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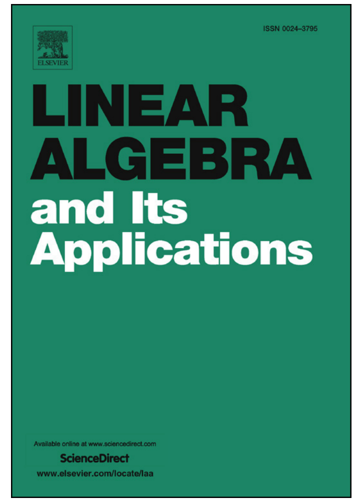

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Majorization for $(0, \pm 1)$-matrices

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Abstract

Matrix majorization is a generalization of the classical majorization for vectors. We study several basic questions concerning matrix majorization for $(0, \pm 1)$-matrices, i.e., matrices whose entries are restricted to 0, 1 and -1. In particular, we characterize when the zero vector is weakly majorized by a matrix, and show related results. Connections to linear programming are discussed. We obtain simpler characterizations of majorization under different assumptions. Also, several results on directional and strong majorization for $(0, \pm 1)$-matrices are shown.

Key words. Matrix majorization, $(0, \pm 1)$-matrices, linear programming.

AMS subject classifications. 06A06, 15B51.

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1 Introduction

The notion of majorization is of great importance in several mathematical areas, such as analysis, probability and combinatorics. It deals with an order notion reflecting "spread", and it is a basis for a theory of mathematical inequalities. As an example, the famous Gale-Ryser theorem on the existence of a $(0, 1)$-matrix with given row and column sums is formulated as a majorization condition, i.e., certain inequalities on the spread of all the given numbers. Some central books on majorization are [15] (a basic reference for theory and applications), [3] (connections to combinatorics and matrix theory), [18] (generalizations and information theory in statistics) and [12] (a classical text on inequalities). In recent years many papers have been written on majorization, by extending the notion and connecting it to various other mathematical questions, see e.g. [6, 7, 8, 9, 13] and references therein. Among the goals of our paper there are

- Investigate the borderline between general results on matrix majorization, typically related to linear optimization duality, and specific combinatorial results obtained for constrained situations where entries are in the set $\{0, \pm 1\}$.
- Establish connections to other notions, such as totally unimodular matrices and network flows, and to qualitative matrix theory.
- Study the combinatorics of strong matrix majorization.

The set of all real $m \times n$ matrices with every element in a set $X \subseteq \mathbb{R}$ is denoted by $M_{m,n}(X)$. In particular, we investigate $M_{m,n}(0, \pm 1)$, $M_{m,n}(0, 1)$, $M_{m,n}(0, -1)$ and $M_{m,n}(\pm 1)$. We denote $M_{m,n} = M_{m,n}(\mathbb{R})$.

For a matrix A its j'th column is denoted by $A^{(j)}$ and its i'th row is denoted by $A_{(i)}$. Note that we view real n-vectors as column vectors and identify them with the corresponding n-tuples. Let $\mathcal{R}(A)$ denote the set of rows of a matrix A. Let $A \in M_{m,n}$ and $\mathcal{I} \subseteq \{1, 2, \ldots, m\}$, $\mathcal{J} \subseteq \{1, 2, \ldots, n\}$. We let $A^{(\mathcal{J})}$ denote the $m \times |\mathcal{J}|$ submatrix of A obtained by deleting all columns of A that are not indexed by $\mathcal{J}$. $A_{(\mathcal{I})}$ denotes the same for rows. $A_{(\setminus \mathcal{I})}$ denotes the submatrix obtained by deleting the $\mathcal{I}$-indexed rows from A. Finally, $A^{(\mathcal{J})}_{(\mathcal{I})}$ denotes the submatrix of A obtained as the intersection of rows of A indexed by $\mathcal{I}$ and columns of A indexed by $\mathcal{J}$.

I denotes the identity matrix, O denotes the zero matrix, and we indicate dimensions by subscripts, if needed. An all ones vector is denoted by e, and the j'th coordinate vector in $\mathbb{R}^n$ is denoted by e_j, so $e = \sum_{j=1}^n e_j$. A zero vector of size n is denoted by 0_n, or simply 0, if n is clear from the context. Let Ω_n, Ω_n^{row} and $\Omega_{m,n}^{row}$ denote, respectively, the sets of all $n \times n$ doubly stochastic matrices, the $n \times n$ row stochastic matrices and the $m \times n$ row stochastic matrices. The transpose of a matrix A is denoted by A^t. The j'th largest component of a vector $x = [x_1\ x_2\ \cdots\ x_n]^t$ is denoted by $x_{[j]}$ $(j \leq n)$. The set of all $n \times n$ permutation matrices is denoted by $P(n)$. The convex hull of a set $X \subseteq \mathbb{R}^n$ is denoted by $\mathrm{conv}(X)$.

We recall some main notions of majorization from the literature.

(i) Let $a, b \in \mathbb{R}^n$:

- Vector majorization is defined as follows: $a \preceq b$ if $\sum_{j=1}^k a_{[j]} \leq \sum_{j=1}^k b_{[j]}$ for any $k = 1, \ldots, n-1$ and $\sum_{j=1}^n a_j = \sum_{j=1}^n b_j$.

(ii) Let $A \in M_{m_1,n}$ and $B \in M_{m_2,n}$:

- Weak matrix majorization: $A \preceq^w B$ if there exists $R \in \Omega_{m_1,m_2}^{row}$ such that $A = RB$. Note that this definition only requires that A and B have the same number of columns. Thus, the number of rows may be different, and we shall permit this in the following.

(iii) Let $A, B \in M_{m,n}$:

- Directional majorization: $A \preceq^d B$ if $Av \preceq Bv$ for all $v \in \mathbb{R}^n$.
- Strong majorization: $A \preceq^s B$ if there exists $D \in \Omega_m$ such that $A = DB$.

It is known that strong majorization implies directional majorization, and directional implies weak. None of the reverse implications holds in general, see Examples 1 and 2 in [16].

Weak matrix majorization has a convenient geometrical characterization.

Proposition 1.1 *Let $A \in M_{m_1,n}$, $B \in M_{m_2,n}$. Then*

$$A \preceq^w B \text{ if and only if } \mathcal{R}(A) \subseteq \mathrm{conv}(\mathcal{R}(B)).$$

This result was proved in [16, Proposition 3.3] for $m_1 = m_2$. The case $m_1 \neq m_2$ can be proved essentially in the same way.

The remaining paper is organized as follows. In Section 2 we establish several criteria that allow us to check whether the zero matrix is weakly majorized by a given matrix. These criteria are based on matrix theory, graph theory, and optimization. In Section 3 we investigate when a (row) vector is majorized by a given matrix and study the structure of the row vector realizing this majorization. In particular, we consider totally unimodular matrices and obtain some special results for them. In Section 4 we present a quantitative approach to the above majorization problems. Section 5 deals with directional and strong majorizations.

We use the standard notations $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ for the set of real, integer and natural numbers, respectively.

2 Basic weak majorization

In this section, we consider the notion of weak majorization. The focus is on results for a basic question concerning majorization – when is the zero (row) vector weakly majorized by a given matrix?

First, we explain the fundamental role of this question. Proposition 1.1 shows that checking if $A \preceq^w B$ can be reduced to checking if each row of A lies in the convex hull of the rows of B. Thus, we have problems of the form: given a (column) vector a, check if $a^t \preceq^w B$ holds. This problem can be "reduced" even further.

Lemma 2.1 *Let $a \in \mathbb{R}^n$, $B \in M_{m,n}$. Then $a^t \preceq^w B$ if and only if $0_n^t \preceq^w B - ea^t$.*

Proof. The majorization $a^t \preceq^w B$ means that $a^t = \sum_i x_i B_{(i)}$ where $x_i \geq 0$ $(i \leq m)$ and $\sum_i x_i = 1$. Now, $\sum_i x_i B_{(i)} = a^t = \sum_i x_i a^t$ is equivalent to

$$0_n^t = \sum_i x_i (B_{(i)} - a^t) = \sum_i x_i (B - ea^t)_{(i)}$$

and the result follows. □

Thus, weak matrix majorization reduces to basic majorization, i.e., checking if the zero row vector is weakly majorized by a certain matrix. By the definition of weak majorization, $0^t \preceq^w B$ means that $x^t B = 0^t$ for some

$x^t = [x_1\ x_2\ \cdots\ x_m]$ satisfying $\sum_i x_i = 1$ and $x_i \geq 0$ $(i \leq m)$. This immediately shows that the rows of B are linearly dependent when $0^t \preceq^w B$ holds.

A useful tool applied in this paper is the classical Farkas' lemma [17, Section 7.3] which we state next.

Lemma 2.2 (Farkas' lemma) *Let $A \in M_{m,n}$ and $b \in \mathbb{R}^n$. Then exactly one of the following two assertions is true:*

(i) There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0_n$.

(ii) There exists $y \in \mathbb{R}^m$ such that $A^t y \leq 0_m$ and $b^t y > 0$.

Corollary 2.3 *Let $B \in M_{m,n}$. Then $0_n^t \preceq^w B$ if and only if the column space of B does not contain a positive vector.*

Proof. By definition of weak majorization, $0_n^t \preceq^w B$ if and only if $x^t B = 0_n^t$, where $x^t = [x_1\ x_2\ \cdots\ x_m]$ satisfies $\sum_i x_i = 1$ and $x_i \geq 0$ $(i \leq m)$.

Let $A = \begin{bmatrix} B^t \\ e^t \end{bmatrix} \in M_{n+1,m}$. That is, A is a matrix obtained from B^t by concatenating a row of ones. Then $0_n^t \preceq^w B$ if and only if $Ax = e_{n+1}$ for some $x \geq 0_m$. By Farkas' lemma the latter holds if and only if there is no $y' \in \mathbb{R}^{n+1}$ such that $A^t y' \leq 0$ and $e_{n+1}^t y' > 0$.

Observe that $e_{n+1}^t y' > 0$ if and only if $y'_{n+1} > 0$. Thus the following conditions are equivalent:

- there is no $y' \in \mathbb{R}^{n+1}$ such that $A^t y' = \begin{bmatrix} B & e \end{bmatrix} y' \leq 0$ and $e_{n+1}^t y' > 0$,
- there is no $y \in \mathbb{R}^n$ such that $By < 0$,
- there is no $z \in \mathbb{R}^n$ such that $Bz > 0$.

The last condition means that the column space of B does not contain a positive vector. $\square$

We remark that the previous result may also be proved using Proposition 1.1, expressed in terms of support functions, see [16, Corollary 3.11].

The majorization question above is closely related to a notion of interest in qualitative matrix theory (QMT) [5]. In QMT one studies linear algebraic properties that depend on the entries of matrices (or vectors) only via their sign. The area was initiated by the economist Paul Samuelson and motivated by qualitative questions in economics. By sign here we mean $+$, $-$ or 0. The *qualitative class* $\mathcal{Q}(A)$ of a matrix A consists of all matrices of the same size and with the same sign in every entry as that of A. An important concept in QMT is the following: consider a (consistent) linear system $Ax = b$, where A is $m \times n$ and $b \in \mathbb{R}^m$. We say $Ax = b$ is *sign-solvable* if for every system $\tilde{A}\tilde{x} = \tilde{b}$ where $\mathcal{Q}(\tilde{A}) = \mathcal{Q}(A)$ and $\mathcal{Q}(\tilde{b}) = \mathcal{Q}(b)$, every solution $\tilde{x}$ (so $\tilde{A}\tilde{x} = \tilde{b}$) satisfies $\mathcal{Q}(\tilde{x}) = \mathcal{Q}(x)$.

Next, let B be a real $m \times n$ matrix. Then B is called *central* ([5]) if it has a nonzero nonnegative vector in its null space, i.e., there is an $x \geq 0_n$ with $x \neq 0_n$ and $Bx = 0_m$. Moreover, B is *sign-central* provided that every matrix $\tilde{B} \in \mathcal{Q}(B)$ is central. A characterization of sign-centrality was given in Theorem 5.4.1 in [5], and a closely related notion was investigated in [4].

We now connect this to our majorization results. If B is central, there exists an $x \geq 0_n$ with $x \neq 0_n$ and $Bx = 0_m$. We may scale such a vector x so that its sum is 1. Therefore, the definition means that the origin 0_m is some convex combination of the columns of B, i.e., 0_m lies in the convex hull of the columns of B. Thus, by Proposition 1.1, we have the following corollary.

Corollary 2.4 *Let $B \in M_{m,n}$. Then $0_n^t \preceq^w B$ if and only if the matrix B^t is central.*

We can combine the last two assertions as follows:

Corollary 2.5 *Let $B \in M_{m,n}$. Then the following are equivalent:*

1. $0_n^t \preceq B$.
2. *The null space V_0 of B^t contains a nonzero nonnegative vector.*
3. *The column space V_1 of B does not contain a positive vector.*

Proof. Item 1. is equivalent to Item 2. by Corollary 2.3. Also Item 1. is equivalent to Item 3. by Corollary 2.4. □

Recall that a linear programming problem (LP) is an optimization problem where we want to maximize (or minimize) a linear function f of finitely many (decision) variables, where the variables are subject to finitely many linear inequalities/equalities. The function f is usually referred to as the objective function. Often we have nonnegativity constraints on the variables.

Computationally one can check efficiently whether $0_n^t \preceq^w B$ holds by solving the LP problem

$$\max\{c^t x : B^t x = 0_n,\ x \geq 0_m,\ \sum_j x_j = 1\} \tag{1}$$

where B is the given $m \times n$ matrix. The vector c and the objective function $c^t x$ plays no role, as the purpose is to decide if there are feasible solutions, i.e., solutions of the system $B^t x = 0_n$ that satisfy all constraints, in our case $x \geq 0_m$, $\sum_j x_j = 1$. So, if there exists a nonnegative x with $B^t x = 0_n$ and $\sum_j x_j = 1$, then $0_n^t \preceq^w B$; otherwise the majorization does not hold. Also, if the majorization holds, it follows from LP theory, or Carathéodory's theorem, that 0_n^t may be written as a convex combination of at most $n+1$ rows in B. For instance, the simplex method for LP will find such $n+1$ rows as there is always an optimal basic feasible solution when the LP has feasible solutions. Alternatively, the LP in the next lemma may be used to check the majorization $0_n^t \preceq^w B$.

Lemma 2.6 *Let $B \in M_{m,n}$. Then $0_n^t \preceq^w B$ if and only if*

$$\max\{e^t x : B^t x = 0_n,\ x \geq 0_m\ \} = \infty, \tag{2}$$

i.e., this LP has optimal value which is infinite.

Proof. If $0_n^t \preceq^w B$, then by the definition of weak majorization there exists $x \in \mathbb{R}^m$ such that $x \geq 0_m$, $e^t x = 1$ and $B^t x = 0_n$. Then for any $\alpha > 0$ the vector αx satisfies the constraints of the linear program (2) and as a consequence the optimal value is infinite.

Conversely, if this optimal value is infinite, there must exist an x with $B^t x = 0_n$, $x \geq 0_m$ and $e^t x > 0$. By positive scaling of x we obtain a nonnegative vector x' with $B^t x' = 0_n$ and $e^t x' = 1$, providing $0_n^t \preceq^w B$. $\square$

The majorization result above has a connection to the theory of network flows, as we now discuss. Let $G = (V, E)$ be a directed graph with vertex set $V = \{v_1, v_2, \ldots, v_m\}$ and edge set E, $n = |E|$. Let $A \in M_{m,n}$ be its vertex-edge incidence matrix. Thus, we order vertices and edges, and associate these with rows and columns of A, respectively. The column associated with the edge (v_i, v_j) has two nonzero entries: -1 in row v_i and 1 in row v_j. In particular, A is a $(0, \pm 1)$-matrix.

Recall that a *circulation* in G is a function $x : E \to \mathbb{R}$ satisfying

$$\sum_{e \in \delta^+(v)} x_e = \sum_{e \in \delta^-(v)} x_e \quad (v \in V)$$

where $\delta^+(v)$ (resp. $\delta^-(v)$) is the set of edges with v as initial vertex (resp. terminal vertex), and we write $x_e = x(e)$.

Proposition 2.7 *Let $A \in M_{m,n}$ be the vertex-edge incidence matrix of a directed graph G, as above. Then the following statements are equivalent:*

(*i*) $0_m^t \preceq^w A^t$;

(*ii*) *G has a nonzero nonnegative circulation;*

(*iii*) *G has a directed cycle.*

Proof. For $x \in \mathbb{R}^n$ the equation $x^t A^t = 0_m^t$ means that x is a circulation, and the equivalence of (i) and (ii) follows, since a nonzero nonnegative circulation x can be scaled so that $\sum_e x_e = 1$. Next, assume (ii) holds. Then $x_{e_1} > 0$ for some edge $e_1 = (v_i, v_j)$. Since x is a circulation there is an edge $e_2 = (v_j, v_k)$ with $x_{e_2} > 0$. We continue like this and eventually find a directed cycle C where each edge has a positive x-value, so (iii) holds. Finally, assume G has a directed cycle C. Define x by $x_e = 1$ for each $e \in C$ and $x_f = 0$ otherwise. Then x is a nonnegative nonzero circulation, and (ii) holds. □

We may always assume that columns of B are linearly independent, as described next.

Lemma 2.8 *Let $B \in M_{m,n}$, rank $B = r$ and let C be an $m \times r$ submatrix of B with rank $C = r$. Then $0_n^t \preceq^w B$ if and only if $0_r^t \preceq^w C$.*

Proof. Let $T \in M_n$ be such invertible matrix that $BT = [C\ O_{m,n-r}]$.

Assume first that $0_n^t \preceq^w B$. Then $0_n^t = x^t B$, where $x^t = [x_1\ x_2\ \cdots\ x_m]$, $\sum_i x_i = 1$ and $x_i \geq 0$ $(i \leq m)$. Then $0_n^t = 0_n^t T = x^t BT$. Thus $0_n^t \preceq^w BT = [C\ O_{m,n-r}]$. It follows that $0_r^t \preceq^w C$.

Next, assume that $0_r^t \preceq^w C$. Then $0_n^t \preceq^w [C\ O_{m,n-r}] = BT$. It follows that $0_n^t = 0_n^t T^{-1} \preceq^w BTT^{-1} = B$. □

3 Reductions and combinatorial results

We continue the study of the basic majorization $0_n^t \preceq^w B$, and the more general majorization $a^t \preceq^w B$. A goal is to show how these problems may be simplified.

The next lemma follows directly from the definition and our discussion above.

Lemma 3.1 *Let $B \in M_{m,n}$. Then $0_n^t \preceq^w B$ if and only if there exist nonnegative reals $x_1, x_2, \ldots, x_m$ such that $x_1 B_{(1)} + x_2 B_{(2)} + \cdots + x_m B_{(m)} = 0_n^t$ and $x_j > 0$ for some $j \leq m$.*

We shall call the following elementary row operations *positive*:

1. *Row interchange.* A row in the matrix can be interchanged with another row : $B_{(i)} \leftrightarrow B_{(j)}$.
2. *Positive row multiplication.* Each element in a row can be multiplied by a positive constant : $B_{(i)}$ becomes $\lambda B_{(i)}$, where $\lambda > 0$.
3. *Positive row addition.* A row can be replaced by the sum of that row and a positive multiple of another row: $B_{(i)}$ is replaced by $B_{(i)} + \lambda B_{(j)}$, where $\lambda > 0$.

Lemma 3.2 *Let $B \in M_{m,n}$ and let $B' \in M_{m,n}$ be obtained from B via positive elementary row operations. If $0_n^t \preceq^w B'$, then $0_n^t \preceq^w B$.*

Proof. For positive elementary row operations of types 1 and 2 the result clearly holds, so we need to show the result for a single positive row addition. Let B' be obtained from B by adding $\lambda B_{(j)}$ to i'th row of B, where $\lambda > 0$. Without loss of generality assume that $i = 1$ and $j = 2$.

Suppose that $0_n^t \preceq^w B'$. By Lemma 3.1 there exist nonnegative real numbers $x_1, x_2, \ldots, x_m$ such that $x_1(B_{(1)} + \lambda B_{(2)}) + x_2 B_{(2)} + \cdots + x_m B_{(m)} = 0_n^t$. Then $x_1 B_{(1)} + (x_2 + x_1\lambda) B_{(2)} + x_2 B_{(2)} + \cdots + x_m B_{(m)} = 0_n^t$, so, again by Lemma 3.1, $0_n^t \preceq^w B$. $\square$

Example 3.3 Let

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \overset{+}{\to} \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \overset{+}{\to} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \overset{+}{\to} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} = B'.$$

Here $0_n^t \preceq^w B'$ as B' contains a zero row. Then, by Lemma 3.2, $0_n^t \preceq^w B$. $\square$

As it was stated in Section 2, when investigating weak matrix majorization $A \preceq^w B$, it suffices to consider the case where the matrix A has a single row.

For $(0,\pm 1)$-vector $a = [a_1\ a_2\ \cdots\ a_n]^t$ define its "support" sets

$$\begin{array}{rcl} \mathrm{supp}^+(a) & = & \{j : a_j = 1\}, \\ \mathrm{supp}^-(a) & = & \{j : a_j = -1\}, \text{ and} \\ \mathrm{supp}^0(a) & = & \{j : a_j = 0\}. \end{array}$$

We use the same notation for row-vectors.

For a pair consisting of a vector $a \in \{0,\pm 1\}^n$ and a matrix $B \in M_{m,n}(0,\pm 1)$ we associate the following set of positive integers:

$$\mathcal{I}(B; a) = \{i \leq m : \mathrm{supp}^+(a) \subseteq \mathrm{supp}^+(B_{(i)}),\ \mathrm{supp}^-(a) \subseteq \mathrm{supp}^-(B_{(i)})\} \subseteq \mathbb{N}.$$

Remark 3.4 Observe that $i \in \mathcal{I}(B; a)$ if and only if for any $j \leq n$ with $a_j \neq 0$ we have $b_{ij} = a_j$. □

Remark 3.5 Let $a \in \{\pm 1\}^n$ and $B \in M_{m,n}(0,\pm 1)$. Then $\mathcal{I}(B; a) = \{i \leq m : B_{(i)} = a^t\}$. □

Lemma 3.6 *Let $a \in \{0,\pm 1\}^n$ and $B \in M_{m,n}(0,\pm 1)$. Let $\mathcal{I} = \mathcal{I}(B; a)$, $\mathcal{J} = supp^0(a)$. Then the following holds:*

1. *If $\mathcal{I} = \emptyset$, then $a^t \not\preceq^w B$.*
2. *If $\mathcal{I} \neq \emptyset$ and $\mathcal{J} = \emptyset$, then $a^t \preceq^w B$.*
3. *If $\mathcal{I} \neq \emptyset$ and $\mathcal{J} \neq \emptyset$, then $a^t \preceq^w B$ if and only if $0^t_{|\mathcal{I}|} \preceq^w B^{(\mathcal{J})}_{(\mathcal{I})}$.*

Proof. By the definition of weak majorization $a^t \preceq^w B$ if and only if $x^t B = a^t$, for some $x^t = [x_1\ x_2\ \cdots\ x_m]$ satisfying $\sum_i x_i = 1$ and $x_i \geq 0$ $(i \leq m)$. Observe that if $i \notin \mathcal{I} = \mathcal{I}(B; a)$, then the only possibility is $x_i = 0$. Indeed, to obtain 1 as a convex combination of numbers ≤ 1 we can use positive weights only for numbers being 1. Similarly, to obtain -1 as a convex combination of numbers ≥ -1 we can use positive weights only for numbers being -1. Thus $\mathcal{I}$ must be nonempty for $a^t \preceq^w B$ to hold, since otherwise $x = 0^t$. This proves Item 1.

If $\mathcal{J} = \emptyset$, then $a \in \{\pm 1\}^n$ and by Remark 3.5 $\mathcal{I}$ is nonempty if and only if a^t is a row of B. This proves Item 2.

If $\mathcal{I}$ and $\mathcal{J}$ are nonempty, then $a^t \preceq^w B$ if and only if a^t is a convex combination of rows of B indexed by $\mathcal{I}(B; a)$. It follows that $a^t \preceq^w B$ if and only if $0^t_{|\mathcal{I}|} \preceq^w B^{(\mathcal{J})}_{(\mathcal{I})}$. □

Example 3.7 Let $a = [1\ \ 1\ \ 0\ \ 0\ \ -1\ \ -1]$, and

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

Here $\mathcal{I} = \{1, 2, 3\}$ and $\mathcal{J} = \{3, 4\}$. Then $a \preceq^w B$ if and only if

$$[0\ \ 0] \preceq^w B_{(\mathcal{I})}^{(\mathcal{J})} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix},$$

and this majorization clearly holds, so $a \preceq^w B$. □

Lemma 3.8 *Let $B \in M_{m,n}$. If there exists $\mathcal{I} \subseteq \{1, 2, \ldots, m\}$ such that $\sum_{i\in\mathcal{I}} B_{(i)} = 0_n^t$, then $0_n^t \preceq^w B$.*

Proof. This is clear, as the assumption means that $x^t B = 0_n^t$ for $x^t = [x_1\ x_2\ \cdots\ x_m] \in \mathbb{R}^m$ given by $x_i = 1$ when $i \in \mathcal{I}$ and $x_i = 0$ otherwise. □

The lemma above is a rather weak sufficient condition in the general case. We note that even in the case $B \in M_{m,n}(0, \pm 1)$ this condition is not necessary, as the next example shows.

Example 3.9

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \preceq^w \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \preceq^w \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}.$$

□

In Lemma 3.8 one needs to find, if possible, a nonzero $(0, 1)$-vector x satisfying $x^t B = 0_n^t$. This leads, however, to a hard combinatorial problem.

Proposition 3.10 *The problem of deciding, for a given $B \in M_{m,n}$, if there exists a nonzero $(0, 1)$-vector x satisfying $x^t B = 0_n^t$ is* NP*-hard. The same is true when B is restricted to be a $(0, \pm 1)$-matrix.*

Proof. It suffices to prove the second statement. Let $B = [b_{ij}] \in M_{m,n}$ be a $(0, \pm 1)$-matrix whose last row is $-e$ and the submatrix B' obtained by deleting the last row is a $(0, 1)$-matrix with no zero row. Any $(0, 1)$-vector x with $x^t B = 0_n^t$ must satisfy $x_m = 1$. Indeed, otherwise $x_m = 0$ and $x_i = 1$ for some $i < m$, and then $x^t B \neq 0_n^t$ as B' is a $(0, 1)$-matrix and $B_{(i)} = B'_{(i)}$ is nonzero.

Thus, there exists a nonzero vector $x \in \{0, 1\}^m$ satisfying $x^t B = 0_n^t$ if and only if there exists a nonzero vector $x' \in \{0, 1\}^{m-1}$ satisfying $(x')^t B' = e$. The last statement is equivalent to deciding if some subclass of the support sets $S_i = \{j \leq n : b_{ij} = 1\}$ $(i < m)$ is a partition of $\{1, 2, \ldots, n\}$, and this is the general exact cover problem [14, Problem 14], [10, page 53], which is known to be *NP*-hard. □

However, the next lemma shows that if $0_n^t \preceq^w B$ for a rational-valued matrix $B \in M_{m,n}(\mathbb{Q})$, then we can always find a convex combination with rational coefficients. For more details on linear algebraic problems with rational data, see [17].

Lemma 3.11 *Let $B \in M_{m,n}(\mathbb{Q})$. Then $0_n^t \preceq^w B$ if and only if there exist nonnegative $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{Q}$ such that $\lambda_1 B_{(1)} + \lambda_2 B_{(2)} + \cdots + \lambda_m B_{(m)} = 0_n^t$ and $\lambda_j > 0$ for some $j \leq m$.*

Proof. This follows, e.g., from our earlier observation that finding an $x \geq 0_m$ such that $x^t B = 0_n^t$ and $\sum_i x_i = 1$ can be done by linear programming. When B is rational, an optimal basic feasible solution x is also rational. This follows from Cramer's rule. □

A natural question is to estimate the denominators of $x_1, x_2, \ldots, x_m \in \mathbb{Q}$ satisfying $0_n^t = x^t B$, $x^t e = 1$. In particular, in Example 3.9 we may use $x^t = [\frac{1}{2}\ \frac{1}{4}\ \frac{1}{4}]$ and $[\frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{2}]$, respectively. If there is a finite set of denominators, then there is a finite exhaustive search.

Let Δ_B denote the maximal possible absolute value of any minor in $\begin{bmatrix} B^t \\ e^t \end{bmatrix}$.

Lemma 3.12 *Let $B \in M_{m,n}(\mathbb{Z})$ and $a \in \mathbb{Z}^n$. If $a^t \preceq^w B$, then there exists an integer $\beta \in \{1, 2, \ldots, \Delta_B\}$ such that $a^t = \lambda_1 B_{(1)} + \lambda_2 B_{(2)} + \cdots + \lambda_m B_{(m)}$ for some $\lambda_i \in \{\frac{0}{\beta}, \frac{1}{\beta}, \ldots, \frac{\beta-1}{\beta}, 1\}$ with $\sum_{i=1}^{m} \lambda_i = 1$.*

Proof. We have $a^t \preceq^w B$ if and only if the linear programming problem (1) $\max\{e^t x \ :\ B^t x = a,\ e^t x = 1,\ x \geq 0_m\}$ has a feasible solution. Its feasible region is given by $\{Ax = b,\ x \geq 0_m\}$, where $A = \begin{bmatrix} B^t \\ e^t \end{bmatrix}$, $b = \begin{bmatrix} a \\ 1 \end{bmatrix}$.

If $\operatorname{rank} A < \operatorname{rank}[A\ b]$, then there are no feasible solutions. Otherwise we exclude redundant equations to assume without loss of generality that rows of $A = \begin{bmatrix} B^t \\ e^t \end{bmatrix}$ are linearly independent.

If this linear problem is feasible, then there exists a basic feasible solution $y \in \mathbb{R}^m$. By the definition of a basic feasible solution the vector z of nonzero coordinates of y is a solution to $Cx = \begin{bmatrix} a \\ 1 \end{bmatrix}$ for some nonsingular square submatrix C of $\begin{bmatrix} B^t \\ e^t \end{bmatrix}$. Let $\beta = |\det(C)|$. Observe that $\beta \in \mathbb{Z}$, $\beta > 0$ and $\beta \leq \Delta_B$. Then, due to the Cramer's rule, we have $y_i = \frac{\alpha_i}{\beta}$ for some $\alpha_i \in \mathbb{Z}$. Finally, conditions $e^t y = 1$ and $y_i \geq 0$ imply that $\alpha_i \in \{0, 1, \ldots, \beta\}$. ∎

If $B \in M_{m,n}(0, \pm 1)$, then we can estimate the maximal possible absolute value of minors of B. The following is the Hadamard determinant bound.

Theorem 3.13 *[11, Paragraphes 2–4] Let A be an $n \times n$ matrix with entries from the complex unit disk. Then $|\det(A)| \leq n^{n/2}$.*

This bound is attainable for Hadamard matrices. A *Hadamard matrix* is a square (± 1)-matrix whose rows are mutually orthogonal. The order of a Hadamard matrix must be 1, 2, or a multiple of 4 and it is conjectured that Hadamard matrices exist for all these orders. More detailed information can be found in a recent survey [1].

Corollary 3.14 *Let $B \in M_{m,n}(0, \pm 1)$. Then $\Delta_B \leq r^{r/2}$, where $r = \operatorname{rank} \begin{bmatrix} B^t \\ e^t \end{bmatrix}$.*

The following class of $(0, \pm 1)$-matrices plays an important role in combinatorial optimization since they give a quick way to verify that a linear program has an integral optimum, when any optimum exists.

Definition 3.15 *A totally unimodular matrix is a matrix for which every square submatrix has determinant 0 or ± 1. In particular, a totally unimodular matrix is a $(0, \pm 1)$-matrix.*

Specifically, if A is totally unimodular, and b is integral, then linear programs of forms like $\min\{c^t x : Ax \geq b,\ x \geq 0\}$ or $\max\{c^t x : Ax \leq b\}$ have integral optima, for any c, provided that b is integral and optimal solutions exist. Hence if A is totally unimodular and b is integral, then every extreme point of the feasible region (e.g., $\{x : Ax \leq b\}$) is integral and thus the feasible region is an integral polyhedron.

Totally unimodular matrices appear in many combinatorial problems and graph theory, see [17]. For example, the incidence matrices of directed graphs or undirected bipartite graphs are totally unimodular [2, Lemma 2.6].

Lemma 3.16 *Let $B \in M_{m,n}$ be a totally unimodular matrix. Then the maximal absolute value of all minors of* $\begin{bmatrix} B^t \\ e^t \end{bmatrix}$ *does not exceed rank* $\begin{bmatrix} B^t \\ e^t \end{bmatrix}$.

Proof. Let C be an arbitrary $r \times r$ submatrix of $\begin{bmatrix} B^t \\ e^t \end{bmatrix}$ for some $r \leq n, m$. If $\det(C) \neq 0$, then $r \leq \operatorname{rank} \begin{bmatrix} B^t \\ e^t \end{bmatrix}$.

If C is a submatrix of B, then $\det(C) \in \{0, \pm 1\}$ since B is totally unimodular. Otherwise we can expand $\det(C)$ along the row e^t to obtain that $-r \leq \det(C) \leq r$. □

The following example shows that the upper bound in Lemma 3.16 is attained.

Example 3.17 It is easy to see that the matrices $\begin{bmatrix} 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ are totally unimodular. Direct computations show that $\det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = 2 = \operatorname{rank} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = 3 = \operatorname{rank} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$. □

We can generalize this example to the matrices of arbitrary size in the following way.

Lemma 3.18 *Let $n \in \mathbb{N}$ and define $B_n = \begin{bmatrix} I_{n-1} & -e \end{bmatrix} \in M_{n-1,n}$. Then the following holds:*

1. *B_n is totally unimodular.*

2. $\det \begin{bmatrix} B_n \\ e^t \end{bmatrix} = n$.

Proof. 1. Consider arbitrary square submatrix $B' = B_n{}^{(\mathcal{J})}_{(\mathcal{I})}$ of B_n, where $\mathcal{I} \in \{1, 2, \dots, n-1\}$ and $\mathcal{J} \in \{1, 2, \dots, n\}$ with $|\mathcal{I}| = |\mathcal{J}|$. If $n \notin \mathcal{J}$, then B' is a submatrix of I_{n-1} and therefore $\det(B') \in \{0, 1\}$.

Assume that $n \in \mathcal{J}$. In this case there exists $i \in \mathcal{I}$ such that $i \notin \mathcal{J}$. As a consequence, some row of B' is $-e^t_{|\mathcal{I}|} = \begin{bmatrix} 0 & \cdots & 0 & -1 \end{bmatrix}$. Then expansion along this row implies that $\det(B') \in \{\pm \det(B_n{}^{(\mathcal{J}\setminus\{n\})}_{(\mathcal{I}\setminus\{i\})})\} \in \{0, \pm 1\}$ since $B_n{}^{(\mathcal{J}\setminus\{n\})}_{(\mathcal{I}\setminus\{i\})}$ is a submatrix of I_{n-1}.

2. If we subtract first $n-1$ rows of $\begin{bmatrix} B_n \\ e^t \end{bmatrix}$ from the last one, we obtain that $\det \begin{bmatrix} B_n \\ e^t \end{bmatrix} = \det \begin{bmatrix} I_{n-1} & -e \\ 0 & n \end{bmatrix} = n.$ ∎

Applying Lemma 3.16 to the result of Lemma 3.12 we obtain the following.

Corollary 3.19 *Let $B \in M_{m,n}$ be a totally unimodular matrix and $a \in \mathbb{Z}^n$. Let $r = rank \begin{bmatrix} B^t \\ e^t \end{bmatrix}$. Then $a^t \preceq^w B$ implies that there exists $\beta \in \{1, 2, \dots, r\}$ such that $a^t = \lambda_1 B_{(1)} + \lambda_2 B_{(2)} + \cdots + \lambda_m B_{(m)}$ for some $\lambda_i \in \{\frac{0}{\beta}, \frac{1}{\beta}, \dots, \frac{\beta-1}{\beta}, 1\}$ with $\sum_{i=1}^{m} \lambda_i = 1$.*

Lemma 3.20 *Let $B \in M_{m,n}$ be a totally unimodular matrix and let $a \in \{0, \pm 1\}^n$. Then the problem of determining whether $a^t \preceq^w B$ holds can be reduced to a linear program with a totally unimodular coefficient matrix.*

More specifically, there exists a totally unimodular $k \times l$ submatrix A of B such that $a^t \preceq^w B$ if and only if the linear program

$$\max\{e^t x : A^t x = 0_l,\ x \geq 0_k\ \}$$

has unbounded optimal value.

Proof. Lemma 3.6 provides a way to find such $k \times l$ submatrix A of B that $a^t \preceq^w B$ if and only if $0_l^t \preceq^w A$. Note that A is totally unimodular as a submatrix of a totally unimodular matrix B.

Finally, by Lemma 2.6 $0_l^t \preceq^w A$ if and only if the linear program $\max\{e^t x : A^t x = 0_l,\ x \geq 0_k\ \}$ has unbounded optimal value. ∎

Lemma 3.21 *Let $B \in M_{m,n}(0, \pm 1)$. Let $j \leq n$ be such that $B^{(j)} \neq \pm e$ and $B^{(j)}$ contains no zeros. Then the following conditions are equivalent.*

1. $0_n^t \preceq^w B$.

2. There exists $a \in \{\pm 1\}^n$ *such that*

$$a^t \preceq^w B_{(supp^+(B^{(j)}))} \text{ and } -a^t \preceq^w B_{(supp^-(B^{(j)}))}.$$

Proof. Define $\mathcal{I} = \text{supp}^+(B^{(j)}) = \{i \le m : b_{ij} = 1\}$. Observe that the sets $\text{supp}^+(B^{(j)})$ and $\text{supp}^-(B^{(j)})$ are nonempty by the choice of $B^{(j)}$.

Assume that Item 2. holds. Then $a^t = \sum_{i\in\mathcal{I}} \alpha_i B_{(i)} = -\sum_{s\notin\mathcal{I}} \alpha_s B_{(s)}$ for some $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ with $\alpha_i \ge 0$ and $\sum_{i\in\mathcal{I}} \alpha_i = \sum_{s\notin\mathcal{I}} \alpha_s = 1$.

Then

$$\frac{1}{2}\sum_{i=1}^{m} \alpha_i = 1 \text{ and } \frac{1}{2}\sum_{i\in\mathcal{I}} \alpha_i B_{(i)} + \frac{1}{2}\sum_{s\notin\mathcal{I}} \alpha_s B_{(s)} = \frac{1}{2}(a^t - a^t) = 0_n^t.$$

Thus $0_n^t \preceq^w B$.

Assume that Item 1. holds. Then $0_n^t = \sum_i \alpha_i B_{(i)}$ for some $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ with $\alpha_i \ge 0$ and $\sum_i \alpha_i = 1$. In particular, $0 = \sum_i \alpha_i b_{ij} = \sum_{i\in\mathcal{I}} \alpha_i - \sum_{s\notin\mathcal{I}} \alpha_s$. It follows that $\sum_{i\in\mathcal{I}} \alpha_i = \sum_{s\notin\mathcal{I}} \alpha_s = \frac{1}{2}$.

Also

$$0_n^t = \sum_{i=1}^{m} \alpha_i B_{(i)} = \sum_{i\in\mathcal{I}} \alpha_i B_{(i)} + \sum_{s\notin\mathcal{I}} \alpha_s B_{(s)}.$$

Thus $\sum_{i\in\mathcal{I}} \alpha_i B_{(i)} = -\sum_{s\notin\mathcal{I}} \alpha_s B_{(s)}$. Let $a^t = 2\sum_{i\in\mathcal{I}} \alpha_i B_{(i)} = -2\sum_{s\notin\mathcal{I}} \alpha_s B_{(s)}$.

Finally, $a^t \preceq^w B_{(\text{supp}^+(B^{(j)}))}$ and $-a^t \preceq^w B_{(\text{supp}^-(B^{(j)}))}$ by Proposition 1.1. Also $a \in \{\pm 1\}^n$. □

4 Criteria based on the number of rows

Lemma 4.1 *Let* $B \in M_{m,n}(0, \pm 1)$ *be such that rows of* B *are distinct and* $m > (3^n - 1)/2$*. Then* $0_n^t \preceq^w B$.

Proof. If 0_n^t is a row in B, then clearly $0_n^t \preceq^w B$. Assume next that 0_n^t is not a row in B. The set S of nonzero $(0, \pm 1)$-vectors of length n has cardinality $3^n - 1$, and it is symmetric in the sense that, for each $x \in S$ we have $-x \in S$. So S may be partitioned as $S = S_1 \cup S_2$ where S_1 and S_2 are disjoint, each set has cardinality $(1/2)(3^n - 1)$ and $S_2 = \{-x : x \in S_1\}$. As $m \geq (1/2)(3^n - 1)$, B must contain some $x \in S_1$ and also $-x \in S_2$. But then $0_n^t = (1/2)x + (1/2)(-x)$, so $0_n^t \preceq^w B$. □

For instance, if $n = 2$ and $B \in M_{m,2}(0, \pm 1)$ has at least 5 distinct rows, then $0_2^t \preceq^w B$. This bound is attained as the following example shows.

Example 4.2 Let $n = 2$. Then for $m = (3^n - 1)/2 = 4$ we consider $B_2 \in M_{4,2}(0, \pm 1)$ having the rows $[1 \;\; 1]$, $[1 \;\; 0]$, $[1 \;\; -1]$, $[0 \;\; 1]$. Then 0_2^t is not a convex combination of these points, so $0_2^t \not\preceq^w B_2$.

Let $n = 3$. Then for $m = (3^n - 1)/2 = 13$ we consider

$$B_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 \end{bmatrix}^t \in M_{13,3}.$$

Then all 13 rows of B_3 are distinct and $[0 \;\; 0 \;\; 0] \not\preceq^w B_3$. □

Let $n \geq 1$ and define

$$\hat{\gamma}(n) = \max\{m : 0_n^t \not\preceq^w B \text{ for some } B \in M_{m,n}(0, \pm 1) \text{ with distinct rows}\}.$$

We generalize Example 4.2 in order to compute $\hat{\gamma}(n)$ exactly and show that the bound from Lemma 4.1 is attained for all $n \geq 1$.

Theorem 4.3 *For each $n \geq 1$ it holds that $\hat{\gamma}(n) = (3^n - 1)/2$.*

Proof. By Lemma 4.1, $\hat{\gamma}(n) \leq (3^n - 1)/2$. Thus we only need to find a matrix $B \in M_{(3^n-1)/2,n}$ with distinct rows such that $0_n^t \not\preceq^w B$.

Let $B_1 = [1] \in M_{1,1}$. Observe that $(3^1 - 1)/2 = 1$ and $0 \not\preceq^w B_1$. For $i = 2, 3, \ldots, n$ we construct $B_i \in M_{(3^i-1)/2,i}$ in the following way. Let the first 3^{i-1} rows of B_i be all $(0, \pm 1)$-vectors x of length i with $x_1 = 1$. Indeed, there are 3^{i-1} of them. Let the last $(3^{i-1} - 1)/2$ rows of B_i be the rows of the matrix $\left[0_{\frac{3^{i-1}}{2}} \;\; B_{i-1}\right]$. Then the total number of rows in B_i is $3^{i-1} + (3^{i-1} - 1)/2 = (3^i - 1)/2$.

We show that $0_i^t \not\preceq^w B_i$. For fixed i define $N = (3^i - 1)/2$. Assume that $x^t B_i = 0_i^t$, where $x^t = [x_1 \; x_2 \; \cdots \; x_N]$, $x_j \geq 0$ and $\sum_j x_j = 1$. Then $x_1 = x_2 = \cdots = x_{3^{i-1}} = 0$; otherwise $(xB_i)_1 > 0$. Thus $0_i^t \preceq^w [0_{i-1} \; B_{i-1}]$ and that is equivalent to $0_{i-1}^t \preceq^w B_{i-1}$, a contradiction.

Finally, $0_n^t \not\preceq^w B_n$, $B_n \in M_{(3^n-1)/2,n}$ and the rows of B_n are distinct. It follows that $\hat{\gamma}(n) = (3^n - 1)/2$. □

The argument in Lemma 4.1 and Theorem 4.3 can be further generalized.

Lemma 4.4 *Let $\mathcal{K} \subseteq \mathbb{R}$ be such that $0 \in \mathcal{K}$, $-\mathcal{K} = \mathcal{K}$ and $k := |\mathcal{K}| < \infty$. Let $B \in M_{m,n}(\mathcal{K})$ be such that the rows of B are distinct and $m > (k^n - 1)/2$. Then $0_n^t \preceq^w B$.*

Proof. If 0_n^t is a row in B, then clearly $0_n^t \preceq^w B$. Assume next that 0_n^t is not a row in B. The set S of nonzero $\mathcal{K}$-vectors of length n has cardinality $k^n - 1$, and it is symmetric in the sense that, for each $x \in S$ we have $-x \in S$. So S may be partitioned as $S = S_1 \cup S_2$ where S_1 and S_2 are disjoint, each set has cardinality $(1/2)(k^n - 1)$ and $S_2 = \{-x : x \in S_1\}$. As $m > (1/2)(k^n - 1)$, B must contain some $x \in S_1$ and also $-x \in S_2$. But then $0_n^t = (1/2)x + (1/2)(-x)$, so $0_n^t \preceq^w B$. □

Let $n \geq 1$ and $\mathcal{K} \subseteq \mathbb{R}$. We define

$$\hat{\gamma}_{\mathcal{K}}(n) = \max\{m : 0_n^t \not\preceq^w B \text{ for some } B \in M_{m,n}(\mathcal{K}) \text{ with distinct rows}\}.$$

Theorem 4.5 *Let $\mathcal{K} \subseteq \mathbb{R}$ be such that $0 \in \mathcal{K}$, $-\mathcal{K} = \mathcal{K}$ and $k := |\mathcal{K}| < \infty$. Then $\hat{\gamma}_{\mathcal{K}}(n) = (k^n - 1)/2$ for any $n \geq 1$.*

Proof. By Lemma 4.4 we obtain that $\hat{\gamma}(n) \leq (k^n - 1)/2$. Thus we only have to find such matrix $B \in M_{(k^n-1)/2,n}$ that rows of B are distinct and $0_n^t \not\preceq^w B$.

Let $B_1 \in M_{\frac{k-1}{2},1}$ be such that rows of B_1 are all distinct positive elements of $\mathcal{K}$. Then indeed B_1 contains $(k-1)/2$ distinct rows and $0 \not\preceq^w B_1$. For $i = 2, 3, \ldots, n$ let us construct $B_i \in M_{(k^i-1)/2,i}$ the following way. Let the first $\frac{k-1}{2}k^{i-1}$ rows of B_i be all $\mathcal{K}$-vectors x with $x_1 > 0$. Indeed, there are $\frac{k-1}{2}k^{i-1}$ of them. Let the last $\frac{k^{i-1}-1}{2}$ rows of B_i be the rows of the matrix $\begin{bmatrix} 0_{\frac{3^{i-1}}{2}} & B_{i-1} \end{bmatrix}$. Then the total number of rows of B_i is $\frac{k-1}{2}k^{i-1} + \frac{k^{i-1}-1}{2} = \frac{k^i-1}{2}$.

We show that $0_i^t \not\preceq^w B_i$. Let $N = \frac{k^i-1}{2}$. Assume that $x^t B_i = 0_i^t$, where $x = [x_1 \ x_2 \ \cdots \ x_N]$, $x_j \geq 0$ and $\sum_j x_j = 1$. Then $x_1 = \cdots = x_{k^{i-1}} = 0$. Indeed, otherwise $([x_1 \ x_2 \ \cdots \ x_N]B_i)_1 > 0$. Thus $0_i^t \preceq^w [0_{i-1} \ B_{i-1}]$ and that is equivalent to $0_{i-1}^t \preceq^w B_{i-1}$, a contradiction.

Finally, $0_n^t \not\preceq^w B_n$, $B_n \in M_{(k^n-1)/2,n}$ and rows of B_n are distinct. It follows that $\hat{\gamma}_{\mathcal{K}}(n) = (k^n - 1)/2$. □

We illustrate the main construction in the above proof by the following example.

Example 4.6 For $n = 2$ and $\mathcal{K} = \{0, \pm 1, \pm 2\}$ let

$$B = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -2 & -1 & 0 & 1 & 2 & -2 & -1 & 0 & 1 & 2 & 1 & 2 \end{bmatrix}^t \in M_{12,2}(\mathcal{K}).$$

Then B has $12 = (5^2 - 1)/2$ distinct rows and it is straightforward to check that $[0 \;\; 0] \not\preceq^w B$. □

5 Strong and directional majorization

We now turn to other majorization results, both for other types of majorization and for more general matrices.

Lemma 5.1 *Let $A, B \in M_{m,n}$ such that $A \preceq^d B$. Then $e^t A = e^t B$.*

Proof. If $A \preceq^d B$, then for every $j \in \{1, 2, \ldots, n\}$ $A^{(j)} = Ae_j \preceq Be_j = B^{(j)}$. Hence by the definition of vector majorization $e^t A^{(j)} = e^t B^{(j)}$. □

Lemma 5.2 *Let $B \in M_{m,n}$. Then the following are equivalent.*

1. $O_{m,n} \preceq^s B$.
2. $O_{m,n} \preceq^d B$.
3. $e^t B = 0_n^t$.

Proof. Assume that $O_{m,n} \preceq^s B$. Then $O_{m,n} \preceq^d B$ and, from this, $e^t O_{m,n} = 0_n^t = e^t B$ by Lemma 5.1. Assume that $e^t B = 0_n^t$. Then $O_{m,n} = \frac{1}{m} JB$ and $O \preceq^s B$. □

Lemma 5.3 *Let $A, B \in M_{m,n}(\pm 1, 0)$ and $A \preceq^w B$. Then the following holds:*

1. *If $B \in M_{m,n}(0, 1)$, then $A \in M_{m,n}(0, 1)$.*
2. *If $B \in M_{m,n}(0, -1)$, then $A \in M_{m,n}(0, -1)$.*

Proof. This follows from Proposition 1.1. □

The next example shows that a similar result to Lemma 5.3 does not hold for (± 1)-matrices.

Example 5.4

$$\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \preceq^w \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

□

Recall that for $x \in \{0, \pm 1\}^n$ and $A \in M_{m,n}(0, \pm 1)$ we defined

$$\mathcal{I}(A; x) = \{i \le m : \operatorname{supp}^+(x) \subseteq \operatorname{supp}^+(A_{(i)}),\ \operatorname{supp}^-(x) \subseteq \operatorname{supp}^-(A_{(i)})\}.$$

- For $x \in \{0, \pm 1\}^n$ and $A \in M_{m,n}(0, \pm 1)$ we denote $x_A^{\simeq} = |\mathcal{I}(A; x)|$.
- For $x \in \mathbb{R}^n$ and $A \in M_{m,n}$ let $x_A^{=} = |\{i \le m : A_{(i)} = x^t\}|$, the number of rows in A that are equal to x^t.

We use the same notations $\mathcal{I}(A; x)$, $x_A^{=}$ and $x_A^{\simeq}$ for a row-vector x.

Remark 5.5 *Let $x \in \{0, \pm 1\}^n$ and $A \in M_{m,n}(0, \pm 1)$. Then*

- $x_A^{\simeq} = |\{i \le m : \text{for all } j \le n \text{ either } x_j = 0 \text{ or } a_{ij} = x_j\}|$.
- *If $x \in \{\pm 1\}^n$, then $x_A^{\simeq} = x_A^{=}$.*
- *If $x = 0_n$, then $x_A^{\simeq} = m$.*

Example 5.6 *Let* $x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.

Here $x_A^{=} = |\{1\}| = 1$ and $x_A^{\simeq} = |\{1, 2, 3\}| = 3$.

We now show that the number $x_A^{=}$ plays a role for directional majorization, in a necessary condition.

Lemma 5.7 *Let $A, B \in M_{m,n}(0, \pm 1)$ and $A \preceq^d B$. Let $x \in \{\pm 1\}^n$. Then $x_A^{=} \le x_B^{=}$.*

Proof. Let y be a $(0, \pm 1)$-vector of length n. It is straightforward to see that $y^t x \le x^t x = n$ and $y^t x = x^t x = n$ if and only if $x = y$. It follows that the maximal possible entries of Ax and Bx are n and the number of such entries in Ax (respectively, Bx) is precisely $x_A^{=}$ (respectively, $x_B^{=}$). Since $Ax \preceq Bx$, we may conclude that $x_A^{=} \le x_B^{=}$. Indeed, otherwise $x_A^{=} > 0$ and $\sum_{i=1}^{x_A^{=}} (Ax)_{[i]} > \sum_{i=1}^{x_A^{=}} (Bx)_{[i]}$, a contradiction. □

Recall that the set of all $m \times m$ permutation matrices is denoted by $P(m)$.

Corollary 5.8 *Let $A \in M_{m,n}(\pm 1)$ and $B \in M_{m,n}(0, \pm 1)$. Then $A \preceq^d B$ if and only if $A = PB$ for some $P \in P(m)$.*

The following example shows that the inequality in Lemma 5.7 can be strict for $(0, \pm 1)$-vectors.

Example 5.9 Let $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Then $(\frac{1}{2}J_2)B = O$. In particular, $O \preceq^s B$ and, as a consequence, $O \preceq^d B$. Let $x = \begin{bmatrix} 1 & -1 \end{bmatrix}$. Then $x_O^{=} = 0 < x_B^{=} = 1$. □

Remark 5.10 Example 5.9 shows that Corollary 5.8 may not hold if $A \in M_{m,n}(0, 1)$ or $A \in M_{m,n}(0, -1)$. □

It turns out that Lemma 5.7 remains true for $(0, \pm 1)$-vectors if we substitute $x^{=}$ by $x^{\simeq}$.

Lemma 5.11 *Let $A, B \in M_{m,n}(0, \pm 1)$ and $A \preceq^d B$. Let $x \in \{0, \pm 1\}^n$. Then $x_A^{\simeq} \leq x_B^{\simeq}$.*

Proof. Let $y \in \{0, \pm 1\}^n$. It is straightforward to see that $y^t x \leq x^t x$ and $y^t x = x^t x$ if and only if for any $k \leq n$ $x_k \neq 0$ implies that $y_k = x_k$. It follows that the maximal possible entry of Ax and Bx is $x^t x$ and the number of such entries in Ax (respectively, Bx) is precisely $x_A^{\simeq}$ (respectively, $x_B^{\simeq}$). Since $Ax \preceq Bx$, we may conclude that $x_A^{\simeq} \leq x_B^{\simeq}$. Indeed, otherwise $x_A^{\simeq} > 0$ and $\sum_{i=1}^{x_A^{\simeq}} (Ax)_{[i]} > \sum_{i=1}^{x_A^{\simeq}} (Bx)_{[i]}$, a contradiction. □

The same is true for strong majorization since $A \preceq^s B$ implies $A \preceq^d B$:

Corollary 5.12 *Let $A, B \in M_{m,n}(0, \pm 1)$ and $A \preceq^s B$. Let $x \in \{0, \pm 1\}^n$. Then $x_A^{\simeq} \leq x_B^{\simeq}$.*

The following lemma allows us to reduce the problem for strong majorization to the case $x_A^{\simeq} < x_B^{\simeq}$ for any x with $x_A^{\simeq} > 0$. This is a strong condition, especially together with the equality $e^t A = e^t B$.

For $A \in M_{n_1}$ and $B \in M_{n_2}$ the matrix $A \oplus B$ denotes the direct sum of the matrices, i.e., the block-diagonal matrix

$$\begin{bmatrix} A & O \\ O & B \end{bmatrix} \in M_{n_1 + n_2}.$$

Lemma 5.13 *Let $x \in \{0, \pm 1\}^n$ and $A, B \in M_{m,n}(0, \pm 1)$. Assume that $m > x_{\widetilde{A}} = x_{\widetilde{B}} > 0$. Then $A \preceq^s B$ if and only if*

$$A_{(\mathcal{I}(A;x))} \preceq^s B_{(\mathcal{I}(B;x))} \text{ and } A_{(\backslash \mathcal{I}(A;x))} \preceq^s B_{(\backslash \mathcal{I}(B;x))}.$$

Proof. Let $k = x_{\widetilde{A}}$. Without loss of generality we may assume that $\mathcal{I}(A;x) = \mathcal{I}(B;x) = \{1, 2, \dots, k\}$.

Assume that $A_{(\mathcal{I}(A;x))} \preceq^s B_{(\mathcal{I}(B;x))}$ and $A_{(\backslash \mathcal{I}(A;x))} \preceq^s B_{(\backslash \mathcal{I}(B;x))}$. It follows that $A_{(\mathcal{I}(A;x))} = Q_1 B_{(\mathcal{I}(B;x))}$ and $A_{(\backslash \mathcal{I}(A;x))} = Q_2 B_{(\backslash \mathcal{I}(B;x))}$ for some $Q_1 \in \Omega_k$ and $Q_2 \in \Omega_{m-k}$. Then $A = (Q_1 \oplus Q_2)B$ and $Q_1 \oplus Q_2 \in \Omega_m$.

Assume that $A \preceq^s B$. Then $A = QB$ for some $Q \in \Omega_m$. We show that Q is block-diagonal. Consider an arbitrary $i \leq k$. Then $A_{(i)} = \sum_{s=1}^{m} q_{is} B_{(s)}$.

Let $l \in \{k+1, k+2, \dots, m\}$. Then $B_{(l)} \notin \mathcal{I}(B;x)$. It follows that for some $j \leq n$ we have $x_j \neq 0$ and $x_j \neq b_{lj}$. We show that $q_{il} = 0$. Without loss of generality assume that $x_j = 1$. If $x_j = -1$, then the same arguments apply. Thus $a_{ij} = 1$ since $i \in \mathcal{I}(A;x)$ and $b_{lj} \leq 0$. It follows that

$$a_{ij} = \sum_{s=1}^{m} q_{is} b_{sj} = \sum_{s=1, s \neq l}^{m} q_{is} b_{sj} + q_{il} b_{lj} \leq \sum_{s=1, s \neq l}^{m} q_{is} + q_{il} b_{lj}.$$

If $q_{il} \neq 0$, then $\sum_{s=1, s \neq l}^{m} q_{is} < 1$ and $a_{ij} \leq \sum_{s=1, s \neq l}^{m} q_{is} + q_{il} b_{lj} \leq \sum_{s=1, s \neq l}^{m} q_{is} < 1$, a contradiction.

Finally, for any $i \leq k$ we have $\sum_{s=1}^{k} q_{is} = 1$. It follows that $\sum_{i=1}^{k} \sum_{s=1}^{k} q_{is} = k$, that is, the top-left $k \times k$-submatrix Q_1 of Q is doubly stochastic. Hence the bottom-right $(m-k) \times (m-k)$-submatrix Q_2 of Q is also doubly stochastic and $Q = Q_1 \oplus Q_2$. It follows that $A_{(\mathcal{I}(A;x))} = Q_1 B_{(\mathcal{I}(B;x))}$ and $A_{(\backslash \mathcal{I}(A;x))} = Q_2 B_{(\backslash \mathcal{I}(B;x))}$ for $Q_1 \in \Omega_k$ and $Q_2 \in \Omega_{m-k}$. □

Example 5.14 Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Then $B \not\preceq^s A$ because for $x = [1 \ \ 1 \ -1]$ we have $x_{\widetilde{B}} = 1 \not\leq x_{\widetilde{A}} = 0$.

Consider $x = [1 \;\; 0 \;\; -1]$. Then $x_{\widetilde{A}} = x_{\widetilde{B}} = 2$. Thus, by Lemma 5.13, $A \preceq^s B$ if and only if

$$A_{(\mathcal{I}(A;x))} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \preceq^s B_{(\mathcal{I}(B;x))} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

and

$$A_2 := A_{(\backslash \mathcal{I}(A;x))} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \preceq^s B_2 := B_{(\backslash \mathcal{I}(B;x))} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Consider $x = [-1 \;\; 0 \;\; 0]$. Then $x_{\widetilde{A_2}} = x_{\widetilde{B_2}}$. Thus $A_2 \preceq B_2$ if and only if $[-1 \;\; 0 \;\; 0] \preceq^s [-1 \;\; 0 \;\; 1]$ and $[0 \;\; 0 \;\; 1] \preceq^s [0 \;\; 0 \;\; 0]$. The latter does not hold. Therefore, $A \not\preceq^s B$. □

Lemma 5.15 *Let $A, B \in M_{m,n}$. Assume that $A^{(j)} = B^{(j)} = \alpha e$ for some $j \leq n$ and $\alpha \in \mathbb{R}$. Then $A \preceq^s B$ if and only if $A^{(\backslash\{j\})} \preceq^s B^{(\backslash\{j\})}$.*

Proof. Let $Q \in \Omega_m$. Then $A = QB$ if and only if $A^{(\backslash\{j\})} = QB^{(\backslash\{j\})}$. □

Lemma 5.15 allows us to assume in Lemma 5.13 that, for $x \neq 0$, $x_{\widetilde{A}} \neq m$. Indeed, assume that $x_{\widetilde{A}} = m$. If $x_{\widetilde{B}} < m$, then $A \not\preceq^s B$ by Corollary 5.12. Otherwise $x_{\widetilde{A}} = x_{\widetilde{B}} = m$ and for any $j \leq n$ with $x_j \neq 0$ we have $A^{(j)} = B^{(j)} = x_j e$.

Classical majorization for $(0, \pm 1)$-vectors can be characterized as follows.

Lemma 5.16 *Let $a, b \in \{0, \pm 1\}^n$. Then $a \preceq b$ if and only if $e^t a = e^t b$ and $|supp^+(a)| \leq |supp^+(b)|$.*

Proof. Define $s_a = |\text{supp}^+(a)|$ and $s_b = |\text{supp}^+(b)|$. If $a \preceq b$, then $e^t a = e^t b$ by the definition of vector majorization, and $s_a \leq s_b$ by Lemma 5.7.

Conversely, assume that $e^t a = e^t b$ and $s_a \leq s_b$. Then $\sum_{i=1}^{k} a_{[i]} \leq \sum_{i=1}^{k} b_{[i]}$ for any $k = \{1, \ldots, n\}$ and $a \preceq b$ by the definition. □

The following example illustrates Lemma 5.16.

Example 5.17

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \preceq \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \preceq \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

□

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Example 5.18 Consider

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ -1 & 1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix}.$$

Here $e^tA = e^tB$ and $x_A < x_B$ for any nonzero $x \in \{0, \pm 1\}^m$ with $x_A \neq 0$. In this case $A \preceq^s B$ since $A = (I_{2,2} \oplus \frac{1}{4} J_{4,4})B$. □

Theorem 5.19 *[16, Theorem 3.9], [6, Corollary 3.5] Let $A, B \in M_{m,n}$. Then $A \preceq^s B$ if and only if for every convex function $f : V \to \mathbb{R}$ we have*

$$\sum_{j=1}^{m} f(A_{(j)}) \leq \sum_{j=1}^{m} f(B_{(j)}),$$

where $V \subseteq \mathbb{R}^n$ is a convex set such that $\mathcal{R}(A) \cup \mathcal{R}(B) \subseteq V$. Here the row space of a matrix is considered as a subspace of $\mathbb{R}^n$.

Corollary 5.20 *Let $A, B \in M_{m,n}$. Assume that $A_{(i)} = B_{(i')}$ for some $i, i' \in \{1, 2, \ldots, m\}$. Then $A \preceq^s B$ if and only if $A_{(\backslash\{i\})} \preceq^s B_{(\backslash\{i'\})}$.*

Example 5.21 Consider

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Then $A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} B$ and $A \preceq^s B$.

Observe that $e^tA = e^tB$ and $x_A < x_B$ for any nonzero $x \in \{0, \pm 1\}$ with $x_A \neq 0$. □

Example 5.22 Consider

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Here for $i = 1, 2$ $e^t A_i = e^t B_i$ and $x_{A_i} < x_{B_i}$ for any nonzero $x \in \{0, \pm 1\}^m$ with $x_{A_i} \neq 0$. Also, no row of A_i is a row of B_i. But $A_i \not\prec^s B_i$ because the row $[1 \;\; 0] \in \mathcal{R}(A_i)$ does not lie in $\mathrm{conv}(\mathcal{R}(B_i))$. □

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