Journal of Functional Analysis 283 (2022) 109620

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Full Length Article

Homogenization of stochastic conservation laws with multiplicative noise

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A R T I C L E I N F O

Article history:
Received 6 April 2021
Accepted 24 June 2022
Available online 8 July 2022
Communicated by Guido De Philippis

MSC:
35L65
35B27
60H15

Keywords:
Stochastic conservation laws
Homogenization
Two-scale Young measures

A B S T R A C T

We consider the generalized almost periodic homogenization problem for two different types of stochastic conservation laws with oscillatory coefficients and multiplicative noise. In both cases the stochastic perturbations are such that the equation admits special stochastic solutions which play the role of the steady-state solutions in the deterministic case. Specially in the second type, these stochastic solutions are crucial elements in the homogenization analysis. Our homogenization method is based on the notion of stochastic two-scale Young measure, whose existence is established here.

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[1] H. Frid gratefully acknowledges the support from CNPq, through grant proc. 305097/2019-9, and FAPERJ, through grant proc. E-26/202.900-2017.
[2] K. H. Karlsen gratefully acknowledges the support from the Research Council of Norway through the project Stochastic Conservation Laws (250674).
[3] D. Marroquin thankfully acknowledges the support from CNPq, through grant proc. 150118/2018-0.

https://doi.org/10.1016/j.jfa.2022.109620
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1. Introduction

We consider two very representative homogenization problems for conservation laws subjected to a stochastic perturbation by a multiplicative noise.

The first problem we consider is the one of the nonlinear transport equation whose deterministic case was first addressed in [24], in the periodic case, and later on in [2,30] in the almost periodic, Fourier-Stieltjes algebras cases, respectively. See also [15,55]. The equation is the following

$$du^\varepsilon + a\left(\frac{x}{\varepsilon}\right)\cdot\nabla_x f(u^\varepsilon)\,dt = \kappa_0\,\sigma(u^\varepsilon)\,dW + \frac{1}{2}\kappa_0^2\,h(u^\varepsilon)\,dt, \tag{1.1}$$

where W is a scalar Brownian motion, dW denotes Itô differential, $a(y) \in (\mathrm{Lip}\cap\mathcal{A})\,(\mathbb{R}^d)^d$ satisfies $\nabla_y\cdot a(y) = 0$, $\mathcal{A}(\mathbb{R}^d)$ is a general ergodic algebra, a concept whose definition we recall subsequently, $f,\sigma,h:\mathbb{R}\to\mathbb{R}$ are smooth functions, with σ and h satisfying $h = \sigma'\sigma$. We also assume that $f',\sigma',h' \in L^\infty(\mathbb{R})$, and $\sigma \ge \delta_0 > 0$. We further assume that the set of zeros of f' has measure zero, namely, $|\{u\in\mathbb{R} : f'(u) = 0\}| = 0$.

Note that by the well-known conversion formula between Stratonovich and Itô differentials (see, e.g., [5]) equation (1.1) may be written as

$$du^\varepsilon + a\left(\frac{x}{\varepsilon}\right)\cdot\nabla_x f(u^\varepsilon)\,dt = \kappa_0\,\sigma(u^\varepsilon)\circ dW,$$

where $\circ dW$ denotes integration in the Stratonovich sense.

The initial condition is given by

$$u^\varepsilon(0,x) = U_0\left(x,\frac{x}{\varepsilon}\right), \tag{1.2}$$

where $U_0(x,y) \in L^\infty(\mathbb{R}^d;\mathcal{A}(\mathbb{R}^d))$. Although we study the homogenization problems here in the general context of ergodic algebras, the results established in this paper are new even in the context of periodic homogenization. So, the reader not familiarized with the concept of ergodic algebras may, in a first reading, just assume the periodic case.

The concept of ergodic algebra was introduced in [57] (see also [40]), motivated by algebras generated by typical realizations of stationary ergodic processes and their selfaveraging property provided by Birkhoff theorem. Namely, an ergodic algebra is an algebra $\mathcal{A}(\mathbb{R}^d)$ of bounded uniformly continuous (BUC) functions in $\mathbb{R}^d$ satisfying the following: (i) $\mathcal{A}(\mathbb{R}^d)$ is invariant by translations, that is, if $f \in \mathcal{A}$, then $f(\cdot+\lambda) \in \mathcal{A}$, for all $\lambda \in \mathbb{R}^d$; (ii) every function $f \in \mathcal{A}(\mathbb{R}^d)$ possesses mean-value, that is, there exists a number $M(f)$ such that $f(\varepsilon^{-1}x) \rightharpoonup M(f)$ as $\varepsilon \to 0$ in the weak–$\star$ topology of $L^\infty(\mathbb{R}^d)$. In particular, we have

$$M(f) := \lim_{R\to\infty} \frac{1}{|B(0;R)|} \int\limits_{B(0;R)} f(x)\,dx,$$

where $B(0;R)$ is the open ball with radius R centered at the origin 0, and $|B(0,R)|$ is its n-dimensional Lebesgue measure. Also, one easily sees that $M(f(\cdot+\lambda)) = M(f)$, for all $\lambda \in \mathbb{R}^n$. We also use the notation $M(f) = \fint f\,dx$; (iii) $\mathcal{A}$ is ergodic in the sense that if we define in $\mathcal{A}$ the semi-norm $[f]_2 := M(|f|^2)^{1/2}$, taking equivalence classes by the relation $f \sim g \iff [f-g]_2 = 0$, and denoting the completion of the quotient space by $\mathcal{B}^2(\mathbb{R}^n)$, the Besicovitch space of exponent 2 associated with $\mathcal{A}(\mathbb{R}^d)$, we have that any $g \in \mathcal{B}^2(\mathbb{R}^d)$, satisfying $g(\cdot+\lambda) = g(\cdot)$, in the sense of $\mathcal{B}^2(\mathbb{R}^d)$, for all $\lambda \in \mathbb{R}^d$, is equal to a constant in $\mathcal{B}^2(\mathbb{R}^d)$. As examples of ergodic algebras, besides the periodic functions, we have $\mathrm{AP}(\mathbb{R}^d)$, the space of almost periodic functions (see, e.g., [11]), the Fourier-Stieltjes algebra $\mathrm{FS}(\mathbb{R}^d)$ (see, e.g., [26,30]), or the larger one $\mathrm{WAP}(\mathbb{R}^d)$, the space of the weak almost periodic functions, see [26,27]. In particular, in [27], Eberlein proved that every function $\phi \in \mathrm{WAP}(\mathbb{R}^d)$ admits a decomposition $\phi = \phi_* + \phi_{\mathcal{N}}$, where $\phi_* \in \mathrm{AP}(\mathbb{R}^d)$ and $\phi_{\mathcal{N}} \in \mathcal{N}(\mathbb{R}^d)$ where

$$\mathcal{N}(\mathbb{R}^d) := \{f \in \mathrm{BUC}(\mathbb{R}^d) \,:\, \lim_{R\to\infty} \frac{1}{|B(0;R)|} \int\limits_{B(0;R)} |f(y)|\,dy = 0\}.$$

This motivates the introduction in [29] of the algebra of the weak–$*$ almost periodic functions, $\mathcal{W}^*\mathrm{AP}(\mathbb{R}^d)$, defined by

$$\mathcal{W}^*\mathrm{AP}(\mathbb{R}^d) := \mathrm{AP}(\mathbb{R}^d) + \mathcal{N}(\mathbb{R}^d),$$

which is clearly an ergodic algebra and contains all the ergodic algebras containing the periodic functions so far known.

In all that follows, we assume that the ergodic algebra $\mathcal{A}(\mathbb{R}^d)$ is a subalgebra of $\mathcal{W}^*\mathrm{AP}(\mathbb{R}^d)$, that is, $\mathcal{A}(\mathbb{R}^d) \subset \mathcal{W}^*\mathrm{AP}(\mathbb{R}^d)$.

Let $\mathcal{B}^2(\mathbb{R}^d)$ denote the L^2-Besicovitch space associated with $\mathcal{A}(\mathbb{R}^d)$. Set

$$\mathcal{T} := \{v \in \mathcal{A}(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n) \,:\, \nabla_a v := a\cdot\nabla v \in \mathcal{A}(\mathbb{R}^n)\}.$$

We define

$$\mathcal{S} := \left\{ v \in \mathcal{B}^2(\mathbb{R}^d) \,:\, \fint_{\mathbb{R}^d} v(y)a(y)\cdot\nabla\varphi(y)\,dy = 0, \quad \text{for all } \varphi \in \mathcal{T} \right\}$$

and its subspaces

$$\mathcal{S}^* := \left\{ v \in \mathcal{A}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d) \,:\, \nabla_a v = 0, \text{ a.e.} \right\},$$

and

$$\mathcal{S}^\dagger := \left\{ v \in \mathcal{S} \,:\, \exists\, (v_k)_{k\in\mathbb{N}} \subset \mathcal{T},\ v_k \overset{\mathcal{B}^2\cap L^2_{\rm loc}}{\longrightarrow} v \text{ and } \nabla_a v_k \overset{\mathcal{B}^2\cap L^2_{\rm loc}}{\longrightarrow} 0 \right\}.$$

In the periodic case we have $\mathcal{S}^\dagger = \mathcal{S}$, as proven in [15] by applying the commutation lemma in [20]. In general, it holds $\mathcal{S}^* \subset \mathcal{S}^\dagger$. In [2] it was shown that for a large collection of fields $a(y) \in (\mathrm{AP} \cap \mathrm{Lip})\,(\mathbb{R}^d;\mathbb{R}^d)$, with $\operatorname{div} a = 0$, the space $\mathcal{S}^*$ is dense in $\mathcal{S}$ in the $\mathcal{B}^2(\mathbb{R}^d)$ topology, when $\mathcal{A}(\mathbb{R}^d) = \mathrm{AP}(\mathbb{R}^d)$. Similarly, in [30] also a large collection of fields $a(y) \in (\mathrm{FS} \cap \mathrm{Lip})\,(\mathbb{R}^d;\mathbb{R}^d)$, with $\operatorname{div} a = 0$, was described for which the space $\mathcal{S}^*$ is dense in $\mathcal{S}$ in the topology of $\mathcal{B}^2(\mathbb{R}^d)$, when $\mathcal{A}(\mathbb{R}^d) = \mathrm{FS}(\mathbb{R}^d)$. Finally, in [55], it was shown that for any $a(y) \in (\mathrm{AP} \cap \mathrm{Lip})\,(\mathbb{R}^d;\mathbb{R}^d)$, $\mathcal{S}^\dagger$ is dense in $\mathcal{S}$, in the topology of $\mathcal{B}^2(\mathbb{R}^d)$, for $\mathcal{A}(\mathbb{R}^d) = \mathrm{AP}(\mathbb{R}^d)$.

We assume that

$$U_0 \in L^\infty(\mathbb{R}^d;\mathcal{A}(\mathbb{R}^d)),\ U_0(x,\cdot) \in \mathcal{S} \text{ for a.e. } x \in \mathbb{R}^d. \tag{1.3}$$

Let $\mathcal{K}$ be the compactification of $\mathbb{R}^d$ associated with the ergodic algebra $\mathcal{A}(\mathbb{R}^d)$, through a classical theorem by Stone (see, e.g., [22,23]). For each $y \in \mathcal{K}$, consider the following auxiliary initial value problem

$$dU + \nabla_x\cdot(\tilde a(y)f(U))\,dt = \kappa_0\,\sigma(U)\,dW + \frac{1}{2}\kappa_0^2\,h(U)\,dt, \quad t>0,\ x\in\mathbb{R}^d, \tag{1.4}$$

$$U(0,x,y) = U_0(x,y), \quad x \in \mathbb{R}^d, \tag{1.5}$$

where $\tilde a(y)$ is the orthogonal projection of $a(y)$ onto $\mathcal{S}$ in $\mathcal{B}^2(\mathbb{R}^d)$. In particular, $\tilde a$ is a Borel function over $\mathcal{K}$. Actually, it has been proven in [55] (see Theorem 3.2 in [55]) that $\tilde a(y) \in C(\mathcal{K}) \sim \mathcal{A}(\mathbb{R}^d)$; we will not make use of this fact here. The stability properties of solutions of the Cauchy problem for stochastic scalar conservation laws imply that $U \in L^2(\Omega; L^\infty((0,T)\times\mathbb{R}^d\times\mathcal{K}))$, for any $T>0$; we will comment further on this point in Section 3.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, $\{\mathcal{F}_t \,:\, 0\le t\le T\}$ be a complete filtration, that is, an increasing family of σ-algebras contained in $\mathcal{F}$, all of them containing all the null sets of $\mathcal{F}$, such that $\mathcal{F}_s = \bigcap_{t\ge s}\mathcal{F}_t$. In this paper, for simplicity, we assume that the σ-algebra $\mathcal{F}$ is countably generated and $\mathcal{F}_t$ is the filtration generated by the Brownian motion $\{W(s) \,:\, 0\le s\le t\}$ and $\mathcal{F}_0$, the σ-algebra generated by the null sets of $\mathcal{F}$.

If X is a Banach space, let $\mathcal{N}_W^2(0,T,X)$ denote the space of the predictable X-valued processes (see, e.g., [17], p.94, [51], p.28). This is the same as the space $L^2([0,T]\times\Omega,X)$ with the product measure $dt\otimes d\mathbb{P}$ on $\mathcal{P}_T$, the predictable σ-algebra, i.e., the σ-algebra generated by the sets $\{0\}\times\mathcal{F}_0$ and the rectangles $(s,t]\times A$ for $A\in\mathcal{F}_s$. We denote $\mathcal{N}_W^2(0,T,L^2_{\mathrm{loc}}(\mathbb{R}^d)) := \bigcap_{R>0}\mathcal{N}_W^2(0,T,L^2(B(0,R)))$, where $B(0,R)$ is the open ball centered at 0 with radius R in $\mathbb{R}^d$. We will say that u is predictable if $u\in\mathcal{N}_W^2(0,T,L^2_{\mathrm{loc}}(\mathbb{R}^d))$. Let us also denote $Q=(0,T)\times\mathbb{R}^d$.

Definition 1.1. We say that a predictable function $u^\varepsilon\in L^2(\Omega;L^\infty(Q))$ is an entropy solution of (1.1)–(1.2) if for all convex $\eta\in C^2(\mathbb{R})$, for $q\in C^2(\mathbb{R})$, such that $q'(u)=\eta'(u)f'(u)$, and for all $0\le\varphi\in C_c^\infty((-\infty,T)\times\mathbb{R}^d)$, a.s. in Ω, we have

$$\int\limits_Q \eta(u^\varepsilon)\partial_t\varphi + q(u^\varepsilon)a\left(\frac{x}{\varepsilon}\right)\cdot\nabla\varphi + \frac{\kappa_0^2}{2}\left(\eta'(u^\varepsilon)h(u^\varepsilon)+\eta''(u^\varepsilon)\sigma^2(u^\varepsilon)\right)\varphi\,dx\,dt$$

$$+\kappa_0\int\limits_0^T\int\limits_{\mathbb{R}^d}\eta'(u^\varepsilon)\sigma(u^\varepsilon)\varphi\,dx\,dW(t)+\int\limits_{\mathbb{R}^d}\eta\left(U_0\left(x,\frac{x}{\varepsilon}\right)\right)\varphi(0,x)\,dx\,dt\ge 0.$$

Definition 1.2. For each $y\in\mathcal{K}$, we say that a predictable function $U(y)\in L^2(\Omega;L^\infty(Q))$ is an entropy solution of (1.4)–(1.5) if for all convex $\eta\in C^2(\mathbb{R})$, for $q\in C^2(\mathbb{R})$, such that $q'(u)=\eta'(u)f'(u)$, and for all $0\le\varphi\in C_c^\infty((-\infty,T)\times\mathbb{R}^d)$, a.s. in Ω, we have

$$\int\limits_Q \eta(U(y))\partial_t\varphi + q(U(y))\tilde{a}(y)\cdot\nabla\varphi + \frac{\kappa_0^2}{2}\left(\eta'(U(y))h(U(y))+\eta''(U(y))\sigma^2(U(y))\right)\varphi\,dx\,dt$$

$$+\kappa_0\int\limits_0^T\int\limits_{\mathbb{R}^d}\eta'(U(y))\sigma(U(y))\varphi\,dx\,dW(t)+\int\limits_{\mathbb{R}^d}\eta(U_0(y))\varphi(0,x)\,dx\,dt\ge 0. \quad (1.6)$$

Theorem 1.1. *Let u^ε be the entropy solution of (1.1)–(1.2), with U_0 satisfying (1.3), and, for each $y\in\mathcal{K}$, let $U(y)$ be the entropy solution (1.4)–(1.5). Assume that $\mathcal{S}^\dagger$ is dense in $\mathcal{S}$ in the topology of $\mathcal{B}^2(\mathbb{R}^d)$. Then, we have that $u^\varepsilon\rightharpoonup u$, in the weak topology of $L^2(\Omega;L^2_{\mathrm{loc}}(Q))$, that is, $L^2(\Omega;L^2((0,T)\times\{|x|<R\})$, for any $R>0$, where*

$$u(t,x)=\int\limits_{\mathcal{K}}U(t,x,y)\,d\mathfrak{m}(y),$$

and $d\mathfrak{m}(y)$ is the measure on $\mathcal{K}$ induced by the mean value on $\mathcal{A}(\mathbb{R}^d)$. Moreover, if $U\in L^2(\Omega;\mathcal{B}^2(\mathbb{R}^d;C_b([0,T]\times\mathbb{R}^d)))$, then $u^\varepsilon(t,x)-U\left(t,x,\frac{x}{\varepsilon}\right)$ strongly converges to zero in $L^2(\Omega;L^2_{\mathrm{loc}}(Q))$.

The second problem is the one of a stiff oscillatory external force whose deterministic case was first addressed in [25], in the periodic one-dimensional case and later on in [2,3] in the almost periodic and ergodic algebras multidimensional case. The corresponding equation is as follows

$$du^\varepsilon + \nabla_x \cdot f(u^\varepsilon)\, dt = \frac{1}{\varepsilon} V'\left(\frac{x_1}{\varepsilon}\right) dt + \kappa_0\, \sigma_{f_1}(u^\varepsilon)\, dW + \frac{1}{2}\kappa_0^2\, h_{f_1}(u^\varepsilon)\, dt, \tag{1.7}$$

where $f = (f_1, \dots, f_d)$, $f_i : \mathbb{R} \to \mathbb{R}$ are smooth functions, $i = 1, \dots, d$, $f_1' \geq \delta_0 > 0$, $f_k' \geq 0$, $k = 2, \dots, d$. We also assume that $f' \in L^\infty(\mathbb{R}; \mathbb{R}^d)$ and $f_1', f_1'', f_1''' \in L^\infty(\mathbb{R})$. $\kappa_0 \in \mathbb{R}$ is a constant. $V : \mathbb{R} \to \mathbb{R}$ is a smooth function belonging to an arbitrary ergodic algebra $\mathcal{A}(\mathbb{R})$, $W : \Omega \times [0, T] \to \mathbb{R}$ is a standard Brownian motion, and σ_{f_1}, h_{f_1} are obtained from f_1 from the expressions

$$\sigma_{f_1}(u) := \frac{1}{f_1'(u)}, \qquad h_{f_1} := -\frac{f_1''(u)}{f_1'(u)^3}.$$

We observe that, from the assumptions on f_1, it follows that $h_{f_1}' \in L^\infty(\mathbb{R})$.

Again, in view of the Stratonovich-Itô conversion formula, we note that equation (1.7) may be written as

$$du^\varepsilon + \nabla_x \cdot f(u^\varepsilon)\, dt = \frac{1}{\varepsilon} V'\left(\frac{x_1}{\varepsilon}\right) dt + \kappa_0\, \sigma_{f_1}(u^\varepsilon) \circ dW.$$

We prescribe an initial data for (1.7) of the form

$$u^\varepsilon(0, x) = u_0\left(x, \frac{x}{\varepsilon}\right), \tag{1.8}$$

which, for simplicity, we may assume to be deterministic, whose hypotheses we will specify later on.

Let $g = f_1^{-1}$ be the inverse of f_1. We assume that, for some $v_0 \in L^\infty(\mathbb{R}^d)$, $u_0(x, y)$ satisfies

$$u_0(x, y) = g(V(y) + v_0(x)). \tag{1.9}$$

Let us consider the auxiliary equation

$$d\bar{u} + \nabla \cdot \bar{f}(\bar{u})\, dt = \kappa_0\, \sigma_{\bar{f}_1}(\bar{u})\, dW + \frac{1}{2}\kappa_0^2\, h_{\bar{f}_1}(\bar{u})\, dt, \tag{1.10}$$

where $\bar{f} = (\bar{f}_1, \bar{f}_2, \dots, \bar{f}_d)$, with $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_d$, satisfying

$$p = \fint_{\mathbb{R}} g\left(\bar{f}_1(p) + V(z_1)\right) dz_1, \tag{1.11}$$

$$\bar{f}_k(p) = \fint_{\mathbb{R}} f_k \circ g\left(\bar{f}_1(p) + V(z_1)\right) dz_1, \quad k = 2, \cdots, d, \tag{1.12}$$

and $\sigma_{\bar{f}_1}(\cdot)$, $h_{\bar{f}_1}(\cdot)$ are defined as σ_{f_1}, h_{f_1} with $\bar{f}_1(\cdot)$ instead of f_1 We remark that, from the assumptions on f and f_1, it follows from (1.11) and (1.12) that $\bar{f}$ and $\bar{f}_1$ also satisfy $\bar{f}' \in L^\infty(\mathbb{R};\mathbb{R}^d)$ and $\bar{f}_1', \bar{f}_1'', \bar{f}_1''' \in L^\infty(\mathbb{R})$.

For (1.10) the following initial condition is prescribed

$$\bar{u}(0,x) = \bar{u}_0(x) := \fint_{\mathbb{R}} u_0(x,z_1)\,dz_1 = \bar{f}_1^{-1}(v_0(x)). \tag{1.13}$$

Definition 1.3. We say that $u^\varepsilon \in \mathcal{N}_W^2(0,T,L^2_{\text{loc}}(\mathbb{R}^d)) \cap L^2(\Omega; L^\infty(Q))$ is an entropy solution of (1.7)–(1.8), with $u_0\left(\cdot, \frac{\cdot}{\varepsilon}\right) \in L^2(\Omega; L^\infty(\mathbb{R}^d))$, satisfying (1.9), if for all convex $\eta \in C^2(\mathbb{R})$, for $q \in C^2(\mathbb{R},\mathbb{R}^d)$, such that $q'(u) = \eta'(u)f'(u)$, and for all $0 \le \varphi \in C_c^\infty((-\infty,T)\times\mathbb{R}^d)$, a.s. in Ω, we have

$$\int_Q \eta(u^\varepsilon)\partial_t\varphi + q(u^\varepsilon)\cdot\nabla\varphi + \eta'(u^\varepsilon)\left(\frac{1}{\varepsilon}V'\left(\frac{x_1}{\varepsilon}\right) + \frac{\kappa_0^2}{2}h_{f_1}(u^\varepsilon)\right)\varphi\,dx\,dt$$
$$+ \frac{\kappa_0^2}{2}\int_Q \sigma_{f_1}^2(u)\eta''(u)\varphi\,dx\,dt + \kappa_0\int_0^T\int_{\mathbb{R}^d}\eta'(u)\sigma_{f_1}(u)\varphi\,dx\,dW(t)$$
$$+ \int_{\mathbb{R}^d}\eta(u)\varphi(0,x)\,dx\,dt \ge 0.$$

Definition 1.4. We say that a predictable function $\bar{u} \in L^2(\Omega; L^\infty(Q))$ is an entropy solution of (1.10)–(1.13) if for all convex $\eta \in C^2(\mathbb{R})$, for $\bar{q} \in C^2(\mathbb{R},\mathbb{R}^d)$, such that $\bar{q}'(u) = \eta'(u)\bar{f}'(u)$, and for all $0 \le \varphi \in C_c^\infty((-\infty,T)\times\mathbb{R}^d)$, a.s. in Ω, we have

$$\int_Q \eta(\bar{u})\partial_t\varphi + \bar{q}(\bar{u})\cdot\nabla\varphi + \frac{\kappa_0^2}{2}\left(\eta'(\bar{u})h_{\bar{f}_1}(\bar{u}) + \eta''(\bar{u})\sigma_{\bar{f}_1}^2(\bar{u})\right)\varphi\,dx\,dt$$
$$+ \kappa_0\int_0^T\int_{\mathbb{R}^d}\eta'(\bar{u})\sigma_{\bar{f}_1}(\bar{u})\varphi\,dx\,dW(t) + \int_{\mathbb{R}^d}\eta(\bar{u}_0)\varphi(0,x)\,dx\,dt \ge 0.$$

We can state our second main result.

Theorem 1.2. *Let u^ε be the entropy solution of (1.7)–(1.8), with u_0 satisfying (1.9), and $\bar{u}$ be the entropy solution of (1.10)–(1.13). Then, $u^\varepsilon \rightharpoonup \bar{u}$ in the weak topology of $L^2(\Omega; L^2_{\text{loc}}(Q))$. Moreover, $u^\varepsilon(t,x) - U\left(t,x,\frac{x}{\varepsilon}\right)$ strongly converges to zero in $L^2(\Omega; L^2_{\text{loc}}(Q))$, as $\varepsilon \to 0$, where $U(t,x,y) = g\left(\bar{f}_1(\bar{u}(t,x)) + V(y)\right)$.*

Before we make an account of earlier works connected to the present one, both in homogenization theory and in the theory of SPDEs, and a brief description of the contents in this paper, we remark for practical purposes that the stochastic perturbation of the deterministic versions, of the equations we deal with herein, are determined by the stochastic equations satisfied by certain special solutions, which in turn are natural stochastic extensions of the stationary solutions of the corresponding deterministic versions, which play a central role in the homogenization process in the deterministic case. Homogenization theory has been useful in many well known cases to derive equations from mechanics and other applied areas, as the Darcy law in two-phase flows in porous media (see, e.g., the famous appendix by Tartar in [53]), and we believe that the way the stochastic perturbations were derived here may be useful in applications.

This paper is concerned with both the theory of homogenization of partial differential equations and the theory of stochastic differential equations. The homogenization theory of partial differential equations has been a field of intense research since the 1970's and we refer to the classical book [10] for an account of this theory up to 1978. We also refer to the other classical book [40] where a section is devoted to the homogenization theory in the context of ergodic algebras, which is the setting adopted in this paper. The homogenization methods used in this paper are based on those developed in [2] and [3], which in turn are mostly based on the concept of two-scale Young measures for almost periodic oscillations and its natural extension to ergodic algebras. Two-scale Young measures were introduced in the periodic case in [24] (see also [25]) as an extension to the notion of two-scale convergence introduced in [49] and further developed in [1] (see also [15]). Two-scale convergence for general oscillations in ergodic algebras were established in [13], and corresponds to the linear case of the two-scale Young measures established in [2], as proved in [31].

The theory of stochastic partial differential equations has experienced intense progress in the last three decades and we cite the treatise [17] for a basic general account of this theory and references. More specifically, concerning the theory of stochastic conservation laws, we mention the first contributions by Kim [43], and Feng and Nualart [28]. The latter was further developed in Chen, Ding, and Karlsen in [14] and Karlsen and Storrøsten in [42]. An inflection in the course of this theory was achieved by Debussche and Vovelle [18] with the introduction of the notion of kinetic stochastic solution, extending the corresponding deterministic concept introduced by Lions, Perthame, and Tadmor [45]. We also mention the independent development in this theory made by Bauzet, Vallet, and Wittbold [7]. Concerning homogenization of stochastic partial differential equations, this has not been a frequently researched topic, although the earliest contribution seems to have appeared already in the early 1990's by Bensoussan in [9]. As to more recent publications on this subject, we mention the contributions of Ichihara [38], Sango [54], Mohammed [47], and Mohammed and Sango [48], among others. Consult also references in these papers.

Concerning our method for proving Theorem 1.1 and Theorem 1.2, the core of our technique is to begin by using two-scale Young measures, as in [24,25,2,3], for instance, then to derive a stochastic kinetic equation satisfied by the generalized kinetic function associated with the two-scale Young measure, and then to apply a uniqueness result for weak solutions of the corresponding stochastic kinetic equation, as is done in [15] in the deterministic periodic case for general conservation laws.

This paper is organized as follows. In Section 2 we state and prove a result on the existence of stochastic two-scale Young measures which will be used in the two subsequent sections. In Section 3, we address the homogenization of the stochastic nonlinear transport equation. In Section 4, we deal with the same problem for the stochastic stiff oscillatory external force equation. In Section 5 we establish a general well-posedness result for stochastic conservation laws, which fits the needs of the present article. Finally, in Section 6, we gather a general comparison principle and the so-called stochastic Kružkov inequality. Both needed for the analysis in Sections 3 and 4.

2. Stochastic two-scale Young measures

In the following sections our analysis will be based on the notion of two-scale Young measures as was done in the deterministic case in, e.g., [24,25,2,3]. For future reference, we next state as a proposition the existence of stochastic two-scale Young measures associated with (generalized) subsequences satisfying bounds such as (3.3) or (4.4) below. The proof follows ideas in [2]. Nevertheless, here there is the probability space Ω and the stochastic integral as new ingredients. Also we need to establish an estimate (*cf.* (2.3)) that will be needed in the following sections. Therefore, we include a detailed proof here for the convenience of the reader. For simplicity, to avoid the use of generalized subsequences, we assume that our ergodic algebra is separable. In practice, this means that if $\Psi_1(t,x,\frac{x}{\varepsilon},u),\dots,\Psi_N(t,x,\frac{x}{\varepsilon},u)$ is the finite family of continuous oscillatory functions involved in our homogenization problem, we consider the closure of the subalgebra of $\mathcal{A}(\mathbb{R}^n)$ (invariant by translations) generated by the functions $g_{\alpha_1,\beta_1,\gamma_1}(y):=\Psi_1(t_{\alpha_1},x_{\beta_1},y,u_{\gamma_1}),\cdots,g_{\alpha_N,\beta_N,\gamma_N}(y):=\Psi_N(t_{\alpha_N},x_{\beta_N},y,u_{\gamma_N})$, $\alpha_i,\beta_i,\gamma_i\in\mathbb{N}$, $i=1,\cdots,N$, where $\{(t_{\alpha_i},x_{\beta_i},u_{\gamma_i})\,:\,\alpha_i,\beta_i,\gamma_i\in\mathbb{N}\}$ is a countable dense subset of $[0,\infty)\times\mathbb{R}^d\times\mathbb{R}$, for $i=1,\cdots,N$.

Proposition 2.1. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, with $\mathcal{F}$ countably generated, let $\mathcal{F}_t$ be the filtration generated by the Brownian motion $W(t)$ and $\mathcal{F}_0$, the σ-algebra generated by the null sets of $\mathcal{F}$. Let $\mathcal{A}(\mathbb{R}^d)$ be a separable ergodic algebra and $\mathcal{K}$ the associated separable compact space such that $\mathcal{A}(\mathbb{R}^d)\sim C(\mathcal{K})$, with associated invariant measure $d\mathfrak{m}(y)$. Let u^ε, $\varepsilon>0$, be a sequence of predictable functions in $L^p(\Omega;L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^d))$, for all $p\geq 1$, satisfying*

$$|u^\varepsilon(\omega,t,x)|\leq C_*(1+|W(\omega,t)|^{N_0}),\quad \textit{for a.e. } (\omega,t,x)\in\Omega\times[0,\infty)\times\mathbb{R}^d, \tag{2.1}$$

for some $C_*>0$ and $N_0\in\mathbb{N}$. Let w_N be defined in (5.11). Then, there exists a subsequence, u^{ε_k}, $\varepsilon_k\to 0$, and a parameterized family of probability measures over $\mathbb{R}$, $\nu_{\omega,t,x,y}$, satisfying the properties:

(1) *$\nu_{\omega,t,x,y}$ is measurable, in the sense that for any $\zeta \in C_c(\mathbb{R})$, $\langle \nu_{\omega,t,x,y}, \zeta\rangle$ is measurable with respect to the sigma-algebra $\mathcal{F} \otimes \mathbb{B}([0,\infty)\times\mathbb{R}^d)\otimes\mathbb{B}(\mathcal{K})$;*

(2) *For any $A \in \mathcal{F}$, denoting by $\mathbb{E}_A$ the conditional expectation with respect to A, for all $\Psi \in C_c([0,\infty)\times\mathbb{R}^d\times\mathbb{R};\mathcal{A}(\mathbb{R}^d)) \sim C_c([0,\infty)\times\mathbb{R}^d\times\mathcal{K}\times\mathbb{R})$,*

$$\lim_{k\to\infty}\mathbb{E}_A \int\limits_{[0,\infty)\times\mathbb{R}^d} \Psi\left(t,x,\frac{x}{\varepsilon_k},u^{\varepsilon_k}(t,x)\right) w_N(x)\,dt\,dx$$
$$= \mathbb{E}_A \int\limits_{[0,\infty)\times\mathbb{R}^d\times\mathcal{K}} \langle \nu_{\omega,t,x,y}, \Psi(t,x,y,\cdot)\rangle\, w_N(x)\,d\mathfrak{m}(y)\,dt\,dx. \quad (2.2)$$

(3) *For a.e. $y\in\mathcal{K}$, for all $T>0$, we have*

$$\mathbb{E}\left(\operatorname*{ess\,sup}_{t\in[0,T]} \iint\limits_{\mathbb{R}^d\times\mathbb{R}} |\xi|^p\, w_N(x)\,\nu_{\omega,t,x,y}(d\xi)\,dx\right) \le C_{T,N,p}, \qquad \forall p\in[1,\infty), \quad (2.3)$$

where $C_{T,N,p}$ is a positive constant depending only on T,N,p.

(4) *If $\Psi \in C([0,\infty)\times\mathbb{R}^d\times\mathbb{R};\mathcal{A}(\mathbb{R}^d)) \sim C([0,\infty)\times\mathbb{R}^d\times\mathcal{K}\times\mathbb{R})$ is such that $|\Psi(t,x,y,\xi)| \le 1_{[0,T_0]}(t)C(1+|\xi|^p)$, for some $p\ge 1$ and $T_0>0$, then (2.2) holds for all $A\in\mathcal{F}$. More generally, for such Ψ, if $\ell \in L^2(\Omega)$ and $\tilde\Psi(\omega,t,x,y,\xi) = \ell(\omega)\Psi(t,x,y,\xi)$, then*

$$\lim_{k\to\infty}\mathbb{E} \int\limits_{[0,\infty)\times\mathbb{R}^d} \tilde\Psi\left(\omega,t,x,\frac{x}{\varepsilon_k},u^{\varepsilon_k}(t,x)\right) w_N(x)\,dt\,dx$$
$$= \mathbb{E} \int\limits_{[0,\infty)\times\mathbb{R}^d\times\mathcal{K}} \left\langle \nu_{\omega,t,x,y}, \tilde\Psi(\omega,t,x,y,\cdot)\right\rangle w_N(x)\,d\mathfrak{m}(y)\,dt\,dx. \quad (2.4)$$

(5) *If $\Psi\in C([0,\infty)\times\mathbb{R}^d\times\mathbb{R};\mathcal{A}(\mathbb{R}^d))$ satisfying $|\Psi(t,x,y,\xi)|\le 1_{[0,T_0]}(t)C(1+|\xi|^p)$, for some $p\ge1$ and $T_0>0$, then*

$$(\omega,t)\mapsto \int\limits_{\mathbb{R}^d}\int\limits_{\mathcal{K}} \langle \nu_{\omega,t,x,y}\,,\Psi(t,x,y,\cdot)\rangle\, w_N(x)\,d\mathfrak{m}(y)\,dx$$

is a predictable process on $\Omega\times[0,\infty)$, and, for any $A\in\mathcal{F}$,

$$\lim_{k\to\infty}\mathbb{E}_A \int\limits_0^\infty\int\limits_{\mathbb{R}^d} \Psi\left(t,x,\frac{x}{\varepsilon_k},u^{\varepsilon_k}(t,x)\right) w_N(x)\,dx\,dW(t)$$
$$= \mathbb{E}_A \int\limits_0^\infty\int\limits_{\mathbb{R}^d}\int\limits_{\mathcal{K}} \langle \nu_{\omega,t,x,y}\,,\Psi(t,x,y,\cdot)\rangle\, w_N(x)\,d\mathfrak{m}(y)\,dx\,dW(t). \quad (2.5)$$

Moreover, for $\mathfrak{m}$*-a.e.* $y \in \mathcal{K}$*,*

$$(\omega, t) \mapsto \int_{\mathbb{R}^d} \langle \nu_{\omega,t,x,y}\,, \Psi(t,x,y,\cdot)\rangle\, w_N(x)\, dx$$

is a predictable process on $\Omega \times [0, \infty)$*.*

Proof. Let $W^*(t) := \max_{0\le s\le t} |W(s)|$ and, given $M > 0$, let $t_M := \inf\{t \ge 0 \,:\, W^*(t) \ge M\}$. Given $T > 0$, for M sufficiently large, $t_M > T$. Therefore, taking $M \in \mathbb{N}$, making $M \to \infty$, and defining $\Omega_M(T) := \{\omega \in \Omega \,:\, t_M(\omega) > T\}$, we see that $\mathbb{P}\left(\Omega \setminus \Omega_M(T)\right) \to 0$. Indeed, $\Omega_M(T)$ is an increasing family of subsets of Ω and if $\mathbb{P}\left(\Omega \setminus \bigcup_{M\in\mathbb{N}} \Omega_M(T)\right) > 0$, then we would be able to find $\omega \in \Omega$ for which $W(t)$ is defined and continuous for $t \in [0, \infty)$ and such that $W^*(\omega, t) \to +\infty$ as $t \to T$, which is absurd. We fix $T > 0$, and, for simplicity we write simply Ω_M instead of $\Omega_M(T)$. So, for each $M \in \mathbb{N}$, we have that u^ε is a bounded sequence in $L^\infty(\Omega_M \times [0,T] \times \mathbb{R}^d)$. Let us consider the countable family of real valued functions over Ω_M, $\mathfrak{F} := \{W(\cdot, r) \,:\, r \in \mathbb{Q} \cap [0,T]\}$. We may assume, without loss of generality, that the functions of the family $\mathfrak{F}$ are defined at every point of Ω_M and that $\mathfrak{F}$ distinguishes between the points of Ω_M, that is, given $\omega_1, \omega_2 \in \Omega_M$, $\omega_1 \ne \omega_2$, then there is $r \in \mathbb{Q} \cap [0,T]$ such that $W(\omega_1, r) \ne W(\omega_2, r)$. The first assertion is clear since we may find a set of null $\mathbb{P}$-measure in Ω out of which the functions in the countable family $\mathfrak{F}$ are defined everywhere, and so we can define them as 0 over this null $\mathbb{P}$-measure subset of Ω. The second assertion follows from the fact that we can define in Ω the equivalence relation $\omega_1 \sim \omega_2$ if and only if $W(\omega_1, r) = W(\omega_2, r)$ for all $r \in \mathbb{Q} \cap [0,T]$. Then we define the quotient space $\tilde{\Omega} := \Omega/\sim$, with the natural projection $\pi_\sim : \Omega \to \tilde{\Omega}$, $\pi_\sim(\omega) = [\omega]$, where $[\omega]$ is the $\sim$-equivalence class of ω. We also define the class $\tilde{\mathcal{F}}$ of subsets of $\tilde{\Omega}$ by $\tilde{A} \in \tilde{\mathcal{F}}$ if and only if $\pi_\sim^{-1}(\tilde{A}) \in \mathcal{F}$, and for $\tilde{A} \in \tilde{\mathcal{F}}$ we define $\tilde{\mathbb{P}}(\tilde{A}) = \mathbb{P}(\pi_\sim^{-1}(\tilde{A}))$. It is easy to check that $\tilde{\mathcal{F}}$ is a sigma-algebra and $\tilde{\mathbb{P}}$ is a probability measure on $\tilde{\Omega}$. Moreover, $W(t)$ is a Brownian motion over $\tilde{\Omega}$, since the distributions of $W(t)$, $t \in [0,T]$, on $(\Omega, \mathcal{F}, \mathbb{P})$ and on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ coincide; therefore, for all purposes, we can assume that the family $\mathfrak{F}$ distinguishes between the points of Ω; otherwise we replace $(\Omega, \mathcal{F}, \mathbb{P})$ by the quotient space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and, once we obtain the result for the latter, it can be automatically lifted up to the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $B(\Omega_M)$ be the algebra of bounded functions over Ω_M. Let $\mathfrak{A}$ be the closed subalgebra of $B(\Omega_M)$ generated by $\{1, \mathfrak{F}\}$. According to a well-known extension of the Stone-Weierstrass theorem (see [22], p.274–276, Theorem 18 and Corollary 19) there exist a compact Hausdorff space $\overline{\Omega_M}$ and an one-to-one embedding of Ω_M as a dense subset of $\overline{\Omega_M}$, such that each $\psi \in \mathfrak{A}$ has a unique continuous extension $\overline{\psi}$ to $\overline{\Omega_M}$, and such that the correspondence $\psi \leftrightarrow \overline{\psi}$ is an isomeric isomorphism between $\mathfrak{A}$ and $C(\overline{\Omega_M})$. Moreover, the relation

$$\int_{\overline{\Omega_M}} \overline{\psi}(\omega)\, d\mathbb{P}(\omega) := \int_{\Omega_M} \psi(\omega)\, d\mathbb{P}(\omega)$$

defines $\mathbb{P}$ as a Radon measure over $\overline{\Omega_M}$. In particular, we can endow Ω_M with the topology induced by the embedding $\Omega_M \to \overline{\Omega_M}$ with respect to which $\mathbb{P}$ is a Radon measure and Ω_M is relatively compact. Therefore, henceforth, for simplicity, we consider Ω_M as compact and $\mathbb{P}$ as a Radon measure on Ω_M, with the referred topology, which coincides with the topology generated by the family $\mathfrak{F}$.

Let $L_M := C_*(1 + M^{N_0})$, where C_* is as in (2.1). Denote by $C_0(\Omega_M \times [0,T] \times \mathbb{R}^d \times [-L_M, L_M]; \mathcal{A}(\mathbb{R}^d))$ the space of functions $\Psi(\omega,t,x,y,\xi)$ continuous in $\Omega_M \times [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, belonging to $\mathcal{A}(\mathbb{R}^d)$, as functions of y, for each fixed $(\omega,t,x,\xi) \in \Omega_M \times [0,T] \times \mathbb{R}^d \times \mathbb{R}$, and such that $\Psi(\cdot,\cdot,x,\cdot,\cdot) \to 0$ as $|x| \to \infty$, uniformly in $\Omega_M \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$. Clearly, $C_0(\Omega_M \times [0,T] \times \mathbb{R}^d \times [-L_M, L_M]; \mathcal{A}(\mathbb{R}^d))$ is isometrically isomorphic to $C_0(\Omega_M \times [0,T] \times \mathbb{R}^d \times \mathcal{K} \times [-L_M, L_M])$, defined similarly. Given $\Psi \in C_0(\Omega_M \times [0,T] \times \mathbb{R}^d \times [-L_M, L_M]; \mathcal{A}(\mathbb{R}^d))$, define

$$\langle \mu_M^\varepsilon, \Psi \rangle := \int\limits_{\Omega_M \times [0,T] \times \mathbb{R}^d} \Psi\left(\omega, t, x, \frac{x}{\varepsilon}, u^\varepsilon\right) w_N \, dt \, dx \, d\mathbb{P}(\omega).$$

Because we are assuming $|u^\varepsilon(\omega,t,x)| \le C_0(1 + |W(t)|^{N_0})$, the above equation defines μ_M^ε as a bounded sequence of Radon measures on $\Omega_M \times [0,T] \times \overline{\mathbb{R}^d} \times \mathcal{K} \times [-L_M, L_M]$, where $\overline{\mathbb{R}^d}$ is the one point compactification of $\mathbb{R}^d$ generated by $C_0(\mathbb{R}^d)$, the continuous functions on $\mathbb{R}^d$ vanishing at ∞. Since the space of the Radon measures on $\Omega_M \times [0,T] \times \overline{\mathbb{R}^d} \times \mathcal{K} \times [-L_M, L_M]$ is compact in the weak-$\star$ topology by the Banach-Alaoglu theorem, we can find a subsequence $\mu_M^{\varepsilon_{M,k}}$ converging to some Radon measure μ_M on $\Omega_M \times [0,T] \times \overline{\mathbb{R}^d} \times \mathcal{K} \times [-L_M, L_M]$. Making $M = 1, 2, \cdots$, we can extract for each $M > 1$ a subsequence $\varepsilon_{M,k}$ from the subsequence obtained for $M-1$, $\varepsilon_{M-1,k}$, inductively, and then take the diagonal subsequence $\varepsilon_{k,k} =: \varepsilon_k$. Observe that μ_M^ε restricted to Ω_{M-1}, coincides with μ_{M-1}^ε. Therefore, the limit measure $\mu = \lim \mu_k^{\varepsilon_k}$, which is well defined in $\Omega_M \times [0,T] \times \mathbb{R}^d \times \mathcal{K} \times \mathbb{R}$, for each $M \in \mathbb{N}$, is then defined in $\Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{K} \times \mathbb{R}$ and coincides with μ_M when restricted to $\Omega_M \times [0,T] \times \mathbb{R}^d \times \mathcal{K} \times \mathbb{R}$. In particular, for all $C_c(\Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}; \mathcal{A}(\mathbb{R}^d))$ we have

$$\lim_{k\to\infty} \int\limits_{\Omega} \int\limits_0^T \int\limits_{\mathbb{R}^d} \Psi\left(\omega, t, x, \frac{x}{\varepsilon_k}, u^{\varepsilon_k}\right) d\mathbb{P} \, dt \, w_N dx = \int\limits_{\Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{K} \times \mathbb{R}} \Psi(\omega,t,x,y,\xi) \, d\mu(\omega,t,x,y,\xi). \quad (2.6)$$

Now it is easy to check that the projection of the measure μ, obtained above, over $\Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{K}$ is equal to $d\mathbb{P} \, dt \, w_N dx \, d\mathfrak{m}(y)$, since this is true for any μ_M^ε. We can then apply the theorem on disintegration of measures (see, e.g., theorem 2.28 in [4], whose extension to the present case is straightforward) to conclude the existence of a

$d\mathbb{P}\,dt\,w_N dx\,d\mathfrak{m}(y)$-measurable family of probability measures $\nu_{\omega,t,x,y}$ such that, for any $\Psi \in L^1(\Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{K} \times \mathbb{R}; \mu)$ we have

$$\int\limits_{\Omega\times[0,T]\times\mathbb{R}^d\times\mathcal{K}\times\mathbb{R}} \Psi(\omega,t,x,y,\xi)\,d\mu(\omega,t,x,y,\xi)$$
$$= \int\limits_{\Omega\times[0,T]\times\mathbb{R}^d\times\mathcal{K}} \left(\int\limits_{\mathbb{R}} \Psi(\omega,t,x,y,\xi)\,d\nu_{\omega,t,x,y}(\xi) \right) d\mathbb{P}\,dt\,w_N dx\,d\mathfrak{m}(y).$$

In particular, item (1) follows.

As for (2), it is enough to prove the result for all $A \in \mathcal{F}_t$, $t \geq 0$. So, take $A \in \mathcal{F}_{T_0}$ for some $T_0 \geq 0$. We can repeat the above construction for $T = 1, 2, \cdots$, starting at $T = k$ with the subsequence obtained in $T = k-1$ and so, using again the diagonal argument, we may define a subsequence which is good for any time interval $[0,T]$, with $T > 0$ arbitrary. In particular, we may assume that, for each $M \in \mathbb{N}$, $A \cap \Omega_M$ is a Borel set in our topology for Ω_M. Therefore, given $M \in \mathbb{N}$, we can find sets K and V with K compact and V open in Ω_M satisfying $K \subset A \cap \Omega_M \subset V$ and such that $\mathbb{P}(A \cap \Omega_M \setminus K)$ and $\mathbb{P}(V \setminus A \cap \Omega_M)$ are arbitrarily small. We can also find $\psi \in C(\Omega_M)$, with $1_K \leq \psi \leq 1_V$. Using a sequence of such $\psi \in C(\Omega_M)$ in (2.6), we get, for any $\Psi \in C_c([0,\infty) \times \mathbb{R}^d \times \mathcal{K} \times \mathbb{R})$,

$$\lim_{k\to\infty} \int\limits_{A\cap\Omega_M} \int_0^T \int\limits_{\mathbb{R}^d} \Psi\left(t,x,\frac{x}{\varepsilon_k},u^{\varepsilon_k}\right) d\mathbb{P}\,dt\,w_N dx$$
$$= \int\limits_{A\cap\Omega_M\times[0,T]\times\mathbb{R}^d\times\mathcal{K}\times\mathbb{R}} \Psi(t,x,y,\xi)\,d\mu(\omega,t,x,y,\xi). \quad (2.7)$$

Making $M \to \infty$, we get (2).

Concerning (3), given $\zeta \in C([0,T])$, with $\|\zeta\|_{L^1([0,T])} = 1$ and $\varphi \in C(\mathcal{K})$, with $\|\varphi\|_{L^1(\mathcal{K})} = 1$, for each M we have

$$\mathbb{E}_{\Omega_M} \int_0^T \zeta(t) \left(\int\limits_{\mathcal{K}} \varphi(y) \left(\int\limits_{\mathbb{R}^d} w_N(x) \left(\int\limits_{\mathbb{R}} |\xi|^p\,d\nu_{\omega,t,x,y}(\xi) \right) dx \right) d\mathfrak{m}(y) \right) dt$$
$$= \lim_{k\to\infty} \mathbb{E}_{\Omega_M} \int_0^T \zeta(t) \left(\int\limits_{\mathbb{R}^d} \varphi\left(\frac{x}{\varepsilon_k}\right) w_N(x)\,|u^{\varepsilon_k}|^p\,dx \right) dt$$
$$\leq \lim_{k\to\infty} C\,\mathbb{E} \int_0^T \zeta(t) \int\limits_{\mathbb{R}^d} \left(1 + |W(t)|^{N_0}\right)^p \left|\varphi\left(\frac{x}{\varepsilon_k}\right)\right| w_N(x)\,dx\,dt$$

$$\leq C\|\zeta\|_{L^1([0,T])}\|\varphi\|_{L^1(\mathcal{K})}\int_{\mathbb{R}^d} w_N(x)\,dx\,\mathbb{E}\sup_{0\leq t\leq T}\left(1+|W(t)|^{N_0}\right)^p.$$

Since the right-hand side does not depend on M we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T \zeta(t)\left(\int_{\mathcal{K}}\varphi(y)\left(\int_{\mathbb{R}^d} w_N(x)\left(\int_{\mathbb{R}}|\xi|^p\,d\nu_{\omega,t,x,y}(\xi)\right)dx\right)d\mathfrak{m}(y)\right)dt\\
&\quad\leq C\|\zeta\|_{L^1([0,T])}\|\varphi\|_{L^1(\mathcal{K})}\int_{\mathbb{R}^d} w_N(x)\,dx\,\mathbb{E}\sup_{0\leq t\leq T}(1+|W(t)|^{N_0})^p\\
&\quad\leq C\|\zeta\|_{L^1([0,T])}\|\varphi\|_{L^1(\mathcal{K})}\int_{\mathbb{R}^d} w_N(x)\,dx\,\mathbb{E}\left(1+|W(T)|^{N_0}\right)^p,
\end{aligned}$$

where the latter inequality follows from Doob's maximal inequality, and the fact that $(1+|W(t)|^{N_0})^p$ is a submatingale (see, e.g., [17]). Taking the sup for $\zeta\in L^1([0,T])$, with $\|\zeta\|_{L^1([0,T])}=1$, and $\varphi\in L^1(\mathcal{K})$, with $\|\varphi\|_{L^1(\mathcal{K})}=1$, we finally get

$$\mathbb{E}\sup_{t\in[0,T]}\sup_{y\in\mathcal{K}}\int_{\mathbb{R}^d} w_N(x)\left(\int_{\mathbb{R}}|\xi|^p\,d\nu_{\omega,t,x,y}(\xi)\right)dx\leq C_{T,N,p},$$

and so (2.3) follows.

Concerning (4), for $\Psi\in C([0,\infty)\times\mathbb{R}^d\times\mathbb{R};\mathcal{A}(\mathbb{R}^d))$ is such that $|\Psi(t,x,y,\xi)|\leq 1_{[0,T_0]}(t)C(1+|\xi|^p)$, for some $p\geq 1$ and $T_0>0$,

$$\begin{aligned}
&\lim_{k\to\infty}\mathbb{E}_A\int_{[0,\infty)\times\mathbb{R}^d}\Psi\left(t,x,\frac{x}{\varepsilon_k},u^{\varepsilon_k}(t,x)\right)w_N(x)\,dt\,dx\\
&\quad=\frac{\mathbb{P}(A\cap\Omega_M)}{\mathbb{P}(A)}\lim_{k\to\infty}\mathbb{E}_{A\cap\Omega_M}\int_{[0,T]\times\mathbb{R}^d}\Psi\left(t,x,\frac{x}{\varepsilon_k},u^{\varepsilon_k}(t,x)\right)w_N(x)\,dt\,dx+R_M\\
&\quad=\frac{\mathbb{P}(A\cap\Omega_M)}{\mathbb{P}(A)}\mathbb{E}_{A\cap\Omega_M}\int_{[0,T]\times\mathbb{R}^d\times\mathcal{K}}\left\langle\nu_{\omega,t,x,y},\tilde\Psi(t,x,y,\cdot)\right\rangle w_N(x)\,dt\,dx\,d\mathfrak{m}(y)+R_M,
\end{aligned}$$

where,

$$\begin{aligned}
|R_M|&\leq\frac{\mathbb{P}(A\cap(\Omega\setminus\Omega_M))}{\mathbb{P}(A)}\mathbb{E}_{A\cap(\Omega\setminus\Omega_M)}\int_{[0,T]}C(1+|W(t)|^{N_0})^p\,dt\int_{\mathbb{R}^d}w_N(x)\,dx\\
&=\int_{\mathbb{R}^d}w_N(x)\,dx\frac{1}{\mathbb{P}(A)}\int_{A\cap(\Omega\setminus\Omega_M)}\int_{[0,T]}C(1+|W(t)|^{N_0})^p\,dt\,d\mathbb{P},
\end{aligned}$$

which yields (2.2), for such $\Psi(t,x,y,\xi)$, by making $M\to\infty$. In particular, (2.4) follows for $\ell(\omega)=\frac{1}{\mathbb{P}(A)}1_A$, for $A\in\mathcal{F}$. Now, given any $\ell\in L^2(\Omega)$, (2.4) follows by approximating ℓ in $L^2(\Omega)$ by finite linear combinations of indicator functions, which concludes the proof of (4).

We now pass to the proof of (5). Let ε_k be the subsequence obtained above. First, we note that (2.1) and the assumed bound on Ψ implies that the sequence

$$(\omega,t)\to\int\limits_{\mathbb{R}^d}\Psi\left(t,x,\frac{x}{\varepsilon_k},u^{\varepsilon_k}(t,x)\right)w_N(x)\,dx,\qquad k\in\mathbb{N}$$

is uniformly bounded in $L^2(\Omega\times[0,T])$ and so it has a subsequence that converges weakly in $L^2(\Omega\times[0,T])$. Since each element of the sequence is predictable, then the limit, which by (2.2) equals

$$\int\limits_{\mathbb{R}^d}\int\limits_{\mathcal{K}}\langle\nu_{\omega,t,x,y},\Psi(t,x,y,\cdot)\rangle\,w_N\,dx\,d\mathfrak{m}(y),$$

is also predictable.

Fix $T>0$, and consider the sequence of random variables

$$X_k:=\int\limits_0^T\int\limits_{\mathbb{R}^d}\Psi\left(t,x,\frac{x}{\varepsilon_k},u^{\varepsilon_k}(t,x)\right)w_N(x)\,dx\,dW(t).$$

Define also

$$X:=\int\limits_0^T\int\limits_{\mathbb{R}^d\times\mathcal{K}}\langle\nu_{\omega,t,x,y},\Psi(t,x,y,\cdot)\rangle\,w_N(x)\,d\mathfrak{m}(y)\,dx\,dW(t).$$

To prove (5) it is enough to show that any subsequence $\{X_{k_j}\}$ has a further subsequence that converges to X weakly in $L^2(\Omega)$.

Take any subsequence $\{X_{k_j}\}_j$. By (2.1) and the Itô isometry, we have that the $\{X_{k_j}\}_j$ is uniformly bounded in $L^2(\Omega)$. Thus, it has a further subsequence which converges weakly to some $\tilde{X}\in L^2(\Omega)$. For simplicity of notation, we denote this sub-subsequence by $\{X_{k_j}\}_j$ as well. In particular, for any predictable square integrable process $C(t)$ we have that

$$\lim_{j\to\infty}\mathbb{E}\left(X_{k_j}\int\limits_0^T C(t)\,dW(t)\right)=\mathbb{E}\left(\tilde{X}\int\limits_0^T C(t)\,dW(t)\right).\tag{2.8}$$

On the other hand, using the Itô isometry and applying (2.2) we see that

$$\lim_{j\to\infty} \mathbb{E}\left(X_{k_j} \int_0^T C(t)\, dW(t) \right)$$
$$= \lim_{j\to\infty} \mathbb{E}\left(\int_0^T C(t) \int_{\mathbb{R}^d} \Psi\left(t, x, \frac{x}{\varepsilon_{k_j}}, u^{\varepsilon_{k_j}}(t,x)\right) w_N(x)\, dx\, dt \right)$$
$$= \mathbb{E}\left(\int_0^T C(t) \int_{\mathbb{R}^d\times\mathcal{K}} \langle \nu_{\omega,t,x,y}, \Psi(t,x,y,\cdot)\rangle\, w_N(x)\, dx\, d\mathfrak{m}(y)\, dt \right)$$

Now, using the Itô isometry once again we see that

$$\lim_{j\to\infty} \mathbb{E}\left(X_{k_j} \int_0^T C(t)\, dW(t) \right) = \mathbb{E}\left(X \int_0^T C(t)\, dW(t) \right). \tag{2.9}$$

Comparing (2.8) and (2.9) we can conclude that $X = \tilde{X}$ a.s.. Indeed, note that X is $\mathcal{F}_T$-measurable, since every X_k is. Also, note that $\mathbb{E}(\tilde{X}) = \lim_{j\to\infty} \mathbb{E}(X_{k_j}) = 0$. Then, we can define the $\mathcal{F}_t$-martingale $Y(t)$ by $Y(t) := \mathbb{E}(\tilde{X}|\mathcal{F}_t)$, $0 \le t \le T$ (which is sometimes called the Doob martingale associated to $\tilde{X}$). By the martingale representation theorem (see, e.g., [52]) we have that there is some predictable integrable process $D(t)$ such that

$$Y(t) = \int_0^t D(s)\, dW(s).$$

Then, choosing

$$C(t) = D(t) - \int_{\mathbb{R}^d\times\mathcal{K}} \langle \nu_{\omega,t,x,y}, \Psi(t,x,y,\cdot)\rangle\, w_N(x)\, dx\, d\mathfrak{m}(y),$$

by virtue of (2.8) and (2.9), we have that

$$\mathbb{E}\left(\left|\tilde{X} - X\right|^2\right) = \mathbb{E}\left((\tilde{X} - X) \int_0^T C(t)\, dW(t) \right) = 0,$$

which proves the claim.

Since this holds for any sub-subsequence of $\{X_k\}_k$ we have that the whole sequence converges to X weakly in $L^2(\Omega)$. In particular, given $A \in \mathcal{F}$ we have that

$$\lim_{k\to\infty} \mathbb{E}\left(1_A X_k\right) = \mathbb{E}\left(1_A X\right),$$

which yields (2.5).

Moreover, to prove the assertion about the predictability of

$$\int_{\mathbb{R}^d} \langle \nu_{\omega,t,x,y}, \Psi(t,x,y,\cdot)\rangle\, w_N\, dx,$$

for $\mathfrak{m}$-a.e. $y \in \mathcal{K}$ we argue as follows. Since we are assuming that the separable ergodic algebra $\mathcal{A}(\mathbb{R}^d)$ is a subalgebra of $\mathcal{W}^*\mathrm{AP}(\mathbb{R}^d)$, we may as well assume that $\mathcal{A}(\mathbb{R}^d)$ contains the trigonometric functions $\sin\lambda\cdot y, \cos\lambda\cdot y$, for all $\lambda \in \mathbb{R}^d$ such that $\int_{\mathcal{K}} g(y)e^{i\lambda\cdot y}\, d\mathfrak{m}(y) \neq 0$, for some $g \in \mathcal{A}(\mathbb{R}^d)$, which is sometimes called the spectrum of the algebra $\mathcal{A}(\mathbb{R}^d)$, which is a countable set; otherwise we can augment $\mathcal{A}(\mathbb{R}^n)$ to a separable ergodic algebra containing such trigonometric functions. In particular, it contains an almost periodic approximation of the unit, that is, a sequence of functions in $\mathrm{AP}(\mathbb{R}^d)$, $\{\rho_k(y) : k \in \mathbb{N}\}$, such that for all $g \in \mathcal{A}(\mathbb{R}^d)$, $\rho_k * g(y) = \int_{\mathcal{K}} \rho_k(y-z)g(z)\, d\mathfrak{m}(z) \to g_*(y)$ in $C(\mathcal{K})$, where g_* is the almost periodic component of g, and so the convergence is a.e. in $\mathcal{K}$ to g; ρ_k may be taken as the Bochner-Fejér polynomials associated with the spectrum of the algebra $\mathcal{A}(\mathbb{R}^d)$ (see, e.g., [11]). Since $C(\mathcal{K})$ is dense in $L^1(\mathcal{K})$, $\rho_k * g(y) \to g(y)$ in $L^1(\mathcal{K})$ for all $g \in L^1(\mathcal{K})$. Now, from what was seen before, for each $y \in \mathcal{K}$,

$$(\omega,t) \mapsto \int_{\mathbb{R}^d}\int_{\mathcal{K}} \rho_k(y-z)\, \langle \nu_{\omega,t,x,z}, \Psi(t,x,z,\cdot)\rangle\, w_N\, dx\, d\mathfrak{m}(z),$$

is predictable, for all $k \in \mathbb{N}$.

Let us fix $T > 0$. Since $\mathcal{F}$ is countably generated, we can find a family $\{\psi_l : l \in \mathbb{N}\}$ in $L^\infty(\Omega \times [0,T])$ dense in $L^2(\Omega \times [0,T])$. Then using the bound for Ψ and (2.3), we have that for all $l \in \mathbb{N}$

$$\lim_{k\to\infty} \int_{\mathcal{K}} \rho_k(y-z) \left(\mathbb{E}\int_0^T \psi_l(\omega,t) \int_{\mathbb{R}^d} \langle \nu_{\omega,t,x,z}, \Psi(t,x,z,\cdot)\rangle\, w_N\, dx\, dt \right) d\mathfrak{m}(z)$$
$$= \mathbb{E}\int_0^T \psi_l(\omega,t) \int_{\mathbb{R}^d} \langle \nu_{\omega,t,x,y}, \Psi(t,x,y,\cdot)\rangle\, w_N\, dx\, dt \qquad (2.10)$$

in $L^1(\mathcal{K})$ and, after passing to a subsequence if necessary, the convergence is also a.e. in $\mathcal{K}$.

Let us now fix, $y \in \mathcal{K}$ in the subset of full measure in $\mathcal{K}$ for which (2.10) holds for all $l \in \mathbb{N}$. Using Jensen inequality, the bound for Ψ and (2.3), we see that the functions

$$\gamma_k(\omega,t) := \int_{\mathcal{K}} \rho_k(y-z) \int_{\mathbb{R}^d} \langle \nu_{\omega,t,x,z}, \Psi(t,x,z,\cdot)\rangle\, w_N\, dx\, d\mathfrak{m}(z), \quad k = 1,2,\cdots,$$

form a bounded sequence in $L^2(\Omega\times[0,T])$. Then, given any subsequence of this sequence, we can find a further subsequence converging weakly in $L^2(\Omega\times[0,T])$, and, because of (2.10), its weak limit in $L^2(\Omega\times[0,T])$ must be

$$\gamma(\omega,t) := \int_{\mathbb{R}^d} \langle \nu_{\omega,t,x,y}, \Psi(t,x,y,\cdot)\rangle\, w_N\, dx,$$

therefore, the whole sequence γ_k converges weakly to $\gamma(\omega,t)$ in $L^2(\Omega\times[0,T])$. Now, since the γ_k's are predictable and $L^2(\Omega\times[0,T];\mathcal{P})$ is a closed subspace of $L^2(\Omega\times[0,T])$, we deduce that $\gamma(\omega,t)$ is also predictable, for a.e. $y\in\mathcal{K}$, which concludes the proof. □

Remark 2.1. We remark that it follows from item (4) of Proposition 2.1 that given $F\in C(\mathbb{R})$ such that $|F(\xi)|\le C(1+|\xi|^p)$, for some $p\ge 1$, and letting $\overline{F}(\omega,t,x,y) := \langle \nu_{\omega,t,x,y}, F\rangle$, then for any $T>0$ we have that $F(u^{\varepsilon_k}) \rightharpoonup \overline{F(u)}$ in the weak topology of $L^2(\Omega; L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^d))$, where

$$\overline{F(u)}(\omega,t,x) = \int_{\mathcal{K}} \overline{F}(\omega,t,x,y)d\mathfrak{m}(y). \tag{2.11}$$

In particular, if $\nu_{\omega,t,x,y} = \delta_{U(\omega,t,x,y)}$, then $\overline{F(u)} = \int_{\mathcal{K}} F(U)d\mathfrak{m}(y)$.

Indeed, it suffices to take $\tilde{\Psi}$ in (2.4) of the form $\Psi(\omega,t,x,y,\xi) = \ell(\omega)\psi(t,x)F(\xi)$ with $\ell\in L^2(\Omega)$, $\psi\in C_c((0,T)\times\mathbb{R}^d)$ arbitrary to deduce (2.11), observing that, by (2.1) and the assumption on F, $F(u^{\varepsilon_k})$ is bounded in $L^2(\Omega;L^\infty((0,T)\times\mathbb{R}^d))$, therefore bounded in $L^2(\Omega;L^2((0,T)\times\{|x|<R\}))$, for each $R>0$ and the functions of the form $\ell(\omega)\psi(t,x)$ with $\ell\in L^2(\Omega)$ and $\psi\in C_c((0,T)\times\mathbb{R}^d)$ are dense in $L^2(\Omega;L^2((0,T)\times\mathbb{R}^d))$.

The next result gives sufficient conditions for the existence of correctors for the weak convergence of the sequence u^{ε_k} established by Remark 2.1.

Proposition 2.2. *Let $\nu_{\omega,t,x,y}$ be the stochastic two-scale Young measure constructed in Proposition 2.1. Assume $\nu_{\omega,t,x,y} = \delta_{U(\omega,t,x,y)}$ for*

(a) $U\in L^2(\Omega;\mathcal{A}(\mathbb{R}^d;L^\infty((0,T)\times\mathbb{R}^d)))$, *or*
(b) $U\in L^2(\Omega;\mathcal{B}^2(\mathbb{R}^d;C_b([0,T]\times\mathbb{R}^d)))$.

Then, $u^{\varepsilon_k} - U\left(\omega,t,x,\frac{x}{\varepsilon_k}\right) \to 0$ *strongly in* $L^2(\Omega;L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^d))$.

Proof. First we observe that, because we only seek to show convergence in $L^2(\Omega;L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^d))$, for item (b), we can just consider $U\in L^2(\Omega;\mathcal{B}^2(\mathbb{R}^d;C_c((0,T)\times\mathbb{R}^d)))$. Second, we see that the result would follow immediately from Proposition 2.1 if we were allowed to use

$$\tilde{\Psi}(\omega,t,x,y,\xi)=|\xi-U(\omega,t,x,y)|^2=\xi^2-2\xi U(\omega,t,x,y)+|U(\omega,t,x,y)|^2, \tag{2.12}$$

as a test function in (2.4). Let us check this possibility for each of the terms in the right-hand of the last equation in (2.12). The first term, ξ^2, is good and, by Remark 2.1, we have

$$\lim_{k\to\infty}\mathbb{E}\int\limits_{(0,T)\times\mathbb{R}^d}|u^{\varepsilon_k}|^2 w_N\,dx\,dt=\mathbb{E}\int\limits_{(0,T)\times\mathbb{R}^d\times\mathcal{K}}|U(\omega,t,x,y)|^2 w_N(x)\,d\mathfrak{m}(y)\,dx\,dt.$$

Concerning the second term, $-2\xi U(\omega,t,x,y)$, we observe first that if

$$U\in L^2(\Omega;\mathcal{A}(\mathbb{R}^d;C_c((0,T)\times\mathbb{R}^d)))$$

we could approximate it in

$$L^2(\Omega;\mathcal{A}(\mathbb{R}^d;C_c((0,T)\times\mathbb{R}^d)))$$

by finite linear combinations of functions of the form $1_A(\omega)\psi(t,x,y)$ with $\psi\in\mathcal{A}(\mathbb{R}^d;C_c((0,T)\times\mathbb{R}^d))$, $A\in\mathcal{F}$, and for such functions we could apply Proposition 2.1 to obtain

$$\lim_{k\to\infty}\mathbb{E}\int\limits_{(0,T)\times\mathbb{R}^d}u^{\varepsilon_k}U(\omega,t,x,\frac{x}{\varepsilon_k})w_N\,dx\,dt$$

$$=\mathbb{E}\int\limits_{(0,T)\times\mathbb{R}^d\times\mathcal{K}}|U(\omega,t,x,y)|^2 w_N(x)\,d\mathfrak{m}(y)\,dx\,dt, \tag{2.13}$$

so this equation holds for $U\in L^2(\Omega;\mathcal{A}(\mathbb{R}^d;C_c((0,T)\times\mathbb{R}^d)))$.

Now, in case (a), for $U\in L^2(\Omega;\mathcal{A}(\mathbb{R}^d;L^\infty((0,T)\times\mathbb{R}^d)))$, we can approximate U in $L^2(\Omega;\mathcal{A}(\mathbb{R}^d;L^2_{\rm loc}((0,T)\times\mathbb{R}^d)))$ by a sequence of functions in $L^2(\Omega;\mathcal{A}(\mathbb{R}^d;C_c((0,T)\times\mathbb{R}^d)))$ to obtain that (2.13) holds also for $U\in L^2(\Omega;\mathcal{A}(\mathbb{R}^d;L^\infty((0,T)\times\mathbb{R}^d)))$.

In case (b), if $U\in L^2(\Omega;\mathcal{B}^2(\mathbb{R}^d;C_c((0,T)\times\mathbb{R}^d)))$, we can approximate U in

$$L^2(\Omega;\mathcal{B}^2(\mathbb{R}^d;C_c((0,T)\times\mathbb{R}^d)))$$

by a sequence of functions in $L^2(\Omega;\mathcal{A}(\mathbb{R}^d;C_c((0,T)\times\mathbb{R}^d)))$ and for the latter we have already shown that equation (2.13) holds, so it also holds $U\in L^2(\Omega;\mathcal{B}^2(\mathbb{R}^d;C_c((0,T)\times\mathbb{R}^d)))$.

Now, concerning the last term in the right-hand side of the last equation in (2.12), it does not depend on ξ, so we just need to use the well known fact that, for a function $\Psi\in L^2(\Omega;\mathcal{A}(\mathbb{R}^d;L^\infty((0,T)\times\mathbb{R}^d)))$, in case (a), and $\Psi\in L^2(\Omega;\mathcal{B}^2(\mathbb{R}^d;C_c((0,T)\times\mathbb{R}^d)))$ in case (b),

$$\Psi(\omega,t,x,\frac{x}{\varepsilon_k}) \rightharpoonup \int_{\mathcal{K}} \Psi(\omega,t,x,y)\,d\mathfrak{m}(y),$$

in the weak topology of $L^2(\Omega; L^2_{\rm loc}((0,T)\times\mathbb{R}^d))$, and so we get

$$\lim_{k\to\infty} \mathbb{E} \int_{(0,T)\times\mathbb{R}^d} \left| U\left(\omega,t,x,\frac{x}{\varepsilon_k}\right)\right|^2 w_N\,dx\,dt$$
$$= \mathbb{E} \int_{(0,T)\times\mathbb{R}^d\times\mathcal{K}} |U(\omega,t,x,y)|^2 w_N\ d\mathfrak{m}(y)\,dx\,dt.$$

Putting together the facts described above, we conclude that

$$\lim_{k\to\infty} \mathbb{E} \int_{(0,T)\times\mathbb{R}^d} \left| u^{\varepsilon_k} - U\left(\omega,t,x,\frac{x}{\varepsilon_k}\right)\right|^2 w_N\,dx\,dt = 0,$$

which finishes the proof. □

3. Stochastic nonlinear transport, proof of Theorem 1.1

In this section we prove Theorem 1.1. By assumption, we have that $h = \sigma'\sigma$. Set

$$\psi_\alpha(t) = g(\alpha + \kappa_0 W(t)), \tag{3.1}$$

where g is a solution of the ODE $g'(\xi) = \sigma(g(\xi))$ and $\alpha \in \mathbb{R}$. We assert that ψ_α is a solution of equation (1.1), for any $\alpha \in \mathbb{R}$. Indeed, since $g''(\xi) = \sigma'(g(\xi))g'(\xi) = \sigma'(g(\xi))\sigma(g(\xi)) = h(g(\xi))$, the assertion follows from the Itô formula.

By the Stochastic Kružkov inequality, cf. Proposition 6.1, a.s. we have

$$\int_0^\infty \int_{\mathbb{R}^d} \Big\{ |u^\varepsilon - \psi_\alpha(t)|\,\phi_t + \operatorname{sgn}(u^\varepsilon - \psi_\alpha(t))\,(f(u^\varepsilon) - f(\psi_\alpha(t)))\,a\left(\frac{x}{\varepsilon}\right)\cdot\nabla_x\phi$$
$$+ \frac{1}{2}\kappa_0^2 \operatorname{sgn}(u^\varepsilon - \psi_\alpha)\,(h(u^\varepsilon) - h(\psi_\alpha))\,\phi \Big\}\,dx\,dt + \int_{\mathbb{R}^d} \left| U_0\left(x,\frac{x}{\varepsilon}\right) - g(\alpha)\right| \phi\,dx$$
$$+ \int_0^\infty \int_{\mathbb{R}^d} \kappa_0 \operatorname{sgn}(u^\varepsilon - \psi_\alpha)\,(\sigma(u^\varepsilon) - \sigma(\psi_\alpha))\,\phi\,dx\,dW(t) \geq 0. \tag{3.2}$$

A similar inequality holds with $(\cdot - \cdot)_+$ instead of $|\cdot - \cdot|$, which easily follows by adding to (3.2) the difference of integral equations defining weak solutions for $u^\varepsilon(t,x)$ and for $\psi_\alpha(t)$. From (3.2) we easily get the comparison principle

$$\mathbb{E}\int_0^\infty\int_{\mathbb{R}^d}\Big\{(u^\varepsilon-\psi_\alpha(t))_+\,\phi_t+\operatorname{sgn}(u^\varepsilon-\psi_\alpha(t))_+\,(f(u^\varepsilon)-f(\psi_\alpha(t)))\,a\left(\frac{x}{\varepsilon}\right)\cdot\nabla_x\phi$$
$$+\frac{1}{2}\kappa_0^2\operatorname{sgn}(u^\varepsilon-\psi_\alpha)_+\,(h(u^\varepsilon)-h(\psi_\alpha))\,\phi\Big\}\,dx\,dt$$
$$+\int_{\mathbb{R}^d}\left(U_0\left(x,\frac{x}{\varepsilon}\right)-g(\alpha)\right)_+\phi(0,x)\,dx\geq 0,$$

which, when $g(\alpha_1)\leq U_0\left(x,\frac{x}{\varepsilon}\right)\leq g(\alpha_2)$, for some $\alpha_1,\alpha_2\in\mathbb{R}$, implies a.s. the following uniform boundedness of the solutions of (1.1)-(1.2)

$$\psi_{\alpha_1}(t)\leq u^\varepsilon(t,x)\leq\psi_{\alpha_2}(t),\quad\text{for a.e. }(t,x).\tag{3.3}$$

We recall that it follows from the definition of entropy solution (see Definition 1.1), for any C^2 convex function $\eta:\mathbb{R}\to\mathbb{R}$, and q satisfying $q'=\eta'f'$, u^ε satisfies

$$\int_0^\infty\int_{\mathbb{R}^d}\Big\{\eta(u^\varepsilon)\phi_t+q(u^\varepsilon)a\left(\frac{x}{\varepsilon}\right)\cdot\nabla\phi+\frac{1}{2}\kappa_0^2\left(\eta'(u^\varepsilon)h(u^\varepsilon)+\eta''(u^\varepsilon)\sigma(u^\varepsilon)^2\right)\phi\Big\}\,dxdt$$
$$\int_0^\infty\int_{\mathbb{R}^d}\kappa_0\eta'(u^\varepsilon)\sigma(u^\varepsilon)\phi\,dx\,dW(t)+\int_{\mathbb{R}^d}\eta\left(U_0\left(x,\frac{x}{\varepsilon}\right)\right)\phi(0,x)\,dx\geq 0.\tag{3.4}$$

Now, in equation (3.4) we take $\phi(t,x)=\varepsilon\varphi\left(\frac{x}{\varepsilon}\right)\zeta(t)\vartheta(x)$, where $0\leq\varphi\in\mathcal{A}(\mathbb{R}^d)$, $\nabla\varphi\in\mathcal{A}(\mathbb{R}^d;\mathbb{R}^d)$, $0\leq\zeta\in C_c^\infty([0,\infty))$ and $0\leq\vartheta\in C_c^\infty(\mathbb{R}^d)$, take conditional expectation with respect to an arbitrary $A\in\mathcal{F}$, and let $\varepsilon\to 0$, along a subsequence for which u^ε generates a two-scale Young measure $\nu_{\omega,t,x,y}$, according to Proposition 2.1, to obtain, since $A\in\mathcal{F}$ is arbitrary and we drop the ω subscript from $\nu_{\omega,t,x,y}$, a.s.,

$$\int_0^\infty\zeta(t)\int_{\mathcal{K}}\left\langle\sigma_{t,y}^\vartheta,q(\cdot)\right\rangle a(y)\cdot\nabla_y\varphi\,d\mathfrak{m}(y)\,dt\geq 0,\tag{3.5}$$

where

$$\sigma_{t,y}^\vartheta:=\int_{\mathbb{R}^d}\vartheta(x)\nu_{t,x,y}\,dx.$$

By applying inequality (3.5) to $C\pm\varphi$, with $C=\|\varphi\|_\infty$, and using the arbitrariness of φ,

$$y\mapsto\left\langle\sigma_{t,y}^\vartheta,q(\cdot)\right\rangle\in\mathcal{S},\quad\text{for a.e. }t\in(0,T).\tag{3.6}$$

Now, for any $\eta\in C^2$, $C|u|^2+\eta(u)$ is convex for C sufficiently large (depending on η), so (3.6) holds for any $\eta\in C^2$ and, by approximation, for any Lipschitz continuous η. Now, if $f' \neq 0$, given any $\eta \in C^1$, defining $\tilde{\eta}' = \eta'/f'$, the entropy-flux associated to $\tilde{\eta}$ is $\tilde{q} = \eta$, so that (3.6) gives

$$y \mapsto \left\langle \sigma^{\vartheta}_{t,y}, \eta(\cdot) \right\rangle \in \mathcal{S}, \qquad \text{for a.e. } t \in (0,T) \text{ and all } \eta \in C^1. \tag{3.7}$$

In the more general case, where $|\{u : f'(u) = 0\}| = 0$, we argue as in [30] to deduce that (3.7) still holds. Namely, for any open interval I with $\bar{I} \subset \mathbb{R} \setminus E_0$, where $E_0 = \{u : f'(u) = 0\}$, we define $\eta_I' = \chi_I/f'$, where χ_I is the indicator function of the interval I, whose corresponding entropy flux is q_I, with $q_I' = \chi_I$. Now, by approximation with convergence everywhere, the property may be extended to any open interval in $\mathbb{R} \setminus E_0$. Also, since the intersection of any open set with $\mathbb{R} \setminus E_0$ is a countable union of intervals in $\mathbb{R} \setminus E_0$, by approximation with convergence everywhere we get the property for any such intersection, and since E_0 has measure zero, the primitive of such intersection is equal to the primitive of the interval itself, so the property holds for q_I, where I is any open interval, and hence for q_I where I is any interval. Since any C^1 function may be uniformly approximated by piecewise linear functions, which are linear combinations of q_I functions, we deduce that (3.7) also holds in this more general case.

Now, we take $\phi(t,x) = \varphi\left(\frac{x}{\varepsilon}\right)\vartheta(t,x)$ in (3.4), where $0 \leq \varphi \in \mathcal{S}^\dagger$ and $0 \leq \vartheta \in C_c^\infty(\mathbb{R}^{d+1})$, and take the conditional expectation with respect to an arbitrary $A \in \mathcal{F}$. Passing to the limit as $\varepsilon \to 0$ in (3.4), along a subsequence which generates a two-scale Young measure according to Proposition 2.1, as above, we get, a.s.,

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathcal{K}} \Big\{ & \langle \nu_{t,x,y}, \eta \rangle \vartheta_t + \langle \nu_{t,x,y}, q \rangle \tilde{a}(y) \cdot \nabla \vartheta + \frac{1}{2}\kappa_0^2 \left\langle \nu_{t,x,y}, (\eta' h + \eta'' \sigma^2) \right\rangle \vartheta \Big\} \\
& \times \varphi(y)\, d\mathfrak{m}(y)\, dx\, dt \\
& + \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathcal{K}} \kappa_0 \langle \nu_{t,x,y}, \eta' \sigma \rangle \vartheta\, \varphi(y)\, d\mathfrak{m}(y)\, dx\, dW(t) \\
& + \int_{\mathbb{R}^d} \int_{\mathcal{K}} \eta\left(U_0(x,y)\right) \vartheta(0,x) \varphi(y)\, d\mathfrak{m}(y)\, dx \geq 0.
\end{aligned} \tag{3.8}$$

Observe that instead of $a(y)$ we have, in (3.8), $\tilde{a}(y)$, the vector field whose components are the orthogonal projections of the corresponding components of $a(y)$ onto $\mathcal{S}$, in $L^2(\mathcal{K})$, which is due to (3.7). Indeed, we use the fact that, for $g \in (\mathcal{S} \cap L^\infty)(\mathcal{K})$ and $r \in L^2(\mathcal{K})$, the orthogonal projection of gr onto $\mathcal{S}$, $\widetilde{gr}$, is equal to $g\tilde{r}$, where $\tilde{r}$ is the orthogonal projection of r onto $\mathcal{S}$ (see Proposition 4.2 in [30]). Since we assume that $\mathcal{S}^\dagger$ is dense in $\mathcal{S}$, we can extend (3.8) from $0 \leq \varphi \in \mathcal{S}$ to all $0 \leq \varphi \in L^2(\mathcal{K})$, where we also use condition (1.3) on the initial data $U_0(x,y)$. Therefore, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $\mathfrak{m}$-a.e. $y \in \mathcal{K}$, we have, for all $\vartheta \in C_c^\infty(\mathbb{R}^{d+1})$,

$$\int_0^{\infty}\int_{\mathbb{R}^d}\Big\{\langle\nu_{t,x,y},\eta\rangle\,\vartheta_t+\langle\nu_{t,x,y},q\rangle\,\tilde a(y)\cdot\nabla\vartheta+\frac{1}{2}\kappa_0^2\left\langle\nu_{t,x,y},(\eta' h+\eta''\sigma^2)\right\rangle\vartheta\Big\}\,dx\,dt$$
$$+\int_0^{\infty}\int_{\mathbb{R}^d}\kappa_0\,\langle\nu_{t,x,y},\eta'\sigma\rangle\,\vartheta\,dx\,dW(t)$$
$$+\int_{\mathbb{R}^d}\eta\left(U_0(x,y)\right)\vartheta(0,x)\,dx\geq 0. \tag{3.9}$$

Now for a convex $\eta\in C^3(\mathbb{R})$, such that $\eta''\in C_c^1(\mathbb{R})$, we have the obvious formulas

$$\eta(\cdot)=\int_{\mathbb{R}}\eta'(\xi)\mathbb{I}_{(-\infty,\,\cdot)}(\xi)\,d\xi,$$
$$q(\cdot)=\int_{\mathbb{R}}f'(\xi)\eta'(\xi)\mathbb{I}_{(-\infty,\,\cdot)}(\xi)\,d\xi,$$
$$\left(\eta' h+\eta''\sigma^2\right)(\cdot)=\int_{\mathbb{R}}\left(\eta' h'+\eta''(h+2\sigma\sigma')+\eta'''\sigma^2\right)(\xi)\mathbb{I}_{(-\infty,\,\cdot)}(\xi)\,d\xi, \tag{3.10}$$
$$\left(\eta'\sigma\right)(\cdot)=\int_{\mathbb{R}}\left(\eta'\sigma'+\eta''\sigma\right)(\xi)\mathbb{I}_{(-\infty,\,\cdot)}(\xi)\,d\xi,$$
$$\eta\left(U_0(x,y)\right)=\int_{\mathbb{R}}\eta'(\xi)\mathbb{I}_{(-\infty,U_0(x,y))}(\xi)\,d\xi.$$

Therefore, for a fixed $y\in\mathcal{K}$, setting $\rho_1(t,x,\xi)=\nu_{t,x,y}((\xi,+\infty))$, we get from (3.9)

$$\int_0^{\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\Big\{\rho_1(t,x,\xi)\eta'(\xi)\vartheta_t+\rho_1(t,x,\xi)f'(\xi)\eta'(\xi)\tilde a(y)\cdot\nabla\vartheta$$
$$+\frac{1}{2}\kappa_0^2\rho_1(t,x,\xi)\left(\eta' h'+\eta''(h+2\sigma\sigma')+\eta'''\sigma^2\right)(\xi)\vartheta\Big\}\,d\xi\,dx\,dt$$
$$+\int_0^{\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\kappa_0\rho_1(t,x,\xi)\left(\eta'\sigma'+\eta''\sigma\right)(\xi)\vartheta\,d\xi\,dx\,dW(t)$$
$$+\int_{\mathbb{R}^d}\int_{\mathbb{R}}\eta'(\xi)\mathbb{I}_{(-\infty,U_0(x,y))}(\xi)\vartheta(0,x)\,d\xi\,dx\geq 0. \tag{3.11}$$

Thus, seeing the left-hand side of the inequality above as a distribution applied to $\eta''(\xi)\vartheta(t,x)$, we conclude that it is indeed a measure, which we denote by $m_1(t,x,\xi)$. We can then extend the identity defining m_1 to any η of the form $\eta(\xi)=\int_{-\infty}^{\xi}\zeta(s)\,ds$, for some $\zeta \in C_c^\infty(\mathbb{R})$. This way we deduce that ρ_1 is a weak solution of the following kinetic equation

$$\frac{\partial \rho_1}{\partial t} + f'(\xi)\tilde{a}(y)\cdot\nabla\rho_1 + \frac{1}{2}\kappa_0^2 h(\xi)\partial_\xi\rho_1 - \frac{1}{2}\kappa_0^2\partial_\xi(\sigma^2\partial_\xi\rho_1) = \partial_\xi m_1 - \kappa_0\sigma\partial_\xi\rho_1\frac{dW(t)}{dt}, \quad (3.12)$$

and we see from (3.11) that ρ_1 verifies

$$\operatorname*{ess\,lim}_{t\to 0+}\int_{\mathbb{R}^{d+1}} \rho_1(t,x,\xi)\zeta(\xi)\phi(x)\,d\xi\,dx = \int_{\mathbb{R}^{d+1}} \mathbb{I}_{(-\infty,U_0(x,y))}(\xi)\zeta(\xi)\phi(x)\,d\xi\,dx, \quad (3.13)$$

for all $\zeta \in C_c^\infty(\mathbb{R})$ and all $\phi \in C_c^\infty(\mathbb{R}^d)$, see also Remark 5.13.

Considering the definition of the measure $m_1(t,x,\xi)$ from (3.11), we may check that m_1 satisfies the conditions of a kinetic measure in Definition 5.1. Also, $\rho_1(t,x,\xi)$ is a generalized kinetic function whose associated Young measure, $\nu_{t,x,y}$, satisfies (5.27), which can be verified without difficulty using the bounds (3.3).

Now, let $U(t,x,y)$ be the entropy solution of (1.4)-(1.5). According to Definition 1.2, recalling (1.6), for each $y \in \mathcal{K}$, for any convex $\eta \in C^2(\mathbb{R})$, and $0 \le \varphi \in C_c^\infty(\mathbb{R}^{d+1})$,

$$\int_Q \eta(U(y))\partial_t\varphi + q(U(y))\tilde{a}(y)\cdot\nabla\varphi + \frac{\kappa_0^2}{2}\left(\eta'(U(y))h(U(y)) + \eta''(U(y))\sigma^2(U(y))\right)\varphi\,dx\,dt$$
$$+ \kappa_0\int_0^T\int_{\mathbb{R}^d} \eta'(U(y))\sigma(U(y))\varphi\,dx\,dW(t) + \int_{\mathbb{R}^d} \eta(U_0(y))\varphi(0,x)\,dx\,dt \ge 0.$$

Setting $\rho_2(t,x,\xi) = 1_{(-\infty,U(t,x,y))}(\xi)$, using the formulas (3.10), we get from (1.6)

$$\int_0^\infty\int_{\mathbb{R}^d}\int_{\mathbb{R}}\Big\{\rho_2(t,x,\xi)\eta'(\xi)\varphi_t + \rho_2(t,x,\xi)f'(\xi)\eta'(\xi)\tilde{a}(y)\cdot\nabla\varphi$$
$$+\frac{1}{2}\kappa_0^2\rho_2(t,x,\xi)\left(\eta'h'(\xi) + \eta''(h+2\sigma\sigma')(\xi) + \eta'''\sigma^2(\xi)\right)\varphi\Big\}\,d\xi\,dx\,dt$$
$$+\int_0^\infty\int_{\mathbb{R}^d}\int_{\mathbb{R}} \kappa_0\rho_2(t,x,\xi)\left(\eta'\sigma'(\xi) + \eta''\sigma(\xi)\right)\varphi\,d\xi\,dx\,dW(t)$$
$$+\int_{\mathbb{R}^d}\int_{\mathbb{R}} \eta'(\xi)\mathbb{I}_{(-\infty,U_0(x,y))}(\xi)\varphi(0,x)\,d\xi\,dx \ge 0. \quad (3.14)$$

Thus, again, the left-hand side of (3.14) defines a measure $m_2(t,x,\xi)$ applied to $\varphi\eta''$. Therefore, as above, we see that ρ_2 is a weak solution of the kinetic equation

$$\frac{\partial \rho_2}{\partial t} + f'(\xi)\tilde{a}(y)\cdot\nabla\rho_2 + \frac{1}{2}\kappa_0^2 h(\xi)\partial_\xi\rho_2 - \frac{1}{2}\kappa_0^2\partial_\xi(\sigma^2\partial_\xi\rho_2) = \partial_\xi m_2 - \kappa_0\sigma\partial_\xi\rho_2\frac{dW(t)}{dt}, \quad (3.15)$$

and we see from (3.14) that ρ_2 verifies

$$\operatorname*{ess\,lim}_{t\to 0+}\int_{\mathbb{R}^{d+1}} \rho_2(t,x,\xi)\zeta(\xi)\phi(x)\,d\xi\,dx = \int_{\mathbb{R}^{d+1}} \mathbb{I}_{(-\infty,U_0(x,y))}(\xi)\zeta(\xi)\phi(x)\,d\xi\,dx, \quad (3.16)$$

for all $\zeta \in C_c^\infty(\mathbb{R})$ and all $\phi \in C_c^\infty(\mathbb{R}^d)$. Since ρ_2 is a standard kinetic function, a well known argument shows that the convergence in (3.16) may be strengthen to a strong convergence in $L^1(\mathbb{R}^d; w_N)$ (see also Remark 5.13).

Again, by the definition of the measure $m_2(t,x,\xi)$ from (3.14), we may check that m_2 satisfies the conditions of a kinetic measure in Definition 5.1. Also, $\rho_2(t,x,\xi)$ is trivially a generalized kinetic function whose associated Young measure $\delta_{U(t,x,y)}$, satisfies (5.27), which can be verified without difficulty using the bounds (3.3).

Due to (3.12)-(3.13) and (3.15)-(3.16) and the properties satisfied by m_1, m_2 and $\mu^1_{t,x} = \nu_{t,x,y}$, $\mu^2_{t,x} = \delta_{U(t,x,y)}$ (see, in particular, Proposition 2.1(3)), we can apply Proposition 5.1 together with the well-posedness result in Theorem 5.1 (see also (6.2) and discussion at the beginning of Section 6) to deduce that

$$\mathbb{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \rho_1(t,x,\xi)\,(1-\rho_2(t,x,\xi))\,w_N\,dx\,d\xi \le C(T)\int_{\mathbb{R}^d}\int_{\mathbb{R}} \rho_1(0,x,\xi)\,(1-\rho_2(0,x,\xi))\,w_N\,dx\,d\xi = 0,$$

for $0 < t \le T$, w_N given by (5.11), and so $\rho_1(t,x,\xi) = \rho_2(t,x,\xi)$, a.e. in $\Omega\times(0,\infty)\times\mathbb{R}^d\times\mathbb{R}$. Clearly, this implies that

$$\nu_{t,x,y} = \delta_{U(t,x,y)},$$

a.e. in $\Omega\times(0,\infty)\times\mathbb{R}^d\times\mathcal{K}$. In particular, due to the uniqueness of the limit, we deduce that the whole sequence $u^\varepsilon(t,x)$ satisfies

$$u^\varepsilon \rightharpoonup u(t,x) := \int_{\mathcal{K}} U(t,x,y)\,d\mathfrak{m}(y),$$

in the weak topology of $L^2(\Omega; L^2_{\text{loc}}((0,T)\times\mathbb{R}^d))$, for each $T>0$. Indeed, if this is not the case, then there would be a sequence $\varepsilon_j \to 0$, a test function $\psi \in L^2(\Omega; L^2((0,T)\times\mathbb{R}^d))$ and a constant $\alpha > 0$ such that

$$\left|\mathbb{E}\int_Q u^{\varepsilon_j}\,\psi dx\,dt - \mathbb{E}\int_Q u\,\psi dx\,dt\right| > \alpha, \quad \text{for all } j\in\mathbb{N}. \quad (3.17)$$

However, by the procedure above, there is a further subsequence ε_{j_k} for which $u^{\varepsilon_{j_k}}$ generates a Young measure $\nu_{\omega,t,x,y}$ that turns out to be equal to $\delta_{U(\omega,t,x,y)}$, which by Remark 2.1, contradicts (3.17).

Finally, concerning the last assertion in Theorem 1.1, it follows directly from Proposition 2.2 (b). This concludes the proof of Theorem 1.1.

4. Stiff oscillatory external force, proof of Theorem 1.2

In this section we prove Theorem 1.2. For the convenience of the reader, we rewrite here the formulas related to the homogenization problem for (1.7), beginning with (1.7) itself

$$du^\varepsilon + \nabla_x \cdot f(u^\varepsilon)\,dt = \frac{1}{\varepsilon}V'\left(\frac{x_1}{\varepsilon}\right)\,dt + \kappa_0\,\sigma_{f_1}(u^\varepsilon)\,dW + \frac{1}{2}\kappa_0^2\,h_{f_1}(u^\varepsilon)\,dt,$$

where $f = (f_1,\dots,f_d)$, $f_i : \mathbb{R} \to \mathbb{R}$ are smooth functions, $i = 1,\dots,d$, $f_1' \geq \delta_0 > 0$, $f_k' \geq 0$, $k = 2,\dots,d$. We also assume that $f' \in L^\infty(\mathbb{R};\mathbb{R}^d)$ and $f_1', f_1'', f_1''' \in L^\infty(\mathbb{R})$. $\kappa_0 \in \mathbb{R}$ is a constant. $V : \mathbb{R} \to \mathbb{R}$ is a smooth function belonging to an arbitrary ergodic algebra $\mathcal{A}(\mathbb{R}^d)$, σ_{f_1}, h_{f_1} are obtained from f_1 from the expressions

$$\sigma_{f_1}(u) := \frac{1}{f_1'(u)}, \qquad h_{f_1} := -\frac{f_1''(u)}{f_1'(u)^3}.$$

We observe that, from the assumptions on f_1, it follows that $h_{f_1}' \in L^\infty(\mathbb{R})$.

We recall that $g = f_1^{-1}$ is the inverse of f_1. We assume that, for some $v_0 \in L^\infty(\mathbb{R}^d)$, the initial data $u_0\left(x, \frac{x}{\varepsilon}\right)$ in (1.8) satisfy

$$u_0(x,y) = g(V(y) + v_0(x)).$$

We recall the auxiliary equation (1.10)

$$d\bar{u} + \nabla \cdot \bar{f}(\bar{u})\,dt = \kappa_0\,\sigma_{\bar{f}_1}(\bar{u})\,dW + \frac{1}{2}\kappa_0^2\,h_{\bar{f}_1}(\bar{u})\,dt,$$

where $\bar{f} = (\bar{f}_1, \bar{f}_2, \dots, \bar{f}_d)$, with $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_d$, satisfying (1.11), (1.12) which we recall here

$$p = \fint_{\mathbb{R}} g\left(\bar{f}_1(p) + V(z_1)\right)\,dz_1,$$

$$\bar{f}_k(p) = \fint_{\mathbb{R}} f_k \circ g\left(\bar{f}_1(p) + V(z_1)\right)\,dz_1, \quad k = 2,\dots,d,$$

and $\sigma_{\bar{f}_1}(\cdot)$, $h_{\bar{f}_1}(\cdot)$ are defined as σ_{f_1}, h_{f_1} with $\bar{f}_1(\cdot)$ instead of f_1 We recall that, from the assumptions on f and f_1, it follows from (1.11) and (1.12) that $\bar{f}$ and $\bar{f}_1$ also satisfy $\bar{f}' \in L^\infty(\mathbb{R};\mathbb{R}^d)$ and $\bar{f}_1', \bar{f}_1'', \bar{f}_1''' \in L^\infty(\mathbb{R})$.

We recall that for (1.10) we have prescribed the initial condition

$$\bar{u}(0,x)=\bar{u}_0(x):=\fint_{\mathbb{R}} u_0(x,z_1)\,dz_1=\bar{f}_1^{-1}(v_0(x)).$$

We begin the proof of Theorem 1.2 by observing that (1.7) admits special solutions of the form

$$\psi_\alpha\left(t,\frac{x_1}{\varepsilon}\right):=g\left(V\left(\frac{x_1}{\varepsilon}\right)+\kappa_0 W(t)+\alpha\right), \tag{4.1}$$

where $\alpha\in\mathbb{R}$, as a consequence of Itô's formula.

The equation (1.10) has the following special solutions

$$\psi_{*\gamma}(t):=\bar{g}\left(\gamma+\kappa_0 W(t)\right), \tag{4.2}$$

where $\bar{g}(\cdot):=\bar{f}_1^{-1}(\cdot)$, the inverse function of $\bar{f}_1(\cdot)$, that is,

$$\psi_{*\gamma}(t)=\fint_{\mathbb{R}} g\left(\gamma+\kappa_0 W(t)+V(z_1)\right)\,dz_1.$$

By the stochastic Kružkov inequality, cf. Proposition 6.1, we get a.s.

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^d}\Big\{\Big|u^\varepsilon-\psi_\alpha\left(t,\frac{x_1}{\varepsilon}\right)\Big|\phi_t+\Big|f_1(u^\varepsilon)-f_1\left(\psi_\alpha\left(t,\frac{x_1}{\varepsilon}\right)\right)\Big|\phi_{x_1}\\
&+\sum_{k=2}^d\Big|f_k(u^\varepsilon)-f_k\left(\psi_\alpha\left(t,\frac{x_1}{\varepsilon}\right)\right)\Big|\phi_{x_k}+\frac{1}{2}\kappa_0^2 S_{u^\varepsilon,\psi_\alpha^\varepsilon}\left(h_{f_1}(u^\varepsilon)-h_{f_1}\left(\psi_\alpha\left(t,\frac{x_1}{\varepsilon}\right)\right)\right)\phi\Big\}\,dx\,dt\\
&\qquad+\int_0^\infty\int_{\mathbb{R}^d}\kappa_0\Big|\sigma_{f_1}(u^\varepsilon)-\sigma_{f_1}\left(\psi_\alpha\left(t,\frac{x_1}{\varepsilon}\right)\right)\Big|\phi\,dx\,dW(t)\\
&\qquad\qquad+\int_{\mathbb{R}^d}|u_0^\varepsilon-\psi_\alpha^\varepsilon|\,\phi(0,x)\,dx\geq 0,
\end{aligned} \tag{4.3}$$

where, $\psi_\alpha^\varepsilon:=\psi_\alpha\left(t,\frac{x_1}{\varepsilon}\right)$, $S_{a,b}=\operatorname{sgn}(a-b)$, as in the last section, and we use the fact that $f_1,f_2,\cdots,f_d,\sigma_{f_1}$ are monotone increasing. A similar inequality holds with $(\cdot-\cdot)_\pm$ instead of $|\cdot-\cdot|$, which easily follows by adding (subtracting) to (4.3) the difference of integral equations defining weak solutions for $u^\varepsilon(t,x)$ and for $\psi_\alpha\left(t,\frac{x_1}{\varepsilon}\right)$. Let w_N be defined as in (5.11). In particular, from (4.3) it follows the comparison principle

$$\mathbb{E}\int_{\mathbb{R}^d}\left(u^\varepsilon(t,x)-\psi_\alpha(t,\frac{x_1}{\varepsilon})\right)_\pm w_N\,dx\leq e^{Ct}\int_{\mathbb{R}^d}\left(u_0(x,\frac{x_1}{\varepsilon})-\psi_\alpha\left(0,\frac{x_1}{\varepsilon}\right)\right)_\pm w_N\,dx,$$

for some $C > 0$.

Thus, if $\alpha_1, \alpha_2 \in \mathbb{R}$ are such that

$$\psi_{\alpha_1}\left(0, \frac{x_1}{\varepsilon}\right) \le u_0(x, \frac{x_1}{\varepsilon}) \le \psi_{\alpha_2}\left(0, \frac{x_1}{\varepsilon}\right),$$

we obtain the following, which provide bounds for u^ε independent of ε:

$$\psi_{\alpha_1}\left(t, \frac{x_1}{\varepsilon}\right) \le u^\varepsilon(t,x) \le \psi_{\alpha_2}\left(t, \frac{x_1}{\varepsilon}\right). \tag{4.4}$$

Taking, in (4.3), $\phi(t,x) = \varepsilon\varphi\left(\frac{x_1}{\varepsilon}\right)\psi(t,x)$, where $\varphi, \varphi' \in \mathcal{A}(\mathbb{R})$, $\varphi \ge 0$ and $0 \le \psi \in C_c^\infty((0,\infty)\times\mathbb{R}^d)$, taking conditional expectation with respect to an arbitrary $A \in \mathcal{F}$, and letting $\varepsilon \to 0$, along a subsequence for which u^ε generates a two-scale Young measure $\nu_{\omega,t,x,y}$ (see Proposition 2.1), we get, a.s., where we again drop the subscript ω from $\nu_{\omega,t,x,y}$,

$$\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathcal{K}} \psi(t,x)\, \langle \nu_{t,x,y}, |f_1(\lambda) - f_1(\psi_\alpha(t,y))|\rangle\, \varphi'(y)\, d\mathfrak{m}(y)\, dx\, dt \ge 0,$$

where $\mathcal{K}$ denotes the compactification of $\mathbb{R}^d$ generated by $\mathcal{A}(\mathbb{R}^d)$, whose invariant measure associated with the mean-value is denoted by $d\mathfrak{m}(y)$.

Applying this inequality to $C \pm \varphi$, with $C = \|\varphi\|_\infty$, we obtain, a.s.,

$$\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathcal{K}} \psi(t,x)\, \langle \nu_{t,x,y}, |f_1(\lambda) - f_1(\psi_\alpha(t,y))|\rangle\, \varphi'(y)\, d\mathfrak{m}(y)\, dx\, dt = 0. \tag{4.5}$$

We define, similarly to [25,2], the family of parameterized measures $\mu_{t,x,y}$ over $\mathbb{R}$ by

$$\langle \mu_{t,x,y}, \theta\rangle := \langle \nu_{t,x,y}, \theta(f_1(\lambda) - \kappa_0 W(t) - V(y))\rangle, \quad \text{for } \theta \in C_c(\mathbb{R}).$$

We see from (4.5) that $\mu_{t,x,y}$ actually does not depend on $y \in \mathcal{K}$, since

$$\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathcal{K}} \psi(t,x)\, \langle \mu_{t,x,y}, \theta\rangle\, \varphi'(y)\, d\mathfrak{m}(y)\, dx\, dt = 0, \tag{4.6}$$

for all θ of the form $|\cdot - \alpha|$, $\alpha \in \mathbb{R}$, and, from the remark made just after (4.3), also for $\theta(\cdot) = (\cdot - \alpha)_\pm$, $\alpha \in \mathbb{R}$, which implies that (4.6) holds for all $\theta \in C(\mathbb{R})$.

Now, taking any nonnegative $\phi \in C_c^1(\mathbb{R}^{d+1})$ in (4.3), taking conditional expectation with respect to an arbitrary $A \in \mathcal{F}$, and making $\varepsilon \to 0$ along a subsequence as above, given by Proposition 2.1, we get a.s.

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathcal{K}} \Big\{ \langle \nu_{t,x,y}, |\lambda - \psi_\alpha(t,y)| \rangle \phi_t + \langle \nu_{t,x,y}, |f_1(\lambda) - f_1(\psi_\alpha(t,y))| \rangle \phi_{x_1} \\
&\qquad + \sum_{k=2}^d \left\langle \nu_{t,x,y}, \left| f^k(\lambda) - f^k(\psi_\alpha(t,y)) \right| \right\rangle \phi_{x_k} \\
&\qquad + \frac{1}{2}\kappa_0^2 \langle \nu_{t,x,y}, S_{\lambda,\psi_\alpha} (h_{f_1}(\lambda) - h_{f_1}(\psi_\alpha(t,y))) \rangle \phi \Big\} \, d\mathfrak{m}(y)\, dx\, dt \\
&\quad + \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathcal{K}} \kappa_0 \langle \nu_{t,x,y}, |\sigma_{f_1}(\lambda) - \sigma_{f_1}(\psi_\alpha(t,y))| \rangle \phi \, d\mathfrak{m}(y)\, dx\, dW(t) \\
&\quad + \int_{\mathbb{R}^d} \int_{\mathcal{K}} |u_0(x,y) - \psi_\alpha(0,y)| \, \phi(0,x)\, d\mathfrak{m}(y)\, dx \geq 0. \qquad (4.7)
\end{aligned}
$$

Due to (4.6) we can write $\mu_{t,x,y} = \mu_{t,x}$. Then, using the substitution formulas $\lambda = g(\rho + \kappa_0 W(t) + V(y))$, $\psi_\alpha(t,y) = g(\alpha + \kappa_0 W(t) + V(y))$, we can rewrite (4.7) a.s. as

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \Big\{ \Big\langle \mu_{t,x}, \int_{\mathcal{K}} |g(\cdot + \kappa_0 W(t) + V(y)) - g(\alpha + \kappa_0 W(t) + V(y))| \; \mathfrak{m}(y) \Big\rangle \phi_t \\
&+ \sum_{k=1}^d \Big\langle \mu_{t,x}, \int_{\mathcal{K}} |p_k(\cdot + \kappa_0 W(t) + V(y)) - p_k(\alpha + \kappa_0 W(t) + V(y))| \; d\mathfrak{m}(y) \Big\rangle \phi_{x_k} \\
&+ \frac{1}{2}\kappa_0^2 \Big\langle \mu_{t,x}, \int_{\mathcal{K}} S_{\cdot,\alpha} (h_{f_1} \circ g(\cdot + \kappa_0 W(t) + V(y)) - h_{f_1} \circ g(\alpha + \kappa_0 W(t) + V(y))) \; d\mathfrak{m}(y) \Big\rangle \phi \Big\} \, dx\, dt \\
&+ \int_0^\infty \int_{\mathbb{R}^d} \kappa_0 \Big\langle \mu_{t,x,y}, \int_{\mathcal{K}} |\sigma_{f_1} \circ g(\cdot + \kappa_0 W(t) + V(y)) - \sigma_{f_1} \circ g(\alpha + \kappa_0 W(t) + V(y))| \; d\mathfrak{m}(y) \Big\rangle \phi \, dx\, dW(t) \\
&+ \int_{\mathbb{R}^d} \int_{\mathcal{K}} |u_0(x,y) - g(\alpha + V(y))| \, \phi(0,x)\, d\mathfrak{m}(y)\, dx \geq 0,
\end{aligned}
$$

where $p_k = f_k \circ g$ (so $p_1(t) = t$), $S_{a,b} = \operatorname{sgn}(a-b) = \operatorname{sgn}(g(\cdot + \kappa_0 W(t) + V(y)) - g(\alpha + \kappa_0 W(t) + V(y)))$, from which it follows

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \Big\{ \Big\langle \mu_{t,x}, \int_{\mathcal{K}} (g(\cdot + \kappa_0 W(t) + V(y)) - g(\alpha + \kappa_0 W(t) + V(y)))_+ \; d\mathfrak{m}(y) \Big\rangle \phi_t \\
&\qquad + \sum_{k=1}^d \Big\langle \mu_{t,x}, \int_{\mathcal{K}} (p_k(\cdot + \kappa_0 W(t) + V(y)) - p_k(\alpha + \kappa_0 W(t) + V(y)))_+ \, d\mathfrak{m}(y) \Big\rangle \phi_{x_k} \\
&\qquad + \frac{1}{2}\kappa_0^2 \Big\langle \mu_{t,x}, \int_{\mathcal{K}} S_{\cdot,\alpha,+} (h_{f_1} \circ g(\cdot + \kappa_0 W(t) + V(y))
\end{aligned}
$$

$$-h_{f_1}\circ g\left(\alpha+\kappa_0 W(t)+V(y)\right)\right)\,d\mathfrak{m}(y)\Big\rangle\phi\Big\}\,dx\,dt$$

$$+\int_0^\infty\int_{\mathbb{R}^d}\kappa_0\Big\langle\mu_{t,x},\int_{\mathcal{K}}\big(\sigma_{f_1}\circ g(\cdot+\kappa_0 W(t)+V(y))$$

$$-\sigma_{f_1}\circ g(\alpha+\kappa_0 W(t)+V(y))\big)_+\,d\mathfrak{m}(y)\Big\rangle\phi\,dx\,dW(t)$$

$$+\int_{\mathbb{R}^d}\int_{\mathcal{K}}\left(u_0(x,y)-g(\alpha+V(y))\right)_+\,d\mathfrak{m}(y)\phi(0,x)\,dx\geq 0, \tag{4.8}$$

where $S_{a,b,+}:=(a-b)_+=\operatorname{sgn}_+(g(a+\kappa_0W(t)+V(y))-g(b+\kappa_0W(t)+V(y)))_+$.

Note that from the formulas for $\sigma_{\bar f_1}$ and $h_{\bar f_1}$, recalled in the beginning of this section, we may verify, and this seems a little miraculous(!), the equations

$$\sigma_{\bar f_1}\circ\bar g(v)=\int_{\mathcal{K}}\sigma_{f_1}\circ g(v+V(y))\,d\mathfrak{m}(y),$$
$$h_{\bar f_1}\circ\bar g(v)=\int_{\mathcal{K}}h_{f_1}\circ g(v+V(y))\,d\mathfrak{m}(y).$$

Also, by the monotonicity of g, p_k, $k=2,\dots,d$ and σ_{f_1} we can pass the integral over $\mathcal{K}$ inside of the positive part in each term of (4.8). Thus, recalling the definition of $\bar f_k$, $k=1,\dots,d$ and $\psi_{*\alpha}$, we obtain

$$\int_0^\infty\int_{\mathbb{R}^d}\Big\{\left\langle\mu_{t,x},\left(\bar g\left(\cdot+\kappa_0W(t)\right)-\bar g(\alpha+\kappa_0W(t))\right)_+\right\rangle\phi_t$$

$$+\sum_{k=1}^d\Big\langle\mu_{t,x},\big(\bar f_k\circ\bar g(\cdot+\kappa_0W(t))-\bar f_k\circ\bar g(\alpha+\kappa_0W(t))\big)_+\Big\rangle\phi_{x_k}$$

$$+\Big\langle\mu_{t,x},\frac{1}{2}\kappa_0^2S_{\cdot,\alpha,+}\left(h_{\bar f_1}\circ\bar g\left(\cdot+\kappa_0W(t)\right)-h_{\bar f_1}\circ\bar g\left(\alpha+\kappa_0W(t)\right)\right)\Big\rangle\phi\Big\}\,dx\,dt$$

$$+\int_0^\infty\int_{\mathbb{R}^d}\kappa_0\Big\langle\mu_{t,x},\big(\sigma_{\bar f_1}\circ\bar g(\cdot+\kappa_0W(t))-\sigma_{\bar f_1}\circ\bar g(\alpha+\kappa_0W(t))\big)_+\Big\rangle\phi\,dx\,dW(t)$$

$$+\int_{\mathbb{R}^d}\left(\bar g(v_0(x))-\bar g(\alpha)\right)_+\,\phi(0,x)\,dx\geq 0.$$

Given $\vartheta\in C_c^\infty(\mathbb{R}^{d+1})$ and $\tilde\varphi\in C_c^\infty(\mathbb{R})$, we define the measure $m_1=m_1(t,x,\xi)$ applied to $\vartheta\tilde\varphi$ by

$$\langle m_1, \vartheta\tilde{\varphi}\rangle := \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}} \Big\{ \big\langle \mu_{t,x}, (\bar{g}\,(\cdot + \kappa_0 W(t)) - \bar{g}(\xi + \kappa_0 W(t)))_+ \big\rangle \vartheta_t$$

$$+ \sum_{k=1}^d \Big\langle \mu_{t,x}, \big(\bar{f}_k \circ \bar{g}(\cdot + \kappa_0 W(t)) - \bar{f}_k \circ \bar{g}(\xi + \kappa_0 W(t))\big)_+ \Big\rangle \vartheta_{x_k}$$

$$+ \Big\langle \mu_{t,x}, \frac{1}{2}\kappa_0^2 S_{\cdot,\alpha,+}\left(h_{\bar{f}_1} \circ \bar{g}\,(\cdot + \kappa_0 W(t)) - h_{\bar{f}_1} \circ \bar{g}\,(\xi + \kappa_0 W(t))\right) \Big\rangle \vartheta \Big\} \tilde{\varphi}(\xi)\, d\xi\, dx\, dt$$

$$+ \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}} \kappa_0 \Big\langle \mu_{t,x}, \big(\sigma_{\bar{f}_1} \circ \bar{g}(\cdot + \kappa_0 W(t)) - \sigma_{\bar{f}_1} \circ \bar{g}(\xi + \kappa_0 W(t))\big)_+ \Big\rangle \vartheta \tilde{\varphi}(\xi)\, d\xi\, dx\, dW(t)$$

$$+ \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left(\bar{g}(v_0(x)) - \bar{g}(\xi)\right)_+ \vartheta(0,x)\tilde{\varphi}(\xi)\, d\xi\, dx. \tag{4.9}$$

We then take $\tilde{\varphi} = \varphi'$, for some $\varphi \in C_c^\infty(\mathbb{R})$ and make an integration by parts in the integral in ξ. Hence, defining $\rho_1(t,x,\xi) := \mu_{t,x}((\xi, +\infty))$, $a_0(\xi) = \bar{g}'(\xi)$, $a_i(\xi) = (\bar{f}_i \circ \bar{g})'(\xi)$, $i = 1, \dots, d$, $H(\xi) := (h_{\bar{f}_1} \circ \bar{g})'(\xi)$, $G(\xi) := (\sigma_{\bar{f}_1} \circ \bar{g})'(\xi)$, setting $a := (a_1, \dots, a_d)$, we get from (4.9)

$$\langle \partial_\xi m_1, \vartheta\varphi\rangle = \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}} \Big\{ a_0(\xi + \kappa_0 W(t))\rho_1(t,x,\xi)\vartheta_t$$

$$+ \sum_{k=1}^d a_k(\xi + \kappa_0 W(t))\rho_1(t,x,\xi)\vartheta_{x_k} + \frac{1}{2}\kappa_0^2 H(\xi + \kappa_0 W(t))\rho_1(t,x,\xi)\vartheta \Big\} \varphi(\xi)\, d\xi\, dx\, dt$$

$$+ \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}} \kappa_0 G(\xi + \kappa_0 W(t))\rho_1(t,x,\xi)\vartheta\varphi(\xi)\, d\xi\, dx\, dW(t)$$

$$+ \int_{\mathbb{R}^d} \int_{\mathbb{R}} \bar{g}'(\xi)\mathbb{I}_{\xi < v_0(x)}\vartheta(0,x)\varphi(\xi)\, d\xi\, dx. \tag{4.10}$$

Therefore, we see that ρ_1 is a weak solution of the stochastic kinetic equation

$$\partial_t(a_0(\xi + \kappa_0 W(t))\rho_1) + a(\xi + \kappa_0 W(t)) \cdot \nabla_x \rho_1 - \frac{1}{2}\kappa_0^2 H(\xi + \kappa_0 W(t))\rho_1$$
$$= \partial_\xi m_1 + \kappa_0 G(\xi + \kappa_0 W(t))\rho_1 \frac{dW(t)}{dt}, \tag{4.11}$$

in the sense of (4.10) extended from test functions of the form $\vartheta\varphi$ to all test functions in $C_c^\infty(\mathbb{R}^{d+2})$. Also, from (4.10), it follows that

$$\operatorname*{ess\,lim}_{t\to 0+} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \rho_1(t,x,\xi)\phi(x,\xi)\, dx\, d\xi = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathbb{I}_{\xi < v_0(x)}\phi(x,\xi)\, dx\, d\xi, \tag{4.12}$$

for all $\phi \in C_c^\infty(\mathbb{R}^{d+1})$, see also Remark 5.13. We observe that ρ_1 is a kinetic function associated to the Young measure $\mu_{t,x}$. Also, it is not difficult to check, by Proposition 2.1(3), that μ satisfies (5.27) and we also may check that m_1, defined by (4.9), satisfies the conditions of a kinetic measure in Definition 5.1.

On the other hand, using the here called stochastic Kružkov inequality (see Proposition 6.1) for (1.10) for the entropy solution of (1.10)-(1.13) and for the special solution $\psi_{*\gamma}(t)$, we get

$$\int_0^\infty \int_{\mathbb{R}^d} \Big\{ |\psi_{*\gamma}(t) - \bar{u}|\, \phi_t + \sum_{k=1}^d \left| \bar{f}_k(\psi_{*\gamma}(t)) - \bar{f}_k(\bar{u}) \right| \phi_{x_k} + \frac{1}{2}\kappa_0^2 S_{\psi_{*\gamma}(t),\bar{u}} \left(h_{\bar{f}_1}(\psi_{*\gamma}) - h_{\bar{f}_1}(\bar{u}) \right) \phi \Big\}\, dx\, dt + \int_{\mathbb{R}^d} |\psi_{*\gamma}(0) - \bar{u}(0,x)|\, \phi(0,x)\, dx + \int_0^\infty \int_{\mathbb{R}^d} \kappa_0 \left| \sigma_{\bar{f}_1}(\bar{u}) - \sigma_{\bar{f}_1}(\psi_{*\gamma}(t)) \right| \phi\, dx\, dW(t) \geq 0, \tag{4.13}$$

for all $\phi \in C_c^\infty(\mathbb{R}^{d+1})$.

Let $X(t,x) = \bar{f}_1(\bar{u}(t,x)) - \kappa_0 W(t)$ and observe $\bar{u}(t,x) = \bar{g}(X(t,x) + \kappa_0 W(t))$. We then get from (4.13) as before,

$$\int_0^\infty \int_{\mathbb{R}^d} \Big\{ (\bar{g}(\gamma + \kappa_0 W(t)) - \bar{g}(X(t,x) + \kappa_0 W(t))_+\, \phi_t + \sum_{k=1}^d \left(\bar{f}_k \circ \bar{g}(\gamma + \kappa_0 W(t)) - \bar{f}_k \circ \bar{g}(X(t,x) + \kappa_0 W(t)) \right)_+ \phi_{x_k} \Big\}\, dt\, dx + \int_0^\infty \int_{\mathbb{R}^d} \frac{1}{2}\kappa_0^2 S_{\gamma,X,+} \left(\bar{h}_{f_1} \circ \bar{g}(\gamma + \kappa_0 W(t)) - \bar{h}_{f_1} \circ \bar{g}(X(t,x) + \kappa_0 W(t)) \right) \phi\, dx\, dt + \int_{\mathbb{R}^d} (\bar{g}(\gamma + V(y)) - \bar{g}(v_0(x) + V(y)))_+\, \phi(0,x))\, dx + \int_0^\infty \kappa_0 \int_{\mathbb{R}^d} \left(\sigma_{\bar{f}_1} \circ \bar{g}(\gamma + \kappa_0 W(t)) - \sigma_{\bar{f}_1} \circ \bar{g}(X(t,x) + \kappa_0 W(t)) \right)_+ \phi\, dx\, dW(t) \geq 0.$$

Hence, given $\vartheta \in C_c^\infty(\mathbb{R}^{d+1})$, $\tilde{\varphi} \in C_c^\infty(\mathbb{R})$ we can similarly define the measure $m_2 = m_2(t,x,\xi)$ applied to $\vartheta\tilde{\varphi}$ by

$$\langle m_2, \psi\tilde{\varphi} \rangle := \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}} \Big\{ (\bar{g}(\gamma + \kappa_0 W(t)) - \bar{g}(X(t,x) + \kappa_0 W(t))_+\, \vartheta_t \tilde{\varphi}(\xi)$$

$$
\begin{aligned}
&+\sum_{k=1}^{d}\left(\bar f_k\circ\bar g(\gamma+\kappa_0W(t))-\bar f_k\circ\bar g(X(t,x)+\kappa_0W(t))\right)_+\theta_{x_k}\tilde\varphi(\xi)\Big\}\,d\xi\,dx\,dt\\
&+\frac12\kappa_0^2\int_0^\infty\int_{\mathbb{R}^d}\int_{\mathbb{R}}\left(\bar h_{f_1}\circ\bar g(\gamma+\kappa_0W(t))-\bar h_{f_1}\circ\bar g(X(t,x)+\kappa_0W(t))\right)\vartheta\tilde\varphi(\xi)\,d\xi\,dx\,dt\\
&+\int_{\mathbb{R}^d}\int_{\mathbb{R}}(\bar g(\gamma+V(y))-\bar g(v_0(x)+V(y)))_+\vartheta(0,x)\tilde\varphi(\xi)\,d\xi\,dx\\
&+\int_0^\infty\kappa_0\int_{\mathbb{R}^d}\int_{\mathcal{K}}\left(\sigma_{\bar f_1}\circ\bar g(\gamma+\kappa_0W(t))-\sigma_{\bar f_1}\circ\bar g(X(t,x)+\kappa_0W(t))\right)_+\vartheta\tilde\varphi(\xi)\,d\xi\,dx\,dW(t).
\end{aligned}
\tag{4.14}
$$

Therefore, we again take $\tilde\varphi=\varphi'$ for some $\varphi\in C_c^\infty(\mathbb{R})$ and make an integration by parts in the integral in ξ. Hence, defining $\rho_2(t,x,\xi):=\mathbb{I}_{(-\infty,X(t,x))}(\xi)$, we see that ρ_2 is a weak solution of the stochastic kinetic equation

$$
\begin{aligned}
\partial_t(a_0(\xi+\kappa_0W(t))\rho_2)+a(\xi+\kappa_0W(t))\cdot\nabla_x\rho_2-\frac12\kappa_0^2H(\xi+\kappa_0W(t))\rho_2&\\
=\partial_\xi m_2+\kappa_0G(\xi+\kappa_0W(t))\rho_2\frac{dW(t)}{dt},&
\end{aligned}
\tag{4.15}
$$

where a_i, $i=0,\dots,d$, H and G are as before. Also, from (4.14), it follows that

$$
\operatorname*{ess\,lim}_{t\to0+}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\rho_2(t,x,\xi)\phi(x,\xi)\,dx\,d\xi=\int_{\mathbb{R}^d}\int_{\mathbb{R}}\mathbb{I}_{\xi<v_0(x)}\phi(x,\xi)\,dx\,d\xi,
\tag{4.16}
$$

for all $\phi\in C_c^\infty(\mathbb{R}^{d+1})$. Since ρ_2 is a standard kinetic function, a well known argument shows that the convergence in (4.16) may be strengthen to a convergence in $L^1(\mathbb{R}^N;w_N)$ (see also Remark 5.13).

Therefore, ρ_1, ρ_2 are weak solutions of identical kinetic equations, (4.11) and (4.15), with possibly distinct kinetic measures m_1 and m_2, and satisfy identical initial conditions (4.12) and (4.16).

Our next goal is to prove that $\mu_{t,x}=\delta_{X(t,x)}$ a.s. and to do that we are going to prove the uniqueness of the weak solution of (4.11)-(4.12) or (4.15)-(4.16), independently of the corresponding kinetic measure. This, in turn, will be a consequence of the next lemma.

Lemma 4.1 *(rigidity/comparison result). Let $\rho_1(t,x,\xi)$, $\rho_2(t,x,\xi)$ be generalized kinetic functions, that is, functions taking values in $[0,1]$ such that $-\partial_\xi\rho_1$ and $-\partial_\xi\rho_2$ are Young measures, which solve equations (4.11) and (4.15) with initial conditions $\rho_{0,1}$ and $\rho_{0,2}$, respectively, where m_1 and m_2 are kinetic measures in the sense of Definition 5.1. Then there is a constant $C>0$ such that*

$$
\begin{aligned}
&\mathbb{E}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}} \alpha_0(\xi+\kappa_0 W(t))\rho_1(1-\rho_2)(t)w_N\,d\xi\,dx \\
&\qquad \le C\mathbb{E}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}} a_0(\xi)\rho_{0,1}(1-\rho_{0,2})w_N\,d\xi\,dx,
\end{aligned}
\tag{4.17}
$$

for a.e. $t\in[0,T]$, where w_N is the weight given by (5.11).

Remark 4.1. Observe that the stochastic kinetic equations (4.11), (4.15) are different from the equations (3.12), (3.15) (which are of the type analyzed in Section 5). In particular, the former two equations do not have *gradient* noise and a second order differential operator. They do, however, contain coefficients that are predictable random fields. We recall that a continuous mapping $H = H(\omega,t,x,u) : \Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ is a called a random field when it is viewed as a random variable $\omega\mapsto H(\omega,t,x,u)$, with (t,x,u) fixed. If, for each fixed (x,u), the stochastic process $(\omega,t)\mapsto H(\omega,t,x,u)$ is $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable, then H is called a predictable random field. Nevertheless, given the crucial observation below, cf. (4.21), the analysis in Section 5 carries over to the stochastic kinetic equations (4.11) and (4.15), see in particular Proposition 5.1, the equation (5.36), and (6.2) and the discussion found at the beginning of Section 6.

To keep this paper at a reasonable length, we will only supply a sketch of the proof of Lemma 4.1, focusing on the formal argument leading up the crucial equation (4.21), from which we can proceed as in Section 5. The rigorous proof relies on the usual regularization procedure, the Itô product formula, and commutator estimates to control regularization errors. In fact, the step involving regularization by convolution (in x,ξ) is simpler (than in Section 5) since there are no error terms that require second-order commutator estimates, like (5.37), that is, all the error terms can be handled using the standard DiPerna-Lions folklore lemma [20]. We refer to Section 5 for details.

Sketch of proof of Lemma 4.1. We will first formally derive stochastic kinetic equations for ρ_1 and $\alpha_0(\xi+\kappa_0 W(t))(1-\rho_2)$. Then, combining the resulting equations, Itô's product formula will provide (at least formally) an equation for $a_0(\xi+\kappa_0 W(t)\rho_1(1-\rho_2)$, which we can use to prove (4.17), along the lines of Section 5.

We observe that we can write (4.11) as a stochastic differential equation of the following form, where we drop the subscript 1 in ρ_1 and m_1,

$$
d(a_0(\xi+\kappa_0 W(t))\rho) = A\,dt + B\,dW + \partial_\xi m,
$$

where

$$
A = -a(\xi+\kappa_0 W(t))\cdot\nabla_x\rho + \frac{1}{2}\kappa_0^2 H(\xi+\kappa_0 W(t))\rho, \quad B = \kappa_0 G(\xi+\kappa_0 W(t))\rho.
$$

Thanks to the Itô formula, $a_0(\xi+\kappa_0 W(t))$ satisfies the stochastic differential equation

$$
da_0(\xi+\kappa_0 W(t)) = \kappa_0 a_0'(\xi+\kappa_0 W(t))\,dW + \frac{1}{2}\kappa_0^2 a_0''(\xi+\kappa_0 W(t))\,dt, \tag{4.18}
$$

and, by the formulas for a_0, $\sigma_{\bar{f}_1}$, and $h_{\bar{f}_1}$, we have

$$a_0(\xi) = \sigma_{\bar{f}_1} \circ \bar{g}(\xi), \qquad a_0' = (\sigma_{\bar{f}_1} \circ \bar{g})'(\xi), \qquad a_0'' = (h_{\bar{f}_1} \circ \bar{g})'(\xi),$$

so

$$a_0'(\xi) = G(\xi), \quad a_0''(\xi) = H(\xi). \tag{4.19}$$

Denoting $\tilde{a}_0(\xi) = \frac{1}{a_0(\xi)}$ we get, also from Itô's formula, the following stochastic differential equation for $\tilde{a}_0(\xi)$:

$$d\tilde{a}_0(\xi + \kappa_0 W(t)) = \kappa_0 \tilde{a}_0'(\xi + \kappa_0 W(t))\, dW + \frac{1}{2}\kappa_0^2 \tilde{a}_0''(\xi + \kappa_0 W(t))\, dt,$$

where, by virtue of (4.19),

$$\tilde{a}_0'(\xi) = -\frac{G(\xi)}{a_0(\xi)^2}, \quad \tilde{a}_0''(\xi) = \frac{2G(\xi)^2 - a_0(\xi)H(\xi)}{a_0(\xi)^3}. \tag{4.20}$$

By the Itô product rule,

$$\begin{aligned} d\rho &= d\big(\tilde{a}_0(\xi + \kappa_0 W(t)) a_0(\xi + \kappa_0 W(t))\rho\big) \\ &= a_0(\xi + \kappa_0 W(t))\rho\, d(\tilde{a}_0(\xi + \kappa_0 W(t))) + \tilde{a}_0(\xi + \kappa_0 W(t))\, d(a_0(\xi + \kappa_0 W(t))\rho) \\ &\qquad + [\tilde{a}_0(\xi + \kappa_0 W(t)), a_0(\xi + \kappa_0 W(t))\rho] \\ &= \kappa_0 a_0(\xi + \kappa_0 W(t)) \tilde{a}_0'(\xi + \kappa_0 W(t))\rho\, dW + \frac{1}{2}\kappa_0^2 a_0(\xi + \kappa_0 W(t))\tilde{a}_0''(\xi + \kappa_0 W(t))\, \rho\, dt \\ &\qquad + \tilde{a}_0(\xi + \kappa_0 W(t)) A\, dt + \tilde{a}_0(\xi + \kappa_0 W(t)) B\, dW + \tilde{a}_0(\xi + \kappa_0 W(t))\partial_\xi m \\ &\qquad + \kappa_0 \tilde{a}_0'(\xi + \kappa_0 W(t)) B\, dt. \end{aligned}$$

In sum, we deduce that ρ_1 satisfies the stochastic kinetic equation

$$\begin{aligned} &\partial_t \rho_1 + \tilde{a}_0(\xi + \kappa_0 W(t)) a(\xi + \kappa_0 W(t)) \cdot \nabla_x \rho_1 \\ &\qquad = \frac{1}{2}\kappa_0^2 \tilde{A}\rho_1 + \kappa_0 \tilde{B}\rho_1 \frac{dW(t)}{dt} + \tilde{a}_0(\xi + \kappa_0 W(t))\partial_\xi m_1, \end{aligned}$$

where

$$\begin{aligned} \tilde{A} &= \tilde{a}_0(\xi + \kappa_0 W(t)) H(\xi + \kappa_0 W(t)) + 2\tilde{a}_0'(\xi + \kappa_0 W(t)) G(\xi + \kappa_0 W(t)) \\ &\qquad + a_0(\xi + \kappa_0 W(t))\tilde{a}_0''(\xi + \kappa_0 W(t)), \end{aligned}$$

and

$$\tilde{B} = a_0(\xi + \kappa_0 W(t))\tilde{a}_0'(\xi + \kappa_0 W(t)) + \tilde{a}_0(\xi + \kappa_0 W(t))G(\xi + \kappa_0 W(t)).$$

At this point, we observe that (4.19) and (4.20) imply that $\tilde{A} = 0$ and $\tilde{B} = 0$. Thus, we conclude that ρ_1 satisfies the (much simpler) equation

$$\partial_t \rho_1 + \tilde{a}_0(\xi + \kappa_0 W(t))a(\xi + \kappa_0 W(t)) \cdot \nabla_x \rho_1 = \tilde{a}_0(\xi + \kappa_0 W(t))\partial_\xi m_1. \tag{4.21}$$

In view of (4.15) and (4.18), we obtain the following equation for $a_0(\xi + \kappa_0 W(t))(1 - \rho_2)$:

$$\begin{aligned} \partial_t \left(a_0(\xi + \kappa_0 W(t))(1 - \rho_2)\right) + a(\xi + \kappa_0 W(t)) \cdot \nabla_x (1 - \rho_2) - \frac{1}{2}\kappa_0^2 H(\xi + \kappa_0 W(t))(1 - \rho_2) \\ = -\partial_\xi m_2 + \kappa_0 G(\xi + \kappa_0 W(t))(1 - \rho_2)\frac{dW(t)}{dt}. \end{aligned} \tag{4.22}$$

Given the stochastic kinetic equations (4.21) and (4.22), we may apply (again formally) Itô's product rule to obtain

$$\begin{aligned} & d\left(a_0(\xi + \kappa_0 W(t))\rho_1(1 - \rho_2)\right) \\ &\quad = a_0(\xi + \kappa_0 W(t))(1 - \rho_2)\, d\rho_1 + \rho_1\, d\left(a_0(\xi + \kappa_0 W(t))(1 - \rho_2)\right) \\ &\qquad\qquad + [\rho_1, a_0(\xi + \kappa_0 W(t))(1 - \rho_2)] \\ &\quad = -a(\xi + \kappa_0 W(t)) \cdot (1 - \rho_2)\nabla_x \rho_1\, dt + (1 - \rho_2)\partial_\xi m_1 \\ &\qquad - a(\xi + \kappa_0 W(t)) \cdot \rho_1 \nabla_x (1 - \rho_2)\, dt + \frac{1}{2}\kappa_0^2 H(\xi + \kappa_0 W(t))\rho_1(1 - \rho_2)\, dt \\ &\qquad\qquad - \rho_1 \partial_\xi m_2 + \kappa_0 G(\xi + \kappa_0 W(t))\rho_1(1 - \rho_2) dW(t). \end{aligned}$$

In other words, we have the following equation for $a_0(\xi + \kappa_0 W(t))\rho_1(1 - \rho_2)$:

$$\begin{aligned} \partial_t \left(a_0(\xi + \kappa_0 W(t))\rho_1(1 - \rho_2)\right) + a(\xi + \kappa_0 W(t)) \cdot \nabla_x \left(\rho_1(1 - \rho_2)\right) \\ = \frac{1}{2}\kappa_0^2 H(\xi + \kappa_0 W(t))\rho_1(1 - \rho_2) + (1 - \rho_2)\partial_\xi m_1 - \rho_1 \partial_\xi m_2 \\ + \kappa_0 G(\xi + \kappa_0 W(t))\rho_1(1 - \rho_2) dW(t). \end{aligned} \tag{4.23}$$

Note that the coefficient $a_0(\xi + \kappa_0 W(t))$ in the equation for $(1 - \rho_2)$ provides a cancellation with the coefficient $\tilde{a}_0(\xi + \kappa_0 W(t))$ which multiplies the measure m_1 on the right-hand-side of (4.21). This cancellation, which results in the term $(1 - \rho_2)\partial_\xi m_2$ on the right-hand side of equation (4.23), is essential for the proof of Lemma 4.1, as it will allow us to discard this term later on in the analysis based on its sign, after integration by parts. Similarly, the term $-\rho_1 \partial_\xi m_2$, which is of the same nature, will also be discarded by its sign. To carry on the proof, we take appropriate test functions in the equation (4.23) and manipulate the remaining terms to conclude by applying Gronwall's inequality.

At last, we reiterate that the above argument can be turned into a rigorous proof using regularization by convolution (in x, ξ), following Section 5. □

Finally, in view of Lemma 4.1, we deduce that

$$\mathbb{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}} a_0(\xi+\kappa_0 W(t))\rho_1(t,x,\xi)\,(1-\rho_2(t,x,\xi))\,w_N\,dx\,d\xi \\ \le C(T)\int_{\mathbb{R}^d}\int_{\mathbb{R}} \alpha_0(\xi)\rho_{1,0}(x,\xi)\,(1-\rho_{2,0}(x,\xi))\,w_N\,dx\,d\xi = 0,$$

for $0 < t \le T$, and so, since ρ_1 and ρ_2 coincide at $t = 0$, both being equal to $\mathbb{I}_{\xi < v_0(x)}$, we obtain

$$\mu_{t,x} = \delta_{\bar{f}_1(\bar{u}(t,x)) - \kappa_0 W(t)},$$

and consequently

$$\nu_{t,x,y} = \delta_{g(\bar{f}_1(\bar{u}(t,x)) + V(y))}, \quad \mathbb{P}\text{-a.e. in } \Omega.$$

In particular, it follows that, $u^\varepsilon \rightharpoonup \int_{\mathcal{K}} g(\bar{f}_1(\bar{u}(\cdot,\cdot) + V(y)))d\mathfrak{m}(y) = \bar{u}$ in the weak–$\star$ topology of $L^2(\Omega; L^2_{\text{loc}}((0,T)\times\mathbb{R}^d))$, for all $T > 0$ (cf. Remark 2.1 above). Note that we used the uniqueness of the limit to conclude that the whole sequence u^ε converges, similarly as in the proof of Theorem 1.1.

Again, the last assertion in Theorem 1.2 follows directly from Proposition 2.2 (a). This concludes the proof of Theorem 1.2.

5. A well-posedness result

In this section, we provide a well-posedness result for a class of stochastic conservation laws that is (more than) general enough to encompass some of the equations encountered earlier in this paper; namely, hyperbolic conservation laws with variable coefficients and deterministic/stochastic source terms, posed on an unbounded spatial domain ($\mathbb{R}^d$), see Remark 5.5 for further details on the class of equations. Since these equations are not all covered by the available well-posedness literature [7,8,14,18,19,21,28,37,42–44,46], we will outline some of the arguments leading to this result, particularly the uniqueness part of it. On a technical level, the approach presented here is somewhat different from the one [18] utilized in many of the references listed above.

The initial–value problem for these SPDEs take the form

$$\begin{aligned}&\partial_t u + \operatorname{div}_x A(t,x,u) = B(t,u)\dot{W}(t) + R(t,x,u), \quad (t,x) \in (0,T)\times\mathbb{R}^d,\\ &u(0,x) = u_0(\omega,x), \quad x\in\mathbb{R}^d,\end{aligned} \tag{5.1}$$

where W is a cylindrical Wiener process [17] with noise amplitude B, $A = (A_1,\dots,A_d)$ is the flux vector, R is the "deterministic" source term, u_0 is the initial function, and $T > 0$ is a fixed final time. We fix a stochastic basic $\mathcal{S}$ consisting of a complete probability space $(\Omega,\mathcal{F},P)$, a complete right-continuous filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, and a sequence $\{W_k\}_{k=1}^{\infty}$ of independent one-dimensional Wiener processes adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$.

We assume that the flux A belongs to $C([0,T];C^2(\mathbb{R}^d\times\mathbb{R};\mathbb{R}^d))$ and

$$\begin{aligned}&|A(t,x,u)| \le m_a(t)\,(1+|u|)\,(1+|x|)\,,\\ &|A(t,x,u)-A(t,x,v)| \le m_a(t)\,|u-v|\,(1+|x|)\,,\end{aligned} \tag{5.2}$$

for $t\in[0,T]$, $x\in\mathbb{R}^d$, and $u,v\in\mathbb{R}$, where $m_a(t)$ is an integrable function. Moreover,

$$|(\operatorname{div}_x A)(t,x,u)| \le m_d(t)\,(1+|u|)\,, \qquad (\operatorname{div}_x A)(t,x,0) = 0, \tag{5.3}$$

for $t\in[0,T]$, $x\in\mathbb{R}^d$, and $u\in\mathbb{R}$, where $m_d(t)$ is another integrable function. Note that, without loss of generality, we may always assume $(\operatorname{div}_x A)(t,x,0)=0$.

We assume that the source function R belongs to $C([0,T];C^1(\mathbb{R}^d\times\mathbb{R}))$, and

$$|R(t,x,u)| \le m_R(t)\,(1+|u|)\,, \quad |R(t,x,u)-R(t,x,v)| \le m_R(t)\,|u-v|\,, \tag{5.4}$$

for $t\in[0,T]$, $x\in\mathbb{R}^d$, and $u,v\in\mathbb{R}$, where $m_R(t)$ is an integrable function.

The driving noise W is a cylindrical Wiener process [17],

$$W(t) = \sum_{k\ge1} W_k(t)\psi_k, \tag{5.5}$$

evolving over a separable Hilbert space $\mathfrak{U}$, equipped with an orthonormal basis $\{\psi_k\}_{k\ge1}$. The series (5.5) converges in an auxiliary (larger) Hilbert space $\mathfrak{U}_0$ with Hilbert-Schmidt embedding $\mathfrak{U}\subset\mathfrak{U}_0$. The (nonlinear) noise amplitude $B = B(\omega,t,u)$ is an operator-valued mapping. For each $u\in L^2(\mathbb{R}^d)$, we define $B(t,u)$ by its action on each ψ_k:

$$B(t,u)\psi_k := b_k(\omega,t,\cdot,u(\cdot)), \qquad b_k\in C([0,T]\times\mathbb{R}^d\times\mathbb{R}), \qquad k\in\mathbb{N}.$$

We then obtain

$$B(t,u)\,dW(t) = \sum_{k\ge1} b_k(t,x,u)\,dW_k(t). \tag{5.6}$$

We assume that the sequence $\{b_k\}_{k\ge1}$ satisfy the following conditions:

$$B^2(t,x,u) := \sum_{k\geq 1} (b_k(t,x,u))^2 \lesssim 1 + |u|^2, \tag{5.7}$$

$$\sum_{k\geq 1} |b_k(t,x,u) - b_k(t,y,v)|^2 \lesssim |x-y|^2 + |u-v|\,\mu(|u-v|), \tag{5.8}$$

for $\omega \in \Omega$, $t \in [0,T]$, $x, y \in \mathbb{R}^d$, and $u, v \in \mathbb{R}$, for some continuous nondecreasing function μ on $\mathbb{R}_+$ with $\mu(0+) = 0$. The "Lipschitz case" corresponds to $\mu(\xi) = \xi$.

Remark 5.1. We have assumed that the coefficients A, B, and R in (5.1) are deterministic. However, this is not necessary. Indeed, the results presented in this section carry over to the case where A, B, R are predictable random fields satisfying conditions similar to those listed above (cf. Remark 4.1 for the notion of predictable random field).

The initial function u_0 is an $\mathcal{F}_0$-measurable random variable satisfying

$$u_0 \in L^\infty\left(\Omega; L^\infty(\mathbb{R}^d)\right). \tag{5.9}$$

Given a convex $S \in C^2(\mathbb{R})$, define $Q_S : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^d$ by $(\partial_u Q_S)(t,x,u) = S'(u)(\partial_u A)(t,x,u)$. We call (S, Q_S) an *entropy/entropy-flux pair* and write $(S, Q_S) \in \mathcal{E}$. For (5.1) the entropy inequalities read

$$\begin{aligned}
&\partial_t S(u) + \operatorname{div}_x Q_S(t,x,u) + S'(u)\left((\operatorname{div}_x A)(t,x,u) - R(t,x,u)\right) - (\operatorname{div}_x Q_S)(t,x,u) \\
&\quad \leq \sum_{k\geq 1} S'(u) b_k(t,x,u)\, \dot{W}_k(t) + \frac{1}{2} S''(u) B^2(t,x,u) \\
&\quad \text{in } \mathcal{D}'([0,T) \times \mathbb{R}^d), \text{ a.s.}, \ \forall (S,Q) \in \mathcal{E}.
\end{aligned} \tag{5.10}$$

Remark 5.2 *(weighted L^p estimates).* For discontinuous solutions, the entropy inequalities act as a replacement for the Itô (temporal) and classical (spatial) chain rules. It follows from (5.10) with $S(u) = u^p$ ($p \geq 2$) and a standard martingale argument that

$$u \in L^p\left(\Omega; L^\infty\left(0,T; L^p(w_N dx)\right)\right),$$

where $L^p(w_N dx)$ denotes the weighted L^p space of functions $v : \mathbb{R}^d \to \mathbb{R}$ for which

$$\int_{\mathbb{R}^d} |v|^p \, w_N dx < \infty.$$

Throughout this section, we make use of the weight function

$$w_N(x) = (1 + |x|^2)^{-N}, \quad N > d/2. \tag{5.11}$$

This function is integrable on $\mathbb{R}^d$ and satisfies

$$\nabla w_N(x) = \frac{-2Nx}{1+|x|^2} w_N(x) \quad \Longrightarrow \quad |\nabla w_N(x)| \lesssim \frac{w_N(x)}{1+|x|}.$$

Note that $L^p(w_N dx)$–bounds with $p \in [1,2)$ follow trivially from the $L^2(w_N dx)$–bound.

Remark 5.3 *(weight-free framework).* The Itô noise term continuously injects "entropy" into the system, cf. the $S''B^2$–term in (5.10). Suppose $B(t,x,0) = 0$. Then the ordinary L^p spaces constitute a natural choice for (5.1), in which case we may drop the weight w_N and obtain $u \in L^p\left(\Omega; L^\infty\left(0,T; L^p(\mathbb{R}^d)\right)\right)$ for all $p \in [2,\infty)$, provided

$$u_0 \in L^\infty\left(\Omega; \left(L^2 \cap L^\infty\right)(\mathbb{R}^d)\right). \tag{5.12}$$

Without this assumption ($B(\omega,t,x,0) \neq 0$), weighted L^p spaces appear to be better suited.

We can also drop the weight w_N at the expense of imposing a stronger condition on B^2 as $|x| \to \infty$, cf. (5.7), namely that

$$B^2(\omega,t,x,u) \leq (b(x))^2 \left(1 + |u|^2\right), \qquad b \in \left(L^2 \cap L^\infty\right)(\mathbb{R}^d), \tag{5.13}$$

for $\omega \in \Omega$, $t \in [0,T]$, $x \in \mathbb{R}^d$, and $u \in \mathbb{R}$. Under this assumption or $B(\omega,t,x,0) \equiv 0$, it is possible to use (5.10), with $S(\cdot) \approx |\cdot|$ and $S''(\cdot) \approx \delta(\cdot)$, to arrive at an L^1 bound, and consequently $u \in L^p\left(\Omega; L^\infty\left(0,T; L^p(\mathbb{R}^d)\right)\right)$ for all $p \in [1,\infty)$, in the event that $u_0 \in L^\infty\left(\Omega; \left(L^1 \cap L^\infty\right)(\mathbb{R}^d)\right)$. At the same time, it is possible to replace the assumptions on the flux function, cf. (5.2) and (5.3), by the following more general ones:

$$\begin{aligned}
&A(t,x,u) = \tilde{A}(t,x,u) + \tilde{\tilde{A}}(t,u),\\
&\left|\tilde{A}(t,x,u)\right| \leq m_a(t)\,(1+|u|)\,(1+|x|)\,, \quad \left|\tilde{\tilde{A}}(t,x,u)\right| \leq m_a(t)\,(1+|u|^{r_a})\,,\\
&\left|\tilde{A}(t,x,u) - \tilde{A}(t,x,v)\right| \leq m_a(t)\,|u-v|\,(1+|x|)\,,\\
&\left|\tilde{\tilde{A}}(t,u) - \tilde{\tilde{A}}(t,v)\right| \leq m_a(t)\left(1 + |u|^{r_a-1} + |v|^{r_a-1}\right)|u-v|\,,\\
&\left|(\operatorname{div}_x \tilde{A})(t,x,u)\right| \leq m_d(t)\,(1+|u|)\,, \qquad (\operatorname{div}_x \tilde{A})(t,x,0) = 0,
\end{aligned} \tag{5.14}$$

for $t \in [0,T]$, $x \in \mathbb{R}^d$, and $u,v \in \mathbb{R}$, where $r_a \geq 1$ is a number and m_a, m_d are integrable functions on $[0,T]$. "Globally Lipschitz" fluxes correspond to setting $\tilde{\tilde{A}} \equiv 0$ in (5.14), while "polynomially growing" (x-independent) fluxes correspond to setting $\tilde{A} \equiv 0$. In the "weight-free" L^p framework it is natural to assume (5.12).

Most of the works on kinetic solutions for stochastic conservation laws have dealt with the torus case ($\mathbb{T}^d$), and x-independent flux / no reaction term. The works on entropy solutions, on the other hand, have considered the unbounded domain case ($\mathbb{R}^d$), often with globally Lipschitz (x-independent) flux and no reaction term. In [28] the authors allow for a polynomially growing flux $A = A(u)$ (and $R \equiv 0$), corresponding to the $\tilde{\tilde{A}} = \tilde{\tilde{A}}(u)$ part of our flux. Existence of an entropy solution is proved in [28] under the assumptions (5.12) and (5.13), whereas uniqueness is established under the weaker condition (5.7). These results, based on entropy solutions, are consistent with ours based on kinetic solutions.

For some specific choices of the noise amplitude B it is possible to construct L^∞ solutions of (5.1), that is, $u \in L^\infty_{\omega,t,x}$, assuming (5.9). Of course, for L^∞ solutions, it is sufficient that A, R, B are merely "locally Lipschitz in u".

In what follows, we mostly lay out the results and proofs in the context of weighted L^p spaces. However, whenever relevant conditions are imposed on the "data" of the problem, cf. (5.12), (5.13), and (5.14), the reader may set "$w_N \equiv 1$" in the stated results.

We are going to rely on the (more precise) "kinetic" interpretation [50] of the entropy inequalities (5.10). The mapping $\chi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\chi(\xi, u) = \begin{cases} \mathbb{I}_{0<\xi<u}, & \text{if } u > 0 \\ 0, & \text{if } u = 0 \\ -\mathbb{I}_{u<\xi<0} & \text{if } u < 0 \end{cases}$$

is called a χ function. Notice that $\chi(\xi, u) = \mathbb{I}_{\xi<u} - \mathbb{I}_{\xi<0}$ for a.e. ξ, for each fixed $u \in \mathbb{R}$. Moreover, χ is compactly supported in the ξ-variable, and thus $\chi(\cdot, u) \in L^1(\mathbb{R})$. For any locally Lipschitz continuous $h : \mathbb{R} \to \mathbb{R}$, we have the following representation formula:

$$h(u) = h(0) + \int_{\mathbb{R}} h'(\xi)\chi(\xi, u)\, d\xi, \qquad u \in \mathbb{R}.$$

We also need the "one-sided" χ-functions $\chi_+(\xi, u) = \mathbb{I}_{\xi<u}$ and $\chi_-(\xi, u) := \chi_+(\xi, u) - 1$ $(= -\mathbb{I}_{\xi \geq u})$. Observe that $\chi_+(\xi, u) = \chi(\xi, u) + \mathbb{I}_{\xi<0}$ and $\chi_-(\xi, u) = \chi(\xi, u) - \mathbb{I}_{\xi\geq 0}$, for a.e. ξ, for each fixed $u \in \mathbb{R}$. In contrast to χ, the one-sided functions $\chi_\pm(\cdot, u)$ are not compactly supported and thus not integrable on $\mathbb{R}$. In most applications, however, it is sufficient that $\chi_\pm(\cdot, u)$ is in $L^1_{\mathrm{loc}}(\mathbb{R})$, for each fixed $u \in \mathbb{R}$.

Remark 5.4 *(properties of χ_+).* The following properties are easy to verify:

(1) $(u - v)_+ = \int_{\mathbb{R}} \chi_+(\xi, u)(1 - \chi_+(\xi, v))\, d\xi$;
(2) $\int_{\mathbb{R}} S'(\xi)\chi_+(\xi, u)(1 - \chi_+(\xi, v))\, d\xi = \mathbb{I}_{u>v}\, (S(u) - S(v))$, $\forall S \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$;
(3) $|u - v| = \int_{\mathbb{R}} |\chi_+(\xi, u) - \chi_+(\xi, v)|\, d\xi$;
(4) Set $g(\xi, u, v) = \frac{1}{2}(\chi_+(\xi, u) + \chi_+(\xi, v))$. Then $\frac{1}{4}|u - v| = \int_{\mathbb{R}} g - g^2\, d\xi$.

Let us introduce the following notations for further use:

$$a_i = a_i(t, x, \xi) := (\partial_u A_i)(t, x, \xi), \quad i = 1, \dots, d,$$

$$a = (a_1, \ldots, a_d), \quad d = d(t,x,\xi) := -(\operatorname{div}_x A)(t,x,\xi),$$

$\overline{a} = \overline{a}(t,x,\xi) = \{a, d\}$ [16], and note that $\operatorname{div}_{(x,\xi)} \overline{a} := \operatorname{div}_x a + \partial_\xi d = 0$. In view of our assumptions (5.2), (5.3), and (5.4), we clearly have

$$\left\| \frac{a(t,x,\xi)}{1+|x|} \right\|_{L^\infty_x} \leq m_a(t), \qquad (\omega,t,\xi) \in \Omega \times [0,T] \times \mathbb{R}, \tag{5.15}$$

$$\|d(t,x,\xi)\|_{L^\infty_x} \leq m_d(t)\,(1+|\xi|), \qquad (\omega,t,\xi) \in \Omega \times [0,T] \times \mathbb{R}, \tag{5.16}$$

and

$$\|R(t,x,\xi)\|_{L^\infty_x} \leq m_R(t)\,(1+|\xi|), \quad \|\partial_\xi R(\omega,t,x,\xi)\|_{L^\infty_x} \leq m_R(t), \tag{5.17}$$

for $(\omega,t,\xi) \in \Omega \times [0,T] \times \mathbb{R}$. These estimates imply, a.s., $\overline{a},\ R \in L^1\left(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R})\right)$. Besides, we will always assume

$$\nabla_{(x,\xi)}\overline{a},\ \nabla_x R \in L^1\left(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R})\right) \quad \text{a.s.}, \tag{5.18}$$

and so, a.s., $\overline{a},\ R \in L^1\left(0,T; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R})\right)$ (for the DiPerna-Lions regularization lemma).

Setting

$$\rho = \rho(\omega,t,x) := \chi_+(\xi, u(\omega,t,x)) = \mathbb{I}_{\xi < u(\omega,t,x)},$$

the kinetic equation reads

$$\begin{aligned} &\partial_t \rho + \operatorname{div}_{(x,\xi)}\left(\overline{a}\rho\right) + R\partial_\xi \rho \\ &\quad + \sum_{k\geq 1} b_k \partial_\xi \rho\, \dot{W}_k(t) = \partial_\xi\left(\frac{B^2}{2}\partial_\xi \rho\right) + \partial_\xi m \quad \text{in } \mathcal{D}'([0,T) \times \mathbb{R}^d \times \mathbb{R}), \text{ a.s.}, \end{aligned} \tag{5.19}$$

where $\overline{a} := \{a, d\}$ satisfies $\operatorname{div}_{(x,\xi)} \overline{a} = 0$, B^2 is defined in (5.7), and $\partial_\xi \rho = -\delta(\xi - u)$. All the coefficients $\overline{a}, R, b_k, B^2$ depend on (t,x,ξ). On the right-hand side of (5.19), m is the so-called *kinetic measure*.

Remark 5.5. Observe that the stochastic kinetic equations (3.12) and (3.15), which arise in our first homogenization problem, are both of the type (5.19). On the other hand, the kinetic equations (4.11) and (4.15) (arising in the second homogenization problem) are not, see also Remark 5.5. However, combining the arguments developed in this section with those used in the proof of Lemma 4.1, we can also handle this (new) type of stochastic kinetic equations.

Definition 5.1 *(kinetic measure).* A nonnegative mapping $m : \Omega \to \mathcal{M}([0,T] \times \mathbb{R}^d \times \mathbb{R})$ is called a (weighted) kinetic measure provided the following three conditions hold:

(1) $m(\phi) : \Omega \to \mathbb{R}$ is measurable for each $\phi \in C_c([0,T] \times \mathbb{R}^d \times \mathbb{R})$, where $m(\phi)$ denotes the action of m on ϕ, i.e., $m(\phi) = \int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}} \phi(t,x,\xi)\, m(dt,dx,d\xi)$;
(2) the process $(\omega,t) \mapsto m(\phi)([0,t] \times \mathbb{R}^d \times \mathbb{R}) = \int_{[0,t]\times M\times\mathbb{R}} \phi(x,\xi)\, m(ds,dx,d\xi)$ is predictable and belongs to $L^2(\Omega \times [0,T])$, for any $\phi \in C_c(\mathbb{R}^d \times \mathbb{R})$;
(3) m exhibits weighted p–moments: $m_N := w_N m$, cf. (5.11), satisfies

$$\mathbb{E} \int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}} |\xi|^p \, m_N(dt,dx,d\xi) \lesssim_{T,N,p} 1, \qquad \forall p \in [0,\infty). \tag{5.20}$$

Definition 5.2 *(kinetic solution).* Given an initial function $u_0 \in L^\infty\left(\Omega, \mathcal{F}_0; L^\infty(\mathbb{R}^d)\right)$, set $\rho_0 := \mathbb{I}_{\xi<u_0}$. A measurable function $u : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is said to be a *kinetic solution* of (5.1) if u is a predictable $L^2(w_N dx)$–valued stochastic process such that

$$\mathbb{E}\left(\operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|^p_{L^p(w_N dx)} \right) \lesssim_{T,N,p} 1, \qquad \forall p \in [2,\infty), \tag{5.21}$$

and there is a kinetic measure m such that $\rho := \mathbb{I}_{\xi<u}$ satisfies (5.19).

Remark 5.6. The property $\partial_\xi \rho = -\delta(\xi - u)$ is satisfied by any kinetic solution ρ (and thus $\rho \in BV_\xi$). Given a function $H = H(t,x,\xi)$ that is continuous in ξ, we assign the following meaning to the distribution $H\partial_\xi\rho$:

$$\langle H\partial_\xi \rho, \phi\rangle_{\mathcal{D}'_\xi,\mathcal{D}_\xi} = -H(t,x,u(\omega,t,x))\phi(t,x,u(\omega,t,x)), \qquad \phi \in \mathcal{D}_{t,x,\xi},$$

for a.e. $(\omega,t,x) \in \Omega \times [0,T] \times \mathbb{R}^d$, thereby explaining the meaning of (5.19).

Remark 5.7 *(entropy & kinetic solutions).* It is equivalent to be a kinetic solution according to Definition 5.2 and an entropy solution, i.e., a weak solution of (5.1) satisfying (5.10).

Remark 5.8 *(weighted p–moments of kinetic measure).* Fix a kinetic solution ρ with kinetic measure m. For later use, let us compute the p-moments of the weighted measure $m_N := w_N m$, where w_N is the weight function (5.11). It follows from (5.19) that

$$\begin{aligned} m(\partial_\xi \varphi)([0,T]) &= \int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}} \partial_\xi \varphi(x,\xi)\, m(dt,dx,d\xi) \\ &= \langle \chi_0, \varphi\rangle - \langle \chi(T), \varphi\rangle + \int_0^T \left\langle \rho(t), \overline{a}(t) \cdot \nabla_{(x,\xi)}\varphi \right\rangle dt \end{aligned}$$

$$
- \int_0^T \left\langle \left(R\partial_\xi \rho\right)(t), \varphi \right\rangle \, dt - \sum_{k\geq 1} \int_0^T \left\langle \left(b_k \partial_\xi \rho\right)(t), \varphi \right\rangle \, dW_k(t)
$$

$$
- \int_0^T \left\langle \left(\frac{B^2}{2}\partial_\xi \rho\right)(t), \partial_\xi \varphi \right\rangle \, dt, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}), \quad (5.22)
$$

where $\chi := \rho - \mathbb{I}_{\xi<0}$ and $\chi_0 := \rho_0 - \mathbb{I}_{\xi<0}$. Fix any convex function $S \in C^2(\mathbb{R})$ with $|S(\xi)| \lesssim |\xi|^{p+2}$, $|S'(\xi)| \lesssim |\xi|^{p+1}$, $|S''(\xi)| \lesssim |\xi|^p$ ($p \geq 0$), i.e., $S \in C^2_{\mathrm{pol}}(\mathbb{R})$. We will utilize the test function $\varphi = \varphi_{\kappa,\ell}(x,\xi) := S'(\xi)\, w_N(x)\, \phi_\kappa(x)\psi_\ell(\xi) \overset{\kappa,\ell\uparrow\infty}{\longrightarrow} S'(\xi)\, w_N(x)$, where $\phi_\kappa(x) = \phi_1\left(\frac{x}{\kappa}\right)$, $\phi_1 \in C_c^\infty(\mathbb{R}^d)$, $0 \leq \phi_1 \leq 1$, $\phi_1 = 1$ on $\{|x| \leq 1\}$, and $\phi_1 = 0$ on $\{|x| \geq 2\}$. Moreover, $\psi_\ell(x) = \psi_1\left(\frac{\xi}{\ell}\right)$, $\psi_1 \in C_c^\infty(\mathbb{R}^d)$, $0 \leq \psi_1 \leq 1$, $\psi_1 = 1$ on $\{|\xi| \leq 1\}$, and $\psi_1 = 0$ on $\{|\xi| \geq 2\}$. We refer to $\{\phi_\kappa(x)\}_{\kappa\geq 1}$, and $\{\psi_\ell(x)\}_{\ell\geq 1}$ as truncation sequences (on, respectively, $\mathbb{R}^d$ and $\mathbb{R}$). Clearly, $|\nabla\phi_\kappa(x)| \lesssim \frac{1}{\kappa}\mathbb{I}_{\kappa\leq|x|\leq 2\kappa}$, $|\psi'_\ell(\xi)| \lesssim \frac{1}{\ell}\mathbb{I}_{\ell\leq|\xi|\leq 2\ell}$, and

$$
\partial_\xi \varphi_{\kappa,\ell} = S''(\xi) w_N(x)\phi_\kappa(x)\psi_\ell(\xi) + S'(\xi) w_N(x)\phi_\kappa(x)\psi'_\ell(\xi) \overset{\kappa,\ell\uparrow\infty}{\longrightarrow} S''(\xi) w_N(x),
$$

$$
\nabla_x \varphi_{\kappa,\ell} = S'(\xi)\nabla w_N(x)\phi_\kappa(x)\psi_\ell(\xi) + S'(\xi) w_N(x)\nabla\phi_\kappa(x)\psi_\ell(\xi) \overset{\kappa,\ell\uparrow\infty}{\longrightarrow} S'(\xi)\nabla w_N(x).
$$

Making use of $\varphi_{\kappa,\ell}$ in (5.22) and sending $\kappa, \ell \to \infty$, we eventually arrive at the following equation satisfied a.s. by the weighted kinetic measure m_N ($= w_N m$):

$$
\begin{aligned}
\int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}} S''(\xi)\, m_N(dt,dx,d\xi) &= \int_{\mathbb{R}^d} S(u_0)\, w_N dx - \int_{\mathbb{R}^d} S(u(T))\, w_N dx \\
&+ \int_0^T \int_{\mathbb{R}^d} \Bigg(-2N \frac{Q_S(t,x,u)\cdot x}{1+|x|^2} + (\mathrm{div}_x\, Q_S)(t,x,u) \\
&\qquad + S'(u)\left(R(t,x,u) - (\mathrm{div}_x\, A)(t,x,u)\right) \Bigg) w_N dx\, dt \\
&+ \sum_{k\geq 1} \int_0^T \int_{\mathbb{R}^d} S'(u) b_k(t,x,u)\, w_N dx\, dW_k(t) \\
&+ \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} S''(u) B^2(t,x,u)\, w_N dx\, dt,
\end{aligned} \quad (5.23)
$$

for any $S \in C^2_{\mathrm{pol}}(\mathbb{R})$, $S(0) = 0$, $S'' \geq 0$. Keeping in mind our assumptions (5.2), (5.3), (5.4), (5.7), and (5.21), choosing $S(\xi) = \frac{1}{(p+1)(p+2)}|\xi|^{p+2}$ in (5.23) gives

$$\mathbb{E} \int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}} |\xi|^p \, m_N(dt,dx,d\xi) \le C, \qquad p \in [0,\infty), \tag{5.24}$$

where C depends on T, N and $\|u\|_{L^{p+2}(\Omega;L^\infty(0,T;L^{p+2}(w_N dx)))}$ (see also next remark).

Regarding the "weight-free" L^p–framework discussed in Remark 5.2, cf. (5.12), (5.13), and (5.14), the equation (5.23) continues to hold with $w_N \equiv 1$ (and thus $m_N = m$), in which case the "$-2N$" term is zero. As a result, $\mathbb{E}\int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}} |\xi|^p \, m(dt,dx,d\xi) \le C$, where C depends on T and $\|u\|_{L^{p+2}(\Omega;L^\infty(0,T;L^{p+2}(\mathbb{R}^d)))}$.

For L^∞–solutions, the bound (5.24) on m_N continues to hold with C depending on T, N, and $K_{\max} := \|u\|_{L^\infty_{\omega,t,x}}$. If $R - (\operatorname{div}_x A)$, b_k, B^2 are zero on $\mathbb{R}_\xi \setminus [-K_{\max}, K_{\max}]$, it follows from (5.23) that the weighted kinetic measure m_N is compactly supported in ξ.

Remark 5.9 *(improvement of integrability via a martingale argument).* By the previous remark, the random variable $\omega \mapsto \int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}} |\xi|^p \, m_N(dt,dx,d\xi)$ belongs to $L^1(\Omega)$. One can improve this to $L^q(\Omega)$ for any finite $q \ge 1$. To this end, we will argue that

$$\mathbb{E}\left(\operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|^r_{L^{p+2}(w_N dx)} \right) + \mathbb{E}\left(\int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}} |\xi|^p \, m_N(dt,dx,d\xi) \right)^{\frac{r}{p+2}} \lesssim_{r,T,N} 1,$$

provided the initial data u_0 satisfy $\mathbb{E}\left(\|u_0\|^r_{L^{p+2}(w_N dx)}\right) < \infty$, for $r > p+2$, a condition that clearly is satisfied due to (5.9). The case $r = p+2$ is covered by the definition of kinetic solution, cf. (5.20) and (5.21). In view of (5.23) with $S(\xi) = \frac{1}{(p+1)(p+2)} |\xi|^{p+2}$ and the growth assumptions (5.15), (5.16), and (5.17), it follows easily that

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\mathbb{R}^d} |u(t)|^{p+2} \, w_N dx + \int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}} |\xi|^p \, m_N(dt,dx,d\xi) \\ &\quad \lesssim \int_{\mathbb{R}^d} |u_0|^{p+2} \, w_N dx + \int_0^T \int_{\mathbb{R}^d} |u(t)|^{p+2} \, w_N dx\, dt + \sup_{t\in[0,T]} |M(t)|, \end{aligned} \tag{5.25}$$

for a.e. $(\omega,t) \in \Omega \times [0,T]$, where

$$M(t) = \sum_{k\ge1} \int_0^t \int_{\mathbb{R}^d} S'(u) b_k(\omega,s,x,u) \, w_N dx \, dW_k(s), \qquad S'(u) = \frac{1}{p+1} |u|^p \, u.$$

We raise both sides of (5.25) to the power $r/(p+2) > 1$, apply Jensen's inequality to the second term on the right-hand side, and take the expectation, eventually arriving at

$$
\begin{aligned}
&\mathbb{E}\left(\operatorname*{ess\,sup}_{t\in[0,T]}\|u(t)\|^r_{L^{p+2}(w_N dx)}\right)+\mathbb{E}\left(\int\limits_{[0,T]\times\mathbb{R}^d\times\mathbb{R}}|\xi|^p\,m_N(dt,dx,d\xi)\right)^{\frac{r}{p+2}}\\
&\quad\lesssim_T \mathbb{E}\left(\|u_0\|^r_{L^{p+2}(w_N dx)}\right)+\int_0^T\mathbb{E}\left(\|u(t)\|^r_{L^{p+2}(w_N dx)}\right)dt\\
&\qquad+\mathbb{E}\sup_{t\in[0,T]}|M(t)|^{\frac{r}{p+2}}.
\end{aligned}
\tag{5.26}
$$

A standard martingale argument (Burkholder-Davis-Gundy inequality [12]) supplies

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}|M(t)|^{\frac{r}{p+2}}\\
&\quad\lesssim_{T,N}\frac{1}{2}\,\mathbb{E}\left(\operatorname*{ess\,sup}_{t\in[0,T]}\|u(t)\|^r_{L^{p+2}(w_N dx)}\right)+\int_0^T\mathbb{E}\left(\|u(t)\|^r_{L^{p+2}(w_N dx)}\right)dt+1.
\end{aligned}
$$

Making use of this estimate in (5.26), followed by an application of Gronwall's inequality, leads to the sought after estimates.

It is easy to make the previous argument operational in the "weight-free" L^p–framework discussed in Remark 5.2, assuming (5.12), (5.13), (5.14). The same applies to L^∞–solutions.

Roughly speaking, the difference between a kinetic solution ρ and a so-called *generalized* kinetic solution ϱ is that the structural property $\partial_\xi\rho=-\delta(\xi-u)$ is replaced by the requirement $\partial_\xi\varrho=-\nu$ for some Young measure ν on $\mathbb{R}_\xi$. We refer to [18] for relevant background material on Young measures.

In what follows, any function of the form $\rho=\rho(z,\xi)=\mathbb{I}_{\xi<u(z)}$ will be called a *kinetic function*. We reserve the term *generalized kinetic function* to functions $\varrho=\varrho(z,\xi)$ taking values in $[0,1]$ such that $-\partial_\xi\varrho$ is a Young measure. For us $z=(\omega,x)$ or $z=(\omega,t,x)$.

Definition 5.3 *(generalized kinetic solution).* Fix a generalized kinetic function $\varrho_0(\omega,x,\xi)$. We call $\varrho:\Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}\to[0,1]$ a *generalized kinetic solution* of (5.1) with initial data ρ_0 if $\widetilde{\varrho}:=\varrho-\mathbb{I}_{\xi<0}$ is $\mathcal{P}/\mathcal{B}(L^2(w_N dx\,d\xi))$ measurable and

$$
\mathbb{E}\left(\operatorname*{ess\,sup}_{t\in[0,T]}\iint\limits_{\mathbb{R}^d\times\mathbb{R}}|\xi|^p\,w_N(x)\,\nu_{\omega,t,x}(d\xi)\,dx\right)\lesssim_{T,N,p}1,\qquad\forall p\in[2,\infty),
\tag{5.27}
$$

where $\nu:=-\partial_\xi\varrho$ is a Young measure, the spatial weight w_N is defined in (5.11), and there is a kinetic measure m such that ϱ satisfies a.s.

$$
\partial_t\varrho+\operatorname{div}_{(x,\xi)}\left(\overline{a}\varrho\right)+R\partial_\xi\rho
$$

$$+\sum_{k\geq 1} b_k \partial_\xi \varrho \, \dot{W}_k(t) = \partial_\xi \left(\frac{B^2}{2} \partial_\xi \varrho \right) + \partial_\xi m \quad \text{in } \mathcal{D}'([0,T) \times \mathbb{R}^d \times \mathbb{R}). \tag{5.28}$$

Remark 5.10. Given a function $H(t,x,\xi)$ that is continuous in ξ and a generalized kinetic solution ϱ, we assign the following meaning to the distribution $H\partial_\xi \varrho$:

$$\langle H \partial_\xi \varrho, \phi \rangle_{\mathcal{D}'_\xi, \mathcal{D}_\xi} = -\int_{\mathbb{R}} H(\omega,t,x,\xi)\phi(t,x,\xi)\, \nu_{\omega,t,x}(d\xi), \qquad \phi \in \mathcal{D}_{t,x,\xi},$$

for a.e. $(\omega,t,x) \in \Omega \times [0,T] \times \mathbb{R}^d$, thereby making precise the meaning of (5.28).

Remark 5.11. Although a generalized kinetic solution ϱ is merely locally integrable in ξ, the associated function $\widetilde{\varrho}$ $(= \varrho - \mathbb{I}_{\xi<0})$ is globally integrable; by (5.27),

$$\iint_{\mathbb{R}^d \times \mathbb{R}} |\widetilde{\varrho}(t)| \, |\xi|^p \, w_N(x) \, d\xi \, dx \lesssim_{T,N,p} 1, \quad t \in [0,T], \quad \forall p \in [1,\infty).$$

Remark 5.12 *(càdlàg / càglàd versions).* There are general theorems [52] ensuring that many real-valued stochastic processes $X(t)$ (discontinuous semimartingales) have a right-continuous version and, what's more, these versions necessarily have left-limits everywhere. Right-continuous processes with left-limits everywhere are referred to as *càdlàg*. Left-continuous processes with right-limits everywhere are referred to as *càglàd*.

A generalized kinetic solution ϱ is clearly not affected by modification of its values on any set of measure zero. In fact, ϱ is an equivalence class of functions. When proving stability and uniqueness results we must work with left/right continuous representatives of each equivalence class. Indeed, a result from [18, Proposition 10] (see also [21, Lemma 1.3.3]), easily generalized to our setting, says that a generalized kinetic solution ϱ possesses weak left and right limits $\varrho^{t,\pm}$ at every instant of time t. We then introduce left and right continuous representatives of ϱ by setting $\varrho^{\pm}(t) := \varrho^{t,\pm}$ for all $t \in [0,T]$. Clearly, $\varrho^{\pm}$ are both predictable since ϱ is. Using the left and right continuous representatives $\varrho^{\pm}$ one can convert the time-space weak formulation (5.28) into a formulation that is weak in space only (and pointwise in time): for *any* $t \in [0,T]$, a.s.,

$$\begin{aligned} \langle \varrho^{\pm}(t), \varphi \rangle &= \langle \varrho_0, \varphi \rangle + \int_0^t \langle \varrho(s), \overline{a}(s) \cdot \nabla_{(x,\xi)} \varphi \rangle \, ds - \int_0^t \langle (R\partial_\xi \varrho)(s), \varphi \rangle \, ds \\ &\quad - \sum_{k\geq 1} \int_0^t \langle (b_k \partial_\xi \varrho)(s), \varphi \rangle \, dW_k(s) - \int_0^t \left\langle \left(\frac{B^2}{2} \partial_\xi \varrho \right)(s), \partial_\xi \varphi \right\rangle ds \\ &\quad - \begin{cases} m(\partial_\xi \varphi)([0,t]), & \text{for } \varrho^+ \\ m(\partial_\xi \varphi)([0,t)), & \text{for } \varrho^- \end{cases}. \end{aligned} \tag{5.29}$$

Be mindful of the fact that $\langle \varrho^+(t) - \varrho^-(t), \varphi \rangle = -m(\partial_\xi \varphi)(\{t\})$. Since the atomic points of $m(\partial_\xi \varphi)(\cdot)$ is at most countable, we have $\langle \varrho^+(t), \varphi \rangle = \langle \varrho^-(t), \varphi \rangle$ for a.e. t and in turn $\varrho^+ = \varrho^-$ almost everywhere. The real-valued stochastic processes $X^\pm(t) := \langle \varrho^\pm(t), \varphi \rangle$, defined by (5.29), are of the form $X^\pm(t) = A^\pm(t) + M(t)$, where $A^\pm(t)$ are finite variation processes and $M(t)$ is a continuous martingale. Moreover, $A^+(0) = \langle \varrho_0, \varphi \rangle - m(\partial_\xi \varphi)(\{0\})$, $A^-(0) = \langle \varrho_0, \varphi \rangle$, and $M(0) = 0$. Below we note that $m(\partial_\xi \varphi)(\{0\}) = 0$ for kinetic initial data $\varrho_0 = \mathbb{I}_{\xi < u_0}$. Whenever convenient, we may assume that X^+ (X^-) are càdlàg (càglàd).

In what follows, we will outline a proof of uniqueness. Although we should work with the left/right continuous representatives $\varrho^\pm$ as in [18, Proposition 10] (see also [21]) and make use of the space-weak formulation (5.29), we will not do so in an attempt to save space and keep the presentation as simple as possible. Instead we refer to [18,19,21,32,33,37] for such details, see also [34,35].

Remark 5.13. Let us make a comment on generalized kinetic solutions and the satisfaction of the initial condition. Suppose $\varrho_0 = \mathbb{I}_{\xi < u_0}$ for some function u_0 satisfying (5.9). It follows from (5.29) that (the right-continuous representative of) ϱ satisfies a.s.

$$\langle \varrho(0), \varphi \rangle = \langle \varrho_0, \varphi \rangle - m(\partial_\xi \varphi)(\{0\}), \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}). \tag{5.30}$$

To conclude $\varrho(0) = \varrho_0$ we argue that $m\left(\{0\} \times \mathbb{R}^d \times \mathbb{R}\right) = 0$. The argument is standard [50], so we merely sketch it. Following Remark 5.8, (5.30) implies a.s. that

$$\iint\limits_{\mathbb{R}^d \times \mathbb{R}} S'(\xi)\left(\widetilde{\varrho}(0) - \chi(\xi, u_0)\right) w_N \, d\xi \, dx + \int\limits_{\{t=0\} \times \mathbb{R}^d \times \mathbb{R}} S''(\xi)\, w_N \, m(dt, dx, d\xi) = 0,$$

for any $S \in C^2(\mathbb{R})$ for which $S'' \geq 0$ and S, S', S'' grow at most polynomially. By Brenier's lemma [50], the first integral is nonnegative. As a result, both integrals must be zero. In other words, a.s., $\varrho(0) = \varrho_0$ and $m\left(\{0\} \times \mathbb{R}^d \times \mathbb{R}\right) = 0$.

Following an approach developed by Perthame [50], later extended to the stochastic case in [18] (see also [18,19,21,32,33,37,44,46]), we establish a rigidity result implying that generalized kinetic solutions are in fact kinetic solutions, at least when the initial function is a kinetic function, $\varrho_0 = \mathbb{I}_{\xi < u_0}$. The proof herein involves a regularization (via convolution) procedure, the Itô formula, and commutator arguments (going beyond the deterministic one by DiPerna-Lions) [36]. Essentially the same proof also shows that kinetic solutions are uniquely determined by their initial data, satisfying an L^1 contraction principle.

Proposition 5.1 *(rigidity result). Suppose that* $b_k, B^2, \overline{a} = \{a, d\}, R$ *satisfy conditions* (5.7)*,* (5.8)*,* (5.15)*,* (5.16)*,* (5.17)*,* (5.18)*, and* $\operatorname{div}_{(x,\xi)} \overline{a} = 0$*. Let* ϱ *be a generalized* *kinetic solution of* (5.1) *with initial data* ϱ_0. *Suppose* $m\left(\{0\}\times\mathbb{R}^d\times\mathbb{R}\right)=0$. *Then, for* $t\in[0,T]$,

$$0\le \mathbb{E}\iint\limits_{\mathbb{R}^d\times\mathbb{R}}\left(\varrho-\varrho^2\right)(t)\,w_N\,d\xi\,dx\lesssim_{T,N}\mathbb{E}\iint\limits_{\mathbb{R}^d\times\mathbb{R}}\left(\rho_0-\rho_0^2\right)\,w_N\,d\xi\,dx. \tag{5.31}$$

If $\varrho_0=\mathbb{I}_{\xi<u_0}$ *for some* u_0 *satisfying* (5.9), *then* $m\left(\{0\}\times\mathbb{R}^d\times\mathbb{R}\right)=0$ *and thus* $\varrho-\varrho^2=0$ *a.e.; whence* $\varrho=\mathbb{I}_{\xi<u}$ *for some function* u *that necessarily is a kinetic solution of* (5.1).

Remark 5.14. Informally speaking, cf. (5.29), we have $\varrho(t)=V(t)+M(t)$, where $V(t)$ is a finite variation process, $M(t)$ is a continuous martingale, and $\varrho(0)=V(0)$. In the proof below we need to determine the equation satisfied by $S(\varrho(t))$, where $S(\varrho)=\varrho-\varrho^2$. Noting that $(\varrho(t))^2=(V(t))^2+2V(t)M(t)+(M(t))^2$, we can calculate the first and second terms using standard calculus, while the third term can be computed using the Itô formula for continuous martingales [52]. Alternatively, we use the Itô formula for discontinuous semimartingales [39] to write $S(\varrho(t))=S(\varrho(0))+\int_0^t S'(\varrho(s-))\,d\varrho(s)+Q_S(t)+J_S(t)$, where $Q_S(t)=\int_0^t\frac{1}{2}S''(\varrho(s-))\,d[\varrho](s)$, $[\varrho](t)=[M](t)+\sum_{s\le t}\left(\Delta\varrho(s)\right)^2$ is the quadratic variation process, and

$$J_S(t)=\sum_{s\le t}\left(S(\varrho(s))-S(\varrho(s-))-S'(\varrho(s-))\Delta\varrho(s)-\frac{1}{2}S''(\varrho(s-))\left(\Delta\varrho(s)\right)^2\right)$$

is the "jump part" coming from the (temporal) discontinuities in ϱ. With $S(\varrho)=\varrho-\varrho^2$ (and $S''=-2$), we have $J_S\equiv 0$ and $Q_S(t)=-[M](t)-\sum_{s\le t}\left(\Delta\varrho(s)\right)^2\le -[M](t)$.

Proof. We will first give an informal proof of (5.31). Recall that ϱ satisfies a.s. (5.28). By the Itô and classical chain rules we arrive at the following equation for $S(\varrho):=\varrho-\varrho^2$:

$$\begin{aligned}&\partial_t S(\varrho)+\operatorname{div}_{(x,\xi)}\left(\overline{a}S(\varrho)\right)+R\partial_\xi S(\rho)\\&\qquad+\sum_{k\ge1}b_k\partial_\xi S(\varrho)\,\dot{W}_k(t)=\partial_\xi\left(\frac{B^2}{2}\partial_\xi S(\varrho)\right)+S'(\varrho)\partial_\xi m+\mathcal{Q},\end{aligned}\tag{5.32}$$

where $\mathcal{Q}$ contains the difference between certain quadratic terms linked to the variation of the martingale part and the second-order differential operator of the equation (5.28):

$$\mathcal{Q}=\frac{S''(\varrho)}{2}\sum_{k\ge1}\left(b_k\partial_\xi\varrho\right)^2-\frac{S''(\varrho)}{2}B^2\left(\partial_\xi\varrho\right)^2\equiv 0.$$

The perfect cancellation (i.e., $Q=0$) is the basic reason why the Proposition 5.1 holds. It follows from (5.32) that $I(\phi)=I_0(\varphi)+\sum_{i=1}^4 I_i(\varphi)$, $t\in[0,T]$, where

$$I(\varphi) = \mathbb{E} \iint\limits_{\mathbb{R}^d \times \mathbb{R}} S(\varrho(t))\varphi \, d\xi \, dx, \quad I_0(\varphi) = \mathbb{E} \iint\limits_{\mathbb{R}^d \times \mathbb{R}} S(\varrho_0)\varphi \, d\xi \, dx,$$

$$I_1(\varphi) = \int_0^t \left(\mathbb{E} \iint\limits_{\mathbb{R}^d \times \mathbb{R}} S(\varrho(s))\overline{a}(s) \cdot \nabla_{(x,\xi)}\varphi \, d\xi \, dx \right) ds,$$

$$I_2(\varphi) = -\frac{1}{2} \int_0^t \left(\mathbb{E} \iint\limits_{\mathbb{R}^d \times \mathbb{R}} B^2(s)\partial_\xi S(\varrho(s))\partial_\xi \varphi \, d\xi \, dx \right) ds,$$

$$I_3(\varphi) = -\int_0^t \left(\mathbb{E} \iint\limits_{\mathbb{R}^d \times \mathbb{R}} R(s)\partial_\xi S(\varrho(s))\varphi \, d\xi \, dx \right) ds,$$

$$I_4(\varphi) = -\,\mathbb{E} \iiint\limits_{[0,t] \times \mathbb{R}^d \times \mathbb{R}} \partial_\xi \left(S'(\varrho(s))\varphi \right) m(ds, dx, d\xi),$$

for any $\phi \in C_c^1(\mathbb{R}^d \times \mathbb{R})$. Let us particularize the test function as

$$\varphi(x, \xi) = \varphi_{\kappa,\ell}(x, \xi) = w_N(x)\phi_\kappa(x)\psi_\ell(\xi), \tag{5.33}$$

where the weight function w_N is defined in (5.11) and $\{\phi_\kappa\}_{\kappa \geq 1}$, $\{\psi_\ell\}_{\ell \geq 1}$ are truncation sequences respectively on $\mathbb{R}^d$, $\mathbb{R}$.

We rely on (5.15) and (5.16) to supply

$$\begin{aligned}
\left|S(\varrho(s))\overline{a}(s) \cdot \nabla_{(x,\xi)}\varphi_{\kappa,\ell}\right| &\lesssim \left(\varrho - \varrho^2\right)(s)\,|a(s)|\,\psi_\ell \left(|\nabla w_N| + \frac{1}{\kappa}\mathbb{I}_{\kappa \leq |\xi| \leq 2\kappa} w_N \right) \\
&\quad + \left(\varrho - \varrho^2\right)(s)\,|d(s)|\,\frac{1}{\ell}\mathbb{I}_{\ell \leq |\xi| \leq 2\ell} w_N \\
&\lesssim \left\| \frac{a(s)}{1 + |x|} \right\|_{L^\infty_x} \left(\varrho - \varrho^2\right)(s)\psi_\ell w_N + m_d(t)\left(\varrho - \varrho^2\right)(s)\left(1 + |\xi|\right)\frac{1}{\ell}\mathbb{I}_{\ell \leq |\xi| \leq 2\ell} w_N \\
&\lesssim \left(m_a(s) + m_d(s)\right)\left(\varrho - \varrho^2\right)(s) w_N \in L^1_{\omega,t,x,\xi},
\end{aligned}$$

and thus

$$|I_1(\varphi_{\kappa,\ell})| \lesssim \int_0^t \left(m_a + m_d\right)(s) \left(\mathbb{E} \iint\limits_{\mathbb{R}^d \times \mathbb{R}} \left(\varrho - \varrho^2\right)(s)\, w_N \, d\xi \, dx \right) ds.$$

Next, since $\varrho \in L^\infty_{\omega,t,x,\xi}$ and $\partial_\xi \varrho = -\nu(d\xi)$,

$$\begin{aligned}&\left|B^2(s)\partial_\xi S(\varrho(s))\partial_\xi \varphi_{\kappa,\ell}\right|\\ &\overset{(5.7)}{\lesssim} \frac{1}{\ell}\mathbb{I}_{\ell\le|\xi|\le 2\ell}\left(1+|\xi|^2\right)|1-2\varrho(s)|\,\phi_\kappa\, w_N\,\nu(d\xi) \lesssim \frac{1}{\ell}\left(1+|\xi|^2\right) w_N\,\nu(d\xi),\end{aligned}$$

and so, recalling (5.27), $|I_2(\varphi_{\kappa,\ell}| \lesssim_{T,N} \frac{1}{\ell} \xrightarrow{\ell\uparrow\infty} 0$.

Evoking (5.17),

$$\begin{aligned}|\partial_\xi\left(R\varphi_{\kappa,\ell}\right)| &\le |\partial_\xi R(s)\psi_\ell + R(s)\psi_\ell'|\;\phi_\kappa\, w_N\\ &\lesssim \left(m_R(s) + m_R(s)\left(1+|\xi|\right)\mathbb{I}_{\ell\le|\xi|\le 2\ell}\frac{1}{\ell}\right) w_N \lesssim m_R(s)\, w_N,\end{aligned}$$

and thus, after an integration by parts,

$$|I_3(\varphi_{\kappa,\ell})| \lesssim \int_0^t m_R(s)\left(\mathbb{E}\iint_{\mathbb{R}^d\times\mathbb{R}}\left(\varrho-\varrho^2\right)(s)\, w_N\, d\xi\, dx\right) ds.$$

Finally, using again that $\partial_\xi\varrho = -\nu$,

$$\begin{aligned}-\partial_\xi\left(S'(\varrho(s))\varphi_{\kappa,\ell}\right) &= -2\phi_\kappa\,\psi_\ell\, w_N\,\nu(d\xi) - \left(1-2\varrho(s)(s)\right)\;\phi_\kappa\,\psi_\ell'\, w_N\\ &\le \left(2\varrho(s)-1\right)\;\phi_\kappa\,\psi_\ell'\, w_N,\end{aligned}$$

and so, putting $\varrho \in L^\infty_{\omega,t,x,\xi}$ and (5.20) to good use,

$$|I_4(\varphi_{\kappa,\ell})| \lesssim \frac{1}{\ell}\,\mathbb{E}\, m_N\left([0,T]\times\mathbb{R}^d\times\{\ell\le|\xi|\le 2\ell\}\right) = O(1/\ell) \xrightarrow{\ell\uparrow\infty} 0.$$

Summarizing our computations (after sending $\kappa\to\infty$),

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\varrho-\varrho^2\right)(t)\,\psi_\ell w_N\, d\xi\, dx &\lesssim \mathbb{E}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\varrho_0-\varrho_0^2\right)\,\psi_\ell w_N\, d\xi\, dx\\ &\quad+\int_0^t M(s)\left(\mathbb{E}\int_{\mathbb{R}^d\times\mathbb{R}}\left(\varrho-\varrho^2\right)(s)\,\psi_\ell w_N\, d\xi\, dx\right) ds + O(1/\ell),\end{aligned}\tag{5.34}$$

where M is an integrable function on $[0,T]$. We arrive at the sought after (5.31) by sending $\ell\uparrow\infty$ and then applying Gronwall's inequality.

Unfortunately the equation (5.32) for $S(\rho)$ is only suggestive as the calculations involving the chain rule are merely formal. To make the calculations rigorous we regularize the "linear" equation (5.28), bringing in several regularization errors that must be controlled. Let $J^x_\varepsilon : \mathbb{R}^d \to \mathbb{R}$, $J^\xi_\delta : \mathbb{R} \to \mathbb{R}$ be standard Friedrich mollifiers, and define

$$\varrho_{\varepsilon,\delta}(\omega,t,x,\xi) = \varrho \star \left(J^x_\varepsilon J^\xi_\delta\right) = \iint\limits_{\mathbb{R}^d\times\mathbb{R}} \varrho(\omega,t,y,\zeta) J^x_\varepsilon(x-y) J^\xi_\delta(\xi-\zeta)\, dy\, d\zeta,$$

$$m_{\varepsilon,\delta}(\omega,t,x,\xi) = m \star \left(J^x_\varepsilon J^\xi_\delta\right) = \iint\limits_{\mathbb{R}^d\times\mathbb{R}} J^x_\varepsilon(x-y) J^\xi_\delta(\xi-\zeta)\, m(t,dy,d\zeta).$$

The mollified quantities $\varrho_{\varepsilon,\delta}, m_{\varepsilon,\delta}$ are smooth in x,ξ but discontinuous in t. However, working with suitable representatives (versions), we can ensure that $\varrho_{\varepsilon,\delta}, m_{\varepsilon,\delta}$ are càdlàg / càglàd in time t, thereby making the Itô formula available to us, and thus the arguments below can be made rigorous (see e.g. [18,21,32–35]). In passing, note that $m_{\varepsilon,\delta}$ is a measure on $[0,T]$ (depending on the "parameters" ω, x, ξ).

The following equation holds a.s.:

$$\begin{aligned} &\partial_t \varrho_{\varepsilon,\delta} + \operatorname{div}_{(x,\xi)}\left(\overline{a}\varrho_{\varepsilon,\delta}\right) + R\partial_\xi \varrho_{\varepsilon,\delta} + \sum_{k\ge1}\left((b_k\partial_\xi\varrho)\star\left(J^x_\varepsilon J^\xi_\delta\right)\right)\dot{W}_k(t) \\ &\quad = \partial_\xi\left(\left(\frac{B^2}{2}\partial_\xi\varrho\right)\star\left(J^x_\varepsilon J^\xi_\delta\right)\right) + \partial_\xi m_{\varepsilon,\delta} + r_{\varepsilon,\delta} \quad \text{in } \mathcal{D}'([0,T)\times\mathbb{R}^d\times\mathbb{R}), \end{aligned} \tag{5.35}$$

where the reminder term $r_{\varepsilon,\delta} = r_{\varepsilon,\delta}(\omega,t,x,\xi)$ takes the form

$$r_{\varepsilon,\delta} := \operatorname{div}_{(x,\xi)}\left(\overline{a}\varrho_{\varepsilon,\delta}\right) - \operatorname{div}_{(x,\xi)}\left((\overline{a}\varrho)\star\left(J^x_\varepsilon J^\xi_\delta\right)\right) + R\partial_\xi\varrho_{\varepsilon,\delta} - (R\partial_\xi\varrho)\star\left(J^x_\varepsilon J^\xi_\delta\right).$$

Our assumptions imply that $\overline{a}, R \in L^1\left(0,T; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d\times\mathbb{R})\right)$, whereas the generalized kinetic solution ϱ belongs a.s. to $L^\infty\left(0,T; L^\infty(\mathbb{R}^d\times\mathbb{R})\right)$. Moreover, $\operatorname{div}_{(x,\xi)}\overline{a} = 0$. Hence, by [20, Lemma II.1], $r_{\varepsilon,\delta}$ converges a.s. to zero in L^1_{loc} as $\varepsilon,\delta\to0$. Given (5.35), we apply the Itô formula as well as the classical (spatial) chain rule. The result is the following equation for $S(\varrho_{\varepsilon,\delta})$ that holds a.s. in $\mathcal{D}'([0,T)\times\mathbb{R}^d\times\mathbb{R})$:

$$\begin{aligned} &\partial_t S(\varrho_{\varepsilon,\delta}) + \operatorname{div}_{(x,\xi)}\left(\overline{a}S(\varrho_{\varepsilon,\delta})\right) + R\partial_\xi S(\varrho_{\varepsilon,\delta}) \\ &\quad + \sum_{k\ge1} S'(\varrho_{\varepsilon,\delta})\left((b_k\partial_\xi\varrho)\star\left(J^x_\varepsilon J^\xi_\delta\right)\right)\dot{W}_k(t) = \partial_\xi\left(\frac{B^2}{2}\partial_\xi S(\varrho_{\varepsilon,\delta})\right) \\ &\quad\quad + S'(\varrho_{\varepsilon,\delta})\partial_\xi m_{\varepsilon,\delta} + S'(\varrho_{\varepsilon,\delta})r_{\varepsilon,\delta} + \partial_\xi\left(S'(\varrho_{\varepsilon,\delta})\tilde{r}_{\varepsilon,\delta}\right) + \mathcal{Q}_{\varepsilon,\delta}, \end{aligned} \tag{5.36}$$

where $\tilde{r}_{\varepsilon,\delta} = \frac{B^2}{2}\partial_\xi\varrho_{\varepsilon,\delta} - \left(\frac{B^2}{2}\partial_\xi\varrho\right)\star\left(J^x_\varepsilon J^\xi_\delta\right)$ and

$$\mathcal{Q}_{\varepsilon,\delta} = \frac{1}{2}S''(\varrho_{\varepsilon,\delta})\sum_{k\ge1}\left((b_k\partial_\xi\varrho)\star\left(J^x_\varepsilon J^\xi_\delta\right)\right)^2 - \frac{1}{2}S''(\varrho_{\varepsilon,\delta})\left((B^2\partial_\xi\varrho)\star\left(J^x_\varepsilon J^\xi_\delta\right)\right)\partial_\xi\varrho_{\varepsilon,\delta}. \tag{5.37}$$

As a result of assumptions (5.7) and (5.8), $B^2 \in L^1\left(0,T; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d\times\mathbb{R})\right)$ (besides, we know $\varrho \in BV_\xi$). Thus, it is not difficult o show that, $\tilde r_{\varepsilon,\delta}$ converges a.s. to zero in L^1_{loc} as $\varepsilon,\delta \to 0$ [32]. Choosing (5.33) as test function in (5.36), recalling that $S(\varrho)=\varrho-\varrho^2$, and carrying on as before (5.34), we deliver

$$\begin{aligned}
\mathbb{E}\int_{\mathbb{R}^d\times\mathbb{R}} \left(\varrho_{\varepsilon,\delta}-\varrho^2_{\varepsilon,\delta}\right)(t)\,\psi_\ell w_N\,d\xi\,dx &\lesssim \mathbb{E}\int_{\mathbb{R}^d\times\mathbb{R}} \left(\varrho_{0,\varepsilon,\delta}-\varrho^2_{0,\varepsilon,\delta}\right)(0)\,\psi_\ell w_N\,d\xi\,dx \\
&+\int_0^t M(s)\left(\mathbb{E}\int_{\mathbb{R}^d\times\mathbb{R}} \left(\varrho_{\varepsilon,\delta}-\varrho^2_{\varepsilon,\delta}\right)(s)\psi_\ell w_N\,d\xi\,dx\right)ds \\
&+\mathbb{E}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}}\left(|r_{\varepsilon,\delta}|+\frac{1}{\ell}\mathbb{I}_{\ell\le|\xi|\le 2\ell}\,|\tilde r_{\varepsilon,\delta}|\right)w_N\,d\xi\,dx\,dt \\
&+\mathbb{E}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}} \mathcal{Q}_{\varepsilon,\delta} w_N\,d\xi\,dx\,dt + O(1/\ell),
\end{aligned} \tag{5.38}$$

for some integrable function M on $[0,T]$, where $\varrho_{0,\varepsilon,\delta} := \varrho_0 \star \left(J^x_\varepsilon J^\xi_\delta\right)$. Provided we show that the "$\varepsilon,\delta\to 0$ limit" of the $\mathcal{Q}_{\varepsilon,\delta}$–term is zero, we obtain the rigidity inequality (5.31) by sending $\varepsilon,\delta\downarrow 0$ and $\ell\uparrow\infty$ in (5.38), followed by an application of Gronwall's inequality.

It remains to compute the limit of the $\mathcal{Q}_{\varepsilon,\delta}$–term. Recalling that $B^2=\sum_{k\ge1} b_k^2$, we write $\mathcal{Q}_{\varepsilon,\delta}(\omega,t,x,\xi)=\sum_{k\ge1}\mathcal{Q}_{\varepsilon,\delta,k}(\omega,t,x,\xi)$, where, for $k=1,2,\dots$,

$$\begin{aligned}
\mathcal{Q}_{\varepsilon,\delta,k}(\omega,t,x,\xi) &:= \left(\left(b_k^2\partial_\xi\varrho\right)\star\left(J^x_\varepsilon J^\xi_\delta\right)\right)\partial_\xi\varrho_{\varepsilon,\delta} - \left(\left(b_k\partial_\xi\varrho\right)\star\left(J^x_\varepsilon J^\xi_\delta\right)\right)^2 \\
&= \iiiint \left(\left(b_k(\omega,t,y,\zeta)\right)^2 - b_k(\omega,t,y,\zeta)b_k(\omega,t,\bar y,\bar\zeta)\right) \\
&\qquad\qquad \times (\partial_\xi\varrho)(\omega,t,y,\zeta)(\partial_\xi\varrho)(\omega,t,\bar y,\bar\zeta) \\
&\qquad\qquad\qquad \times J^x_\varepsilon(x-y)J^x_\varepsilon(x-\bar y)J^\xi_\delta(\xi-\zeta)J^\xi_\delta(\xi-\bar\zeta)\,d\zeta\,dy\,d\bar\zeta\,d\bar y.
\end{aligned}$$

We can switch the roles of y and $\bar y$ as well as ζ and $\bar\zeta$. Add the resulting expression for $\mathcal{Q}_{\varepsilon,\delta,k}$ to the one above and divide by 2, obtaining

$$\begin{aligned}
\mathcal{Q}_{\varepsilon,\delta,k}(\omega,t,x,\xi) = \frac{1}{2}\iiiint &\left|b_k(\omega,t,y,\zeta)-b_k(\omega,t,\bar y,\bar\zeta)\right|^2(\partial_\xi\varrho)(\omega,t,y,\zeta)(\partial_\xi\varrho)(\omega,t,\bar y,\bar\zeta) \\
&\times J^x_\varepsilon(x-y)J^x_\varepsilon(x-\bar y)J^\xi_\delta(\xi-\zeta)J^\xi_\delta(\xi-\bar\zeta)\,d\zeta\,dy\,d\bar\zeta\,d\bar y.
\end{aligned} \tag{5.39}$$

Summing over k, recalling (5.7), and using $\partial_\xi\varrho = -\nu_{\omega,t,x}(d\xi)$ with $\nu(\mathbb{R})=1$, the following estimate eventually materializes:

$$\begin{aligned}
&\iint \mathcal{Q}_{\varepsilon,\delta}(\omega,t,x,\xi)\, w_N(x)\; d\xi\, dx \\
&\quad \lesssim \frac{1}{2} \iiiint\!\!\iint \Big(|y-\bar{y}|^2 + |\zeta-\bar{\zeta}|\,\mu\left(|\zeta-\bar{\zeta}|\right)\Big)(\partial_\xi \varrho)(\omega,t,y,\zeta)(\partial_\xi \varrho)(\omega,t,\bar{y},\bar{\zeta}) \\
&\qquad\qquad \times J^x_\varepsilon(x-y)J^x_\varepsilon(x-\bar{y})J^\xi_\delta(\xi-\zeta)J^\xi_\delta(\xi-\bar{\zeta})\, w_N(x)\, d\zeta\, dy\, d\bar{\zeta}\, d\bar{y}\; d\xi\; dx \\
&\quad \lesssim_N (\varepsilon + \mu(\delta)) \xrightarrow{\varepsilon,\delta\downarrow 0} 0.
\end{aligned}$$

This concludes the proof. □

Remark 5.15. Regarding the "weight-free" L^p–framework discussed in Remark, the proof of Proposition 5.1 remains the same except for a few changes involving the terms $I_1(\varphi_{\kappa,\ell})$ and $I_2(\varphi_{\kappa,\ell})$ to account for the weight-free test function $\varphi_{\kappa,\ell}(x,\xi) = \phi_\kappa(x)\psi_\ell(\xi)$ and the modified assumptions (5.12), (5.13), and (5.14).

The next theorem contains the main result of this section, namely the existence, uniqueness, and L^1 stability of kinetic solutions.

Theorem 5.1 *(well-posedness). Suppose that* $b_k, B^2, \overline{a} = \{a, d\}$ *,* R *satisfy conditions (5.7), (5.8), (5.15), (5.16), (5.17), (5.18) and* $\operatorname{div}_{(x,\xi)} \overline{a} = 0$*. There exists a unique kinetic solution of (5.1) with initial data* u_0 *satisfying (5.9). If* u_1, u_2 *are two kinetic solutions of (5.1) with initial data* $u_{1,0}, u_{2,0}$*, respectively, then*

$$\mathbb{E}\int_{\mathbb{R}^d} |u_1(t,x) - u_2(t,x)|\; w_N\, dx \lesssim_{T,N} \mathbb{E}\int_{\mathbb{R}^d} |u_{1,0}(x) - u_{2,0}(x)|\; w_N\, dx, \tag{5.40}$$

for all $t \in [0,T]$*, where* w_N *is defined in (5.11). Besides, the unique kinetic solution* u *of (5.1) has a representative in the space* $L^p(\Omega; L^\infty(0,T; L^p(w_N dx)))$ *which a.s. exhibits continuous samples paths in* $L^p(w_N dx)$*, for all* $p \in [1,\infty)$*.*

Proof. As in [32,35], we point out that the L^1 contraction principle (5.40) is a simple consequence of Proposition 5.1. Indeed, define $\overline{\varrho} = \frac{1}{2}\left(\mathbb{I}_{\xi<u_1} + \mathbb{I}_{\xi<u_2}\right) =: \frac{1}{2}\left(\rho_1 + \rho_2\right)$ and also $\overline{\varrho}_0 = \frac{1}{2}\left(\mathbb{I}_{\xi<u_{1,0}} + \mathbb{I}_{\xi<u_{2,0}}\right) =: \frac{1}{2}\left(\rho_{0,1} + \rho_{0,2}\right)$. Note that $\overline{\varrho}$ is a generalized kinetic solution with initial data $\overline{\varrho}_0$, kinetic measure $\overline{m} = \frac{1}{2}(m_1 + m_2)$, and $\partial_\xi \overline{\varrho} = -\frac{1}{2}\left(\delta_{u_1} + \delta_{u_2}\right) =: -\overline{\nu}$. Clearly, $\overline{m}(\{0\} \times \mathbb{R}^d \times \mathbb{R}) = 0$ (since m_1, m_2 both vanish at $t = 0$ because of the kinetic initial data) and thus $\varrho(0) = \varrho_0$, cf. Remark 5.13. By Proposition 5.1,

$$\mathbb{E}\iint_{\mathbb{R}^d\times\mathbb{R}} \left(\overline{\varrho} - \overline{\varrho}^2\right)(t)\, w_N\, d\xi\, dx \lesssim_{T,N} \mathbb{E}\iint_{\mathbb{R}^d\times\mathbb{R}} \left(\overline{\varrho}_0 - \overline{\varrho}_0^2\right) w_N\, d\xi\, dx,$$

for a.e. $t \in [0,T]$. A simple computation, exploiting the identities $\rho_i^2 = \rho_i$ $(i = 1,2)$, will reveal that $\overline{\varrho} - \overline{\varrho}^2 = \frac{1}{4}\left(\rho_1 - \rho_2\right)^2 = \frac{1}{4}|\rho_1 - \rho_2|$ and so $\int_{\mathbb{R}}\left(\overline{\varrho} - \overline{\varrho}^2\right) d\xi = \frac{1}{4}|u_1 - u_2|$. In the same way, we have $\int_{\mathbb{R}}\left(\overline{\rho}_0 - \overline{\rho}_0^2\right) d\xi = \frac{1}{4}|u_{1,0} - u_{2,0}|$. Consequently, (5.40) holds.

The sample paths of a kinetic solution u are a.s. continuous as a result of the uniqueness result. The detailed proof is the same as in [18, Corollary 16] (see also [21]). Thanks to the continuity of the sample paths, the contraction inequality (5.40) holds for all $t \in [0,T]$.

The existence part of the theorem can be founded on the vanishing viscosity method [7,14,18,28,42], or operator splitting [6,41] to separate the deterministic and stochastic effects in (5.1). Existence results on $\mathbb{R}^d$ are provided in these references under the assumptions that $R \equiv 0$ and $A = A(u)$ does not depend on t, x. The techniques employed in [6,7,14,18,28,41,42] can be adapted to the general context provided by (5.1). Here we only give a sketch of the proof via the vanishing viscosity method, based on [18].

Given $\varepsilon > 0$ and consider the following parabolic SPDE

$$\begin{aligned}&\partial_t u^\varepsilon + \operatorname{div}_x A(t,x,u^\varepsilon) - \varepsilon\Delta_x u^\varepsilon = B(t,u^\varepsilon)\dot W(t) + R(t,x,u^\varepsilon), \quad (t,x)\in(0,T)\times\mathbb{R}^d,\\ &u^\varepsilon(0,x) = u_0(\omega,x), \quad x\in\mathbb{R}^d.\end{aligned} \tag{5.41}$$

It is not difficult to show that equation (5.41) is well-posed. Indeed, the unique weak solution belonging to the weighted space $L^2(\Omega;(C([0,T]);L^2(\omega_N dx)))\cap L^2(\Omega\times[0,T];H^1(\omega_N dx))$ can be found as a fixed point of the operator

$$\begin{aligned}Kv(t) := S(t)u_0 &+ \int_0^t S(t-s)\Big(R(s,\cdot,v(s)) - \operatorname{div}_x A(s,\cdot,v(s))\Big)\,ds\\ &+ \int_0^t S(t-s)B(s,v(s))\,dW(s),\end{aligned}$$

where $S(t)$ is the semigroup generated by the heat equation in $\mathbb{R}^d$.

Let u^ε be the weak solution of (5.41). Then, for $S\in C^2(\mathbb{R})$, by Itô formula we have that the following equation is a.s. satisfied in the sense of distributions:

$$\begin{aligned}&\partial_t S(u^\varepsilon) + \operatorname{div}_x Q_S(t,x,u^\varepsilon) + S'(u^\varepsilon)\left((\operatorname{div}_x A)(t,x,u^\varepsilon) - R(t,x,u^\varepsilon)\right) - (\operatorname{div}_x Q_S)(t,x,u^\varepsilon)\\ &\quad = -\varepsilon S''(u^\varepsilon)|\nabla u^\varepsilon|^2 + \varepsilon\Delta_x\mathcal{S}(u^\varepsilon) + \sum_{k\ge1} S'(u^\varepsilon)b_k(t,x,u^\varepsilon)\,\dot W_k(t) + \frac12 S''(u^\varepsilon)B^2(t,x,u^\varepsilon),\end{aligned} \tag{5.42}$$

where $Q_S : [0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^d$ is given by $(\partial_u Q_S)(t,x,u) = S'(u)(\partial_u A)(t,x,u)$.

Let $S(\xi) = |\xi|^p$, $p\ge 2$. Then, similarly as in Remark 5.2, taking conveniently chosen test functions, after some manipulation it follows that

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|u^\varepsilon(t)\|^p_{L^p(\omega_N dx)}\right) + \varepsilon\int_0^T\int_{\mathbb{R}^d}|u^\varepsilon(t,x)|^{p-2}|\nabla u^\varepsilon|^2\omega_N(x)\,dx\,dt \le C, \tag{5.43}$$

where $C = C(p, u_0, T)$ is independent of ε.

Moreover, u^ε is a kinetic solution of equation (5.41), in the sense that the function $\varrho^\varepsilon(t,x,\xi) := \mathbb{I}_{\xi<u^\varepsilon(t,x)}$ satisfies the SPDE

$$\begin{aligned} &\partial_t \varrho^\varepsilon + \operatorname{div}_{(x,\xi)}\left(\overline{a}\varrho^\varepsilon\right) + R\partial_\xi\varrho^\varepsilon - \varepsilon\Delta_x\varrho^\varepsilon \\ &\quad + \sum_{k\geq 1} b_k\partial_\xi\varrho^\varepsilon\,\dot W_k(t) = \partial_\xi\left(\frac{B^2}{2}\partial_\xi\varrho^\varepsilon\right) + \partial_\xi m^\varepsilon \quad \text{in } \mathcal{D}'([0,T)\times\mathbb{R}^d\times\mathbb{R}), \text{ a.s.,} \end{aligned} \tag{5.44}$$

where $m^\varepsilon = \varepsilon|\nabla_x u^\varepsilon|^2\delta_{\xi=u^\varepsilon}$, with initial data $\varrho^\varepsilon(0,x,\xi) = \rho_0(x,\xi) := \mathbb{I}_{\xi<u_0(x)}$.

Let us denote $\nu^\varepsilon_{t,x} = -\partial_\xi\varrho^\varepsilon(t,x,\xi) = \delta_{\xi=u^\varepsilon(t,x)}$. Then, ν^ε is a Young measure and by (5.43) we have, in particular, that

$$\mathbb{E}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}} |\xi|^p\, d\nu^\varepsilon_{t,x}(\xi)\,\omega_N dx\,dt \leq C_p, \tag{5.45}$$

for any $p \geq 0$, uniformly in ε. Likewise, (5.43) also implies that

$$\mathbb{E}\int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}} |\xi|^p\, dm^\varepsilon_N(\xi,t,x) \leq C_p,$$

uniformly in ε, where $m^\varepsilon_N = \omega_N m^\varepsilon$. This last estimate can be improved to the following

$$\mathbb{E}\left|\int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}} |\xi|^{2p}\, dm^\varepsilon_N(\xi,t,x)\right|^2 \leq C_p, \qquad p \geq 2. \tag{5.46}$$

Proceeding similarly as in Remark 5.9, it suffices to take convenient test functions (in connection with the weight ω_N) in (5.42) with $S(\xi) = |\xi|^{2p+2}$, squaring the resulting equation and taking expectation. Indeed, note that

$$\begin{aligned} &\mathbb{E}\left|\int_{[0,T]\times\mathbb{R}^d\times\mathbb{R}} |\xi|^{2p}\, dm^\varepsilon_N(\xi,t,x)\right|^2 \\ &\quad = \frac{1}{(p+2)(p+1)}\mathbb{E}\left|\int_0^T\int_{\mathbb{R}^d} \varepsilon S''(u^\varepsilon)\,|\nabla_x u^\varepsilon|^2\,\omega_N dx\,dt\right|^2. \end{aligned}$$

With some manipulation involving the Itô isometry and using (5.43) all the other terms can be bounded appropriately so that (5.46) follows. We omit the details.

Now, by the theory of Young measures and kinetic functions (see e.g. Theorem 5 and Corollary 6 in [18]) (5.45) guarantees the existence of a sequence $\{\varepsilon_n\}_n$, a young measure ν and a generalized kinetic function $\varrho : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R} \to [0,1]$ such that $\varepsilon_n \to 0$, $\nu^{\varepsilon_n} \to \nu$ in the sense of Young measures and $\varrho^{\varepsilon_n} \rightharpoonup \varrho$ weakly-$*$ in $L^\infty(\Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R})$ as $n \to \infty$. Moreover, denoting by $\mathcal{M}_b$ the space of the bounded Borel Measures on $[0,T] \times \mathbb{R}^d \times \mathbb{R}$, by (5.46) there is a kinetic measure m_N such that, up to a subsequence, $m_N^{\varepsilon_n} \rightharpoonup m_N$ weakly-$*$ in $L^2(\Omega; \mathcal{M}_b)$, as $n \to \infty$. Defining $m := \frac{1}{\omega_N} m_N$, then m turns out to be a kinetic measure in the sense of Definition 5.1 and we may pass to the limit as $\varepsilon = \varepsilon^n \to 0$ in equation (5.44) in order to conclude that ϱ is a generalized kinetic solution of equation (5.1). At this point, the rigidity result implies that $\rho = \mathbb{I}_{\xi < u}$ where u is a kinetic solution. □

Remark 5.16 *(strong convergence of the parabolic approximations).* Let ϱ and ϱ^ε be as in the proof of Theorem 5.1. Taking advantage of the particular structure of ϱ^{ε_n} and ϱ we have that

$$\|u^{\varepsilon_n}\|^2_{L^2(\Omega \times [0,T]; L^2(\omega_N dx))} - \|u\|^2_{L^2(\Omega \times [0,T]; L^2(\omega_N dx))} = \int_{[0,T] \times \mathbb{R}^d \times \mathbb{R}} 2\xi(\varrho - \varrho^{\varepsilon_n})\, d\xi\, \omega_N dx\, dt. \quad (5.47)$$

By Chebyshev's inequality and using (5.43) with $p = 3$, for any $R > 0$ we have

$$\mathbb{E} \int_0^T \int_{\mathbb{R}^d} \int_{|\xi| > R} |2\xi(\varrho - \varrho^\varepsilon)|\, d\xi\, \omega_N dx\, dt \le \frac{C}{R}.$$

Thus, taking expectation in (5.47), we may pass to the limit as $\varepsilon_n \to 0$ in order to conclude that

$$\|u^{\varepsilon_n} - u\|_{L^2(\Omega \times [0,T]; L^2(\omega_N dx))} \to 0, \qquad \text{as } n \to \infty.$$

In fact, by uniqueness, the whole sequence u^ε converges strongly to the kinetic solution.

Finally, in light of estimate (5.43), by Hölder inequality we also deduce that

$$\|u^\varepsilon - u\|_{L^p(\Omega \times [0,T]; L^p(\omega_N dx))} \to 0, \qquad \text{as } \varepsilon \to 0,$$

for any $p \ge 1$.

Remark 5.17 *(1/2–Hölder continuous noise coefficient).* Referring to (5.6), consider the simple noise term $b(u)\, dW(t)$, where $W(t)$ is a one-dimensional Wiener process and $b(u)$ is a scalar function. Typical noise functions covered by the regularity condition (5.7) include $b(u) = |u|^\gamma$, $\gamma > \frac{1}{2}$, which is Hölder continuous with exponent $\gamma > \frac{1}{2}$.

Condition (5.7) is the same as the one imposed in the existing literature (see e.g. [18]). Unfortunately, it does not allow for the interesting example $b(u) = \sqrt{|u|}$, or any function b that satisfies $|b(u) - b(v)| \lesssim \mu(|u - v|)$, where

$$\int_0^1 \frac{1}{\left(\mu(\xi)\right)^2}\, d\xi = \infty. \tag{5.48}$$

Condition (5.48) embraces $\frac{1}{2}$–Hölder continuous noise functions b, like $b(u) = \sqrt{|u|}$.

Returning to the general case (5.6), assuming $b_k = b_k(\xi)\ \forall k$, we claim that Proposition 5.1 (and Theorem 5.1) actually holds with (5.8) replaced by

$$\sum_{k\geq 1} |b_k(u) - b_k(v)|^2 \lesssim \left(\mu(|u - v|)\right)^2, \tag{5.49}$$

for some continuous nondecreasing function μ on $\mathbb{R}_+$ satisfying $\mu(0+) = 0$ and (5.48). To allow for (5.49), we will make a more careful choice of the approximate delta function J_δ^ξ in order to handle the key error term (5.39). Inspired by the work [56] of Yamada and Watanabe on stochastic differential equations, we pick a strictly decreasing sequence $\{a_n\}_{n=0}^\infty$ of positive numbers, $a_n \downarrow 0$, recursively defined by $a_0 = 1$ and for $n = 1, 2, \dots$ by $\int_{a_n}^{a_{n-1}} \frac{1}{(\mu(\xi))^2}\, d\xi = n$. For example, with $\mu(\xi) = \sqrt{\xi}$ for $\xi > 0$, $a_n = a_{n-1}e^{-n}$; hence $a_n = e^{-\frac{1}{2}n(n+1)}$. Next, pick positive C_c^∞ functions ψ_n on $\mathbb{R}_+$ with $\operatorname{supp}\psi \subset (a_n, a_{n-1})$ and

$$0 \leq \psi_n(\xi) \leq \frac{2}{n\left(\mu(\xi)\right)^2} \leq \frac{2}{n\xi}, \quad \text{for any } \xi \in \mathbb{R}, \qquad \int_{a_n}^{a_{n-1}} \psi_n(\xi)\, d\xi = 1. \tag{5.50}$$

We introduce the function $\Psi_n(\xi) := \int_0^{|\xi|}\int_0^{\bar\kappa} \psi_n(\kappa)\, d\kappa\, d\bar\kappa$ for $\xi \in \mathbb{R}$, which is a symmetric approximation of $|\xi|$. Since ψ_n (and thus Ψ_n) is zero in a neighborhood of the origin, we have $\Psi_n \in C^\infty(\mathbb{R})$ and $\Psi_n''(\xi) = \psi_n(|\xi|) \leq \frac{2}{n|\xi|}$. Moreover, $\Psi_n(\cdot) \to |\cdot|$ uniformly on $\mathbb{R}$.

Let us now return to (5.38) and the error term (5.39), replacing $J_\delta^\xi(\cdot)$ by $\psi_n(|\cdot|)$ $(= \Psi_n''(\cdot))$ and, at the same time, renaming δ by n. Note that $\sum_{k\geq 1} \left|b_k(\zeta) - b_k(\bar\zeta)\right|^2$ is bounded by a constant times $\left(\mu(|\xi - \zeta|)\right)^2 + \left(\mu(\left|\xi - \bar\zeta\right|)\right)^2$, and thus, cf. (5.50),

$$\sum_{k\geq 1} \left|b_k(\zeta) - b_k(\bar\zeta)\right|^2 \psi_n\left(|\xi - \zeta|\right)\psi_n\left(\left|\xi - \bar\zeta\right|\right) \lesssim \frac{1}{n}\left(\psi_n\left(\left|\xi - \bar\zeta\right|\right) + \psi_n\left(|\xi - \zeta|\right)\right).$$

As a result,

$$\begin{aligned} &\iint \mathcal{Q}_{\varepsilon,\delta}(\omega, t, x, \xi)\, w_N(x)\, d\xi\, dx \\ &\quad \lesssim \frac{1}{n} \iiiiiint \Big(\psi_n\left(\left|\xi - \bar\zeta\right|\right) + \psi_n\left(|\xi - \zeta|\right)\Big) \left|(\partial_\xi \varrho)(\omega, t, y, \zeta)\right| \left|(\partial_\xi \varrho)(\omega, t, \bar y, \bar\zeta)\right| \end{aligned}$$

$$
\begin{aligned}
&\times J_\varepsilon^x(x-y) J_\varepsilon^x(x-\bar y)\, w_N(x)\, d\zeta\, dy\, d\bar\zeta\, d\bar y\; d\xi\; dx \\
&\lesssim \frac{1}{n}\int \left(\iint \left|(\partial_\xi \varrho)(\omega,t,y,\zeta)\right| J_\varepsilon^x(x-y)\, d\zeta\, dy \right) \\
&\qquad \times \left(\iint \left|(\partial_\xi \varrho)(\omega,t,\bar y,\bar\zeta)\right| J_\varepsilon^x(x-\bar y)\, d\bar\zeta\, d\bar y \right) w_N(x)\, dx \lesssim_N \frac{1}{n} \overset{n\uparrow\infty}{\longrightarrow} 0,
\end{aligned}
$$

where we have used $\partial_\xi \varrho = -\nu$ with $\nu(\mathbb{R}) = 1$. Therefore, sending $n \to \infty$, $\varepsilon \to 0$, and then $\ell \to \infty$ in (5.38), we obtain (5.31).

6. Comparison principle & stochastic Kružkov inequality

In a standard way, one can use Theorem 5.1 to deduce a comparison result. Indeed,

$$
\mathbb{E} \int_{\mathbb{R}^d} (u_1(t) - u_2(t))_+ \, w_N \, dx \lesssim \mathbb{E} \int_{\mathbb{R}^d} (u_{1,0} - u_{2,0})_+ \, w_N(x)\, dx, \tag{6.1}
$$

which follows from (5.40) and the identity $2(a-b)_+ = |a-b| + (a-b)$ for all $a, b \in \mathbb{R}$. As a result, $u_{0,1} \le u_{0,2}$ implies $u_1 \le u_2$.

One can also establish (6.1) directly, following the proof of Proposition 5.1 step-by-step, modulo one change. The proof of Proposition 5.1 makes use of the Itô chain rule to compute the equation for $\varrho - \varrho^2 = \varrho(1-\varrho)$. To establish (6.1), we use instead the Itô product formula to deduce that (formally) the functions $\rho_1 = \mathbb{I}_{\xi < u_1}$ and $\rho_2 = \mathbb{I}_{\xi < u_2}$ satisfy the inequality

$$
\begin{aligned}
&\partial_t \Big(\varrho_1(1-\varrho_2) \Big) + \operatorname{div}_{(x,\xi)} \Big(\overline{a}\, \varrho_1(1-\varrho_2) \Big) + R\partial_\xi \Big(\varrho_1(1-\varrho_2) \Big) \\
&\qquad + \sum_{k\ge 1} b_k \partial_\xi \Big(\varrho_1(1-\varrho_2) \Big) \dot W_k(t) \\
&\qquad\quad \le \partial_\xi \left(\frac{B^2}{2} \partial_\xi \Big(\varrho_1(1-\varrho_2) \Big) \right) + \partial_\xi \Big((1-\varrho_2) m_1 - \varrho_1 m_2 \Big),
\end{aligned} \tag{6.2}
$$

where u_1, u_2 are two kinetic solutions with corresponding kinetic measures m_1 and m_2. Of course, the rigorous proof goes through a regularization step that justifies the application of the Itô product formula.

More generally, we can derive a stochastic Kružkov inequality, that may be considered as a comparison inequality which is satisfied a.s.. Particular cases of this inequality have been proven to be extremely useful in Sections 3 and 4.

Proposition 6.1 *(stochastic Kružkov inequality). Let u_1 and u_2 be two kinetic solutions of (5.1) with initial data $u_{1,0}$ and $u_{2,0}$, respectively. Suppose* $\operatorname{div}_x A = 0$. *Then, almost surely,*

$$\int_0^\infty \int_{\mathbb{R}^d} \Big\{ |u_1 - u_2|\,\phi_t + \operatorname{sgn}(u_1 - u_2)\,(A(t,x,u_1) - A(t,x,u_2)) \cdot \nabla_x \phi$$
$$+ \operatorname{sgn}(u_1 - u_2)\,(R(t,x,u_1) - R(t,x,u_2))\,\phi \Big\}\,dx\,dt$$
$$+ \sum_{k\geq 1} \int_0^\infty \int_{\mathbb{R}^d} \operatorname{sgn}\,(u_1 - u_2)\,(b_k(t,x,u_1) - b_k(t,x,u_2))\,\phi\,dx\,dW_k(t)$$
$$+ \int_{\mathbb{R}^d} |u_{1,0} - u_{2,0}|\,\phi(0,x)\,dx \geq 0, \quad (6.3)$$

for any $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$ *with* $\varphi \geq 0$.

Note that, formally, this inequality results by integrating inequality (6.2). Below, we present a straightforward proof using the fact that the unique solutions are obtained through the vanishing viscosity method.

Proof. Following the proof of Theorem 5.1 we have that u_j, $j = 1, 2$, may be found as a limit in $L^p(\Omega \times [0,T] \times \mathbb{R}^d)$ when $\varepsilon \to 0$ of a sequence $\{u_j^\varepsilon\}_{\varepsilon>0}$ of weak solutions to the parabolic SPDEs

$$\partial_t u_j^\varepsilon + \operatorname{div}_x A(t,x,u_j^\varepsilon) - \varepsilon \Delta_x u_j^\varepsilon = B(t,u_j^\varepsilon)\dot{W}(t) + R(t,x,u_j^\varepsilon), \quad (t,x) \in (0,T) \times \mathbb{R}^d,$$
$$u_j^\varepsilon(0,x) = u_{0,j}(\omega,x), \quad x \in \mathbb{R}^d.$$

For fixed $\varepsilon > 0$, we have that $(u_1 - u_2)$ is a weak solution of the following equation

$$\partial_t (u_1 - u_2)^\varepsilon + \operatorname{div}_x (A(t,x,u_1^\varepsilon) - A(t,x,u_2^\varepsilon)) - \varepsilon \Delta_x (u_1^\varepsilon - u_2^\varepsilon)$$
$$= (B(t,u_1^\varepsilon) - B(t,u_2^\varepsilon))\,\dot{W}(t) + R(t,x,u_1^\varepsilon) - R(t,x,u_2^\varepsilon), \quad (t,x) \in (0,T) \times \mathbb{R}^d,$$
$$(u_1^\varepsilon - u_2^\varepsilon)(0,x) = (u_{0,1} - u_{0,2})(\omega,x), \quad x \in \mathbb{R}^d.$$

Let $S_\theta(\xi)$ be a C^2 convex approximation of $|\xi|$, such that $S'_\theta(\xi)$ is monotone nondecreasing, $S'_\theta(\xi) = 1$, for $\xi > \delta$, and $S'_\theta(\xi) = -1$, for $\xi \leq -\delta$. Then, for any nonnegative test function $\varphi(t,x)$, after sending $\theta \to 0$, by Itô formula we have a.s. that

$$\int_0^\infty \int_{\mathbb{R}^d} |u_1^\varepsilon - u_2^\varepsilon|\,\varphi_t\,dx\,dt + \int_0^\infty \int_{\mathbb{R}^d} \operatorname{sgn}(u_1^\varepsilon - u_2^\varepsilon)\,(A(t,x,u_1^\varepsilon) - A(t,x,u_2^\varepsilon)) \cdot \nabla_x \varphi\,dx\,dt$$
$$- \varepsilon \int_0^\infty \int_{\mathbb{R}^d} \operatorname{sgn}(u_1^\varepsilon - u_2^\varepsilon) \nabla (u_1^\varepsilon - u_2^\varepsilon) \cdot \nabla \varphi dx\,dt$$

$$+\int_0^\infty \int_{\mathbb{R}^d} \operatorname{sgn}(u_1^\varepsilon - u_2^\varepsilon)\,(R(t,x,u_1^\varepsilon) - R(t,x,u_2^\varepsilon))\,\varphi\,dx\,dt$$

$$+\sum_{k\geq 1}\int_0^\infty \int_{\mathbb{R}^d} \operatorname{sgn}(u_1^\varepsilon - u_2^\varepsilon)\,(b_k(t,x,u_1^\varepsilon) - b_k(t,x,u_2^\varepsilon))\,\varphi\,dx\,dW_k(t)$$

$$+\int_{\mathbb{R}^d} |u_{0,1} - u_{0,2}|\,\varphi(0,x)\,dx \geq 0, \quad (6.4)$$

where the convergence in the stochastic integral is enabled by (5.8).

Recall that both u_1, u_2 satisfy estimate (5.43), uniformly in ε. Thus, as convergence in mean square implies convergence in probability, which, in turn, implies a.s. convergence along a subsequence, we know that the third term on the left-hand side of (6.4) converges to zero a.s. along a subsequence $\varepsilon_n \to 0$. By the same token, passing to a further subsequence as the case may be, taking the limit as $\varepsilon_n \to 0$ in (6.4), we obtain (6.3). □

References

[1] G. Allaire, Homogenization and two-scale convergence, SIAM J. Math. Anal. 23 (6) (1992) 1482–1518.

[2] L. Ambrosio, H. Frid, Multiscale Young measures in almost periodic homogenization and applications, Arch. Ration. Mech. Anal. 192 (1) (2009) 37–85.

[3] L. Ambrosio, H. Frid, J. Silva, Multiscale Young measures in homogenization of continuous stationary processes in compact spaces and applications, J. Funct. Anal. 256 (6) (2009) 1962–1997.

[4] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[5] L. Arnold, Stochastic Differential Equations: Theory and Applications, Wiley-Interscience, 1974.

[6] C. Bauzet, Time-splitting approximation of the cauchy problem for a stochastic conservation law, Math. Comput. Simul. 118 (2015) 73–86.

[7] C. Bauzet, G. Vallet, P. Wittbold, The Cauchy problem for conservation laws with a multiplicative stochastic perturbation, J. Hyperbolic Differ. Equ. 9 (4) (2012) 661–709.

[8] C. Bauzet, G. Vallet, P. Wittbold, A degenerate parabolic-hyperbolic Cauchy problem with a stochastic force, J. Hyperbolic Differ. Equ. 12 (3) (2015) 501–533.

[9] A. Bensoussan, Homogenization of a class of stochastic partial differential equations, in: Composite Media and Homogenization Theory, Trieste, 1990, in: Progr. Nonlinear Differential Equations Appl., vol. 5, Birkhäuser Boston, Boston, MA, 1991, pp. 47–65.

[10] A. Bensoussan, J.-L. Lions, G. Papanicolaou, Asymptotic Analysis for Periodic Structures, Studies in Mathematics and its Applications, vol. 5, North-Holland Publishing Co., Amsterdam-New York, 1978.

[11] A.S. Besicovitch, Almost Periodic Functions, Dover Publications, Inc., New York, 1955.

[12] D.L. Burkholder, B.J. Davis, R.F. Gundy, Integral inequalities for convex functions of operators on martingales, in: Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Univ. California, Berkeley, Calif., 1970/1971, in: Vol. II: Probability Theory, 1972, pp. 223–240.

[13] J. Casado-Díaz, I. Gayte, The two-scale convergence method applied to generalized Besicovitch spaces, R. Soc. Lond. Proc., Ser. A, Math. Phys. Eng. Sci. 458 (2028) (2002) 2925–2946.

[14] G.-Q. Chen, Q. Ding, K.H. Karlsen, On nonlinear stochastic balance laws, Arch. Ration. Mech. Anal. 204 (3) (2012) 707–743.

[15] A.-L. Dalibard, Homogenization of non-linear scalar conservation laws, Arch. Ration. Mech. Anal. 192 (1) (2009) 117–164.

[16] A.-L. Dalibard, Kinetic formulation for heterogeneous scalar conservation laws, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 23 (4) (2006) 475–498.

[17] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, second edition, Encyclopedia of Mathematics and its Applications, vol. 152, Cambridge University Press, Cambridge, 2014.
[18] A. Debussche, J. Vovelle, Scalar conservation laws with stochastic forcing, J. Funct. Anal. 259 (4) (2010) 1014–1042.
[19] A. Debussche, M. Hofmanová, J. Vovelle, Degenerate parabolic stochastic partial differential equations: quasilinear case, Ann. Probab. 44 (3) (2016) 1916–1955.
[20] R.J. DiPerna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (3) (1989) 511–547.
[21] S. Dotti, J. Vovelle, Convergence of approximations to stochastic scalar conservation laws, Arch. Ration. Mech. Anal. 230 (2) (2018) 539–591.
[22] N. Dunford, J.T. Schwartz, Linear Operators. Part I, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1988. General theory, Reprint of the 1958 original.
[23] N. Dunford, J.T. Schwartz, Linear Operators. Part II, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1988. Spectral theory. Selfadjoint operators in Hilbert space, Reprint of the 1963 original.
[24] W. E, Homogenization of linear and nonlinear transport equations, Commun. Pure Appl. Math. 45 (3) (1992) 301–326.
[25] W. E, D. Serre, Correctors for the homogenization of conservation laws with oscillatory forcing terms, Asymptot. Anal. 5 (4) (1992) 311–316.
[26] W.F. Eberlein, Abstract ergodic theorems and weak almost periodic functions, Trans. Am. Math. Soc. 67 (1949) 217–240.
[27] W.F. Eberlein, The point spectrum of weakly almost periodic functions, Mich. Math. J. 3 (1955) 137–139.
[28] J. Feng, D. Nualart, Stochastic scalar conservation laws, J. Funct. Anal. 255 (2) (2008) 313–373.
[29] H. Frid, A note on the stochastic weakly* almost periodic homogenization of fully nonlinear elliptic equations, Port. Math. 72 (2–3) (2015) 207–227.
[30] H. Frid, J. Silva, Homogenization of nonlinear PDEs in the Fourier-Stieltjes algebras, SIAM J. Math. Anal. 41 (4) (2009) 1589–1620.
[31] H. Frid, J. Silva, H. Versieux, Homogenization of a generalized Stefan problem in the context of ergodic algebras, J. Funct. Anal. 268 (11) (2015) 3232–3277.
[32] L. Galimberti, K.H. Karlsen, Well-posedness theory for stochastically forced conservation laws on Riemannian manifolds, J. Hyperbolic Differ. Equ. 16 (3) (2019) 519–593.
[33] B. Gess, M. Hofmanová, Well-posedness and regularity for quasilinear degenerate parabolic-hyperbolic SPDE, Ann. Probab. 46 (5) (2018) 2495–2544.
[34] B. Gess, M. Maurelli, Well-posedness by noise for scalar conservation laws, Commun. Partial Differ. Equ. 43 (12) (2018) 1702–1736.
[35] B. Gess, S. Smith, Stochastic continuity equations with conservative noise, J. Math. Pures Appl. 9 (128) (2019) 225–263.
[36] I. Gyöngy, N. Krylov, Existence of strong solutions for Itô's stochastic equations via approximations, Probab. Theory Relat. Fields 105 (2) (1996) 143–158.
[37] M. Hofmanová, Degenerate parabolic stochastic partial differential equations, Stoch. Process. Appl. 123 (12) (2013) 4294–4336.
[38] N. Ichihara, Homogenization for stochastic partial differential equations derived from nonlinear filterings with feedback, J. Math. Soc. Jpn. 57 (2) (2005) 593–603.
[39] J. Jacod, A.N. Shiryaev, Limit Theorems for Stochastic Processes, second edition, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 288, Springer-Verlag, Berlin, 2003.
[40] V.V. Jikov, S.M. Kozlov, O.A. Oleĭnik, Homogenization of Differential Operators and Integral Functionals, Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian.
[41] K.H. Karlsen, E.B. Storrøsten, Analysis of a splitting method for stochastic balance laws, IMA J. Numer. Anal. 38 (1) (2018) 1–56.
[42] K.H. Karlsen, E.B. Storrøsten, On stochastic conservation laws and Malliavin calculus, J. Funct. Anal. 272 (2017) 421–497.
[43] J.U. Kim, On a stochastic scalar conservation law, Indiana Univ. Math. J. 52 (1) (2003) 227–256.
[44] K. Kobayasi, D. Noboriguchi, A stochastic conservation law with nonhomogeneous Dirichlet boundary conditions, Acta Math. Vietnam. 41 (4) (2016) 607–632.
[45] P.-L. Lions, B. Perthame, E. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related equations, J. Am. Math. Soc. 7 (1) (1994) 169–191.

[46] G. Lv, J.-L. Wu, Renormalized entropy solutions of stochastic scalar conservation laws with boundary condition, J. Funct. Anal. 271 (8) (2016) 2308–2338.
[47] M. Mohammed, Homogenization of nonlinear hyperbolic stochastic equation via Tartar's method, J. Hyperbolic Differ. Equ. 14 (2) (2017) 323–340.
[48] M. Mohammed, M. Sango, Homogenization of nonlinear hyperbolic stochastic partial differential equations with nonlinear damping and forcing, Netw. Heterog. Media 14 (2) (2019) 341–369.
[49] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal. 20 (3) (1989) 608–623.
[50] B. Perthame, Kinetic Formulation of Conservation Laws, Oxford Lecture Series in Mathematics and its Applications, vol. 21, Oxford University Press, Oxford, 2002.
[51] C. Prévôt, M. Röckner, A Concise Course on Stochastic Partial Differential Equations, Lecture Notes in Mathematics, vol. 1905, Springer, Berlin, 2007.
[52] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, third edition, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 293, Springer-Verlag, Berlin, 1999.
[53] E. Sánchez-Palencia, Nonhomogeneous Media and Vibration Theory, Lecture Notes in Physics, vol. 127, Springer-Verlag, Berlin-New York, 1980.
[54] M. Sango, Homogenization of stochastic semilinear parabolic equations with non-Lipschitz forcings in domains with fine grained boundaries, Commun. Math. Sci. 12 (2) (2014) 345–382.
[55] J. Silva, On the almost periodic homogenization of non-linear scalar conservation laws, Calc. Var. Partial Differ. Equ. 54 (4) (2015) 3623–3641.
[56] T. Yamada, S. Watanabe, On the uniqueness of solutions of stochastic differential equations, J. Math. Kyoto Univ. 11 (1971) 155–167.
[57] V.V. Zhikov, E.V. Krivenko, Averaging of singularly perturbed elliptic operators, Mat. Zametki 33 (4) (1983) 571–582.