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Towards Computing Phase Portrait Objects of Polygonal Hybrid Systems on Surfaces

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Abstract

Polygonal hybrid systems (SPDIs) are a subclass of hybrid systems whose dynamics is defined by constant differential inclusions. We can define SPDIs on surfaces, obtaining a new class of hybrid systems ($\mathrm{SPDI_{2m}}$). In this paper we define and compute various $\mathrm{SPDI_{2m}}$'s phase portrait objects: invariance, controllability and viability kernels and separatrix sets.

1 Introduction

An interesting and still decidable (w.r.t reachability) class of hybrid systems is the so-called Polygonal Hybrid System (SPDI for short, [ASY01]) which is a subclass of hybrid systems on the plane whose dynamics is defined by constant differential inclusions. SPDIs are a generalization of PCDs (deterministic systems with Piece-wise Constant Derivatives) for which it has been shown that the reachability problem is decidable for the planar case [MP93] but undecidable for three dimensions [AMP95]. SPDIs may be defined on surfaces (or two dimensional manifolds) giving rise to a new class of hybrid systems, denoted $\mathrm{SPDI_{2m}}$, for which the reachability problem is an open question [AS02]. One way of providing useful information about the qualitative behavior, including reachability issues, of a hybrid system in general and of $\mathrm{SPDI_{2m}}$ in particular, is through the study of its *phase portrait*. Some works along these lines are [ASY02], [Aub01], [DV95], [KV95], [KdB01], [MS00] and [SJSL00]. In particular, and closely related to this paper, in [MS00] it is shown how to build the phase portrait of PCDs while

in [ASY02] algorithms are given for computing viability and controllability kernels for SPDIs. Moreover, a characterization of viability and invariance kernels was given by [ALQ$^+$01] for impulsive differential inclusions.
An *invariant set* is a set of initial points of trajectories which keep necessarily rotating in a cycle forever. A set is a *viability domain* if for every point in the set, there is at least one trajectory which keep in the set forever. A set such that any two points are reachable one from the other is called *controllable*. Given a cycle, the greatest such sets are called *invariance*, *viability* and *controllability kernels*, respectively. A *separatrix* is a curve which bisects a set into two subsets A and B such that no trajectory starting in A can reach a point in B and vice-versa.
In this paper we give decision procedures for computing the invariance, controllability and viability kernels for $\mathrm{SPDI}_{2\mathrm{m}}$s. Moreover, we define *separatrix sets*, which are closed sets of points dissecting the $\mathrm{SPDI}_{2\mathrm{m}}$ into at least three disjoint sets such that two of them are non-connected w.r.t. reachability. We show how to compute such sets. Even though the computation of all the above $\mathrm{SPDI}_{2\mathrm{m}}$'s phase portrait objects are contributions of this work, we make a qualitative difference between their originality. While the algorithms for computing invariance, controllability and viability kernels for $\mathrm{SPDI}_{2\mathrm{m}}$s are straightforwardly obtained from the given algorithms for the corresponding SPDI's kernels, it is not the case with the computation of separatrix sets. Indeed, the latter have not been computed for SPDIs.
The paper is organized as follows. In next section we give some preliminaries, providing useful notation and definition and recalling the definition of SPDI, $\mathrm{SPDI}_{2\mathrm{m}}$ and of some topological notions needed. In Section 3 we define and compute invariance, controllability and viability kernels for $\mathrm{SPDI}_{2\mathrm{m}}$s while in Section 4 we show how to obtain their separatrix sets. We conclude in the last section.

2 Preliminaries

2.1 SPDI

Let $\mathbf{a} = (a_1, a_2), \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$. The *inner product* of two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{x} = (x_1, x_2)$ is defined as $\mathbf{a} \cdot \mathbf{x} = a_1 x_1 + a_2 x_2$. We denote by $\hat{\mathbf{x}}$ the vector $(x_2, -x_1)$ obtained from $\mathbf{x}$ by rotating clockwise by the angle $\pi/2$. Notice that $\mathbf{x} \cdot \hat{\mathbf{x}} = 0$.
An *angle* $\angle_{\mathbf{a}}^{\mathbf{b}}$ on the plane, defined by two non-zero vectors $\mathbf{a}, \mathbf{b}$ is the set of all positive linear combinations $\mathbf{x} = \alpha\ \mathbf{a} + \beta\ \mathbf{b}$, with $\alpha, \beta \geq 0$, and $\alpha + \beta > 0$. We can always assume that $\mathbf{b}$ is situated in the counter-clockwise direction

from $\mathbf{a}$.

A *polygonal differential inclusion system* (SPDI) is defined by giving a finite partition $\mathbb{P}$ of the plane into convex polygonal sets (called *regions*), and associating with each $P \in \mathbb{P}$ a couple of vectors $\mathbf{a}_P$ and $\mathbf{b}_P$. Let $\phi(P) = \angle_{\mathbf{a}_P}^{\mathbf{b}_P}$, we have that for each $\mathbf{x} \in P$, $\dot{\mathbf{x}} \in \phi(P)$.

Let $E(P)$ be the set of edges of P. We say that $e \in E(P)$ is an *entry* of P if for all $\mathbf{x} \in e$ and for all $\mathbf{c} \in \phi(P)$, $\mathbf{x} + \mathbf{c}\epsilon \in P$ for some $\epsilon > 0$. We say that e is an *exit* of P if the same condition holds for some $\epsilon < 0$. We denote by $In(P) \subseteq E(P)$ the set of all entries of P and by $Out(P) \subseteq E(P)$ the set of all exits of P.

Assumption 1. *All the edges in $E(P)$ are either entries or exits, that is, $E(P) = In(P) \cup Out(P)$.*

A *trajectory segment* of an SPDI is a continuous function $\xi : [0, T] \to \mathbb{R}^2$ which is smooth everywhere except in a discrete set of points, and such that for all $t \in [0, T]$, if $\xi(t) \in P$ and $\dot{\xi}(t)$ is defined then $\dot{\xi}(t) \in \phi(P)$. The *signature*, denoted $\mathsf{Sig}(\xi)$, is the ordered sequence of edges traversed by the trajectory segment, that is, $e_1, e_2, \ldots$, where $\xi(t_i) \in e_i$ and $t_i < t_{i+1}$. If $T = \infty$, a trajectory segment is called a *trajectory*.

Assumption 2. *We will only consider trajectories with infinite signatures.*

2.1.1 Successors and predecessors

Given an SPDI, we fix a one-dimensional coordinate system on each edge to represent points laying on edges. For notational convenience, we will use e to denote both the edge and its one-dimensional representation. Accordingly, we write $\mathbf{x} \in e$ or $x \in e$, to mean "point $\mathbf{x}$ in edge e with coordinate x in the one-dimensional coordinate system of e". The same convention is applied to sets of points of e represented as intervals (e.g., $\mathbf{x} \in I$ or $x \in I$, where $I \subseteq e$) and to trajectories (e.g., "ξ starting in x" or "ξ starting in $\mathbf{x}$").

Now, let $P \in \mathbb{P}$, $e \in In(P)$ and $e' \in Out(P)$. For $I \subseteq e$, $\mathsf{Succ}_{ee'}(I)$ is the set of all points in e' reachable from some point in I by a trajectory segment $\xi : [0, t] \to \mathbb{R}^2$ in P (i.e., $\xi(0) \in I \wedge \xi(t) \in e' \wedge \mathsf{Sig}(\xi) = ee'$). Given $I = [l, u]$, $\mathsf{Succ}_{ee'}(I) = F(I \cap S) \cap J$, where S and J are intervals, $F([l, u]) = \langle f_l(l), f_u(u) \rangle$ and f_l and f_u are affine functions (a function $f : \mathbb{R} \to \mathbb{R}$ is affine iff $f(x) = ax + b$ with $a > 0$).

For $I \subseteq e'$, $\mathsf{Pre}_{ee'}(I)$ is the set of points in e that can reach a point in I by a trajectory segment in P. We have that: $\mathsf{Pre}_{ee'} = \mathsf{Succ}_{ee'}^{-1}$ and $\mathsf{Pre}_{\sigma} = \mathsf{Succ}_{\sigma}^{-1}$.

2.1.2 Qualitative analysis of simple edge-cycles

Let $\sigma = e_1 \cdots e_k e_1$ be a simple edge-cycle, i.e., $e_i \neq e_j$ for all $1 \leq i \neq j \leq k$. Let $\mathsf{Succ}_\sigma(I) = F(I \cap S) \cap J$ with $F = \langle f_l, f_u \rangle$.

Assumption 3. *None of the two functions f_l, f_u is the identity.*

Let l^* and u^* be the fix-points[1] of f_l and f_u, respectively, and $S \cap J = \langle L, U \rangle$. It can be shown that a simple cycle is of one of the following types:

STAY. The cycle is not abandoned neither by the leftmost nor the rightmost trajectory, that is, $L \leq l^* \leq u^* \leq U$.

DIE. The rightmost trajectory exits the cycle through the left (consequently the leftmost one also exits) or the leftmost trajectory exits the cycle through the right (consequently the rightmost one also exits), that is, $u^* < L \vee l^* > U$.

EXIT-BOTH. The leftmost trajectory exits the cycle through the left and the rightmost one through the right, that is, $l^* < L \wedge u^* > U$.

EXIT-LEFT. The leftmost trajectory exits the cycle (through the left) but the rightmost one stays inside, that is, $l^* < L \leq u^* \leq U$.

EXIT-RIGHT. The rightmost trajectory exits the cycle (through the right) but the leftmost one stays inside, that is, $L \leq l^* \leq U < u^*$.

The classification above gives us some information about the qualitative behavior of trajectories. Any trajectory that enters a cycle of type DIE will eventually quit it after a finite number of turns. If the cycle is of type STAY, all trajectories that happen to enter it will keep turning inside it forever. In all other cases, some trajectories will turn for a while and then exit, and others will continue turning forever. This information is very useful for solving the reachability problem for SPDIs.
The above result does not allow us to directly answer other questions about the behavior of the SPDI such as determine for a given point (or set of points) whether any trajectory (if it exists) starting in the point remains in the cycle forever. In order to do this, we need to further study the properties of the system around simple edge-cycles and in particular STAY cycles. See [Sch04] for some important properties of STAY cycles.
A more detailed presentation of SPDIs and their properties may be found in [ASY01] and [Sch02].

[1] The fix-point x^* is computed by solving a linear equation $f(x^*) = x^*$, which can be finite or infinite.

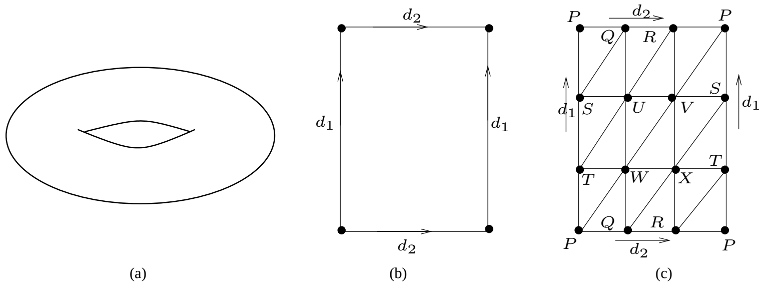

Figure 1: Representations of a Torus: (a) a surface in $\mathbb{R}^3$; (b) a square with identified edges; (c) a triangulated surface.

2.2 Surfaces (Two Dimensional Manifolds)

All the (topological) definitions, examples and results of this section follow the *combinatorial method*, based on [Hen79].
A topological space is *triangulable* if it can be obtained from a set of triangles by the identification of edges and vertexes subject to the restriction that any two triangles are identified either along a single edge or at a single vertex, or are completely disjoint. The identification should be done via an affine bijection.
A *surface* (or *2-dim manifold*) is a triangulable space for which in addition: (1) each edge is identified with exactly one other edge; and (2) the triangles identified at each vertex can always be arranged in a cycle $T_1, \dots, T_k, T_1$ so that adjacent triangles are identified along an edge. Typical examples are the torus (see Fig. 1), the sphere, the Klein bottle and the projective plane (see Fig. 2).
A *surface with boundary* is a topological space obtained by identifying edges and vertexes of a set of triangles as for surfaces except that certain edges may not be identified with another edge. These edges, which violate the definition of a surface, are called *boundary edges*, and their vertexes, which also violate the definition of surface, are called *boundary vertexes*. Typical examples of surfaces with boundary are the cylinder and the Möbi's strip. Indeed, the cylinder is equivalent to a sphere with two disks cut out.
We state now an important theorem in the topological theory of surfaces ([Hen79, p.122]; see also [Xu01]):

Theorem 1 (Classification theorem). • *Every compact, connected surface is topologically equivalent to a sphere, or a connected sum of tori*[2]*,*

[2] The connected sum construction connects two surfaces with a tube (after cutting out holes in the surfaces where the tubes are attached).

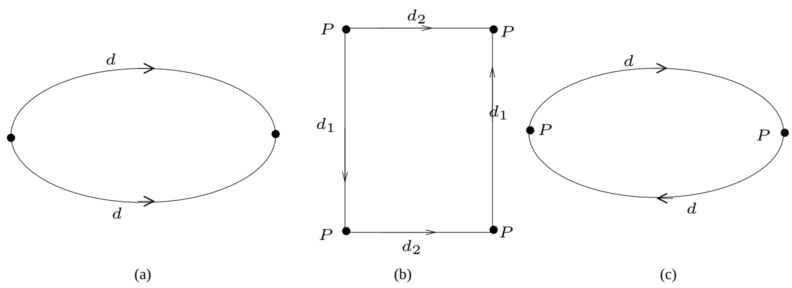

Figure 2: Planar representations: (a) sphere; (b) Klein bottle; (c) projective plane.

or a connected sum of projective planes.

- *Every compact, connected surface with boundary is equivalent to either a sphere, or a connected sum of tori, or a connected sum of projective planes, in any case with some finite number of disks removed.* □

The sphere and a connected sum of tori are called *orientable*, while the (connected sum of) projective planes are *unorientable* surfaces.

Example 1. *The Klein bottle (2-(b)) is the connected sum of two projective planes while the connected sum of two Möbius Strip is a cylinder.* □

When representing a surface in a plane (as in Fig. 1-(c)), some identified edges (vertexes) may be put together while others need to be identified through their name and their orientation (in the case of edges). In Fig. 1-(c), vertex U, V, W and X, as well as the edges they define, are unique and trivially identified (with themselves). However, S, T and Q, R are identified according to the orientation of d_1 and d_2 respectively. We call such edges and vertexes, *directed edges* and *directed vertexes* respectively.
Even though our result can be extended to surfaces with boundaries, we will restrict our analysis only to surfaces without boundaries.

Assumption 4. *We will consider only surfaces without boundaries.*

2.3 Jordan curve theorem for surfaces

By the Classification Theorem we know that it suffices to compute the phase portrait objects for a sphere, a connected sum of tori and a connected sum of projective planes.

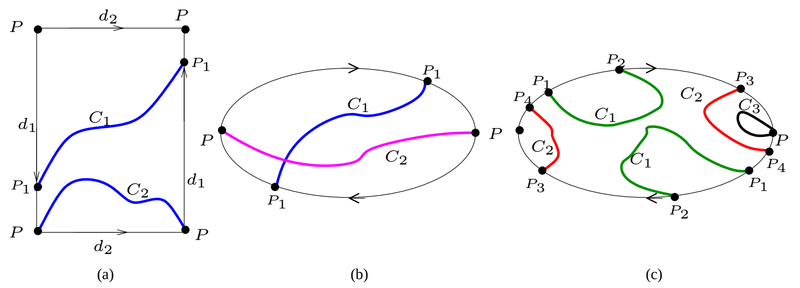

Figure 3: (a) Disjoint closed curves which are not Jordan curves on a Klein bottle; (b) Non-Jordan curves on a projective plane; (c) Jordan curves on a projective plane.

Before showing how to compute the kernels and separatrix sets, we recall here some needed definitions and results. We recall first the *Jordan Curve Theorem* in an informal way: "A simple closed curve in the plane divides the plane in exactly two parts, one bounded (the inside) and one unbounded (the outside). Furthermore the curve is the complete frontier of both parts". Notice that the Jordan curve theorem for the plane holds for the sphere. However, the theorem is not true for the other closed surfaces: there are simple closed curves which do not disconnect the surface. The appropriate generalization of the Jordan curve theorem for arbitrary closed surfaces is given below. It is stated in terms of *genus* of a surface, a concept which we define as follows: A sphere is defined to have genus 0, the connected sum of g tori is defined to have genus g and the connected sum of g projective planes is defined to have genus $g-1$ [Fie].

Theorem 2 (Jordan Curve Theorem for Surfaces)**.** *The maximum number of disjoint simple closed curves which can be cut from an orientable surface of genus g without disconnecting it is g. The maximum number of disjoint simple closed curves which can be cut from an unorientable surface of genus g without disconnecting it is $g+1$.* □

Thus, for a sphere, every closed curve disconnect it, whereas not every closed curve disconnects a torus or a projective plane; we may need two closed curves. Closed curves disconnecting a surface are called *Jordan curves.*

Example 2. *Fig. 3-(a) depicts a Klein bottle with two typical disjoint closed curves which are not Jordan curves. In Fig. 3-(b) none of C_1 nor C_2 are Jordan curves on a projective plane. In Fig. 3-(c) all of C_1, C_2 and C_3 are Jordan curves. In Fig. 4, none of the curves C_1 nor C_2 are Jordan*

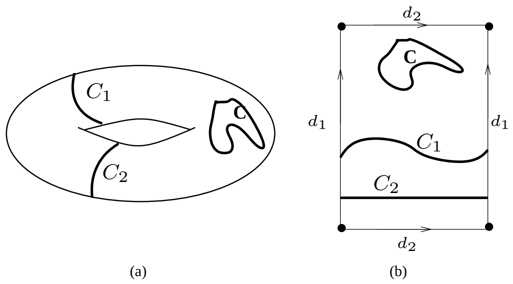

Figure 4: Jordan curves on the torus: (a) 3-dim representation; (b) Planar representation.

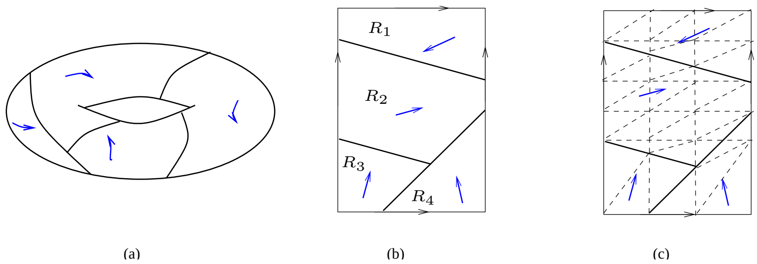

Figure 5: A $\text{SPDI}_{2\text{m}}$ on the torus: three views.

curves. However, the set $\{C_1, C_2\}$ (as well as the curve C) disconnects the surface. □

Jordan curves for surfaces may be characterized using the notion of *linking number* and *homology* cycles. For our purposes, it suffices to know which closed curves are (not) Jordan curves w.r.t. some concept related to the definition of hybrid systems (i.e. the curves containing points of a *directed edge*; see next Section).

2.4 $\text{SPDI}_{2\text{m}}$: SPDIs on Surfaces

To define an SPDI on a triangulated surface $\mathcal{M}$, an SPDI should be defined on each of its triangles. We call this class of systems *SPDI on surfaces* ($\text{SPDI}_{2\text{m}}$).
In Fig. 5 we define an SPDI on a torus and show how to represent it as a family of SPDIs on triangles.
The notion of successor, predecessor as well as the classification of simple cycles given for SPDIs in Section 2.1 hold for $\text{SPDI}_{2\text{m}}$s. One difference between simple cycles of both hybrid systems is that in $\text{SPDI}_{2\text{m}}$s they may have

the occurrence of directed edges. This fact does not change the definition of simple cycle nor the above classification, however, it has a decisive influence on the decidability of the reachability problem.
When defining an SPDI on a surface, directed edges are partitioned into intervals, each corresponding to a different region. We will call the directed edges of the original surface $\mathcal{M}$-*directed edges* while the term "directed edge" will be used for the $\mathrm{SPDI}_{2\mathrm{m}}$s' ones. Each directed edge e is a subinterval of only one $\mathcal{M}$-directed edge d and this is denoted by $e \subseteq d$.

3 Kernels Computation

We state here how to compute the invariance, controllability and viability kernels. However, proofs are omitted since they are similar as for SPDIs, with the additional feature that simple cycles in a $\mathrm{SPDI}_{2\mathrm{m}}$ may contain directed edges[3]. The details of proofs for computing viability and controllability kernels of SPDIs can be found in [ASY02] and in [Sch04] for invariance kernels. In what follows, let K be a subset of a surface $\mathcal{M}$ and given a cyclic signature σ, let K_σ be defined as follows:

$$K_\sigma = \bigcup_{i=1}^{k} (\mathtt{int}(P_i) \cup e_i) \tag{1}$$

where P_i is such that $e_{i-1} \in \mathit{In}(P_i)$, $e_i \in \mathit{Out}(P_i)$ and $\mathtt{int}(P_i)$ is P_i's interior.

3.1 Viability Kernel

We recall the definition of *viability kernel*.

Definition 1. *A trajectory ξ is* viable *in K if $\xi(t) \in K$ for all $t \geq 0$. K is a* viability domain *if for every $\mathbf{x} \in K$, there exists at least one trajectory ξ, with $\xi(0) = \mathbf{x}$, which is viable in K. The* viability kernel *of K, denoted* $\mathsf{Viab}(K)$*, is the largest viability domain contained in K.*

For $I \subseteq e_1$ let us define $\overline{\mathsf{Pre}}_\sigma(I)$ as the set of all $\mathbf{x} \in \mathcal{M}$ for which there exists a trajectory segment ξ starting in $\mathbf{x}$, that reaches some point in I, such that $\mathsf{Sig}(\xi)$ is a suffix of $e_2 \ldots e_k e_1$. It is easy to see that $\overline{\mathsf{Pre}}_\sigma(I)$ is a polygonal subset of the plane which can be calculated using the following procedure. First define

[3] Indeed, the only difference in the proof is that sets, points, etc are defined on $\mathcal{M}$ instead of on $\mathbb{R}^2$.

$$\overline{\mathsf{Pre}}_e(I) \quad = \quad \{\mathbf{x} \mid \exists \xi : [0,t] \to \mathcal{M}, t > 0 \; . \; \xi(0) = \mathbf{x} \wedge \xi(t) \in I \wedge \mathsf{Sig}(\xi) = e\}$$

and apply this operation k times: $\overline{\mathsf{Pre}}_\sigma(I) = \bigcup_{i=1}^{k} \overline{\mathsf{Pre}}_{e_i}(I_i)$ with $I_1 = I$, $I_k = \mathsf{Pre}_{e_k,e_1}(I_1)$ and $I_i = \mathsf{Pre}_{e_i,e_{i+1}}(I_{i+1})$, for $2 \leq i \leq k-1$.
The following result provides a non-iterative algorithmic procedure for computing the viability kernel of K_σ of an SPDI$_{\mathrm{2m}}$.

Theorem 3. *If σ is not DIE,* $\mathsf{Viab}(K_\sigma) = \overline{\mathsf{Pre}}_\sigma(S)$, *otherwise* $\mathsf{Viab}(K_\sigma) = \emptyset$. □

3.2 Controllability Kernel

We say K is *controllable* if for any two points $\mathbf{x}$ and $\mathbf{y}$ in K there exists a trajectory segment ξ starting in $\mathbf{x}$ that reaches an arbitrarily small neighborhood of $\mathbf{y}$ without leaving K. More formally,

Definition 2. *A set K is controllable iff* $\forall \mathbf{x}, \mathbf{y} \in K, \forall \delta > 0, \exists \xi : [0,t] \to \mathcal{M}, t > 0 \; . \; (\xi(0) = \mathbf{x} \wedge |\xi(t) - \mathbf{y}| < \delta \wedge \forall t' \in [0,t] \; . \; \xi(t') \in K)$. *The* controllability kernel *of K, denoted* $\mathsf{Cntr}(K)$, *is the largest controllable subset of K.*

For a given cyclic signature σ, let us define $\mathcal{C}_{\mathcal{D}}(\sigma)$ as follows:

$$\mathcal{C}_{\mathcal{D}}(\sigma) = \begin{cases} \langle L, U \rangle & \text{if } \sigma \text{ is EXIT-BOTH} \\ \langle L, u^* \rangle & \text{if } \sigma \text{ is EXIT-LEFT} \\ \langle l^*, U \rangle & \text{if } \sigma \text{ is EXIT-RIGHT} \\ \langle l^*, u^* \rangle & \text{if } \sigma \text{ is STAY} \\ \emptyset & \text{if } \sigma \text{ is DIE} \end{cases} \tag{2}$$

For $I \subseteq e_1$ let us define $\overline{\mathsf{Succ}}_\sigma(I)$ as the set of all points $\mathbf{y} \in \mathcal{M}$ for which there exists a trajectory segment ξ starting in some point $x \in I$, that reaches $\mathbf{y}$, such that $\mathsf{Sig}(\xi)$ is a prefix of $e_1 \ldots e_k$. The successor $\overline{\mathsf{Succ}}_\sigma(I)$ is a polygonal subset of the plane which can be computed similarly to $\overline{\mathsf{Pre}}_\sigma(I)$. Define

$$\mathcal{C}(\sigma) = (\overline{\mathsf{Succ}}_\sigma \cap \overline{\mathsf{Pre}}_\sigma)(\mathcal{C}_{\mathcal{D}}(\sigma))$$

We compute the controllability kernel of K_σ as follows.

Theorem 4. $\mathsf{Cntr}(K_\sigma) = \mathcal{C}(\sigma)$. □

3.3 Invariance Kernel

In general, an *invariant set* is a set of points such that for any point in the set, every trajectory starting in such point remains in the set forever and the *invariance kernel* is the largest of such sets. In particular, for SPDI, given a cyclic signature, an *invariant set* is a set of points which keep rotating in the cycle forever and the *invariance kernel* is the largest of such sets. More formally,

Definition 3. *We say that a set K is* invariant *iff for any $x \in K$ there exists at least one trajectory starting in it and every trajectory starting in x is viable in K. Given a set K, its largest invariant subset is called the* invariance kernel *of K and is denoted by* $\mathsf{Inv}(K_\sigma)$.

We need some preliminary definitions before stating the main theorem. The *extended* $\forall$*-predecessor* of an output edge e of a region R is the set of points in R such that every trajectory segment starting in such point reaches e without traversing any other edge. More formally, let R be a region and e be an edge in $Out(R)$, then the *e-extended* $\forall$*-predecessor* of I, $\widetilde{\overline{\mathsf{Pre}}}_e(I)$ is defined as:

$$\widetilde{\overline{\mathsf{Pre}}}_e(I) = \{\mathbf{x} \mid \forall \xi \ . \ (\xi(0) = \mathbf{x} \Rightarrow \exists t \geq 0 \ . \ (\xi(t) \in I \wedge \mathsf{Sig}(\xi[0,t]) = e))\}.$$

It is easy to see that $\widetilde{\overline{\mathsf{Pre}}}_\sigma(I)$ is a polygonal subset of the plane which can be calculated using the following procedure. First compute $\widetilde{\overline{\mathsf{Pre}}}_{e_i}(I)$ for all $1 \leq i \leq k$ and then apply this operation k times: $\widetilde{\overline{\mathsf{Pre}}}_\sigma(I) = \bigcup_{i=1}^{k} \widetilde{\overline{\mathsf{Pre}}}_{e_i}(I_i)$ with $I_1 = I$, $I_k = \widetilde{\mathsf{Pre}}_{e_k e_1}(I_1)$ and $I_i = \widetilde{\mathsf{Pre}}_{e_i e_{i+1}}(I_{i+1})$, for $2 \leq i \leq k-1$. We compute the invariance kernel of K_σ as follows.

Theorem 5. *If $\sigma = e_1 \ldots e_n e_1$ is STAY then* $\mathsf{Inv}(K_\sigma) = \widetilde{\overline{\mathsf{Pre}}}_\sigma(\widetilde{\mathsf{Pre}}_\sigma(J))$, *otherwise* $\mathsf{Inv}(K_\sigma) = \emptyset$. $\square$

4 Separatrix Sets Computation

Let $\mathcal{M}$ be a surface with a dynamics ϕ defined on it. In this section we define the notion of *separatrix sets*, which are subsets of $\mathcal{M}$ dissecting the surface into two mutually non-reachable subsets. We relax the notion of separatrix obtaining *semi-separatrix sets* such that some points in one set may be reachable from the other set, but not vice-versa.
We define first the above notions for surfaces, independently of SPDI_{2m}s.

Definition 4. *Let $K \subseteq \mathcal{M}$, a* separatrix *in K is a curve γ partitioning K into three sets K_A, K_B and γ itself, such that $K_A \cap K_B \cap \gamma = \emptyset$, $K = K_A \cup K_B \cup \gamma$ and the following conditions hold:*

1. *For any point $x_0 \in K_A$ and trajectory ξ, with $\xi(0) = x_0$, there is no t such that $\xi(t) \in K_B$; and*

2. *For any point $x_0 \in K_B$ and trajectory ξ, with $\xi(0) = x_0$, there is no t such that $\xi(t) \in K_A$.*

If only one of the above conditions holds then we say that the curve is a semi-separatrix.

We can extend the above notion to sets. A *separatrix set* S of K is a set of closed subsets S_i (with $1 \leq i \leq 2$) of K with the above separation property. We will denote by K_A and K_B the two subsets of K defined by a separatrix set S. The set of all the separatrix sets of a surface $\mathcal{M}$ is denoted by $\mathsf{Sep}(\mathcal{M})$, or simply Sep if $\mathcal{M}$ is understood from the context.
Notice that in some cases a separatrix set contains only one set or curve while in other cases, two are needed, which follows directly from the Jordan curve theorem for surfaces.
The above notions are extended to $\text{SPDI}_{2\text{m}}$s straightforwardly.
Now, let $\sigma = e_1 \dots e_n e_1$ be a simple cycle, $\angle_{\mathbf{a}_i}^{\mathbf{b}_i}$ $(1 \leq i \leq n)$ be the dynamics of the regions for which e_i is an entry edge and $I = [l, u]$ and interval on edge e_1. Remember that $\mathsf{Succ}_{e_1 e_2}(I) = F(I \cap S) \cap J$, where $F = [a_1 l + b_1, a_2 u + b_2]$. Let $\mathbf{l}$ be the vector corresponding to the point on e_1 with local coordinates l and $\mathbf{l}'$ be the vector corresponding to the point on e_2 with local coordinates $F(l)$ (similarly, we define $\mathbf{u}$ and $\mathbf{u}'$ for $F(u)$). We define first $\overline{\mathsf{Succ}}_{e_1}^{\mathbf{b}_1}(I) = \{\mathbf{x} \mid \mathbf{l}' = \alpha\mathbf{x} + \mathbf{l}, 0 < \alpha < 1\}$ and $\overline{\mathsf{Succ}}_{e_1}^{\mathbf{a}_1}(I) = \{\mathbf{x} \mid \mathbf{u}' = \alpha\mathbf{x} + \mathbf{u}, 0 < \alpha < 1\}$. We extend these definitions in a straight way to any (cyclic) signature $\sigma = e_1 \dots e_n e_1$, denoting them by $\overline{\mathsf{Succ}}_{\sigma}^{\mathbf{b}}(I)$ and $\overline{\mathsf{Succ}}_{\sigma}^{\mathbf{a}}(I)$, respectively; we can compute them similarly as for $\overline{\mathsf{Pre}}$. Whenever applied to the fix-point $I^* = [l^*, u^*]$, we denote $\overline{\mathsf{Succ}}_{\sigma}^{\mathbf{b}}(I^*)$ and $\overline{\mathsf{Succ}}_{\sigma}^{\mathbf{a}}(I^*)$ by ξ_σ^l and ξ_σ^u respectively. Intuitively, ξ_σ^l (ξ_σ^u) denotes the piece-wise affine closed curve defined by the leftmost (rightmost) fix-point l^* (u^*). The *inner* of a simple cycle σ is defined as follows: if σ is STAY, then the inner of σ is the set defined by the (possible non-convex) polygon delimited by ξ_σ^l and ξ_σ^u; if σ is EXIT-LEFT, then the inner of σ is the set defined by the non-convex polygon delimited by $\overline{\mathsf{Succ}}_{\sigma}^{\mathbf{b}}([L, U])$ and ξ_σ^u; if σ is EXIT-RIGHT, then the inner of σ is the set defined by the non-convex polygon delimited by ξ_σ^l and $\overline{\mathsf{Succ}}_{\sigma}^{\mathbf{a}}([L, U])$; otherwise, the inner of σ is empty.

Notice that the inner of a simple cycle is non-empty only for those cycles for which at least one of the leftmost and rightmost trajectory limit is in $[L, U]$. We show now how to identify separatrix sets for simple cycles not involving directed edges.

Theorem 6. *Let $\mathcal{M}$ be an* $\text{SPDI}_{2\text{m}}$ *and $\sigma = e_1 \dots e_n e_1$ be a simple cycle not involving directed edges, then the following hold:*

1. *If σ is EXIT-RIGHT then $\{\xi^l_\sigma\}$ is a semi-separatrix set (filtering trajectories from "left" to "right");*

2. *If σ is EXIT-LEFT then $\{\xi^u_\sigma\}$ is a semi-separatrix set (filtering trajectories from "right" to "left");*

3. *If σ is STAY, then set containing the invariance kernel $\mathsf{Inv}(K_\sigma)$ is a separatrix set, i.e. $\{\mathsf{Inv}(K_\sigma)\} \in \mathsf{Sep}$.*

Proof. Notice that by hypothesis, there is no directed edge on σ which means that the reasoning may be conducted as for the planar case.

1. By definition of EXIT-RIGHT, any trajectory is bounded to the left by ξ^l_σ, which is a piece-wise affine closed curve, partitioning $\mathcal{M}$ into three disjoint sets: K_B, the "right" part of ξ^l_σ (i.e. the subset containing the inner of σ); K_A, the "left" part of ξ^l_σ; and ξ^l_σ itself. By Jordan's theorem, any trajectory may pass from K_B to K_A if and only if it cross ξ^l_σ. However, by definition of EXIT-RIGHT, this is only possible from K_A to K_B but not vice-versa. Hence $\{\xi^l_\sigma\}$ is a semi-separatrix set.

2. Symmetric to the previous case.

3. Follows directly from the definition of invariance kernel, since any trajectory arriving to it from the left cannot leave $\mathsf{Inv}(K_\sigma)$ and hence no point on its right can be reached. Similarly for trajectories entering $\mathsf{Inv}(K_\sigma)$ from the right, no point on the left of $\mathsf{Inv}(K_\sigma)$ may be reached. □

□

Notice that in the above result, computing a (semi-) separatrix set depends only on one simple cycle, and the corresponding algorithm is then reduced to find simple cycles in the $\text{SPDI}_{2\text{m}}$ and checking whether it is STAY, EXIT-RIGHT or EXIT-LEFT. In fact, by the Jordan curve theorem for surfaces, the above result holds also for any surface topologically equivalent to a sphere:

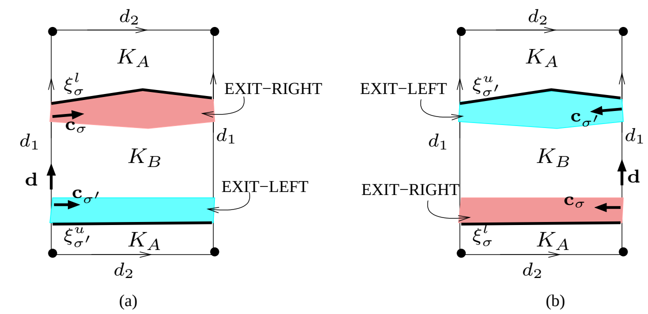

Figure 6: EXIT-LEFT and EXIT-RIGHT cycles with $\mathsf{Sg}(\mathbf{d}.\widehat{\mathbf{c}_\sigma}) = \mathsf{Sg}(\mathbf{d}.\widehat{\mathbf{c}_{\sigma'}})$.

Theorem 7. *Let $\mathcal{M}$ be an SPDI defined on a (surface topologically equivalent to a) sphere and σ be a simple cycle, then conditions 1 to 3 of Theorem 6 hold.*

Given a signature $\sigma = e_1 \dots e_n e_1$, we denote by Dir_σ the set of $\mathcal{M}$-directed edges d such that there exists a directed edge $e_i \subseteq d$ $(1 \le i \le n)$ in σ, and by $NDir_\sigma$ the set of edges $e_i \in \sigma$ but such that $e_i \not\subseteq d$ for any $d \in Dir_\sigma$. For each region P such that there is a directed edge $e_i \subseteq d$ with $e_i \in In(P)$ and $\phi(P) = \angle_{\mathbf{a}_P}^{\mathbf{b}_P}$, $\mathbf{c}_\sigma$ will denote the vector $\mathbf{a}_P + \mathbf{b}_P$; let $\mathbf{d}$ be the director vector of d and $\mathsf{Sg}(\cdot)$ be the usual sign function. The following theorem gives sufficient conditions for obtaining separatrix sets for cycles involving directed edges for SPDIs defined on a connected sum of tori or any topologically equivalent surface to a connected sum of tori.

Theorem 8. *Let $\mathcal{M}$ be an* $\mathrm{SPDI}_{2\mathrm{m}}$ *defined on a (topologically equivalent surface to a) connected sum of tori and let $\sigma = e_1 \dots e_n e_1$ and $\sigma' = e'_1 \dots e'_m e'_1$ be two simple cycles containing one or more directed edges. Let Dir_σ and $Dir_{\sigma'}$ be the sets of $\mathcal{M}$-directed edges of σ and σ' respectively. If $Dir_\sigma = Dir_{\sigma'}$ and $NDir_\sigma \cap NDir_{\sigma'} = \emptyset$, then the following hold:*

1. *If σ is EXIT-LEFT and σ' is EXIT-RIGHT and $\mathsf{Sg}(\mathbf{d}.\widehat{\mathbf{c}_\sigma}) = \mathsf{Sg}(\mathbf{d}.\widehat{\mathbf{c}_{\sigma'}})$, then $\{\xi^l_\sigma, \xi^u_{\sigma'}\}$ is a semi-separatrix set;*

2. *If σ is EXIT-LEFT and σ' is EXIT-LEFT and $\mathsf{Sg}(\mathbf{d}.\widehat{\mathbf{c}_\sigma}) \neq \mathsf{Sg}(\mathbf{d}.\widehat{\mathbf{c}_{\sigma'}})$, then $\{\xi^l_\sigma, \xi^u_{\sigma'}\}$ is a semi-separatrix set;*

3. *If σ is EXIT-RIGHT and σ' is EXIT-RIGHT and $\mathsf{Sg}(\mathbf{d}.\widehat{\mathbf{c}_\sigma}) \neq \mathsf{Sg}(\mathbf{d}.\widehat{\mathbf{c}_{\sigma'}})$, then $\{\xi^l_\sigma, \xi^u_{\sigma'}\}$ is a semi-separatrix set;*

4. *If σ is EXIT-RIGHT and σ' is STAY, then $\{\xi^l_\sigma, \mathsf{Inv}(K_{\sigma'})\}$ is a semi-separatrix set;*

5. *If σ is EXIT-LEFT and σ' is STAY, then $\{\xi^u_\sigma, \mathsf{Inv}(K_{\sigma'})\}$ is a semi-separatrix set;*

6. *If σ and σ' are STAY cycles, then $\{\mathsf{Inv}(K_\sigma), \mathsf{Inv}(K_{\sigma'})\} \in \mathsf{Sep}$.*

Proof. All the above cases follow directly from the definition of each different kind of simple cycle and the characterization of Jordan curves for surfaces; all the above pairs of closed curves are Jordan curves. For lack of space we will prove here only the case 1 for a torus, the other cases being similar.

1. The hypothesis $\mathsf{Sg}(\mathbf{d}.\widehat{\mathbf{c}_\sigma}) = \mathsf{Sg}(\mathbf{d}.\widehat{\mathbf{c}_{\sigma'}})$ guarantees that the inner parts of σ and σ' lie in a same subset of $\mathcal{M}$ delimited by ξ^l_σ and $\xi^u_{\sigma'}$ (see Fig. 6 for examples of such cycles). Let K_B be the open set delimited by the closed curves ξ^l_σ and $\xi^u_{\sigma'}$ and containing the inner part of σ and σ'. Let $K_A = \mathcal{M} \setminus (\{\xi^l_\sigma, \xi^u_{\sigma'}\} \cup K_B)$ (hence, $K_A \cap K_B = \emptyset$). We prove that for any trajectory ξ, with $\xi(0) \in K_B$, $\forall t > 0 \cdot \xi(t) \notin K_A$. Considering the planar representation, by the Jordan curve theorem, the only way to leave K_B is traversing one of the directed edges, ξ^l_σ or $\xi^u_{\sigma'}$. By definition of K_B, EXIT-LEFT and EXIT-RIGHT, for any $t' \geq 0$, $\xi(t) \notin \xi^l_\sigma$ and $\xi(t) \notin \xi^u_{\sigma'}$, which implies that $\forall t > 0 \cdot \xi(t) \in K_B$; thus, $\forall t > 0 \cdot \xi(t) \notin K_A$.

The proof may be generalized to a connected sum of tori. □ □

Notice that for cycles involving directed edges on projective planes the case is slightly different, due to their "twisted" nature. It is possible, however, to give a similar result as the previous theorem, taking into account the closed curves which are not Jordan curves. Notice that in a projective plane it is not possible to draw two (or more) disjoint closed curves containing one directed edge.

Theorem 9. *Let $\mathcal{M}$ be an* $\mathrm{SPDI}_{2\mathrm{m}}$ *defined on a (topologically equivalent surface to a) projective plane and let $\sigma = e_1 \ldots e_n e_1$ be a simple cycle containing at least two different directed edges, then the following hold:*

1. *If σ is EXIT-RIGHT then $\{\xi^l_\sigma\}$ is a semi-separatrix set (filtering trajectories from "left" to "right");*

2. *If σ is EXIT-LEFT then $\{\xi^u_\sigma\}$ is a semi-separatrix set (filtering trajectories from "right" to "left");*

3. *If σ is STAY, then set containing the invariance kernel $\mathsf{Inv}(K_\sigma)$ is a separatrix set, i.e. $\{\mathsf{Inv}(K_\sigma)\} \in \mathsf{Sep}$.*

Proof. The hypothesis of σ containing at least two different directed edges, excludes all the closed curves which are not Jordan curves (as C_1 and C_2 in Fig. 3-(b)). We prove here only the first case, the others being similar.

1. Let K_B be the open set delimited by the closed curve ξ^l_σ and containing the inner part of σ. Let $K_A = \mathcal{M} \setminus (\{\xi^l_\sigma\} \cup K_B)$ (hence, $K_A \cap K_B = \emptyset$). We prove that for any trajectory ξ, with $\xi(0) \in K_B$, $\forall t > 0 \cdot \xi(t) \notin K_A$. Considering the planar representation, by the Jordan curve theorem, the only way to leave K_B is traversing one of the directed edges or ξ^l_σ. By definition of K_B and EXIT-RIGHT, for any $t' \geq 0$, $\xi(t) \notin \xi^l_\sigma$, which implies that $\forall t > 0 \cdot \xi(t) \in K_B$; thus, $\forall t > 0 \cdot \xi(t) \notin K_A$. □

□

Remark. Notice that the restriction of containing at least two different directed edges in the statement of Theorem 9 is to avoid Jordan curves like C_1 and C_2 in Fig. 3-(b). However, this shows that we are not able to distinguish a curve like C_3 in Fig. 3-(c) (which could define a separatrix set) from the closed curve C_2 in Fig. 3-(b) (which cannot define a separatrix set). Our result is thus correct but not complete (we do not identify all the separatrix sets).
An algorithm for computing (semi-) separatrix sets for SPDIs defined on surfaces follows from the above theorems.

5 Final Discussion

We have given an automatic procedure to obtain all the viability, controllability and invariance kernels of simple cycles of polygonal differential inclusion systems defined over surfaces ($\mathrm{SPDI}_{2\mathrm{m}}$). We have also provided an algorithm for computing separatrix sets for such systems. While the computation of the above-mentioned kernels is parameterized by a single simple cycle, it is not the case for separatrix sets. For the latter we could need two simple cycles sharing exactly the same directed edges and disjoint on the non-directed edges. In all the cases the algorithms given depend only on the computation of the fix-points of simple cycles and all the "technology" for obtaining such objects is based on the analysis of SPDIs [ASY01, Sch02, Sch04].
We have here only computed separatrix sets for surfaces topologically equivalent to spheres, projective planes and connected sum of tori. We have not characterized such sets for connected sum of projective planes; we believe this may be done but probably making use of more complex topological notions that the ones used in this work. Moreover, we have given only sufficient conditions for computing the separatrix sets of projective planes, which

means we are not able to compute all the separatrix sets (see last Remark on previous Section). For providing also necessary conditions for detecting separatrix sets we need to give a better characterization of Jordan curves on such surfaces. This could be given, for instance, taking into account the *improper* points (i.e. points on a $\mathcal{M}$-directed edge) and considering the form of trajectories on their neighborhood.
The decidability of the reachability problem for $\mathrm{SPDI}_{2\mathrm{m}}$s is an open question [AS02]. However the result of this work may be further explored to give partial (or semi-) decision procedures for solving the reachability problem for $\mathrm{SPDI}_{2\mathrm{m}}$s. The assumption of considering surfaces without boundary is not a restriction of our result, but was introduced to simplify the presentation.

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