ISSN 2590-9770
The Art of Discrete and Applied Mathematics
https://doi.org/10.26493/2590-9770.1536.8db
(Also available at http://adam-journal.eu)

Permutation graphs and the weak Bruhat order*

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Received 23 March 2022, accepted 5 January 2023

Abstract

Permutation graphs are graphs associated with permutations where edges represent inversions. We study different classes of permutation graphs and isomorphic permutation graphs. A complete answer of a basic isomorphism question is given for trees. A connection to the majorization order for integer partitions is established, in connection with the notion of inversion vectors. Finally, we study special transitivity properties related to inversions of permutations.

Keywords: Permutation, inversion, partially ordered set (poset), comparability graph, permutation graph, weak Bruhat order.

Math. Subj. Class.: 05A05, 05B20, 05C05, 05C38

1 Introduction

This paper is concerned with both some old and some new aspects of permutation graphs. We discuss how graphs are naturally associated with permutations and the properties of such graphs. A main new result concerns trees that are permutations graphs, and shows that only two permutations correspond to a given tree. Moreover, we investigate inversions and related orders, both the Bruhat order and the majorization order. Finally, we discuss some new problems related to permutation graphs.

Let S_n denote the set of permutations of $\{1, 2, \ldots, n\}$, and let $\sigma = (i_1, i_2, \ldots, i_n)$ (sometimes written as $i_1 i_2 \ldots i_n$) be in S_n with corresponding $n \times n$ permutation matrix P_σ. Note that we mostly use the notation $\sigma = (i_1, i_2, \ldots, i_n)$, with a parenthesis. This

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must not be confused with the cycle notation for permutations which we do not use at all in this paper. Let

$$\mathcal{I}(\sigma) = \{(i_k, i_l) : i_k > i_l, 1 \le k < l \le n\}$$

be the *inversion set* of σ. The function that maps a permutation in $\mathcal{S}_n$ into its inversion set is injective, since the set of inversions of a permutation in $\mathcal{S}_n$ uniquely determines the permutation. Each inversion (i_k, i_l) of σ has an associated unordered pair $\{i_k, i_l\}$ called an *unordered inversion* of σ, the set of which is denoted by $E(\sigma)$. The permutation σ thus determines

(a) a labeled graph $G(\sigma)$ with vertex set $\{1, 2, \ldots, n\}$ and edge set $E(\sigma)$ with the edges of $G(\sigma)$ corresponding to the unordered inversions of the permutation σ;

(b) a labeled digraph $\Gamma(\sigma)$ with vertex set $\{1, 2, \ldots, n\}$ and edge set $\mathcal{I}(\sigma)$ whose (directed) edges thus correspond to the inversions of σ.

Thus $\Gamma(\sigma)$ is an *orientation* of $G(\sigma)$.

The set $\mathcal{I}(\sigma)$ of inversions also determines a partially ordered set $(P(\sigma), \preceq_\sigma)$ on $\{1, 2, \ldots, n\}$, since

$$1 \le k < l < t \le n \text{ and } (i_k, i_l), (i_l, i_t) \in \mathcal{I}(\sigma) \text{ imply } (i_k, i_t) \in \mathcal{I}(\sigma),$$

and thus transitivity is satisfied:

$$i_t \preceq_\sigma i_l, \; i_l \preceq_\sigma i_k \text{ implies } i_t \preceq i_k.$$

This implies that graph $G(\sigma)$ is transitively orientable with the result being $\Gamma(\sigma)$.

Recall that a graph is a *comparability graph* provided that its edges can be oriented to give a transitive directed graph. Thus the permutation graph $G(\sigma)$ is a comparability graph. Note that the graph complement of $G(\sigma)$ is $G(\overleftarrow{\sigma})$ where $\overleftarrow{\sigma}$ is the permutation obtained from σ by reversing the sequence of its elements to get $\overleftarrow{\sigma} = (i_n, \ldots, i_2, i_1)$. Thus the complementary graph of $G(\sigma)$ is also a comparability graph. We call the graph $G(\sigma)$ with its vertices specifically labeled as $1, 2, \ldots, n$ an *inversion graph*. A *permutation graph* [1] is any graph that is isomorphic to the inversion graph of a permutation of $\{1, 2, \ldots, n\}$. In particular, a permutation graph does not come with the specific labels $1, 2, \ldots, n$ attached to its vertices. An induced subgraph of a permutation graph is also a permutation graph. Determining whether or not a graph is a permutation graph is equivalent to determining whether the labels $1, 2, \ldots, n$ can be assigned to its vertices to give the inversion graph of a permutation. In summary, if σ is a permutation of $\{1, 2, \ldots, n\}$, then

(a) $\mathcal{I}(\sigma) \cup \mathcal{I}(\overleftarrow{\sigma})$ is a partition of the set of $\binom{n}{2}$ pairs of $\{1, 2, \ldots, n\}$.

(b) $G(\sigma)$ and $G(\overleftarrow{\sigma})$ are *complementary graphs*, that is, their edge sets partition the set of pairs of $\{1, 2, \ldots, n\}$:

$$E(\sigma) \cup E(\overleftarrow{\sigma}) = \binom{\{1, 2, \ldots, n\}}{2}, E(\sigma) \cap E(\overleftarrow{\sigma}) = \emptyset.$$

(c) the edge sets of each of the graphs $G(\sigma)$ and $G(\overleftarrow{\sigma})$ can be oriented to give transitive digraphs.

(d) Moreover, a subset X of unordered pairs of $\{1, 2, \ldots, n\}$ is the set of unordered inversions of a permutation if and only if X is the set of edges of a permutation graph.

It is known that a permutation determines a graph such that both it and its complement are comparability graphs characterizes permutation graphs; see Proposition 5.10 in [2] (as noted in [3]), [1], and Theorem 7.1 in [11]). The result seems to be due to Dushnik and Miller [6] (Theorem 3.61).

Theorem 1.1 ([6, 2])**.** *A graph G is a permutation graph if and only if G and its complementary graph $\overline{G}$ are comparability graphs.*

This theorem may be used to construct a recognition algorithm for permutation graphs that runs in $O(n^3)$ time, see [11]. The algorithm is based on finding a transitive orientation of a comparability graph.

The graph $G(\sigma)$, where $\sigma = (i_1, i_2, \ldots, i_n)$, can also be viewed geometrically as follows: Put the integers $1, 2, \ldots, n$ in that order in a column (row) and the integers $i_1, i_2, \ldots, i_n$ in that order in a neighboring parallel column (row), and then join by a line segment each pair of equal integers (one in each column). The edges of $G(\sigma)$ are those pairs $\{i_k, i_l\}$ such that the line segment joining the two k's and the line segment joining the two l's intersect.

In general, a poset $\mathcal{P} = (X, \leq)$ determines a graph $G(\mathcal{P})$ with vertex set X whose edges are all those pairs $\{a, b\}$ such that $a \neq b$ and either $a \leq b$ or $b \leq a$. The transitive property of a partial order implies that $G(\mathcal{P})$ is a *comparability graph*. A *linear extension* of a poset $\mathcal{P} = (X, \leq)$ is a linearly ordered set $\mathcal{P} = (X, \leq')$ on the same set X of elements such that $x \leq y$ always implies $x \leq' y$. The *dimension* of a poset $(\mathcal{P}, \leq)$ is the smallest number of its linear extensions whose intersection is $(\mathcal{P}, \leq)$. A poset of dimension 1 is a linearly ordered set. Posets of dimension 2 are characterized via the following theorem from [6] (see also Theorem 10.30 in [17]).

Theorem 1.2 ([6])**.** *A poset $\mathcal{P} = (X, \leq)$ has dimension at most 2 if and only if the complement of its comparability graph is also a comparability graph.*

Corollary 1.3. *Let σ be a permutation of $\{1, 2, \ldots, n\}$ not equal to the identity permutation $(1, 2, \ldots, n)$ or its reversal $(n, \ldots, 2, 1)$. Then the dimension of $(\mathcal{P}(\sigma), \preceq)$ equals 2.*

In the literature (see e.g. [18]) a permutation graph has been defined more generally using two permutations $\sigma = (i_1, i_2, \ldots, i_n)$ and $\pi = (j_1, j_2, \ldots, j_n)$ of $\{1, 2, \ldots, n\}$, but has been considered primarily in the case of one permutation (equivalently, two permutations where one is the identity permutation ι_n). Put the integers $i_1, i_2, \ldots, i_n$ in that order in a column and the integers $j_1, j_2, \ldots, j_n$ in that order in a neighboring parallel column, and then join by a line segment ℓ_k each pair of equal integers k. The set $E(\sigma, \pi)$ of edges of $G(\sigma, \pi)$ are those pairs $\{p, q\}$ of integers p and q such that ℓ_p and ℓ_q intersect, equivalently, those pairs $\{p, q\}$ such that p precedes q in one of the permutations and q precedes p in the other, thereby determining an inversion in exactly one of σ and τ. Thus $E(\sigma, \pi)$ is the symmetric difference of $E(\sigma)$ and $E(\pi)$:

$$E(\sigma, \pi) = E(\sigma) \,\Delta\, E(\pi) = (E(\sigma) \setminus E(\pi)) \cup (E(\pi) \setminus E(\sigma)).$$

If $\pi = \iota_n$, then $G(\sigma, \iota_n) = G(\sigma)$. If $\pi = \overleftarrow{\sigma}$, then the permutation graph $G(\sigma, \overleftarrow{\sigma})$ is the complete graph K_n.

After this brief background exposition, we now summarize the content of this paper. In Section 2 we consider the weak Bruhat order for permutations. Moreover we characterize the trees that are permutation graphs and show that each such tree is the permutation graph of exactly two permutations. Next, in Section 3 we study other permutation graphs than trees and construction procedures for permutation graphs. Inversions may also be represented by certain inversion vectors and these are studied in Section 4. A connection to the majorization order is shown. In Section 5 we consider the notion of weighted inversion matrices and relations to transitive tournaments.

Notation: In a directed graph a directed edge from u to v is denoted either as the ordered pair (u, v) or as $u \to v$. We let ι_n, or simply ι, denote the identity permutation $(1, 2, \ldots, n)$. For vectors $x, y \in \mathbb{R}^n$ we let $x \preceq y$ denote the classical majorization order [16], i.e., the sum of the k largest elements in x is at most the sum of the k largest elements in y ($k < n$), and the sum of all elements is the same in x and y.

2 Permutation graphs and trees

We begin with special pairs of permutations. Consider two permutations σ and π of $\{1, 2, \ldots, n\}$ where σ is less than or equal to π in the *weak Bruhat order*, written as $\sigma \preceq_b \pi$. By definition this means that $\mathcal{I}(\sigma) \subseteq \mathcal{I}(\pi)$, and it follows that σ can be obtained from π by a sequence of adjacent transpositions each of which reduces the number of inversions by exactly 1. For convenience, in the next lemma, which is directly verified, we summarize the effect on the inversion set when a transposition is applied to a permutation.

Lemma 2.1. *Assume that* $1 \leq k < l \leq n$. *Let* $\pi = (i_1, \ldots, i_{k-1}, i_k, \ldots, i_l, i_{l+1}, \ldots, i_n)$ *where* $i_k > i_l$, *and let* $\sigma = (i_1, \ldots, i_{k-1}, i_l, \ldots, i_k, i_{l+1}, \ldots, i_n)$ *be obtained from* π *by the transposition that interchanges* i_k *and* i_l. *Consider the partition of* $L = \{k, k+1, \ldots, l\}$ *given by* $L = L_1 \cup L_2 \cup L_3 \cup \{k, l\}$ *where*

$$L_1 = \{s \in L : i_s > i_k\}, L_2 = \{s \in L : i_k > i_s > i_l\}, L_3 = \{s \in L : i_l > i_s\}.$$

Consider the graphs $G(\sigma)$ *and* $G(\pi)$ *with edge sets* $E(\sigma)$ *and* $E(\pi)$. *Then*

$$\begin{aligned} E(\sigma) \setminus E(\pi) &= \{(i_k, i_l)\} \cup \{(i_k, i_s) : s \in L_2 \cup L_3\}, \\ E(\pi) \setminus E(\sigma) &= \{(i_l, i_s) : s \in L_3\} \cup \{(i_s, i_k) : s \in L_1\}. \end{aligned} \tag{2.1}$$

In particular, $\sigma \preceq_b \pi$ *if and only if*

$$i_k > i_s > i_l \quad (k < s < l),$$

and thus $\sigma \preceq_b \pi$ *if and only if* σ *can be obtained from* π *by a sequence of adjacent transpositions.*

When the weak Bruhat order $\sigma \preceq_b \pi$ holds, the permutation graph $G(\sigma, \pi)$ has vertex set $\{1, 2, \ldots, n\}$ and set of edges $E(\sigma, \pi) = E(\pi) \setminus E(\sigma)$.

Example 2.2. Let $\sigma = (1, 3, 4, 2)$ and $\pi = (3, 4, 2, 1)$. Then $\mathcal{I}(\sigma) = \{(3, 2), (4, 2)\}$ and $\mathcal{I}(\pi) = \{(3, 2), (3, 1), (4, 2), (4, 1), (2, 1)\}$ and so $\sigma \preceq_b \pi$. The permutation σ is obtained from π by three adjacent transpositions:

$$\pi = (3, 4, 2, 1) \to (3, 4, 1, 2) \to (3, 1, 4, 2) \to (1, 3, 4, 2) = \sigma$$

each of which removes exactly one inversion. The graph $G(\sigma,\pi)$ has vertex set $\{1,2,3,4\}$ with

$$E(\sigma,\pi) = E(\pi) \setminus E(\sigma) = \{\{3,1\},\{4,1\},\{2,1\}\}.$$

Note that $\mathcal{I}(\pi) \setminus \mathcal{I}(\sigma) = \{(3,1),(4,1),(2,1)\}$ is the set $\mathcal{I}(\tau)$ of inversions of the permutation $\tau = (2,3,4,1)$ so that $E(\sigma,\pi) = E(\tau)$. □

Remark 2.3. It is not always true that $\mathcal{I}(\pi) \setminus \mathcal{I}(\sigma)$ is the set of inversions of a permutation if $\sigma \preceq_b \pi$. Consider permutations $\sigma = (1,3,4,2)$ and $\pi = (3,4,1,2)$ with $\mathcal{I}(\sigma) = \{(3,2),(4,2)\}$ and $\mathcal{I}(\pi) = \{(3,1),(3,2),(4,1),(4,2)\}$. We then have that $\sigma \preceq_b \pi$ and $\mathcal{I}(\pi) \setminus \mathcal{I}(\sigma) = \{(3,1),(4,1)\}$, but $\mathcal{I}(\pi) \setminus \mathcal{I}(\sigma)$ is not the inversion set of a permutation of $\{1,2,3,4\}$. If it were, then 3 and 4 would have to precede 1 in the permutation, and 2 would have to precede 3 and 4, implying that $(2,1)$ would be an inversion. Thus $G(\sigma,\pi)$ on the vertex set $\{1,2,3,4\}$ has edge set $E(\sigma,\pi) = \{\{3,1\},\{4,1\}\}$ and is not a permutation graph of one permutation, but we note that $G(\sigma,\pi)$ is isomorphic to the permutation graph of the permutation $\tau = (3,1,2,4)$ where $\mathcal{I}(\tau) = \{(3,1),(3,2)\}$ and $E(\tau) = \{\{3,1\},\{3,2\}\}$. □

In general, if $\sigma \preceq_b \pi$, $I(\pi) \setminus I(\sigma)$ can always be identified with the set of inversions of some permutation in the sense that every interval $[a,b]$ in the weak Bruhat order is isomorphic to a lower interval, that is, one of the form $[\iota,\tau]$. We have the following Proposition 3.1.6 in [3].

Theorem 2.4 ([3]). *Let σ and π be permutations of $\{1,2,\ldots,n\}$ such that $\sigma \preceq_b \pi$, (that is, $\mathcal{I}(\sigma) \subseteq \mathcal{I}(\pi)$). Then the interval $[\sigma,\pi]$ in the weak Bruhat order is isomorphic to the interval $[\iota_n, \sigma^{-1}\pi]$. Thus $G(\sigma,\pi)$ is isomorphic to $G(\sigma^{-1}\pi)$.*

This theorem holds, more generally, for Coxeter groups [3] of which permutation groups are special cases.

For a permutation $\sigma \in \mathcal{S}_n$, let $\mathcal{S}_n(\sigma)$ be the number of permutations in $\mathcal{S}_n$ whose permutation graph is isomorphic to that of σ.

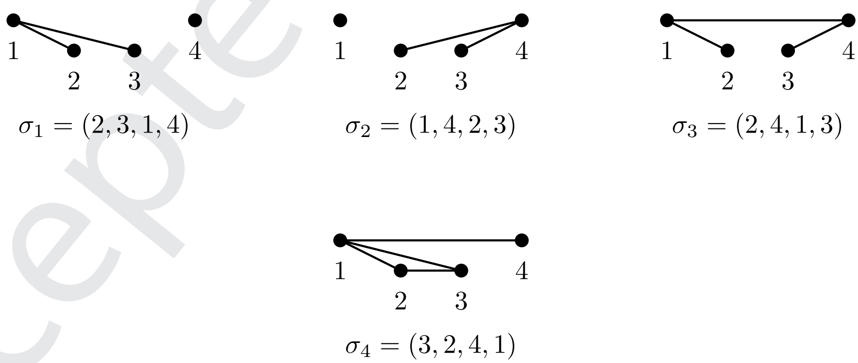


Figure 1: Permutation graphs.

Example 2.5. Let $n = 4$. Figure 1 shows four permutations in $\mathcal{S}_4$ of which the graphs $G(\sigma_1)$ and $G(\sigma_2)$ are isomorphic. There are $4! = 24$ permutations in $\mathcal{S}_4$, but only 11 different permutations graphs occur. For instance, there are four permutations with graph isomorphic to $G(\sigma_1)$, and two permutations with graph isomorphic to $G(\sigma_3)$. The maximum of $|\mathcal{S}_4(\sigma)|$ is 4 and occurs for the graph $G(\sigma_1)$ and also for the graph $G(\sigma_4)$. In general, $\max\{|\mathcal{S}_n(\sigma)| : \sigma \in \mathcal{S}_n\}$ seems to be hard to determine for general n.

In [15] a characterization of permutation graphs is given by families of forbidden subgraphs and their complements. In [13, 14] these families are reduced by using Seidel complementation. We now investigate some classes of permutation graphs and their representations.

We now consider bipartite graphs. Bipartite permutation graphs have a linear time recognition algorithm, and allow a nice characterization [18]. To describe this characterization, consider a bipartite graph $G = (I, J, E)$ with color classes I and J. A *strong ordering* of the vertices in G is an ordering of I and an ordering of J satisfying the following condition:

- if $\{i, j\}, \{i', j'\} \in E$ where $i, i' \in I$, $j, j' \in J$, $i < i'$ and $j' < j$, then also $\{i, j'\}$ and $\{i', j\}$ are in E.

Thus, if vertices in I and J are ordered via the strong ordering in a left and a right column, respectively, with increasing order downwards, then, if two edges cross, those four vertices induce a bipartite clique $K_{2,2}$ in G. The following result is Theorem 1 in [18].

Theorem 2.6 ([18]). *Let G be a bipartite graph. Then G is a permutation graph if and only if G has a strong ordering of its vertices.*

It follows that every complete bipartite graph $K_{n-p,p}$ $(1 \leq p \leq n-1)$ has a strong ordering and so is a permutation graph. In fact, a corresponding permutation is $(p+1, p+2, \ldots, n, 1, 2, \ldots, p)$. More generally, a permutation σ for which $G(\sigma)$ is bipartite is of the form (σ_1, σ_2) where σ_1 and σ_2 are both increasing.

Finding a permutation directly from a strong ordering seems complicated. We therefore focus on finding explicit permutations by combinatorial methods for a special class of (bipartite) graphs, namely trees. A *caterpillar* is a tree such that if all pendent vertices are deleted, then a path remains. The next result may be found in [9], but we first give a different proof using strong ordering, and then give an algorithm for determining a corresponding permutation.

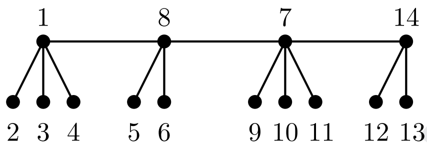
Corollary 2.7 ([9]). *A tree T is a permutation graph if and only if T is a caterpillar.*

Proof. Assume first that $T_k(a_1, a_2, \ldots, a_k)$ is the caterpillar consisting of a path P of k vertices given by $(v_1, v_2, \ldots, v_k)$ where for $1 \leq i \leq k$, a set of $a_i \geq 0$ pendent vertices is joined to the vertex v_i of P.

Let I consist of those v_i where i is odd and those pendent vertices attached to some v_i with i even. The remaining vertices are in J, so then each edge joins a vertex in I and a vertex in J. Now, order the vertices as follows: first v_1 and then its adjacent pendent vertices, next v_2 and its adjacent pendent vertices, etc. From this we obtain consistent orderings of I and J, respectively. It is easy to see that this is a strong ordering, by checking the drawing in two columns with the mentioned orderings (the path then forms a zig-zag curve with the pendent vertices "in between", so there are no crossing edges. Therefore, by Theorem 2.6, G is a permutation graph.

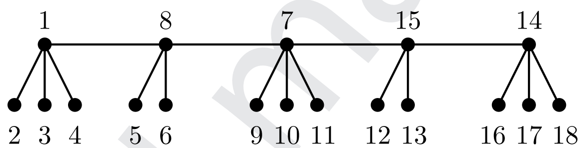
Conversely, a tree which is not a caterpillar must contain an induced subgraph $K^*_{1,3}$ obtained from the star $K_{1,3}$, by subdividing each edge into two edges. But $K^*_{1,3}$ is not a permutation graph; this can be checked directly and it is also one of the forbidden graphs given in [15]. Therefore T is not a permutation graph. Here we used that an induced subgraph of a permutation graph is also a permutation graph. □

In fact, to show that a caterpillar is a permutation graph, there is a simpler algorithm that does not make use of strong ordering to determine a permutation (indeed two in general) whose permutation graph is a given caterpillar. An instance of the result of this algorithm is given in Figure 2.



$$\sigma = (2, 3, 4, 8, 1, 5, 6, 9, 10, 11, 14, 7, 12, 13).$$

Figure 2: A caterpillar and its corresponding permutation



$$\sigma = (2, 3, 4, \mathbf{8}, \mathbf{1}, 5, 6, 9, 10, 11, \mathbf{15}, \mathbf{7}, 12, 13, 16, 17, 18, \mathbf{14}).$$

Figure 3: A caterpillar and its corresponding permutation

Note that the pendent vertices alternate in having larger or smaller labels at their adjacent path vertex. To proceed by induction with the example in Figure 2, suppose we extend the path by a new vertex, with three pendent vertices joined to it. Then we relabel 14 with 15, call the new vertex 14 with pendent edges to vertices 16, 17, and 18, and we get the caterpillar shown in Figure 3.

We introduce the following notation. Let $k \geq 2$ and consider a path $\gamma_k = u_1, u_2, \ldots, u_k$ of length $k-1$ from vertex u_1 to vertex u_k. For each i with $1 \leq i \leq k$, let m_i be a nonnegative integer, and let $U_1, U_2, \ldots, U_k$ be pairwise disjoint sets with $U_i = m_i$ for $i = 1, 2, \ldots, k$. Let $C_k(m_1, m_2, \ldots, m_k)$ be the caterpillar obtained from γ_k by inserting, for each $i = 1, 2, \ldots, k$, edges between u_i and each vertex in U_i.

Theorem 2.8. *The caterpillar $C_k(m_1, m_2, \ldots, m_k)$ is a permutation graph, and the number of permutations with $C_k(m_1, m_2, \ldots, m_k)$ as permutation graph is at least* 2.

Proof. The caterpillar $C_k(m_1, m_2, \ldots, m_k)$ has $n = k + \sum_{i=1}^{k} m_i$ vertices and has exactly two transitive orientations labeled (i) and (ii) below. Indeed any tree with at least one edge has exactly two transitive orientations, since orienting any edge in either direction determines a transitive orientation uniquely. The edges on the path γ_k are alternately oriented:

(i) u_2 to u_1, u_2 to u_3, u_4 to u_3, and so on;

(ii) u_1 to u_2, u_3 to u_2, u_3 to u_4, and so on.

In both cases the edge(s) between the vertices in U_i and u_i are oriented in the same direction as the edge(s) meeting u_i above $(1 \leq i \leq k)$. Thus e.g., in case of (i) the edges between u_1 and the pendent vertices in U_1 are oriented from the vertices in U_1 to u_1, as implicit in Figures 2 and 3.

We have to assign the integers $1, 2, \ldots, n$ to the vertices of $C_k(m_1, m_2, \ldots, m_k)$ so that if an edge $\{x, y\}$ is oriented from x to y, then the integer assigned to x has to be greater than that assigned to y and precedes it in the permutation. We can proceed inductively. If $k = 1$, we assign the integer 1 to u_1 and the integers $2, 3, \ldots, m_1 + 1$ to the vertices in U_1, giving the permutation $2, 3, m_1 + 1, 1$. Proceeding inductively from k to $k + 1$ with the resulting permutation, consider first k even. We increase the integer p assigned to u_k by 1 and assign p to u_{k+1} and then assign the integers $p + 2, p + 3, \ldots$ to the pendent vertices joined to u_{k+1}. Thus the permutation is updated by first including the new numbers in the order

$$p + 2, p + 3, \ldots, p + 1$$

and then interchanging the numbers p and $p + 1$. Now suppose that k is odd. Then with $p_k = k + (m_1 + m_2 + \cdots + m_k)$ we then assign the integers $p_k + 1, p_k + 2, \ldots, p_k + m_{k+1}$ to the m_{k+1} pendent vertices joined to u_{k+1} and the integer $p_k + m_{k+1} + 1$ to u_{k+1}. In this case the permutation is updated by first including the new numbers in the order

$$p_k + m_{k+1} + 1, p_k + 1, p_k + 2, \ldots, p_k + m_{k+1}$$

and then interchanging the numbers corresponding to vertices u_k and u_{k+1}. □

Since a caterpillar has been shown to be a permutation graph G, it follows that its complement $\overline{G}$ is also a permutation graph. For the caterpillar and permutation in Figure 2, the reverse permutation is $(13, 12, 7, 14, 11, 10, 9, 6, 5, 1, 8, 4, 3, 2)$ whose graph is the complement of the caterpillar.

A special case of Theorem 2.8 occurs when the caterpillar is a path.

Corollary 2.9. *A path is a permutation graph.*

Example 2.10. Consider the path γ of n vertices labeled in order as

$$2, 1, 4, 3, 6, 5, \ldots, n, n - 1 \quad (n \text{ even}),$$

and

$$2, 1, 4, 3, 6, 5, \ldots, n - 1, n - 2, n \quad (n \text{ odd}).$$

Then this path is the permutation graph of the permutation

$$(2, 4, 1, 6, 3, 8, 5, \ldots, n - 2, n - 5, n, n - 3, n - 1) \quad (n \text{ even}),$$

and

$$(2, 4, 1, 6, 3, \ldots, n - 3, n - 6, n - 1, n - 4, n, n - 2) \quad (n \text{ odd}).$$

The permutation is not unique in general. For example, the permutation $(2, 4, 1, 3)$ gives the path $2, 1, 4, 3$; the permutation $(3, 1, 4, 2)$ also gives the path $1, 3, 2, 4$. □

For $n = 1, 2$ there is trivially a unique permutation whose graph is a path on n vertices. The case $n \geq 3$ is different, as the following result on isomorphic permutation graphs shows.

Theorem 2.11. *For each $n \geq 3$ there are exactly two permutations in $\mathcal{S}_n$ whose permutation graph is a path on n vertices.*

Proof. Let $n \geq 3$, and let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{S}_n$ be such that $G(\sigma)$ is a path with n vertices. Observe the following properties

(a) $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ is *not* a permutation in $\mathcal{S}_k$ $(k < n)$;

(b) $i - 2 \leq \sigma_i \leq i + 2$ $(i \leq n)$.

Proof of (a): Let $k < n$. If the first k numbers in σ were a permutation in $\mathcal{S}_k$, then there would be no inversion (p, q) where $q \leq k < p$. This means that $G(\sigma)$ is not connected, which contradicts that that $G(\sigma)$ is a path. So, (a) holds.

Proof of (b): Assume first that $\sigma_i \leq i - 3$. So at most $i - 4$ numbers are smaller than σ_i among the $i - 1$ components in σ before σ_i. Thus, at least 3 of these numbers are bigger than σ_i which means that the corresponding vertex has degree at least 3; a contradiction. The upper bound is proved similarly (if σ_i is too large, there will be 3 inversions starting with σ_i). This proves (b).

We now give a combinatorial discussion based on the established properties.

By (a) and (b), $\sigma_1 \in \{2, 3\}$. We first treat the case $\sigma_1 = 2$. Then the corresponding vertex, i.e., vertex 2, has degree 1, so it is pendent. Also, again by (a), (b) $\sigma_2 \in \{3, 4\}$. If $\sigma_2 = 3$, the graph contains the path $2, 1, 3$ where vertices 1 and 3 have degree 1. This can only happen when $n = 3$ (otherwise $G(\sigma)$ would be disconnected). Next, consider $\sigma_2 = 4$. This implies $\sigma_3 = 1$, otherwise vertex 1 would have degree ≥ 3. So $\sigma = (2, 4, 1, \ldots)$. By (a), (b) $\sigma_4 \in \{5, 6\}$. If $\sigma_4 = 5$, vertex 5 has degree 1, and this can only happen if $n = 5$ and then $\sigma = (2, 4, 1, 5, 3)$. Otherwise, $\sigma_4 = 6$ so $\sigma = (2, 4, 1, 6, \ldots)$. Then $\sigma_5 = 3$ (otherwise a vertex has degree ≥ 3) and then $\sigma_6 \in \{7, 8\}$. In general, one gets in the odd positions from position 3: $1, 3, 5, \ldots$, and in the even positions from position 2: $4, 6, 8, \ldots$ until the proper termination. One can repeat these arguments and prove by induction that σ is uniquely determined.

Next, consider the alternative case, $\sigma_1 = 3$. Then $\sigma_2 \in \{1, 2, 4\}$. We can rule out $\sigma_2 = 2$, because then σ would contain a subsequence $3, 2, 1$ which gives a cycle. Also, $\sigma_2 = 4$ would mean that the graph contains edges $\{3, 1\}, \{3, 2\}, \{4, 1\}, \{4, 2\}$, a 4-cycle. Therefore $\sigma_2 = 1$. By (a), (b) $\sigma_3 \in \{4, 5\}$. If $\sigma_3 = 4$, then $\sigma_4 = 2$ (otherwise the vertex 2 would have degree ≥ 3) so $n = 4$ and $\sigma = (3, 1, 4, 2)$. Otherwise, $\sigma_3 = 5$ which implies $\sigma_4 = 2$, so $\sigma = (3, 1, 5, 2, \ldots)$. We can then proceed very similar to above to show uniqueness.

Thus, we conclude that there are exactly two permutations whose permutation graph is a path on n vertices. □

The two permutations in Theorem 2.11 may be found in the proof, and one was described above.

Example 2.12. The two permutations whose permutation graph is a path with 10 vertices are

$$\sigma_1 = (2, 4, 1, 6, 3, 8, 5, 10, 7, 9), \quad \text{and} \quad \sigma_2 = (3, 1, 5, 2, 7, 4, 9, 6, 10, 8).$$

The permutation graphs of σ_1 and σ_2 are shown in Figure 4, and these illustrate the general pattern.

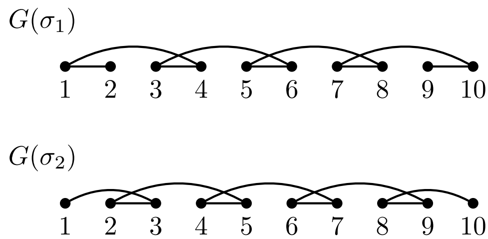


Figure 4: Permutation graphs of σ_1 and σ_2, $n = 10$.

Corollary 2.13. *Let $\sigma \in \mathcal{S}_n$ be such that its permutation graph is a path P. Then the sequence of vertices in P gives a permutation σ^* consisting of consecutive adjacent transpositions (a graph matching with a possible isolated vertex at the ends). Moreover, $\sigma^* \preceq_b \sigma$.*

Proof. This follows from the proof of Theorem 2.11, and the fact that σ^* is obtained from σ by a sequence of adjacent transpositions each removing an inversion. □

For instance, the two permutations in Example 2.12 gives the permutations

$$\sigma_1^* = (2,1,4,3,6,5,8,7,10,9), \quad \text{and} \quad \sigma_2^* = (1,3,2,5,4,7,6,9,8,10).$$

Corollary 2.14. *For each $n \geq 3$ and let C be a caterpillar with n vertices. Then there are exactly two permutations in $\mathcal{S}_n$ whose permutation graph is isomorphic to C.*

Proof. We may assume that the central path has maximum length (by including pendant vertices, so $m_1 = m_k = 0$). Then the central path is unique. Let N be the number of permutations in $\mathcal{S}_n$ whose permutation graph is isomorphic to C. It follows from Theorem 2.8 that $N \geq 2$. Next, let π be a permutation whose permutation graph is isomorphic to C. Then one can see that π is uniquely determined by the ordering of the vertices of the central path of C. This is because the pendent vertices must be ordered increasingly to avoid inversions and extra edges. Since, by Theorem 2.11 there are exactly two such orderings for the path, it follows that $N \leq 2$. So, $N = 2$, as desired. □

3 Constructions and other permutation graphs

Let G_i be two graphs with disjoint vertex sets and let v_i be a vertex in G_i $(i = 1, 2)$. Let $G_1 \oplus_{v_1}^{v_2} G_2$ be the graph obtained from the disjoint union of G_1 and G_2 by adding an edge $\{v_1, v_2\}$. Also, let e be the all ones vector (of suitable length).

Proposition 3.1. *Let $G_i = G(\pi_i)$ be the permutation graph associated with permutation $\pi_i \in \mathcal{S}_{n_i}$ $(i = 1, 2)$. Let v_1 be the vertex corresponding to n_1 in π_1, and let v_2 be the vertex corresponding to 1 in π_2. Let π be obtained from the permutation $(\pi_1, \pi_2 + n_1 e) \in \mathcal{S}_{n_1+n_2}$ by interchanging n_1 and $n_1 + 1$. Then $G = G_1 \oplus_{v_1}^{v_2} G_2$ is the permutation graph $G(\pi)$.*

Proof. Consider $\sigma = (\pi_1, \pi_2 + n_1 e)$. Here $\pi_1 \in \mathcal{S}_{n_1}$ and $\pi_2 + n_1 e$ contains a permutation of the numbers $1+n_1, 2+n_1, \ldots, n_2+n_1$. Thus, σ is also a permutation and $\sigma \in \mathcal{S}_{n_1+n_2}$. The inversion set of σ consists of all inversions in π_1 and all inversions in π_2 by adding n_1 to the components in each inversion.

Now, π is obtained from σ by interchanging n_1 and $n_1 + 1$, so π is a permutation in $\mathcal{S}_{n_1+n_2}$. Since n_1 is the largest number in π_1 and $1 + n_1$ is the smallest number in $\pi_2 + n_1 e$, the change in the inversion set when we go from σ to π is just to add the inversion $(n_1 + 1, n_1)$. This means that the permutation graph of π is the disjoint union of G_1 and G_2 plus the edge $\{v_1, v_2\}$, i.e., $G(\pi) = G_1 \oplus_{v_1}^{v_2} G_2$, as desired. □

The construction above of a permutation whose permutation graph is a path may be seen as an application of Proposition 3.1. In fact, this proposition shows how permutation graphs may be "glued" together into more complex permutation graphs, as the next corollary says.

Corollary 3.2. *Let $G_i = (V_i, E_i)$ be permutation graphs $(i = 1, 2)$ with disjoint vertex sets. Then there exists $v_i \in V_i$ $(i = 1, 2)$ such that the graph G obtained from the disjoint union of G_1 and G_2 by adding the edge $e = \{v_1, v_2\}$ is a permutation graph.*

Proof. This follows immediately from Proposition 3.1 by choosing v_1 as the vertex with the largest (permutation) label in G_1 and v_2 as the vertex with the smallest (permutation) label in G_1. □

Let G be a permutation graph, so $G = G(\pi)$ for some permutation π. A natural and apparently difficult question is to determine all permutations σ such that $G(\sigma)$ is isomorphic to G. If $G(\sigma)$ and $G(\pi)$ are isomorphic we write $\sigma \cong \pi$. We consider the following special case.

Proposition 3.3. *Let G be a graph consisting of a clique of size $k \geq 2$ and $n - k$ isolated vertices. Let $\pi = (i_1, i_2, \ldots, i_n)$ be a permutation. Then $G(\pi)$ is isomorphic to G if and only if there is a p such that*

$$\pi = (1, 2, \ldots, p, \underline{p + k, p + k - 1, \ldots, p + 1}, p + k + 1, p + k + 2, \ldots, n). \tag{3.1}$$

There are $n - k + 1$ such permutations.

Proof. First, if π has the form (3.1), then $G(\pi)$ contains a clique on the vertices $p + 1, p + 2, \ldots, p + k$ plus $n - k$ isolated vertices, so $G(\pi)$ is isomorphic to G, as desired.

Conversely, assume $G(\pi)$ is isomorphic to G. Then there must exist a $j_1 < j_2 < \cdots < j_k$ with

$$i_{j_1} > i_{j_2} > \cdots > i_{j_k}$$

so that vertices $i_{j_1}, i_{j_2}, \ldots, i_{j_k}$ forms a clique. Now, $j_1 = j_2 - 1$. Otherwise there exists a j with $j_1 < j < j_2$ and for any value of i_j there will be another inversion and therefore an edge in $G(\pi)$; a contradiction. By a similar argument we conclude that $j_1, j_2, \ldots, j_k$ are consecutive integers. Similarly, if $i_{j_1} \geq i_{j_2} + 2$, there exists a k with $i_{j_1} > k > i_{j_2} + 2$, which again would introduce another edge in $G(\pi)$. It follows from a similar argument that the subsequence $i_{j_1}, i_{j_2}, \ldots, i_{j_k}$ are consecutive integers in a decreasing order. But then all integers in π before i_{j_1} must be smaller than i_{j_k} and increasing, and all integers after i_{j_k} must be larger than i_{j_1} and increasing (otherwise an edge is introduced), and from this the desired form is obtained. □

For instance, if $\pi = (1, 2, 3, 4, \mathbf{9}, \mathbf{8}, \mathbf{7}, \mathbf{6}, \mathbf{5}, 10, 11) \in S_{11}$, then $G(\pi)$ is isomorphic to a 5-clique and 6 isolated vertices. Here the numbers generating the 5-clique are shown in boldface, and $p = 4$, $k = 5$.

The previous proposition leads to a more general construction as described next. Let $\kappa = (n, \mathcal{I}_k, \tau_k)$ where

(i) $n \geq 1$ and $\mathcal{I}_k = (I_1, I_2, \ldots, I_k)$ is an ordered partition of $\{1, 2, \ldots, n\}$ into intervals (each interval consists of consecutive integers), and

(ii) $\tau_k = (s_1, s_2, \ldots, s_k)$ is a (± 1)-vector.

Associated with κ is a permutation $\pi = \pi^\kappa \in \mathcal{S}_n$ defined as follows: π contains the interval I_1, then the interval $I_2, \ldots,$ and finally the interval I_k. Here the interval I_t is ordered increasingly (resp. decreasingly) if $s_t = 1$ (resp. $s_t = -1$) for $t = 1, 2, \ldots, k$. The graph $G = G(\pi)$ is as follows:

(i) the induced subgraph $G(I_t)$ (with vertices in I_t) has no edges if $s_t = 1$ and it is a clique if $s_t = -1$,

(ii) if the integers in an I_p are larger than the integers in an I_q, then G contains the edge $\{u, v\}$ for every $u \in I_p$ and $v \in I_q$.

Example 3.4. The example above was $\pi = (1, 2, 3, 4, \mathbf{9}, \mathbf{8}, \mathbf{7}, \mathbf{6}, \mathbf{5}, 10, 11) \in \mathcal{S}_{11}$. This is of the form $\pi = \pi^\kappa$ where $n = 11$, $\mathcal{I}_3 = (I_1, I_2, I_3)$ with $I_1 = \{1, 2, 3, 4\}$, $I_2 = \{6, 7, 8, 9\}$, $I_1 = \{10, 11\}$, and $\tau_3 = (1, -1, 1)$. Then $G(\pi)$ contains a clique I_2 and otherwise isolated vertices (and no further edges). If we here change the order of the two intervals I_1 and I_2, the graph is changed by adding every edge between a vertex in I_1 and I_2.

We remark that *every* permutation π is of the form π^κ by choosing the ordered partition of intervals as one-point sets. So, the simpler permutation graphs are obtained when k, the number of intervals in the partition, is small.

Note that, for the class of permutation graphs considered in Proposition 3.3, the maximum number of permutations with the same permutation graph is $n-1$ and this corresponds to all adjacent transpositions. The graph then consists of an edge and isolated vertices.

4 Inversion vectors and majorization

Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a permutation in $\mathcal{S}_n$. Its *inversion vector* is defined by $a(\sigma) = a = (a_1, a_2, \ldots, a_n)$ where a_j is the number of inversions whose second coordinate is j. Therefore a is a partition of the inversion set $\mathcal{I}(\sigma)$, and

$$\kappa_\sigma := \sum_{j=1}^{n} a_j = |\mathcal{I}(\sigma)|$$

is the number of inversions in σ. Let $\mathcal{V}_n$ be the set of inversion vectors of permutations in $\mathcal{S}_n$. A proof of the following elementary fact may be found in [5].

Theorem 4.1. *The inversion vector uniquely determines the permutation. Moreover, $\mathcal{V}_n$ equals the set of integer vectors $a = (a_1, a_2, \ldots, a_n)$ satisfying*

$$\begin{aligned} 0 \leq a_i \leq n - i \quad (1 \leq i \leq n-1), \\ a_n = 0. \end{aligned} \tag{4.1}$$

Due to this theorem, the mapping from $\mathcal{S}_n$ into $\mathcal{V}_n$ that sends a permutation $\sigma \in \mathcal{S}_n$ into $a(\sigma) \in \mathcal{V}_n$ is a bijection.

Let $n \geq 1$ and let $0 \leq N \leq n(n-1)/2$ be an integer. Define

$$\mathcal{S}_n^N = \{\sigma \in \mathcal{S}_n : \kappa_\sigma = N\}$$

as the set of permutations with N inversions. We now introduce a partial order on the set $\mathcal{S}_n^N$, and establish a connection to the majorization order for integer partitions (and Ferrers diagrams).

Let $\pi, \sigma \in \mathcal{S}_n^N$ and define

$$\pi \preceq^* \sigma \quad \text{whenever} \quad a(\pi) \preceq a(\sigma),$$

where $\preceq$ denotes classical vector majorization. We define two special vectors associated with the pair n, N. An example is shown in Example 4.3. First, let I be the set of positions strictly below the main diagonal of an $n \times n$ matrix, i.e., $I = \{(i,j) : 2 \leq i \leq n-1,\ j < i\}$. So $|I| = n(n-1)/2$. Let $A_1 = A_1(n,N)$ be the $n \times n$ $(0,1)$-matrix obtained from the zero matrix by placing N ones in positions in I, working row wise starting from the last row and going up, from left to right. Let $\bar{\alpha} = \bar{\alpha}(n,N) = (\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n)$ be the column sum vector of A_1. Next, let $A_2 = A_2(n,N)$ be the $n \times n$ $(0,1)$-matrix obtained from the zero matrix by placing N ones in positions in I, working column wise starting with the first column, and with increasing row numbers. Let $\hat{\alpha} = \hat{\alpha}(n,N) = (\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$ be the column sum vector of A_2. Then, clearly, $\bar{\alpha}, \hat{\alpha} \in \mathcal{V}_n$, so we may define the two permutations $\bar{\pi}$ and $\hat{\pi}$ by $a(\bar{\pi}) = \bar{\alpha}$ and $a(\hat{\pi}) = \hat{\alpha}$. With this notation we have the next result.

Theorem 4.2. *Let $\sigma \in \mathcal{S}_n^N$. Then $\bar{\pi} \preceq^* \sigma \preceq^* \hat{\pi}$.*

Proof. Let $b = (b_1, b_2, \ldots, b_n)$ be obtained from $a(\sigma)$ by reordering components such that b is monotone (nonincreasing). It suffices to show

$$\bar{\alpha} = a(\bar{\pi}) \preceq b \preceq a(\hat{\pi}) = \hat{\alpha}.$$

To do so, let A be the $n \times n$ $(0,1)$-matrix such that the last b_j rows in column j contain a 1. So the column sum vector of A is b and its ones are in a Ferrers pattern (rotated 90 degrees). Assume that A has two nonzero rows i and i', with $i < i'$, such that both rows contains a zero in some position in I. Then it is possible to move the last 1 in row i to the position of the first zero in row i'. The resulting matrix A' also has support corresponding to a Ferrers pattern. Moreover, the column sum vector of A' is (strictly) majorized by b, as a larger component has been reduced by 1 and a smaller component has been increased by 1 (this is often called a Robin Hood transfer). Clearly we can repeat this process (on A') a number of times until, eventually, the current matrix A'' has at most one row with a zero in a position in I. But this means that $A'' = A_1(n,N)$, and it follows (by transitivity of majorization) that $a(\pi_L) \preceq b$.

The other majorization, $b \preceq a(\hat{\pi})$ is shown in a similar way. The only difference is that we shift a 1 from a column j to some column $j' < j$, which gives a new column sum vector that majorizes the previous one. Eventually we obtain the matrix $A_2(n,N)$. □

Example 4.3. Let $n = 8$ and $N = 20$. Then, in the notation above,

$$A_1 = \left[\begin{array}{c|c|c|c|c|c|c|c} \cdot & & & & & & & \\ \hline & \cdot & & & & & & \\ \hline & & \cdot & & & & & \\ \hline & & & \cdot & & & & \\ \hline 1 & 1 & & & \cdot & & & \\ \hline 1 & 1 & 1 & 1 & 1 & \cdot & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot \end{array}\right], \quad A_2 = \left[\begin{array}{c|c|c|c|c|c|c|c} \cdot & & & & & & & \\ \hline 1 & \cdot & & & & & & \\ \hline 1 & 1 & \cdot & & & & & \\ \hline 1 & 1 & 1 & \cdot & & & & \\ \hline 1 & 1 & 1 & & \cdot & & & \\ \hline 1 & 1 & 1 & & & \cdot & & \\ \hline 1 & 1 & 1 & 1 & & & \cdot & \\ \hline 1 & 1 & 1 & 1 & & & & \cdot \end{array}\right]$$

where zeros are not shown, and dots indicate the main diagonal. Then

$$\bar{\alpha} = a(\bar{\pi}) = (4, 4, 3, 3, 3, 2, 1, 0), \quad a(\hat{\pi}) = (7, 6, 5, 2, 0, 0, 0, 0) = \hat{\alpha},$$

and

$$\bar{\pi} = (8, 7, 6, 3, 1, 2, 4, 5), \quad \hat{\pi} = (5, 6, 4, 7, 8, 3, 2, 1).$$

Let $\sigma = (8, 7, 5, 4, 1, 2, 3, 6)$, so $a(\sigma) = (4, 4, 4, 3, 2, 2, 1, 0)$. Then

$$\bar{\alpha} = (4, 4, 3, 3, 3, 2, 1, 0) \preceq a(\sigma) = (4, 4, 4, 3, 2, 2, 1, 0) \preceq (7, 6, 5, 2, 0, 0, 0, 0) = \hat{\alpha}.$$

The $(0, 1)$-matrix with column sum $a(\sigma)$ and 1's in the last rows is

$$A = \left[\begin{array}{c|c|c|c|c|c|c|c} \cdot & & & & & & & \\ \hline & \cdot & & & & & & \\ \hline & & \cdot & & & & & \\ \hline & & & \cdot & & & & \\ \hline 1 & 1 & 1 & & \cdot & & & \\ \hline 1 & 1 & 1 & 1 & & \cdot & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot \end{array}\right].$$

Theorem 4.2 implies that if we restrict to permutations σ such that $a(\sigma)$ is monotone, then $\bar{\pi}$ and $\hat{\pi}$ are minimum and maximum elements in the partial order $\preceq^*$.

5 Further related discussion

In this section we discuss different concepts related to permutation graphs. Section 2 of [12] contains some related concepts and results.

For the convenience of the reader, we first comment on a, perhaps not well-known, discussion in Berge [2] (pages 136–139).

First, let $\sigma_n = (i_1, i_2, \ldots, i_n)$ be a permutation of $\{1, 2, \ldots, n\}$. Then σ_n determines an orientation $\Gamma(\sigma_n)$ of the complete graph K_n as follows: The edge $\{k, l\}$ with $k < l$ is oriented as $k \to l$ provided that $i_k > i_l$ (so an inversion of σ_n) and as $l \to k$ otherwise (so an inversion of $\overleftarrow{\sigma}_n$). Then $\Gamma(\sigma_n)$ is a tournament (an orientation of K_n without any (directed) cycles).

Now let E be an arbitrary subset of pairs $\{p, q\}$ of the edges of the complete graph K_n with $p < q$ (so possible positions of the set of inversions of a permutation), and let E^* be the complement of E. Orient the edge $\{p, q\}$ of K_n as $p \to q$ if $\{p, q\}$ is in E and as $q \to p$ if $\{p, q\} \in E^*$, thereby obtaining a tournament $\Gamma(E)$. Berge shows that if $\Gamma(E)$ has no (directed) cycles, that is, is a transitive tournament, then there is a unique Hamilton path, thus defining a unique permutation $\sigma = (i_1, i_2, \ldots, i_n) \in \mathcal{S}_n$. He also shows that the weak Bruhat order (without using that name) is a lattice, a sublattice of the lattice of all subsets of $\{1, 2, \ldots, n\}$ partially ordered by inclusion.

Theorem 5.1. *Let E be a subset of the pairs of integers taken from $\{1, 2, \ldots, n\}$ and let E^* be the complement of E. Assume that the pairs in E have been oriented from smallest to largest and those in E^* from largest to smallest. Let Γ be the resulting tournament on $\{1, 2, \ldots, n\}$. Then Γ is a transitive tournament if and only if the digraph $\Gamma(E)$ with oriented edges in E and the digraph $\Gamma(E^*)$ with oriented edges in E^* are both transitive.*

Proof. First assume that Γ is transitive. Let $i \to j$ and $j \to k$ be in E so that $i < j$ and $j < k$. Since Γ is transitive, $i \to k$ is in Γ. But $i < k$ and so $i \to k$ is in E. Thus $\Gamma(E)$ is transitive. Similarly $\Gamma(E^*)$ is transitive.

Conversely, assume that the digraphs $\Gamma(E)$ and $\Gamma(E^*)$ are both transitive. Let $i \to j$ and $j \to k$ be edges in Γ. There are four possibilities:

(i) $i \to j \in E$ and $j \to k \in E$,

(ii) $i \to j \in E$ and $j \to k \in E^*$,

(iii) $i \to j \in E^*$ and $j \to k \in E$, ,

(iv) $i \to j \in E^*$ and $j \to k \in E^*$.

We need only consider (i) and (i) by symmetry.

(i) $i \to j \in E$ and $j \to k \in E$: Since $\Gamma(E)$ is transitive, $i \to k$ is in E and so in $\Gamma(E)$.

(ii) $i \to j \in E$ and $j \to k \in E^*$: We need to show that $i \to k$ is in Γ. Suppose to the contrary, that $k \to i$ is in Γ and so either in E or E^*. If in E, this violates transitivity of $\Gamma(E)$. So $k \to i$ is in E^*. But then with $j \to k \in E^*$, we conclude from the transitivity of $\Gamma(E^*)$, that $j \to i$ is in E^*, a contradiction since $i \to j$ is in E. □

Again consider a permutation $\sigma = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$ and construct an $n \times n$ nonnegative integral matrix T_σ as follows. Let $1 \le k < l \le n$. If $i_k > i_l$ put the *inversion weight* $(i_k - i_l)$ of σ in the position (k, l) and put 0 in position (l, k). Otherwise, if $i_l > i_k$, put the *inversion weight* $i_l - i_k$ of $\overleftarrow{\sigma}$ in position (l, k) and then put 0 in position (k, l). Finally put 0's on the main diagonal. For example, with $\sigma = (3, 4, 1, 5, 2)$ we get

$$T_\sigma = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 2 & 0 & 1 \\ \hline 1 & 0 & 3 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 2 & 1 & 4 & 0 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

The entries above the main diagonal correspond to the inversion weights of σ. The entries below the main diagonal correspond to the inversion weights of $\overleftarrow{\sigma}$. Replacing the nonzeros of T_σ with 1's we get a *transitive tournament matrix*, that is the adjacency matrix of a transitive tournament as defined above:

$$T'_\sigma = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

Simultaneously permuting these matrices, we get, respectively,

$$L^*_5 = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ \hline 2 & 1 & 0 & 0 & 0 \\ \hline 3 & 2 & 1 & 0 & 0 \\ \hline 4 & 3 & 2 & 1 & 0 \end{array}\right] \text{ and } L'_5 = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 \end{array}\right].$$

In general, there are $n!$ simultaneous permutations of the rows and columns of L^*_n (and L'_n), and the resulting matrices correspond to the $n!$ permutations of $\{1, 2, \ldots, n\}$. Thus T_σ is a *weighted transitive tournament matrix.*

Now the set of inversions of a permutation σ uniquely determines σ. So the set of *positions* of T_σ above the main diagonal determine T_σ uniquely. (Cf. Lemma 5.7 which is basically this fact; only the positions of the nonzeros are used.) So the matrix T_σ contains a lot of redundant information.

We know that a characterization of a permutation graph G is that G and its complement $\overline{G}$ are transitively orientable giving transitive digraphs T_1 and T_2. In fact, this is equivalent by Theorem 5.1 to the union of the edges of T_1 and T_2 giving a transitive tournament. *Thus there are $n!$ ways to partition the edges of the standard transitive tournament on $\{1, 2, \ldots, n\}$ (edges are all $i \to j$ with $1 \le i < j \le n$) into two transitive tournaments T_1 and T_2 on $\{1, 2, \ldots, n\}$.* In terms of the adjacency matrix, with suitable ordering of its vertices, T_1 corresponds to the upper triangular part and T_2 corresponds to the lower triangular part.

A digraph may be transitive because it never contains two consecutive edges in the same direction: $u \to v$ and $v \to w$. Call such a transitive tournament (or directed graph) *trivially transitive*.

A permutation σ of $\{1, 2, \ldots, n\}$ or its reverse $\overleftarrow{\sigma}$ may give trivially transitive tournaments T_1 or T_2. As we have seen permutations whose permutation graphs are caterpillars have trivially transitive digraphs. Their complements, while transitive, need not be trivially transitive. In fact, we have the following. Let P_k denote a path with k vertices.

Theorem 5.2. *Let G be a permutation graph. Then the following are equivalent:*

(i) *Both G and its complement can be oriented into trivially transitive directed graphs;*

(ii) *G is one of the following types: (a) the cycle C_4, (b) the path P_k with $k \le 4$, or (c) the disjoint union of P_1 and P_2 (so, the permutation is e.g., $\sigma = (1, 3, 2)$).*

Proof. First, observe that a directed graph is trivially transitive if and only if each vertex has indegree or outdegree equal to 0, and this gives a 2-coloring of the vertices (where isolated vertices are included in any of these sets). Thus the associated graph is bipartite.

Assume that (i) holds, i.e., both G and its complement can be oriented into trivially transitive directed graphs. By what we just observed, G is bipartite. There are two cases.

Case 1*: G is bipartite, but not a tree.* Then is color class (in the bipartition) has at most two vertices, otherwise, the complement of G would contain a triangle (contradicting the observation). Since G is not a tree, the only possibility is that G equals C_4, the cycle with four vertices.

Case 2*: G is a tree.* Then, by Corollary 2.7, G is a caterpillar. But G can not contain the star $K_{1,3}$ is an induced subgraph, because, if so, the complement would contain a 3-cycle. Therefore each vertex in G has degree at most 2, so G is the disjoint union of paths. Each such path contains at most 4 vertices (otherwise the complement contains a 3-cycle), and by the same argument, there are at most two (weak) components. If there are two components, the only possibility is that these components are the path P_2 and an isolated vertex (P_1). Finally, if G is connected, it is a path P_k with $k \leq 4$.

So, we have shown that (i) implies (ii). The converse implication is easy to verify, by checking each of these small graphs and their complements explicitly. □

Example 5.3. Consider the permutation $(3, 4, 1, 2)$. Then its permutation graph G is a 4-cycle C_4 of vertices $1, 2, 3, 4$ and edges $\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}$. The transitive orientation corresponding to its inversion set is given by $(1, 3), (1, 4), (2, 3), (2, 4)$ giving a trivially transitive digraph. The complement of G is the graph of the permutation $(2, 1, 4, 3)$ with only two edges $\{1, 2\}$ and $\{3, 4\}$ whose corresponding digraph with edges are $(2, 1), (4, 3)$ is also clearly trivially transitive. The tournament whose edges are their union is not trivially transitive, although it is transitive. □

Let T be a subset of $\Gamma_n := \{(p, q) : 1 \leq q < p \leq n\}$. Then T may, or may not be, the inversion set of a permutation, as we have seen in previous examples. We now study a minimal extension of T into an inversion set. Let $K \subseteq \Gamma_n$. We say that K is an *inv-extension* of T if $T \subseteq K$ and $K = \mathcal{I}(\sigma)$ for some permutation $\sigma \in S_n$. An inv-extension always exists as $\mathcal{I}(\sigma) = \Gamma_n$ when $\sigma = (n, \ldots, 2, 1)$. We seek one or more *minimal* inv-extensions, meaning that the removal of any element gives a set which is not an inv-extension. It is also of interest to determine the minimum cardinality of an inv-extension of T, denoted by $\kappa(T)$. Thus, $\kappa(T) \geq |T|$ with equality if and only if T is an inversion set.

Example 5.4. Let $n = 4$ and $T = \{(3, 1), (4, 1)\}$. Then, as we saw before, T is not the inversion set of a permutation. Some permutations whose inversion set contains T are

$$\pi_1 = (2, 3, 4, 1),\ \pi_2 = (3, 2, 4, 1),\ \ \pi_3 = (3, 4, 1, 2).$$

The corresponding inversion sets consist of T and the following sets, respectively

$$\mathcal{I}_1 = \{(2, 1)\},\ \mathcal{I}_2 = \{(3, 2), (2, 1)\},\ \mathcal{I}_3 = \{(3, 2), (4, 2)\}.$$

There are also permutations where 4 precedes 3, but those inversion sets strictly contain those sets just listed. Therefore, there are two minimal inv-extensions, corresponding to $\mathcal{I}_1 = \mathcal{I}_3$ and $\mathcal{I}_4$, and they are

$$K_1 = \{(2, 1), (3, 1), (4, 1)\} \ \text{ and } \ K_3 = \{(3, 2), (4, 2), (3, 1), (4, 1)\}.$$

We see that the minimum cardinality of an inv-extension is $\kappa(T) = 3$.

Let $K \subseteq \Gamma_n$. Define K_1 as the set of those integers i that occur in some pair in K (as first or second component) and let

$$K^* = \{r : p > r > q \text{ for some } (p, q) \in K\} \setminus K_1.$$

Lemma 5.5. *Let $K \subseteq \Gamma_n$. Then*

$$\kappa(K) \geq |K| + \lceil K^*/2 \rceil. \tag{5.1}$$

If $K = \{(p, q)\}$ where $p > q$, then $\kappa(K) = p - q$.

Proof. Let $r \in K^*$. Then there exists $(p, q) \in K$ with $p > r > q$. Let $\sigma = (j_1, j_2, \ldots, j_n) \in S_n$ be such that $K \subseteq \mathcal{I}(\sigma)$. So there are $k < l$ with $j_k = p$ and $j_l = q$. Define i by $j_i = r$. If $i < l$, σ contains the inversion (r, q). If $i > l$, σ contains the inversion (p, r). So, in any case there is an inversion with r as an end point. This inversion is not in K, by definition of K^*. So, each $r \in K^*$ is in an inversion. But two such inversions might coincide, and this accounts for the second term in (5.1).

For the second part, consider $K = \{(p, q)\}$. Then $|K^*| = p - q - 1$ and none of the inversions mentioned above (for $r \in K^*$) can coincide, so $\kappa(K) \geq 1 + (p - q - 1) = p - q$. Next, consider the permutation $\sigma \in S_n$ given by

$$\sigma = (1, 2, \ldots, q - 1, q + 1, q + 2, \ldots, p - 1, p, q, p + 1, p + 2, \ldots, n).$$

Then $\mathcal{I}(\sigma) = \{(q+1, q), (q+2, q), \ldots, (p, q)\}$, so $|\mathcal{I}(\sigma)| = \kappa(K) = p - q$, which implies that $\kappa(K) = p - q$. □

Example 5.6. Let $n = 8$ and $K = \{(7, 3)\}$. Then $K^* = \{4, 5, 6\}$. So, Lemma 5.5 gives $\kappa(K) = p - q = 4$. Consider $\sigma = (1, 4, 5, 6, 7, 3, 8) \in S_8$. Then

$$\mathcal{I}(\sigma) = \{(7, 3), (4, 3), (5, 3), (6, 3)\}$$

so $K \subseteq \mathcal{I}(\sigma)$ and $|\mathcal{I}(\sigma)| = \kappa(K) = 4$.

Let $\sigma = (j_1, j_2, \ldots, j_n) \in S_n$. consider the $n \times n$ matrix $A'_\sigma = [a_{kl}]$ by $a_{kl} = j_k - j_l$ when $k < l$ and $j_k > j_l$, i.e., (j_k, j_l) is an inversion. All other entries are zero. For instance, if $\sigma = (3, 4, 1, 5, 2)$ we get

$$A' = A'_\sigma = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 2 & 0 & 1 \\ \hline 0 & 0 & 3 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This matrix is obtained from the weighted inversion matrix by replacing all nonzeros below the diagonal with zeros. This matrix shows where the inversions of σ occur (the positions) and their magnitude, the *weighted inversion matrix* of σ. Note that $A_\sigma = O$ if and only if $\sigma = \iota = (1, 2, \ldots, n)$. Let $\sigma = (j_1, j_2, \ldots, j_n) \in S_n$ and consider its weighted inversion matrix A_σ. Assume $j_s = n$. If we delete $j_s = n$ from σ we obtain a permutation $\sigma' \in S_{n-1}$. The resulting weighted inversion matrix is obtained from A'_σ simply by deleting row and column s. This fact leads to an algorithm for reconstructing σ from its weighted inversion matrix A'.

Algorithm 1:
Step 1. Let s be maximal such that column s of A' is the zero vector. Then $\sigma(s) = n$. Delete row and column s in A'.
Step 2. Repeat the procedure for the updated A' and update σ accordingly.

Lemma 5.7. *The function that maps $\sigma \in S_n$ into its weighted inversion matrix is injective.* Algorithm 1 *reconstructs a permutation from its inversion matrix.*

Proof. This follows from the discussion above. If A' is a weighted inversion matrix of some permutation σ, then it must have (at least) one zero column, and the last among these must correspond to the entry n. Deleting the corresponding row and column then gives the inversion matrix of the permutation obtained from σ by deleting n. We then proceed by induction and the algorithm eventually reconstructs σ. This implies the the function in the first statement is injective (which also can be seen directly). □

Note that Algorithm 1 only depends on the pattern of the given matrix A, which again specifies the inversions of the underlying permutation.

Example 5.8. Consider the example above where $\sigma = (3, 4, 1, 5, 2)$, and apply Algorithm 1 to the given matrix A'. This gives $s = 4$, so $\sigma(4) = 5$ and the new matrix is

$$\left[\begin{array}{c|c|c|c} 0 & 0 & 2 & 1 \\ \hline 0 & 0 & 3 & 2 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

This gives $s = 2$, so $\sigma(2) = 4$, and the new matrix is

$$\left[\begin{array}{c|c|c} 0 & 2 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right].$$

Then $s = 1$ and $\sigma(1) = 3$, and the new matrix is the zero matrix. Thus the last two entries, 1 and 2, are ordered increasingly. This gives $\sigma = (3, 4, 1, 5, 2)$, as desired. Finally, note that in general, all entries in the strict upper triangular part are positive if and only if $\sigma = (n, \ldots, 2, 1)$. The class $S_n^{(k)}$ defined above corresponds to the inversion matrices satisfying $A' \leq kJ$.

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