{2,4}$ together with an assignment of affine spaces along its edges that determine a real phase structure. The induced necklace ordering of the facets is $\sigma_2, \sigma_3, \sigma_1, \sigma_4$. From Figure 3, we see that the point p is contained in the intersection $\mathcal{E}(\sigma_2) \cap \mathcal{E}(\sigma_3)$. If we set $p = (0, 0, 0, 0) \in \mathbb{Z}_2^4/\langle(1, 1, 1, 1)\rangle$, then the corresponding necklace of lines is the last of the three arrangements in $\mathbb{Z}_2^3 \cong \mathbb{Z}_2^4/\langle(1, 1, 1, 1)\rangle$ depicted in Figure 2.

Example 2.13. For the matroid $M = U_{n-1,n}$ we will show that there is a unique real phase structure on $\mathbb{P}\Sigma_M$ up to reorientation. Such real phase structures were considered in [RS18].

By [RS18, Lemma 3.14], a real phase structure $\mathcal{E}$ on $\mathbb{P}\Sigma_M$ satisfies

$$|\bigcup_{\sigma} \mathcal{E}(\sigma)| = 2^{n-1} - 1.$$

Therefore there is exactly one element ε in the complement $\mathbb{Z}_2^{n-1} \setminus \cup_\sigma \mathcal{E}(\sigma)$. Up to reorientation we can suppose that $\varepsilon = (0, \dots, 0)$.

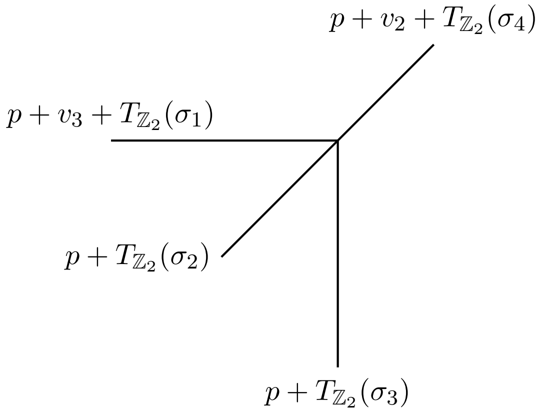

FIGURE 3. The projective fan of the matroid $U_{2,4}$ from Example 2.12 drawn in $\mathbb{R}^4/\langle(1,\dots,1)\rangle$ with the labelling of the faces indicating the assignment of an affine space $\mathcal{E}(\sigma_i)$ parallel to $T_{\mathbb{Z}_2}(\sigma_i)$.

Since $\mathbb{P}\Sigma_M$ is of codimension one in $\mathbb{R}^{n-1}$, for each facet σ of $\mathbb{P}\Sigma_M$ there is a choice of exactly two affine subspaces of $\mathbb{Z}_2^{n-1}$ which are parallel to the reduction of the span of σ in $\mathbb{Z}_2^{n-1}$. One of these spaces is an honest vector subspace and hence contains $(0,\dots,0)$. Therefore, if we are to associate to each σ an affine subspace $\mathcal{E}(\sigma)$ and wish to avoid that it contains $(0,\dots,0)$, then the choice of affine space at each top dimensional face is determined. This demonstrates that there is at most one real phase structure on $\mathbb{P}\Sigma_M$, up to reorientation in the sense of Definition 2.3.

2.4. **Deletion and contraction of real phase structures.** We briefly recall the notion of minors of a matroid. Let M be a matroid with ground set E and rank function r. For $S\subset E$, then the *deletion* of S is the matroid $M\backslash S$ with ground set $E\setminus S$ and rank function $\mathrm{r}_{M\backslash S}(A)=\mathrm{r}_M(A)$. The *contraction* of M by S is the matroid M/S whose ground set is again $E\setminus S$ and rank function $\mathrm{r}_{M/S}(A)=\mathrm{r}_M(A\cup S)-\mathrm{r}_M(A)$. Lastly, the *restriction* of M to S is the matroid $M|_S$ whose ground set is S and rank function $\mathrm{r}_{M|_S}$ is the restriction of r_M. Notice that $M|_S=M\backslash S^c$, where $S^c=E\setminus S$. A *minor* of a matroid M is any matroid obtained from M by a sequence of deletions and contractions.

For any subset $A\subset E$, we denote by $p_A\colon \mathbb{R}^E\to\mathbb{R}^{E\backslash A}$ the projection which forgets the coordinates x_i for all $i\in A$. If the matroid M has loops $L\subseteq E$, we use the same notation for the projection $p_A\colon\mathbb{R}^{E\backslash L}\to\mathbb{R}^{E\backslash(L\cup A)}$. We also use the shorthand p_i in the case $A=\{i\}$.

If i is a loop or coloop of M, then $M\backslash i=M/i$, so deletion and contraction are equivalent. The support of $\Sigma_{M\backslash i}$ is the image of the projection of the matroid fan Σ_M under the projection p_i. Note that this is also true if i is a loop, in which case, according to our conventions, $\Sigma_{M\backslash i}=\Sigma_M$ and $p_i=\mathrm{id}$. If i is not a coloop, then the facets of $\Sigma_{M\backslash i}$ are the projections of facets of Σ_M whose dimensions are preserved under p_i.

Suppose that i is not a loop of M. Note that by our convention regarding loops, we have $\Sigma_{M/i}=\Sigma_{M/\operatorname{cl}(i)}$. The support of the matroid fan of M/i is the set $\{x \in \mathbb{R}^{E\setminus \mathrm{cl}(i)} \mid |p_{\mathrm{cl}(i)}^{-1}(x)| > 1\}$. The facets of $\Sigma_{M/i}$ are the images of facets of Σ_M whose dimensions are *not* preserved under the projection by $p_{\mathrm{cl}(i)}$. More details on the geometry of Σ_M, $\Sigma_{M\setminus i}$, and $\Sigma_{M/i}$ and their relations under p_i can be found in [Sha13, Section 2] and also [FR13, Section 3].

A real phase structure on the fan of a matroid M induces canonical real phase structures on the fans of all minors of M. We will describe the induced real phase structures for elementary deletions and contractions of a matroid M. The geometric idea is very simple: Given a facet σ of the matroid fan of a minor, we pick a facet $\tilde{\sigma}$ of Σ_M that projects to σ. Then the affine space associated to σ is the projection of $\mathcal{E}(\tilde{\sigma})$. Given that we work with the fine subdivision of Σ_M induced by the lattice of flats, the choice of $\tilde{\sigma}$ is in general not unique. To simplify the proofs in the following sections, we make a specific choice for $\tilde{\sigma}$. However, Definition 2.14 is independent of this choice, as discussed after the definition. For σ a facet of $\Sigma_{M\setminus i}$, let $p_i^*(\sigma)$ be the facet of Σ_M which is obtained by taking the closure in M of all the flats of $M\setminus i$ occurring in the chain of flats describing σ when i is not a coloop of M. If i is a coloop of M, prolongate the chain by one piece by adding i everywhere. Note that $p_i(p_i^*(\sigma)) = \sigma$. Moreover, if i is not a coloop $p_i^*(\sigma)$ is the unique facet that projects to σ. For σ a facet in $\Sigma_{M/i}$ corresponding to the chain of flats

$$\emptyset = \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E \setminus \mathrm{cl}(i),$$

set $p^{\diamond}_{\mathrm{cl}(i)}(\sigma)$ to be the facet of Σ_M given by the chain

$$\mathrm{cl}(\emptyset) \subseteq \mathrm{cl}(i) \subsetneq F_1 \sqcup \mathrm{cl}(i) \subsetneq \cdots \subsetneq F_l \sqcup \mathrm{cl}(i) \subsetneq E.$$

Note that $p_{\mathrm{cl}(i)}(p^{\diamond}_{\mathrm{cl}(i)}(\sigma)) = \sigma$. By abuse of notation, we use the same letter p_A for the reduction mod 2 counterpart $p_A \colon \mathbb{Z}_2^E \to \mathbb{Z}_2^{E\setminus A}$.

Definition 2.14. *Let $\mathcal{E}$ be a real phase structure for the matroid fan Σ_M and choose $i \in E$. The* deletion *$\mathcal{E}\setminus i$ is the real phase structure on $\Sigma_{M\setminus i}$ given by*

$$(\mathcal{E}\setminus i)(\sigma) = p_i(\mathcal{E}(p_i^*(\sigma)))$$

for any facet σ of $\Sigma_{M\setminus i}$. The contraction *$\mathcal{E}/i$ is the real phase structure on $\Sigma_{M/i}$ given by*

$$(\mathcal{E}/i)(\sigma) = p_{\mathrm{cl}(i)}(\mathcal{E}(p^{\diamond}_{\mathrm{cl}(i)}(\sigma)))$$

for any facet σ of $\Sigma_{M/i}$.

As previously mentioned, for σ a facet of either $\Sigma_{M\setminus i}$ or $\Sigma_{M/i}$, we are free to replace $p_i^*(\sigma)$ or $p^{\diamond}_{\mathrm{cl}(i)}(\sigma)$ in the above definition with any facet of Σ_M which projects onto σ under p_i or $p_{\mathrm{cl}(i)}$, respectively. Any such facet will be contained in the same facet of the coarsest subdivision of the support of Σ_M as the facets $p_i^*(\sigma)$ or $p^{\diamond}_{\mathrm{cl}(i)}(\sigma)$, respectively. Therefore, Conditions (1) and (2) of a real phase structure imply that $\mathcal{E}$ must assign the same affine space to any such choice of face.

Proposition 2.15. *The maps $\mathcal{E}\setminus i$ and $\mathcal{E}/i$ from Definition 2.14 define real phase structures on $\Sigma_{M\setminus i}$ and $\Sigma_{M/i}$ respectively.*

Proof. We must show that the maps $\mathcal{E}\backslash i$ and $\mathcal{E}/i$ satisfy Conditions (1) and (2) of Definition 2.2. We set $p = p_i$ or $p = p_{\mathrm{cl}(i)}$ depending on whether we consider the deletion or the contraction. Let σ be a facet of either $\Sigma_{M\backslash i}$ or $\Sigma_{M/i}$. Let $\tilde{\sigma} = p^*(\sigma)$ or $\tilde{\sigma} = p^\diamond(\sigma)$. The projection of an affine space along p remains an affine space. Moreover, since we have $\sigma = p(\tilde{\sigma})$ and hence also $T(\sigma) = p(T(\tilde{\sigma}))$, Condition (1) holds for both $\mathcal{E}\backslash i$ and $\mathcal{E}/i$.

Let us now check Condition (2). Let τ be a codimension one face of either $\Sigma_{M\backslash i}$ or $\Sigma_{M/i}$ and set $\tilde{\tau} = p^*(\tau)$ or $\tilde{\tau} = p^\diamond(\tau)$, respectively. If $\dim \tilde{\tau} = \dim \tau + 1$, then the arrangements of affine subspaces around τ and $\tilde{\tau}$ agree after quotienting by $T_{\mathbb{Z}_2}(\tau)$ and $T_{\mathbb{Z}_2}(\tilde{\tau})$, respectively.

Now let us assume that $\dim \tilde{\tau} = \dim \tau$. Let $\tilde{\sigma}_1, \dots, \tilde{\sigma}_k$ denote the facets of Σ_M adjacent to $\tilde{\tau}$. Let $J = \{j \mid 1 \leq j \leq k, \dim \tilde{\sigma}_j = \dim p(\tilde{\sigma}_j)\}$. Then the arrangement of affine subspaces around τ in the induced real phase structure consists of the affine spaces $p(\mathcal{E}(\tilde{\sigma}_j))$ for $j \in J$. Given a point $\varepsilon \in p(\mathcal{E}(\tilde{\sigma}_{j_0}))$ for some $j_0 \in J$, first notice that

$$\begin{aligned} |\{j \in J \colon \varepsilon \in p(\mathcal{E}(\tilde{\sigma}_j))\}| &= |\{(\tilde{\varepsilon}, j) \mid \tilde{\varepsilon} \in p^{-1}(\varepsilon) \cap \mathcal{E}(\tilde{\sigma}_j) \text{ and } j \in J\}| \\ &\equiv |\{(\tilde{\varepsilon}, i) \mid \tilde{\varepsilon} \in p^{-1}(\varepsilon) \cap \mathcal{E}(\tilde{\sigma}_i), 1 \leq i \leq k\}| \mod 2. \end{aligned}$$

The equality and congruence follow from the fact that $\#\{\mathcal{E}(\tilde{\sigma}_i) \cap p^{-1}(\varepsilon)\} = 1$ if $\dim \tilde{\sigma}_i = \dim p(\tilde{\sigma}_i)$ and $\#\{\mathcal{E}(\tilde{\sigma}_i) \cap p^{-1}(\varepsilon)\}$ is even otherwise. The last expression is a sum of even numbers by the fact that $\mathcal{E}$ is a real phase structure on Σ_M. Hence, any point $\varepsilon \in p(\mathcal{E}(\tilde{\sigma}_{j_0}))$ for any $j_0 \in J$ is covered an even number of times by the affine spaces around τ which proves condition (2). □

Note that we can iterate the operations of deletion and contraction to construct general *minors* $\mathcal{E}\backslash A/B$. To justify the notation, we need to show that the result is invariant under reordering the sequence of deletions and contractions.

Proposition 2.16. *Let $\mathcal{E}$ be a real phase structure for the matroid fan Σ_M of the matroid M and choose $i \neq j \in E$. Then*

$$\begin{aligned} \mathcal{E}\backslash i\backslash j &= \mathcal{E}\backslash j\backslash i, \\ \mathcal{E}\backslash i/j &= \mathcal{E}/j\backslash i, \\ \mathcal{E}/i/j &= \mathcal{E}/j/i. \end{aligned}$$

Proof. We start with the first two equalities. Note that by definition $\mathcal{E}\backslash i = \mathcal{E}/i$ if i is a coloop. Hence we may assume that not both i and j are coloops in the first equation and i not a coloop in the second equation, the exceptions being covered by the third equation. Under these assumptions, the first two equations hold since $p_i^* \circ p_j^* = p_j^* \circ p_i^*$ and $p_i^* \circ p_j^\diamond = p_j^\diamond \circ p_i^*$.

For the last equality, notice that for any i, j the projections $p_{\mathrm{cl}(i)}$ and $p_{\mathrm{cl}(j)}$ commute and the composition is $p_{\mathrm{cl}(i,j)}$. We are asked to compare the projections under $p_{\mathrm{cl}(i,j)}$ of the affine spaces $\mathcal{E}(\sigma_1), \mathcal{E}(\sigma_2)$ where σ_1, σ_2 are two faces of Σ_M associated to the chains of flats

$$\mathrm{cl}(\emptyset) \subseteq \mathrm{cl}(i) \subseteq \mathrm{cl}(i,j) \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E, \tag{2.1}$$

$$\mathrm{cl}(\emptyset) \subseteq \mathrm{cl}(j) \subseteq \mathrm{cl}(i,j) \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E. \tag{2.2}$$

We may assume $\mathrm{r}(i,j) = 2$, since otherwise $\sigma_1 = \sigma_2$. Then σ_1 and σ_2 are adjacent to the codimension one face τ associated to

$$\mathrm{cl}(\emptyset) \subsetneq \mathrm{cl}(i,j) \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E.$$

Note that $p_{\mathrm{cl}(i,j)}(T(\sigma)) = p_{\mathrm{cl}(i,j)}(T(\tau))$ for every facet σ adjacent to τ. In particular, if two affine subspaces in the necklace arrangement intersect, their projections under $p_{\mathrm{cl}(i,j)}$ agree. Therefore the necklace condition (2') implies that $p_{\mathrm{cl}(i,j)}(\mathcal{E}(\sigma))$ is the same affine space for every σ adjacent to τ. In particular, we have $p_{\mathrm{cl}(i,j)}(\mathcal{E}(\sigma_1)) = p_{\mathrm{cl}(i,j)}(\mathcal{E}(\sigma_2))$. This proves the claim. □

3. Matroid orientations and real phase structures

3.1. **From oriented matroids to real phase structures.** Here we will produce a real phase structure on a matroid fan from an oriented matroid. We will use the covector description of oriented matroids. For an oriented matroid $\mathcal{M}$, on ground set E, the covectors of $\mathcal{M}$ are a subset $\mathcal{C} \subseteq \{0, +1, -1\}^E$. Let $X \in \mathcal{C}$. For $i \in E$, the i-th coordinate of X is denoted by X_i. The positive and negative parts of X are respectively

$$X^+ := \{i \in E \mid X_i = +1\},$$

and

$$X^- := \{i \in E \mid X_i = -1\}.$$

The support of X is

$$\mathrm{Supp}(X) := \{i \in E \mid X_i \neq 0\}.$$

The composition operation $\circ$ on covectors X and Y is defined by

$$(X \circ Y)_i = \begin{cases} X_i & \text{if } X_i \neq 0 \\ Y_i & \text{if } X_i = 0. \end{cases}$$

The separation set $S(X,Y)$ is defined by

$$S(X,Y) := \{i \in E \mid X_i = -Y_i \neq 0\}.$$

The covectors of an oriented matroid satisfy the following axioms:

(1) $0 \in \mathcal{C}$
(2) $X \in \mathcal{C}$ if and only if $-X \in \mathcal{C}$
(3) $X, Y \in \mathcal{C}$ implies that $X \circ Y \in \mathcal{C}$
(4) If $X, Y \in \mathcal{C}$ and $i \in S(X,Y)$ then there exists a $Z \in \mathcal{C}$ such that $Z_i = 0$ and $Z_j = (X \circ Y)_j = (Y \circ X)_j$ for all $j \notin S(X,Y)$.

The set of covectors $\mathcal{C}$ forms a lattice under the partial order $0 < +1, -1$ considered coordinatewise. There is a forgetful map ϕ from oriented matroids to matroids which preserves rank and the size of the ground set. Given an oriented matroid $\mathcal{M}$, we let $\underline{\mathcal{M}} = \phi(\mathcal{M})$ denote its underlying matroid. We can describe the forgetful map on the level of the covector lattice $\mathcal{C}$ of $\mathcal{M}$ and the lattice of flats $\mathcal{L}$ of M. Given a covector $X \in \mathcal{C}$ the forgetful map assigns $\phi(X) = \mathrm{Supp}(X)^c \in \mathcal{L}$ where A^c denotes $E \setminus A$. The image of a covector of the oriented matroid under the forgetful map is a flat of the underlying matroid [BLVS+99, Proposition 4.1.13].

The set of *topes* $\mathcal{T}$ are the maximal covectors with respect to the partial order on $\mathcal{C}$. If the underlying matroid of $\mathcal{M}$ has no loops we have $\mathcal{T} \subseteq$

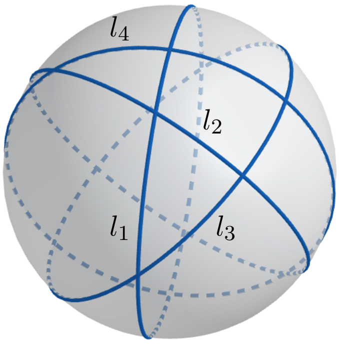

FIGURE 4. The intersection of a real arrangement of 4 generic planes in $\mathbb{R}^3$ with the unit sphere. Assigning the covector $(+,+,+,+)$ to the region bounded by the spherical triangle facing the viewer formed by l_1, l_2, l_3 determines the covectors of all regions.

$\{+1,-1\}^{|E|}$. Let $\mathcal{M}$ be an oriented matroid with collection of topes $\mathcal{T}$ and underlying lattice of flats $\mathcal{L}$. For $F \in \mathcal{L}$ and $T \in \mathcal{T}$, we denote by $T \setminus F \in \{0,+1,-1\}^E$ the vector obtained by setting all coordinates in F to 0. We say F is *adjacent to* T if $T \setminus F \in \mathcal{C}$. More generally, given a flag $\mathcal{F} := F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ of flats in $\mathcal{L}$, we define the set of topes adjacent to $\mathcal{F}$ by

$$\mathcal{T}(\mathcal{F}) = \{T \in \mathcal{T} \mid T \setminus F_i \in \mathcal{C} \text{ for all } i = 0,\dots,k\}.$$

Example 3.1. A set $H_1,\dots,H_n$ of hyperplanes in $\mathbb{R}^r$ defined by linear forms $l_1,\dots,l_n$ produces an oriented matroid on $\{1,\dots,n\}$. The underlying matroid on $\{1,\dots,n\}$ is given by the rank function $\mathrm{r}(A) = \mathrm{codim}(\cap_{i\in A} H_i)$. A covector corresponds to a cell of the decomposition of $\mathbb{R}^r$ induced by the positive regions $H_i^+ = \{l_i(x) \geq 0\}$ and negative regions $H_i^- = \{l_i(x) \leq 0\}$. Assuming that none of the linear forms are identically equal to zero, the topes are in bijection with the cells in the complement of the arrangement. The flat associated to a covector is in bijection with the set of hyperplanes containing the corresponding cell. Figure 4, shows the intersection of an arrangement of four planes in $\mathbb{R}^3$ with a sphere. The underlying matroid of this arrangement is the uniform matroid $U_{3,4}$. There are 14 cells of dimension two in the subdivision of the sphere induced by the intersections of the four planes. These are the topes of the oriented matroid. Each cell of the complement is labelled by a tuple $\{+,-\}^4$ corresponding to the sign of the linear forms $l_1,\dots,l_4$ evaluated at a point in the open cell.

Lemma 3.2. *Let $\mathcal{M}$ be an oriented matroid of rank d and set $M = \underline{\mathcal{M}}$. For any flag $\mathcal{F} = \{\mathrm{cl}(\emptyset) = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E\}$ of M we have*

$$|\mathcal{T}(\mathcal{F})| = |\chi_{M_{\mathcal{F}}}(-1)|,$$

where $\chi_{M_{\mathcal{F}}}(t)$ denotes the characteristic polynomial of the matroid

$$M_{\mathcal{F}} = \bigoplus_{i=1}^{k+1} M|F_i/F_{i-1}.$$

In particular, if $\mathcal{F}$ is a maximal flag, then $|\mathcal{T}(\mathcal{F})| = 2^d$.

Proof. The number of topes of an oriented matroid $\mathcal{N}$ is counted by $|\chi_{\underline{\mathcal{N}}}(-1)|$ [Zas75]. Moreover, by [BLVS+99, Proposition 3.7.11] the set $\mathcal{T}(\mathcal{F})$ is equal to the set of topes of the oriented matroid $\mathcal{M}_{\mathcal{F}} = \bigoplus_{i=1}^{k+1} \mathcal{M}|F_i/F_{i-1}$. Since $\underline{\mathcal{M}}_{\mathcal{F}} = M_{\mathcal{F}}$, the statement follows. □

Remark 3.3. *We would like to make the following remark on our choice of conventions. In this paper, we use both the multiplicative and additive notation on the group of two elements $(\{0,1\},+)$ and $(\{1,-1\},\cdot)$. When speaking of real phase structures we work with vector spaces over $\mathbb{Z}_2$, therefore it is preferable to use the additive notation and denote the field of two elements by $\{0,1\}$. On the other hand it is tradition that the covectors of oriented matroids take values in $\{0,+,-\}$ and we also make use of the group structure on $\{+,-\}$. We routinely use the notation ε to denote elements of the field $\{0,1\}$ or of vector spaces over this field. We use uppercase roman letters, for example, X, Y, T, to denote covectors. Covectors can be multiplied entry by entry and this operation is denoted by $T \cdot T'$.*

To go from the additive group notation to the multiplicative group notation for vectors, we use $(-1)^\varepsilon = ((-1)^{\varepsilon_1}, \dots, (-1)^{\varepsilon_n})$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$. This defines a bijection

$$\begin{aligned} \mathbb{Z}_2^E &\to \{+1,-1\}^E, \\ \varepsilon &\mapsto (-1)^\varepsilon. \end{aligned} \tag{3.1}$$

We avoid going backwards as much as possible to avoid writing such perversities as $\log_{-1}$, even though this map makes sense as a discrete logarithm for groups.

Definition 3.4. *Let $\mathcal{M}$ be a loopfree oriented matroid on the ground set E and let $\Sigma_M \subseteq \mathbb{R}^E$ the fan of the underlying matroid $M = \underline{\mathcal{M}}$. For every facet $\sigma_{\mathcal{F}}$ of Σ_M corresponding to the maximal flag of flats $\mathcal{F}$, we set*

$$\mathcal{E}_{\mathcal{M}}(\sigma_{\mathcal{F}}) = \{\varepsilon \mid (-1)^\varepsilon \in \mathcal{T}(\mathcal{F})\} \subseteq \mathbb{Z}_2^E.$$

If $\mathcal{M}$ has loops $L = \mathrm{cl}(\emptyset)$, we set $\mathcal{E}_{\mathcal{M}}(\sigma) := \mathcal{E}_{\mathcal{M}\setminus L}(\sigma) \subseteq \mathbb{Z}_2^{E\setminus L}$.

In the next two lemmata, we show that $\mathcal{E}_{\mathcal{M}}$ defines a real phase structure on Σ_M. We start with a few useful observations. If T is a tope then $T \circ X = T$ for any X and $X \circ T$ is always another tope, which is distinct from T if and only if $X^+ \nsubseteq T^+$ or $X^- \nsubseteq T^-$. Given a subset $F \subset E$, the *reflection* $r_F(X)$ of a covector X in F is given by flipping the signs for all $e \in F$ while keeping the signs for $e \in E \setminus F$. The reflection of a covector X in a flat F is not always a covector of the oriented matroid. However, note that if F is adjacent to the tope T, then $r_F(T)$ is also a tope. Indeed, setting $X = T \setminus F$, note that we can rewrite $r_F(T) = X \circ (-T)$, hence the statement.

Recall that given any flat F of M, there is a vector in Σ_M defined by $v_F := \sum_{i \in F} v_i$, where $v_i = -e_i$, see Section 2.2. We denote by ε_F the reduction of v_F modulo 2. Note that $r_F((-1)^\varepsilon) = (-1)^{\varepsilon + \varepsilon_F}$.

Lemma 3.5. *The set $\mathcal{E}_{\mathcal{M}}(\sigma_{\mathcal{F}})$ from Definition 3.4 is a d-dimensional affine subspace parallel to $T_{\mathbb{Z}_2}(\sigma_{\mathcal{F}})$ for every facet $\sigma_{\mathcal{F}}$ of Σ_M.*

Proof. We may assume that $\mathcal{M}$ is loopfree. Let $\sigma_{\mathcal{F}}$ be a facet of Σ_M with associated maximal flag

$$\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_{d-1} \subsetneq F_d = E\}.$$

Let $T = (-1)^\varepsilon \in \mathcal{T}(\mathcal{F})$ be a tope adjacent to $\mathcal{F}$. We need to show that the bijection from (3.1) maps $\varepsilon + \langle \varepsilon_{F_1}, \dots, \varepsilon_{F_{d-1}}, \varepsilon_E \rangle_{\mathbb{Z}_2}$ to $\mathcal{T}(\mathcal{F})$. Note that $|\mathcal{T}(\mathcal{F})| = 2^d$ by Lemma 3.2, so the two sets have the same size. Therefore, it suffices to show containment of the image in $\mathcal{T}(\mathcal{F})$. For this, it is enough to prove that $T' = r_{F_i}(T) \in \mathcal{T}(\mathcal{F})$ for every $i = 1, \dots, d$. Note that $T' \setminus F_j = T \setminus F_j$ for $j \geq i$ and $T' \setminus F_j = (T \setminus F_i) \circ (-T \setminus F_j)$ for $j \leq i$. In both cases, $T' \setminus F_j$ is a covector, and hence $T' \in \mathcal{T}(\mathcal{F})$ as required. □

Lemma 3.6. *Let $\mathcal{M}$ be a oriented matroid and set $M = \underline{\mathcal{M}}$. Let τ be a codimension one face of Σ_M and let $\sigma_1, \dots, \sigma_k$ be the adjacent facets. Then the subspaces $\mathcal{E}_{\mathcal{M}}(\sigma_i)$ for $i = 1, \dots, k$ form an even covering.*

Proof. Again, we can reduce to the case when $\mathcal{M}$ is loopfree. Let $\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E\}$ be the chain of flats associated to τ and pick $T \in \mathcal{T}(\mathcal{F})$. Since τ is of codimension one, there is exactly one rank two jump in $\mathcal{F}$. We need to show that there is an even number of completions of $\mathcal{F}$ to maximal chains $\mathcal{F}'$ such that $T \in \mathcal{T}(\mathcal{F}')$ Clearly, such completions are in bijection with the completions of the chain of covectors $(0, \dots, 0) < X_k < X_{k-1} < \cdots < X_1 < T$ with $X_i := T \setminus F_i$. Now, recall from [BLVS+99, Theorem 4.1.14 (ii))] that the covector lattice of an oriented matroid satisfies the diamond property, that is, all intervals of length 2 consist of 4 elements. It follows that there are exactly two such completions, which proves the claim. □

Proposition 3.7. *The map $\mathcal{E}_{\mathcal{M}} \colon \mathit{Facets}(\Sigma_M) \to \mathit{Aff}_d(\mathbb{Z}^{E \setminus \mathrm{cl}(\emptyset)})$ from Definition 3.4 defines a real phase structure on Σ_M.*

Proof. By Lemma 3.5 the first axiom of a real phase structure is satisfied. By Lemma 3.6 the second axiom is satisfied. □

To summarise, we have constructed a map from orientations of M (that is, oriented matroids $\mathcal{M}$ such that $\underline{\mathcal{M}} = M$) to real phase structures on Σ_M,

$$\{\text{Orientation of } M\} \xrightarrow{\mathbf{E}} \{\text{Real phase structure on } \Sigma_M\}, \tag{3.2}$$

which is given by $\mathbf{E}(\mathcal{M}) = \mathcal{E}_{\mathcal{M}}$. The main result of this paper claims that this map is bijective. We note that the map is injective, since the topes of $\mathcal{M}$ can be recovered from $\mathcal{E}$ as

$$\mathcal{T} = \{(-1)^\varepsilon \mid \varepsilon \in \bigcup_\sigma \mathcal{E}(\sigma)\},$$

where σ runs through all facets of Σ_M and moreover, an oriented matroid is determined by its collection of topes [dS95].

Remark 3.8. *Given an oriented matroid $\mathcal{M}$ and a subset $S \subset E$, the reorientation of $\mathcal{M}$ along S is the oriented matroid $\mathcal{M}'$ whose topes are the covectors $r_S(T)$ for any tope T of $\mathcal{M}$. Clearly, in this case $\mathcal{E}_{\mathcal{M}'}$ is a reorientation of $\mathcal{E}_{\mathcal{M}}$ (in the sense of Definition 2.3) with translation vector ε_S. Hence the map $\mathbf{E}$ from (3.2) descends to a map modulo reorientations on both sides.*

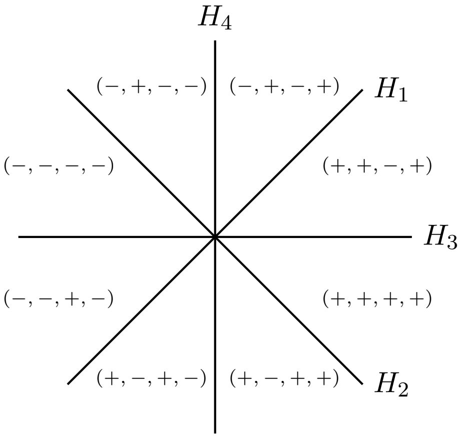

FIGURE 5. An arrangement of 4 lines through the origin in $\mathbb{R}^2$ as used in Example 3.9 to go from real phase structures to oriented matroids. The necklace ordering of the lines corresponds to the necklace ordering of the facets of $\mathbb{P}\Sigma_M$ defined by the real phase structure in Figure 3.

Example 3.9. Here we consider the matroid $M = U_{2,n}$ and describe how we can construct an inverse to the map in (3.2). By Example 2.12, a real phase structure $\mathcal{E}$ on Σ_M determines a necklace ordering of $\{1, \dots, n\}$. Note that Example 2.12 described the projective matroid fan which can be obtained from Σ_M by quotienting by $(1, \dots, 1)$. To determine an orientation of M from the real phase structure $\mathcal{E}$, let $H_1, \dots, H_n \subset \mathbb{R}^2$ be a collection of pairwise distinct lines passing through the origin in $\mathbb{R}^2$, arranged so that the clockwise/anticlockwise appearance of these lines when making a turn around the origin defines the same necklace ordering on $\{1, \dots, n\}$ as the real phase structure $\mathcal{E}$. Two lines H_i, H_j border a chamber of $\mathbb{R}^2$ if and only if $\mathcal{E}(\sigma_i) \cap \mathcal{E}(\sigma_j) \neq \emptyset$. Moreover, if this is satisfied then H_i, H_j border exactly two chambers which are related by the antipodal map. If the intersection of the two affine spaces $\mathcal{E}(\sigma_i)$ and $\mathcal{E}(\sigma_j)$ is non-empty, then it consists of two points $\varepsilon, \varepsilon' = \varepsilon + (1, \dots, 1)$. Assigning $(-1)^{\varepsilon}$ to one of the chambers bordered by H_i, H_j and $(-1)^{\varepsilon'}$ to the other chamber determines an orientation of $U_{2,n}$. See Figure 5 for the example of the 4 lines in $\mathbb{R}^2$ corresponding to the real phase structure on the projective fan of $U_{2,4}$ from Figure 3. This procedure associates to any real phase structure on Σ_M, an orientation of M and it is easily checked that it provides an inverse map to $\mathbf{E}$ from (3.2).

Example 3.10. Here we consider the matroid $M = U_{n-1,n}$. Orientations of M can be constructed from generic real hyperplane arrangements like in Example 3.1. Moreover, in Example 2.13 we showed that Σ_M carries a unique real phase structure up to reorientation. By Remark 3.8, the map (3.2) is surjective and hence bijective. An analogous discussion shows that (3.2) is bijective for $M = U_{1,n}$.

Our main theorem is to prove the equivalence of real phase structures on matroid fans and matroid orientations. In the more general framework of matroids over hyperfields, oriented matroids are identical to matroids over the sign hyperfield $\mathbb{S} = \{0, +, -\}$ [BB18]. More precisely, a chirotope describing an oriented matroid can be interpreted as Grassmann-Plücker function on E^r with values in $\mathbb{S}$. Correspondingly, the affine subspaces provided by real phase structures live in a vector space over $\mathbb{Z}_2 \cong \mathbb{S}^*$. For a general hyperfield $\mathbb{H}$ the non-zero elements form a group $\mathbb{H}^*$ under multiplication and we can exchange the role of affine subspaces of $\mathbb{Z}_2^n$ for cosets of subgroups of $(\mathbb{H}^*)^n$.

Question 3.11. *Can matroids over a general hyperfield $\mathbb{H}$ be equivalently formulated by specifying cosets of the group $(\mathbb{H}^*)^n$ on top dimensional faces of a matroid fan $\Sigma \subseteq \mathbb{R}^n$?*

We finish this subsection by showing that the operations of deletion and contraction on both oriented matroids and real phase structures commute with the map $\mathbf{E}$ from Equation (3.2). We will use the more convenient notation of $\mathcal{E}_{\mathcal{M}}$ to denote the real phase structure $\mathbf{E}(\mathcal{M})$.

Proposition 3.12. *Let $\mathcal{M}$ be an oriented matroid on E and $i \in E$. Then*

$$\mathcal{E}_{\mathcal{M}/i} = \mathcal{E}_{\mathcal{M}}/i \quad \text{and} \quad \mathcal{E}_{\mathcal{M}\backslash i} = \mathcal{E}_{\mathcal{M}}\backslash i.$$

Proof. We set $\mathcal{E} = \mathcal{E}_{\mathcal{M}}$ and denote by $\mathcal{M}'$, $\mathcal{E}'$, M' the contraction or deletion by i, respectively. It suffices to show that $\mathcal{E}'(\sigma) \subseteq \mathcal{E}'_{\mathcal{M}}(\sigma)$ for every facet σ of $\Sigma_{M'}$, because both sets are affine subspace of equal dimension. By [BLVS$^+$99, Proposition 3.7.11] the covectors of $\mathcal{M}\backslash i$ are the projections of the covectors of $\mathcal{M}$ under p_i. Analogously, the covectors of $\mathcal{M}/i$ are the projections under p_i of the covectors of $\mathcal{M}$ with $X_i = 0$ (hence they are zero on $\text{cl}(i)$). Comparing with Definitions 2.14 and 3.4, the inclusion then follows. □

3.2. **Real subfans and oriented matroid quotients.** In this subsection, we introduce the concept of *real subfans* and study their relationship to oriented matroid quotients. Besides the intrinsic importance of these constructions, they will be used in the following subsection in the induction step in the proof of Theorem 1.1.

Given a real phase structure $\mathcal{E}$ on a polyhedral fan Σ, we extend the definition of $\mathcal{E}$ to non-maximal cones $\tau \in \Sigma$ by setting

$$\mathcal{E}(\tau) = \bigcup_{\substack{\sigma \text{ facet} \\ \tau \subset \sigma}} \mathcal{E}(\sigma).$$

As an example, note that if $\mathcal{E} = \mathcal{E}_{\mathcal{M}}$ is induced by an oriented matroid $\mathcal{M}$, the description of $\mathcal{E}(\sigma_{\mathcal{F}})$ as the set of elements ε such that $(-1)^{\varepsilon}$ is adjacent to $\mathcal{F}$ extends to non-maximal cones $\sigma_{\mathcal{F}}$, that is,

$$\mathcal{E}_{\mathcal{M}}(\sigma_{\mathcal{F}}) = \{\varepsilon \mid (-1)^{\varepsilon} \in \mathcal{T}(\mathcal{F})\} \subseteq \mathbb{Z}_2^E. \tag{3.3}$$

The set above no longer has the structure of an affine subspace over $\mathbb{Z}_2^E$ when $\sigma_{\mathcal{F}}$ is not a top dimensional face of the fan.

Definition 3.13. *Given a real phase structure $\mathcal{E}'$ on a fan Σ', we say that $(\Sigma', \mathcal{E}')$ is a* real subfan *of $(\Sigma, \mathcal{E})$ if $\Sigma' \subseteq \Sigma$ and for any $\tau \in \Sigma'$ we have $\mathcal{E}'(\tau) \subseteq \mathcal{E}(\tau)$.*

Real subfans occur naturally when considering the pair of contraction and deletion of a real phase structure along the some subset.

Lemma 3.14. *Let M be a matroid on the ground set E and let $\mathcal{E}$ be real phase structure on Σ_M. Let F be a flat of M. Then $(\Sigma_{M/F}, \mathcal{E}/F)$ is a real subfan of $(\Sigma_{M\backslash F}, \mathcal{E}\backslash F)$.*

Proof. By definition of contraction and deletion for matroids, all flats of M/F are also flats of $M\backslash F$. In the language of matroids the contraction M/F is a quotient of the deletion $M\backslash F$. This implies $\Sigma_{M/F} \subseteq \Sigma_{M\backslash F}$, see [FR13] or [Sha13]. For the inclusions $\mathcal{E}/F(\tau) \subseteq \mathcal{E}\backslash F(\tau)$, by recursion we may reduce to the case where M is loopfree, $\mathrm{r}(F) = 1$ and τ a facet of $\Sigma_{M/F}$.

Let $\{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_l \subsetneq F_{l+1} = E \setminus F\}$ be the chain of flats in M/F corresponding to the face τ. Then $\sigma_0 := p_F^{\diamond}(\tau)$ is given by $\mathcal{F}_0 = \{\emptyset \subseteq F \subsetneq F_1 \sqcup F \subsetneq \cdots \subsetneq F_l \sqcup F \subsetneq E\}$. Fix an element $\varepsilon \in \mathcal{E}(\sigma_0)$. Removing F from this chain gives rise to a codimension one face τ_0 of σ_0 such that $p_F(\tau_0) = \tau$. By condition (2) of Definition 2.2 there exists another facet σ_1 adjacent to τ_0 such that $\varepsilon \in \mathcal{E}(\sigma_1)$.

We now distinguish two cases: If $p_F(\sigma_1) \supsetneq \tau$ we are done, since we proved that $p_F(\varepsilon) \in \mathcal{E}\backslash F(p_F(\sigma_1)) \subseteq \mathcal{E}\backslash F(\tau)$. If $p_F(\sigma_1) = \tau$, this implies that F_1 is a flat in M and σ_1 is given by $\mathcal{F}_1 = \{\emptyset \subseteq F_1 \subsetneq F_1 \sqcup F \subsetneq \cdots \subsetneq F_l \sqcup F \subsetneq E\}$. In this case we may follow a similar procedure as above, this time removing the flat $F_1 \sqcup F$ to obtain a face τ_1 of codimension one of σ_1 in Σ_M. Notice that $p_F(\tau_1) = \tau$. Repeating the argument above we have another facet σ_2 of Σ_M adjacent to τ_1 such that $\varepsilon \in \mathcal{E}(\sigma_2)$. Once again, if $p_F(\sigma_2) \supsetneq \tau$ we are done. Otherwise the facet σ_2 corresponds to a flag $\mathcal{F}_2 = \{\emptyset \subseteq F_1 \subsetneq F_2 \subsetneq F_3 \sqcup F \subsetneq \cdots \subsetneq F_l \sqcup F \subsetneq E\}$. We see that we can continue to repeat the procedure above finding facets σ_k of Σ_M until $p(\sigma_k) \supsetneq \tau$, unless it is the case that all F_i, and in particular $E \setminus F$, are flats of M. This special case is equivalent to F being a connected component of M. In this case, $(\Sigma_{M/F}, \mathcal{E}/F) = (\Sigma_{M\backslash F}, \mathcal{E}\backslash F)$ and the statement is trivial. ☐

We denote by $\mathcal{M}_1$ and $\mathcal{M}_2$ two oriented matroids on the same ground set E. For $i = 1, 2$, we denote by M_i, Σ_i, $\mathcal{E}_i$ the underlying matroids, associated matroid fans, and associated real phase structures, respectively.

We call $\mathcal{M}_1$ a *quotient* of $\mathcal{M}_2$ if all covectors of $\mathcal{M}_1$ are covectors of $\mathcal{M}_2$, see [BLVS+99, Section 7.7]. In this case, M_1 is a quotient of M_2, that is, all flats of M_1 are flats of M_2, see [BLVS+99, Corollary 7.7.3].

Proposition 3.15. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two loopfree oriented matroids on E. Then $\mathcal{M}_1$ is a quotient of $\mathcal{M}_2$ if and only if $(\Sigma_1, \mathcal{E}_1)$ is a real subfan of $(\Sigma_2, \mathcal{E}_2)$.*

Proof. In general, given a real phase structure $\mathcal{E}_{\mathcal{M}}$ induced by an oriented matroid $\mathcal{M}$, the inclusion of matroid fans $\Sigma_1 \subseteq \Sigma_2$ is equivalent to M_1 being a (non-oriented) quotient of M_2, that is, the flats of M_1 are flats of M_2, see [FR13] or [Sha13]. In particular, if $\mathcal{M}_1$ is a quotient of $\mathcal{M}_2$, then $\Sigma_1 \subseteq \Sigma_2$. Together with applying Equation (3.3) to $\mathcal{E}_1(\tau)$ and $\mathcal{E}_2(\tau)$, the "only if" direction follows.

For the "if" direction, let X be a covector of $\mathcal{M}_1$. Let $F = \phi(X)$ denote the underlying flat of M_1. Again, since $\Sigma_1 \subseteq \Sigma_2$, the flat F is also a flat of

M_2. We consider the chain $\mathcal{F} = \{\emptyset \subsetneq F \subsetneq E\}$ and the corresponding ray $\sigma_{\mathcal{F}}$. Let T be a tope of $\mathcal{M}_1$ such that $X \leq T$. It follows that $T = (-1)^{\varepsilon}$ with $\varepsilon \in \mathcal{E}_1(\sigma_{\mathcal{F}})$, hence by assumption $\varepsilon \in \mathcal{E}_2(\sigma_{\mathcal{F}})$. This, however, implies that $X = T \setminus F$ is a covector of $\mathcal{M}_2$, and we are done. □

Example 3.16. For $M = U_{3,4}$, the choice of a real phase structure on $\Sigma_M \subset \mathbb{R}^4$ is equivalent to the choice of $\varepsilon \in \mathbb{Z}_2^4$ such that $\varepsilon, \varepsilon' \notin \cup_\sigma \mathcal{E}(\sigma)$, where $\varepsilon' = \varepsilon + (1, \dots, 1)$.

The uniform matroid $M' = U_{2,4}$ is an ordinary matroid quotient of $U_{3,4}$. Given a real phase structure $\mathcal{E}$ on M, there are 12 real phase structures on M' such that $\Sigma_{M'}$ equipped with one of these real phase structures produces a real subfan of $(\Sigma_M, \mathcal{E})$. We now describe them. For every $i \in \{1, \dots, 4\}$ consider the chain of flats

$$\mathcal{F}_i := \{\emptyset \subset \{i\} \subset \{1, 2, 3, 4\}\}.$$

The corresponding cone σ_i is in both Σ_M and $\Sigma_{M'}$. The set $\mathcal{E}(\sigma_i)$ consists of three affine spaces of dimension 3 in $\mathbb{Z}_2^4$ which form a necklace arrangement. Moreover, the complement $\mathbb{Z}_2^4 \backslash \mathcal{E}(\sigma_i)$ is the unique affine space of dimension two parallel to $T_{\mathbb{Z}_2}(\sigma_i)$ containing the points $\varepsilon, \varepsilon'$.

Contained in $\mathcal{E}(\sigma_i)$ there are precisely 3 affine spaces of dimension 2 parallel to $T_{\mathbb{Z}_2}(\sigma_i)$. These three affine spaces arise as the intersections of the 3-dimensional affine spaces in the necklace arrangement at σ_i. It follows from this that if $\mathcal{E}'(0) \subset \mathcal{E}(0)$, then $(\Sigma_{M'}, \mathcal{E}')$ is a real subfan of $(\Sigma_M, \mathcal{E})$. This is because if $\mathcal{E}'(0) \subset \mathcal{E}(0)$ then for each face σ_i the affine space $\mathcal{E}'(\sigma_i)$ cannot be equal to the 2-dimensional affine space $\mathbb{Z}_2^4 \backslash \mathcal{E}(\sigma_i)$, and hence $\mathcal{E}'(\sigma_i) \subset \mathcal{E}(\sigma_i)$.

Following the description from Example 2.12, there are a total of 24 real phase structures on $\Sigma_{M'}$ given by combining the choice of 3 necklace orderings and a choice of the points in the intersection $\mathcal{E}'(\sigma_{i_1}) \cap \mathcal{E}'(\sigma_{i_2})$ where i_1 and i_2 are consecutive faces in the necklace ordering. Of the 24 real phase structures there are exactly 12 which contain $\varepsilon, \varepsilon'$ and 12 which do not contain them. Therefore, for $M' = U_{2,4}$ there are a total of 12 real subfans $(\Sigma_{M'}, \mathcal{E}')$ of $(\Sigma_M, \mathcal{E})$ for a fixed real phase structure $\mathcal{E}$.

The uniform matroid $M'' = U_{1,4}$ is also an ordinary matroid quotient of $U_{3,4}$. The matroid fan of $M'' = U_{1,4}$ is just a point which we denote by 0. A real phase structure $\mathcal{E}'$ on $\Sigma_{M''}$ produces a real subfan of $(\Sigma_M, \mathcal{E})$ if and only if $\mathcal{E}''(0) \neq \{\varepsilon, \varepsilon'\}$.

Following our main theorem, the containment $\mathcal{E}'(0) \subset \mathcal{E}(0)$ corresponds to containment of topes of the corresponding oriented matroids. This relation between oriented matroids corresponds to weak maps, [BLVS+99, Proposition 7.7.5]. The condition of being a real phase subfan is not always equivalent to having $\mathcal{E}'(0) \subset \mathcal{E}(0)$ as the next example shows.

Example 3.17. Consider again the uniform matroid $M = U_{3,4}$. The rank 2 matroid N on $E = \{1, \dots, 4\}$ where $1, 2$ are parallel and $3, 4$ are parallel is also an ordinary matroid quotient of $U_{3,4}$. The matroid fan of N is an affine space of dimension 2 in $\mathbb{R}^4$ and hence there are 4 possible real phase structures on Σ_N. However, only one of these possible real phase structures produces a real subfan of $(\Sigma_M, \mathcal{E})$. Indeed, if ρ_1 and ρ_2 denote the two half spaces which are top dimensional cones of Σ_M, then $\mathcal{E}(\rho_1)$ and $\mathcal{E}(\rho_2)$ are transversely intersecting affine subpaces of dimension 3 in $\mathbb{Z}_2^4$, and hence their intersection $\mathcal{E}(\rho_1) \cap \mathcal{E}(\rho_2)$ gives the unique real phase structure on Σ_N yielding a real subfan. Yet three out of the four real phase structures on Σ_N satisfy $\mathcal{E}'(0) \subset \mathcal{E}(0)$ and hence correspond to weak maps of oriented matroids.

3.3. **The proof of Theorem 1.1.** In this section we prove Theorem 1.1. The idea of the proof is a follows: We use double induction on rank and corank of M. The induction step is governed by corank 1 quotients of oriented matroids and (real) tropical modifications, respectively. The crucial ingredient on the oriented matroid side is the positive answer to the *factorization problem* in corank 1. We start by recalling the related facts.

Let $\mathcal{M}_1$ be a quotient of $\mathcal{M}_2$. The factorization problem asks the question whether there exists an oriented matroid $\mathcal{M}$ on a larger ground set $E' \supset E$ such that $\mathrm{r}_{\mathcal{M}}(E^c) = \mathrm{r}(M_2) - \mathrm{r}(M_1)$, $\mathcal{M}_1 = \mathcal{M}/E^c$ and $\mathcal{M}_2 = \mathcal{M}\backslash E^c = \mathcal{M}|E$.

Interestingly, the general answer to this question is no, see [Ric93]. For our purposes, however, it sufficient to consider $\mathrm{r}(M_2) - \mathrm{r}(M_1) = 1$, in which case the answer is positive [RZ94]. Here we present a slight generalization of the statement, which allows for parallel elements.

Lemma 3.18. *Let $E' = E \sqcup F$ be a finite set. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be oriented matroids on E such that $\mathcal{M}_1$ is a quotient of $\mathcal{M}_2$ and $\mathrm{r}(\mathcal{M}_2) - \mathrm{r}(\mathcal{M}_1) = 1$. Then there exists an oriented matroid $\mathcal{M}$ on E' such that $r_{\mathcal{M}}(F) = 1$, F contains no loops of $\mathcal{M}$, $\mathcal{M}/F = \mathcal{M}_1$ and $\mathcal{M}\backslash F = \mathcal{M}_2$. Moreover, the oriented matroid $\mathcal{M}$ is unique up to reorientation of elements in F.*

Proof. Let e be an element of F. By [RZ94, Theorem 4.1] the statement holds true for $E \cup \{e\}$ and the corresponding oriented matroid $\mathcal{M}'$ is unique up to reorientation of e. It is easy to check that $\mathcal{M}_1 \oplus U_{0,1}$ (that is, adding e as a loop) is a quotient of $\mathcal{M}'$. Successively adding all elements of F in this way produces a suitable $\mathcal{M}$ and the uniqueness statement follows by recursion. □

Lemma 3.19. *Let M be a matroid on E and let F be a flat of rank 1. Let $\mathcal{E}$ and $\mathcal{E}'$ be two real phase structures on Σ_M such that $\mathcal{E}/F = \mathcal{E}'/F$ and $\mathcal{E}\backslash F = \mathcal{E}'\backslash F$. Then $\mathcal{E}$ and $\mathcal{E}'$ agree up to reorientation by an element ε in the kernel of p_F.*

Proof. Note that $\mathcal{E}/F$ determines the real phase structure on $M|F \oplus M/F$ induced by $\mathcal{E}$ up to reorientation along the kernel (which is equivalent to picking a real phase structure for $M|F$). In particular, if F is a connected component of M, the claim follows.

Let us now assume that F is not a connected component of M. We pick a fixed facet $\overline{\sigma}$ of $\Sigma_{M\backslash F}$ and set $\sigma := p_F^*(\overline{\sigma})$. In particular, $\dim(p_F(\sigma)) = \dim(\sigma)$ and we will refer to such facets as *non-contracted* in the remainder of this proof. Facets not satisfying this are called contracted.

Since $p_F(\mathcal{E}(\sigma)) = \mathcal{E}\backslash F(\overline{\sigma}) = \mathcal{E}'\backslash F(\overline{\sigma}) = p_F(\mathcal{E}'(\sigma))$, for a non-contracted face σ, there exists an element ε in the kernel of p_F such that $\mathcal{E}(\sigma) = \mathcal{E}'(\sigma)+\varepsilon$. Hence after reorienting $\mathcal{E}'$ by ε we may assume $\mathcal{E}(\sigma) = \mathcal{E}'(\sigma)$. Assume that a facet σ' is non-contracted. Since $\Sigma_{M\backslash F}$ is connected in codimension one, we can connect σ and σ' by a sequence of non-contracted facets such that successive pairs intersect in codimension one. Therefore, it is sufficient to show that given a codimension one face τ of σ, the data of the fixed affine space $\mathcal{E}(\sigma)$ together with the real phase structures $\mathcal{E}/F$ and $\mathcal{E}\backslash F$ determine $\mathcal{E}$ for all facets adjacent to τ. Hence the real phase structure is unique up to reorientation. By the first paragraph of the proof, this also determines $\mathcal{E}(\sigma')$ for all contracted facets $\sigma' \in \Sigma_{M|F\oplus M/F}$.

Let us formulate the given data in terms of the necklace arrangement around τ. Knowing $\mathcal{E}\backslash F$ is equivalent to knowing the projection of the necklace arrangement under p_F. If one of the facets adjacent to τ is contracted, this is equivalent to one of the affine spaces in the necklace arrangement being contracted to a subspace of dimension one less under p_F. In this case, the image of the projection is the intersection of two of the affine subspaces in the corresponding necklace arrangement for $\mathcal{E}\backslash F$. The real phase structure $\mathcal{E}/F$ tells us which intersection of affine spaces in $\mathcal{E}\backslash F$ the contracted face maps to. This data together with the non-contracted affine space $\mathcal{E}(\sigma)$ of the necklace arrangement uniquely determines the arrangement. If none of the facets adjacent to τ are contracted, then knowing the projection of the necklace arrangement under p_F and one of the affine space of the necklace arrangement completely determines the necklace arrangement. This finishes the proof. □

Example 3.20. In this example, we illustrate the main step in the proof of Lemma 3.19 for the matroid $M = U_{3,5}$ and $F = \{5\}$. Then $M\backslash 5 = U_{3,4}$ and $M/5 = U_{2,4}$. Consider the real phase structure $\mathcal{E}_\backslash$ on $\Sigma_{M\backslash 5}$ with $(0,\dots,0),(1,\dots,1) \notin \mathcal{E}_\backslash(\sigma)$ for any face σ of $\Sigma_{M\backslash 5}$.

Let τ be the codimension one face of Σ_M corresponding to the chain of flats

$$\mathcal{F}_1 := \{\emptyset \subset \{1\} \subset \{1,2,3,4,5\}\}.$$

Let σ_2 be the facet of Σ_M corresponding to the chain of flats

$$\mathcal{F}_{12} := \{\emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3,4,5\}\}.$$

Equip $\Sigma_{M/5}$ with a real phase structure $\mathcal{E}_/$ having $\mathcal{E}_/(p_5(\tau)) = (0,0,0,1) + \langle(1,0,0,0),(1,1,1,1)\rangle$. Notice that then $\mathcal{E}_/(p_5(\tau))) \subset \mathcal{E}_\backslash(p_5(\tau))$. Hence $\mathcal{E}_/$ can be completed in such a way as to give rise to a real subfans of $(\Sigma_{M\backslash 5}, E_\backslash)$, see Example 3.17. Suppose that $\mathcal{E}$ is a real phase structure on Σ_M such that $\mathcal{E}\backslash 5 = \mathcal{E}_\backslash$ and $\mathcal{E}/5 = \mathcal{E}_/$. Since we consider real phase structures on Σ_M up to reorientation we are free to fix the affine space for a facet of this fan. Let us set

$$\mathcal{E}(\sigma_2) = (0,0,0,1,0) + \langle(1,0,0,0,0),(0,1,0,0,0),(1,1,1,1,1)\rangle.$$

There are three other facets adjacent to τ in Σ_M namely $\sigma_3, \sigma_4, \sigma_5$ where σ_i corresponds to the chain of flats

$$\mathcal{F}_{1i} := \{\emptyset \subset \{1\} \subset \{1,i\} \subset \{1,2,3,4,5\}\}.$$

The facet σ_5 is contracted under p_5, namely $p_5(\sigma_5) = p_5(\tau)$.

Since there are only three facets adjacent to a codimension one face of $\Sigma_{M\backslash 5}$ the necklace ordering of facets induced by $\mathcal{E}_\backslash$ at $p_5(\tau)$ is unique. The above specification of $\mathcal{E}_/(p_5(\tau))$ means that $\mathcal{E}_/(p_5(\tau)) = \mathcal{E}_\backslash(p_5(\sigma_2) \cap$

$\mathcal{E}_\backslash(p_5(\sigma_3))$. Then the necklace ordering induced on the facets in Σ_M adjacent to τ induced by $\mathcal{E}$ is $\sigma_2, \sigma_5, \sigma_3, \sigma_4$. The affine space $\mathcal{E}(\sigma_5)$ must be the preimage of $\mathcal{E}_/(p_5(\tau))$ under the map p_5. Since we have also fixed $\mathcal{E}(\sigma_2)$ above, we see that

$$\mathcal{E}(\sigma_5) \cap \mathcal{E}(\sigma_2) = (0,0,0,1,0) + \langle (1,0,0,0,0), (1,1,1,1,1) \rangle.$$

This intersection together with the necklace ordering determines completely the necklace arrangement about τ and hence the other affine spaces $\mathcal{E}(\sigma_i)$.

Lemma 3.19 and Lemma 3.14 tell us that deletion and contraction of real phase structures behave as expected in relation to oriented matroid quotients and real subfans in Proposition 3.15. Combining this with the existence of rank 1 extensions proved in Lemma 3.18, we are able to prove Theorem 1.1.

Proof of Theorem 1.1. We need to show that any real phase structure $\mathcal{E}$ on a matroid fan Σ_M can be represented as $\mathcal{E} = \mathcal{E}_{\mathcal{M}}$ for some oriented matroid $\mathcal{M}$. We proceed by double induction on rank and corank. The base cases for the induction are $U_{0,n}$ and $U_{n,n}$ and they are trivial.

In the general case, let F be an arbitrary flat of M of rank 1. Without loss of generality we may assume that M is loopfree. By the induction assumption, the real phase structures $\mathcal{E}/F$ and $\mathcal{E}\backslash F$ are represented by oriented matroids, say $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. By Lemma 3.14 and Proposition 3.15 we know that $\mathcal{M}_1$ is a quotient of $\mathcal{M}_2$. If F is a connected component of M, then the claim follows from the induction assumption applied to each connected component. Otherwise, we have $\mathrm{r}(\mathcal{M}_2) - \mathrm{r}(\mathcal{M}_1) = 1$. By Lemma 3.18, there exists an oriented matroid $\mathcal{M}$ on E such that $r_{\mathcal{M}}(F) = 1$, the flat F contains no loops of $\mathcal{M}$, $\mathcal{M}/F = \mathcal{M}_1$ and $\mathcal{M}\backslash F = \mathcal{M}_2$. Since $\underline{\mathcal{M}}/F = M/F$ and $\underline{\mathcal{M}}\backslash F = M\backslash F$, the uniqueness of this extension for ordinary matroids implies that $\underline{\mathcal{M}} = M$, see [Whi86, Proposition 8.3.1]. It follows that $\mathcal{E}' = \mathcal{E}_{\mathcal{M}}$ and $\mathcal{E}$ are two real phase structures on Σ_M whose deletion and contraction along F agree. By Lemma 3.19, they agree up to reorientation along some ε. Since reorientations of oriented matroids and real phase structures are compatible, this shows that the corresponding reorientation of $\mathcal{M}$ represents $\mathcal{E}$, which proves the claim. □

Remark 3.21. *The previous discussion shows that oriented matroid quotients (or real subfans of codimension* 1 *of matroid fans) play a special role. In general, if* $(\Sigma_1, \mathcal{E}_1)$ *is a real subfan of* $(\Sigma_2, \mathcal{E}_2)$*, we may ask whether this inclusion can be completed to a chain of real subfans whose dimensions increase by one in each step. Interestingly, there is a counter-example of Richter-Gebert* [Ric93, Corollary 3.4] *which shows that this is in general not the case. It tropical language, it gives rise to a pair of real matroid subfans* $(\Sigma_1, \mathcal{E}_1) \subset (\Sigma_2, \mathcal{E}_2)$ *with* $\dim \Sigma_1 = 1$ *and* $\dim \Sigma_2 = 3$ *such that there exists no real matroid fan* $(\Sigma, \mathcal{E})$ *such that* $(\Sigma_1, \mathcal{E}_1) \subsetneq (\Sigma, \mathcal{E}) \subsetneq (\Sigma_2, \mathcal{E}_2)$.

This is in contrast to non-oriented matroids (equivalently, matroid fans without real phase structures), where the factorization problem can be answered affirmatively and hence such chains always exist [Whi86, Chapter 8.2]

3.4. **From real phase structures to sign circuits.** From our main theorem, a real phase structure on a matroid fan is equivalent to specifying an orientation on the underlying matroid and the topes of the oriented matroid are the points in the real phase structure. Signed circuits are a cryptomorphic description of oriented matroids, and in this section, we will describe explicitly how to directly construct the signed circuit vectors of the oriented matroid arising from real phase structure on a matroid fan Σ_M.

The signed circuits of an oriented matroid consists of a collection of sign vectors X_C and $-X_C$ for every circuit C of the underlying matroid such that X_C has support C and satisfying the signed circuit axioms [BLVS+99]. Describing the signed circuits of an oriented matroid $\mathcal{M}$ is equivalent to choosing for each circuit C of M and for each pair of elements $i, j \in C$ a sign $\gamma(\mathcal{M})^C_{ij} \in \{\pm 1\}$, such that for all triples $i, j, k \in C$ we have

$$\gamma(\mathcal{M})^C_{ij}\gamma(\mathcal{M})^C_{jk}\gamma(\mathcal{M})^C_{ik} = +1. \tag{3.4}$$

The signed circuits X_C and $-X_C$ can be recovered from the assignments $\gamma(\mathcal{M})^C_{ij}$. If $\gamma(\mathcal{M})^C_{ij} = +1$, then i, j have the same sign in X_C and $-X_C$, whereas if $\gamma(\mathcal{M})^C_{ij} = -1$, then i, j have opposite signs in X_C and $-X_C$.

Example 3.22. Let $\mathcal{M}$ be the oriented matroid on $E = \{1, \dots, n\}$ associated to the (non-zero) linear forms $l_i \colon \mathbb{R}^d \to \mathbb{R}$, $i = 1, \dots, n$, see Example 3.1. Let M be the underlying matroid. A subset $C \subset E$ is a circuit of M if there exists a linear relation among the $l_i, i \in C$, say

$$\sum_{k \in C} a_k l_k = 0,$$

with all only non-zero coefficients a_k and such that this is the unique relation among the linear forms $\{l_k\}_{k \in C}$, up to multiplying by a constant. We can assign a signed vector X_C to C by setting $(X_C)_i$ equal to the sign of a_k if $k \in C$ and 0 otherwise. Multiplying the relation by -1 produces $-X_C$.

Now, fix $i, j \in C$. Let x be a generic point in $\bigcap_{k \in C \setminus \{i,j\}} \{l_k = 0\}$, and consider $X = (\mathrm{sign}(l_s(x)))_s$ the covector of $\mathcal{M}$ associated to the cell containing x. Note that

$$0 = \sum_{k \in C} a_k l_k(x) = a_i l_i(x) + a_j l_j(x).$$

It follows that the sign of $a_i a_j$ is opposite to the sign of $l_i(x) l_j(x)$, or equivalently,

$$\gamma(\mathcal{M})^C_{ij} = (X_C)_i (X_C)_j = -X_i X_j.$$

Hence $\gamma(\mathcal{M})^C_{ij}$ can be determined by comparing entries in certain covectors, with an extra minus sign. Perturbing x slightly to a generic point x', we obtain a tope $T = (\mathrm{sign}(l_s(x')))$. Since the values $l_k(x')$, for $k \in C \setminus \{i, j\}$ are arbitrarily small, we still have

$$\gamma(\mathcal{M})^C_{ij} = -T_i T_j$$

for such a tope. In order for a covector to be suitable for computing γ^C_{ij} we require that $\mathrm{Supp}(X)^c = \mathrm{cl}(C \setminus \{i, j\})$. Moreover, for a tope to be suitable for computing $\gamma(\mathcal{M})^C_{ij}$ we must have $X < T$. So the suitable topes T for computing the signs $\gamma(\mathcal{M})^C_{ij}$ are the ones that are adjacent to $\mathrm{cl}(C \setminus \{i, j\})$.

The previous example extends directly to oriented matroids $\mathcal{M}$, as follows. For an oriented matroid $\mathcal{M}$, we denote by $\gamma(\mathcal{M})$ the associated γ description of the signed circuits, For any tope T of $\mathcal{M}$ that is adjacent to $\mathrm{cl}(C \setminus \{i,j\})$. we have

$$\gamma(\mathcal{M})^C_{ij} = -T_i T_j. \tag{3.5}$$

The following proposition follows directly from the discussion above and the fact that the points in the real phase structure correspond to topes of the oriented matroid associated by Theorem 1.1.

Proposition 3.23. *Let M be matroid with real phase structure $\mathcal{E}$ on Σ_M. The signed circuits with support C of the orientation $\mathcal{M}_\mathcal{E}$ of M arising from $\mathcal{E}$ are described by*

$$\gamma(\mathcal{M}_\mathcal{E})^C_{ij} = (-1)^{\varepsilon_i}(-1)^{\varepsilon_j}$$

for all $i,j \in C$, where $\varepsilon \in \mathcal{E}(\sigma_\mathcal{F})$ for $\mathcal{F}$ any flag of flats containing the flat $cl(C \backslash ij)$.

Following our extension of $\mathcal{E}$ to arbitrary faces of the matroid fan in Equation (3.3) we do not require $\mathcal{F}$ to be a maximal flag of flats. In particular, we can choose the flag of flats $\mathcal{F} = \{\emptyset \subsetneq cl(C\backslash ij) \subsetneq E\}$ to determine the signs in the above proposition.

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