Divergent potentialism: A modal analysis with an application to choice sequences

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Abstract

Modal logic has been used to analyze potential infinity and potentialism more generally. However, the standard analysis breaks down in cases of divergent possibilities, where there are two or more possibilities that can be individually realized but which are jointly incompatible. This paper has three aims. First, using the intuitionistic theory of free choice sequences, we motivate the need for a modal analysis of divergent potentialism and explain the challenges this involves. Then, using Beth-Kripke semantics for intuitionistic logic, we overcome those challenges. Finally, we apply our modal analysis of divergent potentialism to make choice sequences comprehensible in classical terms.

1 Potentialism

In mathematics, we are often interested in *potential existence*, namely in what can be constructed or generated. Here are some examples that derive from Aristotle and that remained highly influential right up until the Cantorian revolution of the late nineteenth century:

(1) Necessarily, for any number m, possibly there is a successor
$\Box\forall m\Diamond\exists n\,\textsc{Succ}(m,n)$

(2) Necessarily, for any line segment l, possibly l has bisects l_1 and l_2

Such potential existence claims are particularly important in connection with *potential infinity*. Each claim can be combined with the rejection of the corresponding *actual infinity*, for example:

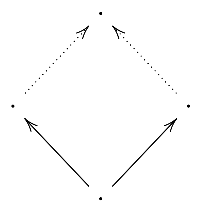
(3) For any number m, there is a successor
$\forall m\exists n\,\textsc{Succ}(m,n)$

Potentialism is the view that potential existence, and modality more generally, have a role to play in mathematics, either explicitly or implicitly.

The view comes in many different forms. One of the more radical versions is *Aristotelian potentialism*, which holds that there are potential but no actual infinities. There are also more relaxed forms of potentialism, such as *set-theoretic potentialism*, which holds that any objects possibly form a set, but (to avoid paradox) denies that every plurality of objects in fact forms a set. Writing $\textsc{Set}(xx, y)$ to mean that y is the set of xx's, this would be:[1]

$$\Box \forall xx \, \Diamond \exists y \, \textsc{Set}(xx, y)$$
$$\neg \Diamond \forall xx \, \exists y \, \textsc{Set}(xx, y)$$

There are now good modal analyses of potential infinity and potentialism more generally: (Linnebo, 2013), (Studd, 2013), (Linnebo and Shapiro, 2019). However, all these analyses assume that the possibilities are *convergent* in the sense that, when confronted with two different generative possibilities, it doesn't matter which possibility we choose to realize first, as the other possibility can always be realized later. For example, given two line segments, we can bisect one first and subsequently bisect the other one; the result will be precisely the same as it would have been had we chosen to proceed in the reverse order. Representing possible worlds by dots and accessibility by arrows, the assumption of convergence can be represented as follows:



As the example of line bisection illustrates, this assumption is often plausible in mathematics. Without convergence, however, the mentioned analyses of potentialism are known to break down (Brauer, 2020b).

This failure prompts the question of how safe the assumption of convergence is. Outside of mathematics, the assumption often fails, because by choosing to realize one of two possible circumstances, you may destroy the possibility of the other. For example, you can bake bread or bake a cake; but, given your ingredients, if you bake one, you will no longer be able to bake the other. Or, suppose that it is currently possible for you to marry

[1]Another way of formulating set-theoretic potentialism is a thesis that there is no set that contains all possible objects (Hellman, 1989). If we assume some simple facts about sets, this is entailed by our two principles of set-theoretic potentialism. Our second principle is that it is not possible that every plurality in fact form a set. Given the first principle, it follows that it is possible for there to be a set which does not actually exist. Since the actual sets can only contain members that actually exist, this merely possible set is not a member of any actual set. And hence there does not exist a set of all possible objects.

either of two people. Given human affections, if you choose to marry one, the other is likely to hate you for doing so and shut down the possibility of the other marriage. More surprisingly, we will argue that even in mathematics, there are cases where the assumption of convergence fails. This raises a hard question, which sets the agenda for our paper. *How should the resulting form of divergent potentialism be analyzed?*

The paper is structured as follows. In Section 2 we explain how the currently available potentialist analysis of a mathematical theory proceeds. This analysis requires two main tools: a translation of mathematical discourse into potentialist terms, and a mirroring theorem showing that the translation preserves entailments. As we explain, however, these tools only work properly with convergent potentialism. Thus, if there are any interesting examples of divergent potentialism, they would require a new translation and a new mirroring theorem. In Section 3, we appeal to the intuitionistic theory of *free choice sequences* to show that there are indeed such examples of divergent potentialism. Then in Section 4 we develop a new translation that works properly with this example of divergent potentialism and prove the corresponding mirroring theorem.

Finally, in Section 5 we apply the resulting modal analysis of divergent potentialism to a simple theory of choice sequences, while in Section 6 we show how these choice sequences can be used to develop an intuitionistic theory of real analysis, with the characteristic property that all functions are continuous. This supports our modal analysis of divergent potentialism, and also has the further benefit of making choice sequences comprehensible in classical terms. We conclude in Section 7.

2 The desire for a translation

Ordinary mathematics—whether classical or intuitionistic—tends to use a non-modal language, say $\mathcal{L}$. By contrast, potentialist analyses of the relevant mathematics use a corresponding modal language, say $\mathcal{L}^\Diamond$, that results from $\mathcal{L}$ by adding $\Box$ and $\Diamond$ (possibly defining one in terms of the other, although this depends on whether the modal logic is classical or intuitionistic), and perhaps other modal operators.

For the potentialist analysis in $\mathcal{L}^\Diamond$ to be relevant to the ordinary mathematics done in $\mathcal{L}$, there should be some connection between the languages. The most direct and easily understood such connection is simply a recursive translation $*$ from $\mathcal{L}$ to $\mathcal{L}^\Diamond$.

For the translation to provide the desired connection, though, it must satisfy the following two desiderata.

1. The translation $* : \mathcal{L} \to \mathcal{L}^\Diamond$ should interpret (perhaps even: faithfully interpret) the logic of $\mathcal{L}$:

$$\varphi_1, \ldots, \varphi_n \vdash \psi \quad \text{only if (or even: iff)} \quad \varphi_1^*, \ldots, \varphi_n^* \vdash^* \psi^*.$$

where $\vdash$ is deducibility in $\mathcal{L}$ and $\vdash^*$ is deducibility in $\mathcal{L}^\Diamond$, perhaps supplemented with acceptable non-logical assumptions.

2. The axioms of the mathematical theory T in $\mathcal{L}$ should translate as theorems of the potentialist theory $T^\Diamond$ in $\mathcal{L}^\Diamond$.

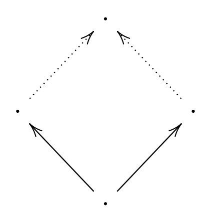
If both desiderata are satisfied, we obtain *an interpretation*, in the usual sense, of T in $T^\Diamond$.

2.1 The potentialist translation

Let us now describe the translation utilized by the currently available analysis of convergent potentialism mentioned above. The central idea is that the existential quantifier, as used in the non-modal language $\mathcal{L}$ of ordinary mathematics, is concerned with potential existence and accordingly should be translated as '$\Diamond\exists$'. The universal quantifier is taken to be dual to the existential one and is therefore translated as '$\Box\forall$'. Thus, in the modal language $\mathcal{L}^\Diamond$ we use the modal-operator-quantifier hybrids '$\Box\forall$' and '$\Diamond\exists$' to generalize over all entities across all stages of the process of generation (Linnebo, 2010). Finally, the connectives are translated homophonically, i.e. as themselves. The resulting mapping from $\mathcal{L}$ to $\mathcal{L}^\Diamond$ is known as *the potentialist translation*.

As mentioned, earlier studies have revealed that the potentialist translation $\varphi \mapsto \varphi^\Diamond$ works really well—*provided that the modal logic is sufficiently strong*. Let us explain.

It is generally agreed that the modal logic for any form of potentialism should be at least S4.[2] The reason is that accessibility among the stages of the generation is surely reflexive and transitive. As noted, however, the crucial question is whether the accessibility relation can additionally be assumed to have the following *convergence property*:



If this is assumed, it will licence the adoption of one more axiom in addition to the usual ones of S4, namely:

(G) $$\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$$

The modal logic that is obtained by adding axiom G to S4 is known as S4.2.

[2]The Appendix contains a standard axiomatization for S4.

One final assumption too will typically be plausible, namely that atomic predicates have the following positive and negative stability properties:

(Stb$^+$-P) $\qquad \Box\forall\vec{u}(P(\vec{u}) \to \Box P(\vec{u}))$

(Stb$^-$-P) $\qquad \Box\forall\vec{u}(\neg P(\vec{u}) \to \Box\neg P(\vec{u}))$

That is, an atomic predicate is never allowed to "change its mind" about whether or not it applies to a string of objects: whatever verdict it makes at one stage it will stick by at all later stages. For example, if one number immediately precedes another (or not), then this will remain so at all later stages; and likewise with membership in a set.

We are now ready to state the theorem that underpins the nice connection that is available between the non-modal and the modal languages in cases where the convergence assumption is acceptable.[3]

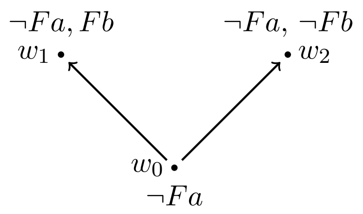
Theorem 1 (First-order mirroring)**.** *Let $\vdash^\Diamond$ be provability by $\vdash$, S4.2, and axioms stating that every atomic predicate is stable, but with no higher-order comprehension. Then we have:*

$$\varphi_1, \ldots, \varphi_n \vdash \psi \quad \textit{iff} \quad \varphi_1^\Diamond, \ldots, \varphi_n^\Diamond \vdash^\Diamond \psi^\Diamond.$$

To explain the challenge posed by divergent potentialism, let us now observe that the potentialist translation breaks down when the accessibility relation cannot be assumed to be convergent and we thus do not have axiom G. For example, consider the following three formulas:

- $\exists x(x \neq a \wedge Fx)$
- $\exists x(x \neq a \wedge \neg Fx)$
- $\exists x\exists y\forall z(z = x \vee z = y)$

Clearly, the formulas are classically inconsistent. However, a divergent model can jointly satisfy their potentialist translations as well as the stability axioms. Consider, for example, a model based on three worlds $w_0 \leq w_1, w_2$, with a single object a at w_0 and a single additional object b at each of the other two worlds, where b is F at w_1 but not at w_2 and where a is not F at any world. Visually:



[3] See (Linnebo, 2010) for a proof. An analogous result holds where (a) the propositional logic of each side is intuitionistic rather than classical, and (b) we add on both sides the decidability of each atomic predicate. See (Linnebo and Shapiro, 2019), Theorem 2. Note, however, that in the case of intuitionistic real analysis, identity is not decidable.

Thus, the existing analyses of potentialism work only in cases of convergent possibilities. If there are well-motivated examples of divergent potentialism, new tools will be needed to incorporate these into the modal-potentialist framework. In the next section, we will offer one such example of divergent potentialism.

3 Free choice sequences as an example of divergence

One topic well-suited to a modal-potentialist treatment is that of a *free choice sequence*, a notion that comes from intuitionistic analysis. Consider an infinite sequence of natural numbers, understood as being merely *potentially* infinite.[4] Furthermore, imagine a mathematician constructing this infinite sequence by successively choosing each value one after another. Of course, this mathematician must be idealized so that she has sufficient time, attention, memory and so on, but these are familiar sorts of idealizations.[5] The resulting object created by this idealized mathematician is a free choice sequence.

Since free choice sequences are explicitly thought of as potentially infinite, it is reasonable to expect that a potentialist theory should be able to account for them. If not, then the potentialist would seem to owe us an explanation of why choice sequences are somehow not proper objects of mathematical study. Without such an explanation, the potentialist would be open to a charge of being too restrictive.

There is a further, independent reason for the potentialist to give an account of free choice sequences. Although free choice sequences have been extensively studied in intuitionistic mathematics, they have not been well-received outside of intuitionistic circles.[6] For instance, Feferman writes:

> Brouwer introduced . . . a novel conception, that of *free choice sequences* . . . With this step Brouwer struck off into increasingly alien territory, and he found few to follow him even among those sympathetic to the constructive position. (Feferman, 1998, 28)

Similarly, as an example of those sympathetic to Brouwer's constructivist position, Bishop writes, with characteristic aplomb:

[4] Free choice sequences are distinctive among the potentialist panoply in being single, particular objects that are merely potential. This is in contrast with older views that might identify a *process*—such as bisecting a line segment—or a *collection*—such as the natural numbers—as the locus of potential existence.

[5] Compare the idealization involved in the definition of a Turing machine. It never malfunctions, it never runs out of memory, when performing a computation it has sufficient time, etc.

[6] See (Troelstra, 1977) and (Troelstra and van Dalen, 1988, Ch. 12) for overviews of work on choice sequences.

> [Brouwer] seems to have [had] a nagging suspicion that unless he personally intervened to prevent it the continuum would turn out to be discrete. He therefore introduced the method of free-choice sequences ... This makes mathematics so bizarre it becomes unpalatable to mathematicians, and foredooms the whole of Brouwer's program. (Bishop, 1967, 6)

Our suggestion is that free choice sequences can be given a modal analysis using the tools of potentialism outlined above. Because the process of choosing the values of a sequence of natural numbers is naturally conceived of as a temporal process, a modal explication of free choice sequences almost suggests itself. If such an account can be accomplished, it would show that free choice sequences need not be located in such 'alien territory' and would, we hope, make them somewhat more palatable to mathematicians.[7] Thus, we see two reasons for providing a modal-potentialist theory of free choice sequences. First, since choice sequences are by their nature potentially infinite, a theory of potential infinity should accommodate them. Second, there is the possible benefit of a potentialist theory of free choice sequence making them comprehensible in classical terms.

Let us consider, then, how a modal-potentialist account of free choice sequences might proceed. The logic S4 is a natural choice for reasoning about temporal processes, so this will be the logic we work in.[8] While the temporal interpretation of modals is a helpful, evocative heuristic, it is intended as a mere heuristic. We are not committed to a particular interpretation of the modality, and all the main conclusions of this paper rely only on the formal properties of the modal logic. As notation, we will let small roman letters x, y, z range over natural numbers and small Greek letters α, β, γ range over free choice sequences. Since choice sequences are growing with time, a term $\alpha(n)$ may not be defined at a given moment; to accommodate terms that may not denote, we will assume a negative free logic.[9] This idea of a sequence growing in time can be captured with four axioms. First, we want to say that choice sequences have no gaps; if α is defined up to n, then it's defined on every argument less than n:

[7](Kripke, 2019) makes a similar point about how to make choice sequences classically comprehensible. Kripke's idea is that choice sequences can be understood as temporal objects in classical terms. This is similar to our approach here, although Kripke's paper is largely programmatic and does not develop a theory of choice sequences in any detail. (Moschovakis, 2017) implements a theory inspired by Kripke's ideas, but in contrast to the approach pursued here, her theory is not explicitly potentialist and takes place against the background of intuitionistic logic.

[8]We will eventually adopt a more expressive extension of S4, but this will be a conservative extension and will not require changing the frame conditions for an S4 model.

[9]On free logic see, e.g., (Hughes and Cresswell, 1996, 292ff). In the setting of a free logic, there are some issues with the philosophical application of the Mirroring Theorem 1 for S4.2 cited above. By adding explicit existence assumptions, however, these issues can be sidestepped, cf. (Brauer, 2020a).

1. $\Box\forall x(\exists y\alpha(x) = y \to \forall x_i < x\exists y_i\alpha(x_i) = y_i)$

Next, we want to say that every argument can be assigned a value. There is no upper limit to how far the sequence will extend:

2. $\Box\forall x\Diamond\exists y\alpha(x) = y$

Next, we stipulate that once values are chosen, they are never changed:

3. $\forall x\forall y(\alpha(x) = y \to \Box\alpha(x) = y)$

Finally, because choice sequences are merely potentially infinite, they can never be completed:

4. $\Box\exists x\neg\exists y\alpha(x) = y$

These axioms capture the general idea of a potentially infinite sequence of natural numbers. In intuitionistic analysis, however, there are distinctions between several more specific types of choice sequences. These different types of choice sequences have to do with whether the process of choosing values for the sequence is subject to constraints. At one extreme, the values of the sequence may be entirely predetermined. For instance, the idealized mathematician may choose the values of a sequence by following a recursive rule, so that the values of the sequence are entirely determined by that rule. The sequence α defined so that $\alpha(n)$ is the n^{th} prime number would be an example. Such sequences are called *lawlike*. At the other extreme *lawless sequences* are subject to no constraints: the idealized mathematician may choose any value for any argument. In between these extremes are sequences that are subject to some constraints without being entirely predetermined. For instance, a sequence α may be constructed which only ever takes value 0 or 1, but at each stage the idealized mathematician is entirely free to choose either of these two options. Or, as another example, a sequence α may be defined as the pointwise sum of two lawless sequences β and γ. Because β and γ are lawless, the values of α are not uniquely predetermined, so α is not lawlike. On the other hand, the values of α are constrained by the values of β and γ, so α is not lawless, either.

For lawless sequences, we may postulate the following axiom:

5. $\Box\forall x(\neg\exists y\alpha(x) = y \to \forall y\Diamond\alpha(x) = y)$

Crucially, this axiom entails that the possible extensions of the sequence are *divergent* rather than *convergent*. If α is a lawless sequence that has been defined up to n, then on $n + 1$ α could take the value 0 or 1, or any other number. But these values are incompatible and once chosen they are fixed, so there is no mutual extension of these different possibilities. If $\alpha(n + 1)$ gets set as 0, then it cannot be 1 and if $\alpha(n+1)$ gets set as 1 it cannot be 0.

Thus, the tools provided by the modal explication of potential infinity are well-suited to give a modal account of free choice sequences. But the detailed analyses that have been worked out thus far all depend on the possibilities in question being convergent. With free choice sequences we have instead divergent possibilities, so we cannot simply take a modal analysis off the shelf and apply it here. In the next section we will develop a new version of the potentialist analysis by providing a potentialist translation and mirroring theorem that are well-suited to deal with free choice sequences. On the one hand, this shows that the framework of divergent potentialism is both coherent and adequate for understanding free choice sequences. On the other hand, the potentialist translation we will provide only applies, at least directly, to intuitionistic theories.[10] Other forms of divergent potentialism have been proposed in classical arithmetic and set theory, however, and the tools we develop below do not yield a clear treatment of those forms of potentialism.[11] Thus, this paper constitutes a first step in understanding divergent potentialism.

4 The Beth-Kripke translation

In this section we provide a translation from intuitionistic logic into our modal logic. Since the modal logic in question is S4, it is natural to try to use the Gödel translation. As we will see, however, this translation is not well-suited to our philosophical goals.

4.1 The Gödel translation into S4

As we have seen, we need a translation that requires only the modal logic S4, not full S4.2. One such translation is famous and available from the literature, namely Gödel's translation g of intuitionistic first-order logic into

[10]Of course, our new translation can be composed with the double negation translation which provides an interpretation of classical logic in intuitionistic logic. This composite translation would be formally adequate in the sense that it provides a faithful interpretation of classical logic in our modal logic. However, this composite translation is rather contrived and has no philosophical motivation in the ideas of potentialism. For instance, the result of applying the composite translation to an atom P would be $\Box\Diamond\mathcal{I}\Box P$ (see §4.4 below). We can see no reason for the potentialist to regard this as capturing their commitments with respect to P.

[11]See (Hamkins, 2018) for divergent potentialism in arithmetic and (Hamkins and Woodin, 2018) for divergent potentialism in set theory. The Hamkinsian approach to divergent potentialism is motivated by studying relationships between non-standard models of a theory, and thus is of a somewhat different flavor from that pursued here. Accordingly, we think it is reasonable to leave consideration of these examples for another occasion.

first-order S4. The clauses of the translation g are:

$$\begin{aligned} P &\mapsto \Box P \qquad \text{for } P \text{ atomic} \\ \neg\varphi &\mapsto \Box\neg\varphi^g \\ \varphi \vee \psi &\mapsto \varphi^g \vee \psi^g \\ \varphi \wedge \psi &\mapsto \varphi^g \wedge \psi^g \\ \varphi \to \psi &\mapsto \Box(\varphi^g \to \psi^g) \\ \forall x\, \varphi &\mapsto \Box\forall x\, \varphi^g \\ \exists x\, \varphi &\mapsto \exists x\, \varphi^g \end{aligned}$$

Theorem 2 (Intuitionistic mirroring)**.** *Let $\vdash_{int}$ be intuitionistic first-order deducibility. Let $\vdash_{S4}$ be deducibility in classical first-order logic plus S4. Then we have:*

$$\varphi_1, \ldots, \varphi_n \vdash_{int} \psi \quad \textit{iff} \quad \varphi_1^g, \ldots, \varphi_n^g \vdash_{S4} \psi^g.$$

This result, for first-order intuitionistic logic, is due to (Rasiowa and Sikorski, 1953).

With Kripke semantics available, however, this theorem follows easily from the following lemma, which records how the Gödel translation can be seen as a syntactic analogue of Kripke semantics for intuitionistic logic.[12]

Lemma 1. *If M is a Kripke model suitable for intuitionistic logic, then for any world $w \in M$ and any formula φ in the language of predicate logic, $M, w \Vdash \varphi$ as an intuitionistic model iff $M, w \models \varphi^g$ as an S4 model.*

Proof sketch. Induction on the complexity of φ. Negation provides a good example of how this goes:

- $w \Vdash \neg\varphi$ iff $\forall w' \geq w$ we have $w \nVdash \varphi$.
- $w \models (\neg\varphi)^g$ iff $w \models \Box\neg(\varphi^g)$ iff $\forall w' \geq w$ we have $w \not\models \varphi^g$.

□

Returning to our main discussion, our question is whether the Gödel translation might underpin an analysis of divergent potentialism. Regrettably, the answer is a resounding *no.* We contend that this translation is *hopeless* in an explication of potentialism. To see this, consider the following truth from the intuitionistic theory of choice sequences:

$$\forall x \exists y\, \alpha(x) = y$$

Its Gödel translation is:

$$\Box\forall x \exists y\, \alpha(x) = y$$

[12] We will use a similar idea in proposing our new translation below.

This is unacceptable to the potentialist, as it contradicts (4) above.

The problem with the Gödel translation stems from its homophonic treatment of the existential quantifier, which is antithetical to the key potentialist idea that in mathematics existence is always implicitly potential. It is natural, therefore, to try to solve the problem by tweaking the translation of this quantifier. Two natural options would be to translate $\exists$ as $\Diamond\exists$ or as $\Box\Diamond\exists$. But again, the verdict is negative: these translations fail to validate even some of the laws of intuitionistic logic.

Consider the first translation, denoted g_1, which is like the g except that it translates the existential quantifier '$\exists$' as '$\Diamond\exists$'. Consider the model based on two worlds $w_0 \leq w_1$, with a single object a at each world. Suppose that at w_1 we have Fa and p, while at w_0 neither of these formulas is forced. Then we have $w_0 \models (\forall x(Fx \to p))^{g_1}$ but not $w_0 \models (\exists x Fx \to p)^{g_1}$, although the latter formula is an intuionistic consequence of the former. The same model construction works for the second translation as well.

Thus, we are still in need of a better translation to underpin our analysis of divergent potentialism. This translation must be logically well-behaved even when the modal logic is no stronger than S4 and it must in some way capture the potential character of mathematical existence. To get there, we will first need to expand the expressive resources of our logic.

4.2 Adding to S4

So far we have been working with the familiar modal operators $\Box$ and $\Diamond$. As it turns out, however, these operators do not have sufficient power to express all the claims that the potentialist will want to make in the context of a divergent potentialism. For instance, one claim that we might want to include in a potentialist account of free choice sequences is that a choice sequence is *eventually* assigned a value for each argument. There is never a time when the choice sequence has a value on every argument, of course, but for each argument there should come a time when it gets assigned a value. The idealized mathematician should always keep extending a choice sequence.

This, however, cannot be expressed using just $\Box$ and $\Diamond$. Obviously, $\forall x \exists y \alpha(x) = y$ is much too strong, saying not that α will *eventually* be defined on any argument but that α is *already* defined on every argument. On the other hand, $\Box\forall x \Diamond\exists y \alpha(x) = y$ is too weak: it says that α *could* be defined on any argument, but not that it *will* be defined on any argument. Similarly, $\Box\forall x \Box\Diamond\exists y \alpha(x) = y$ is also too weak. This says that it is never ruled out that α gets defined on any argument. But saying something is never ruled is not a guarantee that it will eventually happen. As an illustration, consider the following model, with worlds w_i and v_i ($i \in \mathbb{N}$); at each w_i, α is not defined on any argument, but at each v_i α is defined on exactly the arguments $\leq i$:

$$\begin{tikzcd}
\alpha(0)\downarrow & \alpha(1)\downarrow & \alpha(2)\downarrow & \alpha(3)\downarrow & \\
v_0 \arrow[r] & v_1 \arrow[r] & v_2 \arrow[r] & v_3 \arrow[r] & \cdots \\
w_0 \arrow[r] \arrow[u] & w_1 \arrow[r] \arrow[u] & w_2 \arrow[r] \arrow[u] & w_3 \arrow[r] \arrow[u] & \cdots \\
\alpha(0)\uparrow & \alpha(0)\uparrow & \alpha(0)\uparrow & \alpha(0)\uparrow &
\end{tikzcd}$$

As long as α follows the path through the w worlds, it will never be defined on any argument. Thus, it is not inevitable that α gets defined on any given argument. But since it is always possible for α to veer into the v worlds, it is *necessarily possible* for α to be defined on any given argument. This illustrates how $\Box\forall x\Box\Diamond\exists y\alpha(x) = y$ differs from the claim that α will *inevitably* be defined on any given argument.

Let us consider another example of a claim that involves something eventually happening. In the intuitionistic theory of lawless sequences, one of the axioms is that for every finite sequence of numbers $\langle n_1, \ldots, n_k\rangle$ there is a lawless sequence that has $\langle n_1, \ldots, n_k\rangle$ as an initial segment. When n is the code of the sequence $\langle n_1, \ldots, n_k\rangle$, the claim that α has n as a finite initial segment can be abbreviated as $\alpha \in n$. This claim is a form of density principle or maximality principle: the universe of lawless sequences is so rich as to include every possible way of beginning a choice sequence.

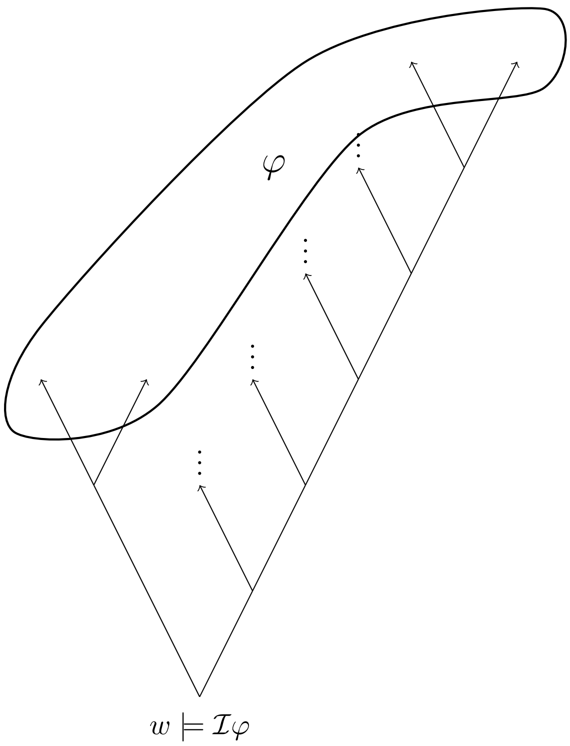
When we cast the theory of lawless sequences in potentialist terms, this can be neatly paraphrased by saying that for every n, somewhere in the potential universe of choice sequences there is a lawless sequence $\alpha \in n$. But this need not require that we already or actually have such an $\alpha \in n$. All that we really want to say is that for every n there will *eventually* be some $\alpha \in n$. Or in other words, for every n, there will *inevitably* be some $\alpha \in n$. And just like the claim that a choice sequence will eventually be defined on each argument, this claim about the inevitably eventual behavior of lawless sequences cannot be expressed in terms of $\Box$ and $\Diamond$.

To overcome this expressive deficiency, we will add a new modal operator to our language, $\mathcal{I}\varphi$, pronounced 'inevitably φ'.[13] The intended interpretation of $\mathcal{I}\varphi$ is that in every possible future branch of time, there comes a moment when φ is true. Formally, for a model M and world w in M, define an R-chain above w to be a set of worlds accessible from w that is linearly ordered by the accessibility relation R. An R-chain is maximal if there are no proper supersets that are R-chains. When the relation R is clear from context, we will sometimes just speak of a chain. Then the truth conditions for $\mathcal{I}$ are as follows:

[13]This operator was first introduced in tense logic by (McCall, 1979), where it was called the 'strong future tense'; its significance for potentialism was noted in (Brauer, 2020b).

- $M, w \models \mathcal{I}\varphi$ iff for every maximal R-chain C above w, there is a $u \in C$ such that $M, u \models \varphi$.

This condition is visualized in the image below: every path above w through the model M eventually intersects the set of φ-worlds.



The inevitability operator enables us to express interesting properties of choice sequences. We have already seen two examples of this. The first example was that every argument eventually gets a value. This allows us to refine our preliminary axioms from Section 3: we replace the principle 2 with:

2′. $\Box\forall x\mathcal{I}\exists y\alpha(x) = y$

The second example was that every possible initial segment is eventually realized by some choice sequence, an idea we will return to in Section 5.

A third example of the usefulness of the inevitability operator's expressive power is that we can use it to say that the future graph of a choice sequence is predetermined:

$$\forall x\forall y(\Diamond\alpha(x) = y \to \mathcal{I}\alpha(x) = y)$$

Let S4+ denote the result of adding the operator $\mathcal{I}$ to S4. This will be a conservative extension of S4. S4+ is not axiomatizable, however.[14]

[14]See (Brauer, 2020b) for a proof as well as references to the earlier literature. This result guarantees that that $\mathcal{I}$ cannot in fact be defined from $\Box$ or $\Diamond$, since S4 is axiomatizable.

Remarkably, we can nevertheless axiomatize a sufficient fragment of the logic for our purposes in this paper. When we introduce our new translation in Section 4.4, we will see that the following axiomatization is sound and complete for the fragment of S4+ that is in the range of our translation.

M0 The axioms and rules of free S4 with Converse Barcan Formula. (See the Appendix.)

M1 $\Box(\varphi \to \psi) \to (\mathcal{I}\varphi \to \mathcal{I}\psi)$

M2 $\varphi \to \mathcal{I}\varphi$

M3 $\Box\varphi \to \neg\mathcal{I}\neg\varphi$

M4 $\mathcal{I}\mathcal{I}\varphi \to \mathcal{I}\varphi$

M5 $\mathcal{I}\Box\varphi \to \Box\mathcal{I}\varphi$

M6 $\mathcal{I}\forall x\varphi \to \forall x\mathcal{I}\varphi$

M7 $\exists x\mathcal{I}\varphi \to \mathcal{I}\exists x\varphi$

It is easy to verify that these axioms are sound. As we noted in Section 2, the use of a free logic is important because we will focus on models where the domain can grow from one world to the next, and thus it might happen that there is a term which does not denote in one world, but which denotes in another world accessible from the first.

4.3 Beth-Kripke semantics for intuitionistic logic

The next step towards defining our translation will be to introduce a somewhat atypical semantics for intuitionistic logic. It is a version of the familiar Beth semantics with a few minor tweaks.

Definition 1. *Given a partial order $\langle T, \leq\rangle$ and a family $\{D_w : w \in T\}$ of domains that are non-shrinking along $\leq$, let V assign a subset of D_w to each predicate P at w and assign the graph of a (not necessarily total) n-ary function on D_w to each n-ary function symbol f at w. If $\vec{x} \in V(P, w)$ and $w \leq u$, then $\vec{x} \in V(P, u)$; and likewise for graphs of functions. We will assume we have a constant available for every $a \in \bigcup_w D_w$ (adding constants to the underlying language if necessary). The forcing relation is defined inductively as follows:*

- *$w \Vdash Pa$ iff every maximal $\leq$-chain above w includes a world u s.t. $a \in V(P, u)$*

- *$w \Vdash t_1 = t_2$, for any closed terms t_1 and t_2, iff every maximal $\leq$-chain above w includes a world u s.t. $V(t_1, u) = V(t_2, u)$*

- $w \Vdash \neg\varphi$ *iff for all* $u \geq w$, $u \nVdash \varphi$
- $w \Vdash \varphi \wedge \psi$ *iff* $w \Vdash \varphi$ *and* $w \Vdash \psi$
- $w \Vdash \varphi \vee \psi$ *iff every maximal* $\leq$*-chain above* w *includes some world* u *s.t.* $u \Vdash \varphi$ *or* $u \Vdash \psi$
- $w \Vdash \varphi \to \psi$ *iff for every* $u \geq w$, *if* $u \Vdash \varphi$ *then* $u \Vdash \psi$
- $w \Vdash \exists x \varphi(x)$ *iff every maximal* $\leq$*-chain above* w *includes some world* u *such that for some* $a \in D_u$, $u \Vdash \varphi(a)$
- $w \Vdash \forall x \varphi(x)$ *iff for every* $u \geq w$ *and for every* $a \in D_u$, $u \Vdash \varphi(a)$

Remark 1. *If* $M = \langle T, \leq, D, V \rangle$ *is a model for a language that includes function symbols or individual constants (which we may regard as 0-place function symbols), it is allowed that not every term has a denotation at each world. What we do require, however, is that for every closed term and every maximal chain in* T*, there is a world* w *in that chain where the term has a denotation. This is required so that every world forces* $t = t$ *for each term* t*.*

There are three differences here from the standard Beth semantics as found in, for instance, (Dummett, 2000). The first is that standard Beth semantics uses a constant domain. The variant of Beth semantics using a growing domain is sometimes called Beth-Kripke semantics. The second difference in our definition is that we allow $\langle T, \leq \rangle$ to be any partial order rather than requiring it to be a tree as in the standard version. Though not particularly common, neither of these two tweaks are novel—both are remarked upon in (Troelstra and van Dalen, 1988, 679). The third difference from the standard Beth semantics is in our treatment of function symbols. In (Troelstra and van Dalen, 1988), when a language includes individual constants, they are all assumed to have a denotation at the root node of T. By contrast, we are only assuming that for each term t, there is a bar where t has a denotation. This suffices for every node to force $t = t$, as desired for a semantics of intuitionistic logic.[15] We doubt this third tweak is novel either, although we are not aware of any explicit precedents.[16]

Let us call a model in the sense defined above a *generalized Beth-Kripke model.* Models that satisfy the ordinary definitions of Beth semantics—that

[15] Note although our modal logic is free, we are working with standard intuitionistic logic, not any of its free logic variants such as E-logic (Troelstra and van Dalen, 1988, Ch.2). This is because our aim is to interpret intuitionistic theories of choice sequences, which are formulated in standard (non-free) intuitionistic logic.

[16] The reason for this third tweak is so that our generalized Beth-Kripke models behave similarly to free S4+ models, allowing us to prove Lemma 2 and Theorem 4. If we wanted to use an unfree modal logic, for instance by treating choice sequences as relations rather than functions, then we could follow Troelstra and van Dalen in requiring terms to have denotations at the root node of T. Then in the statement of the Mirroring Theorems 4 and 5 below, we could omit the assumption IE from the left side of the turnstile.

is, models that do not allow for our three tweaks—may be called *strict Beth models*. Despite the differences between generalized Beth-Kripke models and strict Beth models, the soundness and completeness results are routine.

Theorem 3. *Let $\vdash_I$ be deducibility in intuitionistic logic and let $\models_{BK}$ be entailment in Beth-Kripke models as defined above. Then $\Delta \vdash_I \varphi$ iff $\Delta \models_{BK} \varphi$.*

Proof sketch. Soundness can be proved by a straightforward induction on the length of derivation.

For completeness, suppose that $\Delta \nvdash_I \varphi$. Then by the completeness of standard Beth semantics, there is a strict Beth model M such that $M \Vdash \Delta$ and $M \nVdash \varphi$.[17] But every strict Beth model is also a generalized Beth-Kripke model, so M witnesses $\Delta \not\models_{BK} \varphi$. $\square$

4.4 The Beth-Kripke translation

With the inevitability operator and Beth-Kripke semantics in hand, it is now straightforward to define our desired translation of intuitionistic logic into S4+. The basic insight is that we can treat the partial order $\langle T, \leq \rangle$ as a frame for a Kripke model of S4+; then the requirements in the semantic clauses for a world to force a sentence in the Beth-Kripke model are expressible in terms of inevitability and necessity in the S4+ model, in the same way that the Gödel translation exploits that Kripke models for intuitionistic logic can also be seen as models of S4. For instance, the condition that an atom evaluated as true at some world w must also be evaluated true at all $u \geq w$ is equivalent to saying that atoms are positively stable. Similarly, the condition that every maximal chain above w includes a world u satisfying some condition is equivalent to saying that w thinks that that condition is inevitable. This leads us to what we may call the *Beth-Kripke Translation*:

φ	φ^{BK}
P	$\mathcal{I}\Box P$
$\neg\varphi$	$\Box\neg\varphi^{\mathsf{BK}}$
$\varphi \wedge \psi$	$\varphi^{\mathsf{BK}} \wedge \psi^{\mathsf{BK}}$
$\varphi \vee \psi$	$\mathcal{I}(\varphi^{\mathsf{BK}} \vee \psi^{\mathsf{BK}})$
$\varphi \rightarrow \psi$	$\Box(\varphi^{\mathsf{BK}} \rightarrow \psi^{\mathsf{BK}})$
$\exists x\varphi(x)$	$\mathcal{I}\exists x\varphi^{\mathsf{BK}}(x)$
$\forall x\varphi(x)$	$\Box\forall x\varphi^{\mathsf{BK}}(x)$

We can now prove an analogue of Lemma 1.

Lemma 2. *Let $M = \langle T, \leq, D, V \rangle$, where $\langle T, \leq \rangle$ is a partial order, D is a function assigning non-shrinking domains to members of T, and V is a valuation monotonically assigning subsets of $D(w)^n$ to each n-place predicate*

[17] For completeness of strict Beth models, see (Troelstra and van Dalen, 1988, §13.2).

in a given language and graphs of partial functions on $D(w)^n$ to each n-ary function symbol. Furthermore, assume that for every term t and maximal chain in T, there is a w in that chain where t has a denotation. Then for any sentence φ in the language and any $w \in T$, we have that $M, w \Vdash \varphi$ as a Beth-Kripke model iff $M, w \models \varphi^{\mathsf{BK}}$ as an S4+ model.

Proof. Routine induction on the complexity of φ. $\square$

For a given language and term $t(\vec{x})$ in that language, consider the following principle, expressing the claim that for any objects $\vec{x}$, $t(\vec{x})$ *inevitably exists*:

$$(\mathrm{IE}_t) \qquad \Box\forall\vec{x}\mathcal{I}[t(\vec{x}) = t(\vec{x})]$$

Given a language $\mathcal{L}$, let $\mathrm{IE} := \{\mathrm{IE}_t : t \in \mathcal{L}\}$ be the set of all inevitable existence claims for terms in $\mathcal{L}$. (As is well-known, our standard axiomatization of S4 allows the derivation of $t = t \rightarrow \Box t = t$, so IE also entails that every term inevitably necessarily exists.)

The previous lemma immediately gives us the following theorem.

Theorem 4. *For every sentence φ and set of sentences Γ of first-order logic, $\Gamma \models_{BK} \varphi$ iff* $\mathrm{IE}, \Gamma^{\mathsf{BK}} \models_{\mathbf{S4+}} \varphi^{\mathsf{BK}}$

This theorem shows that the Beth-Kripke translation provides a faithful interpretation of intuitionistic logic in S4+, as we hoped. Furthermore, it does so in a way that avoids the problems afflicting the Gödel translation. The problem with the Gödel translation, recall, was that it translates $\exists x\varphi(x)$ as $\exists x\varphi^g(x)$; existential quantification is treated as a matter of *actual* existence, whereas the potentialist wants to treat existential quantification as a matter of *potential* existence. And the Beth-Kripke translation does this. If the potentialist interprets mathematical discourse under the Beth-Kripke translation, she will take $\exists x\varphi(x)$ to 'really' mean $\mathcal{I}\exists x\varphi^{\mathsf{BK}}(x)$, which does not commit her to the *actual* existence of some x, but merely to the guarantee that such an x will eventually be at hand.

That being said, Theorem 4 only guarantees that the Beth-Kripke translation faithfully preserves model-theoretic entailment from intuitionistic logic in S4+. It would be nice to further show that the Beth-Kripke translation provided a faithful embedding of *deductive reasoning* in intuitionistic logic into S4+. After all, the goal of a mirroring theorem is to show that the deductive reasoning of a non-modal mathematical theory can be exactly captured in the potentialist's modal reconstruction. If we had a sound and complete axiomatization of S4+, this would follow immediately from Theorem 4. But since S4+ is not axiomatizable, it will take a little more work.

We will use the following axiomatization of intuitionistic logic, based on (Kleene, 1952). We follow the convention that $\neg$ binds tighter than $\wedge$ and $\vee$, which in turn bind tighter than $\rightarrow$.

I1 $\varphi \to (\psi \to \varphi)$

I2 $(\varphi \to \psi) \to ((\varphi \to (\psi \to \theta) \to (\varphi \to \theta))$

I3 $\varphi \to (\psi \to \varphi \wedge \psi)$

I4 $\varphi \wedge \psi \to \varphi$ and $\varphi \wedge \psi \to \psi$

I5 $\varphi \to \varphi \vee \psi$ and $\varphi \to \psi \vee \varphi$

I6 $(\varphi \to \theta) \to ((\psi \to \theta) \to (\varphi \vee \psi \to \theta))$

I7 $(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$

I8 $\neg\varphi \to (\varphi \to \psi)$

I9 $\forall x \varphi(x) \to \varphi(t)$

I10 $\varphi(t) \to \exists x \varphi(x)$

I11 $t = t$, for any term t

I12 $t = s \wedge \varphi(t) \to \varphi(s)$

I13 From φ and $\varphi \to \psi$ infer ψ

I14 From $\varphi \to \psi(x)$ infer $\varphi \to \forall x \psi(x)$, provided x does not occur free in φ

I15 From $\varphi(x) \to \psi$ infer $\exists x \varphi(x) \to \psi$, provided x does not occur free in ψ

In what follows, let $\vdash_{S4+}$ be the deducibility relation using the axioms given in Section 4.2. We begin with a lemma.

Lemma 3. *For every formula φ in the language of predicate logic:*

1. *If $\vdash_{S4+} \varphi \to \Box\varphi$, then $\vdash_{S4+} \mathcal{I}\varphi \to \Box\mathcal{I}\varphi$,*
2. *$\vdash_{S4+} \Box(\varphi^{\mathsf{BK}} \leftrightarrow \Box\varphi^{\mathsf{BK}})$,*
3. *$\vdash_{S4+} \Box(\varphi^{\mathsf{BK}} \leftrightarrow \mathcal{I}\varphi^{\mathsf{BK}})$.*

Proof. For brevity we omit the subscript on the turnstile.

(1) Assume $\vdash \varphi \to \Box\varphi$. We get $\vdash \Box(\varphi \to \Box\varphi)$ by necessitation. Then by M1 and modus ponens we get $\vdash \mathcal{I}\varphi \to \mathcal{I}\Box\varphi$ and as an instance of M5 we have $\vdash \mathcal{I}\Box\varphi \to \Box\mathcal{I}\varphi$. This gives us $\vdash \mathcal{I}\varphi \to \Box\mathcal{I}\varphi$ as desired.

Claims (2) and (3) can now be proved by an easy induction on the complexity of φ by using claim (1). We consider two cases as examples.

Suppose φ^{BK} is $\Box\neg\psi^{\mathsf{BK}}$. Then claim (2) is a simple theorem of S4. For claim (3), the left-to-right conditional holds by axiom M2. Conversely, using

(1) we have $\mathcal{I}\Box\neg\psi^{\mathsf{BK}} \to \Box\mathcal{I}\Box\neg\psi^{\mathsf{BK}}$. Since $\mathcal{I}\Box\neg\psi^{\mathsf{BK}} \to \Diamond\neg\psi^{\mathsf{BK}}$, this gives us $\mathcal{I}\Box\neg\psi^{\mathsf{BK}} \to \Box\Diamond\neg\psi^{\mathsf{BK}}$. But by (2) of the i.h., we have $\Diamond\neg\psi^{\mathsf{BK}} \to \neg\psi^{\mathsf{BK}}$, which entails $\Box\Diamond\neg\psi^{\mathsf{BK}} \to \Box\neg\psi^{\mathsf{BK}}$. So we have $\mathcal{I}\Box\neg\psi^{\mathsf{BK}} \to \Box\neg\psi^{\mathsf{BK}}$.

As another example, consider the case where φ^{BK} is $\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}})$. Again, claim (2) and the left-to-right direction of (3) are immediate. For the converse, we know $\mathcal{I}\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}}) \to \Box\mathcal{I}\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}})$, so it will suffice to show $\neg\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}}) \to \neg\Box\mathcal{I}\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}})$ Now, by the i.h. we know $(\psi^{\mathsf{BK}} \wedge \neg\theta^{\mathsf{BK}}) \leftrightarrow (\Box\psi^{\mathsf{BK}} \wedge \neg\mathcal{I}\theta^{\mathsf{BK}})$. Note that we can prove $(\Box\psi^{\mathsf{BK}} \wedge \neg\mathcal{I}\theta^{\mathsf{BK}}) \to \neg\mathcal{I}\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}})$ as follows:

$$\begin{array}{rcl}
\Box\psi^{\mathsf{BK}} & \to & \Box(\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}}) \to \theta^{\mathsf{BK}}) \\
& \to & \mathcal{I}\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}}) \to \mathcal{I}\theta^{\mathsf{BK}} \\
& \to & \neg\mathcal{I}\theta^{\mathsf{BK}} \to \neg\mathcal{I}\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}}) \\
(\Box\psi^{\mathsf{BK}} \wedge \neg\mathcal{I}\theta^{\mathsf{BK}}) & \to & \neg\mathcal{I}\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}})
\end{array}$$

This is equivalent to $\neg(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}}) \to \neg\mathcal{I}\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}})$, giving us $\Diamond\neg(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}}) \to \Diamond\neg\mathcal{I}\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}})$, as required.

The other cases are straightforward. $\Box$

Theorem 5 (BK mirroring)**.** *For every sentence φ and set of sentences Γ of first-order logic, $\Gamma \vdash_I \varphi$ iff $\mathrm{IE}, \Gamma^{\mathsf{BK}} \vdash_{S4+} \varphi^{\mathsf{BK}}$.*

Proof. The essence of the proof is summarized in this diagram:

$$\begin{tikzcd}
\varphi_1, \ldots, \varphi_n \vdash_{\text{int}} \psi \arrow[r, "\text{Induction on proof}"] & \varphi_1^{\mathsf{BK}}, \ldots, \varphi_n^{\mathsf{BK}}, \mathrm{IE} \vdash_{\mathrm{S4+}} \psi^{\mathsf{BK}} \arrow[d, "\text{Soundness}"] \\
\varphi_1, \ldots, \varphi_n \models_{\text{int}} \psi \arrow[u, "\text{Completeness}"] & \varphi_1^{\mathsf{BK}}, \ldots, \varphi_n^{\mathsf{BK}}, \mathrm{IE} \models_{\mathrm{S4+}} \psi^{\mathsf{BK}} \arrow[l, "\text{Thm. 4}"]
\end{tikzcd}$$

We now fill in the details of this picture.

($\Rightarrow$) We proceed by induction on the length of the intuitionistic proof. It will suffice to show that for each of I1-I10, their BK translation is provable in S4+, and the BK translations of the rules I13-I15 are admissible in S4+.

I1: As a tautology, $\varphi^{\mathsf{BK}} \to (\psi^{\mathsf{BK}} \to \varphi^{BK})$ will be a theorem. Then by necessitation we have $\Box(\varphi^{\mathsf{BK}} \to (\psi^{\mathsf{BK}} \to \varphi^{\mathsf{BK}}))$. Further, we have $\Box(\varphi^{\mathsf{BK}} \to (\psi^{\mathsf{BK}} \to \varphi^{\mathsf{BK}})) \to (\Box\varphi^{\mathsf{BK}} \to \Box(\psi^{\mathsf{BK}} \to \varphi^{\mathsf{BK}}))$ as an instance of the K axiom. Thus by modus ponens we have $\Box\varphi^{\mathsf{BK}} \to \Box(\psi^{\mathsf{BK}} \to \varphi^{\mathsf{BK}})$. By Lemma 3, we know $\varphi^{\mathsf{BK}} \to \Box\varphi^{\mathsf{BK}}$, which gives us $\varphi^{\mathsf{BK}} \to \Box(\psi^{\mathsf{BK}} \to \varphi^{\mathsf{BK}})$, and then an application of necessitation gives the desired formula: $\Box(\varphi^{\mathsf{BK}} \to \Box(\psi^{\mathsf{BK}} \to \varphi^{\mathsf{BK}}))$. I2-I4 are similar.

For I5, we argue similarly to get $\varphi^{\mathsf{BK}} \to \varphi^{\mathsf{BK}} \vee \psi^{\mathsf{BK}}$. Then by M2, $\varphi^{\mathsf{BK}} \vee \psi^{\mathsf{BK}} \to \mathcal{I}(\varphi^{\mathsf{BK}} \vee \psi^{\mathsf{BK}})$. So we infer $\varphi^{\mathsf{BK}} \to \mathcal{I}(\varphi^{\mathsf{BK}} \vee \psi^{\mathsf{BK}})$, and then by necessitation we have $\Box(\varphi^{\mathsf{BK}} \to \mathcal{I}(\varphi^{\mathsf{BK}} \vee \psi^{\mathsf{BK}}))$.

For I6 we begin similarly to get $\Box(\varphi^{\mathsf{BK}} \to \theta^{\mathsf{BK}}) \to \Box(\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}}) \to \Box(\varphi^{\mathsf{BK}} \vee \psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}})))$. But as an instance of M1 we have $\Box(\varphi^{\mathsf{BK}} \vee \psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}})) \to (\mathcal{I}(\varphi^{\mathsf{BK}} \vee \psi^{\mathsf{BK}}) \to \mathcal{I}\theta^{\mathsf{BK}})$. But by Lemma 3, $\Box(\theta^{\mathsf{BK}} \leftrightarrow \mathcal{I}\theta^{\mathsf{BK}})$. So, by the substitution of necessary equivalents we infer $\Box(\varphi^{\mathsf{BK}} \to \theta^{\mathsf{BK}}) \to \Box(\Box(\psi^{\mathsf{BK}} \to \theta^{\mathsf{BK}}) \to (\mathcal{I}(\varphi^{\mathsf{BK}} \vee \psi^{\mathsf{BK}}) \to \theta^{\mathsf{BK}})))$.

I7 we begin the same way, with $(\varphi^{\mathsf{BK}} \to \psi^{\mathsf{BK}}) \to ((\varphi^{\mathsf{BK}} \to \neg\psi^{\mathsf{BK}}) \to \neg\varphi^{\mathsf{BK}})$ being a theorem. And by a series of applications of necessitation, K, and modus ponens, we get $\Box(\Box(\varphi^{\mathsf{BK}} \to \psi^{\mathsf{BK}}) \to \Box(\Box(\varphi^{\mathsf{BK}} \to \neg\psi^{\mathsf{BK}}) \to \Box\neg\varphi^{\mathsf{BK}}))$. From here we easily infer $\Box(\Box(\varphi^{\mathsf{BK}} \to \psi^{\mathsf{BK}}) \to \Box(\Box(\varphi^{\mathsf{BK}} \to \Box\neg\psi^{\mathsf{BK}}) \to \Box\neg\varphi^{\mathsf{BK}}))$. I8 is similar.

I9, I10, and I11 follow easily from an appeal to IE.

I12 and I13 are straightforward on their own.

For I14, suppose we have $\Box(\varphi^{\mathsf{BK}} \to \psi^{\mathsf{BK}}(x))$. Infer $\varphi^{\mathsf{BK}} \to \psi^{\mathsf{BK}}(x)$, and apply the standard quantifier inference to get $\varphi^{\mathsf{BK}} \to \forall x\psi^{\mathsf{BK}}(x)$, and some simple modal reasoning to get $\Box(\varphi^{\mathsf{BK}} \to \Box\forall x\psi^{\mathsf{BK}}(x))$.

For I15, we similarly have $\varphi^{\mathsf{BK}}(x) \to \psi^{\mathsf{BK}}$, and infer to $\Box(\exists x\varphi^{\mathsf{BK}}(x) \to \psi^{\mathsf{BK}})$. Using M1 we can get $\Box(\mathcal{I}\exists x\varphi^{\mathsf{BK}}(x) \to \mathcal{I}\psi^{\mathsf{BK}})$, and by appeal to Lemma 3 we have $\Box(\mathcal{I}\exists x\varphi^{\mathsf{BK}}(x) \to \psi^{\mathsf{BK}})$.

($\Leftarrow$) Assume $\Gamma^{BK} \vdash_{S4+} \varphi^{BK}$. Since the axioms given in §4.1 are sound (though necessarily incomplete) for S4+, we have $\Gamma^{BK} \models_{S4+} \varphi^{BK}$. By Theorem 4, we then have $\Gamma \models_{BK} \varphi$. Finally by the completeness of intuitionistic logic we have $\Gamma \vdash_I \varphi$. $\Box$

One interesting consequence of Theorem 5 is that, although S4+ is not axiomatizable, nevertheless the fragment of S4+ that is in the range of the Beth-Kripke translation is axiomatizable.

This completes our basic toolkit for divergent potentialism. We introduced a translation from a non-modal language $\mathcal{L}$ into the modal language $\mathcal{L}^\Diamond$, which includes the modal operators $\Box$, $\Diamond$, and $\mathcal{I}$. Moreover, the mirroring theorem we have just proved shows this translation to be faithful.

4.5 Avoiding $\mathcal{I}$?

We present here a brief account of an approach to developing a modal theory of choice sequences that avoids use of the inevitability operator.[18] Instead of insisting that every choice sequence eventually be defined on each argument, one would only require that any choice sequence *could* be defined on any given argument. Informally, the creating mathematician is always free to keep extending a choice sequence, but is also free to stop.

We agree that this is a coherent way to conceive of the idealized mathematician's activity, and a theory formalizing this idea would likely be of mathematical interest in its own right. If, however, this alternative approach

[18]This alternative was suggested to us by Geoffrey Hellman.

were meant as a way to interpret intuitionistic mathematics in a classical modal theory—as our aim is—then it faces several challenges.

The first, and most serious challenge, is to give a faithful translation of intuitionistic logic. The kernel of this alternative approach is that, instead of saying something *will* inevitably happen, it says merely that it is always possible for that thing to happen. This suggests using a translation that is like the BK translation, but simply replaces $\mathcal{I}$ with $\Box\Diamond$. Call this the *alternative translation*:

φ	φ^{A}
P	$\Box\Diamond\Box P$
$\neg\varphi$	$\Box\neg\varphi^{\mathsf{A}}$
$\varphi \wedge \psi$	$\varphi^{\mathsf{A}} \wedge \psi^{\mathsf{A}}$
$\varphi \vee \psi$	$\Box\Diamond(\varphi^{\mathsf{A}} \vee \psi^{\mathsf{A}})$
$\varphi \to \psi$	$\Box(\varphi^{\mathsf{A}} \to \psi^{\mathsf{A}})$
$\exists x\varphi(x)$	$\Box\Diamond\exists x\varphi^{\mathsf{A}}(x)$
$\forall x\varphi(x)$	$\Box\forall x\varphi^{\mathsf{A}}(x)$

This translation is not faithful, however, because it validates the intuitionistically objectionable Independence of Premises:

- $P \to \exists xQx \nvdash_I \exists x(P \to Qx)$
- $(P \to \exists xQx)^{\mathsf{A}} \vdash_{S4} (\exists x(P \to Qx))^{\mathsf{A}}$

To see that this inference is valid in S4, suppose w is some world (in some model M) such that $w \models (P \to \exists xQx)^{\mathsf{A}}$, but $w \not\models (\exists x(P \to Qx))^{\mathsf{A}}$. So, applying the definition of A, we have $w \models \Box(\Box\Diamond\Box P \to \Box\Diamond\exists x\Box\Diamond\Box Qx)$ and $w \not\models \Box\Diamond\exists x\Box(\Box\Diamond\Box P \to \Box\Diamond\Box Qx)$. So for some wRw' and for all $w'Ru$, for all $a \in D(u)$, $u \not\models \Box(\Box\Diamond\Box P \to \Box\Diamond\Box Qa)$. Hence there is some uRu' such that $u' \models \Box\Diamond\Box P$ but for all $a \in D(u)$, $u' \not\models \Box\Diamond\Box Qa$. Note that this entails that $u \not\models \Box\Diamond\Box Qa$. So for all $w'Ru$, $u \not\models \exists x\Box\Diamond\Box Qx$. It follows in particular that $u' \not\models \Box\Diamond\exists x\Box\Diamond\Box Qx$. But $u' \models \Box\Diamond\Box P$. Since wRu', this contradicts the claim that $w \models \Box(\Box\Diamond\Box P \to \Box\Diamond\exists x\Box\Diamond\Box Qx)$.

This example leaves open the possibility that there is some other translation in the spirit of the alternative approach that is faithful. But such a translation does not immediately fall out of the ideas of this paper.

The second challenge for the alternative approach is that it seems to abandon the traditional constructive readings of the quantifiers. In the BHK interpretation, $\forall x\exists y\alpha(x) = y$ means that for any x I have a method of producing a y such that $\alpha(x) = y$. The alternative approach translates this formula as $\Box\forall x\Box\Diamond\exists y\alpha(x) = y$. Informally, this says only that for any possible x it is never ruled out that there will be a y such that $\alpha(x) = y$; but there is also no guarantee that there is now or ever will be such a y. This seems to be at odds with the intended interpretation of the intuitionistic statement, making the alternative approach seem less than natural philosophically.

The third challenge is similar, observing that the alternative approach seems to abandon the standard intuitionistic view of what it takes to assert $\varphi(\alpha)$. On the standard view, it is not sufficient that $\varphi(\alpha)$ holds with probability 1. Rather, there must exist a dispositive guarantee that $\varphi(\alpha)$. On the alternative approach, however, if $\varphi(\alpha)$ holds with probability 1, then that would entail that it is always possible that $\varphi(\alpha)$, and hence the alternative approach would have us asserting $\varphi(\alpha)$. For instance, the intuitionistic theorist would ascent to $\exists x \alpha(x) = 0$ only if we have a *guarantee* that there is or will be such an x. By contrast, the alternative approach would have us note that, since there is probability 1 that such an x will exist, $\Box\Diamond\exists x \alpha(x) = 0$; thus, the alternative approach would have us assent to this. So in addition to revising the traditional constructive reading of the quantifier, this alternative approach abandons the intuitionistic view on the standard of assent.

Neither the second nor third challenge is a knockdown objection to this alternative approach. After all, it can sometimes be technically or philosophically fruitful to offer a reinterpretation of some theory. But, given our goal of making choice sequences comprehensible to classical mathematicians, it would be preferable to have an interpretation that is more faithful to the intended meaning of the intuitionistic theory. And the BK translation provides exactly such an interpretation. One might be generally uncomfortable with using the inevitability operator because it makes the logic unaxiomatizable. But as we have seen, the fragment of the logic in the range of the BK translation is axiomatizable. So in this particular setting we see no reason to avoid use of the inevitability operator, and the interpretation of intuitionistic logic that it allows is quite attractive.

5 Troelstra's theory of lawless sequences

Let us take stock. In Section 2 we formulated two desiderata on the translation $* : \mathcal{L} \to \mathcal{L}^{\Diamond}$:

1. $*$ should be a (faithful) interpretation of the logic of $\mathcal{L}$
2. $*$ should map the axioms of the $\mathcal{L}$-theory to theorems of the $\mathcal{L}^{\Diamond}$-theory

In short, $*$ should be an interpretation of the mathematical theory in $\mathcal{L}$ in that of $\mathcal{L}^{\Diamond}$.

Having proved the first desideratum in the previous section, we now turn to the second, using lawless choice sequences as our example. Our non-modal theory will be Troelstra's LS (Troelstra, 1977), which is the standard axiomatization of lawless sequences. Our mathematical theory in $\mathcal{L}^{\Diamond}$ will be developed as we go along, adding to the 5 principles already formulated in Section 3. This theory provides a natural explication of lawless sequences understood as potentially infinite, "growing" sequences.

In the non-modal language of the theory LS, there are two notions of identity between choice sequences. *Intensional identity*, symbolized $\alpha \equiv \beta$, is understood as two choice sequences being *given as* one and the same process of choosing the values of the sequence. *Extensional identity* is understood as a matter of simple coextensionality between two sequences; that is, $\alpha = \beta$ is $\forall x(\alpha(x) = \beta(x))$. To accommodate this latter understanding, we would treat the identity predicate between choice sequences as non-logical and translate it as we would translate the corresponding coextensionality statement. We will return to this issue briefly in Section 5.2.

5.1 Density of choice sequences

The first axiom of Troelstra's LS says that for every finite initial segment n, there is a sequence α beginning in that way:

(LS1) $$\forall n \exists \alpha \, \alpha \in n$$

This is the density or maximality principle discussed in Section 4.2. We adopt the analogous principle as an axiom:

(Density) $$\Box \forall n \, \mathcal{I} \exists \alpha \, \alpha \in n$$

This is really an abbreviation, however; the unabbreviated formula is:

$$\Box \forall n[Seq(n) \rightarrow \mathcal{I} \exists \alpha \forall x(x \leq lh(n) \rightarrow \forall y(y = (n)_x \rightarrow \alpha(x) = y))]$$

Here we are assuming enough arithmetic in the background to formalize standard coding of finite sequences. $Seq(x)$ is a formula saying that x is the code of a sequence, $lh(x)$ denotes the length of the sequence coded by x, and $(x)_y$ is the y^{th} element of that sequence. Now we want to observe that one can derive $LS1^{BK}$ from Density (alongside the original 5 axioms from Section 3).

Say that a formula is *arithmetical* when it contains no choice sequence variables and no modal operators.

Lemma 4. *Assume that all arithmetical formulas are positively and negatively stable, in the sense defined in §2.1. Then for any arithmetical formula φ we have $\Box(\varphi \leftrightarrow \varphi^{BK})$.*

Proof. Induction on the complexity of φ. □

In our setting it makes sense to assume that arithmetical formulas are indeed both positively and negatively stable.[19] We will take this to be part of the background theory of arithmetic and no longer flag it as an explicit assumption.

[19] Assuming that arithmetical formulas are both positively and negatively stable will have the consequence that arithmetical formulas satisfy the law of excluded middle, so the background arithmetical portion of our theory will behave classically. Since our aim is to make choice sequences classically comprehensible, we are comfortable with this result.

Proposition 1. *One can derive $LS1^{\mathsf{BK}}$ from* Density.

Proof. $LS1^{\mathsf{BK}}$ is:

$$\Box\forall n\Box[Seq(n)^{\mathsf{BK}} \to \mathcal{I}\exists\alpha\Box\forall x\Box((x \leq lh(n))^{\mathsf{BK}} \to \\ \Box\forall y\Box((y = (n)_x)^{\mathsf{BK}} \to \mathcal{I}\Box\alpha(x) = y))]$$

Everything following '$\exists\alpha$' in this formula is what gets abbreviated as $(\alpha \in n)^{\mathsf{BK}}$. Now, using the facts noted above, it is a routine exercise to show that $\Box\forall\alpha(\alpha \in n \to (\alpha \in n)^{\mathsf{BK}})$. Then using the modal logical axiom M1, we can infer $\mathcal{I}\exists\alpha(\alpha \in n) \to \mathcal{I}\exists\alpha(\alpha \in n)^{\mathsf{BK}}$. Now a few more routine inferences using Lemma 4 will get us from Density to $LS1^{\mathsf{BK}}$. $\Box$

5.2 Compossibility of choice sequences

Our next modal axiom says something about when possibilities are compossible (that is, jointly possible). Consider two distinct α and β. Suppose α could continue in one way and that β could continue in some way. Then it is possible for both sequences to continue as described.

To generalize this idea and make matters formally precise, we need some notation. Let $\neq(\alpha, \beta_1, \ldots, \beta_n)$ abbreviate $\alpha \neq \beta_1 \wedge \ldots \wedge \alpha \neq \beta_n$. Let $\#(\alpha_0, \ldots, \alpha_n)$ formalize the claim that $\alpha_i \neq \alpha_j$ for each $0 \leq i < j \leq n$. As noted earlier, there are two different possible interpretations of this identity relation between choice sequences. One option is to follow (Troelstra, 1977) in adding a notion of intensional identity between choice sequences. Another option is to focus on extensional identity, regarding $\alpha = \beta$ as an abbreviation of $\Box\forall x(\alpha(x) = \beta(x))$. Either option would serve our purposes here.

We adopt an axiom scheme stating that possibilities concerning distinct choice sequences are compossible:

(Comp)

$$\#(\alpha_0, \ldots, \alpha_n) \wedge \Diamond\varphi_0(\alpha_0) \wedge \ldots \wedge \Diamond\varphi_n(\alpha_n) \to \Diamond(\varphi_0(\alpha_0) \wedge \ldots \wedge \varphi_n(\alpha_n))$$

where each φ_i has only α_i as a parameter and no other choice sequence variables either free or bound.

For example, if it is possible for the next entry of α to be 0 and it is possible for the next entry of β to be 1, then it is possible for both of these entries simultaneously to be as described.

Let $CoExt(\alpha, \beta)$ state that α and β are coextensive: $\forall n(\alpha(n) = \beta(n))$.

Proposition 2. *One can derive the Beth-Kripke translation of Troelstra's second axiom:*

(LS2) $$Coext(\alpha, \beta) \vee \neg Coext(\alpha, \beta)$$

Proof. Suppose $\alpha = \beta$. Then the BK translation of the first disjunct is provable. So suppose instead (using the fact that the modal logic is classical) that $\alpha \neq \beta$. Then it is possible that α should continue in one way and that β should continue in an extensionally different way. So the two developments are jointly possible. Thus, we get

$$\Box\neg\Box\forall n(\alpha(n) = \beta(n))$$

which is (equivalent to) the BK translation of the second disjunction. $\Box$

5.3 The "open data" axiom

Troelstra's next axiom, LS3, is:

$$A(\alpha, \beta_1, ...\beta_n) \wedge \neq (\alpha, \beta_1, ...\beta_n) \rightarrow \exists n(\alpha \in n \wedge \forall \gamma \in n(\neq (\gamma, \beta_1, ...\beta_n) \rightarrow A(\gamma, \beta_1, ..., \beta_n))$$

This has the simple one-variable version:

$$A(\alpha) \rightarrow \exists n(\alpha \in n \wedge \forall \beta \in nA(\beta))$$

The idea is that if we make the judgment $A(\alpha)$, then this is based on only a finite amount of information about α, that is, on some finite initial segment coded by n (thus '$\alpha \in n$'). This is called the "principle of open data" because the finite initial segment of α coded by n defines an open region in Baire space.

The following modal axiom scheme makes sense in our setting:

$$\text{(OD)}\quad \varphi(\alpha, \beta_1, \ldots, \beta_k) \wedge \neq (\alpha, \beta_1, \ldots, \beta_k) \wedge \text{“initial segment}(\alpha) = n\text{”} \rightarrow \Box(\forall \gamma \in n)(\neq (\gamma, \beta_1, \ldots, \beta_k) \rightarrow \varphi(\gamma, \beta_1, \ldots, \beta_k))$$

That is, φ only "looks at" the initial segment of α that is available at the relevant world. Whatever φ says about α, it also says about any γ that shares the mentioned initial segment. Our axiom (OD) entails (the Beth-Kripke translation of) Troelstra's axiom open data axiom, LS3.

Troelstra goes on to introduce some further axioms, which we will not discuss here, although we are hopeful they can be handled in a similar way.

6 Real numbers and continuity

We submit that the development of choice sequences is sufficient for our purpose of motivating a framework for non-convergent procedures. However, it would be nice to extend the treatment to an analogue of the intuitionistic treatment of real analysis. A hallmark of this would be the Brouwerian theorem that all total (bounded) functions on real numbers are (uniformly) continuous (see, for example, (Heyting, 1971)). This, of course, is easily shown to be false in classical analysis. We plan to provide a full treatment in future work. Here we sketch how it might go.

Let us assume a coding of the rational numbers into the natural numbers. We will take p, q, r as variables ranging over rational numbers or, strictly speaking, natural numbers construed as codes of rational numbers. The addition, multiplication, and inequality signs will thus be used ambiguously to denote the functions and relations on natural numbers and functions and relations on rational numbers. Context will serve to disambiguate. Division is used only for rational numbers.

Say that a choice sequence α is *Cauchy* if

$$\Box\forall p > 0\mathcal{I}\exists x\Box\forall y(|\alpha(x) - \alpha(x+y)| < p).$$

This is the usual definition of a Cauchy sequence of rational numbers, stating that, for every positive rational number p, the values of α, construed as rational numbers, are eventually (i.e., inevitably) within p of each other.

This definition does not prescribe the rate of convergence, and, in typical cases, the rate may not be known, unless the quantifiers are somehow interpreted constructively (e.g., via the BK-translation). We define a special class of Cauchy sequences, those which have a rate of convergence fixed in advance.

Say that a choice sequence α is a *real number generator* if

$$\Box\forall x\forall y(|\alpha(x) - \alpha(x+y)| < 1/2^x).$$

Clearly, every real number generator is Cauchy.

Let α and β be Cauchy sequences. Say that $\alpha \sim \beta$ if

$$\Box\forall p > 0\mathcal{I}\exists x\Box\forall y > x(|\alpha(x+y) - \beta(x+y)| < p).$$

In words, $\alpha \sim \beta$ holds just in case, for any positive rational number p, the values of α and β are eventually within p of each other. Clearly, this is an equivalence relation. Intuitively, it holds just in case α and β converge to the same real number. As with the definition of Cauchy, it might be better, for present purposes, to specify the bound in advance. We can leave that matter open here.

As in classical mathematics, the intuitionist can identify the real numbers with equivalence classes of Cauchy sequences (or real number generators). Equivalence classes can be construed as either sets or properties, in which case, the background framework would be expanded to include either set theory or higher-order logic. Here, we will work with real number generators directly.

Say that a finite sequence $\langle p_0, p_1, \ldots, p_k\rangle$ of rational numbers is SFSG (so far, so good) just in case:

$$\forall n\forall m((n \leq k \land n < m \leq k) \to |p_m - p_n| < 1/2^n).$$

The idea is that $\langle p_0, p_1, \ldots, p_k \rangle$ is SFSG just in case it could be the start of a real number generator.

Our next chore is to simulate talk of functions on real number generators. To this end, let R be a relation between choice sequences and finite sequences of natural numbers (thought of as coding rational numbers). The idea is that, as more and more information about α becomes available, R will relate α to longer and longer finite sequences, which provide better and better approximations of the value of a function (represented by R) on α as argument.

Say that R is *functional* if:

1. Necessarily, if $R\alpha b$ then α is a real number generator and b is SFSG.

2. Necessarily, if $R\alpha b$ and $R\alpha b'$ then either b is an initial segment of b' or b' is an initial segmenet of b.

3. $\Box(R\alpha b \rightarrow \Box R\alpha b)$. R is positively stable (but not necessarily negatively stable).

4. $\Box(R\alpha b \rightarrow \mathcal{I}\exists p R\alpha b^\frown p)$. If $R\alpha b$ then it is inevitable that b will be extended by one value.

These conditions guarantee that if α is a real number generator, then, for each path through the future, the values of the members of the sequences $\{b \,|\, R\alpha b\}$ along that path are themselves a real number generator. We can think of that as the value of R for the argument α along that path.

Our final condition is that, in effect, R should define a function on real numbers, not just real number generators: if $\alpha \sim \beta$ then it is inevitable that the values of members of the sequences $\{b \,|\, R\alpha b\}$ are arbitrarily close to the corresponding members the sequences $\{b \,|\, R\beta b\}$:

5 Necessarily, if $\alpha \sim \beta$ then for any rational number $p > 0$ there is a number n such that it is inevitable that, necessarily, if $R\alpha b$ and $R\beta b'$ then for any $m \geq n$, the m^{th} elements of b are within p of the m^{th} elements of b'.

Recall that the background modal logic is classical. If we restrict attention to *lawlike* real number generators, then we can define a functional relation S where $S\alpha b$ if either α converges to a rational number and b is a finite sequence of 0's, or else α converges to an irrational number and b is a finite sequence of 1's. This corresponds to a function on real numbers whose value is zero on rational numbers and whose value is one on irrational numbers. In effect, this is an extreme case of a function that "looks ahead" in the argument in order to set the initial values of the output sequence. And, of course, this function is (very) discontinuous.

In contrast, if there is branching in the input real number generator α, then it may not be possible to "look ahead" like this to define a function S. Suppose, for example, that there is a branch on which α converges to a rational number and a branch on which α converges to an irrational number.

This can be made precise. We envision a background that consists of all sorts of choice sequences: lawless, lawlike, and mixed. Clearly, a lawless sequence cannot be a real number generator, since, at any stage, its future values are unconstrained.

Let α be any choice sequence (lawless, lawlike, or mixed). Define a choice sequence α' as follows. We stipulate that for every natural number n, any world in which $\alpha(0), \ldots, \alpha(n)$ are all defined, $\alpha'(0), \ldots, \alpha'(n)$ are also defined. The definition is intended to guarantee that each sequence $\langle \alpha'(0), \ldots \alpha'(n) \rangle$ is SFSG (in all worlds):

> $\alpha'(0) = \alpha(0)$. That is, in any world where $\alpha(0)$ is defined, then, in that world, $\alpha'(0)$ is defined and identical to $\alpha(0)$
>
> Suppose that $\alpha(0), \ldots, \alpha(n), \alpha(n+1)$ are all defined:
>
>> If $\langle \alpha'(0), \ldots, \alpha'(n), \alpha(n{+}1) \rangle$ is SFSG, then $\alpha'(n{+}1) = \alpha(n+1)$.
>> If $\langle \alpha'(0), \ldots, \alpha'(n), \alpha(n{+}1) \rangle$ is not SFSG, then $\alpha'(n+1) = \alpha'(n)$

Informally, here is the recipe for α'. Suppose that, in a given world, $\alpha(0), \ldots,$ $\alpha(n)$ are all defined. To get the corresponding values of α', first set $\alpha'(0) = \alpha(0)$. Then go through the sequence. For each $i + 1 < n$, if the sequence up to (and including) that value is not SFSG, then replace that value with the value at i.

This procedure allows us to associate a real number generator with every choice sequence. Let α be the real number generator associated with a lawless sequence. It is evident that, at each world w, there is a path through w on which α converges to a rational number, and there is a path through w on which α converges to an irrational number.

Recall the "open data" principle for lawless sequences: Suppose that α is a lawless and a certain formula φ holds of α at a given world w. Then there is an initial segment n of α (i.e., $\alpha \in n$) such that if β is any other lawless sequence that agrees with α on n (i.e., if $\beta \in n$), then φ holds of β. It follows that it is impossible to define a functional relation like S on lawless sequences. No initial segment of any lawless α is sufficient to determine whether α' converges to a rational number, and no initial segment α is sufficient to determine whether α' converges to a irrational number. Indeed, we will not even be able to give a one-element sequence b with $S\alpha b$, in any world.

It is a theorem of Brouwerian intuitionistic analysis that every total function defined on all real numbers is continuous, and that every function defined on closed and bounded set of real numbers is uniformly continuous. Our goal is to recapitulate at least the first of these results. By invoking the BK translations of the Brouwerian principles, we plan to show that if a functional relation R is defined on all real number generators, then the function, so defined, is continuous. The idea is that the presence of lawless sequences precludes one from "looking ahead", even in the case of lawlike sequences.

7 Concluding summary

A brief summary may be useful. Potentialism takes mathematical existence to be inherently potential. As we explained, extant analyses of potentialism are based on a translation and an associated mirroring theorem which assumes that all the possibilities in question are convergent. We used the intuitionistic theory of *free choice sequences* to motivate the need for a modal analysis of divergent potentialism as well.

We proceeded to explain the challenge of connecting the ordinary theory of choice sequences with our modal explication. Just as in the case of convergent potentialism, this requires a translation from the ordinary non-modal language into its modal analogue. Our main contribution has been to overcome this challenge. Inspired by the so-called *Beth-Kripke semantics* for intuitionistic logic, we first defined a novel translation from a non-modal language to a corresponding modal language, which adds not only the usual modal operators but also an inevitability operator. Then we showed that this translation provides a faithful interpretation of intuitionistic logic in a modal logic for inevitability, which is a conservative extension of ordinary S4. This novel mirroring theorem promises to do for divergent potentialism what the earlier mirroring theorem did for convergent potentialism.

Finally, to begin to redeem this promise, we applied our modal analysis of divergent potentialism to provide an interpretation of Troelstra's theory of free choice sequences in a classical modal logic. This work has the substantial side benefit of making intuitionistic choice sequences comprehensible in thoroughly classical terms, on the basis of our classical modal logic, including the profoundly non-classical feature that every total function on the real numbers is continuous. We thus demystify this frequently maligned intuitionistic theory.

A Axioms for free S4

Here is a standard axiomatization of free S4 with the Converse Barcan Formula. We are using $x = x$ as an existence predicate.

M0.0 Any instance of a propositional **S4** theorem schema.

M0.1 $(\forall x A x \wedge t = t) \rightarrow At$.

M0.2 $\forall x(A \rightarrow B) \rightarrow (\forall x A \rightarrow \forall x B)$.

M0.3 $A \leftrightarrow \forall x A$, when x is not free in A.

M0.4 $\forall x(x = x)$

M0.5 $(t_1 = t_1 \wedge t_2 = t_2 \wedge t_1 \neq t_2) \rightarrow \Box t_1 \neq t_2$

M0.6 $t_1 = t_2 \rightarrow (At_1 \rightarrow At_2)$

M0.7 $Pt \rightarrow t = t$, for P any atom.

M0.8 $\Box \forall x A \rightarrow \forall x \Box A$.

M0.9 From $\vdash A \rightarrow B$ and $\vdash A$, infer $\vdash B$.

M0.10 From $\vdash A$ infer $\vdash \Box A$.

M0.11 From $\vdash A$ infer $\vdash \forall x A$.

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