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Optimal liquidation of market moving portfolios

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

This Master's thesis addresses how to optimise market moving portfolio liquidation through a theoretical approach to liquidity cost optimisation with an exponential bounce-back model in discrete time. We use numerical methods such as Fixed Point method and Gradient Ascent method, as well as analytical methods like Lagrange Multiplier method and more general mathematical theory. Moreover, we evaluate the investor's investment choices when managing a market moving portfolio with respect to liquidity, market value, sales value, liquidity cost and liquidity cost ratio. A normalised sales value model is used to address liquidation in one, two and three dimensions, where the dimensions represent the number of liquidation steps the investor can utilise in order to liquidate the entire portfolio. The thesis explores investment strategy spaces such as buy, sell and hold, as well as pumping, dumping and short selling. The model and market are investigated from analytical, numerical and naive perspectives based on mathematical and financial theory. On this note, we find that the liquidation cost ratio is constant in one dimension; that the naive analysis seems to be severely insufficient compared to an analysis using numerical methods; and that the sequence of market types does not commute, i.e., a BullFlat market does not behave equal to a FlatBull market. Moreover, we shed light on some perspectives as to why most struggle to read financial markets correctly.

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CHAPTER 1

Introduction

This thesis investigates how an investor should liquidate a market moving portfolio part by part in order to reduce the liquidity cost C. That is, to maximise the sales value S given certain constraints. A part of a portfolio will be denominated as a *block*. The liquidation of a portfolio will be studied through a sales value model S. The model includes a bounce-back function ψ that models the market's reaction to different block sizes relative to the portfolio size. On this basis, we aim to uncover the circumstances where different liquidation strategies should be chosen in order to optimise the sales value S and reduce the liquidity cost C.

1.1 Background and relevance

A familiar problem in finance is 'slippage' – the difference between the expected price of a portfolio and the executed price of the trade [Che20c]. This difference is relevant when stock owners want to liquidate market moving portfolios into the market. Several factors contribute to the end result such as the liquidity of the share, the market sentiment, trading volumes, as well as the general macro picture of the industry and political climates.

Furthermore, the size of an investor's capital base plays a pivotal role. The amount of capital needed affects the time period for the liquidation process. Consequently, different investors may have different liquidity costs on the same portfolio in the same market as they must choose different liquidation strategies on the basis of their individual context.

In addition, liquidating a market moving portfolio is not an unusual event, thus strategies and models for such activities are sought after as they seek to safeguard the initial portfolio value. Even though the factors are many, we are interested in models that can do more than infer what the discounted price of the portfolio may be. We want to study models that may reduce the slippage and find strategies that, in an ideal world, let the slippage tend towards zero.

A rule of thumb in the finance markets is that the level of risk in a transaction should imply the level of return to be expected. The higher the risk, the greater the reward. This is called the risk-return trade off [Sco20]. Consequently, quantifying risk is essential as it works as an anchor[1] for expected returns.

[1] Defined by the anchoring bias from the works of the psychologists Kahnemann and Tversky in economy

As the last decade of research suggests, mainly kicked forward by Acerbi and Scandolo, that risk measures might have been too strict compared to the actual behaviour of the financial markets. Continued research into redefining assumptions will offer the field additional and necessary advances [AS08].

On a grander scale, the business world is primarily about mitigating unnecessary risk, and at the same time optimise return in an attempt to find a fulcrum where it all balances. In order to converge towards these points the field is in need of more accurate risk measures, for then to set more accurate anchors which in return will give a better expected return estimate. It would have been interesting to work on a continuation of this aspect, however the scope of this project is limited to an exploration of a model in discrete time, which includes risk as a part of all other market parameters.

1.2 An overview of the problem

This thesis is an investigation of *how* an investor can liquidate a market moving portfolio with B shares over the discrete times $t_1, t_2, \dots t_N$ in order to maximise the total sales value of the portfolio.

The model presented in the following is a procedure for portfolio liquidation starting at time t_1. The expected future prices of the share at time t_n for $n \in \{1, 2, \dots, N\}$, given that no shares from the investor's portfolio are sold, is denominated by P_n.

Before we continue, let us refresh some financial jargon. A *Bull market* ($P_{n+1} > P_n$) is a market in which prices are rising or expected to rise [Che20b], and a *Bear market* ($P_{n+1} < P_n$) is a market in which prices are declining or expected to decline [Che20a]. Consequently, we define a *Flat market* ($P_{n+1} = P_n$) as a market in which prices are expected to stay constant.

Classical economic theory states that a share price will decline when a stock creates a supply surplus as demand is unable to consume the block of shares [Kvi21].

Firstly, we use a function of two variables $\psi(x_n, k) \in [0, 1]$ in order to model the percentage price reduction after k units of time has passed since x_n shares were sold in the nth block sale. We call ψ the *bounce-back* function. The function has the following properties:

- Selling x_1 shares at time t_1 yields price

$$E_1 = P_1\psi(x_1, 0)$$

  and total sales value $S(x_1) = x_1E_1$. *If no more shares are sold*, then k units of time after the first sale, the share price will recover to the value $P_1\psi(x_1, k)$.

- *If we continue to sell shares*, then at time t_2 we sell x_2 shares at price

$$E_2 = P_2\psi(x_1, 1)\psi(x_2, 0)$$

  with total sales value $S(x_1, x_2) = x_1E_1 + x_2E_2$.

- *In general*: If we sell x_1 shares at time t_1, and x_2 shares at time t_2 up to x_m shares at time t_m, then the total expected price for $n = m$ is

$$E_m = P_m \psi(x_1, m-1)\psi(x_2, m-2)\cdots\psi(x_m, 0) \tag{1.1}$$

with total sales value of

$$S(x_1, x_2, \ldots, x_N) = \sum_{n=1}^{N} x_n E_n \tag{1.2}$$

$$= \sum_{n=1}^{N} x_n P_n \psi(x_1, n-1)\psi(x_2, n-2)\cdots\psi(x_n, 0). \tag{1.3}$$

Secondly, the thesis investigates an exponential bounce-back model ψ with the following properties

- $\psi(x, k)$ is decreasing in x, and
- $\psi(x, k)$ is increasing in k.

Based on these properties the chosen model is given by

$$\psi(x, k) = e^{-\alpha_k x} \tag{1.4}$$

where $\{\alpha_k\}_{k=0}^{\infty}$ is a decreasing sequence of real numbers, and $\alpha_0 < \frac{1}{B}$, where $B \in \mathbb{N}$ is the portfolio size. In general, α_k is a parameter that describes the market reaction to a stock sale including psychology, micro and macro trends and politics that govern the financial markets. The parameter $\alpha_k > 0$ describes to what extent the market reacts negatively to a given sales volume, where a larger α_k entails a bigger market impact. The parameter α_0 must be less than $\frac{1}{B}$ because $x \cdot \frac{1}{B}$, where x is the block size, is the largest possible influence that block size can have on the market in terms of weighted contribution.

On this note, a negative α_k describes how the market reacts positively to a stock sale, where a larger negative α_k entails a bigger positive market impact. This is contrary to empirical data on dumping stock, and in general the α_k parameter is bounded below by 0. Also, a small $\hat{\alpha}_k$ means that the market will recover quickly. Consequently, the portfolio size B must be defined on $\mathbb{N}$ as described above.

Furthermore, in the case of a sale without long term effects $\alpha_k = 0$ for $k > 0$.

Subsequently, the expected prices are given by

$$\begin{aligned} E_1 &= P_1 e^{-\alpha_0 x_1}, \\ E_2 &= P_2 e^{-(\alpha_1 x_1 + \alpha_0 x_2)}, \\ &\vdots \\ E_m &= P_m e^{-(\alpha_{m-1} x_1 + \alpha_{m-2} x_2 + \cdots + \alpha_1 x_{m-1} + \alpha_0 x_m)}, \\ &\vdots \\ E_N &= P_N e^{-(\alpha_{N-1} x_1 + \alpha_{N-2} x_2 + \cdots + \alpha_1 x_{N-1} + \alpha_0 x_N)}. \end{aligned}$$

Finally, the portfolio consists of B shares in total, and we want to liquidate the stock completely x_n shares at a time over N periods. Consequently, the portfolio can be divided into blocks of undetermined sizes x_n such that

$$x_1 + x_2 + \cdots + x_N = B. \tag{1.5}$$

As we do not know the size of a given market moving portfolio in terms of numbers of stocks we choose to look at a normalisation of the model.

1.2.1 Normalisation of variables

To normalise a variable means to adjust the variable to a given scale. It seems reasonable to normalise the different variables, as the model can be applied to a wide range of market situations, rather than a specific case. In this case we shall normalise to 1 in order to evaluate the percentage size of the portfolio that is sold for each x_n.

In order to normalise we introduce a new variable $y_n = \frac{x_n}{B}$, where each y_n is a percentage of the total portfolio. Then $x_1 + x_2 + \cdots + x_N = B$ turns into

$$y_1 + y_2 + \cdots + y_N = 1, \tag{1.6}$$

as we divide across by B, and the expression for the sales value turns into

$$S(y_1, \ldots, y_N) = \sum_{n=1}^{N} By_n P_n \psi(By_1, n-1)\psi(By_2, n-2)\cdots\psi(By_n, 0).$$

As $\psi(x,k) = e^{-\alpha_k x}$ we get $\psi(y,k) = e^{-B\alpha_k y}$ such that

$$S(y_1, \ldots, y_N) = \sum_{n=1}^{N} y_n BP_n e^{-B\alpha_{n-1}y_1 - B\alpha_{n-2}y_2 - \cdots - B\alpha_0 y_n}.$$

If we now introduce $\hat{P}_n = BP_n$ and $\hat{\alpha}_k = B\alpha_k \leq 1$ we eliminate the actual portfolio size such that

$$S(y_1, \ldots, y_N) = \sum_{n=1}^{N} y_n \hat{P}_n e^{-\hat{\alpha}_{n-1}y_1 - \hat{\alpha}_{n-2}y_2 - \cdots - \hat{\alpha}_0 y_n} \tag{1.7}$$

$$= \sum_{n=1}^{N} y_n \hat{E}_n \tag{1.8}$$

where

$$\hat{E}_n = \hat{P}_n e^{-\hat{\alpha}_{n-1}y_1 - \hat{\alpha}_{n-2}y_2 - \cdots - \hat{\alpha}_0 y_n} \tag{1.9}$$

is the normalised expected price at t_n.

Remark 1.2.1. A priori, the original purchase price of the portfolio is unknown. The investor's goal is to liquidate the portfolio and reduce the liquidity cost based on today's market price. For simplicity, and without loss of generality, we will assume that today's purchasing price P_1 is 1.

The task is now to optimise equation (1.7) contingent to equation (1.6), where $x_n = By_n$ will give the actual amount of shares that the investor has to buy, borrow or sell given a certain portfolio size B.

Furthermore, we have to define the domain of the new parameters $\hat{P}_n$ and $\hat{\alpha}_n$. Firstly we sat $\hat{P}_1 = 1$ as it is the basis for the price development. Now, the purchase price $\hat{P}_n > 0$, $n > 1$ may either increase, decrease or remain unchanged. If $\hat{P}_n = 0$ then the buyer gets the company for free. This is a highly unlikely scenario, as even a bankruptcy estate usually has fixed or current assets they can sell in order to raise capital to pay off debt. Also, for example, Norwegian legislation governs that assets may not be sold for unrealistic prices as it is seen as an act of avoidance according to Norwegian tax law, [Fin99, Chapter 13 §13-2(3)].

Also, the parameter $\hat{\alpha}_n$, as described previously with α_k, describes the stock market with all its elements such as macro and micro economic factors, the market psychology, political influence and the market gossip, as well as the portfolio size. This of course makes the α_n parameter extremely powerful but also difficult to control. As this thesis does not intend to explain the holistic mystery of the stock markets, we only use the parameter in order to model the factors of the stock market. This is a flaw that makes the model a theoretical exercise, rather than a model for investment decisions in real life.

Ideally, $\hat{\alpha}_n$ should be defined by multiple parameters in order to look at different aspects of the stock market, such as the volume of shares that are dumped into the market in relation to the number of shares traded. Since this is not the case, we view different sales volumes as a percentage of the number of shares that the investor holds, and the sales values as a percentage of the market price of the investors total portfolio.

On this note, this thesis will try to optimise the sales function (1.7) under the constraint of equation (1.6). This optimisation problem will not be analytically solvable in general, however we are able to investigate special cases for $\psi(y, k)$ and N, subsequently we will examine the problem for small values of N such as $N = 1$, $N = 2$ and $N = 3$, with support from numerical methods.

1.2.2 The validity of the model

The model is an expression for the sales value $S(y_1, \ldots, y_N)$, which entails that $S \geq 0$ and increases as additional shares y_n are sold. Hence, S must be a strictly increasing function. Moreover, a sales value $S < 0$ indicates that the investor must pay money in order to sell shares. This does not, of course, make any sense as the lower bound of a sale would either be scrap value or a non-tradable share. The domain of S is subsequently the areas that adhere to these constraints.

There are also validity restrictions on $\hat{\alpha}_n$ as it defines market sentiments and its influence on our model. We mentioned earlier that a negative $\hat{\alpha}_n$ influences the market positively. A negative $\hat{\alpha}_n$ thus pushes prices up as shares are dumped into the market, which contradicts the expected behaviour. Hence, a deeper inquiry into the influence of $\hat{\alpha}_n$ seems to be in order. We use $N = 1$, the case where an investor only has one step to liquidate the portfolio, to describe and illustrate why $\hat{\alpha}_n$ has restrictions on $\mathbb{R}$.

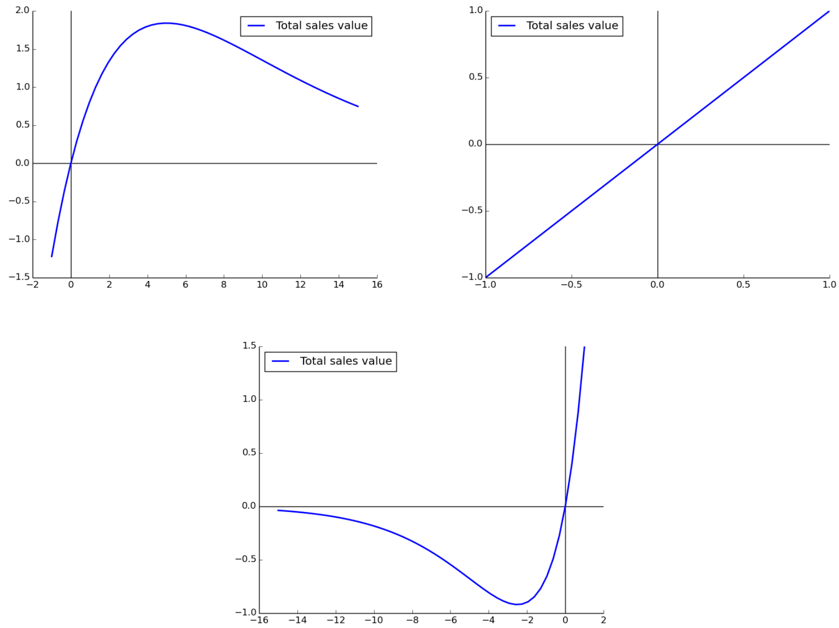


Figure 1.1: All cases in $N = 1$. The upper left graph displays $S(y)$ with $\hat{\alpha}_0 > 0$. The upper right graph displays $S(y)$ with $\hat{\alpha}_0 = 0$. The lower graph displays $S(y)$ with $\hat{\alpha}_0 < 0$.

Firstly, we analyse $\hat{\alpha}_0 > 0$ which shows that we have a maximum for $y_1 > 0$, as displayed in Figure 1.1, but also that there is an upper bound on $\hat{\alpha}_n$. This maximum illustrates what portion of the portfolio should be sold in order to optimise the sales value S. The optimum is given by $S'(y_1) = 0$, which yields $y_1 = {}^1\!/\!_{\alpha_0}$. If the investor chooses to sell more than ${}^1\!/\!_{\alpha_0}$, then each additional share sold will generate negative value. This entails that the share is sold for a negative price, hence the buyer gets the share and some additional money. This will not happen in the stock market as the worst case is defined by selling a share for scrap value, or get stuck with a non-tradable share. An upper boundary on $\hat{\alpha}_0$ seems thus to be reasonable. Since, $\hat{\alpha}_0 < {}^1\!/\!_B$ and $B = 1$ in the normalised case we get an upper boundary of $\hat{\alpha}_0 < 1$.

Secondly, the case $\hat{\alpha}_0 = 0$ indicates that the more you sell the more you earn, as seen in Figure 1.1. This situation includes the possibility for short selling. However, there is no boundary for how big the short position can be. The theoretical upper boundary in this case is the amount of shares issued by the company less the amount of shares the investor already holds.

Finally, we discuss the case of $\hat{\alpha}_0 < 0$, also displayed in Figure 1.1. Here there is a minimum at $y_1 = {}^1\!/\!_{\hat{\alpha}_0}$. There are issues with this situation, as the model yields negative sale value. This means that for every share sold we have to include capital in order to liquidate the portfolio, as the sales value only turns negative as long as negative numbers are added. This situation makes no sense in the real world stock market, and are thus outside the scope of the model. Subsequently, the parameter $\hat{\alpha}_0$ should be bound below by zero.

The analysis of the model will consequently be viewed on the interval $0 < \hat{\alpha}_n < 1$ for $n = 1, 2, \dots N$.

1.3 Thesis structure

This thesis has seven chapters including this introduction. Chapter 2 gives a short run-through of the mathematics and finance theory that forms the basis for the thesis. It is presented in a comprised fashion just to give a theoretical backdrop. Chapter 3 presents a short introduction to the different financial strategies that surfaces throughout the body of this text. Following this are the three chapters on how our model behaves in $N = 1$, $N = 2$ and $N = 3$. The case $N = 1$ gives the investor just one step to liquidate the portfolio, namely to sell all shares in one go. We have thus chosen to look at the model without constraints in order to acquaint ourselves with its possibilities. The case $N = 2$ adds another step and the investor can then utilise several investment strategies in order to liquidate the portfolio, hereunder pumping, dumping and short selling, as well as buy, hold and sell strategies. The case $N = 3$ adds yet another step, which complexifies the liquidation by offering up 27 different strategy spaces. In Chapters 4 to 6 we look to optimise our sales value model S, where the case $N = 3$ also includes a naive analysis in order to make a comparative analysis with the numerical solutions of the model. This comparative analysis is made in order to enlighten how difficult it is to say something reasonable about the stock markets based on lay reasoning. Penultimately, we sum up our findings and some reflections in the closing remarks in Chapter 7. At the end of the thesis in Appendix A we have gathered the six different scripts used to complete the numerical analysis.

All in all, I hope the forthcoming text will have some interest, and not bore you to an early intellectual grave with platitudes. I wish you a fun journey!

CHAPTER 2

Scientific background and prerequisites

This chapter aims to give a scientific context and set the theoretical background for the finance and mathematics for this thesis. It is structured as a short run through of known theory that is required to understand the content and analysis of the thesis' aim:

Optimal liquidation of market moving portfolios

We begin with the mathematical theory and continue on to the finance theory connected to this thesis.

2.1 Scientific context

This scientific context will give a short overview of this theoretical space. It is, however, important to emphasise that this thesis will focus on an analysis based on undisturbed markets as well as a simplified sales value model that includes both risk measures and overall market influence into one parameter $\hat{\alpha}_n$.

The current research of risk measures aims to combine the following theories into a dynamical framework for finding optimal strategies for liquidation of portfolios under different kinds of time restraints, and to use this theory to shed new light on risk measures. The first step is to combine market supply curves and resilience functions into a common framework that is simple enough to analyse.

2.1.1 The insufficient theoretical beginning

Consider a market with $N + 1$ assets where asset number 0 is risk-less. As usual, a portfolio is a vector $\mathbf{p} = (p_0, p_1, \ldots, p_N)$ with the interpretation that p_n is the number of stocks in asset n. As short-selling is allowed, the values of some of the p_n's may be negative. If the best bid price for option n is m_n^+ and the best asking price is m_n^-, the value of the portfolio is usually computed as

$$V(\mathbf{p}) = p_0 + \sum_{n\in L} m_n^+ p_n + \sum_{n\in S} m_n^- p_n \tag{2.1}$$

where L is the set of assets where we have a long position and S is the set of assets where we have a short position. As this valuation method may not be optimal in regards to liquidating portfolios, we choose to investigate a simple discrete time model with a bounce-back rate in order to assess the liquidity cost.

2.1.2 Risk measures

Portfolios carry risks, and the same portfolio may carry different risks to different traders according to the overall situation they find themselves in. Risk measures form a general framework for assessing the risk of portfolios. We follow the definitions in Föllmer and Schied [FS04] where Ω is a probability space and a *financial position* is a random variable $X \colon \Omega \to \mathbb{R}$. We let $\mathcal{X}$ be a set of financial positions, and assume that $\mathcal{X}$ is a linear space of bounded functions including the constants. It is often helpful to think of the financial positions X as the (random) values of portfolios under different scenarios.

Definition 2.1.1. A mapping $\rho \colon \mathcal{X} \to \mathbb{R}$ is a *risk measure* if for all $X, Y \in \mathcal{X}$:

(i) (Monotonicity) If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.

(ii) (Cash Invariance) If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.

We say that ρ is *positive homogeneous* if

(iii) for all $\lambda \geq 0$ and $X \in \mathcal{X}$, we have $\rho(\lambda X) = \lambda \rho(X)$,

and that it is *subadditive* if

(iv) $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{X}$,

A risk measure satisfying (i)–(iv) is called a *coherent* risk measure.

Subadditivity is often criticised for being too strict, and it is therefore common to replace (iii) and (iv) by a fifth axiom:

Definition 2.1.2. We say that a risk measure ρ is *convex* if it satisfies (i), (ii) and

(v) $\rho(\lambda X + (1 - \lambda) Y) \leq \lambda \rho(X) + (1 - \lambda) \rho(Y)$ for all $\lambda \in [0, 1]$ and $X, Y \in \mathcal{X}$.

In the usual arguments against coherent risk measures, it is usually assumed that the value of a portfolio is given by formula (2.1), but Acerbi and Scandolo [AS08] have argued that in a risk setting, this is not the right value to use, and that instead of using the current market value in (2.1), one should use the 'liquidation value' that is describe in the next section.

2.1.3 Liquidation value

In a pressed situation where one needs to liquidate a market moving portfolio immediately, one can not expect to get the market value for the entire portfolio as there usually is a limited supply of traders who are willing to pay the full market price following the theory of supply and demand. In such situations, it is more natural to base the calculation of the portfolio value on *market supply curves* for the stocks. A market supply curve for stock number n is a function $m_n \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that:

(i) m_n is non-increasing

(ii) m_n is right-continuous on $(0, \infty)$ and left-continuous on $(-\infty, 0)$

The idea is that $m(0^+) = \lim_{x \to 0^+} m(x)$ is the current bid price for the stock (i.e. what we referred to as m_n^+ above), and that $m(x)$, $x > 0$, is the bid price after a quantity x of stocks have already been sold. Hence, $\int_0^y m_n(x)\,dx$ is the profit from selling a total quantity y of stock number n. The interpretation for $x < 0$ is similar: $m(0^-) = \lim_{x \to 0^-} m(x)$ is the current asking price for the stock (i.e. what we referred to as m_n^- above), and $m(x)$ is the asking price after a quantity x has already been traded. Hence, the cost of buying a total of y stocks is $\int_{-y}^0 m_n(x)\,dx$. The liquidation value of the portfolio can now be defined as

$$L(\mathbf{p}) = p_0 + \sum_{n=1}^{N} \int_0^{p_n} m_n(x)\,dx. \tag{2.2}$$

Acerbi and Scandalo [AS08] argue quite convincingly that if the liquidation value is used to assess the worth of a portfolio, then the arguments against coherent risk measures lose much of their force (see, however, Lee et al. [LOC15] and Weber et al. [Web+13] for other perspectives).

2.1.4 Resilience

In many situations where a portfolio needs to be liquidated in order to meet a demand, the full portfolio need not be liquidated immediately – one has some time at one's disposal. This means that one can exploit the market's resilience, i.e. the tendency to revert to 'normal' prices after a drop in the market. To model the resilience, authors like Alfonsi et al and Schied and Slynko have introduced *resilience functions* as a counterpart to market supply curves. A resilience function is a non-increasing function $G \colon [0, \infty) \to [0, \infty)$ with the following interpretation: If trades $\xi_{t_0}, \ldots, \xi_{t_n}$ are made at times $t_0, \ldots, t_n$, the asset price at time t is given by

$$S_t = S_t^0 = \sum_{t_n < t} \xi_{t_n} G(t - t_n),$$

where S_t^0 is the undisturbed market price, i.e. the market price we have if no trades are taking place.

The model in this thesis does not allow for the utilisation of resilience as we are looking at what happens when an investor has to liquidate a portfolio over a time span that does not allow for the market to return to its original state. Even though we are investigating an up to three-step liquidation scenario, these liquidation events are too close in time for the market to recover. This thesis tries to describe how to liquidate a portfolio as optimally as possible when time is not on the investor side and can utilise resilience theory.

2.2 Mathematical theory

2.2.1 Optimisation in $N = 2$

This thesis enjoys the study of Fixed Point theory in $N = 2$ in order to optimise the sales function S numerically, since we are unable to find a general expression for the derivatives of $S(y_1, \ldots, y_N)$. We also include Gradient Ascent, however this is covered in Section 2.2.2. Also, in the work with optimising $N = 2$ we make use of The Lagrange Multiplier method. Hence, this section explains the theory connected to these two mathematical concepts.

A fixed point of a function is a value that is unchanged by repeated application of the function. *Banach's fixed point theorem* assures that there is a unique fixed point for contraction mappings. Let's start off by defining a fixed point:

Definition 2.2.1 [Lin17, p. 61], **Fixed point.** For a function $f \colon X \to X$, a *fixed point* $x^* \in X$ is a point where $f(x^*) = x^*$.

In order to use *Banach's fixed point theorem* we need to understand the definitions of Metric space, Cauchy sequence, Complete metric space and Contraction mapping. The following contains these definitions:

Definition 2.2.2 [Lin17, Definition 3.1.1.], **Metric space.** A *metric space* (X, d) consists of a nonempty set X and a function $d \colon X \times X \to [0, \infty)$ such that:

(Positivity) For all $x, y \in X$, we have $d(x, y) \geq 0$ with equality if and only if $x = y$.

(Symmetry) For all $x, y \in X$, we have $d(x, y) = d(y, x)$.

(Triangle Inequality) For all $x, y, z \in X$, we have $d(x, y) \leq d(x, z) + d(z, y)$.

Definition 2.2.3 [Lin17, Definition 2.2.4.], **Cauchy sequence.** A sequence $\{x_n\}$ in $\mathbb{R}^m$ is called a *Cauchy sequence* if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $d(x_n, x_k) < \epsilon$ when $n, k \geq N$.

Definition 2.2.4 [Lin17, Definition 3.4.3.], **Complete metric space.** A metric space is called *complete* if all Cauchy sequences converge.

Definition 2.2.5 [Lin17, p. 61], **Contraction mapping.** Let (X, d) be a metric space and $T \colon X \to X$ a mapping. We say that f is a *contraction mapping* if there exists a positive number $s < 1$ with

$$d(f(x), f(y)) \leq s\, d(x, y) \text{ for all } x, y \in X.$$

We call s a *contraction factor* for f.

We are now ready to introduce *Banach's fixed point theorem*:

Theorem 2.2.6 [Lin17, Theorem 3.4.5.], **Banach's fixed point theorem.** *Assume that (X, d) is a complete metric space and that $f\colon X \to X$ is a contraction. Then f has a unique fixed point a, and no matter which starting point $x_0 \in X$ we choose, the sequence*

$$x_0, x_1 = f(x_0), x_2 = f^{\circ 2}(x_0), \ldots, x_n = f^{\circ n}(x_0), \ldots \tag{2.3}$$

converges to a.

We want to utilise the Fixed Point theory based on the Lagrange Multiplier method, which is a way to optimise a function subject to constraints. Our constraints are given by the limitations on block size y_n, which must sum to 1.

Theorem 2.2.7 [Ada03, Theorem 13.3.4], **Lagrange Multiplier Method.** *Suppose that f and g have continuous first partial derivatives near the point $P_0 = (x_0, y_0)$ on the curve $\mathcal{C}$ with equation $g(x, y) = 0$. Suppose also that, when restricted to points on $\mathcal{C}$, the function $f(x, y)$ has a local maximum or minimum value at P_0. Finally, suppose that*

2.2.7.1. *P_0 is not an endpoint of $\mathcal{C}$,*

2.2.7.2. $\nabla g(P_0) \neq \mathbf{0}$.

Then there exists a number λ_0 such that (x_0, y_0, λ_0) is a critical point of the Lagrangian function

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y).$$

The methods of *Fixed Point theory* and *Lagrange Multiplier* is used to find the optimum by iterating towards the maximum a of S under different sentiments determined by $\hat{\alpha}_n$. Hence, we find the size of the block of shares that we need to sell in each step in order to maximise the sales value S, and reduce the liquidity cost of selling market moving portfolios. Finding the fixed point is done by numerical method and the corresponding Python scripts that utilises these theories can be found in Appendices A.2.1 and A.2.2. The crux of the method is that the maximum of S is the same as the fixed point of a function $d\mathcal{L}$, which we describe below.

We use the constraint $g(y_1, y_2) = 1 - y_1 - y_2$ to reduce the function $S(y_1, y_2)$ into the concave, one-variable function $S(y_1)$. Hence, the algorithm finds the maximum of $S(y_1)$ by using the property of the fixed point contraction as it iterates to the maximum.

Starting off, we use the function S and the partial derivatives of the Lagrangian function $\mathcal{L}(y_1, y_2, \lambda) = S(y_1, y_2) + \lambda g(y_1, y_2)$ to find the maximum of S. In order to reduce the Lagrangian we substitute the partial derivatives of $\mathcal{L}(y_1, y_2, \lambda)$ and the constraint $g(y_1, y_2)$ into the partial derivative $\frac{\partial \mathcal{L}}{\partial y_1}$, yielding a one-variable function $d\mathcal{L}$.

In order to find the maximum of the concave function S we used the fact that $d\mathcal{L}$ tends to 0 as it moves towards the maximum, and this iteration is the use of the Fixed Point theorem in the algorithm. Consequently, the fixed point iteration of the for-loop terminates at the maximum of S.

2.2.2 Optimisation in $N = 3$

As we move on to $N = 3$ the Fixed Point theory is replaced by the method of Gradient Ascent in order to optimise the sales function S. It is worth mentioning that an advantage of Gradient Ascent to Fixed Point is that it has no demand on convexity of the objective function.

In $N = 3$ we choose to change our optimisation method as the Fixed Point method proves to be unstable as it struggles to find the correct optimum and gives deviant results. Hence, we made a change to the Gradient Ascent method, which was easily implemented. The basic idea of Gradient Ascent is fairly simple. The idea is to take repeated steps of sufficient size in the direction of the gradient of a function f at a given point, until $f(\mathbf{a}_n) \nleq f(\mathbf{a}_{n+1})$. Consequently, moving in the direction of the gradient leads to a local maximum of function f in this case. Before we get to the method of Gradient Ascent we need to define the gradient and give it some context through a proposition.

Furthermore, it is worth mentioning that most literature refers to Gradient *Descent*, a method for finding local minima. However, Gradient Ascent is just the reverse, so making the adjustment to Gradient Ascent entails moving in the opposite direction.

Definition 2.2.8 [LH11, Definition 2.4.3], **Gradient.** Assume that the partial derivatives of f exist in a point $\mathbf{a} \in \mathbb{R}^n$. Then

$$\nabla f(\mathbf{a}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{a}), \frac{\partial f}{\partial x_2}(\mathbf{a}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{a}) \right)$$

is the *gradient* of f at $\mathbf{a}$.

Proposition 2.2.9 [LH11, Proposition 2.4.7]. *Assume that f is differentiable at $\mathbf{a}$. Then the gradient $\nabla f(\mathbf{a})$ points in the direction where f increases the fastest in $\mathbf{a}$, and the slope of f in that direction is $|\nabla f(\mathbf{a})|$.*

We now have the necessary background to look into the Gradient Ascent Method.

Theorem 2.2.10 [Wik22], **Gradient Ascent.** *If a multi-variable function $f(\mathbf{x})$ is defined and differentiable in a neighbourhood around point $\mathbf{a}$, then $f(\mathbf{x})$ increases fastest in the direction of the gradient $\nabla f(\mathbf{a})$ from $\mathbf{a}$ at $\mathbf{a}$. It follows that, if*

$$\mathbf{a}_{n+1} = \mathbf{a}_n + \gamma \nabla f(\mathbf{a}_n)$$

for a small enough step size $\gamma \in \mathbb{R}_+$, then $f(\mathbf{a}_n) \leq f(\mathbf{a}_{n+1})$.

The method of Gradient Ascent depends on the step γ not being too large, as it may send the iterations in the wrong direction. Furthermore, the step size may vary from case to case as we perform our numerical analysis dependent on the curvature of the surface.

In the Python script in Appendix A.3 for $N = 3$ we use the Gradient Ascent Method on the function $S(y_1, y_2)$. As it only has two variables we have reduced it by the help of our constraint $y_1 + y_2 + y_3 = 1$, and substituted this into y_3 through out $S(y_1, y_2, y_3)$. Subsequently, we use a pre-made partial derivative algorithm to find the partial derivatives of $S(y_1, y_2)$.

In addition, we use the Intermediate Value Theorem to determine the existence of boundary points between different optimal investment solutions in $N = 3$.

Theorem 2.2.11 [Ada03, Theorem 1.4.9], **Intermediate Value Theorem.** *If $f(x)$ is continuous on the interval $[a, b]$ and if s is a number between $f(a)$ and $f(b)$, then there exists a number c on $[a, b]$ such that $f(c) = s$.*

2.3 Financial theory

2.3.1 Liquidity

Liquidity refers to the ease with which an asset or security can be converted into ready cash without affecting its market price [Jam20]. An investor wants to trade securities in the stock market at an acceptable price and at the desired time. Hence, a transaction of ownership in a market is optimised when no friction occurs and both price and time are as intended. This circumstance is called *liquidity*. A liquid market is thus a market where all players are able to buy or sell an unlimited amount of securities with immediacy at the price close to the last traded price.

A stock market consists of two types or prices. The *bid price* is defined by a market participant's willingness to pay for a number of shares. The *ask price* is the price at which a market participant is willing to sell a number of stocks. For every bid price, there is a specified *bid volume*, which is the number of shares that can be bought at that bid price. Likewise, the *ask volume* is the number of shares that can be sold at that asking price. The difference between the highest ask price and the lowest bid price is defined as the *spread*. When the spread is 0, ask equals bid, and a transaction will occur.

An *illiquid* market is one where market participants are able to trade, but only at prices different from the last traded price. In an illiquid market, transactions cause shifting in the price from the observed price. Markete moving transactions may cause long time lags as the market reacts and consumes the portfolio blockwise. A perfectly liquid market is defined by no price movement irrespective of transaction time, transaction quantity and transaction type. The perfectly liquid market is unobtainable. Hence, market friction comes at a price. This liquidity friction is defined as the *liquidity cost*.

The degree of liquidity in the market is the primary parameter to the liquidity cost.

Definition 2.3.1. The *estimated market value* $\hat{M}_t$ of an asset at time t, is defined by the estimated market price $\hat{P}_t$ per share multiplied by the number of shares B_t in the asset:

$$\hat{M}_t := \hat{P}_t \cdot B_t.$$

Definition 2.3.2. The *sales value* S_t of an asset at time t, is defined by the actual market price P_t per share multiplied by the number of shares B_t in the asset:

$$S_t := P_t \cdot B_t.$$

Definition 2.3.3. The *liquidity cost* C_t at time t, is defined as the difference between the sales value S_t and the estimated market value $\hat{M}_t$:

$$C_t := S_t - \hat{M}_t = (\hat{P}_t - P_t)B_t.$$

Definition 2.3.4. The *liquidity cost ratio* LCR_t at time t, is defined as the ratio between the liquidity cost $C_t(y)$ and the estimated market value $\hat{M}_t(y)$:

$$\mathrm{LCR}_t = \frac{C_t(y)}{\hat{M}_t(Y)}.$$

This theory is utilised when we do a comparative analysis of the situation where the investor must liquidate the entire portfolio versus the situation where the portfolio need not be liquidated completely or we can borrow shares.

CHAPTER 3

Financial strategy

This chapter addresses sales and purchase events – *transactions* – in the stock market. A transaction is an execution of an underlying trading strategy, of which the financial markets have a myriad. In the following we will look at a small selection of these strategies, including investments in the stock market through buying stock with equity, or selling stock the investor already owns. These two situations will be defined as *trades*. Also, we will address the strategies of *pumping* and *dumping*, where the investor either buys more stock or sells stock to push the price up or down in order to capitalise on the market movement. In addition, we will describe the financial strategy of *short selling*, which is the act of selling borrowed stock in order to repurchase the position in the future at a lower price, for then to return the stock to its original owner, and discuss how dumping is a special case of short selling.

Recall that the nth block of shares sold is y_n of the entire portfolio. So in general, a sale in this model is defined on the interval $0 \leq y_n \leq 1$, hence a positive y_n defines a sell request. A sale of $y_n = 1$ in the normalised model describes the complete sale of the portfolio. A classical purchase is defined on the interval $y_n < 0$, as negative y_n defines a buy request. A sale of $y_n = -1$ is the case where the investor doubles the size of the initial portfolio. If $y_1 > 1$ the optimal trading strategy is a short position, as the investor needs to sell more stock then the initial portfolio size.

Furthermore, an investor may utilise pumping or dumping strategies. These strategies influence the stock price by affecting the supply and demand mechanisms with trades and/or true and available market information. It is not particularly popular in all milieus, nonetheless it serves as a trading strategy. Hence, we shall look into pumping and dumping by analysing a data set generated by our model. We will address these strategies both analytically and numerically.

Let's take a look at the pumping strategy, which may be enforced prior to when an investor opts to sell a block of shares, as the goal is to achieve the highest possible sales price by pumping the price upwards.

Definition 3.0.1 Pumping [Lin20]**.** To buy a stock at time t_1 with the intent to push up the price for then to sell the same stock at t_2 within a short time interval.

The opposite strategy of pumping is dumping, which is utilised when the investor sells a larger portion of the stock than they initially want in order to pull down the price, for then to buy back the stock at a lower price hoping it will surge in the future to make a profit. This strategy is, amongst other, favourable when the investor is involved in a short position as described in Definition 3.0.4.

Definition 3.0.2 Dumping [Lin20]**.** To sell a larger block of shares at t_1 than initially wanted with the intent to pull down the price, for then to repurchase the same stock at t_2.

Before we continue it seems important to mention *The Pump and Dump Investment Scheme*, which is illegal. It entails pumping and dumping in situations where an investor attempts to boost the stock price by planting false, misleading or greatly exaggerated information in order to sell the stock at an inflated price and increase the revenue. It is illegal because the stock price is based on fraudulent information, subsequently yielding a stock price that lacks basis in fundamentals or expected growth.

Definition 3.0.3 The Pump and Dump Investment Scheme [Dhi19]**.** The act of intentionally increasing the price of a stock based on fraudulent information at one stage in order to sell the stock at a later stage with increased revenue.

Finally, there is the trading strategy of *short selling*. This is a well established manoeuvre in the stock markets. The strategy is relevant for the analysis of our model, and is defined as follows:

Definition 3.0.4 Short selling a stock [Che19]**.** To borrow shares in order to sell in the market at time t_1 with the expectations of a market decline, for then to buy back the shares at a lower price at t_2 and return the borrowed shares.

It seems important to mention that short selling and dumping are overlapping, and how is dependent on whether the investor must liquidate the portfolio or not. In our case, the investor must liquidate the entire portfolio, and thus borrow shares in order to complete a dumping strategy. Consequently, in this thesis we may view the dumping strategy as a special case of the short selling strategy.

With this backdrop, the thesis analyse extrema of the sales revenue function S, and determine for which strategy S has a maximum, and consequently reduce the liquidity cost as much as possible. The following will then determine which strategy to choose:

- A pumping strategy can be optimal if there is a maximum in the buy request domain.
- A dumping strategy can be optimal if there is a maximum in the sell request domain.
- A short selling strategy can be optimal if there is a maximum in the sell request domain.

These domains depend on which space we operate, and a further analysis will be described in Chapters 4 to 6. In the following chapters, we wish to maximise S and minimise the liquidity cost C given the sales value $S(y_1, \ldots, y_N)$ of selling N blocks of shares.

CHAPTER 4

A dissection of model S with help of case $N = 1$

The case $N = 1$ may at first glance seem rather uninteresting, as the only investment strategy available is the sell strategy. This follows by the constraint of complete liquidation of the portfolio. However, if we look closer we see that the case of $N = 1$ may reveal some interesting perspectives of the relation between the bounce-back function $\psi = e^{-\hat{\alpha}_0 y_1}$ and the market sentiment $\hat{\alpha}_0$. Hence, in this section we analyse how the $\hat{\alpha}_0$-value impacts the sales value $S(y_1) = y_1 \hat{P}_1 e^{-\hat{\alpha}_0 y_1}$ and the bounce-back function ψ's slippage. As we only have one time-step we will never see how the function recuperates. It is however interesting to see how the slippage behaves. Also, a look at $N = 1$ yields some interesting insights into the model S itself, if we ignore the constraint of complete liquidation. We start off by investigating how the $\hat{\alpha}_0$-parameter influences the market and subsequently the sales value S analytically, and then dissect numerically.

4.1 Market states and the analytical approach

The market for $N = 1$ is a snapshot in time because we only evaluate what happens here and now. Subsequently, the value of the parameter $\hat{\alpha}_0$ is the only market information available to us. As described earlier, if $\hat{\alpha}_0 > 0$ the market will react negatively to a sell event, and the larger the value of $\hat{\alpha}_0$ the bigger the negative impact will be.

Remark 4.1.1. Generally, if we had the opportunity to see a period in the market and not only a snapshot in time, a sell event may influence the market quite differently in a Bear market than if it is a Bull market. Bull markets in general handle negative news less dramatically than Bear markets. In other words, in a Bull market, the effect of $\hat{\alpha}_0$ is smaller than in a Bear market. Even though we are looking at $0 < \hat{\alpha}_0 < 1$ it is interesting to see that if $\hat{\alpha}_0 < 0$ the market will react positively to a sell event, and the larger the absolute value of $\hat{\alpha}_0$ is the more positively the market will react. In terms of a Bear market, again such an event can influence the market differently than if it is a Bull market. In this case, Bear markets in general handle positive news less optimistically than Bull markets. Hence, in a Bear market, the effect of the absolute value of $\hat{\alpha}_0$ is smaller than in a Bull market.

As initially mentioned the investor *has* to sell the entire portfolio in the case $N = 1$, due to the liquidation constraint in this one-step scenario. Let's see how the model behaves if we neglect this constraint.

In order to determine the optimal sales value S^* we must localise the maximum. If the maximum is located at $(-\infty, 0)$ we are in the buy request domain for $N = 1$, where $y_1 = -1$ is a doubling of the portfolio. If the maximum is located at $(0, \infty)$ we are in the sell request domain, where $y_1 = 1$ is a complete liquidation of the portfolio. Hence, $y_1 = 0$ is the situation where the investor does nothing. For $N = 1$ our model $S(y_1)$ with $\hat{P}_1 = 1$ and derivative $S'(y_1)$ is given by

$$\begin{aligned} S(y_1) &= y_1 e^{-\hat{\alpha}_0 y_1}, \\ S'(y_1) &= e^{-\hat{\alpha}_0 y_1} - \hat{\alpha}_0 y_1 e^{-\hat{\alpha}_0 y_1} \\ &= (1 - \hat{\alpha}_0 y_1) e^{-\hat{\alpha}_0 y_1}, \end{aligned}$$

with constraint $0 < \hat{\alpha}_0 < 1$.

We find the optimum by solving $S'(y_1) = 0$:

$$\begin{aligned} (1 - \hat{\alpha}_0 y_1) e^{-\hat{\alpha}_0 y_1} &= 0 \\ 1 - \hat{\alpha}_0 y_1 &= 0 \\ y_1 &= \frac{1}{\hat{\alpha}_0}. \end{aligned}$$

Hence, the maximum is given by $y^* = \frac{1}{\hat{\alpha}_0}$ for $N = 1$. As $0 < \hat{\alpha}_0 < 1$ we discover that

$$\frac{1}{\hat{\alpha}_0} > 1.$$

This entails that the optimum is outside the possible sales volume under our original constraints. Also, in this one-step situation we are unable to short sell as there is no opportunity to buy back the shares. Consequently, the obtainable sales value is

$$S(1) = \frac{1}{e^{\hat{\alpha}_0}}.$$

Generally, the optimal sales value looks like this:

$$\begin{aligned} S(y^*) &= y^* e^{-\hat{\alpha}_0 y^*} \\ &= \frac{1}{\hat{\alpha}_0} e^{-\hat{\alpha}_0 \cdot \frac{1}{\hat{\alpha}_0}} \\ &= \frac{1}{\hat{\alpha}_0 e}. \end{aligned}$$

With $\hat{\alpha}_0 > 0$, the optimal sales value $S^* = {}^1\!/\!_{\hat{\alpha}_0 e}$ increases as $\hat{\alpha}_0$ tends towards 0. However, since we have established that $y^* = {}^1\!/\!_{\hat{\alpha}_0} > 1$ the optimal sales value $S(y^*)$ tends to the obtainable sales value $S(1)$ as $\hat{\alpha}_0 \to 1^-$. So, as $\hat{\alpha}_0$ tends to 1 $S(y^*)$ tends to ${}^1\!/\!_e$. Furthermore, when $\hat{\alpha}_0 \to 0$ then $S(y^*) \to \infty$ which entails that the sales value can grow limitlessly outside the scope of any given market.

Let's continue our investigation and evaluate the sales value S for different block sizes y_1, against the estimated market value $\hat{M}$ of that block size, as described in Definition 2.3.1. First we have

$$\hat{M}(y_1) = y_1 \hat{P}_1 = y_1$$

with $\hat{P}_1 = 1$. Then we look at the liquidity cost ratio LCR, in order to understand the market reaction to a flooding of too many shares:

$$\mathrm{LCR}(y_1) = \frac{C(y_1)}{\hat{M}(y_1)} \tag{4.1}$$

where $C(y_1)$ is the actual liquidity cost given by

$$C(y_1) = \hat{M}(y_1) - S(y_1). \tag{4.2}$$

Substituting Equation (4.2) into Equation (4.1) we get

$$\begin{aligned}\mathrm{LCR}(y_1) &= \frac{\hat{M}(y_1) - S(y_1)}{\hat{M}(y_1)}\\ &= \frac{y_1 - y_1 e^{-\hat{\alpha}_0 y_1}}{y_1}\\ &= 1 - e^{-\hat{\alpha}_0 y_1}.\end{aligned}$$

Thus, the liquidity cost ratio LCR, given $\hat{\alpha}_0$ constant, is dependent on y_1. This is as expected as the block size released into the market is market moving. Subsequently, it is interesting to evaluate the liquidity cost ratio LCR in the case of the optimal investment strategy y^*. On this note, we observe that it is constant for all values of $\hat{\alpha}_0$:

$$\mathrm{LCR}(y^*) = 1 - e^{-\hat{\alpha}_0 \cdot \frac{1}{\hat{\alpha}_0}} = 1 - \frac{1}{e} \approx 63.2\%.$$

This is also illustrated with the three black dots coinciding on a line in Figure 4.1, which is based on the script in Appendix A.1. Furthermore, the liquidity cost C is used as a measure of risk, the bigger the potential liquidity cost the bigger the risk. Theoretically, the same figure shows that the most favourable optimisation in terms of risk is $\hat{\alpha}_0 = 0$, as it is the situation where the absolute distance between the estimated market value $\hat{M}$ and sales value S is 0. This is substantiated by

$$\lim_{\hat{\alpha}_0 \to 0^+} S(y_1) = \hat{M}(y_1), \quad \forall y_1.$$

Thus, for $0 < \hat{\alpha}_0 < 1$ the monetary loss for liquidating the portfolio $y_1 = 1$ is the smallest as $\hat{\alpha}_0$ tends to 0. However, it is worth mentioning that all $\hat{\alpha}_0$ on the interval $(0, 1)$ will for some liquidation volume y_1 reach the optimal liquidity cost ratio LCR $= 63.2\%$, but the volume may be bigger than $y_1 = 1$. It's only in the constraint situation of $N = 1$ there is only one possible volume to liquidate, namely $y_1 = 1$, which tells the investor to hope for a market with an $\hat{\alpha}_0$ as close to 0 as possible under the constraint. Even though $y_1 = 1$ is not the maximum in this instance, the difference between $S(1)$ and the estimated marked value $\hat{(M)}(1)$ is as small as possible.

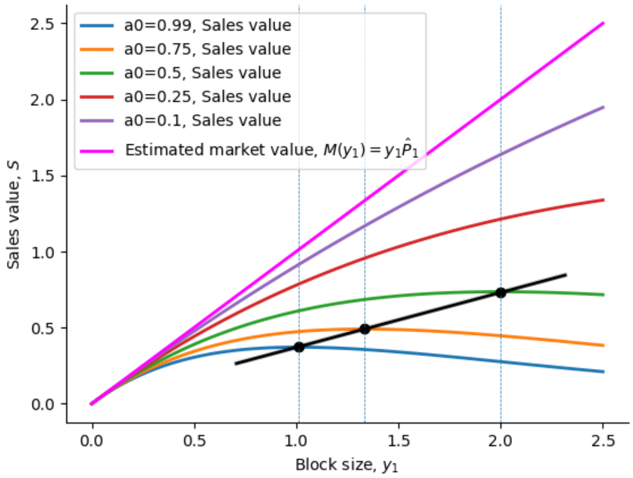


Figure 4.1: The upper graph displays the estimated market value $\hat{M}(y_1)$, which is the estimated share price multiplied by the number of shares. It is also the case of $\hat{\alpha}_0 = 0$. The lower graphs show the sales values S for different values of $\hat{\alpha}_0$, ranging from $\hat{\alpha}_0 = 0.99$ to $\hat{\alpha}_0 = 0.10$. The dotted vertical lines mark the extremum for different values of $\hat{\alpha}_0$. The black line illustrates that the liquidity cost ratio LCR is constant for optimal y^*.

As touched upon, it is important to mention that the case $N = 1$ has neither a pumping, dumping or short selling strategy available to the investor, as those investment strategies demand a two-step scenario. The graphs in Figure 4.1 are merely there to illustrate the point of constant liquidity cost ratio LCR for y^*.

Remark 4.1.2. This thesis does not investigate the element of timing t_n, other than refer to it to explicitly describe that events are happening at different points in time. Timing is, however, essential when operating in the actual stock market.

4.2 The numerical analysis

The model for the sales value $S(y_1) = y_1 e^{-\hat{\alpha}_0 y_1}$ describes the situation where an investor sells the entire portfolio in one block. This entails that the entire portfolio is released into the market at the same time. The blue curves in Figures 4.2 to 4.4 based on the script in Appendix A.1 show how the classic supply and demand theory comes into play. The investor must accept a step-wise price reduction if supply is larger than demand, and a step-wise price increase if supply is less than demand. As one would expect we see that supply outperforms demand as we flood the market with shares. The best bid subsequently reacts as subblocks are picked up in the market, and sold at whatever market price is offered until the portfolio is consumed, see Section 2.3.1. The function $\psi = e^{-\hat{\alpha}_0 y_1}$ in the model aims to incorporate this event as the bounce-back rate, as a slippage estimate since we cannot see the recovery in $N = 1$.

Now, let's take a look at our model displayed in Figures 4.2 to 4.4 for different values of $\hat{\alpha}_0$. Here, the red curve is the estimated market value $\hat{M}(y_1) = y_1$, where y_1 is the number of shares. The blue curves in each figure are the sales values $S(y_1)$ for three different values of $\hat{\alpha}_0$ ($\hat{\alpha}_0 = 0.1$, $\hat{\alpha}_0 = 0.5$ and the boundary case $\hat{\alpha}_0 = 1$) which determines different market states. A market with $\hat{\alpha}_0 = 0.1$ has a small liquidity cost C (Definition 2.3.3) as shown in Figure 4.2 by the small divergence between the red and the blue graphs. To the contrary, a market with $\hat{\alpha}_0 = 1$ has a much bigger liquidity cost C as seen in Figure 4.4 by the larger divergence between the red and the blue curve. Hence, the closer to 1 $\hat{\alpha}_0$ moves the bigger the liquidity cost C, as described in Section 4.1. In the latter case the $S(y_1)$ curve diverges the most from the estimated value $\hat{M}(y_1)$ negatively, and subsequently the liquidity cost C is highest for the model, with an approximate loss of 60% of the portfolio value. Actually, when $\hat{\alpha}_0 = 1$ the curve has its maximum at $y_1 = 1$ so we know from Section 4.1 that this liquidity cost equals the liquidity cost ratio $1 - \frac{1}{e} \approx 63.2\%$.

Furthermore, the model suggests in this scenario that selling one more share after the maximum has zero or negative contribution to the sales value $S(y_1)$. This degrades the stock price to zero or negative value, which is highly unlikely. Consequently, the model is constrained by $\hat{\alpha}_0 < 1$.

In general, as $\hat{\alpha}_0 \to 0^+$ the sales value $S(y_1)$ will converge towards the estimated value $\hat{M}(y_1) = y_1$, and $S(y_1) = \hat{M}(y_1)$ for $\hat{\alpha}_0 = 0$:

$$\lim_{\hat{\alpha}_0 \to 0^+} S(y_1) = \lim_{\hat{\alpha}_0 \to 0^+} y_1 e^{-\hat{\alpha}_0 y_1} = y_1 = \hat{M}(y_1).$$

It is no surprise that the sales functions $S(y_1)$ in these cases are strictly increasing within the scope of the model, since we keep adding new value. Also, the deviation between the sales value S and the estimated market value $\hat{M}$ increases after the portfolio is consumed, independently of the $\hat{\alpha}_0$-value, since we keep adding decreasing values.

The case of $N = 1$ reveals that the model $S(y_1)$ for selling a market moving portfolio is affected by the market sentiment at t_1. Also, the sales value is affected in all market sentiments $\hat{\alpha}_0$ except when $\hat{\alpha}_0 = 0$. Lastly, the investor is not in a situation where time can be utilised in order to optimise the liquidity cost C and must thus accept the market sentiments as is.

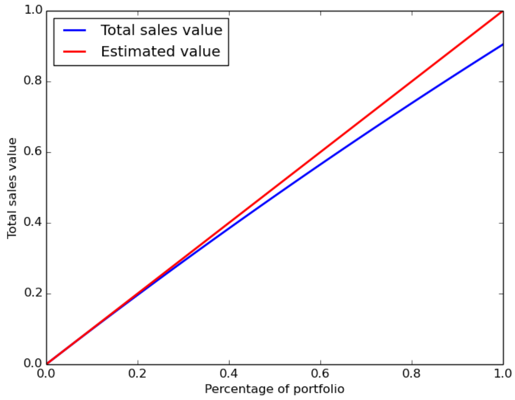


Figure 4.2: Estimated market value $\hat{M}$ versus the sales value $S(y)$ for $\hat{\alpha}_0 = 0.1$.

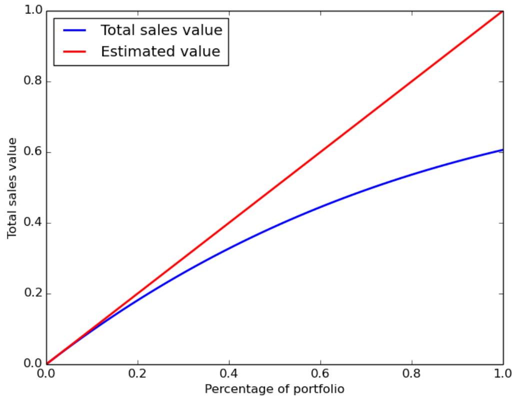


Figure 4.3: Estimated market value $\hat{M}$ versus the sales value $S(y)$ for $\hat{\alpha}_0 = 0.5$.

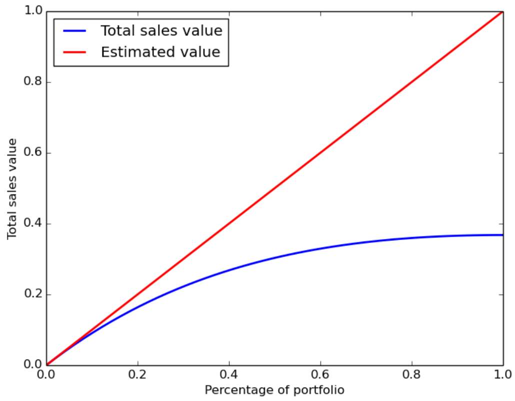


Figure 4.4: Estimated market value $\hat{M}$ versus the sales value $S(y)$ for $\hat{\alpha}_0 = 1$.

CHAPTER 5

The case $N = 2$

In this chapter we want to optimise our model S for $N = 2$, which also introduces additional parameters $\hat{\alpha}_1$, $\hat{P}_2$ and a variable y_2. We will also establish pumping, dumping and short selling strategies as we are in a two-step scenario which introduces an additional transaction opportunity. Moreover, we evaluate different market types: Bear market $\hat{P}_1 > \hat{P}_2$, Flat market $\hat{P}_1 = \hat{P}_2$ and Bull market $\hat{P}_1 < \hat{P}_2$, and how different levels of the estimated price $\hat{P}_2$ may determine which investment strategy is optimal. Let's start off by introducing the model for $N = 2$, and set forth when either pumping, dumping or short selling is the optimal strategy.

With a two-step strategy we can buy or sell y_1 and y_2 shares at times t_1 and t_2, as long as it liquidates the portfolio. In order to find the optimal investment strategy for a two-variable function we could look at the partial derivatives of the model S. However, as $y_1 + y_2 = 1$ is a constraint we have the opportunity to reduce the two-variable sales function $S(y_1, y_2)$ for $N = 2$ into a one-variable function $S(y_1)$. This will simplify the analysis. Thus, we substitute $y_2 = 1 - y_1$ into $S(y_1, y_2)$, remembering the normalised price estimate $\hat{P}_1 = 1$. Subsequently, we turn:

$$\begin{aligned} S(y_1, y_2) &= y_1 \hat{P}_1 e^{-\hat{\alpha}_0 y_1} + y_2 \hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} \\ &= y_1 e^{-\hat{\alpha}_0 y_1} + y_2 \hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} \end{aligned}$$

into,

$$\begin{aligned} S(y_1) &= y_1 e^{-\hat{\alpha}_0 y_1} + (1 - y_1) \hat{P}_2 e^{-\hat{\alpha}_0 (1 - y_1) - \hat{\alpha}_1 y_1} && (5.1) \\ &= y_1 e^{-\hat{\alpha}_0 y_1} + (1 - y_1) \hat{P}_2 e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_1 - \hat{\alpha}_0}. && (5.2) \end{aligned}$$

In order to find where the function $S(y_1)$ increases/decreases, and subsequently has stationary points, we need to find $S'(y_1)$ for our future endeavours:

$$\begin{aligned} S'(y_1) &= e^{-\hat{\alpha}_0 y_1} - \hat{\alpha}_0 y_1 e^{-\hat{\alpha}_0 y_1} - \hat{P}_2 e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_1 - \hat{\alpha}_0} \\ &\quad + (1 - y_1) \hat{P}_2 (\hat{\alpha}_0 - \hat{\alpha}_1) e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_1 - \hat{\alpha}_0} \\ &= (1 - \hat{\alpha}_0 y_1) e^{-\hat{\alpha}_0 y_1} + \hat{P}_2 \left((1 - y_1)(\hat{\alpha}_0 - \hat{\alpha}_1) - 1 \right) e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_1 - \hat{\alpha}_0}. \qquad (5.3) \end{aligned}$$

Before we continue it seems timely for a well-intended reminder that the buy request domain in $N = 1$ is $(-\infty, 0)$ and the sell request domain is $(0, \infty)$.

This is also applicable in this case as we reduced the function into a one-variable function.

Now, we reintroduce the pumping strategy from Definition 3.0.1 on page 16 which can be optimal when $S(y_1)$ has its maximum in the buy request domain $y_1 < 0$. Likewise, the dumping strategy from Definition 3.0.2 on page 17 can be optimal as $S(y_1)$ has a maximum in the sell request domain for $y_1 > 1$. The dumping boundary is equal to 1 as the dumping strategy demands the investor to sell more than initially intended, which is the entire portfolio. In order to fulfil the dumping strategy, in the case of liquidating the portfolio, the investor has to engage in a short selling strategy as defined in Definition 3.0.4 on page 17. When an investor sells on the interval $0 \leq y_1 \leq 1$ the trade is defined as a sell strategy. However, if $S(y_1)$ has a maximum on $y_1 > 1$ the investor should borrow shares from a third party and undertake a short selling strategy, which is a special case of the dumping strategy.

In order to decide on optimal investment strategies we need to perform a deeper analysis of the model S and unveil some key characteristics. This next section is an attempt to do so.

5.1 Analysis of the model S

In order to choose the optimal investment strategy we do an in-depth investigation of the model S. Hence, this section contains general propositions, corollaries and lemmas to complete the proofs for when a pumping and dumping strategy is optimal.

First we need to understand how the model acts as $y_1 \to \pm\infty$. Lemmas 5.1.1 and 5.1.2 ensures that the model does not tend towards infinity in neither case.

Lemma 5.1.1. *The function*

$$S(y_1) = y_1 e^{-\hat{\alpha}_0 y_1} + (1 - y_1)\hat{P}_2 e^{(\hat{\alpha}_0 - \hat{\alpha}_1)y_1 - \hat{\alpha}_0}$$

tends towards $-\infty$ *as* $y_1 \to -\infty$ *for* $\hat{\alpha}_0 \geq 0$ *and* $\hat{\alpha}_0 > \hat{\alpha}_1$*, and tends towards* 0 *as* $y_1 \to -\infty$ *for* $0 > \hat{\alpha}_0 > \hat{\alpha}_1$.

Proof. We start off by splitting the limit into two terms

$$\begin{aligned}\lim_{y_1 \to -\infty} S(y_1) &= \lim_{y_1 \to -\infty} y_1 e^{-\hat{\alpha}_0 y_1} + (1 - y_1)\hat{P}_2 e^{(\hat{\alpha}_0 - \hat{\alpha}_1)y_1 - \hat{\alpha}_0} \\ &= \underbrace{\lim_{y_1 \to -\infty} y_1 e^{-\hat{\alpha}_0 y_1}}_{(1)} + \underbrace{\lim_{y_1 \to -\infty} (1 - y_1)\hat{P}_2 e^{(\hat{\alpha}_0 - \hat{\alpha}_1)y_1 - \hat{\alpha}_0}}_{(2)}\end{aligned}$$

We divide the proof into the cases $\hat{\alpha}_0 \geq 0$ and $\hat{\alpha}_0 < 0$, where the latter is included for a deeper insight into the model even though it is outside the scope.

The case $\hat{\alpha}_0 \geq 0$ (negative market impact):

(1): $$\lim_{y_1 \to -\infty} y_1 e^{-\hat{\alpha}_0 y_1} = -\infty$$

and

(2): $$\lim_{y_1 \to -\infty} (1 - y_1)\hat{P}_2 e^{(\hat{\alpha}_0 - \hat{\alpha}_1)y_1 - \hat{\alpha}_0} = \lim_{y_1 \to -\infty} \frac{(1 - y_1)\hat{P}_2}{e^{-(\hat{\alpha}_0 - \hat{\alpha}_1)y_1 + \hat{\alpha}_0}}$$

$$\overset{\text{L'H}}{=} \lim_{y_1 \to -\infty} \frac{\hat{P}_2}{(\hat{\alpha}_0 - \hat{\alpha}_1)e^{-(\hat{\alpha}_0-\hat{\alpha}_1)y_1+\hat{\alpha}_0}}$$
$$= 0,$$

where L'H signals that we used L'Hôpital's rule. Hence,

$$\lim_{y_1 \to -\infty} S(y_1)_{\hat{\alpha}_0 \geq 0} = -\infty + 0 = -\infty.$$

The case $\hat{\alpha}_0 < 0$ (positive market impact):

$$(1): \quad \lim_{y_1 \to -\infty} y_1 e^{-\hat{\alpha}_0 y_1} = \lim_{y_1 \to -\infty} \frac{y_1}{e^{\hat{\alpha}_0 y_1}} \overset{\text{L'H}}{=} \lim_{y_1 \to -\infty} \frac{1}{\hat{\alpha}_0 e^{\hat{\alpha}_0 y_1}} = 0$$

and

$$\begin{aligned}(2): \quad \lim_{y_1 \to -\infty} (1-y_1)\hat{P}_2 e^{(\hat{\alpha}_0-\hat{\alpha}_1)y_1-\hat{\alpha}_0} &= \lim_{y_1 \to -\infty} \frac{(1-y_1)\hat{P}_2}{e^{-(\hat{\alpha}_0-\hat{\alpha}_1)y_1+\hat{\alpha}_0}} \\ &\overset{\text{L'H}}{=} \lim_{y_1 \to -\infty} \frac{\hat{P}_2}{(\hat{\alpha}_0 - \hat{\alpha}_1)e^{-(\hat{\alpha}_0-\hat{\alpha}_1)y_1+\hat{\alpha}_0}} \\ &= 0.\end{aligned}$$

Hence,

$$\lim_{y_1 \to -\infty} S(y_1)_{\hat{\alpha}_0 < 0} = 0 + 0 = 0.$$

■

Lemma 5.1.2. *The function*

$$S(y_1) = y_1 e^{-\hat{\alpha}_0 y_1} + (1-y_1)\hat{P}_2 e^{(\hat{\alpha}_0-\hat{\alpha}_1)y_1-\hat{\alpha}_0}$$

tends towards $-\infty$ as $y_1 \to \infty$, if $\hat{\alpha}_0 > 0$ and $\hat{\alpha}_0 > \hat{\alpha}_1$.

Proof. First we split the limit into two terms:

$$\begin{aligned}\lim_{y_1 \to \infty} S(y_1) &= \lim_{y_1 \to \infty} y_1 e^{-\hat{\alpha}_0 y_1} + (1-y_1)\hat{P}_2 e^{(\hat{\alpha}_0-\hat{\alpha}_1)y_1-\hat{\alpha}_0} \\ &= \underbrace{\lim_{y_1 \to \infty} y_1 e^{-\hat{\alpha}_0 y_1}}_{(1)} + \underbrace{\lim_{y_1 \to \infty} (1-y_1)\hat{P}_2 e^{(\hat{\alpha}_0-\hat{\alpha}_1)y_1-\hat{\alpha}_0}}_{(2)}\end{aligned}$$

We evaluate each limit by itself:

$$\begin{aligned}(1): \quad \lim_{y_1 \to \infty} y_1 e^{-\hat{\alpha}_0 y_1} &= \lim_{y_1 \to \infty} \frac{y_1}{e^{\hat{\alpha}_0 y_1}} \\ &\overset{\text{L'H}}{=} \lim_{y_1 \to \infty} \frac{1}{\hat{\alpha}_0 e^{\hat{\alpha}_0 y_1}} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}(2): \quad \lim_{y_1 \to \infty} (1-y_1)\hat{P}_2 e^{(\hat{\alpha}_0-\hat{\alpha}_1)y_1-\hat{\alpha}_0} &= -\lim_{y_1 \to \infty} (y_1-1)\hat{P}_2 e^{(\hat{\alpha}_0-\hat{\alpha}_1)y_1-\hat{\alpha}_0} \\ &= -\infty,\end{aligned}$$

where L'H signals that we used L'Hôpital's rule. Hence,

$$\lim_{y_1\to\infty} S(y_1)_{\hat{\alpha}_0>0} = 0 - \infty = -\infty.$$

■

Secondly, we want to understand how the function acts between the limits. Lemma 5.1.3 tell us that the function S has at most one maximum on $\mathbb{R}$.

Lemma 5.1.3. *The function*

$$S(y_1) = y_1 e^{-\hat{\alpha}_0 y_1} + (1-y_1)\hat{P}_2 e^{(\hat{\alpha}_0-\hat{\alpha}_1)y_1-\hat{\alpha}_0}$$

has at most one maximum on $\mathbb{R}$ *for* $\hat{\alpha}_0 > \hat{\alpha}_1$ *and* $\hat{P}_2 > 0$.

Proof. We will show that $S(y_1)$ has no more than one maximum, by showing that S has no more than two stationary points. We start off by finding the number of solutions of $S'(y_1) = 0$:

$$\begin{aligned} S'(y_1) &= 0 \\ (1-\hat{\alpha}_0 y_1)\, e^{-\hat{\alpha}_0 y_1} + \hat{P}_2\left((1-y_1)(\hat{\alpha}_0-\hat{\alpha}_1) - 1\right) e^{(\hat{\alpha}_0-\hat{\alpha}_1)y_1-\hat{\alpha}_0} &= 0. \end{aligned}$$

By rearranging the terms we get an equation with a rational function with linear polynomials in the numerator and the denominator on the left-hand side, and an exponential function on the right-hand side:

$$\begin{aligned} (1-\hat{\alpha}_0 y_1)\, e^{-\hat{\alpha}_0 y_1} &= \hat{P}_2\left(1-(1-y_1)(\hat{\alpha}_0-\hat{\alpha}_1)\right) e^{(\hat{\alpha}_0-\hat{\alpha}_1)y_1-\hat{\alpha}_0} \\ (1-\hat{\alpha}_0 y_1) &= \hat{P}_2\left((\hat{\alpha}_0-\hat{\alpha}_1)y_1 + (1-\hat{\alpha}_0+\hat{\alpha}_1)\right) e^{(2\hat{\alpha}_0-\hat{\alpha}_1)y_1-\hat{\alpha}_0} \\ \frac{-\hat{\alpha}_0 y_1 + 1}{(\hat{\alpha}_0-\hat{\alpha}_1)y_1 + (1-\hat{\alpha}_0+\hat{\alpha}_1)} &= \hat{P}_2 e^{(2\hat{\alpha}_0-\hat{\alpha}_1)y_1-\hat{\alpha}_0} \end{aligned} \tag{5.4}$$

The rational expression on the left-hand side of (5.4) and the exponential expression on the right-hand side are both strictly decreasing or strictly increasing functions for $\hat{\alpha}_0, \hat{\alpha}_1 \neq 0$.

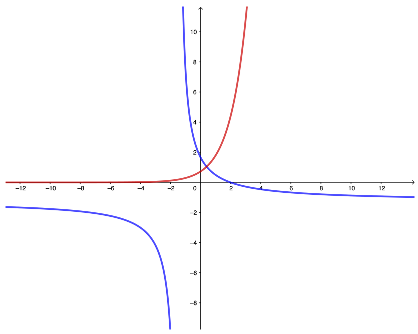
The graph defined by the rational function is a hyperbola denoted $\mathcal{H}$. The graph defined by the exponential function is denoted $\mathcal{E}$. The two graphs, $\mathcal{H}$ and $\mathcal{E}$, intersects in either zero, one, two, or infinitely many points. The different cases are illustrated in Figures 5.1(a) to 5.1(d). The number of intersections may be proved by looking at the asymptotes of the hyperbola. The horizontal asymptote of the hyperbola is given by

$$\frac{-\hat{\alpha}_0}{\hat{\alpha}_0-\hat{\alpha}_1},$$

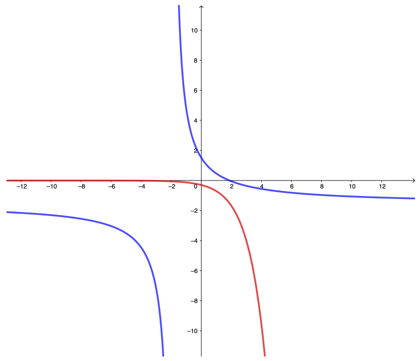
and the vertical asymptote of the hyperbola is given by

$$\begin{aligned} (\hat{\alpha}_0-\hat{\alpha}_1)y_1 + (1-\hat{\alpha}_0+\hat{\alpha}_1) &= 0 \\ y_1 &= \frac{\hat{\alpha}_0-\hat{\alpha}_1-1}{\hat{\alpha}_0-\hat{\alpha}_1} \\ y_1 &= 1 - \frac{1}{\hat{\alpha}_0-\hat{\alpha}_1}. \end{aligned}$$

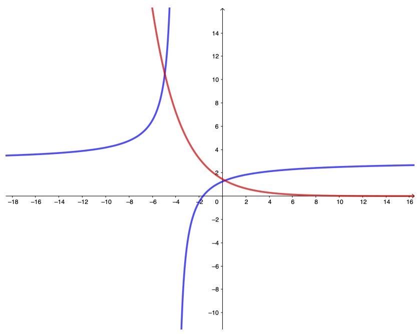
Now, based on these expressions $\mathcal{H}$ and $\mathcal{E}$ have:



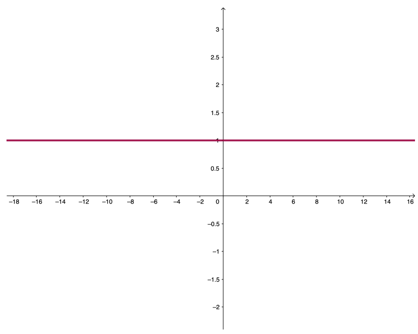
(a) The case of one solution as described in Item i.).



(b) The case of zero solutions as described in Item ii.).



(c) The case of two solutions as described in Item iii.) on the following page.



(d) The case of infinitely many solutions as described in Item iv.) on the following page.

Figure 5.1: The different scenarios of the number of possible solutions in $N = 2$ for $\mathcal{E}$ and $\mathcal{H}$.

i.) *One point of intersection if* $\hat{\alpha}_0 \geq 0$ *and* $\hat{\alpha}_0 > \hat{\alpha}_1$.

When $\hat{\alpha}_0 \geq 0$ and $\hat{\alpha}_0 > \hat{\alpha}_1$, the expression

$$\frac{-\hat{\alpha}_0}{\hat{\alpha}_0 - \hat{\alpha}_1} < 0,$$

because $-\hat{\alpha}_0 < 0$ and $\hat{\alpha}_0 - \hat{\alpha}_1 > 0$ by the constraint criteria. Thus, the graph $\mathcal{E}$ with $\hat{P}_2 > 0$ only intersects the part of $\mathcal{H}$ that lies above the horizontal asymptote. Hence, we have one point of intersection.

See Figure 5.1(a) for an illustration of this situation.

The following three cases are again outside our scope, however they are included for the reader to get a deeper understanding of the model and its general behaviour.

ii.) *Zero points of intersection if* $\hat{P}_2 < 0$ *and* $\hat{\alpha}_0 = \hat{\alpha}_1$ *or the vertical asymptote of* $\mathcal{H}$ *is at* $y_1 \leq 0$.

There are two different scenarios here. The first case is when $\hat{\alpha}_0 = \hat{\alpha}_1 > 0$, and the hyperbola $\mathcal{H}$ degenerates into a line with negative slope. The linear function does not intersect the graph of the exponential function, as $\mathcal{E}$ tends to $-\infty$ much faster than $\mathcal{H}$ in this case.

The second case is when the vertical asymptote of $\mathcal{H}$ is at $y_1 \leq 0$, which is the case that is displayed in Figure 5.1(b) on the preceding page. Either case is relevant in the general situation, however as we demand that $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$ both of these cases are outside our scope. Furthermore, as $\hat{P}_2 < 0$ the scope is also breached.

iii.) *Two points of intersection if* $\hat{\alpha}_0 < 0$ *and* $\hat{\alpha}_0 > \hat{\alpha}_1$.

We have two intersection points when the expression

$$\frac{-\hat{\alpha}_0}{\hat{\alpha}_0 - \hat{\alpha}_1} \geq 0,$$

as the exponential function $\hat{P}_2 e^{(2\hat{\alpha}_0 - \hat{\alpha}_1)y_1 - \hat{\alpha}_0} > 0$ given $\hat{P}_2 > 0$. Consequently, the graph $\mathcal{E}$ intersects the hyperbola $\mathcal{H}$ on both sides of the horizontal asymptote, and we have two points of intersection.

See Figure 5.1(c) on the previous page for an illustration of this situation.

iv.) *Infinitely many or zero intersection points if* $\hat{\alpha}_0 = \hat{\alpha}_1 = 0$.

We have zero points of intersection in the case where $\hat{\alpha}_0 = \hat{\alpha}_1 = 0$ and the graph $\mathcal{E}$ is given by $\hat{P}_2 \neq 1$ as the hyperbola $\mathcal{H}$ intersects the second axis at 1. There are infinitely many intersection points if $\hat{P}_2 = 1$ and the hyperbola $\mathcal{H}$ is defined as above. In both cases we are in breach of the constraint $\hat{\alpha}_0 > \hat{\alpha}_1$ in the model.

See Figure 5.1(d) on the preceding page for an illustration of this situation.

Consequently, $S(y_1)$ has at most one maximum on $\mathbb{R}$. ■

Item i.) describes the situation $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$, which we are investigating. As this is crucial to our constraints it seems reasonable to create a corollary in its honour.

Corollary 5.1.4. *If* $\hat{\alpha}_0 > 0$ *and* $\hat{\alpha}_0 > \hat{\alpha}_1$*, the function* S *has a single extremum, which is a global maximum.*

Proof. As we saw in Item i.), we only have a single extremum when $\hat{\alpha} > 0$. This is a global maximum, as $\lim_{y_1 \to -\infty} S(y_1) = -\infty$, and $\lim_{y_1 \to \infty} S(y_1) = -\infty$. ■

We end this section by restating that the model S has a global maximum, since S tends to $-\infty$ in both directions. Also, as S only has one extremum we have shown the existence of a unique optimum. Now, let's continue on to the investment strategies that are available in $N = 2$.

5.1.1 Pumping

For $N = 2$, pumping entails buying more stock at time t_1 (increasing the portfolio) and selling at time t_2 (liquidating the entire portfolio). Subsequently, a boundary condition for pumping is the situation where *no* purchase is made at t_1, and all shares are sold at t_2. In the normalised case in the original two-variable function $S(y_1, y_2)$, the boundary case for pumping is thus given by $y_1 = 0$, i.e. no shares are bought at t_1, and $y_2 = 1$, i.e. all shares are sold at time t_2. Since we are investigating the one-variable case we restate the boundary case which is $y_1 = 0$.

Also, we must assess when the model S demands a buy event and when it is correct to choose a sell event. This may be determined by evaluating where the model has an extremum, and under what sentiments (parameter values of $\hat{\alpha}_n$) these extremas are located. Proposition 5.1.5 ensures us that the model S for $N = 2$ has one maximum. Consequently, we want to determine when the stationary point is found at $y_1 < 0$.

When looking for stationary points it seems natural to begin with an optimisation analysis based on the derivative, in this case $S'(y_1)$. If the maximum of S is located at $y_1 < 0$, then we have a buy event and possibly a pumping situation. Conversely, if the maximum of S is located at $y_1 > 0$, then we have a sell event and possibly a dumping situation. These locations do affect the slope $S'(0)$. If $S'(0) > 0$, then the maximum is located at $y_1 > 0$, and if $S'(0) < 0$, then the maximum is located at $y_1 < 0$. Hence, investigating $S'(0)$ may reveal the location of the maximum.

We know from Corollary 5.1.4 that our model S has a single maximum. There are two possibilities for the location of the maximum:

i.) If $S'(0) \leq 0$, then the maximum is located at $y_1 < 0$ and there is an optimal pumping strategy.

ii.) If $S'(0) > 0$, then the maximum is located at $y_1 > 0$ and there is no optimal pumping strategy.

On this note, we wish to prove that if $S'(0) \geq 0$ for $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$, then there is no optimal pumping strategy. The following proposition summarises the finding:

Proposition 5.1.5. *There is no optimal pumping strategy for $N = 2$ in the model*

$$S(y_1, \ldots, y_N) = \sum_{n=1}^{N} y_n \hat{P}_n e^{-\hat{\alpha}_{n-1} y_1 - \hat{\alpha}_{n-2} y_2 - \cdots - \hat{\alpha}_0 y_n},$$

subject to $y_1 + y_2 + \cdots + y_N = 1$ and $1 > \hat{\alpha}_0 > \hat{\alpha}_1 > 0$, if $S'(0) > 0$ for the rewritten one variable function $S(y_1)$.

Proof. In order to show that a pumping strategy will not be the optimal trading strategy when $S'(0) \geq 0$, we must show that i.) The function $S(y_1)$ does not tend towards ∞ as $y_1 \to -\infty$. This is to ensure that the model does not demand the investor to pump the stock indefinitely.

ii.) There is at most one maximum on $\mathbb{R}$.

iii.) If there is a maximum, then it is located at $y_1 \geq 0$.

Lemma 5.1.1 ensures us that $S(y_1)$ does not tend towards ∞ as $y_1 \to -\infty$.

Lemma 5.1.3 ensures that $S(y_1)$ has at most one maximum on $\mathbb{R}$.

Lastly, we have to prove that if there is a maximum then it is located at $y_1 \geq 0$. Since there exists a maximum by Corollary 5.1.4, we use the assumption $S'(0) \geq 0$ to prove that it must be located at $y_1 \geq 0$. Lemma 5.1.2 ensures that $S(y_1)$ tends to $-\infty$ as y_1 tends to ∞ under the constraint of $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$.

Hence, since $S'(0) \geq 0$ and the function does not tend towards infinity we must have a maximum for $y_1 > 0$. ■

In conclusion, there is no available optimal pumping strategy for $S(y_1)$ in $N = 2$ under the constraint $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$ and given $S'(0) \geq 0$. However, there may be optimal pumping strategies available under certain market sentiments as long as $S'(0) < 0$. We will take a closer look at this in Section 5.3 where we look at the model from a numerical point of view. Now, let's take a look at the dumping investment strategy.

5.1.2 Dumping

We continue in the same manner as in the pumping case. However, in the dumping scenario the investor sells a market moving block of shares in order to pull down the price for then to repurchase the same stock at a later stage. This repurchase is only possible in a multi-step scenario if the investor utilises a short selling strategy, as the repurchased stock then will be returned to the lender. That way the investor liquidates the entire portfolio, which is our task.

Now, in the two variable function the boundary case for dumping is given by $y_1 = 1$ and $y_2 = 0$. This is the case where the investor liquidates the entire portfolio at t_1, and does not buy back any shares at t_2. As we reduced the two-variable model $S(y_1, y_2)$ into a one-variable model $S(y_1)$ it seems polite to remind the reader of the one-variable model:

$$S(y_1) = y_1 e^{-\hat{\alpha}_0 y_1} + (1 - y_1)\hat{P}_2 e^{(\hat{\alpha}_0 - \hat{\alpha}_1)y_1 - \hat{\alpha}_0}$$

with its derivative

$$S'(y_1) = (1 - \hat{\alpha}_0 y_1)\, e^{-\hat{\alpha}_0 y_1} + \hat{P}_2\left((1 - y_1)(\hat{\alpha}_0 - \hat{\alpha}_1) - 1\right) e^{(\hat{\alpha}_0 - \hat{\alpha}_1)y_1 - \hat{\alpha}_0}.$$

Likewise, we also redefine the boundary conditions.

So, the boundary condition for the one-variable model $S(y_1)$ for the dumping strategy is $y_1 = 1$, as this is the complete liquidation of the portfolio. It is also the boundary between a sell and the dumping strategy with short selling. This means that we dump stock if the maximum should be located at $y_1 > 1$. In our search for the possible dumping scenarios we start off by using the derivative at the boundary $S'(1)$, as its sign helps us determine the location of the maximum. Based on Corollary 5.1.4 our model S has a single maximum. Once again, there are two possible locations for the maximum:

i.) If $S'(1) \leq 0$, then the maximum is not located at $y_1 > 1$ and there is no optimal dumping strategy.

ii.) If $S'(1) > 0$, then the maximum is located at $y_1 > 1$ and there is an optimal dumping strategy.

Subsequently, the investor should never choose a dumping strategy if $S'(1) \leq 0$. We summarise this as a proposition:

Proposition 5.1.6. *There is no dumping strategy that is the optimal investment strategy for $N = 2$ in the model*

$$S(y_1, \ldots, y_N) = \sum_{n=1}^{N} y_n \hat{P}_n e^{-\hat{\alpha}_{n-1} y_1 - \hat{\alpha}_{n-2} y_2 - \cdots - \hat{\alpha}_0 y_n},$$

subject to $y_1 + y_2 + \cdots + y_N = 1$ and $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$, if $S'(1) \leq 0$ for the rewritten one variable function $S(y_1)$.

Proof. Let's build this proof on the same premises as in the proof of Proposition 5.1.5. Hence, we must show that

i.) The function $S(y_1)$ does not tend towards ∞ as $y_1 \to \infty$, to ensure that the function does not demand infinite dumping.

ii.) There is at most one maximum on $\mathbb{R}$.

iii.) If there exists a maximum, then it is not located at $y_1 > 1$.

Lemma 5.1.2 ensures us that $S(y_1)$ does not tend towards ∞ as $y_1 \to \infty$.

Lemma 5.1.3 ensures that S has at most one maximum on $\mathbb{R}$.

Lastly, we have to prove that if there is a maximum then it is located at $y_1 \leq 1$. There exists a maximum by Corollary 5.1.4, and we use the assumption $S'(1) \leq 0$ to prove that it must be located at $y_1 \leq 1$. Lemma 5.1.2 ensures that $S(y_1)$ tends to $-\infty$ as y_1 tends to ∞ under the constraint of $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$.

Hence, since $S'(1) \leq 0$ and the function does not tend towards infinity we must have a maximum for $y_1 < 1$. ∎

In conclusion, there is no dumping strategy for $S(y_1)$ in $N = 2$ under the constraint $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$ and given $S'(1) \leq 0$. However, there may exist dumping strategies under certain market sentiments as long as $S'(1) > 0$. We will deep dive into this in Section 5.3 where we look at the model from a numerical point of view. In the next section we dive into how market strategies are influenced by market sentiments.

5.2 Market sentiments and investment strategies

From empirical observations we know that different investment strategies are utilised in order to optimise return in different markets. In some cases the investment strategies are easily implemented, while in others it may demand access to sufficiently large funds. On that note, it is fun to mention Black Wednesday [Ken20]. One of the most famous short positions in history, when George Soros decided to bet against the British Pound in August and September of 1992, which made him a billion dollars and almost broke the Bank of England. Even the Queen was involved as the British government tried to fend off the attack.

Now, in light of different investment strategies we also need to define different market sentiments as these two are closely linked. Hence, in this numerical analysis we want to evaluate the model S for different market sentiments $\hat{\alpha}_0$ and $\hat{\alpha}_1$ and estimated prices $\hat{P}_2$ to determine when different investment strategies should be implemented. We start by restating the three types of markets a stock can operate within at a given point in time:

1.) Bear market as $\hat{P}_1 = 1 > \hat{P}_2$,

2.) Flat market as $\hat{P}_2 = \hat{P}_1 = 1$ and

3.) Bull market as $\hat{P}_1 = 1 < \hat{P}_2$.

Naturally, the estimated price $\hat{P}_2$ has to be positive. Also, the different market sentiments $\hat{\alpha}_n$ can all operate within the different market types.

In general, a given strategy can be used in different types of markets to achieve different outcomes, however without being the optimal solution in each case. For instance, a pumping strategy can be used in the following way. Given a Bull market $\hat{P}_2 > 1$ an investor may wish to contribute to a price increase and thus initiate a pumping strategy. The same may be the case in a Flat market in order to get the stock price moving. A pumping strategy may also be adopted if the investor wishes to slow down a negative price development. Hence, there are several situations where a certain investment strategy can be utilised in order to optimise a trade.

Based on these descriptions we now look further into the estimated price $\hat{P}_2$, which determines the market type in terms of the optimal trading volumes y_1^* represented by a percentage of portfolio. We know that different optimal trading volumes defined by the model S yields different investment strategies as illustrated in Figure 5.2. The figure shows different estimated prices $\hat{P}_2$ as a function of the optimal trading volume as a percentage of portfolio y_1^* for four different market sentiments $\hat{\alpha}_1$ given a fixed $\hat{\alpha}_0$ depicted with different colours. A first glance analysis reveals that if the optimal trading volume $y_1^* \in (-\infty, 0)$ then $\hat{P}_2$ must be at a certain level (intersection of the second axis and the graph) in order to pump the stock, rather than initiate in a buy strategy. Figure 5.2 only illustrate some values for $\hat{\alpha}_0$ and $\hat{\alpha}_1$, but it does give an impression of how the model behaves in the different domains. However, if the optimal trading volume $y_1^* \in (0, \infty)$ then $\hat{P}_2$ must be under another certain level (less than the intersection of the vertical dotted line and the graph) in order to dump and short sell the stock, rather than engage in a sell strategy. In the following we deep dive into $\hat{P}_2(y_1^*)$ for a more rigorous analysis of the market type and investment strategies.

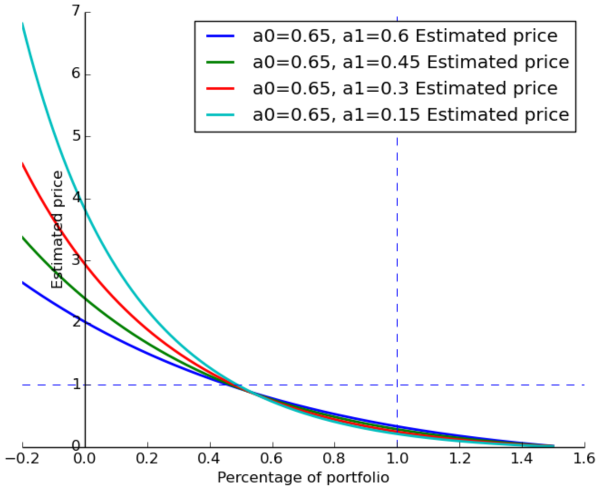


Figure 5.2: The estimated price $\hat{P}_2(y_1^*)$ for illustrative $\hat{\alpha}$-values on the interval $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$. The horizontal dotted line $\hat{P}_2 = 1$ is a Flat market, where the area above is a Bull market and the area below is a Bear market. The vertical dotted line $y_1^* = 1$ is the dumping boundary line. The second axis is the estimated price $\hat{P}_2(y_1^*)$ and also represents the pumping boundary line.

So, in order to evaluate $\hat{P}_2$ we use the derivative of the model $S(y_1)$ in Equation (5.3) to find an expression for $\hat{P}_2$ in terms of the optimised transaction volume y^*:

$$S'(y^*) = (1 - \hat{\alpha}_0 y^*)\, e^{-\hat{\alpha}_0 y^*} + \hat{P}_2(y^*)\, ((1 - y^*)(\hat{\alpha}_0 - \hat{\alpha}_1) - 1)\, e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y^* - \hat{\alpha}_0} = 0$$

which implies that

$$\hat{P}_2(y^*) = \frac{(1 - \hat{\alpha}_0 y^*) e^{-\hat{\alpha}_0 y^*}}{\Big(1 - (1 - y^*)(\hat{\alpha}_0 - \hat{\alpha}_1)\Big) e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y^* - \hat{\alpha}_0}} \tag{5.5}$$

$$= \frac{1 - \hat{\alpha}_0 y^*}{(\hat{\alpha}_0 - \hat{\alpha}_1) y^* + 1 - \hat{\alpha}_0 + \hat{\alpha}_1} e^{(-2\hat{\alpha}_0 + \hat{\alpha}_1) y^* + \hat{\alpha}_0}. \tag{5.6}$$

Remember that $\hat{P}_2$ is the estimated price, so keeping $\hat{\alpha}_0$ and $\hat{\alpha}_1$ constant we are looking to establish at what levels (type of market) it favours certain investment strategies.

From our previous endeavours we know that if S has its maximum at $y_1^* > 1$ the investor should short sell or dump the stock. If S has its maximum at $y_1^* < 0$ the investor should buy or pump the stock, while if $0 < y_1^* < 1$ the investor should initiate a sell transaction. Also, the boundary case in the pumping situation is $y_1^* = 0$, and the dumping situation has boundary $y_1^* = 1$. Subsequently, the estimated prices at the boundaries $\hat{P}_2(0)$ and $\hat{P}_2(1)$

yield the condition for when to implement the different investment strategies. Equation (5.6) is used to find the boundary estimated prices $\hat{P}_2(0)$ and $\hat{P}_2(1)$:

$$\hat{P}_2(0) = \frac{1}{1-\hat{\alpha}_0+\hat{\alpha}_1} e^{\hat{\alpha}_0}, \tag{5.7}$$

and

$$\hat{P}_2(1) = (1-\hat{\alpha}_0)\, e^{\hat{\alpha}_1-\hat{\alpha}_0}, \tag{5.8}$$

where

$$\hat{P}_2(0) = \frac{1}{1-\hat{\alpha}_0+\hat{\alpha}_1} e^{\hat{\alpha}_0} > \frac{1}{1-(\hat{\alpha}_0-\hat{\alpha}_1)} > 1$$

and

$$\hat{P}_2(1) = (1-\hat{\alpha}_0)\, e^{\hat{\alpha}_1-\hat{\alpha}_0} < (1-\hat{\alpha}_0) < 1.$$

Based on Equations (5.7) and (5.8) we get the following relations between market type and investment strategies:

- If $y_1^* < 0$, which is the pumping request domain, the estimated price $\hat{P}_2(y^*) > \hat{P}_2(0) > 1$. Then the pumping strategy is optimal in a Bull market since $\hat{P}_2(y_1^*) > 1$ and $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$.
- If $0 < y_1^* < 1$, which is the traditional sell domain, the estimated price $\hat{P}_2(y_1^*) \in \left(\hat{P}_2(1), \hat{P}_2(0)\right) = (1-\epsilon_1, 1+\epsilon_2)$ for some $\epsilon_1, \epsilon_2 > 0$. Thus, the trading strategy is available in all three markets (Bear $\hat{P}_2 < 1$, Flat $\hat{P}_2 = 1$ and Bull markets $\hat{P}_2 > 1$) for $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$.
- If $y_1^* > 1$, which is the dumping request domain, the estimated price is $\hat{P}_2(y_1^*) < \hat{P}_2(1) < 1$. Then the dumping strategy is optimal in a Bear market since $\hat{P}_2(y_1^*) < 1$ and $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$.

Consequently, we confirmed and elaborated on the first glance analysis: There are levels where the estimated price $\hat{P}_2$ triggers a certain investment strategy, and these investment strategies are found in distinct types of markets as displayed in Figure 5.2.

5.3 The numerical analysis

As previously mentioned the case $N = 2$ with $n = 1, 2$ models the situation where an investor sells the entire portfolio in two, not necessarily equal, blocks. As stated in the case $N = 1$ the investor is restricted to a certain liquidation time period. This time restriction limits the possibility for the price to bounce back to its original level between trades. The bounce-back function ψ in the model is to be viewed as a discount factor.

Our numerical investigation of the case $N = 2$ starts with a visit to the Lagrange Multiplier method in Theorem 2.2.7. It is used to maximise the function $S(y_1, y_2)$, as we have constraints, in order to use the Fixed Point method Theorem 2.2.6. Based on our model in (1.7) we get

$$S(y_1, y_2) = y_1 e^{-(\hat{\alpha}_0 y_1)} + y_2 \hat{P}_2 e^{-(\hat{\alpha}_0 y_2 + \hat{\alpha}_1 y_1)}$$

subject to

$$g(y_1, y_2) = y_1 + y_2 - 1$$

where y_1 and y_2 are weighted blocks of shares as a percent of the total portfolio.

The Lagrangian is thus given by

$$\begin{aligned}
\mathcal{L}(y_1, y_2, \lambda) &= S(y_1, y_2) + \lambda g(y_1, y_2) \\
&= y_1 e^{-(\hat{\alpha}_0 y_1)} + y_2 \hat{P}_2 e^{-(\hat{\alpha}_0 y_2 + \hat{\alpha}_1 y_1)} + \lambda(y_1 + y_2 - 1) \\
&= y_1 \hat{E}_1 + y_2 \hat{E}_2 + \lambda(y_1 + y_2 - 1)
\end{aligned}$$

where

$$\begin{aligned}
\hat{E}_1 &= e^{-\hat{\alpha}_0 y_1} \\
\hat{E}_2 &= \hat{P}_2 e^{-(\hat{\alpha}_0 y_2 + \hat{\alpha}_1 y_1)}
\end{aligned}$$

as described on page 4. In order to optimise the sales value $S(y_1, y_2)$ contingent to the constraint $g(y_1, y_2) = 1 - y_1 - y_2$ we find the derivative of the Lagrangian, $\mathcal{L}(y_1, y_2, \lambda)$ with respect to y_1, y_2 and λ, and equate these expressions to 0 in order to solve the system of equations. Then,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y_1} &= \hat{E}_1 + y_1 \frac{\partial \hat{E}_1}{\partial y_1} + y_2 \frac{\partial \hat{E}_2}{\partial y_1} + \lambda, \\
\frac{\partial \mathcal{L}}{\partial y_2} &= y_1 \frac{\partial \hat{E}_1}{\partial y_2} + \hat{E}_2 + y_2 \frac{\partial \hat{E}_2}{\partial y_2} + \lambda, \text{ and} \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= y_1 + y_2 - 1,
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial \hat{E}_1}{\partial y_1} &= -\hat{\alpha}_0 e^{-\hat{\alpha}_0 y_1} = -\hat{\alpha}_0 \hat{E}_1, \\
\frac{\partial \hat{E}_1}{\partial y_2} &= 0, \\
\frac{\partial \hat{E}_2}{\partial y_1} &= -\hat{\alpha}_1 \hat{P}_2 e^{-(\hat{\alpha}_0 y_2 + \hat{\alpha}_1 y_1)} = -\hat{\alpha}_1 \hat{E}_2, \text{ and} \\
\frac{\partial \hat{E}_2}{\partial y_2} &= -\hat{\alpha}_0 \hat{P}_2 e^{-(\hat{\alpha}_0 y_2 + \hat{\alpha}_1 y_1)} = -\hat{\alpha}_0 \hat{E}_2.
\end{aligned}$$

Thus, we get the following system of equations

$$\hat{E}_1 - \hat{\alpha}_0 \hat{E}_1 y_1 - \hat{\alpha}_1 \hat{E}_2 y_2 + \lambda = 0, \tag{5.9}$$

$$\hat{E}_2 - \hat{\alpha}_0 \hat{E}_2 y_2 + \lambda = 0, \text{ and} \tag{5.10}$$

$$y_1 + y_2 - 1 = 0. \tag{5.11}$$

Equating the two first equations by eliminating λ gives

$$\hat{E}_1 - \hat{\alpha}_0 \hat{E}_1 y_1 - \hat{\alpha}_1 \hat{E}_2 y_2 = \hat{E}_2 - \hat{\alpha}_0 \hat{E}_2 y_2.$$

Solving this for y_2 gives

$$y_2 = \frac{\hat{E}_1(1 - \hat{\alpha}_0 y_1) - \hat{E}_2}{\hat{E}_2(\hat{\alpha}_1 - \hat{\alpha}_0)}.$$

Substituting y_2 into the third equation (5.11) yields an expression for y_1:

$$\begin{aligned} 1 &= y_1 + \frac{\hat{E}_1(1 - \hat{\alpha}_0 y_1) - \hat{E}_2}{\hat{E}_2(\hat{\alpha}_1 - \hat{\alpha}_0)} \\ y_1 &= \frac{\hat{E}_2(\hat{\alpha}_1 - \hat{\alpha}_0) + \hat{E}_2 - \hat{E}_1}{\hat{E}_2(\hat{\alpha}_1 - \hat{\alpha}_0) - \hat{\alpha}_0 \hat{E}_1} \qquad (5.12) \\ &= \frac{(\hat{\alpha}_1 - \hat{\alpha}_0 + 1)\, \hat{P}_2 e^{-(\hat{\alpha}_0 y_2 + \hat{\alpha}_1 y_1)} - e^{-\hat{\alpha}_0 y_1}}{(\hat{\alpha}_1 - \hat{\alpha}_0) \hat{P}_2 e^{-(\hat{\alpha}_0 y_2 + \hat{\alpha}_1 y_1)} - \hat{\alpha}_0 e^{-\hat{\alpha}_0 y_1}} \end{aligned}$$

Now, we find an expression for y_2:

$$\begin{aligned} y_2 &= 1 - y_1 \\ y_2 &= \frac{\hat{E}_1(1 - \hat{\alpha}_0) - \hat{E}_2}{\hat{E}_2(\hat{\alpha}_1 - \hat{\alpha}_0) - \hat{\alpha}_0 \hat{E}_1}, \qquad (5.13) \\ &= \frac{(1 - \hat{\alpha}_0) e^{-\hat{\alpha}_0 y_1} - \hat{P}_2 e^{-(\hat{\alpha}_0 y_2 + \hat{\alpha}_1 y_1)}}{(\hat{\alpha}_1 - \hat{\alpha}_0) \hat{P}_2 e^{-(\hat{\alpha}_0 y_2 + \hat{\alpha}_1 y_1)} - \hat{\alpha}_0 e^{-\hat{\alpha}_0 y_1}} \qquad (5.14) \end{aligned}$$

Thus we managed to find an analytic solutions for y_1 and y_2, if we neglect that $\hat{E}_1$ is a function of y_1 and that $\hat{E}_2$ is a function of y_1 and y_2. However, this dependency is destructive for an analytic solution, as the standard method of separating variables is pivotal to get the right answer. Nonetheless, Equations (5.13) and (5.14) are useful together with the Lagrangian to optimise the model S in the Fixed Point method. The reason is that the Fixed Point method iterates recursively, and thus works well when separating variables is problematic. While running the Fixed Point method with Equations (5.13) and (5.14) we found that the it became unstable in a Bull market due to the complexity of the system of equations and how they intertwine. Subsequently, we introduce the Gradient Ascent method Theorem 2.2.10 as we work with the Bull market.

Appendices A.2.1 to A.2.4 shows the script for the Fixed Point method and Gradient Ascent method that is used to investigate different values for $\hat{\alpha}_0$, $\hat{\alpha}_1$, $\Delta\hat{\alpha}$ and the bounce-back function ψ for cases $\hat{P}_1 > \hat{P}_2$ (Bear market), $\hat{P}_1 = \hat{P}_2$ (Flat market) and $\hat{P}_1 < \hat{P}_2$ (Bull market) in order to find optimal solutions for y_1^*, y_2^* and $S(y_1^*, y_2^*)$. The following numerical sections also contain tables describing the development of $\hat{\alpha}_0$, $\hat{\alpha}_1$, $\Delta\hat{\alpha} = \hat{\alpha}_0 - \hat{\alpha}_1$, y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1,0)$ and ΔS, where the analysis deep dive into the first five parameters. The parameters $S(1,0)$ and $\Delta S = S(y_1^*, y_2^*) - S(1,0)$ are included as supporting material to shed further light on the analysis and for the reader to enjoy. The following analysis is based on the results of these scripts and tables.

5.3.1 The case of $\hat{P}_1 > \hat{P}_2$: Bear market

A Bear market is defined by $\hat{P}_1 > \hat{P}_2$, where the market price is expected to decline in the near future. The following scenarios look at a Bear market with strategically chosen $\hat{\alpha}_0$s, $\hat{\alpha}_1$s and $\hat{P}_1 = 1$.

There are three parameters that influence the behaviour of the model: The market sentiment $\hat{\alpha}_0$ at t_1, the market sentiment $\hat{\alpha}_1$ at t_2 and the estimated price $\hat{P}_2$ given as the growth factor relative to $\hat{P}_1 = 1$. Also, the difference $\Delta\hat{\alpha} = \hat{\alpha}_0 - \hat{\alpha}_1$ plays an important role when determining what actions to take to optimise the sales value S in order to minimise the liquidity cost C. The parameters influence the model to different degrees, where $\Delta\hat{\alpha}$ seems to be more important than $\hat{\alpha}_n$ independently, and $\hat{P}_2$ is a scaling factor. Also, it seems friendly to remind the reader that $\hat{\alpha}_n > 0$ means that the market will react negatively to a sell event.

5.3.1.1 The behaviour of model S in a Bear market

To analyse the model S numerically we have used the PYTHON script for Fixed Point method in Appendices A.2.1 and A.2.2. For illustrative purposes we have used the market drop of 50% to make a general description of the model in a Bear market to emphasise the outcomes. We tested numerous values for $\hat{P}_2$ without loss of generality. Limitations of numerical methods are discussed in Chapter 2.

The rest of this Bear market section will be divided into a general view point of the model through, *The effect of changes in $\hat{\alpha}_0$, $\hat{\alpha}_1$ and $\hat{P}_2$*, and a short bullet point list of examples describing *Investment decisions for $\hat{\alpha}_0$, $\hat{\alpha}_1$ and $\hat{P}_2$* based on specific data sets.

The effect of changes in $\hat{\alpha}_0$, $\hat{\alpha}_1$ and $\hat{P}_2$

In a Bear market the model S displays one type of behaviour, and it is determined by the difference $\Delta\hat{\alpha}$ and the location of $\hat{\alpha}_0$ on the $\hat{\alpha}$-domain $(0, 1)$. Also, the market drop $(1 - \hat{P}_2)$ and the size of $\Delta\hat{\alpha}$ determine the location of the transition points b where $S(y^*)$ changes from increasing to decreasing. In general, we discovered the following: Given the $\hat{\alpha}$-domain $(0, 1)$ with $0 < a < b < c < 1$, for some transition points a, b and c, and $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$, then for a, b and c defined by $\hat{P}_2$, $\Delta\hat{\alpha}$ and $\hat{\alpha}_0$ we have that

i.) if $\hat{\alpha}_0 \in (0, a)$ and $\hat{\alpha}_0$ and $\Delta\hat{\alpha}$ decrease then the sales value $S(y^*)$ increases. See Table 5.1 and Figures 5.7 and 5.8.

ii.) if $\hat{\alpha}_0 \in (a, c)$ then the sales value $S(y^*)$ increases as $\Delta\hat{\alpha}$ moves from the transition point b to 0, and as $\Delta\hat{\alpha}$ moves from b to $\hat{\alpha}_0$. See Table 5.2 and Figure 5.5.

iii.) if $\hat{\alpha}_0 \in (c, 1)$ and $\hat{\alpha}_1$ increases as $\Delta\hat{\alpha} \to 0$ then the sales value $S(y^*)$ decreases. See Table 5.3 and Figures 5.3, 5.4 and 5.6.

Furthermore, we discovered that S behaves differently dependent on the location of $\hat{\alpha}_0$ and $\hat{\alpha}_1$ on the $\hat{\alpha}$-domain $(0, 1)$. We also discovered that, given a fixed $\Delta\hat{\alpha}$ with $\hat{\alpha}_0$ close to 1 on the $\hat{\alpha}$-domain then S has the bigger liquidity

Table 5.1: Chosen differences $\Delta\hat{\alpha} = \hat{\alpha}_0 - \hat{\alpha}_1$ on the lower interval $(0, a)$ for $\hat{P}_1 > \hat{P}_2 = 0.5$ with calculated values for y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1, 0)$ and ΔS.

$\mathbf{\hat{P}_1 > \hat{P}_2 = 0.5}$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1, 0)$	ΔS
0.0001	$1e-07$	0.0001	1725.4129	-1724.4129	427.6768	0.9999	426.6769
0.0001	$1e-06$	0.0001	1732.5281	-1731.5281	429.2733	0.9999	428.2734
0.0001	$9e-05$	0.0000	2912.1954	-2911.1954	677.9729	0.9999	676.9730
0.1	$1e-07$	0.1000	2.1793	-1.1793	1.0891	0.9048	0.1843
0.1	0.0001	0.0999	2.1801	-1.1801	1.0892	0.9048	0.1844
0.1	0.001	0.0990	2.1871	-1.1871	1.0906	0.9048	0.1857
0.1	0.05	0.0500	2.6755	-1.6755	1.1809	0.9048	0.2761
0.1	0.09	0.0100	3.3257	-2.3257	1.2970	0.9048	0.3922
0.3	$1e-06$	0.3000	1.0274	-0.0274	0.7411	0.7408	0.0003
0.3	0.0001	0.2999	1.0275	-0.0275	0.7411	0.7408	0.0003
0.3	0.12	0.1800	1.1423	-0.1423	0.7461	0.7408	0.0053
0.3	0.135	0.1650	1.1600	-0.1600	0.7473	0.7408	0.0065
0.3	0.15	0.1500	1.1786	-0.1786	0.7486	0.7408	0.0078
0.3	0.29	0.0100	1.4124	-0.4124	0.7696	0.7408	0.0288

Table 5.2: Chosen differences $\Delta\hat{\alpha} = \hat{\alpha}_0 - \hat{\alpha}_1$ on the middle interval (a, c) for $\hat{P}_1 > \hat{P}_2 = 0.5$ with calculated values for y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1, 0)$ and ΔS.

$\mathbf{\hat{P}_1 > \hat{P}_2 = 0.5}$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1, 0)$	ΔS
0.35	$1e-09$	0.3500	0.9447	0.0553	0.7058	0.7047	0.0012
0.35	0.0001	0.3499	0.9447	0.0553	0.7058	0.7047	0.0012
0.35	0.12	0.2300	1.0239	-0.0239	0.7049	0.7047	0.0002
0.35	0.135	0.2150	1.0358	-0.0358	0.7051	0.7047	0.0004
0.35	0.15	0.2000	1.0482	-0.0482	0.7054	0.7047	0.0007
0.35	0.3	0.0500	1.2080	-0.2080	0.7136	0.7047	0.0090
0.35	0.3449	0.0051	1.2729	-0.2729	0.7185	0.7047	0.0138
0.4	$1e-05$	0.4000	0.8824	0.1176	0.6761	0.6703	0.0058
0.4	0.0001	0.3999	0.8825	0.1175	0.6761	0.6703	0.0058
0.4	0.1	0.3000	0.9289	0.0711	0.6721	0.6703	0.0018
0.4	0.3	0.1000	1.0643	-0.0643	0.6713	0.6703	0.0010
0.4	0.35	0.0500	1.1110	-0.1110	0.6731	0.6703	0.0027
0.4	0.37	0.0300	1.1316	-0.1316	0.6740	0.6703	0.0037
0.4	0.39	0.0100	1.1534	-0.1534	0.6751	0.6703	0.0048
0.45	$1e-05$	0.4500	0.8339	0.1661	0.6501	0.6376	0.0124
0.45	0.0001	0.4499	0.8339	0.1661	0.6500	0.6376	0.0124
0.45	0.1	0.3500	0.8684	0.1316	0.6444	0.6376	0.0067
0.45	0.3	0.1500	0.9649	0.0351	0.6380	0.6376	0.0003
0.45	0.35	0.1000	0.9968	0.0032	0.6376	0.6376	0.0000
0.45	0.4	0.0500	1.0328	-0.0328	0.6379	0.6376	0.0003
0.45	0.449	0.0010	1.0728	-0.0728	0.6388	0.6376	0.0011

Table 5.3: Chosen differences $\Delta\hat{\alpha} = \hat{\alpha}_0 - \hat{\alpha}_1$ on the upper interval $(c, 1)$ for $\hat{P}_1 > \hat{P}_2 = 0.5$ with calculated values for y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1,0)$ and ΔS.

$\hat{\mathbf{P}}_1 > \hat{\mathbf{P}}_2 = 0.5$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1,0)$	ΔS
0.47	0.0001	0.4699	0.8173	0.1827	0.6404	0.6250	0.0154
0.47	0.01	0.4600	0.8201	0.1799	0.6398	0.6250	0.0148
0.47	0.1	0.3700	0.8481	0.1519	0.6343	0.6250	0.0093
0.47	0.3	0.1700	0.9332	0.0668	0.6263	0.6250	0.0013
0.47	0.4	0.0700	0.9921	0.0079	0.6250	0.6250	0.0000
0.47	0.469	0.0010	1.0416	-0.0416	0.6254	0.6250	0.0004
0.9	0.0001	0.8999	0.6352	0.3648	0.4900	0.4066	0.0834
0.9	0.01	0.8900	0.6354	0.3646	0.4891	0.4066	0.0826
0.9	0.1	0.8000	0.6373	0.3627	0.4819	0.4066	0.0753
0.9	0.5	0.4000	0.6549	0.3451	0.4544	0.4066	0.0478
0.9	0.75	0.1500	0.6748	0.3252	0.4408	0.4066	0.0342
0.9	0.89	0.0100	0.6894	0.3106	0.4343	0.4066	0.0277
0.99	0.0001	0.9899	0.6161	0.3839	0.4660	0.3716	0.0945
0.99	0.01	0.9800	0.6161	0.3839	0.4652	0.3716	0.0937
0.99	0.1	0.8900	0.6164	0.3836	0.4582	0.3716	0.0866
0.99	0.5	0.4900	0.6243	0.3757	0.4313	0.3716	0.0597
0.99	0.75	0.2400	0.6358	0.3642	0.4176	0.3716	0.0461
0.99	0.9	0.0900	0.6454	0.3546	0.4105	0.3716	0.0389
0.99	0.98	0.0100	0.6514	0.3486	0.4070	0.3716	0.0354

cost C than if $\hat{\alpha}_0$ is located close to 0 on the $\hat{\alpha}$-domain as shown in Tables 5.4, 5.5 and 5.7. Hence, there is a sliding decrease in S which is an increase in the liquidity cost as the location of $\hat{\alpha}_0$ given the fixed $\Delta\hat{\alpha}$ moves towards 1 on the $\hat{\alpha}$-domain. Table 5.8 illustrate an example of the finding.

We also see that the volume y_1^* the investor has to acquire at t_1 is increasing as $\hat{\alpha}_0$ with fixed $\Delta\hat{\alpha}$ moves towards 0. This is illustrated in Table 5.8 and can also be seen comparatively across the rows in Tables 5.4, 5.5 and 5.7. Moreover, in the lower half of the $\hat{\alpha}$-domain the model suggests a short selling position when $\hat{\alpha}_0$ given fixed $\Delta\hat{\alpha}$ is located there.

Tables 5.4, 5.5 and 5.7 further illustrate that the range of the volume y_1^* is much greater as the location of $\hat{\alpha}_0$ given fixed $\Delta\hat{\alpha}$ decreases and stays close to 0 in the lower range (illustratively described by $\Delta y_{1,l} = 163.345$) than in the middle range ($\Delta y_{1,m} = 0.3753$) and upper range ($\Delta y_{1,u} = 0.032$). This lends evidence to more stable markets as $\hat{\alpha}_0$ given fixed $\Delta\hat{\alpha}$ is located at the upper end of the $\hat{\alpha}$-domain. At the same time the liquidity cost C is at its highest as the sales value S is at its lowest. The respective ranges in S from lower range to upper range on the $\hat{\alpha}$-domain in the illustrated tables are: $S_l = (0.4864, 41.0106)$, $S_m = (0.4724, 0.6065)$ and $S_u = (0.4590, 0.4037)$.

Moreover, as $\hat{\alpha}_0$ increases the slippage of S displayed by the bounce-back function ψ increases as well. Consequently, the investor may not benefit from a short position as the market struggles to recover in this situation. Contrary, if $\hat{\alpha}_1$ decreases the market will bounce back faster, which in turn speaks for a delay in sales as the sales value S increases. This is illustrated in Figure 5.3.

Table 5.6 alludes to generality of the list on page 39. Numerous runs suggest that as long as $\hat{P}_2 + \Delta\hat{\alpha} \approx 0.9$ the model S will fluctuate as described in Item ii.) on page 39. We have been unable to produce this fluctuation in all cases, however as these are disbursed unevenly we believe it is due to our inability to hit the correct values for $\Delta\hat{\alpha}$, rather than lack of fluctuations. This suspicion is also based on the continuity and pretty behaviour of the model in itself. Figures 5.3 to 5.8 on pages 46–48 illustrate this behaviour, both in terms of the fluctuation and monotonicity of $S(y^*)$.

Table 5.4: Chosen differences $\Delta\hat{\alpha} = \hat{\alpha}_0 - \hat{\alpha}_1$ in the lower range of interval $(0, 1)$ for $\hat{P}_1 > \hat{P}_2 = 0.5$ with calculated values for y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1, 0)$ and ΔS.

$\mathbf{\hat{P}_1 > \hat{P}_2 = 0.5}$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1,0)$	ΔS
0.91	0.01	0.9000	0.6330	0.3670	0.4864	0.4025	0.0839
0.81	0.01	0.8000	0.6585	0.3415	0.5149	0.4449	0.0701
0.71	0.01	0.7000	0.6905	0.3095	0.5463	0.4916	0.0546
0.61	0.01	0.6000	0.7324	0.2676	0.5813	0.5434	0.0380
0.51	0.01	0.5000	0.7902	0.2098	0.6216	0.6005	0.0211
0.41	0.01	0.4000	0.8757	0.1243	0.6701	0.6637	0.0064
0.31	0.01	0.3000	1.0162	-0.0162	0.7335	0.7334	0.0001
0.21	0.01	0.2000	1.2922	-0.2922	0.8317	0.8106	0.0212
0.11	0.01	0.1000	2.0890	-1.0890	1.0590	0.8958	0.1632
0.011	0.001	0.0100	16.80	-15.80	4.722	0.9891	3.7333
0.0011	0.0001	0.0010	163.9780	-162.9780	41.0106	0.9989	40.0117

Table 5.5: Chosen differences $\Delta\hat{\alpha} = \hat{\alpha}_0 - \hat{\alpha}_1$ in the mid range of interval $(0, 1)$ for $\hat{P}_1 > \hat{P}_2 = 0.5$ with calculated values for y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1, 0)$ and ΔS.

$\mathbf{\hat{P}_1 > \hat{P}_2 = 0.5}$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1,0)$	ΔS
0.95	0.05	0.9000	0.6246	0.3754	0.4724	0.3867	0.0857
0.9	0.1	0.8000	0.6373	0.3627	0.4819	0.4066	0.0753
0.85	0.15	0.7000	0.6529	0.3471	0.4920	0.4274	0.0646
0.8	0.2	0.6000	0.6720	0.3280	0.5028	0.4493	0.0535
0.75	0.25	0.5000	0.6960	0.3040	0.5146	0.4724	0.0423
0.7	0.3	0.4000	0.7263	0.2737	0.5277	0.4966	0.0311
0.65	0.35	0.3000	0.7657	0.2343	0.5424	0.5220	0.0204
0.6	0.4	0.2000	0.8184	0.1816	0.5595	0.5488	0.0107
0.55	0.45	0.1000	0.8918	0.1082	0.5802	0.5769	0.0032
0.505	0.495	0.0100	0.9870	0.0130	0.6035	0.6035	0.0000
0.5005	0.4995	0.0010	0.9987	0.0013	0.6062	0.6062	0.0000
0.50005	0.49995	0.0001	0.9999	0.0001	0.6065	0.6065	0.0000

Table 5.6: Model S with different $\hat{P}_1 > \hat{P}_2$ and $\hat{\alpha}_0$ to illustrate transition areas with calculated values for y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1,0)$ and ΔS.

$\hat{\mathbf{P}}_1 > \hat{\mathbf{P}}_2$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1,0)$	ΔS
$\hat{\mathbf{P}}_2 = 0.7$							
0.2	0.1999	0.0001	1.2902	-0.2902	0.8304	0.8187	0.0117
0.2	0.1	0.1000	1.0515	-0.0515	0.8193	0.8187	0.0006
0.2	0.01	0.1900	0.9319	0.0681	0.8200	0.8187	0.0013
0.2	0.001	0.1990	0.9227	0.0773	0.8204	0.8187	0.0017
0.2	$1e-05$	0.2000	0.9217	0.0783	0.8205	0.8187	0.0018
0.2	$1e-06$	0.2000	0.9217	0.0783	0.8205	0.8187	0.0018
$\hat{\mathbf{P}}_2 = 0.5$							
0.4	0.3999	0.0001	1.1647	-0.1647	0.6757	0.6703	0.0054
0.4	0.37	0.0300	1.1316	-0.1316	0.6740	0.6703	0.0037
0.4	0.3	0.1000	1.0643	-0.0643	0.6713	0.6703	0.0010
0.4	0.01	0.3900	0.8866	0.1134	0.6756	0.6703	0.0053
0.4	$1e-06$	0.4000	0.8824	0.1176	0.6761	0.6703	0.0058
$\hat{\mathbf{P}}_2 = 0.3$							
0.6	0.5999	0.0001	1.1270	-0.1270	0.5522	0.5488	0.0034
0.6	0.5	0.1000	1.0784	-0.0784	0.5503	0.5488	0.0015
0.6	0.3	0.3000	0.9954	0.0046	0.5488	0.5488	0.0000
0.6	0.1	0.5000	0.9301	0.0699	0.5506	0.5488	0.0018
0.6	0.001	0.5990	0.9032	0.0968	0.5527	0.5488	0.0039
0.6	$1e-06$	0.6000	0.9030	0.0970	0.5527	0.5488	0.0039
$\hat{\mathbf{P}}_2 = 0.2$							
0.8	0.7999	0.0001	1.0000	0.0000	0.4493	0.4493	0.0000
0.8	0.7	0.1000	0.9794	0.0206	0.4494	0.4493	0.0001
0.8	0.6	0.2000	0.9592	0.0408	0.4497	0.4493	0.0004
0.8	0.5	0.3000	0.9398	0.0602	0.4503	0.4493	0.0010
0.8	0.1	0.7000	0.8719	0.1281	0.4552	0.4493	0.0059
0.8	$1e-06$	0.8000	0.8575	0.1425	0.4573	0.4493	0.0079

Table 5.7: Chosen differences $\Delta\hat{\alpha} = \hat{\alpha}_0 - \hat{\alpha}_1$ in the upper range of the interval $(0, 1)$ for $\hat{P}_1 > \hat{P}_2 = 0.5$ with calculated values for y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1, 0)$ and ΔS.

$\mathbf{\hat{P}_1 > \hat{P}_2 = 0.5}$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1, 0)$	ΔS
0.99	0.09	0.9000	0.6163	0.3837	0.4590	0.3716	0.0874
0.99	0.19	0.8000	0.6172	0.3828	0.4515	0.3716	0.0800
0.99	0.29	0.7000	0.6187	0.3813	0.4446	0.3716	0.0730
0.99	0.39	0.6000	0.6210	0.3790	0.4380	0.3716	0.0664
0.99	0.49	0.5000	0.6240	0.3760	0.4319	0.3716	0.0603
0.99	0.59	0.4000	0.6278	0.3722	0.4261	0.3716	0.0545
0.99	0.69	0.3000	0.6325	0.3675	0.4207	0.3716	0.0491
0.99	0.79	0.2000	0.6381	0.3619	0.4157	0.3716	0.0441
0.99	0.89	0.1000	0.6446	0.3554	0.4109	0.3716	0.0394
0.99	0.98	0.0100	0.6514	0.3486	0.4070	0.3716	0.0354
0.999	0.998	0.0010	0.6486	0.3514	0.4040	0.3682	0.0358
0.9999	0.9998	0.0001	0.6483	0.3517	0.4037	0.3679	0.0358

Table 5.8: Model S with sliding constant $\Delta\hat{\alpha}$ on the interval $(0, 1)$ for $\hat{P}_1 > \hat{P}_2 = 0.5$ with calculated values for y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1, 0)$ and ΔS.

$\mathbf{\hat{P}_1 > \hat{P}_2 = 0.5}$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1, 0)$	ΔS
0.999999	0.8	0.2000	0.6348	0.3652	0.4127	0.3679	0.0449
0.9	0.7	0.2000	0.6702	0.3298	0.4433	0.4066	0.0367
0.8	0.6	0.2000	0.7112	0.2888	0.4774	0.4493	0.0281
0.7	0.5	0.2000	0.7596	0.2404	0.5158	0.4966	0.0192
0.6	0.4	0.2000	0.8184	0.1816	0.5595	0.5488	0.0107
0.5	0.3	0.2000	0.8920	0.1080	0.6102	0.6065	0.0037
0.4	0.2	0.2000	0.9879	0.0121	0.6704	0.6703	0.0000
0.3	0.1	0.2000	1.1200	-0.1200	0.7448	0.7408	0.0040
0.2	$1e-07$	0.2000	1.3160	-0.3160	0.8432	0.8187	0.0244

Resulting investment decisions for $\hat{P}_2 = 0.5$ and $\hat{P}_2 = 0.9$

This section summarises some specific examples from $\hat{P}_2 = 0.5$ and $\hat{P}_2 = 0.9$ in order to show how the investor should utilise different trading strategies dependent on the market sentiments. It also shows actual transition points b that we discussed in *The effect of change in $\hat{\alpha}_0$, $\hat{\alpha}_1$ and $\hat{P}_2$*.

- Figure 5.11 show that as $\hat{\alpha}_0 \to 0$ there is a point where the investor must change strategy, and undertake a short position in order to minimise the liquidity cost. The point of strategy change depends on the value of $\hat{\alpha}_0$ and $\hat{\alpha}_1$. For example, given $\hat{\alpha}_0 = 0.315$ and $\hat{\alpha}_1 = 0.0001$ with a 50% market drop, $y_1^* = 1.0$ is a transition point b from sell to short selling. However, the investor is still experiencing a liquidity cost on the trade. Not until $\hat{\alpha}_0 = 0.123$ and $\hat{\alpha}_1 = 0.0001$ do we reach a break-even point of $S(y_1^*) = 1$. At this point the investor must also take a short position of more than 85% of the initial portfolio size in order to make a profit, that might not be possible in itself.

- Figures 5.11 and 5.12 display that the transition point b for taking a short position is at $\hat{\alpha}_0 = 0.052$ and $\hat{\alpha}_1 = 0.0001$ in the 10% market drop, while at $\hat{\alpha}_0 = 0.315$ and $\hat{\alpha}_1 = 0.0001$ in the 50% market drop. In comparison to the 50% market drop where the liquidity cost was 0.27, the liquidity cost in the 10% market decline is 0.051, a difference of 0.219. This is as expected since $\hat{\alpha}_0$ is much larger in the cracked market compared to the 10% decline. So, in a short position the gain should be larger in a 50% drop than in a 10% drop, which is the case if the $\hat{\alpha}_0$s and $\hat{\alpha}_1$s were equal.

- Also, Figures 5.11 and 5.12 show that the market crack allows for negative liquidity cost at $\hat{\alpha}_0 = 0.1$, while the market correction of 10% decline gives a negative liquidity cost for $\hat{\alpha}_0 = 0.0215$, which means that the investor earns money on the model. In the first instance the investor has to take a short position of $y_1^* = 1.857$, while in the latter case $y_1^* = 1.721$. Again, the market crack demands a larger short position due to unequal $\hat{\alpha}$s.

- Figure 5.11 also displays the transition point of the market crack, $\hat{\alpha}_0 = 0.315$. It is easy to see that the transition point b from a sell to a short position y_1^* changes from 1.0 in the cracked market to 0.576 in the corrected market. At the same time the $S(y_1^*) = 0.73$ in the cracked market while at $S(y_1^*) = 0.814$ in the corrected market, as shown in Figure 5.12. Empirical observations expect us to see a faster transition in a cracked marked.

- Figure 5.10 is the case of a market correction of 10% decline with $y_1^* = 0.472$ and $S(y_1^*) = 0.473$ given $\hat{\alpha}_0 = 0.99$ and $\hat{\alpha}_1 = 0.98$, and a difference $\Delta\hat{\alpha} = 0.01$. If we change $\hat{\alpha}_1$ to 0.0001 giving a difference $\Delta\hat{\alpha} = 0.9899$, then $y_1^* = 0.518$ and $S(y_1^*) = 0.579$. In this case a larger difference between $\hat{\alpha}_0$ and $\hat{\alpha}_1$ yields a smaller liquidity cost. This can be seen by the change in sales values of 0.106 in this case.

- Also, Figure 5.9 is the case of a market crack of 50% with $y_1^* = 0.651$ and $S(y_1^*) = 0.407$ given $\hat{\alpha}_0 = 0.99$ and $\hat{\alpha}_1 = 0.98$, and a difference $\Delta\hat{\alpha} = 0.01$. If we change $\hat{\alpha}_1$ to 0.0001 giving a difference $\Delta\hat{\alpha} = 0.9899$, then $y_1^* = 0.616$ and $S(y_1^*) = 0.466$. In this case a larger $\Delta\hat{\alpha}$ yields again a smaller liquidity cost, with a change in $S(y_1^*)$ of 0.059. We notice that the change in $S(y_1^*)$ is almost halved, from the change i the previous bullet point 0.106 to this, between a market decline of 10% and 50%. In general, as we ran the script we saw that as $\hat{P}_2 \to 0$ then the change in $S(y_1^*) \to 0$.

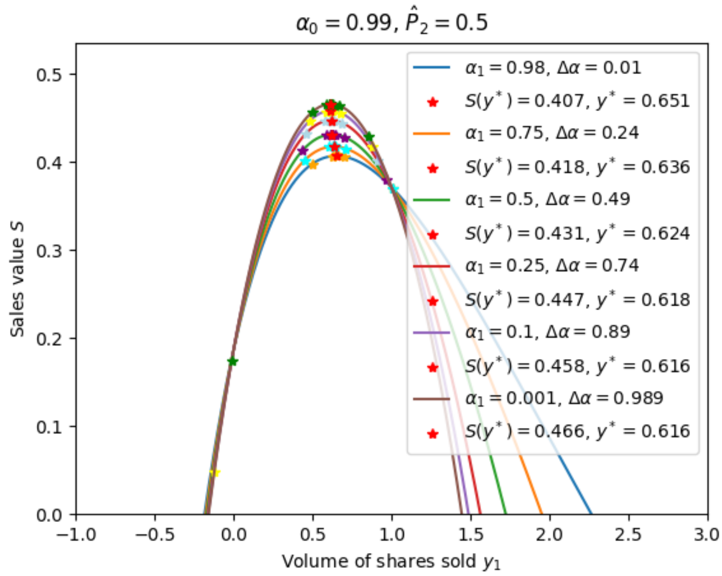


Figure 5.3: Graphs of $S(y_1)$ with $\hat{P}_2 = 0.5$ and $\hat{\alpha}_0 = 0.99$ with six different $\hat{\alpha}_1$. The graphs show the bigger the difference $\Delta\hat{\alpha}$ the smaller the liquidity cost and the smaller the y_1^* volume. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y_1^*)$.

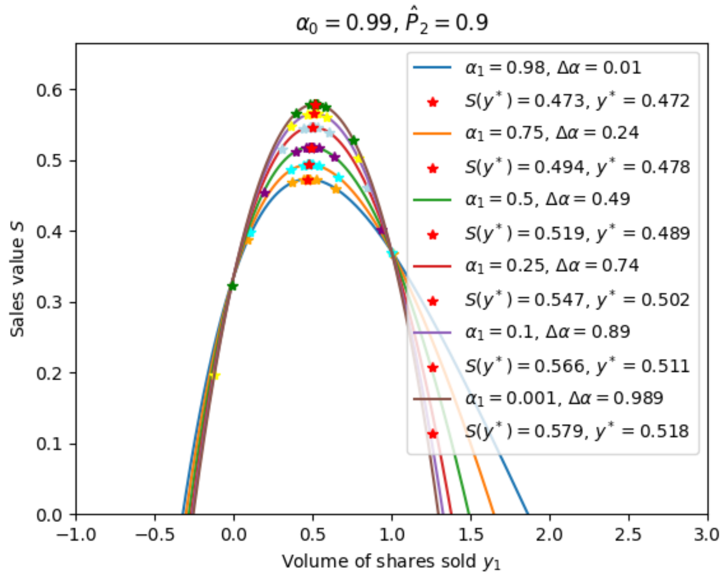


Figure 5.4: Graphs of $S(y_1)$ with $\hat{P}_2 = 0.9$ and $\hat{\alpha}_0 = 0.99$ with six different $\hat{\alpha}_1$. The graphs show the bigger the difference $\Delta\hat{\alpha}$ the smaller the liquidity cost and the bigger the y_1^* volume. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y_1^*)$.

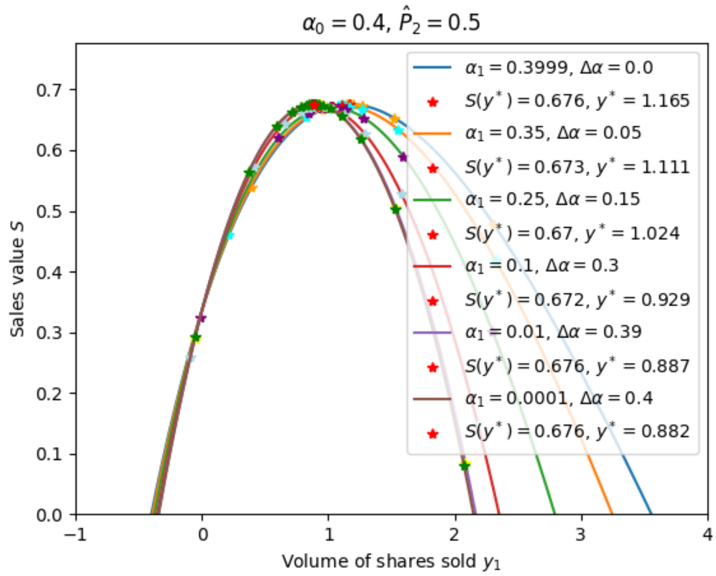


Figure 5.5: Graphs of $S(y_1)$ with $\hat{P}_2 = 0.5$ and $\hat{\alpha}_0 = 0.4$ with six different $\hat{\alpha}_1$. The graphs show the bigger the difference $\Delta\hat{\alpha}$ the smaller the y_1^* volume, while the liquidity cost first decreases for then to increase again. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y_1^*)$. The figure reveals a transition point $b \in (0.01, 0.25)$ in the $\hat{\alpha}_1$-domain where $S(y_1^*)$ decreases as $\hat{\alpha}_1$ moves from 0 to b, and increases from b to 1.

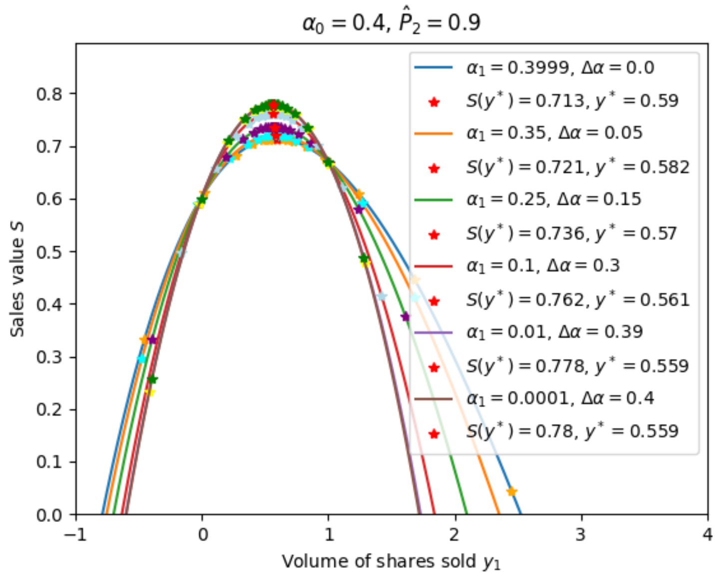


Figure 5.6: Graphs of $S(y)$ with $\hat{P}_2 = 0.9$ and $\hat{\alpha}_0 = 0.4$ with six different $\hat{\alpha}_1$. The graphs show the bigger the difference $\Delta\hat{\alpha}$ the smaller the y_1^* volume and the smaller the liquidity cost. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y^*)$. The figure does *not* reveal any transition points.

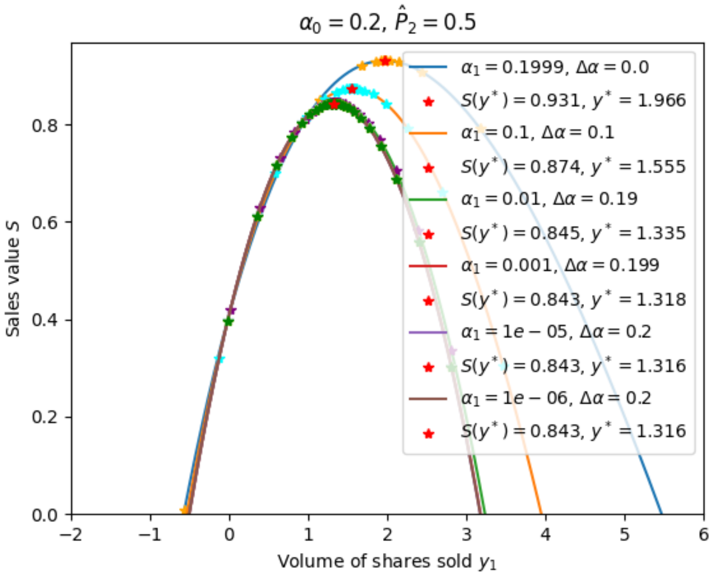


Figure 5.7: Graphs of $S(y_1)$ with $\hat{P}_2 = 0.5$ and $\hat{\alpha}_0 = 0.2$ with six different $\hat{\alpha}_1$. The graphs show the bigger the difference $\Delta\hat{\alpha}$ the smaller the y_1^* volume, while the liquidity cost first decreases for then to increase again. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y_1^*)$.

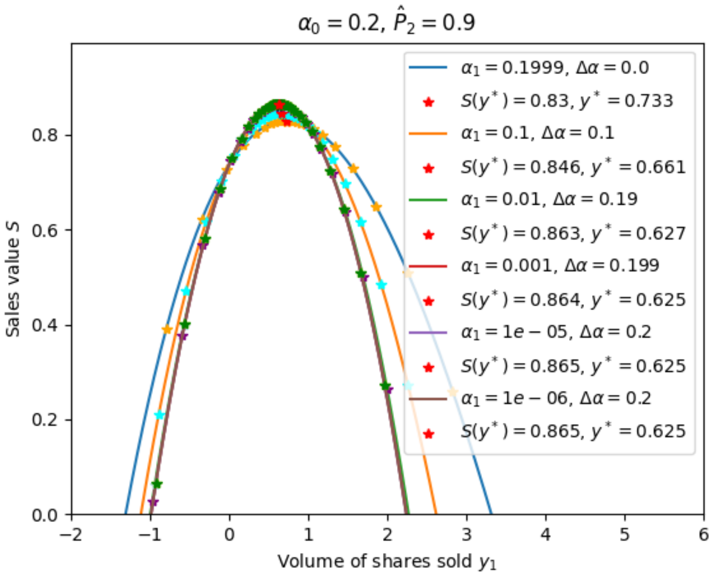


Figure 5.8: Graphs of $S(y_1)$ with $\hat{P}_2 = 0.9$ and $\hat{\alpha}_0 = 0.2$ with six different $\hat{\alpha}_1$. The graphs show the bigger the difference $\Delta\hat{\alpha}$ the smaller the y_1^* volume and the smaller the liquidity cost. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y_1^*)$.

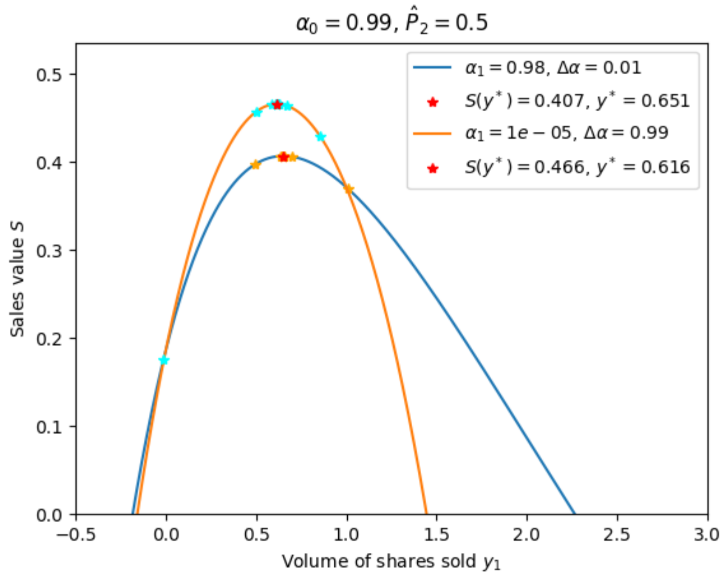


Figure 5.9: Graphs of $S(y_1)$ with $\hat{P}_2 = 0.5$ and $\hat{\alpha}_0 = 0.99$ with two different $\hat{\alpha}_1$. The development of the graphs show that the bigger the difference between $\hat{\alpha}_0$ and $\hat{\alpha}_1$ the smaller the liquidity cost and the smaller the y_1^* volume. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y_1^*)$.

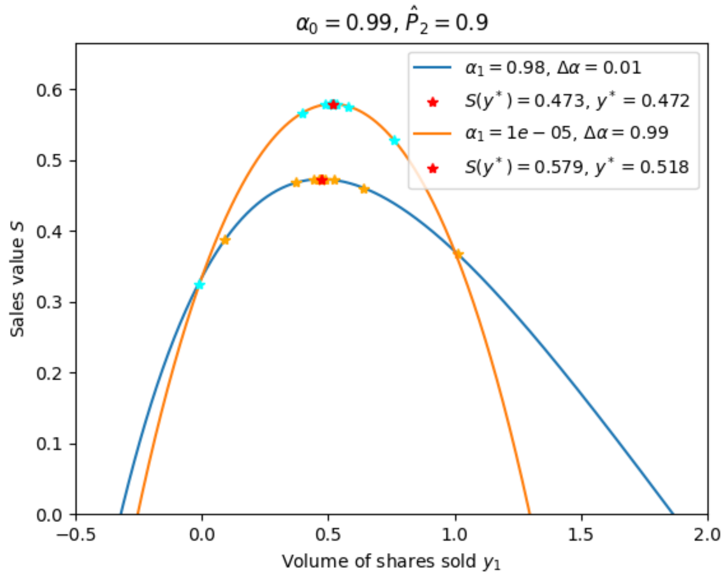


Figure 5.10: Two graphs of $S(y_1)$ with $\hat{P}_2 = 0.9$, $\hat{\alpha}_0 = 0.99$ and $\hat{\alpha}_1 = 0.98$ and $\hat{\alpha}_1 = 0.00001$. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y_1^*)$.

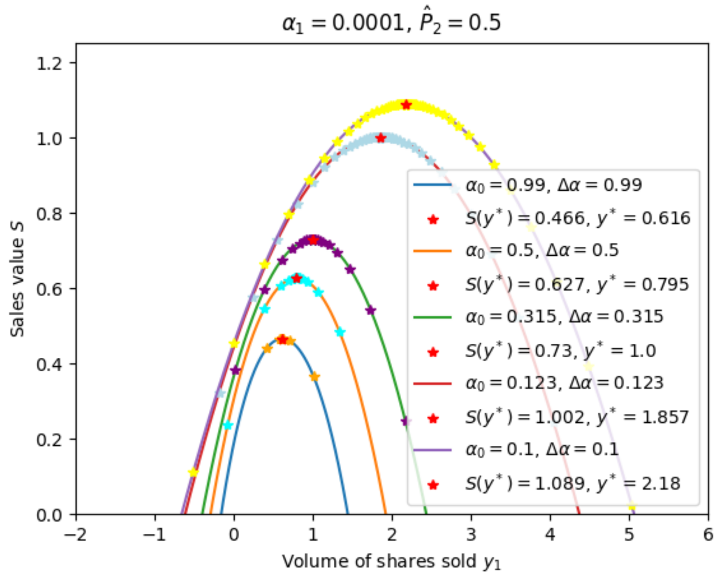


Figure 5.11: Five graphs of $S(y_1)$ with $\hat{\alpha}_0$ from 0.1 to 0.99 and $\hat{\alpha}_1 = 0.0001$ for $\hat{P}_2 = 0.5$ showing their sales values S. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y^*)$. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y^*)$.

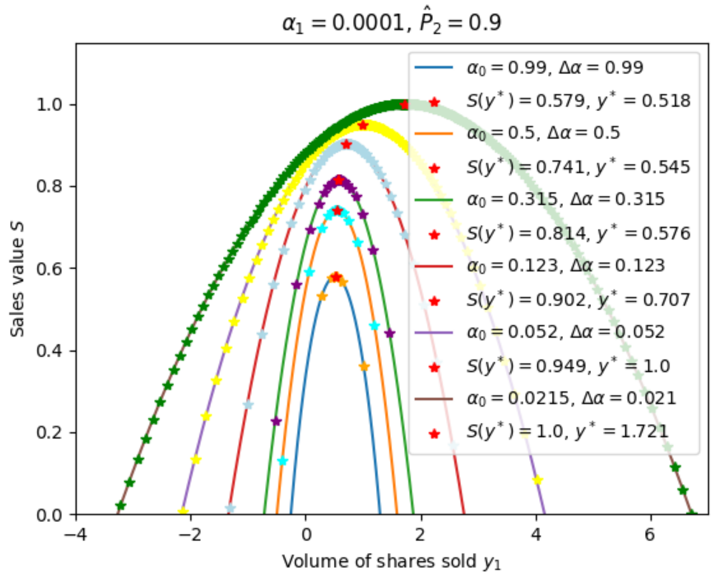


Figure 5.12: Six graphs of $S(y_1)$ with $\hat{\alpha}_0$ from 0.0215 to 0.99 and $\hat{\alpha}_1 = 0.0001$ for $\hat{P}_2 = 0.9$ showing their sales value S. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y_1^*)$.

5.3.2 The case of $\hat{P}_1 = \hat{P}_2$: Flat market

A Flat market is described by $\hat{P}_1 = \hat{P}_2$, where the market price is expected to stay unchanged in the near future. The following scenarios look at a Flat market with strategically chosen $\hat{\alpha}_0$s and $\hat{\alpha}_1$s. In this section we use the Fixed Point method in Appendices A.2.1 and A.2.2.

5.3.2.1 The behaviour of model S in a Flat market

Starting out, the greater the difference $\Delta\hat{\alpha}$ the smaller the liquidity cost C incurred on the investor. This is supported by the increasing $S(y_1^*)$ no matter the difference between $\hat{\alpha}_0$ and $\hat{\alpha}_1$ and the location of $\hat{\alpha}_0$, see Figures 5.13 to 5.15 and Table 5.9. Now, the smaller the difference $\Delta\hat{\alpha}$ the smaller the stock sale y_1^* in t_1. This is supported by the decreasing y_1^* for all differences $\Delta\hat{\alpha}$. As $\hat{\alpha}_0$ moves close to 0 it yields a faster recovery. These findings are as expected.

We know that the location of $\hat{\alpha}_0$ on the $\hat{\alpha}$-domain influences the sales value S. Also, as $\hat{\alpha}_1$ decreases the market recovers faster than if $\hat{\alpha}_1$ is closer to 1. As the location of $\hat{\alpha}_0$ moves to 0 the sales value S increases, consequently $\hat{\alpha}_1$ gets smaller. Moreover, as $\hat{\alpha}_0$ given a fixed $\Delta\hat{\alpha}$ is located close to 0 then $S \to 1$ in a Flat market. Subsequently, as $\hat{\alpha}_0$ and $\hat{\alpha}_1$ are located close to 1 such that $\Delta\hat{\alpha}$ decreases, then S has its highest liquidity cost C. An illustration of this is displayed in Table 5.9. The Flat market displays the effects of the $\hat{\alpha}_n$s on the model, as described in the introduction.

Resulting investment decisions for $\hat{P}_1 = \hat{P}_2$

Again we summarise specific examples from $\hat{P}_1 = \hat{P}_2$. In this case the possibility space for trading strategies is limited to a sell strategy.

- We find no transition point where the investor should change to a short position in a Flat market. In general, regardless of $\Delta\hat{\alpha}$ the model advises the investor to sell $y^* = 0.5$ at t_1 as $\Delta\hat{\alpha}$ increases. However, the different market sentiments of $\hat{\alpha}_0$ yield a spread in $S(y_1^*)$ from 0.5 in Figure 5.13 to 0.905 in Figure 5.15. Numerous runs showed that $S(y_1^*) \to 1$ as $\hat{\alpha}_0 \to 0$.
- Table 5.9 shows $y_1^* = 0.4385$ and $S(y^*) = 0.4996$ given $\Delta\hat{\alpha}_0 = 0.1099$. If we change $\Delta\hat{\alpha}$ to 0.9999, then $y_1^* = 0.5$ and $S(y^*) = 0.6066$. In this case a larger $\Delta\hat{\alpha}$ yields a smaller liquidity cost C, because $\hat{\alpha}_1$ is closer to 0.

Table 5.9: The table displays chosen values for $\hat{\alpha}_0$ and $\hat{\alpha}_1$ when $\hat{P}_1 = \hat{P}_2 = 1$ with calculated values for $\Delta\hat{\alpha}$, y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1,0)$ and ΔS.

$\mathbf{\hat{P}_1 = \hat{P}_2 = 1}$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1,0)$	ΔS
0.2	0.1999	0.0001	0.4873	0.5127	0.8618	0.8187	0.0431
0.2	0.1	0.1000	0.4935	0.5065	0.8828	0.8187	0.0641
0.2	0.01	0.1900	0.4993	0.5007	0.9026	0.8187	0.0839
0.2	0.001	0.1990	0.4999	0.5001	0.9046	0.8187	0.0859
0.2	0.0001	0.1999	0.5000	0.5000	0.9048	0.8187	0.0861
0.2	$1e-05$	0.2000	0.5000	0.5000	0.9048	0.8187	0.0861
0.5	0.4999	0.0001	0.4675	0.5325	0.6930	0.6065	0.0865
0.5	0.4	0.1000	0.4734	0.5266	0.7085	0.6065	0.1020
0.5	0.2	0.3000	0.4862	0.5138	0.7418	0.6065	0.1353
0.5	0.1	0.4000	0.4930	0.5070	0.7598	0.6065	0.1533
0.5	0.001	0.4990	0.4999	0.5001	0.7786	0.6065	0.1721
0.5	$1e-05$	0.5000	0.5000	0.5000	0.7788	0.6065	0.1723
0.9999	0.89	0.1099	0.4385	0.5615	0.4996	0.3679	0.1317
0.9999	0.8	0.1999	0.4434	0.5566	0.5084	0.3679	0.1405
0.9999	0.6	0.3999	0.4555	0.5445	0.5292	0.3679	0.1613
0.9999	0.3	0.6999	0.4763	0.5237	0.5647	0.3679	0.1968
0.9999	0.1	0.8999	0.4918	0.5082	0.5918	0.3679	0.2239
0.9999	$1e-05$	0.9999	0.5000	0.5000	0.6066	0.3679	0.2386

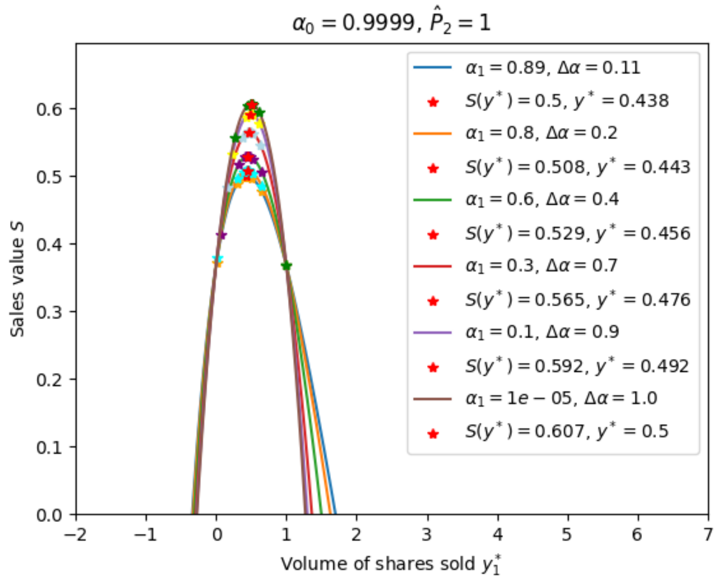


Figure 5.13: Six graphs of $S(y_1)$ with $\hat{P}_2 = \hat{P}_1$, $\hat{\alpha}_0 = 0.9999$ and six different $\hat{\alpha}_1$s.

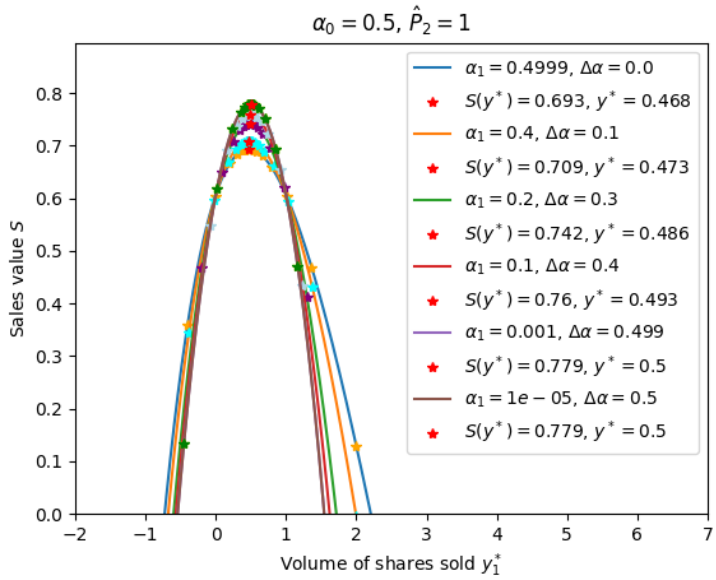


Figure 5.14: Six graphs of $S(y_1)$ with $\hat{P}_2 = \hat{P}_1$, $\hat{\alpha}_0 = 0.5$ and six different $\hat{\alpha}_1$s.

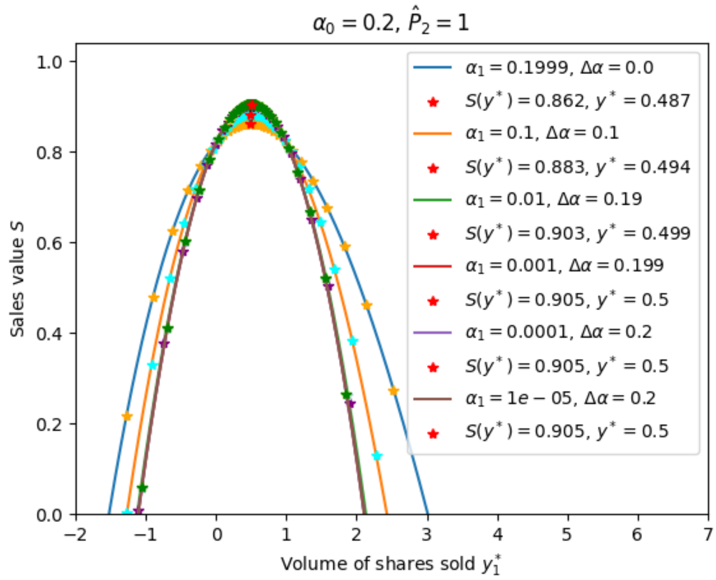


Figure 5.15: Six graphs of $S(y_1)$ with $\hat{P}_2 = \hat{P}_1$, $\hat{\alpha}_0 = 0.2$ and six different $\hat{\alpha}_1$s.

5.3.3 The case of $\hat{P}_1 < \hat{P}_2$: Bull market

A Bull market is described by $\hat{P}_1 < \hat{P}_2$, where the market price is expected to increase in the near future. The following scenarios look at a Bull market with strategically chosen $\hat{\alpha}_0$s, $\hat{\alpha}_1$s and $\hat{P}_2$.

While working on the numerical method for the Bull market we found that the Fixed Point iterations had limitations as $\hat{\alpha}_0$ moved below a transition point on $(0, 1)$ as $\Delta\hat{\alpha} \to 0$. As the model is a smooth concave curve it seemed curious that the algorithm had problems finding the stationary point in this situation. However, a more rigorous analysis showed that the complexity of the expressions in themselves accompanied by how they are intertwined made the method unstable. Consequently, we decided to switch to the Gradient ascent method, which we keep for the rest of $N = 2$ and the whole of $N = 3$. Now, why did we not use Gradient Ascent all along? Because, to begin with the Fixed Point method seemed to do the job as the curves and surfaces are simple and smooth in its kind. When we discovered the more intricate problems it seemed important to show that things are not always as they appear, and being able and prepared to use several methods are in itself an important skill when faced with a challenge.

5.3.3.1 The behaviour of model S in a Bull market

Again we have used a Python script, this time for the Gradient Ascent method in Appendices A.2.3 and A.2.4 to analyse the model S numerically. For illustrative purposes we use the market surge of 50% to make a general description of the model in a Bull market to emphasise the outcomes. Also, this time we tested numerous values for $\hat{P}_2$ without loss of generality. For a review into the limitations of numerical methods please see Chapter 2.

The effect of changes in $\hat{\alpha}_0$, $\hat{\alpha}_1$ and $\hat{P}_2$

In a Bull market the model S displays the same type of behaviour as in the Bear market, however with different transition points for trading strategies. Also, this time it is determined by $\hat{\alpha}_0$ and $\hat{\alpha}_1$. Moreover, the market surge $(\hat{P}_2 - 1)$ and the distance between $\hat{\alpha}_0$ and $\hat{\alpha}_1$ determine the location of the transitions points (b) where S changes from decreasing to increasing. In general, and analogous to the Bear market, we see: Given the $\hat{\alpha}$-domain $(0, 1)$ with $0 < a < b < c < 1$, for some transition points a, b and c, and $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$, then for a, b and c defined by $\hat{P}_2$, $\hat{\alpha}_0$ and $\Delta\hat{\alpha}$ we have that

i.) if $\hat{\alpha}_0 \in (0, a)$ and $\hat{\alpha}_0$ and $\Delta\hat{\alpha}$ decreases then the sales value S increases. See Table 5.11 and Figure 5.20.

ii.) if $\hat{\alpha}_0 \in (a, c)$ then the sales value S increases as $\Delta\hat{\alpha}$ moves from the transition point b to 0, and as $\Delta\hat{\alpha}$ moves from b to $\hat{\alpha}_0$. See Table 5.12 and Figure 5.18.

iii.) if $\hat{\alpha}_0 \in (c, 1)$ and $\hat{\alpha}_1$ increases as $\Delta\hat{\alpha} \to 0$ then the sales value S decreases. See Table 5.13 and Figures 5.16, 5.17, 5.19 and 5.21.

As the model S behaves more or less identical in both Bear and Bull markets, in this respect, we keep the following bullet list short and efficient by reiterating the findings in this list:

- The model S behaves differently dependent on the location of $\hat{\alpha}_0$ and $\hat{\alpha}_1$ on the $\hat{\alpha}$-domain $(0,1)$, as the sales value S fluctuates as certain locations of $\hat{\alpha}_0$ yield a decrease followed by an increase as $\hat{\alpha}_1$ moves towards $\hat{\alpha}_0$, in Tables 5.6 and 5.11 and Figures 5.16 to 5.21.
- Given a fixed $\Delta\hat{\alpha}$ with $\hat{\alpha}_0$ close to 1 on the $\hat{\alpha}$-domain, then S has the bigger liquidity cost C than if $\hat{\alpha}_0$ is located close to 0 on the $\hat{\alpha}$-domain, as shown in Table 5.17.
- There is a sliding decrease in S as the location of $\hat{\alpha}_0$ given the fixed $\Delta\hat{\alpha}$ moves towards 1. Table 5.17 illustrates an example of the finding.
- The range of the volume y_1^* is much greater as the location of $\hat{\alpha}_0$ given fixed $\Delta\hat{\alpha}$ decreases and stays close to 0 in the lower range ($\Delta y_{1,l} = 39.794$) than in the middle ($\Delta y_{1,m} = 0.3259$) and upper range ($\Delta y_{1,u} = 0.1443$), which lends evidence to more stable markets as $\hat{\alpha}_0$ given fixed $\Delta\hat{\alpha}$ is located at the upper end of the $\hat{\alpha}$-domain. The respective ranges from lower to upper on the $\hat{\alpha}$-domain in terms of sales value S in Tables 5.14 to 5.16 are: $S_l = (0.7977, 17.042)$, $S_m = (0.7728, 0.9139)$ and $S_u = (0.6090, 0.7489)$.

The model's behaviour differs from the Bear market, as the volume the investor has to sell at t_1 is decreasing as $\hat{\alpha}_0$ with fixed $\Delta\hat{\alpha}$ moves towards 0. This is illustrated in Table 5.17 and can also be seen comparatively across the rows in Tables 5.14 to 5.16. Further contrary to the Bear market, there is no short selling opportunity when $\hat{\alpha}_0$ decreases.

Moreover, as $\hat{\alpha}_0$ increases the slippage of $S(y_1)$ displayed by the bounce-back function ψ increases as well. However, the slippage in a Bull market is much less than the slippage in a Bear market, it can even be negative. For the illustrative case of a 50% increase or decrease in the market, the change in slippage is 0.3113, as seen by taking the difference between the sales values of the first row in the lower intervals of Tables 5.4 and 5.14. Consequently, the investor should not take a short position in a Bull market – which is no surprise. However, there are pumping opportunities as $\hat{\alpha}_0$ resides closer to 0 with $\Delta\hat{\alpha}$ moving towards $\hat{\alpha}_0$, as illustrated in Figures 5.18 and 5.20. These pumping situations also include a negative slippage as the sales value $S > 1$.

Contrary, if $\hat{\alpha}_1$ decreases the market will bounce back faster, which in turn speaks for a delay in sales as the sales value S increases. This is illustrated in Figure 5.16.

Resulting investment decisions for $\hat{P}_2 = 1.1$ and $\hat{P}_2 = 1.5$

This section merely summarises some specific examples from $\hat{P}_2 = 1.1$ and $\hat{P}_2 = 1.5$ in order to show how the investor should utilise different trading strategies dependent on the market sentiments. It also shows an actual transition points b that we discuss in *The effect of change in $\hat{\alpha}_0$, $\hat{\alpha}_1$ and $\hat{P}_2$*. Moreover, we look at Bull markets compared to Bear markets.

- In Figure 5.25 with a 10% market increase we see that the transition point for taking a buy position is at $\hat{\alpha}_0 = 0.0471$ with $\hat{\alpha}_1 = 0.00001$. This yields a sales value of $S(0) = 1.049$ as all shares are sold in t_2. This transition point is at $\hat{\alpha}_0 = 0.192$ in the case of a 50% increase. Also, the 50% sales value for $\hat{\alpha}_0 = 0.0471$ is 1.725 compared to 1.049 in the 10% market. While the first situation is a pure sell case, the latter situation is a pumping position.

Table 5.10: Model S with different $\hat{P}_1 < \hat{P}_2$ and $\hat{\alpha}_0$ to illustrate transition areas with calculated values for y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1,0)$ and ΔS.

$\hat{\mathbf{P}}_1 < \hat{\mathbf{P}}_2$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1,0)$	ΔS
$\hat{\mathbf{P}}_2 = \mathbf{0.7}$							
0.2	0.1999	0.0001	1.2902	-0.2902	0.8304	0.8187	0.0117
0.2	0.1	0.1000	1.0515	-0.0515	0.8193	0.8187	0.0006
0.2	0.01	0.1900	0.9319	0.0681	0.8200	0.8187	0.0013
0.2	0.001	0.1990	0.9227	0.0773	0.8204	0.8187	0.0017
0.2	$1e-05$	0.2000	0.9217	0.0783	0.8205	0.8187	0.0018
0.2	$1e-06$	0.2000	0.9217	0.0783	0.8205	0.8187	0.0018
$\hat{\mathbf{P}}_2 = \mathbf{0.5}$							
0.4	0.3999	0.0001	1.1647	-0.1647	0.6757	0.6703	0.0054
0.4	0.37	0.0300	1.1316	-0.1316	0.6740	0.6703	0.0037
0.4	0.3	0.1000	1.0643	-0.0643	0.6713	0.6703	0.0010
0.4	0.01	0.3900	0.8866	0.1134	0.6756	0.6703	0.0053
0.4	$1e-06$	0.4000	0.8824	0.1176	0.6761	0.6703	0.0058
$\hat{\mathbf{P}}_2 = \mathbf{0.3}$							
0.6	0.5999	0.0001	1.1270	-0.1270	0.5522	0.5488	0.0034
0.6	0.5	0.1000	1.0784	-0.0784	0.5503	0.5488	0.0015
0.6	0.3	0.3000	0.9954	0.0046	0.5488	0.5488	0.0000
0.6	0.1	0.5000	0.9301	0.0699	0.5506	0.5488	0.0018
0.6	0.001	0.5990	0.9032	0.0968	0.5527	0.5488	0.0039
0.6	$1e-06$	0.6000	0.9030	0.0970	0.5527	0.5488	0.0039
$\hat{\mathbf{P}}_2 = \mathbf{0.2}$							
0.8	0.7999	0.0001	1.0000	0.0000	0.4493	0.4493	0.0000
0.8	0.7	0.1000	0.9794	0.0206	0.4494	0.4493	0.0001
0.8	0.6	0.2000	0.9592	0.0408	0.4497	0.4493	0.0004
0.8	0.5	0.3000	0.9398	0.0602	0.4503	0.4493	0.0010
0.8	0.1	0.7000	0.8719	0.1281	0.4552	0.4493	0.0059
0.8	$1e-06$	0.8000	0.8575	0.1425	0.4573	0.4493	0.0079

Table 5.11: Chosen differences $\Delta\hat{\alpha} = \hat{\alpha}_0 - \hat{\alpha}_1$ on the lower interval $(0, a)$ for $\hat{P}_1 < \hat{P}_2 = 1.5$ with calculated values for y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1, 0)$ and ΔS.

$\hat{\mathbf{P}}_1 < \hat{\mathbf{P}}_2 = 1.5$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1,0)$	ΔS
0.0001	$9e-05$	0.0000	-50.3898	51.3898	26.3938	2.4991	23.8948
0.0001	$1e-06$	0.0001	-49.7024	50.7024	25.7227	2.4983	23.2244
0.0001	$1e-07$	0.0001	-49.6955	50.6955	25.7160	2.4983	23.2178
0.1	0.09	0.0100	-1.3811	2.3811	1.6018	1.5483	0.0535
0.1	0.05	0.0500	-0.8480	1.8480	1.4810	1.2415	0.2396
0.1	0.001	0.0990	-0.4915	1.4915	1.4119	0.8976	0.5144
0.1	0.0001	0.0999	-0.4866	1.4866	1.4111	0.8915	0.5196
0.1	$1e-07$	0.1000	-0.4861	1.4861	1.4110	0.8909	0.5201
0.2	0.1999	0.0001	-0.5263	1.5263	1.2896	-0.5554	1.8451
0.2	0.1	0.1000	-0.1645	1.1645	1.2368	-1.8272	3.0640
0.2	0.01	0.1900	0.0063	0.9937	1.2281	-2.6883	3.9164
0.2	0.001	0.1990	0.0193	0.9807	1.2283	-2.7623	3.9905
0.2	$1e-05$	0.2000	0.0207	0.9793	1.2283	-2.7703	3.9986
0.2	$1e-06$	0.2000	0.0207	0.9793	1.2283	-2.7704	3.9987

Table 5.12: Chosen differences $\Delta\hat{\alpha} = \hat{\alpha}_0 - \hat{\alpha}_1$ on the middle interval (a, c) for $\hat{P}_1 < \hat{P}_2 = 1.5$ with calculated values for y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1, 0)$ and ΔS.

$\hat{\mathbf{P}}_1 < \hat{\mathbf{P}}_2 = 1.5$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1,0)$	ΔS
0.24	0.2399	0.0001	-0.3516	1.3516	1.2122	0.3070	0.9052
0.24	0.2	0.0400	-0.2245	1.2245	1.1950	0.0355	1.1595
0.24	0.15	0.0900	-0.1069	1.1069	1.1839	-0.2754	1.4593
0.24	0.1	0.1400	-0.0188	1.0188	1.1801	-0.5568	1.7369
0.24	0.01	0.2300	0.0951	0.9049	1.1843	-0.9975	2.1818
0.24	0.0001	0.2399	0.1052	0.8948	1.1854	-1.0413	2.2267
0.3	0.2999	0.0001	-0.1780	1.1780	1.1212	-0.3112	1.4325
0.3	0.25	0.0500	-0.0776	1.0776	1.1134	-0.6278	1.7412
0.3	0.15	0.1500	0.0614	0.9386	1.1129	-1.1746	2.2875
0.3	0.1	0.2000	0.1118	0.8882	1.1174	-1.4096	2.5270
0.3	0.01	0.2900	0.1833	0.8167	1.1306	-1.7777	2.9083
0.3	0.0001	0.2999	0.1900	0.8100	1.1323	-1.8143	2.9467
0.35	0.3499	0.0001	-0.0797	1.0797	1.0593	-0.8570	1.9164
0.35	0.3	0.0500	-0.0053	1.0053	1.0570	-1.1582	2.2152
0.35	0.15	0.2000	0.1451	0.8549	1.0682	-1.9019	2.9700
0.35	0.1	0.2500	0.1806	0.8194	1.0757	-2.1041	3.1798
0.35	0.01	0.3400	0.2335	0.7665	1.0923	-2.4210	3.5133
0.35	0.0001	0.3499	0.2386	0.7614	1.0944	-2.4525	3.5469

Table 5.13: Chosen differences $\Delta\hat{\alpha} = \hat{\alpha}_0 - \hat{\alpha}_1$ on the upper interval $(c, 1)$ for $\hat{P}_1 < \hat{P}_2 = 1.5$ with calculated values for y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1,0)$ and ΔS.

$\hat{\mathbf{P}}_1 < \hat{\mathbf{P}}_2 = \mathbf{1.5}$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1,0)$	ΔS
0.36	0.3499	0.0101	-0.0480	1.0480	1.0474	-1.0321	2.0795
0.36	0.3	0.0600	0.0193	0.9807	1.0467	-1.3243	2.3710
0.36	0.15	0.2100	0.1583	0.8417	1.0602	-2.0460	3.1062
0.36	0.1	0.2600	0.1917	0.8083	1.0680	-2.2423	3.3104
0.36	0.01	0.3500	0.2418	0.7582	1.0852	-2.5498	3.6350
0.36	0.0001	0.3599	0.2467	0.7533	1.0873	-2.5804	3.6677
0.9	0.89	0.0100	0.2594	0.7406	0.6582	-10.3060	10.9642
0.9	0.75	0.1500	0.2900	0.7100	0.6756	-10.7439	11.4196
0.9	0.5	0.4000	0.3380	0.6620	0.7115	-11.2772	11.9887
0.9	0.1	0.8000	0.4047	0.5953	0.7830	-11.7299	12.5129
0.9	0.01	0.8900	0.4188	0.5812	0.8018	-11.7908	12.5926
0.9	0.0001	0.8999	0.4203	0.5797	0.8040	-11.7968	12.6008
0.99	0.98	0.0100	0.2745	0.7255	0.6147	-1.5876	2.2023
0.99	0.75	0.2400	0.3159	0.6841	0.6424	-1.8144	2.4567
0.99	0.5	0.4900	0.3566	0.6434	0.6776	-2.0083	2.6859
0.99	0.25	0.7400	0.3947	0.6053	0.7188	-2.1594	2.8782
0.99	0.1	0.8900	0.4169	0.5831	0.7469	-2.2335	2.9804
0.99	0.001	0.9890	0.4314	0.5686	0.7670	-2.2766	3.0436

- The greater the $\Delta\hat{\alpha}$ the smaller liquidity cost C for the investor in the upper range, see Tables 5.7 and 5.16. This is substantiated by the increase in $S(y^*)$ values for the same $\hat{\alpha}_0$ for decreasing $\hat{\alpha}_1$. This is equal to a Bear market with the exception of the size of the liquidity cost. As the market declines 50% the liquidity cost is much larger ($S(y^*) = 0.4590$) than if the market increases by 50% ($S(y^*) = 0.7489$), however not equal to the marekt movement. Now, the smaller the difference between $\hat{\alpha}_0$ and $\hat{\alpha}_1$ the smaller the stock sale at t_1. This is supported by the decreasing y^* values for the same $\hat{\alpha}_0$. Also, this is contrary to the Bear market, in accordance with empirical observations.
- In the middle and lower range our model S behaves differently. The greater the difference between $\hat{\alpha}_0$ and $\hat{\alpha}_1$ the bigger liquidity cost C for the investor, see Tables 5.4, 5.5, 5.14 and 5.15. This is substantiated by the decrease in $S(y^*)$ values for the same $\hat{\alpha}_0$ for decreasing $\hat{\alpha}_1$. This is again equal to a Bear market with the exception of the size of the liquidity cost. As the market declines 50% the liquidity cost is much larger ($S_m(y^*) = 0.4724$, $S_l(y^*) = 0.4864$) than if the market increases by 50% ($S_m(y^*) = 0.7728$, $S_l(y^*) = 0.7977$). Now, the smaller the difference between $\hat{\alpha}_0$ and $\hat{\alpha}_1$ the smaller the stock sale at t_1. This is supported by the decreasing y^* values for the same $\hat{\alpha}_0$. Again, this is contrary to the Bear market.

- Now, as $\hat{\alpha}_0$ moves towards 0 there is a transition point where the investor must change strategy. As the difference between $\hat{\alpha}_0$ and $\hat{\alpha}_1$ is smaller than 0.1 the investor should initiate a pumping strategy in a market with 50% surge. This will incur a negative liquidity cost C and the investor manages to avoid a loss on the portfolio. As can be seen in Table 5.14. This is contrary to the Bear market where the investor should change strategy from sell to short as $\hat{\alpha}_0$ moves towards 0 and the difference between $\hat{\alpha}_0$ and $\hat{\alpha}_1$ is smaller than 0.3 with a market drop of 50% Table 5.4.

Table 5.14: Chosen differences$\Delta\hat{\alpha}$ in the lower range of the interval $(0, 1)$ for $\hat{P}_1 < \hat{P}_2 = 1.5$ with calculated values for $\Delta\hat{\alpha}$, y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1, 0)$ and ΔS.

$\mathbf{\hat{P}_1 < \hat{P}_2 = 1.5}$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1,0)$	ΔS
0.91	0.01	0.9000	0.4201	0.5799	0.7977	-12.0443	12.8420
0.81	0.01	0.8000	0.4052	0.5948	0.8407	-9.7020	10.5427
0.71	0.01	0.7000	0.3865	0.6135	0.8867	-7.7287	8.6154
0.61	0.01	0.6000	0.3618	0.6382	0.9364	-6.0379	6.9743
0.51	0.01	0.5000	0.3278	0.6722	0.9906	-4.5523	5.5429
0.41	0.01	0.4000	0.2775	0.7225	1.0513	-3.1991	4.2504
0.31	0.01	0.3000	0.1947	0.8053	1.1226	-1.9065	3.0291
0.21	0.01	0.2000	0.0317	0.9683	1.2163	-0.5988	1.8152
0.11	0.01	0.1000	-0.4409	1.4409	1.3899	0.8084	0.5815
0.011	0.001	0.0100	-9.0877	10.0877	3.6228	2.3182	1.3047
0.0011	0.0001	0.0010	-39.3736	40.3736	17.0419	2.4817	14.5602

Table 5.15: Chosen differences for $\Delta\hat{\alpha}$ in the middle range of the interval $(0, 1)$ for $\hat{P}_1 < \hat{P}_2 = 1.5$ with calculated values for $\Delta\hat{\alpha}$, y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1, 0)$ and ΔS.

$\mathbf{\hat{P}_1 < \hat{P}_2 = 1.5}$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1,0)$	ΔS
0.95	0.05	0.9000	0.4193	0.5807	0.7728	-13.0841	13.8570
0.9	0.1	0.8000	0.4047	0.5953	0.7830	-11.7299	12.5129
0.85	0.15	0.7000	0.3885	0.6115	0.7938	-10.4736	11.2674
0.8	0.2	0.6000	0.3701	0.6299	0.8054	-9.2971	10.1024
0.75	0.25	0.5000	0.3485	0.6515	0.8178	-8.1814	8.9992
0.7	0.3	0.4000	0.3225	0.6775	0.8315	-7.1063	7.9378
0.65	0.35	0.3000	0.2898	0.7102	0.8467	-6.0493	6.8960
0.6	0.4	0.2000	0.2468	0.7532	0.8643	-4.9848	5.8490
0.55	0.45	0.1000	0.1864	0.8136	0.8856	-3.8827	4.7683
0.505	0.495	0.0100	0.1049	0.8951	0.9106	-2.8292	3.7398
0.5005	0.4995	0.0010	0.0945	0.9055	0.9136	-2.7194	3.6331
0.50005	0.49995	0.0001	0.0934	0.9066	0.9139	-2.7084	3.6223

- Figures 5.24 and 5.25 show that the 50% market increase has a negative liquidity cost of 0.725 for $\hat{\alpha}_0 = 0.0471$, the negative liquidity cost in the 10% market increase is 0.049. This is as expected given that a larger market increase will induce a larger gain than a smaller increase.
- Figures 5.22 and 5.23 show that a smaller $\hat{\alpha}_1$ yields a greater sales value $S(y^*)$. This is to be expected, because a smaller $\hat{\alpha}_1$ means that the market will recover more quickly from the first sell event. Moreover, the figures show that the sales value is greater for $\hat{P}_2 = 1.5$ than for $\hat{P}_2 = 1.1$, again as expected.

Table 5.16: Chosen differences for $\Delta\hat{\alpha}$ in the upper range of the interval $(0, 1)$ for $\hat{P}_1 < \hat{P}_2 = 1.5$ with calculated values for $\Delta\hat{\alpha}$, y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1, 0)$ and ΔS.

$\mathbf{\hat{P}_1 < \hat{P}_2 = 1.5}$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1, 0)$	ΔS
0.99	0.09	0.9000	0.4184	0.5816	0.7489	-14.2091	14.9580
0.99	0.19	0.8000	0.4036	0.5964	0.7298	-14.1479	14.8776
0.99	0.29	0.7000	0.3887	0.6113	0.7118	-14.0732	14.7850
0.99	0.39	0.6000	0.3736	0.6264	0.6950	-13.9819	14.6768
0.99	0.49	0.5000	0.3581	0.6419	0.6791	-13.8704	14.5495
0.99	0.59	0.4000	0.3423	0.6577	0.6643	-13.7342	14.3984
0.99	0.69	0.3000	0.3260	0.6740	0.6503	-13.5678	14.2182
0.99	0.79	0.2000	0.3091	0.6909	0.6373	-13.3646	14.0019
0.99	0.89	0.1000	0.2914	0.7086	0.6250	-13.1165	13.7415
0.99	0.98	0.0100	0.2745	0.7255	0.6147	-12.8465	13.4612
0.999	0.998	0.0010	0.2741	0.7259	0.6095	-13.0948	13.7043
0.9999	0.9998	0.0001	0.2741	0.7259	0.6090	-13.1199	13.7289

Table 5.17: Model S with sliding constant $\Delta\hat{\alpha}$ on the interval $(0, 1)$ for $\hat{P}_1 < \hat{P}_2 = 1.5$ with calculated values for $\Delta\hat{\alpha}$, y_1^*, y_2^*, $S(y_1^*, y_2^*)$, $S(1, 0)$ and ΔS.

$\mathbf{\hat{P}_1 < \hat{P}_2 = 1.5}$							
$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\Delta\hat{\alpha}$	y_1^*	y_2^*	$S(y_1^*, y_2^*)$	$S(1, 0)$	ΔS
0.999999	0.8	0.2000	0.3099	0.6901	0.6325	-13.6684	14.3009
0.9	0.7	0.2000	0.3002	0.6998	0.6823	-10.8729	11.5552
0.8	0.6	0.2000	0.2872	0.7128	0.7371	-8.5507	9.2878
0.7	0.5	0.2000	0.2699	0.7301	0.7974	-6.6125	7.4099
0.6	0.4	0.2000	0.2468	0.7532	0.8643	-4.9848	5.8490
0.5	0.3	0.2000	0.2157	0.7843	0.9387	-3.6070	4.5457
0.4	0.2	0.2000	0.1728	0.8272	1.0222	-2.4291	3.4513
0.3	0.1	0.2000	0.1118	0.8882	1.1174	-1.4096	2.5270
0.2	$1e-07$	0.2000	0.0207	0.9793	1.2283	-0.5140	1.7423

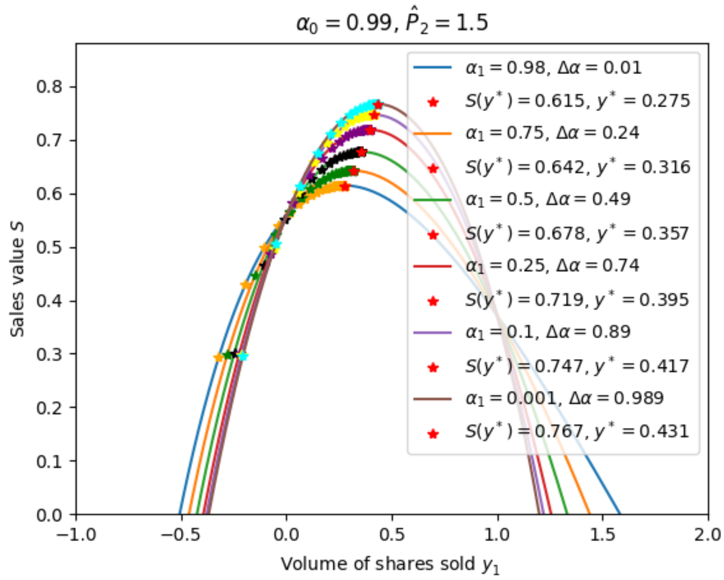


Figure 5.16: Graphs of $S(y)$ with $\hat{P}_2 = 1.5$ and $\hat{\alpha}_0 = 0.99$ with six different $\hat{\alpha}_1$. The graphs show the bigger the difference $\Delta\hat{\alpha}$ the smaller the liquidity cost and the smaller the y_1^* volume. The stars show the Gradient ascent iterations locating maximum and the optimal sales value $S(y^*)$.

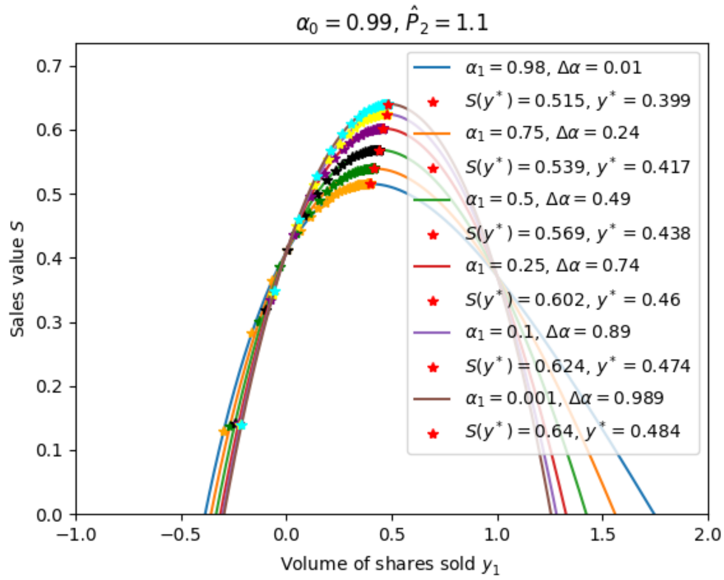


Figure 5.17: Graphs of $S(y)$ with $\hat{P}_2 = 1.1$ and $\hat{\alpha}_0 = 0.99$ with six different $\hat{\alpha}_1$. The graphs show the bigger the difference $\Delta\hat{\alpha}_0$ the smaller the liquidity cost and the bigger the y_1^* volume. The stars show the Gradient ascent iterations locating maximum and the optimal sales value $S(y^*)$.

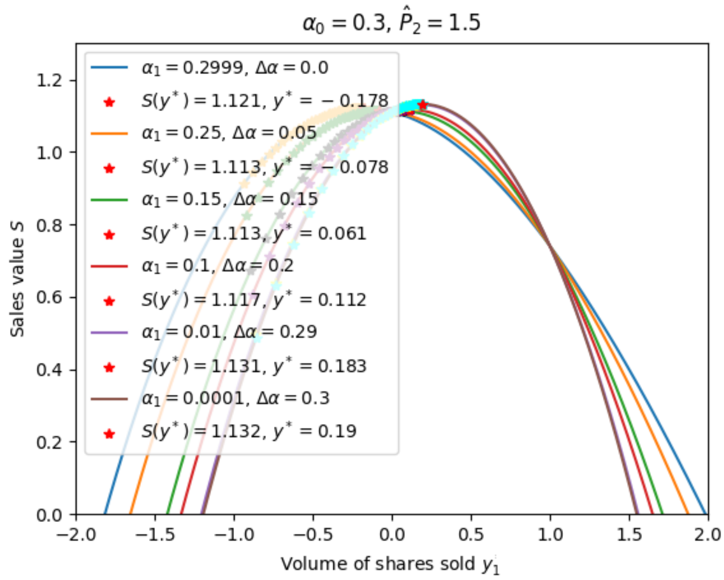


Figure 5.18: Graphs of $S(y)$ with $\hat{P}_2 = 1.5$ and $\hat{\alpha}_0 = 0.3$ with six different $\hat{\alpha}_1$. The graphs show the bigger the difference $\Delta\hat{\alpha}$ the smaller the y_1^* volume, while the liquidity cost first decreases for then to increase again. The stars show the Gradient ascent iterations locating maximum and the optimal sales value $S(y^*)$. The figure reveals a transition point $b \in (0.1, 0.25)$ in the $\hat{\alpha}_1$-domain where $S(y_1^*)$ decreases as $\hat{\alpha}_1$ moves from 0 to b, and increases from b to 1

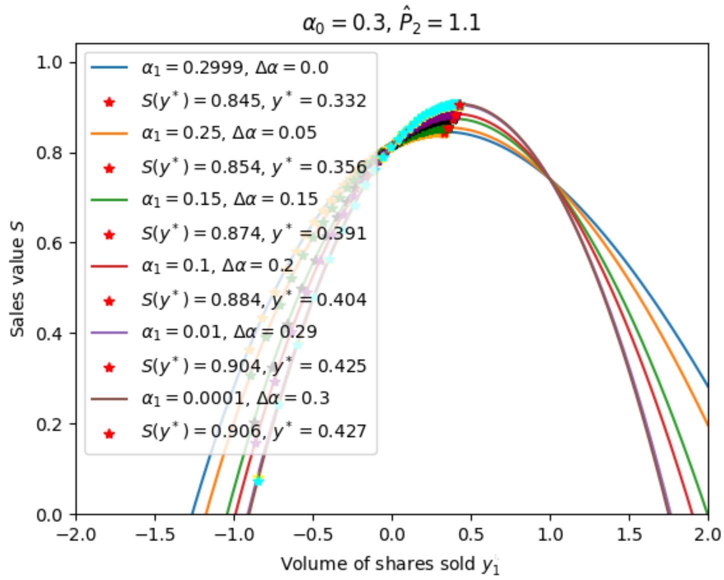


Figure 5.19: Graphs of $S(y^*)$ with $\hat{P}_2 = 1.1$ and $\hat{\alpha}_0 = 0.4$ with six different $\hat{\alpha}_1$. The graphs show the bigger the difference $\Delta\hat{\alpha}_0$ the smaller the y_1^* volume and the smaller the liquidity cost. The stars show the Gradient ascent iterations locating maximum and the optimal sales value $S(y^*)$. The figure does *not* reveal any transition points.

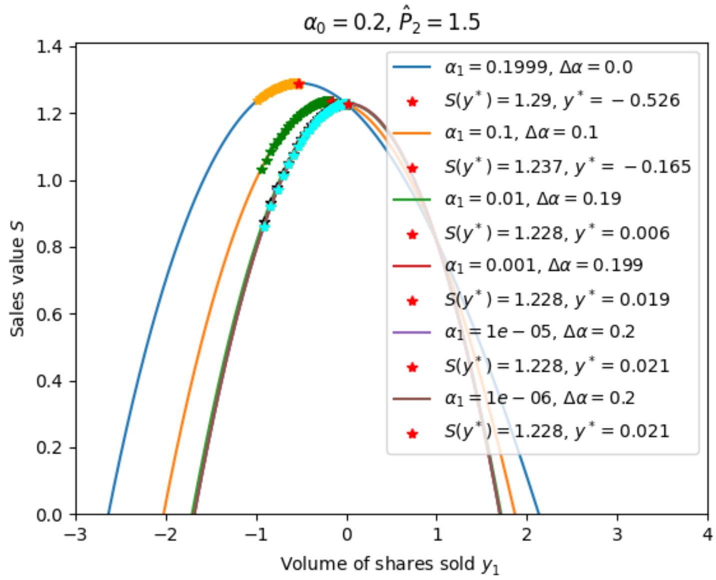


Figure 5.20: Graphs of $S(y)$ with $\hat{P}_2 = 1.5$ and $\hat{\alpha}_0 = 0.2$ with six different $\hat{\alpha}_1$. The graphs show the bigger the difference $\Delta\hat{\alpha}$ the smaller the y_1^* volume, while the liquidity cost first decreases for then to increase again. The stars show the Gradient ascent iterations locating maximum and the optimal sales value $S(y^*)$.

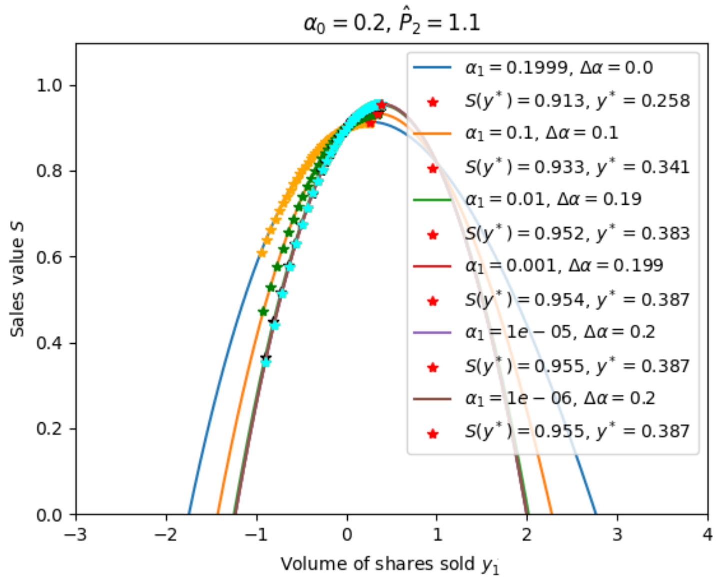


Figure 5.21: Graphs of $S(y)$ with $\hat{P}_2 = 1.1$ and $\hat{\alpha}_0 = 0.2$ with six different $\hat{\alpha}_1$. The graphs show the bigger the difference $\Delta\hat{\alpha}_0$ the smaller the y_1^* volume and the smaller the liquidity cost. The stars show the Gradient ascent iterations locating maximum and the optimal sales value $S(y^*)$.

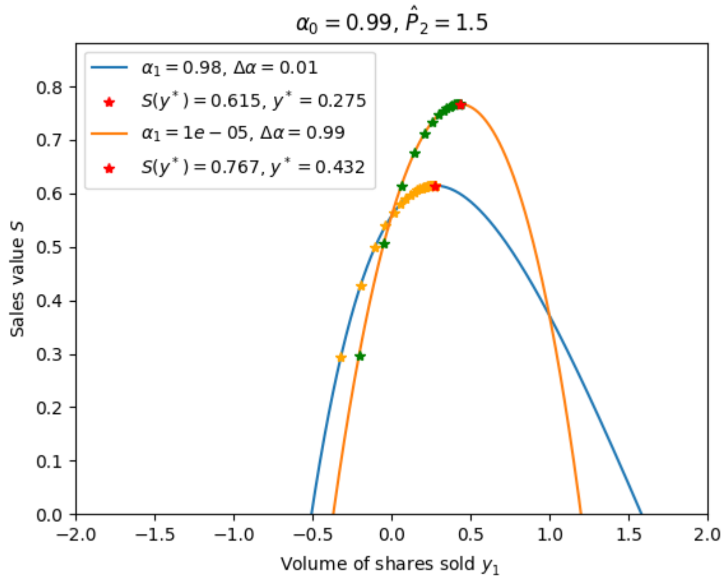


Figure 5.22: Two graphs of $S(y)$ with $\hat{P}_2 = 1.5$, $\hat{\alpha}_0 = 0.99$ and $\hat{\alpha}_1 = 0.98$ and $\hat{\alpha}_1 = 0.00001$.

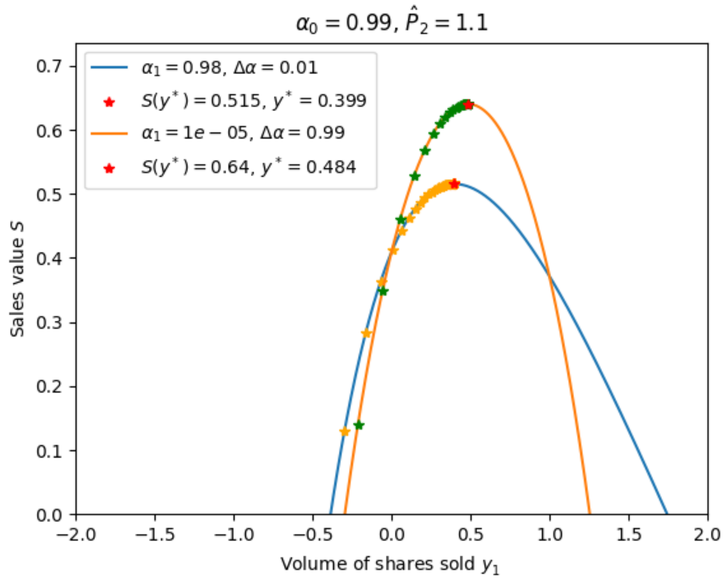


Figure 5.23: Two graphs of $S(y)$ with $\hat{P}_2 = 1.1$, $\hat{\alpha}_0 = 0.99$ and $\hat{\alpha}_1 = 0.98$ and $\hat{\alpha}_1 = 0.00001$.

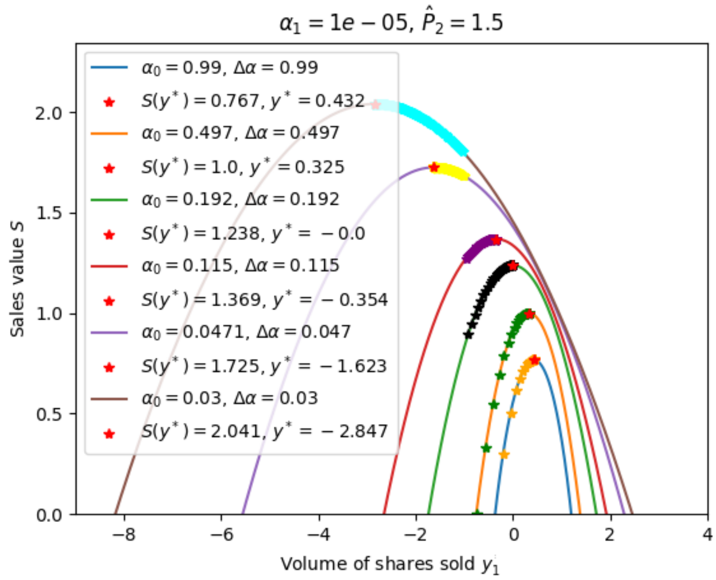


Figure 5.24: Five graphs of $S(y)$ with $\hat{\alpha}_0$ from 0.01 to 0.99 and $\hat{\alpha}_1 = 0.01$ for $\hat{P}_2 = 1.5$ showing their sales value. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y^*)$.

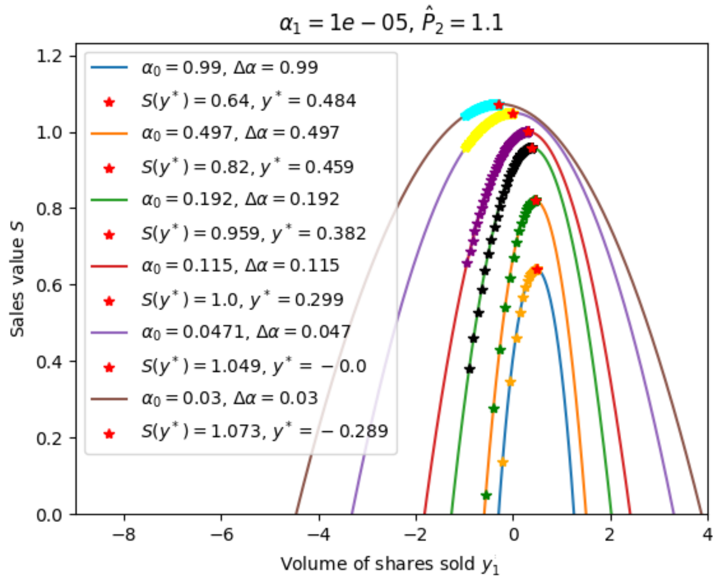


Figure 5.25: Five graphs of $S(y)$ with $\hat{\alpha}_0$ from 0.01 to 0.99 and $\hat{\alpha}_1 = 0.001$ for $\hat{P}_2 = 1.5$ showing their sales value. The stars show the Fixed Point iterations locating maximum and the optimal sales value $S(y^*)$.

CHAPTER 6

The case $N = 3$

This chapter evaluates trading strategies for liquidating a stock portfolio for $N = 3$. In Section 6.1 we show that the system of equations in the Lagrange Multiplier method is also unsolvable for $N = 3$. Secondly, in Section 6.2 we use *naive analysis* from psychology to determine which strategy spaces are feasible and optimal feasible in the different market types. In this case a naive analysis follows from the American Psychological Association's definition: 'A process of reasoning or intuiting by which laypersons determine whether "an actor" caused a certain action', [APA22]. In this definition 'an actor' is to be understood as the blocks of shares and 'action' is to be understood as market reaction to the actor. Hence, the naive feasible and optimal naive feasible solutions do not take the market's reaction to block volumes into consideration. Thirdly, in Section 6.3 we solve the problem of optimising the sales value by numerical iteration in Python, based on the method of Gradient Ascent, Theorem 2.2.10. Lastly, in Section 6.4 we do a comparative analysis of strategies in chosen market types based on the numeric results.

Remark 6.0.1. As narrowing $N = 3$ to fit the scope of the thesis had a myriad of possibilities we chose the twist of naive analysis to show how first glance intuition can diverge from deeper analysis. This seems interesting as the world is experiencing disbelief in science, so showing examples of deviations may be a super tiny step in bridging the gap between feelings and science.

6.1 An analytical approach for $N = 3$

Here we show that the optimisation of model S is unsolvable analytically. We also start by introducing a third transaction opportunity t_3 as we are in $N = 3$. Hence, we have the option to execute a one-step, two-step or three-step strategy; to buy, sell or hold y_1, y_2 and y_3 shares at times t_1, t_2 and t_3, where pumping, dumping and short selling are combinations of buy and sell actions.

So, our model S is now a three-variable sales function $S(y_1, y_2, y_3)$. Again, we wish to simplify the analysis and reduce the function $S(y_1, y_2, y_3)$ into a two-variable function $S(y_1, y_2)$. This time, we substitute $y_3 = 1 - y_1 - y_2$ into S, remembering that $\hat{P}_1 = 1$. Subsequently, we turn:

$$\begin{aligned} S(y_1, y_2, y_3) &= y_1 \hat{P}_1 e^{-\hat{\alpha}_0 y_1} + y_2 \hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} + y_3 \hat{P}_3 e^{-\hat{\alpha}_0 y_3 - \hat{\alpha}_1 y_2 - \hat{\alpha}_2 y_1} \\ &= y_1 e^{-\hat{\alpha}_0 y_1} + y_2 \hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} + y_3 \hat{P}_3 e^{-\hat{\alpha}_0 y_3 - \hat{\alpha}_1 y_2 - \hat{\alpha}_2 y_1} \end{aligned}$$

into

$$\begin{aligned} S(y_1, y_2) &= y_1 e^{-\hat{\alpha}_0 y_1} + y_2 \hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} \\ &\quad + (1 - y_1 - y_2)\hat{P}_3 e^{-\hat{\alpha}_0(1-y_1-y_2) - \hat{\alpha}_1 y_2 - \hat{\alpha}_2 y_1} \qquad (6.1) \\ &= y_1 e^{-\hat{\alpha}_0 y_1} + y_2 \hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} \\ &\quad + (1 - y_1 - y_2)\hat{P}_3 e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_2 + (\hat{\alpha}_0 - \hat{\alpha}_2) y_1 - \hat{\alpha}_0}. \qquad (6.2) \end{aligned}$$

Analytically, we now find where the function $S(y_1, y_2)$ has stationary points, which we do by an investigation of the partial derivatives of $S(y_1, y_2)$. We start out by taking the partial derivatives, equating them to 0 and see if it is possible to solve for y_1 and y_2. We were unable to find analytical solutions for y_1 and y_2 in $N = 2$, so the chance of success in this case seems rather small. But, let's see what happens:

$$\begin{aligned} \frac{\partial S}{\partial y_1} &= e^{-\hat{\alpha}_0 y_1} - \hat{\alpha}_0 y_1 e^{-\hat{\alpha}_0 y_1} - y_2 \hat{P}_2 \alpha_1 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} \\ &\quad - \hat{P}_3 e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_2 + (\hat{\alpha}_0 - \hat{\alpha}_2) y_1 - \hat{\alpha}_0} \\ &\quad + (1 - y_1 - y_2)\hat{P}_3(\hat{\alpha}_0 - \hat{\alpha}_2) e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_2 + (\hat{\alpha}_0 - \hat{\alpha}_2) y_1 - \hat{\alpha}_0} \\ &= (1 - \hat{\alpha}_0 y_1) e^{-\hat{\alpha}_0 y_1} - \hat{\alpha}_1 y_2 \hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} \\ &\quad - \Big(1 - (1 - y_1 - y_2)(\hat{\alpha}_0 - \hat{\alpha}_2)\Big)\hat{P}_3 e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_2 + (\hat{\alpha}_0 - \hat{\alpha}_2) y_1 - \hat{\alpha}_0} \qquad (6.3) \end{aligned}$$

$$\begin{aligned} \frac{\partial S}{\partial y_2} &= 0 + \hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} - y_2 \hat{P}_2 \hat{\alpha}_0 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} \\ &\quad - \hat{P}_3 e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_2 + (\hat{\alpha}_0 - \hat{\alpha}_2) y_1 - \hat{\alpha}_0} \\ &\quad + (1 - y_1 - y_2)\hat{P}_3(\hat{\alpha}_0 - \hat{\alpha}_1) e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_2 + (\hat{\alpha}_0 - \hat{\alpha}_2) y_1 - \hat{\alpha}_0} \\ &= (1 - \hat{\alpha}_0 y_2)\hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} \\ &\quad - \Big(1 - (1 - y_1 - y_2)(\hat{\alpha}_0 - \hat{\alpha}_1)\Big)\hat{P}_3 e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_2 + (\hat{\alpha}_0 - \hat{\alpha}_2) y_1 - \hat{\alpha}_0}. \qquad (6.4) \end{aligned}$$

These equations are on the same form, but more complicated than their $N = 2$ siblings. We tried to solve the equations analytically, but as expected the system of equations $\frac{\partial S}{\partial y_1} = 0$ and $\frac{\partial S}{\partial y_2} = 0$ is unsolvable. Hence, this is done numerically in Section 6.3 by the use of Python, after the naive analysis of how trading strategies may be chosen by reasoning rather than numerical analysis.

6.2 Trade actions for y_1, y_2 and y_3, and strategy spaces

We start this section by looking at trade strategy spaces: Sell, buy, short selling, dumping, pumping and no trade, described by 3-tuples with +, − and 0. Afterwards, we investigate different markets and hypothesise which strategy spaces are feasible and optimal feasible in each market by a naive analysis. Not all strategy spaces complete the task of liquidating an already existing portfolio, hence we will exclude them from further analysis.

As the end game is to liquidate the entire portfolio the sum of the trades must equal to 1. For example, the strategy $(0, 0, 0)$ states that the investor should do nothing. This of course does not sum to 1, and is not only suboptimal, it is outside the scope of the task of liquidating the portfolio. On this note, it is only the no trade strategy and the pure buy strategy space that are outside the scope, as the investor is unable to liquidate the portfolio. In all other cases a combination of pumping, dumping, buy, sell, no trade and short selling can liquidate the portfolio. Furthermore, the dumping strategy space is a subspace of the short selling strategy space, as it is one out of several ways to short sell in a market.

6.2.1 Defining the possible strategy spaces

We use combinatorics to find the different strategy spaces for trading stocks in $N = 3$ steps. We are looking at an ordered situation with the possibility of returning the elements for y_1, y_2 and y_3. As the investor may either buy $(-)$, sell $(+)$ or do nothing (0), we have three possible actions. We also have three steps to carry out each action. Hence, we have $n^r = 3^3 = 27$ different strategy spaces to choose from, where n is the number of possible actions in each position r. So, in the following, m is defined as the number of trade actions the investor utilises in a strategy space.

$\boldsymbol{m = 1}$ is the situation where the investor only engage in one type of action. The cases where the investor either chooses to sit still and not engage in any trades $(0, 0, 0)$, sell some stock in each step $(+, +, +)$, or buy some stock in each step $(-, -, -)$ are the only possibilities. The total number of trade actions for $m = 1$ is three. Of these three permutations we must exclude those marked in red, as they do not liquidate the portfolio. Hence, only **one** strategy space is within the scope of our assignment.

$\boldsymbol{m = 2}$ is the situation where the investor can engage in only two types of actions. The only possible cases are those where the investor either sells and buys $(+, +, -)$, $(-, -, +)$, $(+, -, +)$, $(-, +, -)$, $(-, +, +)$, $(+, -, -)$, sells and holds $(+, 0, 0)$, $(0, +, 0)$, $(0, 0, +)$, $(+, 0, +)$, $(+, +, 0)$, $(0, +, +)$, or buys and holds $(-, 0, 0)$, $(0, -, 0)$, $(0, 0, -)$, $(-, 0, -)$, $(-, -, 0)$, $(0, -, -)$. The total number of trade actions for $m = 2$ is 18. Of these 18 permutations we must exclude the six marked in red, as they do not liquidate the portfolio. Hence, only **twelve** strategy spaces are within the scope.

$\boldsymbol{m = 3}$ is the situation where the investor engages in all three actions. The cases where the investor either sells, buys or holds stock at either step are $(+, -, 0)$, $(-, +, 0)$, $(+, 0, -)$, $(-, 0, +)$, $(0, +, -)$ and $(0, -, +)$. The number of trade actions for $m = 3$ is **six**, where all permutations may liquidate the portfolio.

Of the 27 strategy spaces above 19 are within the scope of our task, namely to liquidate the portfolio. The aim now is to find the strategy spaces that yields the smallest liquidity cost C through maximising the sales value S, so we must investigate these 19 strategy spaces further.

At a first glance it seems tempting to include the *hold action* as a part of the sell strategy space. But, let's wait to see how this unravels. On this note, we may now classify the different strategy spaces in terms of the areas in $\mathbb{R}^3$. For a geometric representation, see Figure 6.1.

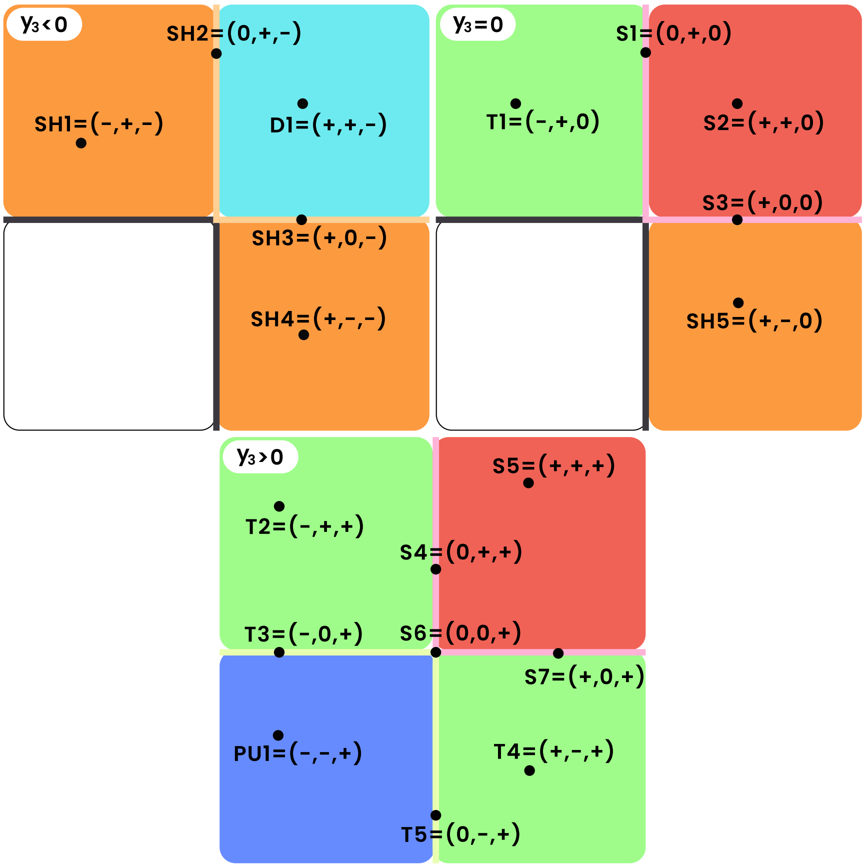


Figure 6.1: The three planes illustrate $\mathbb{R}^3$ sliced into $y_3 < 0$, $y_3 = 0$ and $y_3 > 0$. The points are randomly placed within its strategy space within the scope of the model S. The orange quadrants are short selling spaces. The light blue quadrant is the dumping space. The green quadrants are the buy and sell spaces. The red quadrants are the pure sell spaces. The dark blue quadrant is the pumping space. The pale lines in the strategy space colours indicate which strategy space the points on the axes belongs to. The origin is outside the scope of the model.

Orange quadrants: Short selling strategy spaces, SH.

$$(-\infty, 0] \times (0, \infty) \times (-\infty, 0) \cup (0, \infty) \times (-\infty, 0] \times (-\infty, 0]$$

Light blue quadrant: Dumping strategy space, D.

$$(0, \infty) \times (0, \infty) \times (-\infty, 0)$$

Green quadrants: Sell and buy strategy space, T.

$$(-\infty, 0) \times [0, \infty) \times [0, \infty) \cup [0, \infty) \times (-\infty, 0) \times (0, \infty)$$

Red quadrants: Pure sell strategy space, S.

$$[0, \infty) \times [0, \infty) \times [0, \infty)$$

Dark blue quadrant: Pumping strategy space, PU.

$$(-\infty, 0) \times (-\infty, 0) \times (0, \infty)$$

We have now defined the locations in $\mathbb{R}^3$ for the different strategy spaces. However, there is also a white domain which is outside the scope of our model. In order to operate in these spaces the investor must have the opportunity to do nothing or just buy stock. We do not allow those strategy spaces, as they do not liquidate the portfolio. Furthermore, it seems important to emphasise that Figure 6.1 is $\mathbb{R}^3$ sliced into three layers defined by $y_3 < 0$, $y_3 = 0$ and $y_3 > 0$. In the following we perform a naive analysis to decide which strategy spaces are feasible and optimal feasible in different market types.

6.2.2 Market types and feasible strategy spaces

The strategy spaces displayed in Figure 6.1 must be evaluated based on market types. As both $\hat{P}_2$ and $\hat{P}_3$ may be either Bull, Flat or Bear markets we have nine different market types, all having specific naive optimal market strategies: Bull, BullFlat, BullBear, Flat, FlatBull, FlatBear, Bear, BearFlat and BearBull. We now naively analyse which strategy spaces are feasible and which are optimal feasible for each market type. Furthermore, we provide arguments for the rationale of each strategy space in every market type.

Remark 6.2.1. The *optimal feasible strategy space* is the one that results in the largest sales value S, under the assumption of an undisturbed market, i.e., no bounce-back function ψ. Also, *feasible strategy spaces* are those that are not guarantied a loss, hereunder estimated market value or legal action. An illustration of all strategy spaces based on market types is depicted in Figure 6.2, where the optimal feasible strategy spaces are marked with a golden star.

6.2.2.1 Bull market: $\hat{P}_1 < \hat{P}_2 < \hat{P}_3$

Feasible strategy spaces: $PU_1 = (-, -, +)$, $T_2 = (-, +, +)$, $T_3 = (-, 0, +)$, $S_4 = (0, +, +)$, $S_5 = (+, +, +)$ and $S_6 = (0, 0, +)$.

Optimal feasible strategy space: Buy more shares at t_1 and sell all shares at t_3, $T_3 = (-, 0, +)$.

Rationale:

- The pumping strategy space where the investor buys more shares at t_1 or t_2 for then to sell all shares at t_3. This strategy space will yield a profit under these market sentiments.

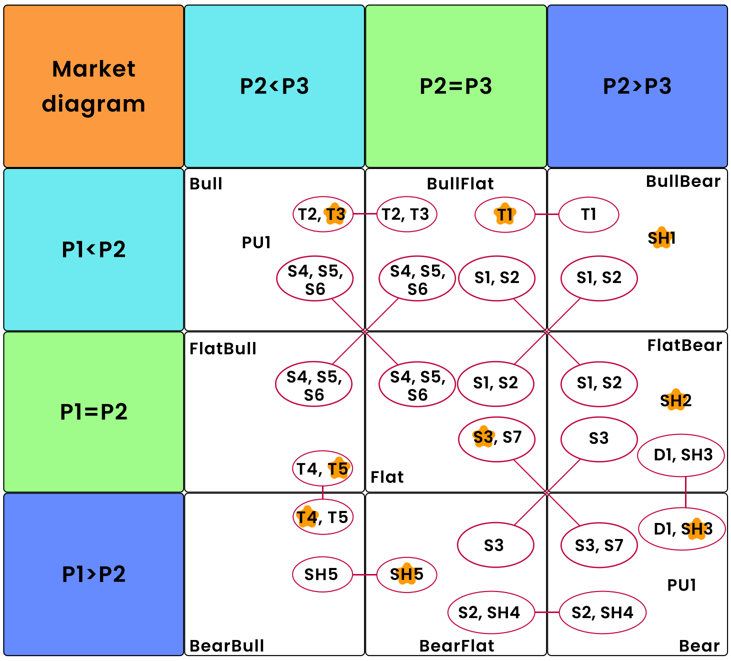


Figure 6.2: The different candidates for feasible and optimal feasible strategy spaces placed into market types. Candidates marked with a golden star are optimal feasible strategy spaces. See Figure 6.1 for strategy space definitions.

- The buy and sell strategy spaces where the investor will buy all stock at t_1 or t_2 for then to sell all shares at t_3. In these spaces the investor is guaranteed to buy low and sell higher and no loss can occur.
- The pure sell strategy spaces include liquidating the stock step-wise in t_1, t_2 and t_3, assuring no loss of sales value on the portfolio as the market is rising.
- The strategy space $T_3 = (-, 0, +)$ is optimal feasible as the investor buys stock when the price is cheapest and sells all when the price is highest.

6.2.2.2 BullFlat market: $\hat{P}_1 < \hat{P}_2 = \hat{P}_3$

Feasible strategy spaces: $S_1 = (0, +, 0)$, $S_2 = (+, +, 0)$, $S_4 = (0, +, +)$, $S_5 = (+, +, +)$, $S_6 = (0, 0, +)$, $T_1 = (-, +, 0)$, $T_2 = (-, +, +)$ and $T_3 = (-, 0, +)$.

Optimal feasible strategy space: Buy more shares at t_1 and sell all at t_2, $T_1 = (-, +, 0)$.

Rationale:

- The pure sell strategy spaces have the investor liquidate the stock step-wise in t_1, t_2 or t_3, assuring a possibility of gain on the estimate market value on the portfolio.

- The buy and sell strategy spaces have the investor buy stock at t_1 for then to sell all shares at t_2 or t_3. In these cases the investor is guaranteed to buy low and sell higher and no loss can occur.
- The optimal strategy space is $T_1 = (-, +, 0)$ as the full gain is taken as soon as possible, reducing the inflation risk and foregoing other investment opportunities.

6.2.2.3 BullBear market: $\hat{P}_1 < \hat{P}_2 \cup \hat{P}_2 > \hat{P}_3$

Feasible strategy spaces: $T_1 = (-, +, 0)$, $S_1 = (0, +, 0)$, $S_2 = (+, +, 0)$, $SH_1 = (-, +, -)$ and $SH_2 = (0, +, -)$.

Optimal feasible strategy space: Buy more shares at t_1, borrow shares and sell those and the original portfolio at t_2, for then to buy back borrowed shares at t_3, $SH_1 = (-, +, -)$.

Rationale:

- The pure sell strategy spaces have the investor liquidate the stock in t_1 or t_2, assuring the possibility of a gain on the estimated market value.
- The buy and sell strategy space have the investor buy stock at t_1 for then to sell all shares at t_2. In this case the investor is guaranteed to buy low and sell higher and no loss can occur.
- The short selling strategy spaces where the investor buys more and borrows stock at t_1 for then to sell it all at t_2. In order to capitalise on the Bear market the investor buys back the borrowed portion of the sold stock, for then to return the borrowed shares. In this case the investor earns the price delta of the market increase from P_1 to P_2 and on the decrease from P_2 to P_3.
- The optimal strategy space $SH_1 = (-, +, -)$ is thus where the investor earns money on the market increase by buying more stock at t_1 and on the market decrease by utilising a short selling strategy space on the entire portfolio plus borrowed shares.

6.2.2.4 FlatBull market: $\hat{P}_1 = \hat{P}_2 < \hat{P}_3$

Feasible strategy spaces: $S_4 = (0, +, +)$, $S_5 = (+, +, +)$, $S_6 = (0, 0, +)$, $T_4 = (+, -, +)$ and $T_5 = (0, -, +)$.

Optimal feasible strategy space: Buy more share at t_2 for then to sell the entire portfolio at t_3, $T_5 = (0, -, +)$.

Rationale:

- The pure sell strategy spaces have the investor liquidate the stock step-wise in t_1, t_2 or t_3, assuring a possibility of a gain compared to the estimated market value.

- The buy and sell strategy spaces have the investor buy stock at t_2 for then to sell all shares at t_3. In these cases the investor is guaranteed to buy low and sell higher and no loss can occur.
- The optimal strategy space is $T_5 = (0, -, +)$ as the entire portfolio is liquidated in the Bull part.

6.2.2.5 Flat market: $\hat{P}_1 = \hat{P}_2 = \hat{P}_3$

Feasible strategy spaces: $S_1 = (0, +, 0)$, $S_2 = (+, +, 0)$, $S_3 = (+, 0, 0)$, $S_4 = (0, +, +)$, $S_5 = (+, +, +)$, $S_6 = (0, 0, +)$ and $S_7 = (+, 0, +)$.

Optimal feasible strategy space: Sell entire portfolio at t_1, $S_3 = (+, 0, 0)$.

Rationale:

- The feasible strategy spaces have the investor sell all strategy spaces as the investor needs to get out of the market in order to reinvest in different assets.
- The optimal strategy space is to sell the entire portfolio at t_1 in order to release funds to make other investments, $S_3 = (+, 0, 0)$.

6.2.2.6 FlatBear market: $\hat{P}_1 = \hat{P}_2 > \hat{P}_3$

Feasible strategy spaces: $D_1 = (+, +, -)$, $SH_2 = (0, +, -)$, $SH_3 = (+, 0, -)$, $S_1 = (0, +, 0)$, $S_2 = (+, +, 0)$ and $S_3 = (+, 0, 0)$.

Optimal feasible strategy space: Sell original portfolio and borrowed shares at t_2 and repurchase the borrowed shares at t_3, $SH_2 = (0, +, -)$.

Rationale:

- The dumping strategy space have the investor sell original and borrowed stock at t_1 and t_2 for then to repurchase the borrowed shares at t_3.
- The pure sell strategy spaces have the investor liquidate the stock in t_1 or t_2, assuring a possibility of a gain compared to the estimated market value.
- The short selling strategy spaces have the investor sell the original and borrowed stock at t_1 or t_2 for then to repurchase the borrowed shares at t_3. In these spaces the investor earns the price delta of the market decline from P_1 or P_2 to P_3.
- The optimal strategy space is $SH_2 = (0, +, -)$ as the investor holds the short position a shorter amount of time and thus reduces the exposure risk.

6.2.2.7 BearBull market: $\hat{P}_1 > \hat{P}_2 \cup \hat{P}_2 < \hat{P}_3$

Feasible strategy spaces: $SH_5 = (+,-,0)$, $T_4 = (+,-,+)$ and $T_5 = (0,-,+)$.

Optimal feasible strategy space: Sell stock at t_1, buy more stock at t_2 and sell the complete position at t_3, $T_4 = (+,-,+)$.

Rationale:

- The short selling strategy spaces have the investor sell original and borrowed stock at t_1 for then to repurchase the borrowed shares at t_2. In this case the investor earns the price delta of the market decline from P_1 to P_2.
- The buy and sell strategy spaces where the investor sells or keeps stock at t_1, then buys more stock as the market is at a low at t_2, for then to sell it all when the market is back up again at t_3.
- The optimal strategy space is $T_4 = (+,-,+)$ as the investor capitalises before the market is in decline. Then the investor buys more stock when the market is at the lowest for then to sell at t_3 and collect a gain on the Bull market.

6.2.2.8 BearFlat market: $\hat{P}_1 > \hat{P}_2 = \hat{P}_3$

Feasible strategy spaces: $S_2 = (+,+,0)$, $S_3 = (+,0,0)$, $SH_4 = (+,-,-)$ and $SH_5 = (+,-,0)$.

Optimal feasible strategy space: Sell original and borrowed stock at t_1 for then to repurchase the borrowed stock at t_2, $SH_5 = (+,-,0)$.

Rationale:

- The pure sell strategy spaces have the investor liquidate the stock in t_1 or t_2, assuring a possibility of a gain compared to the estimated market value.
- The short selling strategy spaces have the investor sell original and borrowed stock at t_1 for then to repurchase the borrowed shares at t_2 or t_3. In this case the investor earns the price delta of the market decline from P_1 to P_2 or P_3.
- The optimal strategy space is $SH_5 = (+,-,0)$ as the market potential is released in the shortest amount of time, reducing the exposure risk.

6.2.2.9 Bear market: $\hat{P}_1 > \hat{P}_2 > \hat{P}_3$

Feasible strategy spaces: $PU_1 = (-,-,+)$, $D_1 = (+,+,-)$, $S_2 = (+,+,0)$, $S_3 = (+,0,0)$, $S_7 = (+,0,+)$, $SH_3 = (+,0,-)$ and $SH_4 = (+,-,-)$.

Optimal feasible strategy space: Sell short at t_1 and repurchase at t_3, $SH_3 = (+, 0, -)$.

Rationale:

- The pumping strategy space have the investor buy stock at t_1 and t_2 in order to support a declining stock. This is not a strategy space that must yield a loss compared to the estimated market value, however it is used in order to dampen the price reduction. It is thus not optimal in terms of sales value.
- The dumping strategy space have the investor sell stock at t_1 and t_2 in order to reduce the liquidity cost. This strategy space will often incur a loss compared to the estimated market value. However, if the stock was originally bought in a market that came from a lower price point than the sell price in the Bear market, the investor may make a profit. This profit will though be lower than if the investor takes a short position.
- The pure sell strategy spaces have the investor try to liquidate the stock in order to dampen a decline in the price development. These strategy spaces, are of course, no-profit strategies unless the stock was bought in a market with a lower price point. Nonetheless, these strategy spaces cannot compete with a short selling strategy space in terms of a profit opportunity.
- The short selling strategy spaces have the investor sell stock at t_1 or t_2 and repurchase borrowed stock at t_3. In this case the investor earns the price delta of the market decline. Under these circumstances the investor is guaranteed to sell high and buy back low and no loss can occur.
- The optimal strategy space $SH_3 = (+, 0, -)$ is thus where original and borrowed shares are sold at t_1 and the borrowed shares are repurchased and returned at t_3.

Remark 6.2.2. It is necessary to mention that some of these optimal strategy spaces may demand that the investor pump or dump shares indefinitely. This is of course outside the scope of our model, so we limit the scope by the individual investor's access to capital and shares, as well as the amount of shares issued by a given company.

Consequently, we have defined strategy spaces in $\mathbb{R}^3$ where we naively found optimal strategy spaces for different market types. In the next chapter we try to determine the location of the actual maxima numerically in either market type, and do a comparative analysis between the naive analysis without bounce-back considerations and our model S with the bounce-back function ψ. Finally, in the next section, we do a comparative analysis of the numeric results across the different market types.

6.3 The numerical analysis

In the numerical approach we plot the model and optimise by Gradient Ascent, and describe the different *actual optimal strategy spaces* and numerical strategies for all nine markets defined in the previous section. An actual optimal strategy space in this case is one with a numerical strategy that our model S generates under certain market sentiments $\hat{\alpha}_0$, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ and market types $\hat{P}_2$ and $\hat{P}_3$.

The case $N = 3$ and $n = 1, 2, 3$ models the situation where the investor sells the entire portfolio of stock in three – not necessarily equal – blocks. As mentioned before, the market will not have time to stabilise between each trade, and the function ψ describes the discount factor generated by the market's lacking ability to bounce-back.

6.3.1 The mathematics behind the Python scene

Our deep dive into the case $N = 3$ starts this time with a visit to the Gradient Ascent through partial derivatives and the gradient ∇S. The model is given by $S(y_1, y_2, y_3)$, as introduced in Equation (1.7):

$$S(y_1, y_2, y_3) = y_1 \hat{P}_1 e^{-\hat{\alpha}_0 y_1} + y_2 \hat{P}_2 e^{-(\hat{\alpha}_0 y_2 + \hat{\alpha}_1 y_1)} + y_3 \hat{P}_3 e^{-(\hat{\alpha}_0 y_3 + \hat{\alpha}_1 y_2 + \hat{\alpha}_2 y_1)},$$

where y_1, y_2 and y_3 are weighted blocks of shares as a percentage of the total portfolio. This also entails that $y_1 + y_2 + y_3 = 1$, which we use to reduce the function $S(y_1, y_2, y_3)$ to a function of two variables by substituting $y_3 = 1 - y_1 - y_2$, just like before. This way we may plot the surface, and find the optimum by Gradient Ascent. We found this expression in (6.2) in the previous section:

$$\begin{aligned} S(y_1, y_2) = y_1 \hat{P}_1 e^{-\hat{\alpha}_0 y_1} &+ y_2 \hat{P}_2 e^{-(\hat{\alpha}_0 y_2 + \hat{\alpha}_1 y_1)} \\ &+ (1 - y_1 - y_2) \hat{P}_3 e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_2 + (\hat{\alpha}_0 - \hat{\alpha}_2) y_1 - \hat{\alpha}_0} \end{aligned}$$

The partial derivatives (6.3) and (6.4) are

$$\begin{aligned} \frac{\partial S}{\partial y_1} &= (1 - \hat{\alpha}_0 y_1) e^{-\hat{\alpha}_0 y_1} - \hat{\alpha}_1 y_2 \hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} \\ &\quad - \Big(1 - (1 - y_1 - y_2)(\hat{\alpha}_0 - \hat{\alpha}_2)\Big) \hat{P}_3 e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_2 + (\hat{\alpha}_0 - \hat{\alpha}_2) y_1 - \hat{\alpha}_0} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial S}{\partial y_2} &= (1 - \hat{\alpha}_0 y_2) \hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} \\ &\quad - \Big(1 - (1 - y_1 - y_2)(\hat{\alpha}_0 - \hat{\alpha}_1)\Big) \hat{P}_3 e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_2 + (\hat{\alpha}_0 - \hat{\alpha}_2) y_1 - \hat{\alpha}_0}. \end{aligned}$$

Hence, the gradient ∇S is as follows.

$$\begin{aligned} \nabla S(y_1, y_2) = \Bigg(&(1 - \hat{\alpha}_0 y_1) e^{-\hat{\alpha}_0 y_1} - \hat{\alpha}_1 y_2 \hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} \\ &- \Big(1 - (1 - y_1 - y_2)(\hat{\alpha}_0 - \hat{\alpha}_2)\Big) \hat{P}_3 e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_2 + (\hat{\alpha}_0 - \hat{\alpha}_2) y_1 - \hat{\alpha}_0}, \\ &(1 - \hat{\alpha}_0 y_2) \hat{P}_2 e^{-\hat{\alpha}_0 y_2 - \hat{\alpha}_1 y_1} \\ &- \Big(1 - (1 - y_1 - y_2)(\hat{\alpha}_0 - \hat{\alpha}_1)\Big) \hat{P}_3 e^{(\hat{\alpha}_0 - \hat{\alpha}_1) y_2 + (\hat{\alpha}_0 - \hat{\alpha}_2) y_1 - \hat{\alpha}_0} \Bigg). \end{aligned}$$

As Python has packages that calculate the partial derivatives, we choose to utilise these in the numerical script. We want, however, to display the gruelling expressions to show what is going on behind the scene.

6.3.2 Numerical plots, Gradient Ascent and other mathematical stars

In this section we find the actual optimal strategy spaces and compare them to the naive optimal strategy spaces. We use the Method of Gradient Ascent to visualise the actual optimal strategy spaces in 3D plots defined by the reduced model $S(y_1, y_2)$.

A weakness of this section is that it only includes one set of $\hat{\alpha}$s: $\hat{\alpha}_0 = 0.9$, $\hat{\alpha}_1 = 0.5$ and $\hat{\alpha}_2 = 0.1$. It would have been very interesting to compare and contrast several combinations of $\hat{\alpha}$s, sadly the scope of this thesis does not allow for that. This limitation does restrict our insight into the behaviour in $N = 3$ of our model, nonetheless we do see some interesting things as we compare the model to the naive analysis.

We continue onto numerical calculations to find different optima dependent on market sentiments $\hat{\alpha}_n$, as well as the estimated prices $\hat{P}_n$. As market sentiments are impossible to determine, but merely are functions of political, psychological and events of the World, we optimise based on different estimated prices $\hat{P}_n$, as an investor's trading decisions often are based on the stock price development rather than market sentiment. Yes, the stock price is of course a result of the overall market, however stock prices can move counter-cyclical to market sentiments. An example in these times of war is the oil price.

Remark 6.3.1. When the naive analysis was made there was no inference to any restrictions on the market type or its price development.

6.3.2.1 Bull: $\hat{P}_1 < \hat{P}_2 < \hat{P}_3$

In Section 6.2.2.1 on page 70 we found the naive optimum to be the buy and sell strategy space $T_3 = (-, 0, +)$ for a pure Bull market. In this case the investor buys more stock at t_1, does nothing at t_2, for then to sell the entire portfolio at t_3.

The naive optimum $T_3 = (-, 0, +)$, approximated by Figure 6.3, is a boundary case between $T_2 = (-, +, +)$ and $PU_1 = (-, -, +)$, Figures 6.4 and 6.5 display two examples of actual optimal strategy spaces for the cases $T_2 = (-, +, +)$ and $PU_1 = (-, -, +)$. Based on the Intermediate Value Theorem 2.2.11 we know that the investment strategy space $T_3 = (-, 0, +)$ also can be found as an actual optimum under certain price conditions given the continuity of the model. It is also important to emphasise that the strategy space chosen in the naive case, $T_3 = (-, 0, +)$, is just that: A naive approach to making investment decisions, as there is no one solution to fit all price developments in a certain type of market. Our numerical analysis unravels that our naive optimum is not the optimal strategy space in most cases.

So, why is it that $T_3 = (-, 0, +)$ is not optimal at all times? It seems that the bounce-back function ψ in the last step t_3 does not have enough bounce. Hence, the price drop of flooding the market at t_3 will outperform the profit from the extra stocks bought in step t_1. It may however be argued that an insignificant portfolio could benefit from the T_3 strategy space as the bounce element will be negligible, but this is outside the scope of this thesis.

Furthermore, numerous iterations within the scope of conceivable price changes found several incidents where many of the naive feasible strategy spaces were the actual optimum, such as the example $T_4 = (+, -, +)$ shown in Figure 6.3. In addition, Figures 6.6 and 6.7 display counter-examples of the naive optimum $T_3 = (-, 0, +)$, and thus another naive feasible strategy space turned actual optimal strategy space. Both of these solutions belong to the trading strategy space $S_5 = (+, +, +)$ where the investor should sell a portion of the portfolio in each step, rather than buy even more stock at t_1 for then to sell all at the last step t_3.

Finally, our model shows that in a moderate market increase, displayed in Figure 6.6, the investor manages to salvage only 69.33% of the portfolio value with this three-step strategy. Even in a hot market with 50% increase shown in Figure 6.7, the investor only manages to escape with 83.16% of the portfolio value, incurring a 16.84% liquidity cost due to the size of the portfolio. So, a market change of 40 percentage points gives a sales value change of 13.83 percentage points. Also, the market increase of 50% steals more value (66.84%) from the investor than the market increase of 10% (40.70%) even though the investor is able to salvage more of the portfolio, compared to the case where the portfolio size was unable to move the market. Hence, the bounce-back function ψ influence less by a bigger market increase than a smaller market increase.

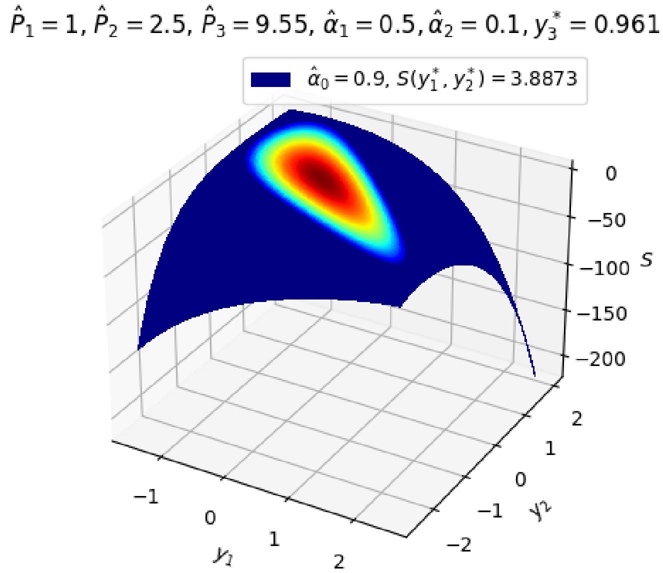
6.3.2.2 BullFlat: $\hat{P}_1 < \hat{P}_2 = \hat{P}_3$

In Section 6.2.2.2 on page 71 we found the naive optimal strategy space to be the buy and sell strategy space $T_1 = (-, +, 0)$, where the investor buys more stock in step t_1, sells all in t_2 and does nothing in t_3.

The numerical analysis shows that the naive optimum, also in this case, is not the actual optimal strategy space. Iterations within the scope of conceivable price changes disclosed options where several of the naive feasible buy and sell strategy spaces were actual optimal strategy spaces, such as $S_5 = (+, +, +)$ shown in Figures 6.8 and 6.9. The figures display a BullFlat market with 10% and 50% increase before levelling out, respectively.

Furthermore, the naive optimal strategy space $T_1 = (-, +, 0)$ is a boundary case for $T_2 = (-, +, +)$ and $SH_1 = (-, +, -)$. It is clear that we are unable to find SH_1, as this is a short selling strategy space, which is not naive feasible in a non-declining market. We were however able to find another optimal strategy space in addition to $S_5 = (+, +, +)$ among the naive feasible strategy spaces, namely the buy and sell strategy space $T_2 = (-, +, +)$. Figure 6.11 shows an approximation to the boundary case $S_4 = (0, +, +)$ between S_5 and T_2, and Figure 6.10 displays an example of T_2.

Based on numerical iterations of $\hat{P}_2 = \hat{P}_3$ we see that the second and third coordinate of our 3-tuple (y_1, y_2, y_3) seem to stay positive independent of whether we make $\hat{P}_2$ go towards 1 or some big positive number even outside of a possible scope. It is interesting to see that the naive optimal strategy space $T_1 = (-, +, 0)$ is thus degraded to a non-feasible strategy space. Furthermore, several of the naive feasible strategy spaces hypothesised in the BullFlat scenario are after the numerical analysis degraded to naive non-feasible strategy space. Of the naive feasible strategy spaces the following are now degraded to naive



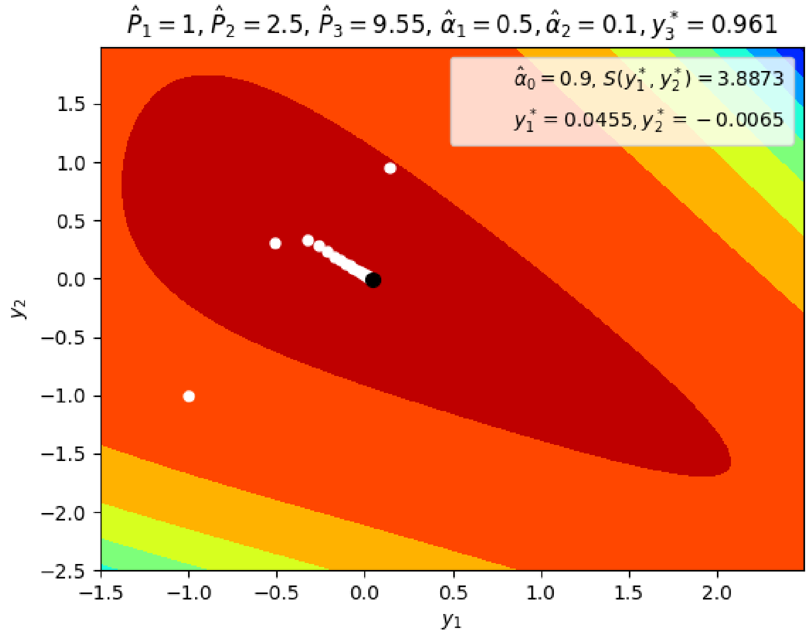


Figure 6.3: An example of $T_4 = (+, -, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 2.5$ and $\hat{P}_3 = 9.55$ in a Bull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

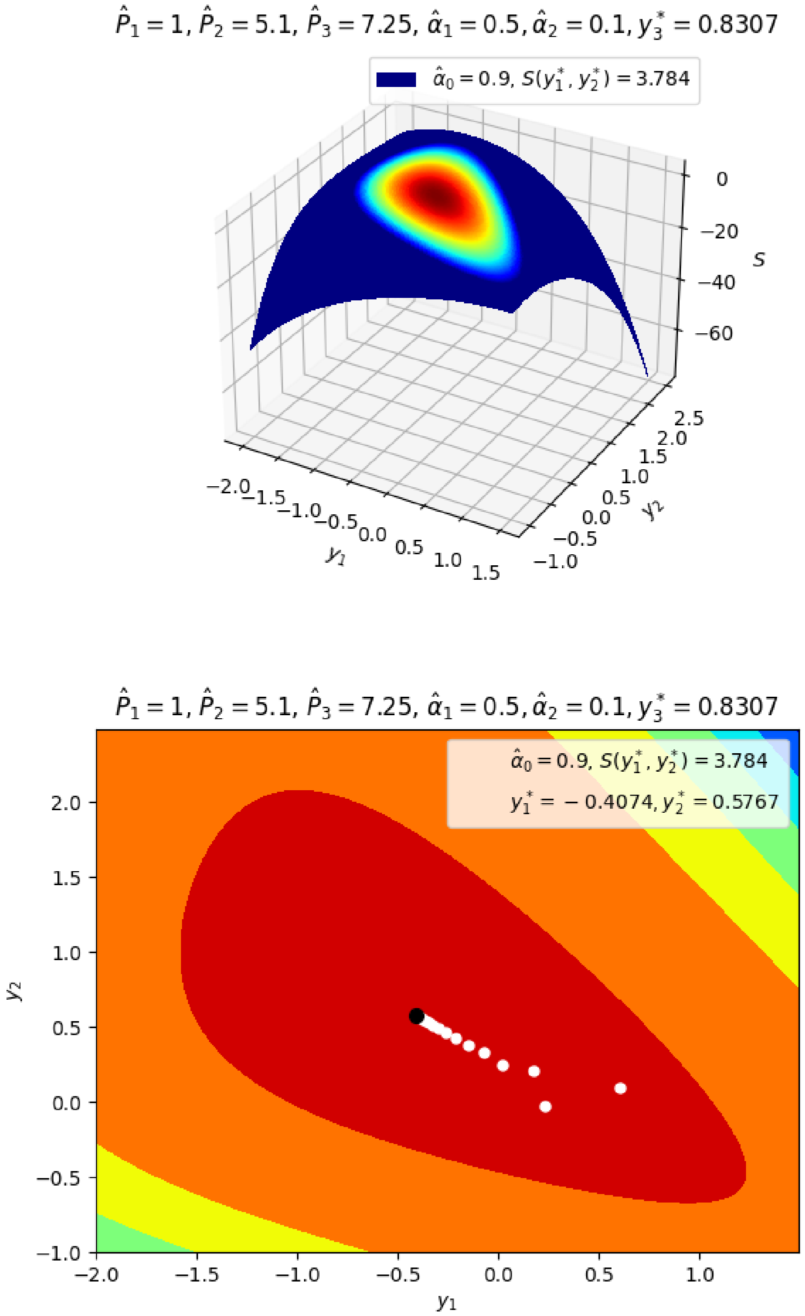
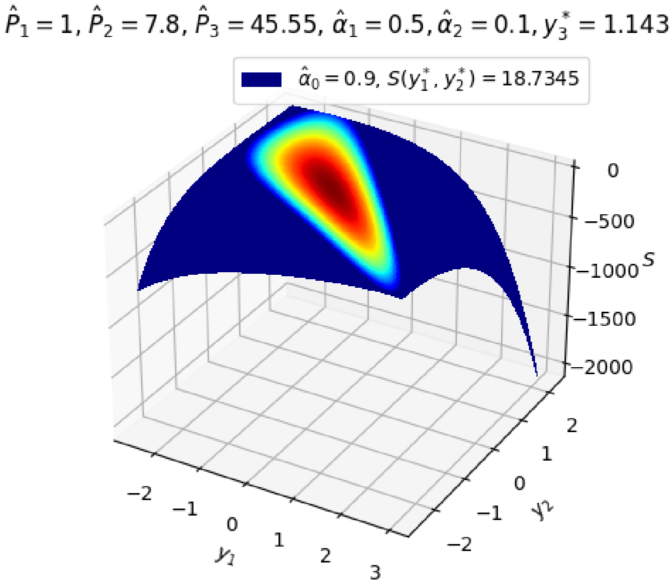


Figure 6.4: An example of $T_2 = (-, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 5.1$ and $\hat{P}_3 = 7.25$ in a Bull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.



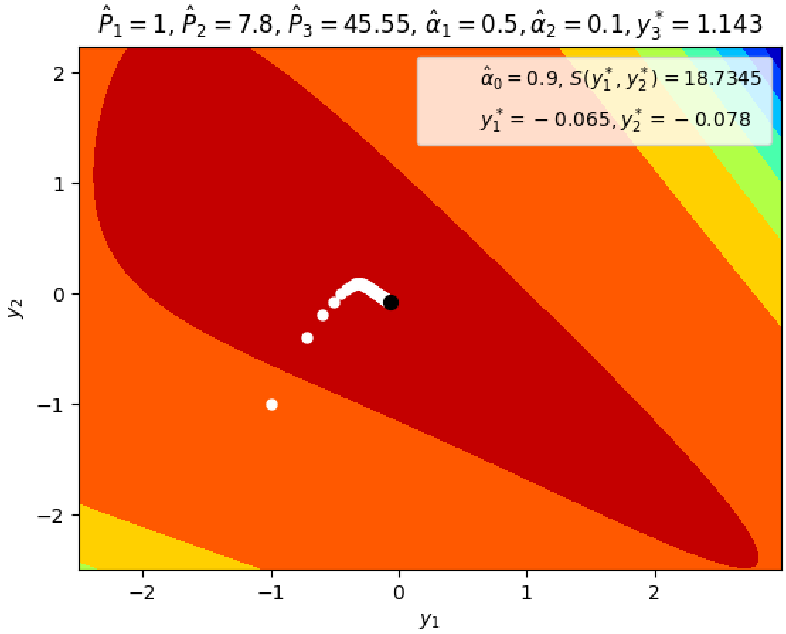
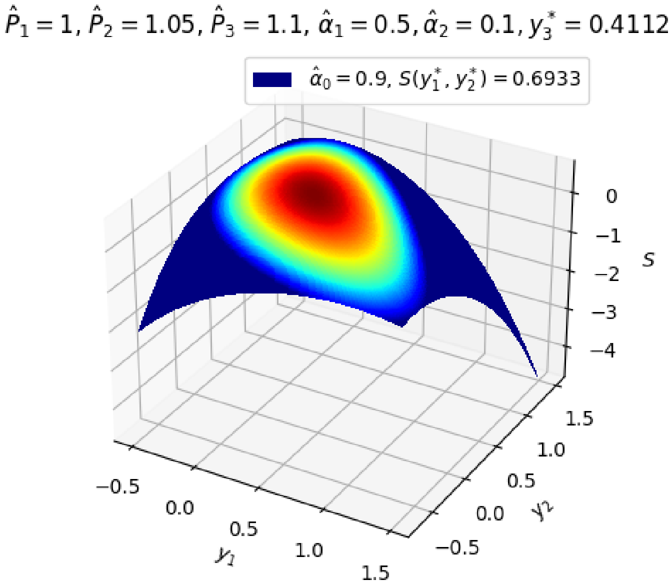


Figure 6.5: An example of $PU_1 = (-, -, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 7.8$ and $\hat{P}_3 = 45.55$ in a Bull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.



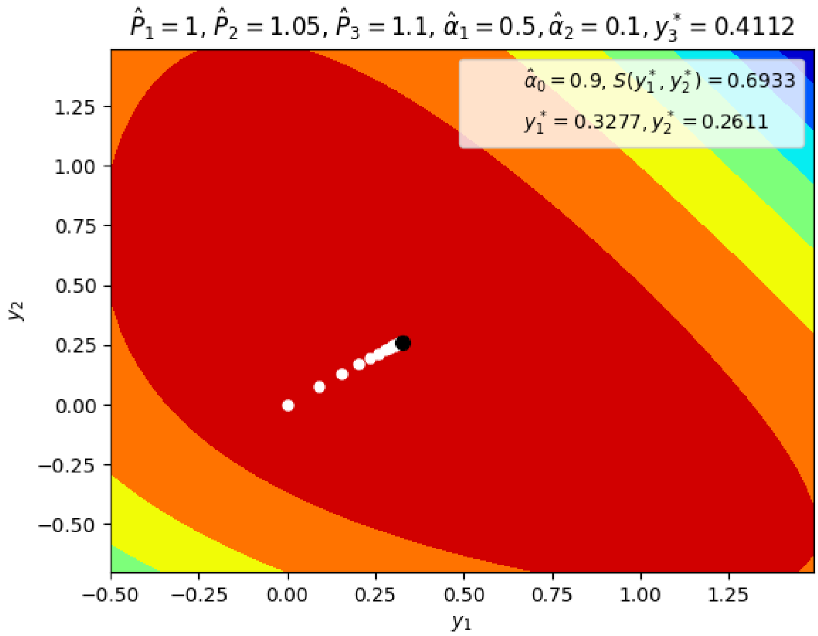


Figure 6.6: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 1.05$ and $\hat{P}_3 = 1.1$ in a Bull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

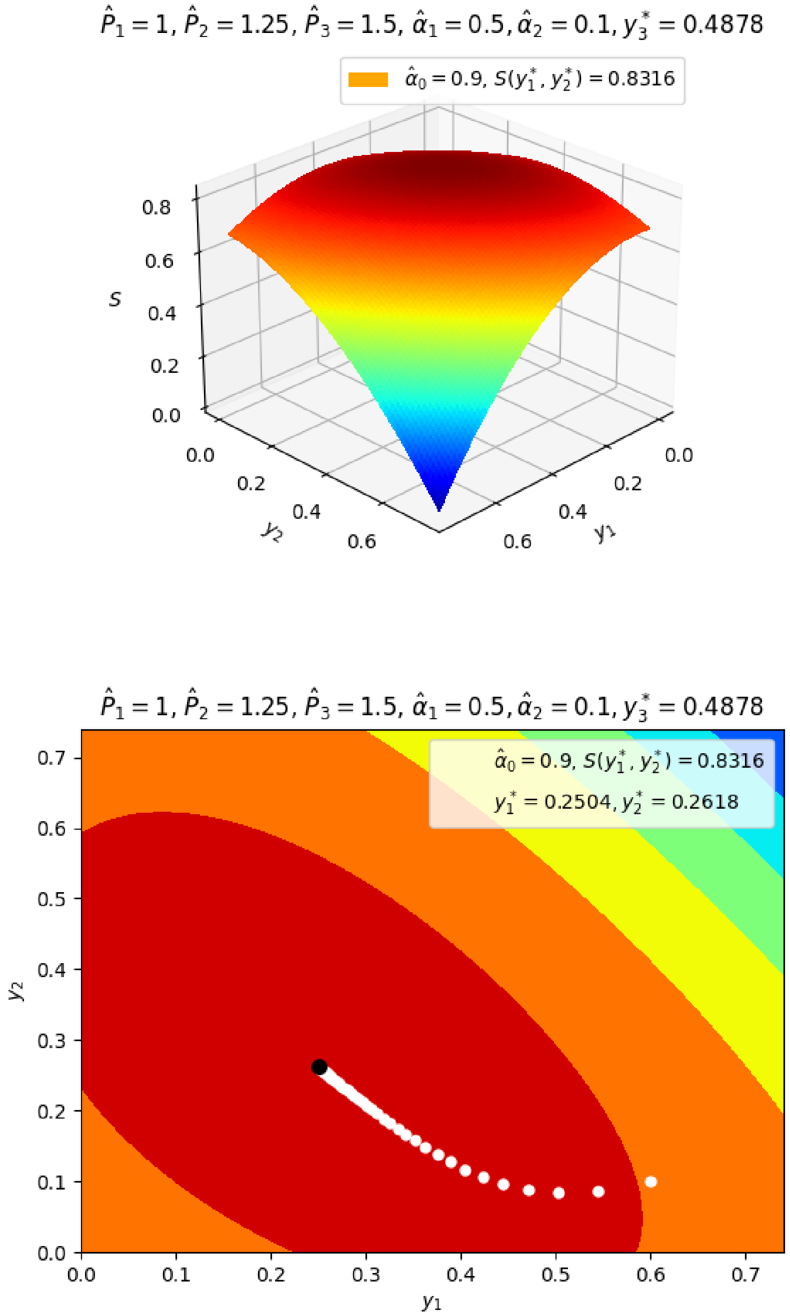


Figure 6.7: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 1.25$ and $\hat{P}_3 = 1.5$ in a Bull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

non-feasible strategy spaces: $S_1 = (0,+,0)$, $S_2 = (+,+,0)$, $S_4 = (0,+,+)$, $S_6 = (0,0,+)$, $T_1 = (-,+,0)$ and $T_3 = (-,0,+)$.

So, why is it that $T_1 = (-,+,0)$ is not optimal at all, not even feasible in the model? It seems that the bounce-back function ψ in step t_2 does not have enough bounce in this case. Hence, the price drop of flooding the market at t_2 will outperform the profit from the extra stocks bought in step t_1. Again, it may however be argued that an insignificant portfolio could benefit from the T_1 buy and sell strategy space as the bounce-back function ψ will be negligible.

Finally, our model shows that in the moderate market (10% increase) displayed in Figure 6.8 the investor manages to salvage only 70.24% of the portfolio value with this S_5 three-step strategy. Even in the extreme market (50% increase) shown in Figure 6.9 the investor only manages to escape with 88.43% of the portfolio value, incurring a 29.76% liquidity cost in the moderate market and a liquidity cost of 11.57% in the extreme market due to the size of the portfolio. Consequently, a bigger market increase only helps reduce the liquidity cost by 18.19 percentage points, while the market moved 40 percentage points. Hence, the bigger the market increase the bigger the effect from the bounce-back function ψ.

6.3.2.3 BullBear: $\hat{P}_1 < \hat{P}_2 \cup \hat{P}_2 > \hat{P}_3$

In this case we know that $\hat{P}_1 < \hat{P}_2$ and $\hat{P}_2 > \hat{P}_3$, however we do not know whether $\hat{P}_3$ is smaller, greater or equal to $\hat{P}_1$. Consequently, we have to evaluate several different situations in order to say something sensible about a BullBear market.

In Section 6.2.2.3 on page 72 we found the naive optimal strategy space to be the short selling strategy space $SH_1 = (-,+,-)$, where the investor buys more stock at t_1, sells stock at t_2, for then to buy back the borrowed shares at t_3.

The numerical analysis shows that the naive optimal strategy space was not found. Numerous iterations within a scope of conceivable price changes found several incidents where no representatives from the naive feasible buy and sell strategy spaces were the actual optimal strategy spaces, such as the example $D_1 = (+,+,-)$, $S_5 = (+,+,+)$ and $T_2 = (-,+,+)$ shown in Figures 6.12, 6.14 and 6.16.

Interestingly, $D_1 = (+,+,-)$, which is the space where the investor sells stock at t_1 and t_2 for then to buy back stock in t_3, represents both a short selling strategy space, as well as a dumping strategy space. As discussed previously, the dumping strategy space is a special case in the set of short selling strategy spaces. It is exciting that this strategy space surfaced as an actual optimal strategy space, as it may seem counter-intuitive that dumping stock into the market, which in itself will drive down the price, actually gives the largest sales value.

Furthermore, since $S_2 = (+,+,0)$ is the boundary case of $S_5 = (+,+,+)$ and $D_1 = (+,+,-)$, we know by the Intermediate Value Theorem 2.2.11 that S_1 can give an actual optimal strategy space, which is what we naively hypothesised as a feasible strategy space. Figure 6.13 shows an approximation of $S_1 = (+,+,0)$.

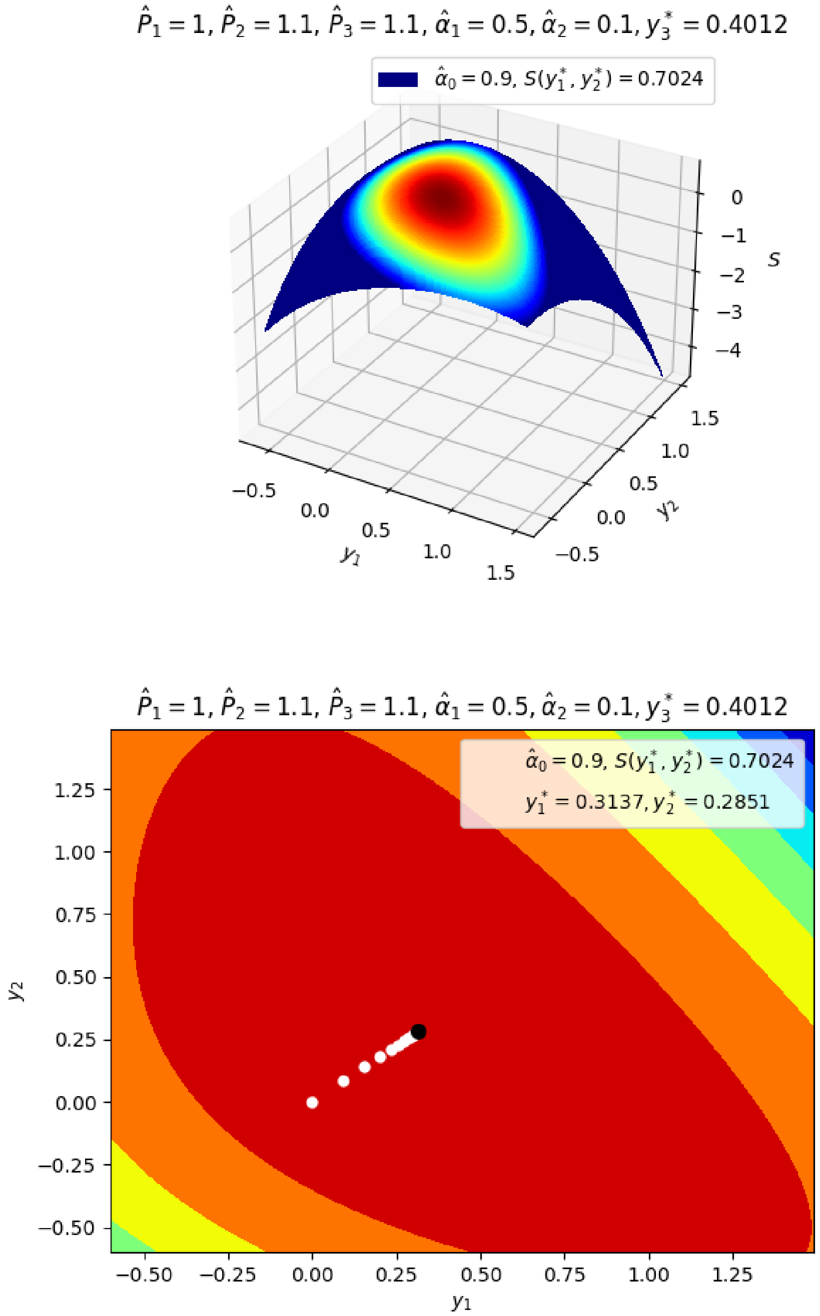


Figure 6.8: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = \hat{P}_3 = 1.1$ in a BullFlat market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

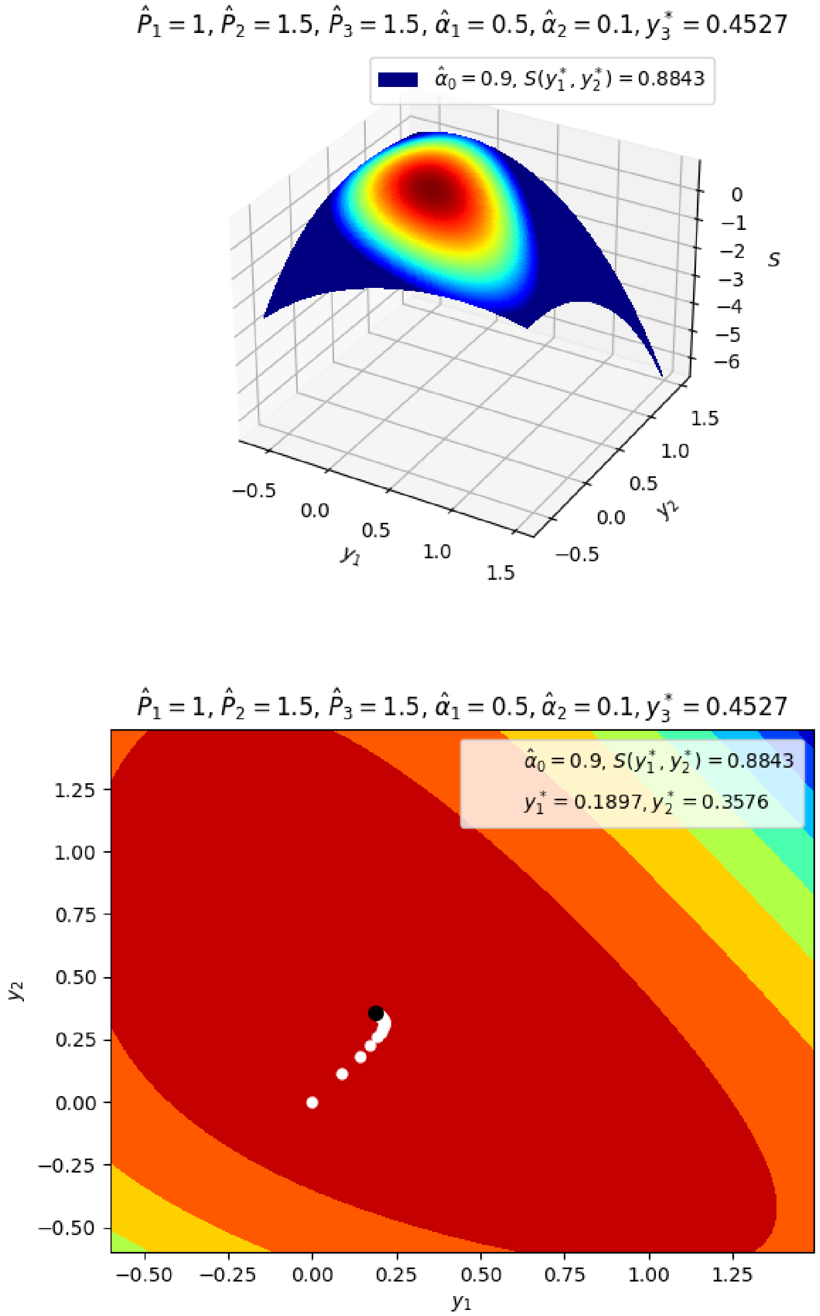
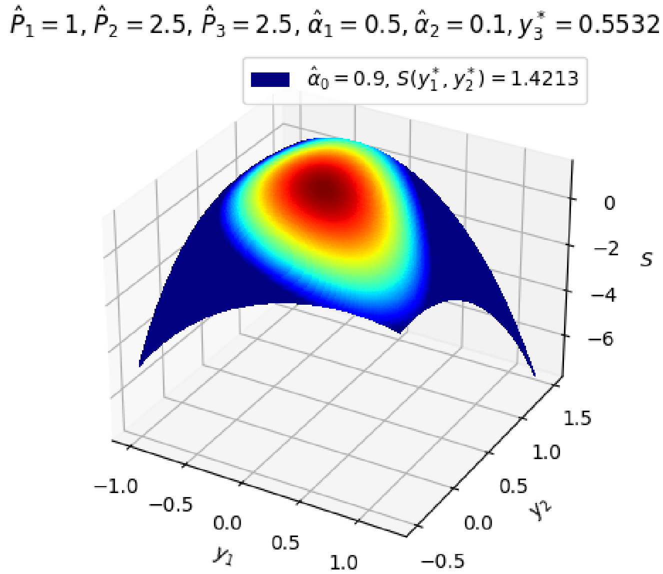


Figure 6.9: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = \hat{P}_3 = 1.5$ in a BullFlat market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.



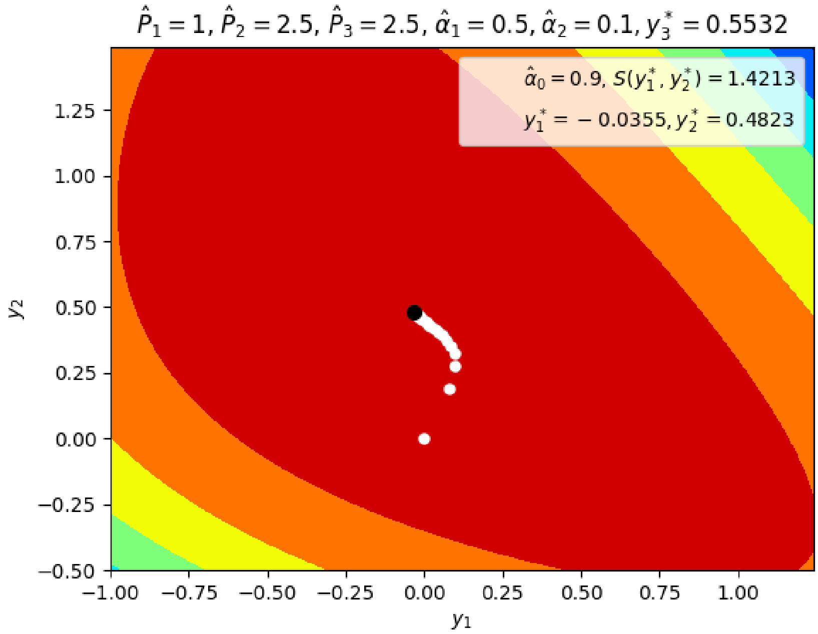
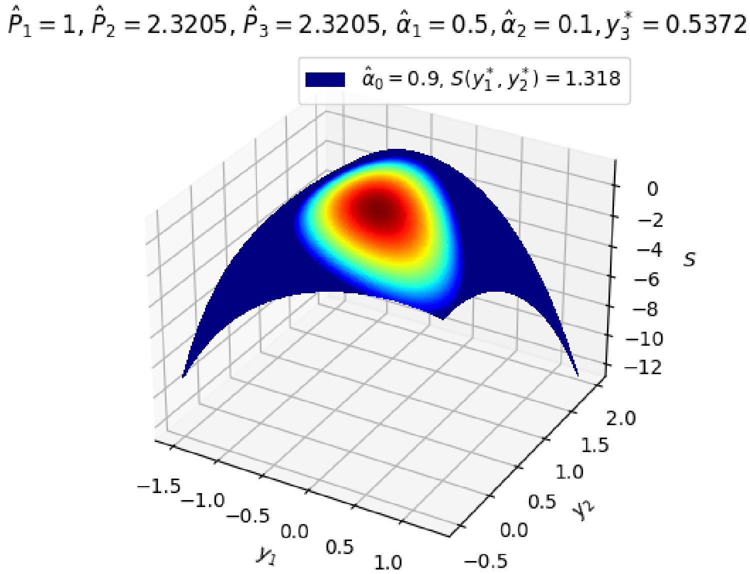


Figure 6.10: An example of $T_2 = (-, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = \hat{P}_3 = 2.5$ in a BullFlat market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.



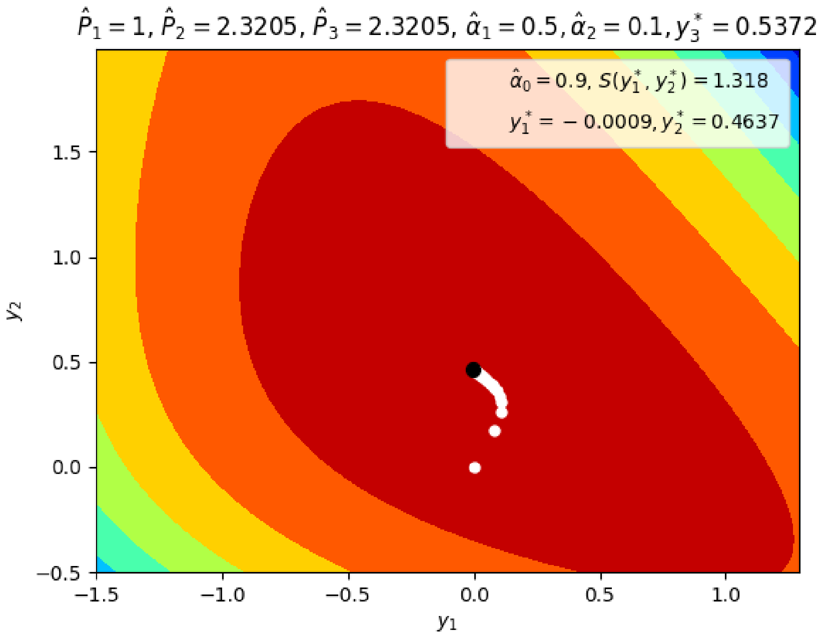


Figure 6.11: An example of $S_4 = (0, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = \hat{P}_3 = 2.3205$ in a BullFlat market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

In addition, Figure 6.15 can be viewed as an approximation of $S_4 = (0, +, +)$, which is also the boundary case of $T_2 = (-, +, +)$ and $S_5 = (+, +, +)$. Interestingly enough, neither of these strategy spaces were included in the naive feasible strategy space.

Our model shows some interesting profits. The sales value is 128.28% of the portfolio value when the market moves 200% on the second step and falls back to its original price point. Also, in the even more extreme market where the market moves 900% for then to fall back to a 300% increase, the sales value is 545.12% of the portfolio value, incurring a negative liquidity cost in both cases, as displayed in Figures 6.15 and 6.16.

In a more moderate market these numbers are quite different. As the market moves 10% and falls back to its original price point, the investor only manages to salvage 67.97% of the portfolio with the optimal trading strategy $S_5 = (+, +, +)$. If the market falls further, for instance to 10% of original value the investor manages to save 63.09% of the portfolio value with the dumping strategy space $D_1 = (+, +, -)$, here shown in Figures 6.12 and 6.14.

So, why were only one of the naive feasible strategy spaces a strategy space in the numerical case? Again, it seems reasonable to suggest the behaviour of the bounce-back function ψ. It has already shown us that it is difficult to embed its behaviour in the naive analysis.

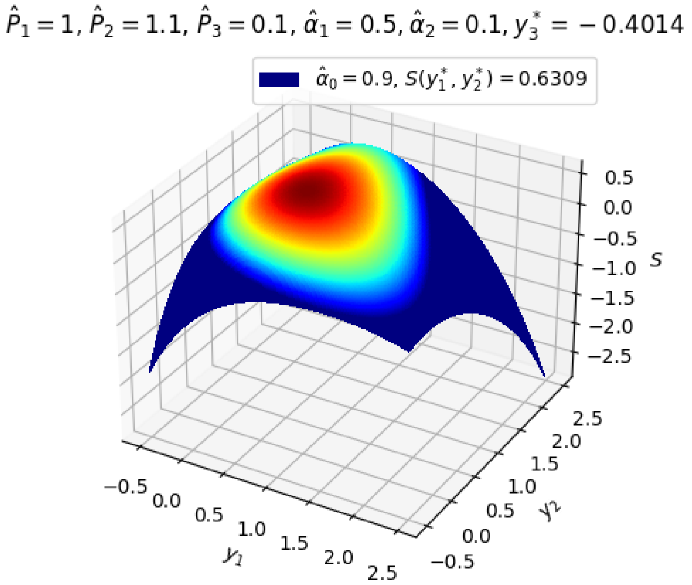
6.3.2.4 FlatBull: $\hat{P}_1 = \hat{P}_2 < \hat{P}_3$

In Section 6.2.2.4 on page 72 we found the naive optimal strategy space to be the buy and sell strategy space $T_5 = (0, -, +)$, where the investor does nothing in step t_1, buys more shares in t_2 and sells all in t_3.

The numerical analysis shows that the naive optimal strategy space, also in this case, is not the actual optimal strategy space, even though it is an edge case of a naive optimal strategy space, namely $T_4 = (+, -, +)$. Iterations within the scope revealed that two of the naive feasible strategy spaces gives an actual optimum, namely $S_5 = (+, +, +)$ and $T_4 = (+, -, +)$. In addition, we found an approximation to $S_7 = (+, 0, +)$ which also gives an optimum. These are shown in Figures 6.17 to 6.20.

It is interesting to see that the naive optimal strategy space $T_5 = (0, -, +)$ is thus degraded to a non-feasible strategy space, as in the BullFlat case. Furthermore, several of the naive feasible strategy spaces in the FlatBull scenario are after the numerical analysis degraded to non-feasible strategy spaces. Of the naive feasible strategy spaces, the following are now degraded to non-feasible: $S_4 = (0, +, +)$, $S_6 = (0, 0, +)$ and $T_5 = (0, -, +)$.

Then, why is it that $T_5 = (0, -, +)$ is not an actual optimal strategy space, not even feasible in this model? It looks like the bounce-back function ψ in step t_3 does not have enough bounce. Hence, the price drop of oversupplying the market at t_3 will outperform the profit from the extra stocks bought in step t_2. Again, it may however be argued that an insignificant portfolio could benefit from the T_5 buy and sell strategy space as the bounce-back function ψ will be negligible.



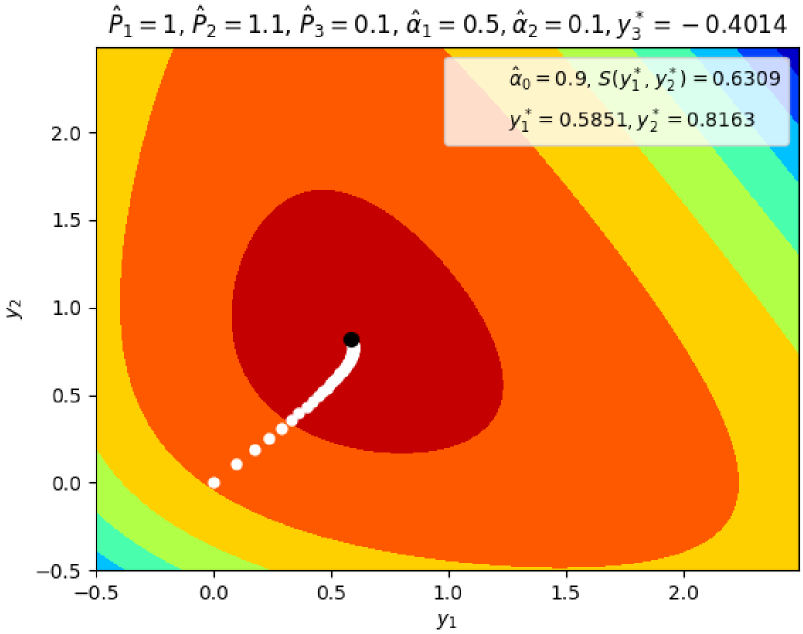
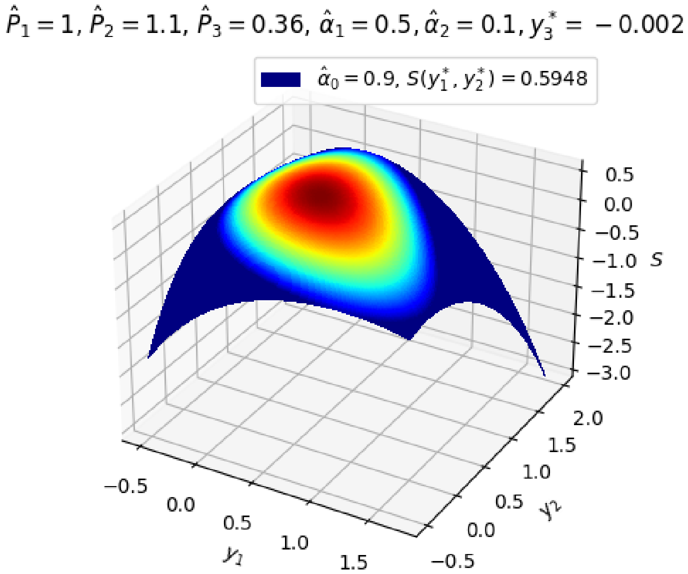


Figure 6.12: An example of $D_1 = (+, +, -)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 1.1$ and $\hat{P}_3 = 0.1$ in a BullBear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.



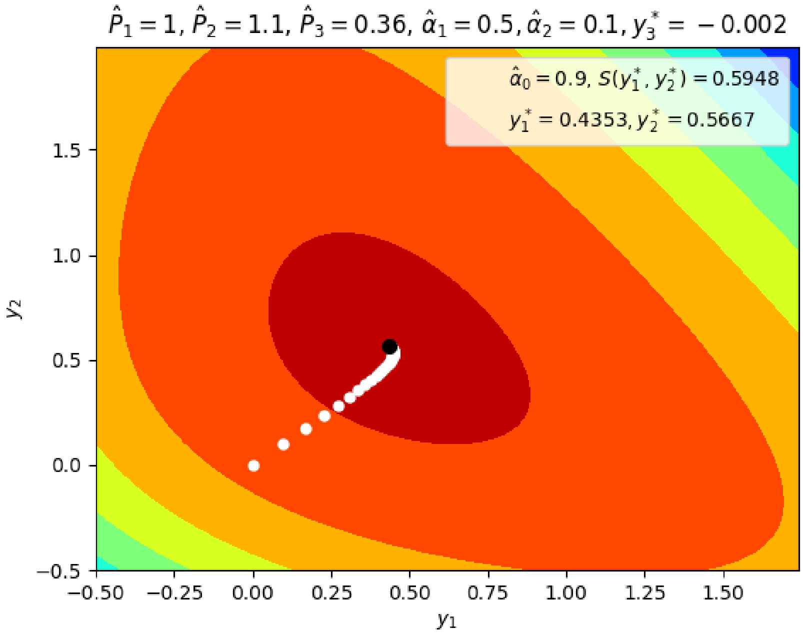


Figure 6.13: An example of $S_2 = (+, +, 0)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 1.1$ and $\hat{P}_3 = 0.36$ in a BullBear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

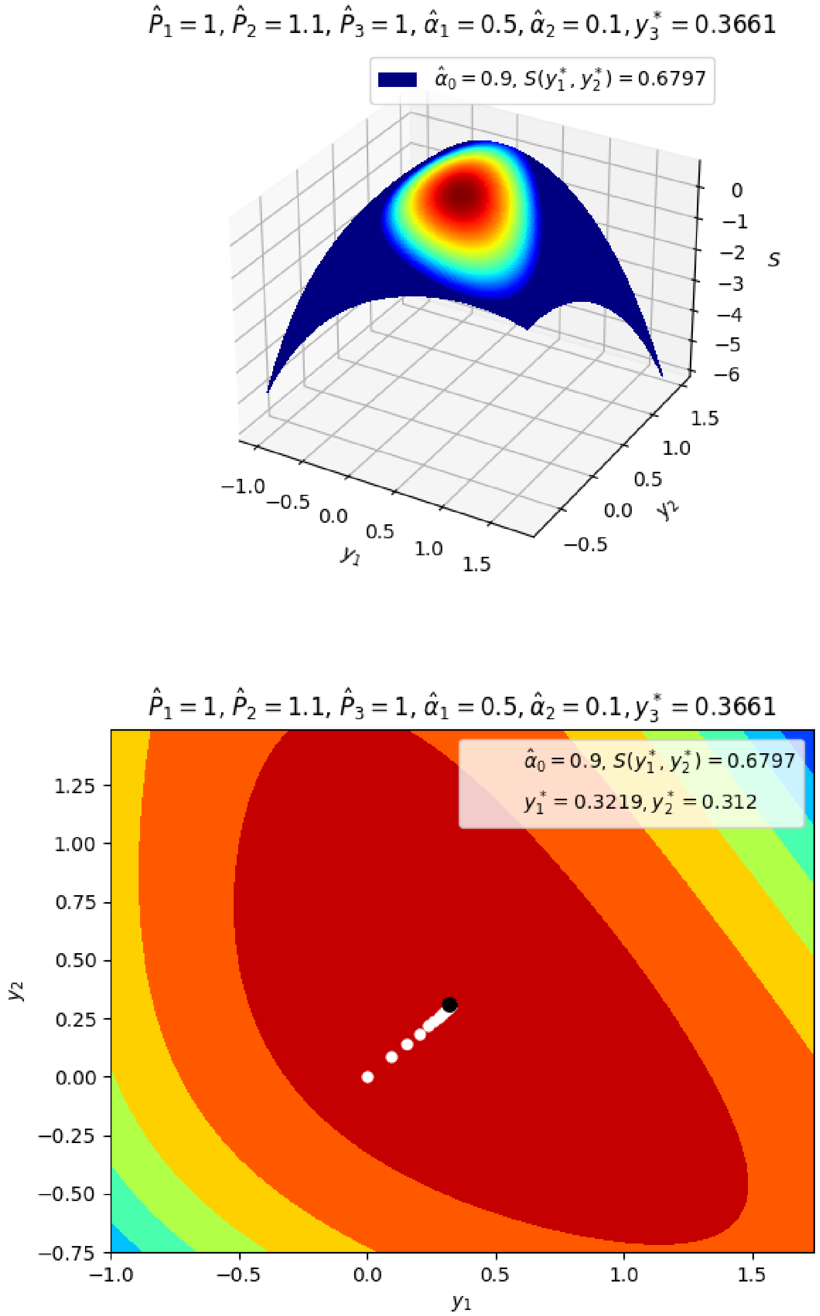


Figure 6.14: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 1.1$ and $\hat{P}_3 = 1$ in a BullBear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

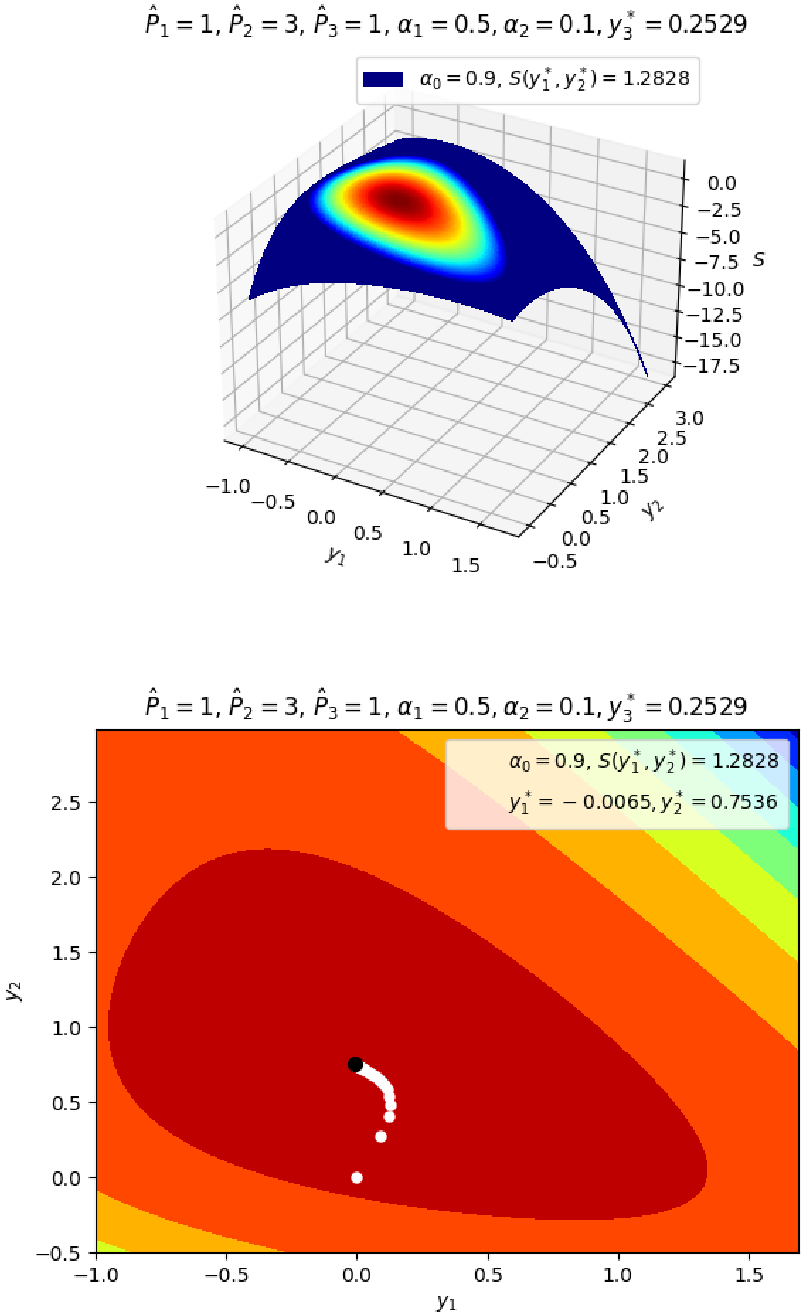
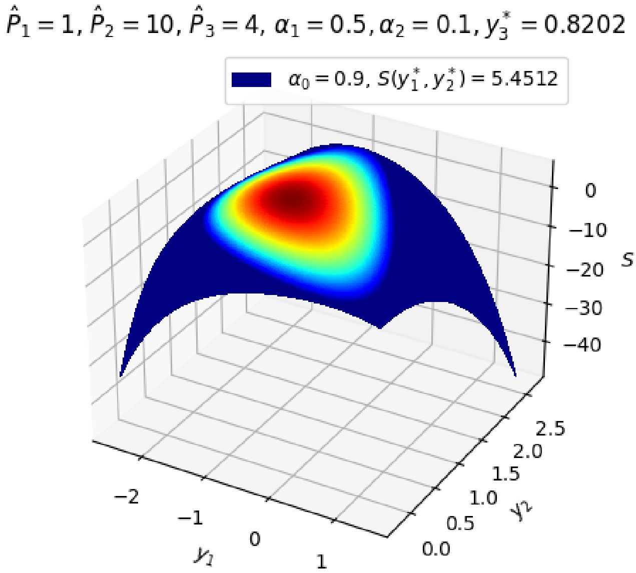


Figure 6.15: An example of $S_4 = (0, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 3$ and $\hat{P}_3 = 1$ in a BullBear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.



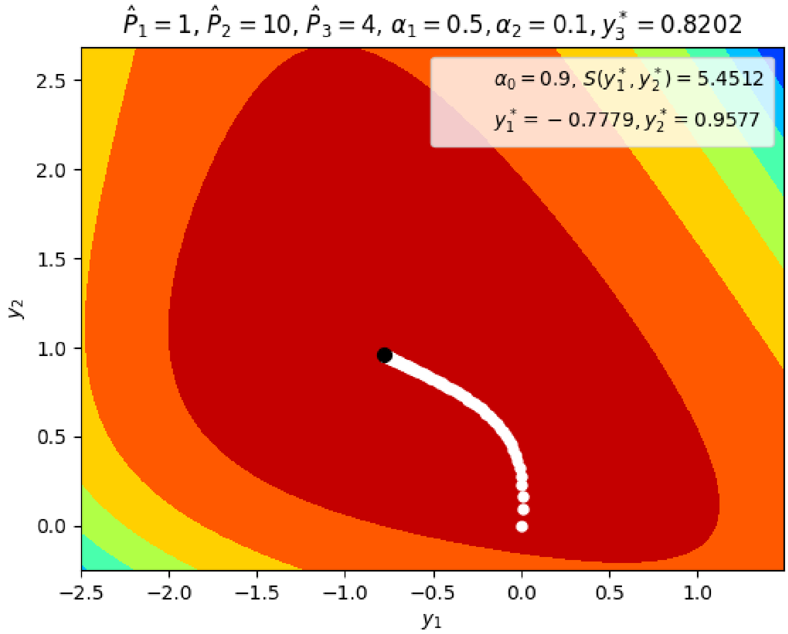


Figure 6.16: An example of $T_2 = (-, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 10$ and $\hat{P}_3 = 4$ in a BullBear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

The investigation shows that if $\hat{P}_3 < 2.2$, then $S_5 = (+,+,+)$ is the actual optimal strategy space. If $\hat{P}_3 > 2.25$, then $T_4 = (+,-,+)$ is the optimal strategy space. Hence, at some point on the interval $2.2 < \hat{P}_3 < 2.25$ the optimal strategy space is given by $S_7 = (+,0,+)$. Figure 6.20 shows an approximation of $S_7 = (+,0,+)$ with a $\hat{P}_3 = 2.23$. Based on numerical iterations of $\hat{P}_1 = \hat{P}_2$, we see that the first and third coordinate of our 3-tuple (y_1, y_2, y_3) seem to stay positive independent of whether we make $\hat{P}_3$ go towards 1 – as in the case of the Flat market (Section 6.3.2.5) – or some big positive number even outside of a possible scope.

Finally, our model shows that in the moderate market (10% increase) displayed in Figure 6.17, the investor manages to salvage only 68.49% of the portfolio value with this three-step strategy, incurring a 31.51% liquidity cost. Even in the extreme market (50% increase) shown in Figure 6.18, the investor only manages to escape with 79.50% of the portfolio value, yielding a liquidity cost of 20.50%. Subsequently, in a FlatBull market, a bigger market increase only helps reduce the liquidity cost by 11.01 percentage points, while the market moves 40 percentage points. Hence, the bigger the market increase the bigger the effect from the bounce-back function ψ.

6.3.2.5 Flat: $\hat{P}_1 = \hat{P}_2 = \hat{P}_3$

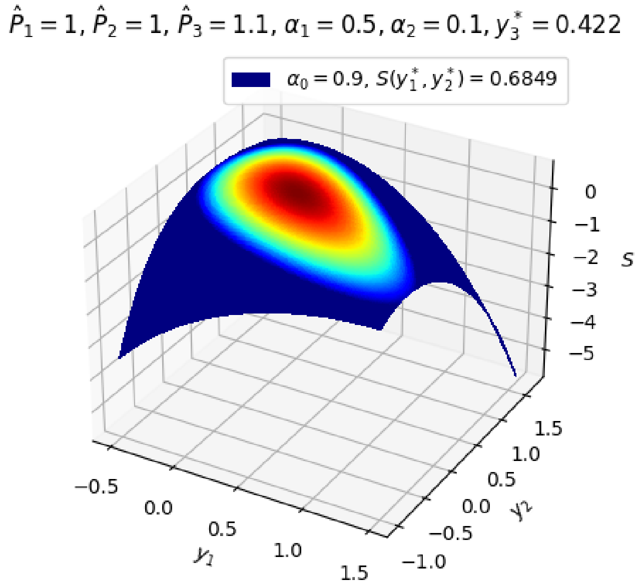
In Section 6.2.2.5 on page 73 we found the naive optimal strategy space to be the sell strategy space $S_3 = (+,0,0)$, where the investor sells the complete market moving portfolio at t_1, and does nothing at t_2 and t_3.

This numerical analysis shows that our naive optimal strategy space is not the actual optimal strategy space in any case when it comes to market moving portfolios. As the market is Flat we have $\hat{P}_1 = \hat{P}_2 = \hat{P}_3 = 1$, so only one run is needed. In this case the sell strategy space $S_5 = (+,+,+)$ is again the actual optimal strategy space, as seen in Figure 6.21.

Our model shows that in the Flat market (0% increase) displayed in Figure 6.21, the investor manages to salvage only 66.09% of the portfolio value with the optimal strategy. The optimum incurs a 33.91% liquidity cost on the portfolio.

Also, in a Flat market the feasible strategy spaces were defined as: $S_1 = (0,+,0)$, $S_2 = (+,+,0)$, $S_3 = (+,0,0)$, $S_4 = (0,+,+)$, $S_5 = (+,+,+)$, $S_6 = (0,0,+)$ and $S_7 = (+,0,+)$. Interestingly, according to our model under the restriction of the $\hat{\alpha}_n$s, all naive feasible strategy spaces except for $S_5 = (+,+,+)$ are degraded to non-feasible. It is interesting to see that all degraded feasible strategy spaces are boundary cases for other strategy space. Nonetheless, the naive optimal strategy space is now non-feasible as well.

Naively, we did not determine which feasible strategy space gave the largest sales value. Actually, we hypothesised that they would be exactly the same. The reason for choosing $S_3 = (+,0,0)$ was that it would take the capital out of the market in order to invest in other opportunities and have a chance for increased profit. On that note, the above argument may be valid in the other feasible strategy spaces, however the model does not take reinvestment into consideration.



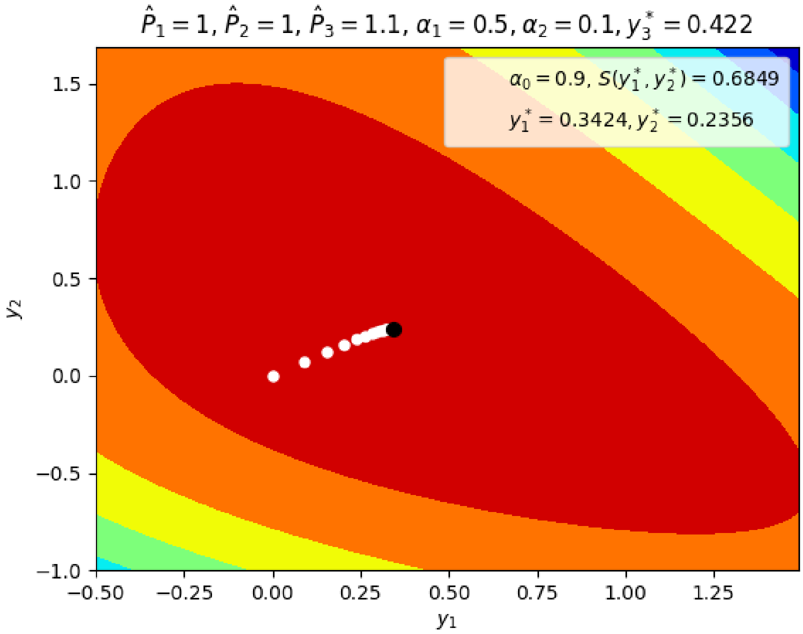
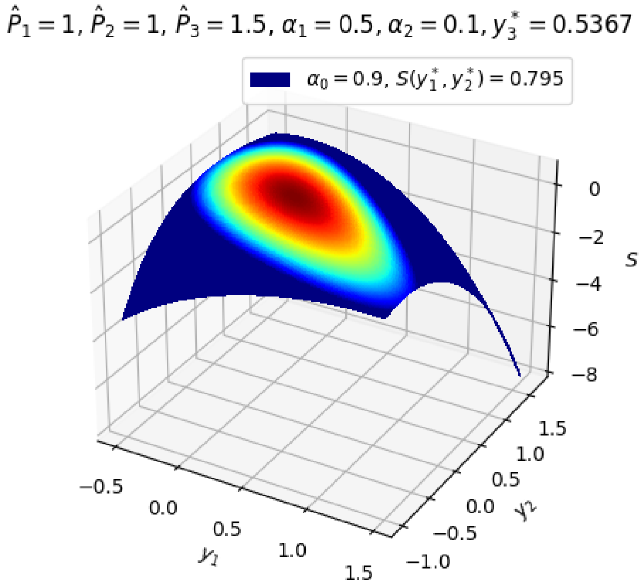


Figure 6.17: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = \hat{P}_2 = 1$ and $\hat{P}_3 = 1.1$ in a FlatBull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.



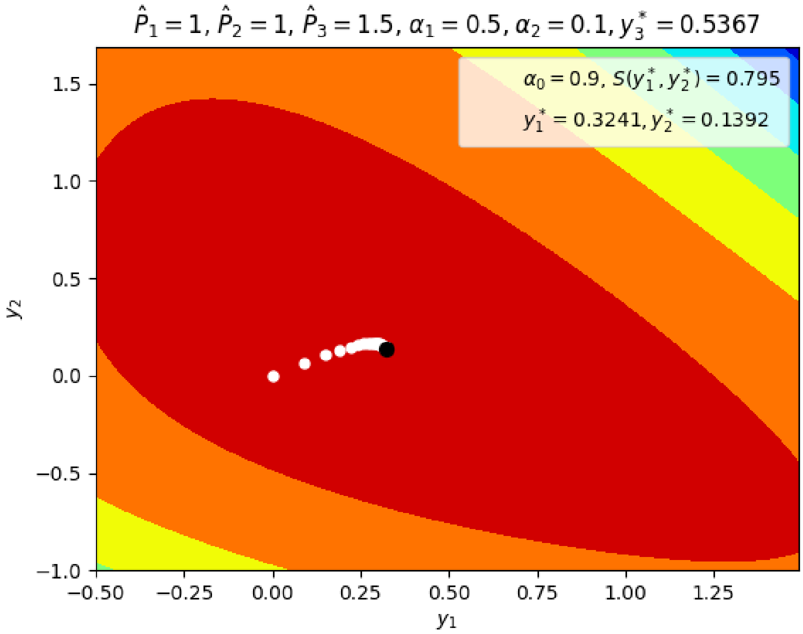


Figure 6.18: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = \hat{P}_2 = 1$ and $\hat{P}_3 = 1.5$ in a FlatBull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

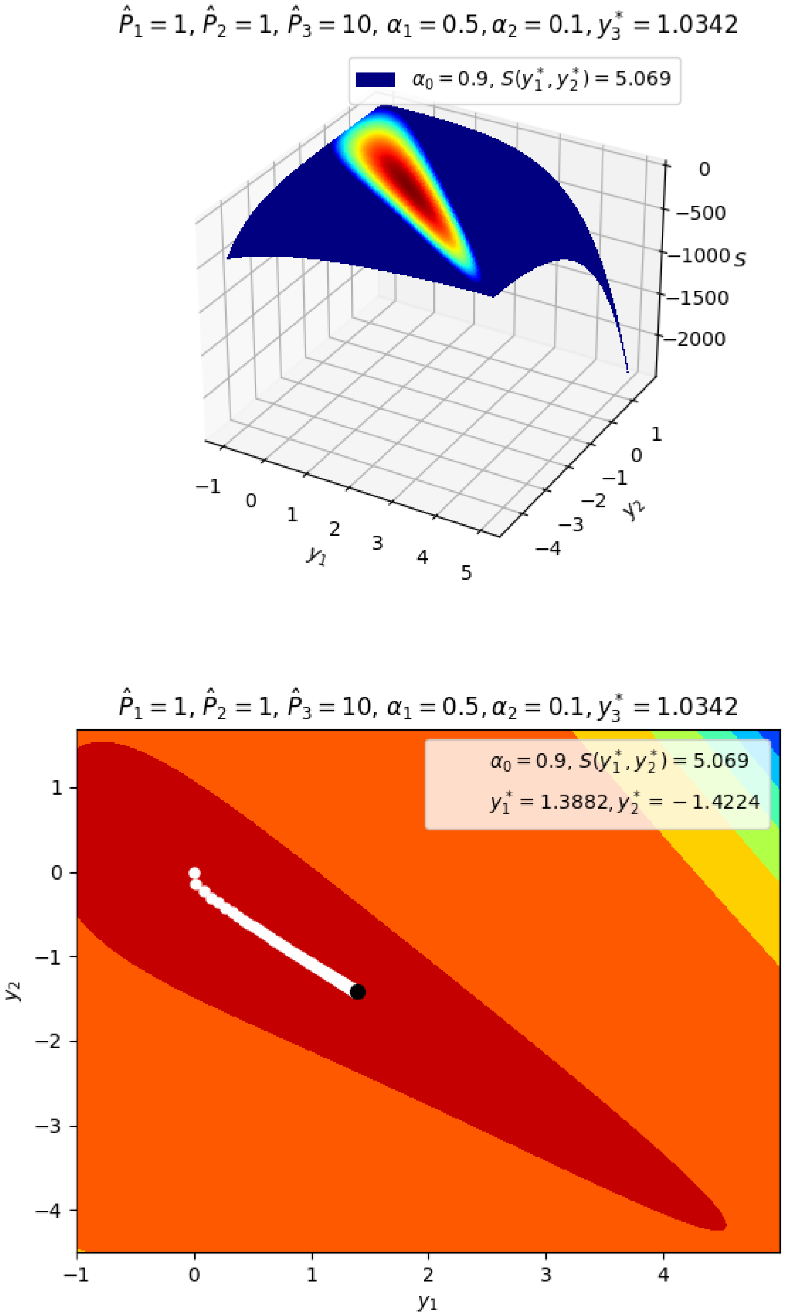


Figure 6.19: An example of $T_4 = (+, -, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = \hat{P}_2 = 1$ and $\hat{P}_3 = 10$ in a FlatBull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

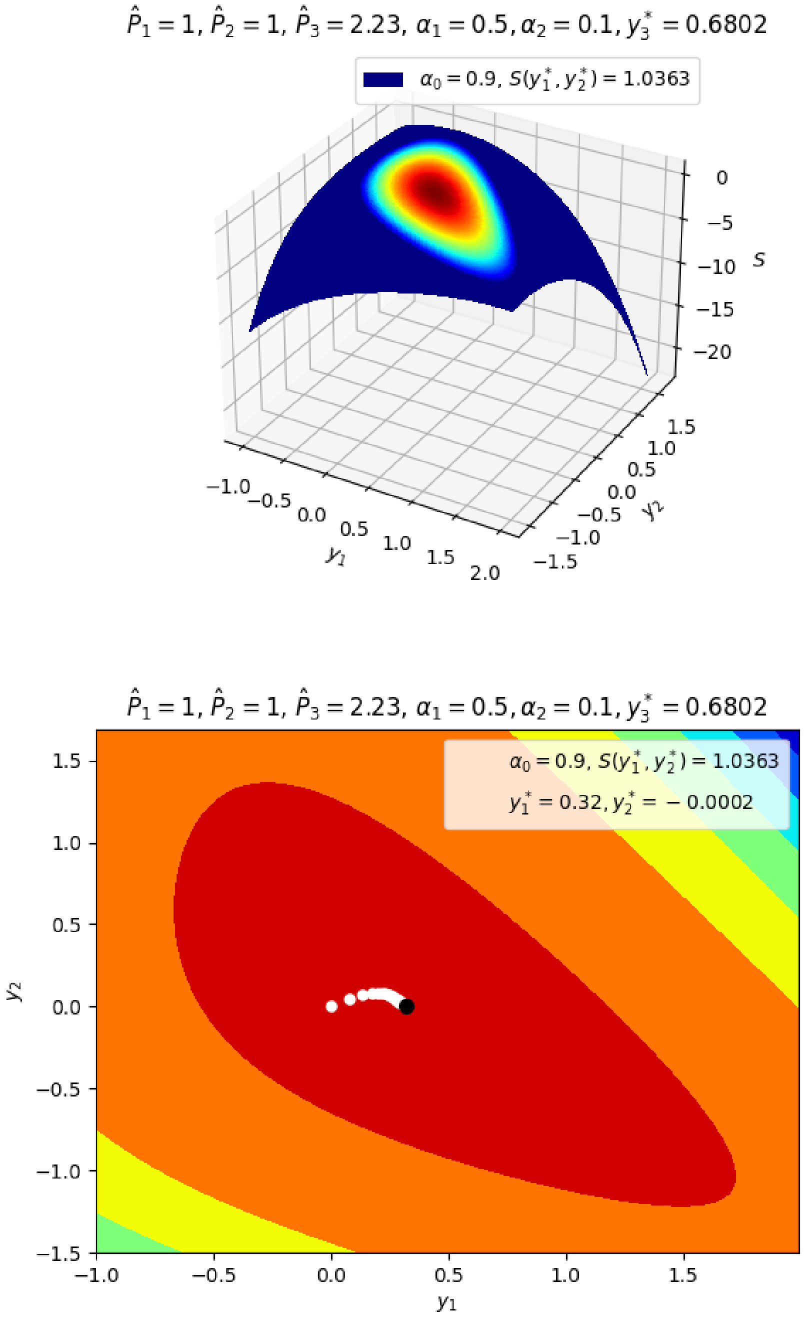


Figure 6.20: An example of $S_7 = (+, 0, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = \hat{P}_2 = 1$ and $\hat{P}_3 = 2.23$ in a FlatBull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

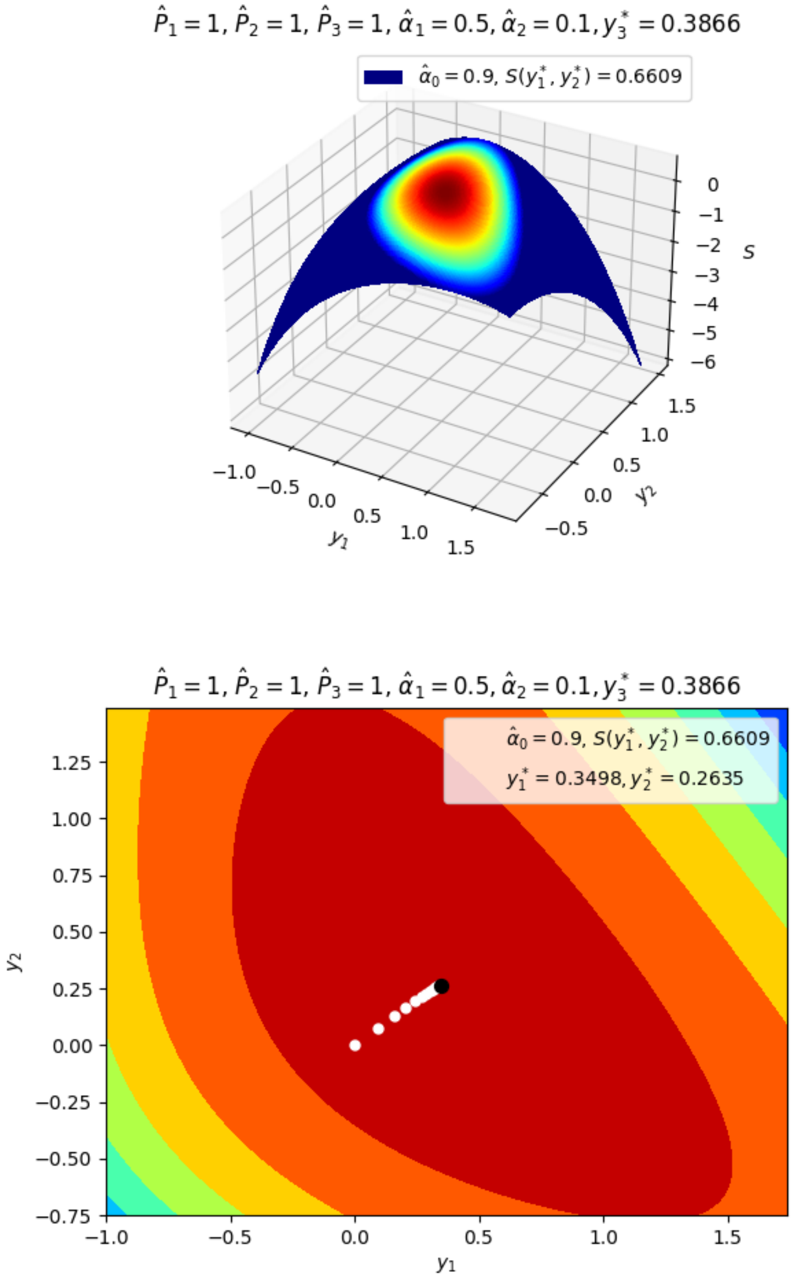


Figure 6.21: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface in a Flat market, $\hat{P}_1 = \hat{P}_2 = \hat{P}_3 = 1$. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

6.3.2.6 FlatBear: $\hat{P}_1 = \hat{P}_2 > \hat{P}_3$

In Section 6.2.2.6 on page 73 we found the naive optimal strategy space to be the short selling strategy space $SH_2 = (0, +, -)$, where the investor does nothing in step t_1, sells original and borrowed shares in t_2 and buys back the borrowed shares in t_3.

The numerical analysis shows that the naive optimal strategy space, also in this case, is not the actual optimal strategy space, even though it is an edge case of an actual optimal strategy space, namely $D_1 = (+, +, -)$. Further iterations on the scope showed that one of the naive feasible strategy spaces gave an actual optimal strategy space, $D_1 = (+, +, -)$. In addition, we found an approximation to $S_2 = (+, +, 0)$ which also gave an actual optimal strategy space. Again, $S_5 = (+, +, +)$ came up as a dark horse and gave another actual optimal strategy space on the interval $0.348 < \hat{P}_3 < 1$. These are shown in Figures 6.22 to 6.25.

The analysis also shows that if $\hat{P}_3 < 0.347$ then $D_1 = (+, +, -)$ is the actual optimal strategy space. If $\hat{P}_3 > 0.348$ then $S_5 = (+, +, +)$ is the actual optimal strategy space. Hence, at some point on the interval $0.347 < \hat{P}_3 < 0.348$ the actual optimal strategy space is given by $S_2 = (+, +, 0)$. Based on numerical iterations of $\hat{P}_1 = \hat{P}_2$, we see that the first and second coordinate of the 3-tuple (y_1, y_2, y_3) seem to stay positive independent of whether we make $\hat{P}_3$ go towards 0 or 1 in a FlatBear market.

Again, we see that the naive optimal strategy space $SH_2 = (0, +, -)$ is degraded to a non-feasible one. Furthermore, several of the naive feasible strategy spaces in the FlatBear scenario are after the numerical analysis degraded to non-feasible, such as: $SH_2 = (0, +, -)$, $SH_3 = (+, 0, -)$, $S_1 = (0, +, 0)$ and $S_3 = (+, 0, 0)$.

Now, why is it that $SH_2 = (0, +, -)$ is not an optimal strategy space, not even feasible in this model? It looks like the bounce-back function ψ has taken us for a spin yet again. Hence, the price drop of oversupplying in the market at t_2 will outperform the profit from the extra stocks bought in step t_3, even with the help of the shorted stock. Also, it may again be argued that an insignificant portfolio size could benefit from the SH_2 trading strategy as the bounce-back function ψ will be negligible.

Our model shows that in an extreme market (90% decrease) displayed in Figure 6.22, the investor manages to salvage only 60.20% of the portfolio value with this three-step strategy, incurring a 39.80% liquidity cost. Even in the less extreme market (50% decrease) shown in Figure 6.24, the investor only manages to escape with 57.54% of the portfolio value, yielding a liquidity cost of 42.46%. Hence, an additional 40 percentage point drop in the market yields only a marginally increased sales value (2.66 percentage points). Consequently, the bounce-back function ψ is stronger the bigger the market drop.

6.3.2.7 BearBull: $\hat{P}_1 > \hat{P}_2 \cup \hat{P}_2 < \hat{P}_3$

In the BearBull case we know that $\hat{P}_1 > \hat{P}_2$ and $\hat{P}_2 < \hat{P}_3$, however this is also a case where we do not know whether $\hat{P}_3$ is smaller, greater or equal to $\hat{P}_1$. Hence, we have to evaluate each situation in order to analyse the BearBull market.

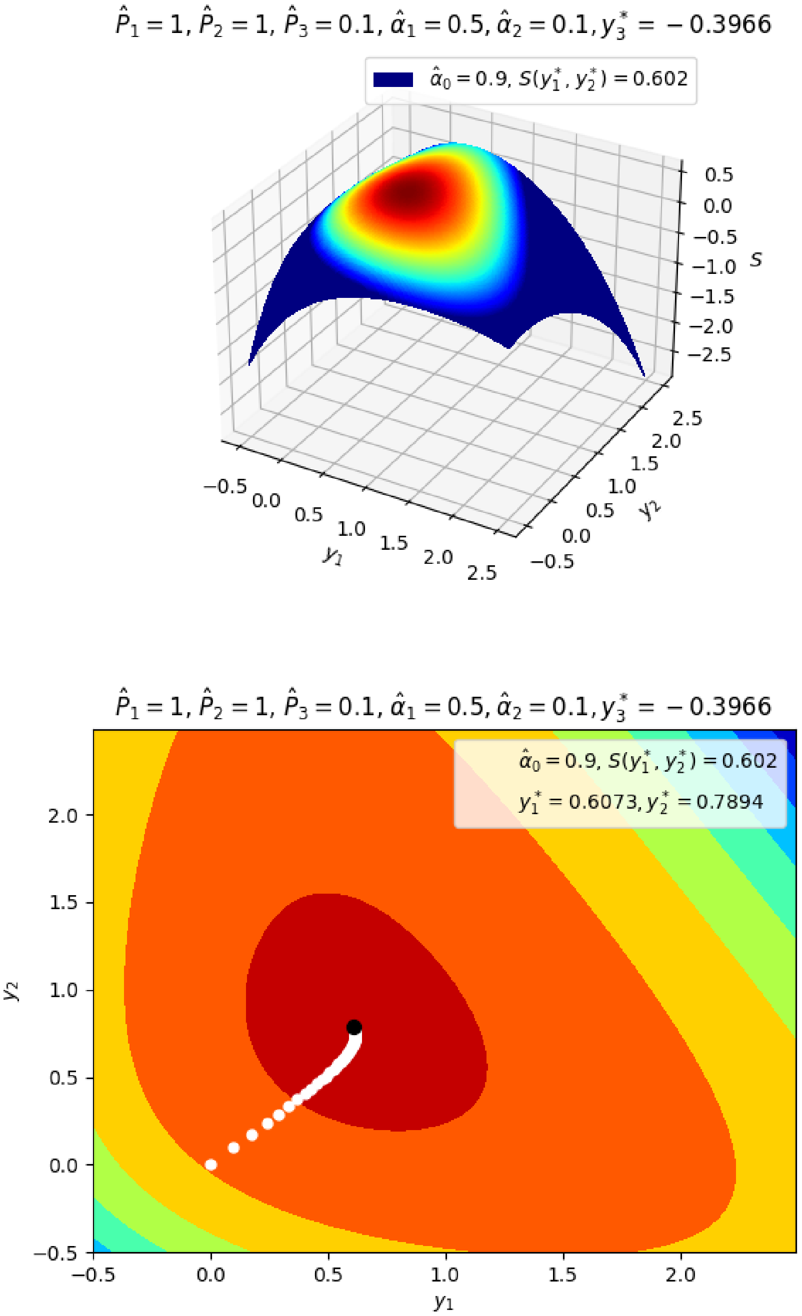


Figure 6.22: An example of $D_1 = (+, +, -)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = \hat{P}_2 = 1$ and $\hat{P}_3 = 0.1$ in a FlatBear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

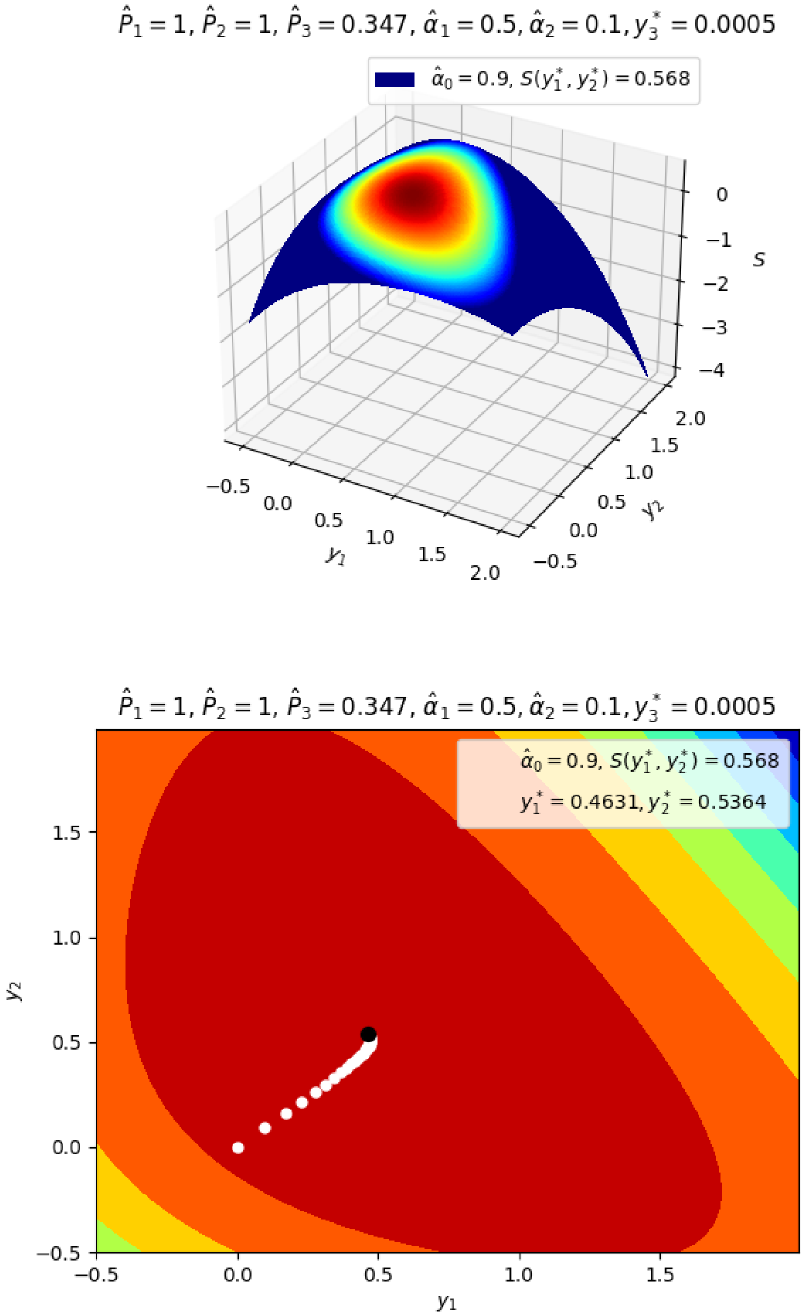


Figure 6.23: An example of $S_2 = (+, +, 0)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = \hat{P}_2 = 1$ and $\hat{P}_3 = 0.347$ in a FlatBear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

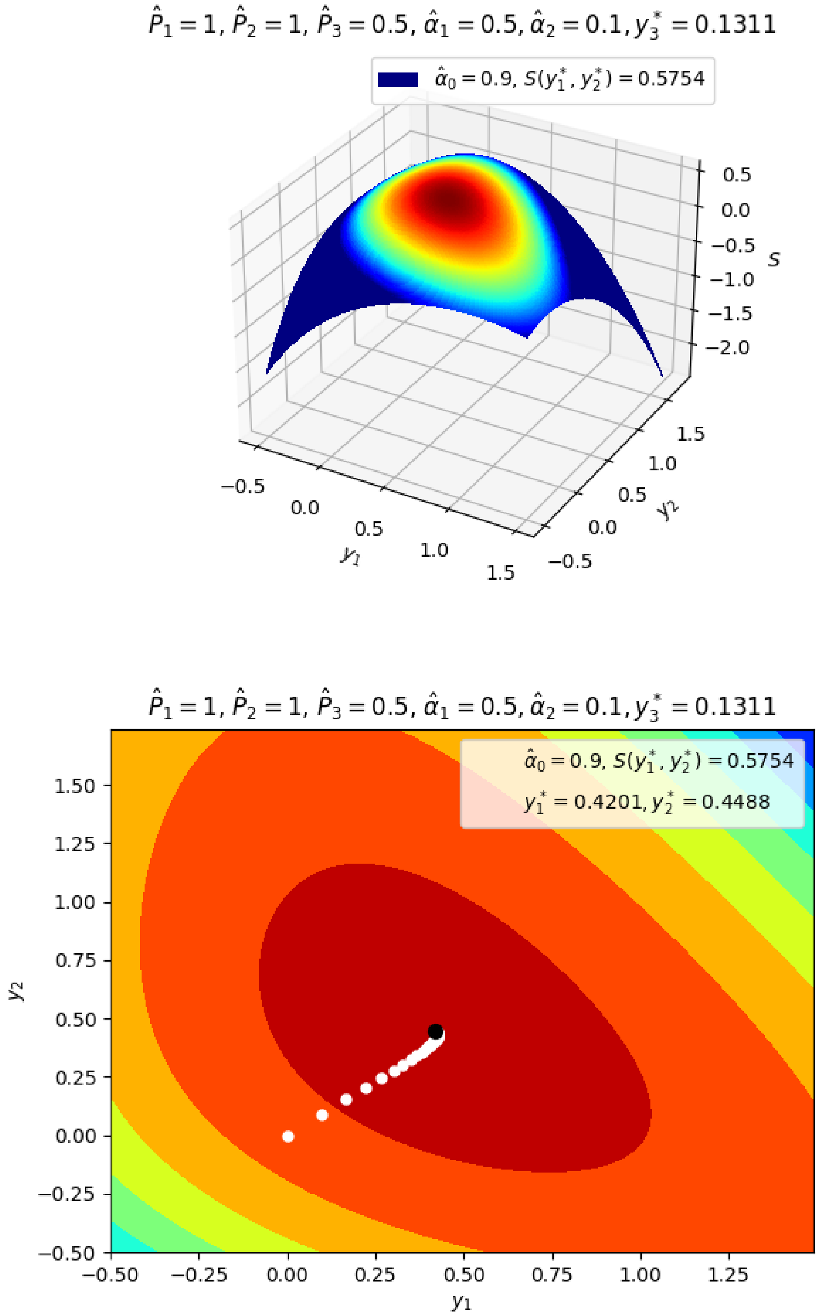


Figure 6.24: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = \hat{P}_2 = 1$ and $\hat{P}_3 = 0.5$ in a FlatBear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

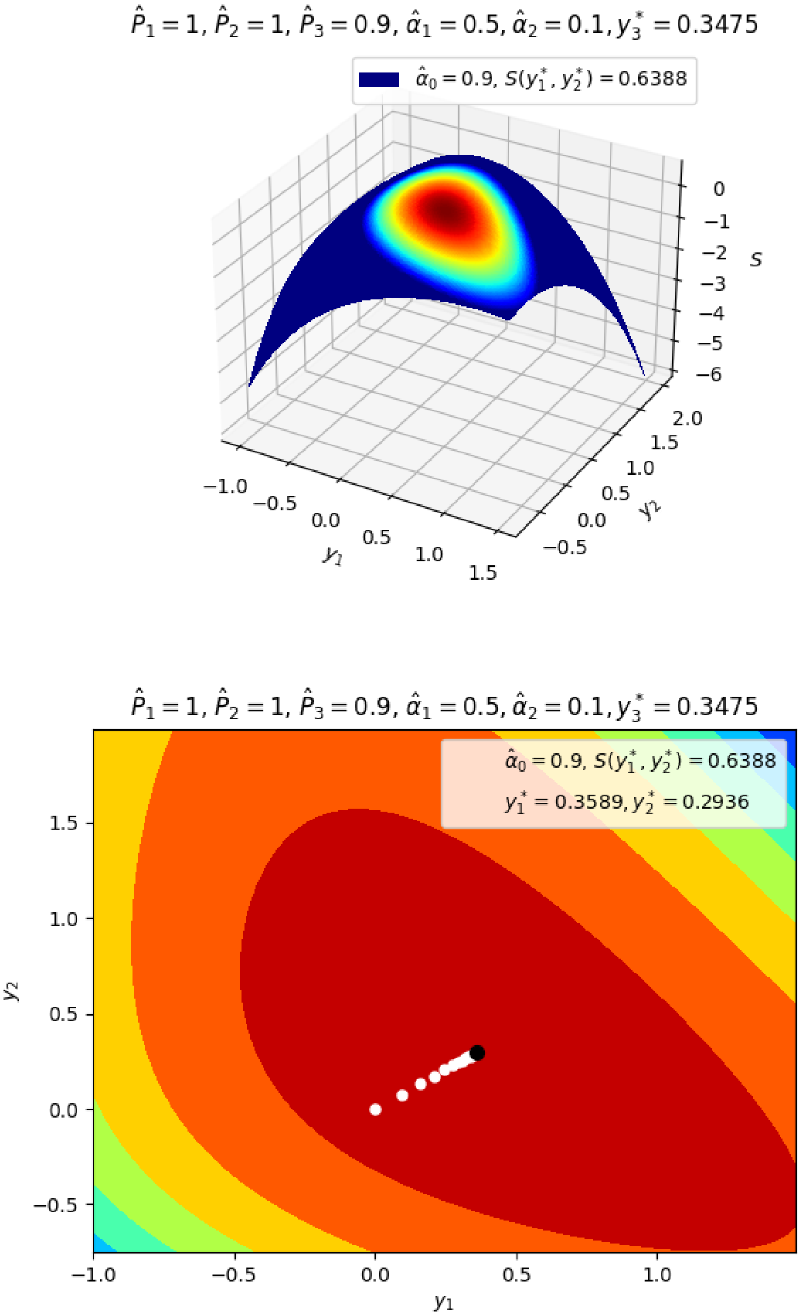


Figure 6.25: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = \hat{P}_2 = 1$ and $\hat{P}_3 = 0.9$ in a FlatBear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

In Section 6.2.2.7 on page 74, we found the naive optimal strategy space to be the buy and sell strategy space $T_4 = (+, -, +)$, where the investor sells stock at t_1, buys more stock at t_2, for then to sell it all at t_3.

The numerical analysis shows that the naive optimal strategy space was indeed found, however only in extreme market sentiments. Numerous iterations within the scope found other strategy spaces than the naive feasible strategy spaces. These are $S_5 = (+, +, +)$ and $S_7 = (+, 0, +)$, as shown in Figures 6.28 and 6.29. We were unable to locate the naive feasible strategy spaces: $T_5 = (0, -, +)$ and $SH_5 = (+, -, 0)$. Hence, the only naive feasible strategy space located was the one that we chose as the naive optimal strategy space.

Also, in this case we were able to find an approximation to the boundary case of $S_5 = (+, +, +)$ and $T_4 = (+, -, +)$, namely $S_7 = (+, 0, +)$. By the Intermediate Value Theorem 2.2.11 we are assured its existence. The approximation of the boundary case is shown in Figure 6.29.

Our model shows some interesting sales values. The investor earns 41.14% of the portfolio value when the market declines 90% in the second step, for then to increase 10% in t_3. The liquidity cost of 58.86% beats the overall market drop of 89%, actually salvaging 30.14 percentage points on the market. See Figure 6.26.

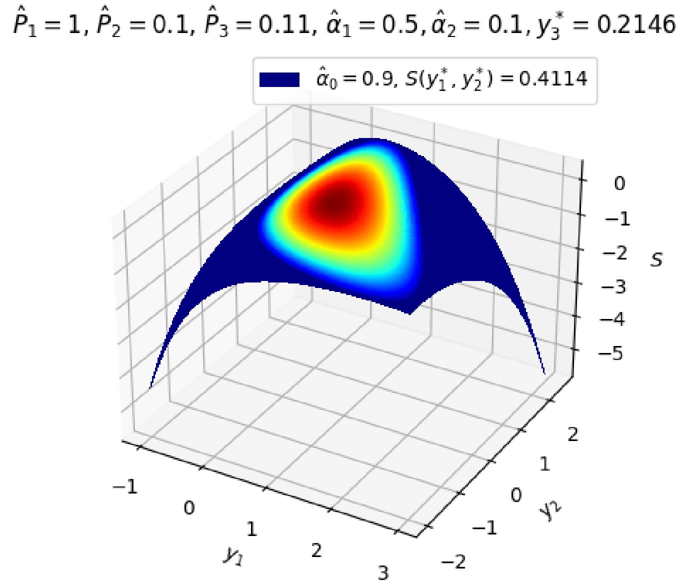
In the case where the market has a 90% decline in t_2, for then to return to its original state in step t_3, the investor experiences a liquidity cost of 12.72% as only 87.28% of the initial portfolio value is rescued. See Figure 6.27. Furthermore, in the case where the market has only a 10% decline in t_2, for then to return to its original state in step t_3, the investor experience a liquidity cost of 35.50%, as only 64.50% of the initial portfolio value is rescued. See Figure 6.28. It is interesting to see that a dramatically larger decline in step t_2 yields a much smaller liquidity cost on the sales value S. The liquidity cost delta is 22.78 percentage points in favour of the dramatic decline in t_2.

A closer look shows that the case of a 90% decline in t_2 follows the trading strategy $T_4 = (+, -, +)$ shown in Figure 6.26, while the 10% decline in t_2 optimises $S_5 = (+, +, +)$ shown in Figure 6.28. As T_4 lends opportunity to buy more stock as the drop occurs, the investor seems to capitalise on the surge of 900% from $\hat{P}_2 = 0.1$ to $\hat{P}_3 = 1$. In that respect it seems rather poor that the actual optimal strategy only gives a sales value less than 1. This again lends evidence to the strength of the bounce-back function ψ. Or maybe it should be called the 'lack-of-bounce-back function'?

Finally, in Figure 6.30 we notice that the market declines 10% in step t_2, for then to return to a doubling of the original state in t_3, the investor incurs a 4.48% liquidity cost due to the bounce-back function. The market doubles, however the investor does not even manage to salvage the original value of the portfolio. The model tells a story of the power of oversupplying the market and how much this influences the actual purchasing price in an unstable market.

6.3.2.8 BearFlat: $\hat{P}_1 > \hat{P}_2 = \hat{P}_3$

In Section 6.2.2.8 on page 74 we found the naive optimal strategy space to be the short selling strategy $SH_5 = (+, -, 0)$, where the investor borrows and sells



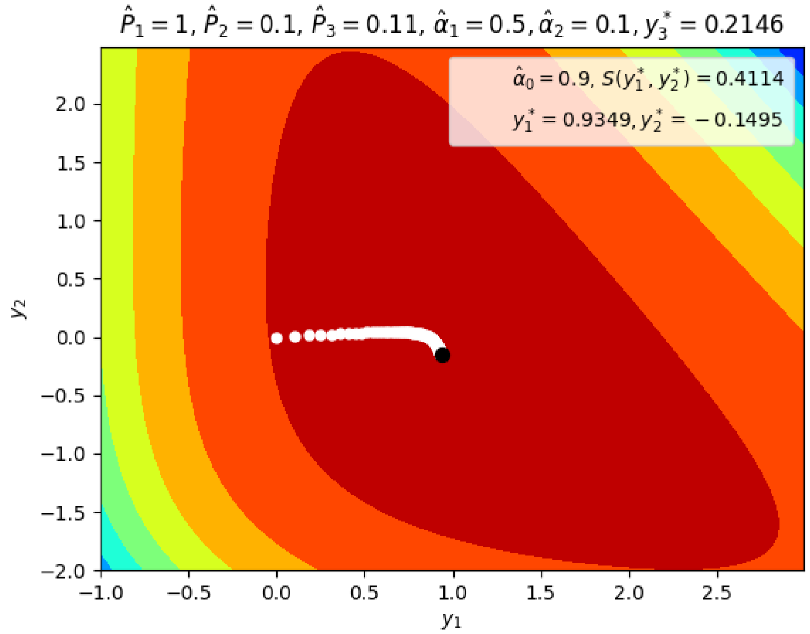


Figure 6.26: An example of $T_4 = (+, -, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 0.1$ and $\hat{P}_3 = 0.11$ in a BearBull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

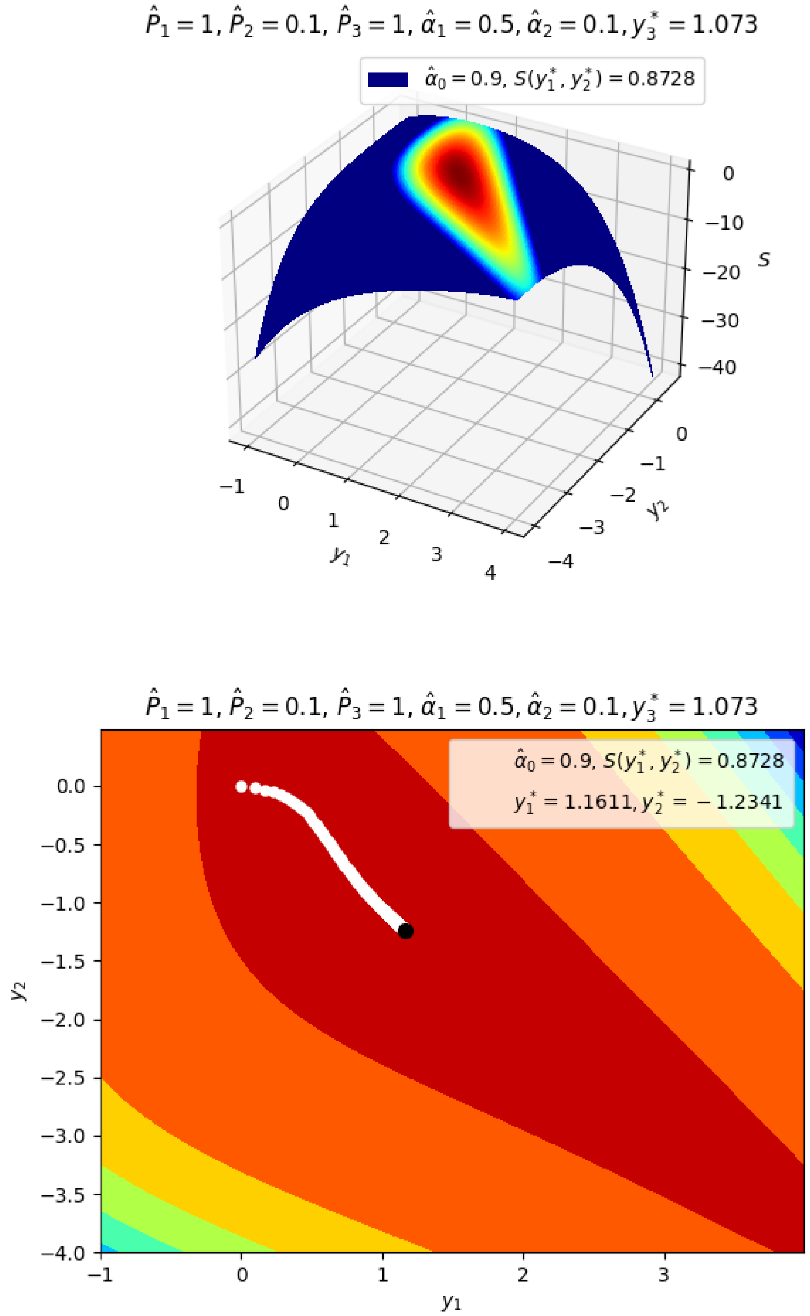


Figure 6.27: An example of $T_4 = (+, -, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 0.1$ and $\hat{P}_3 = 1$ in a BearBull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

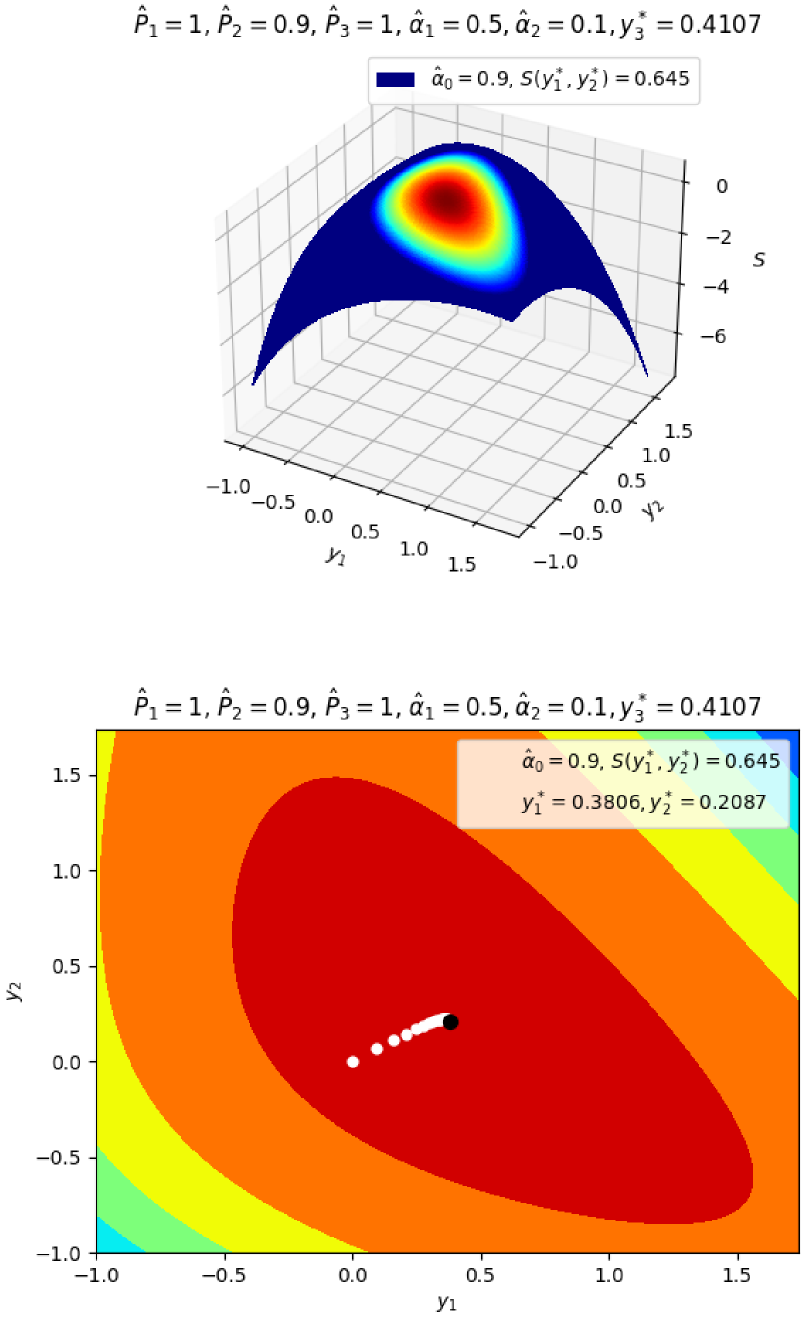


Figure 6.28: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 0.9$ and $\hat{P}_3 = 1$ in a BearBull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

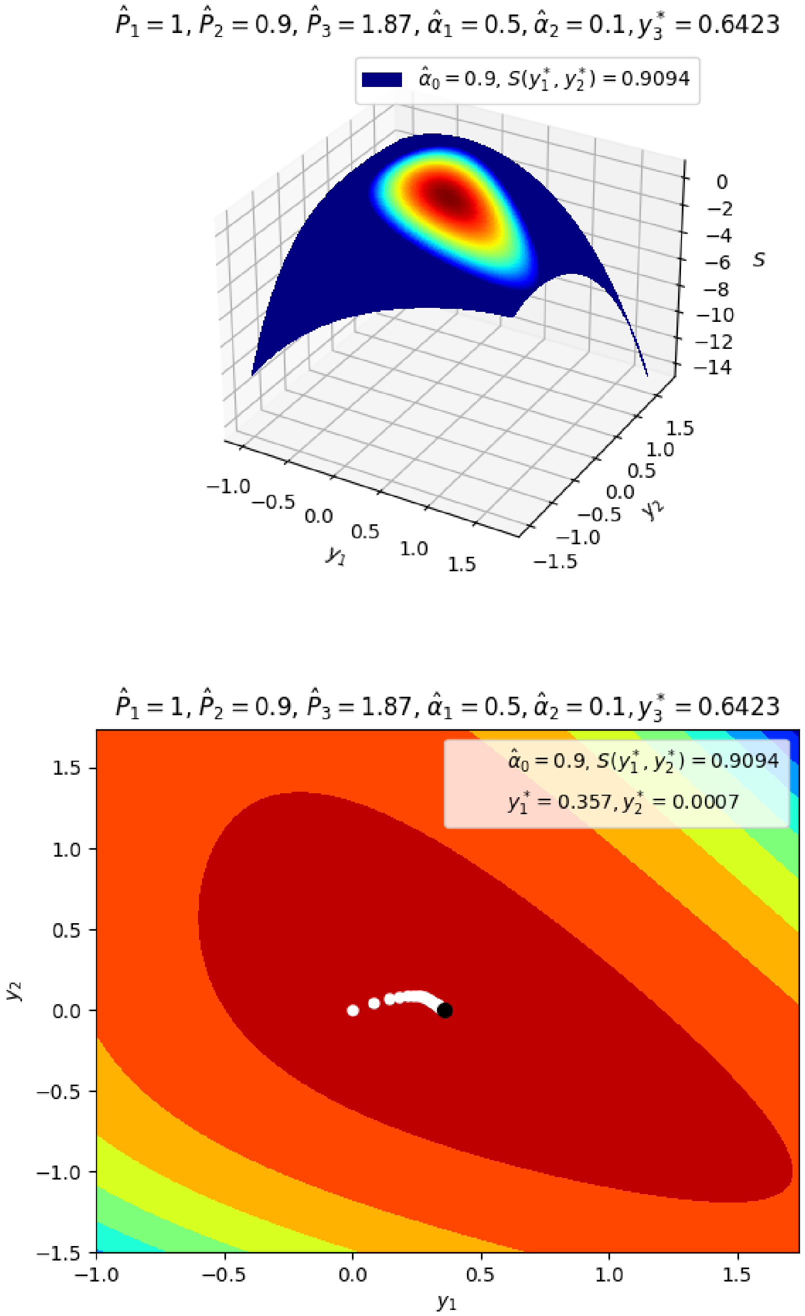
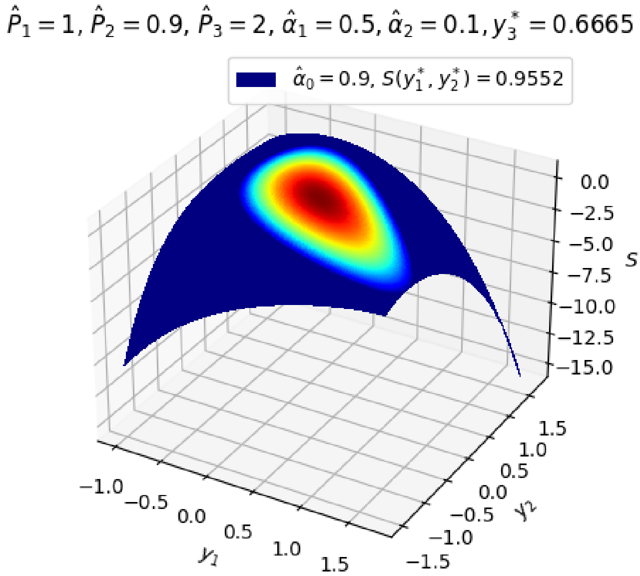


Figure 6.29: An example of $S_7 = (+, 0, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 0.9$ and $\hat{P}_3 = 1.87$ in a BearBull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.



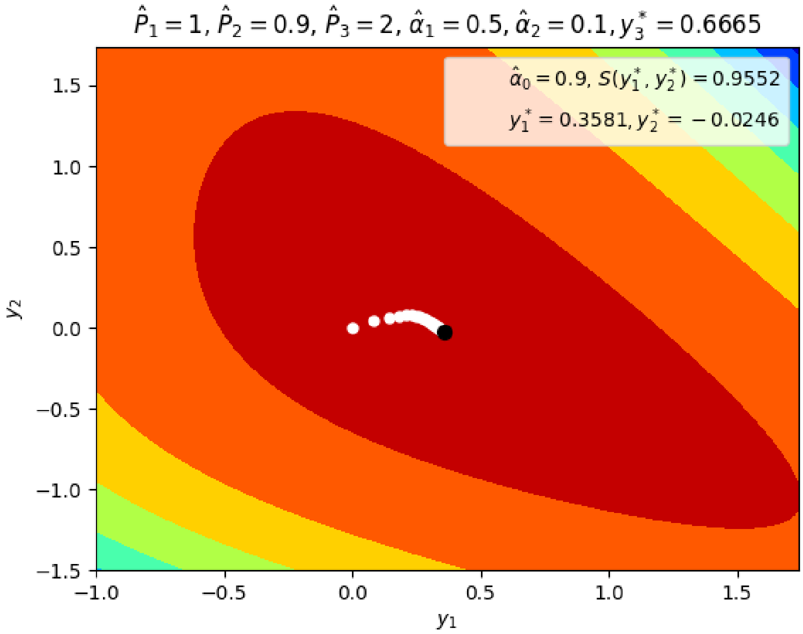


Figure 6.30: An example of $T_4 = (+, -, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 0.9$ and $\hat{P}_3 = 2$ in a BearBull market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

both borrowed and original stock in step t_1, buys back the borrowed stock in t_2 for then to do nothing in t_3.

The numerical analysis shows that the naive optimal strategy space $SH_5 = (+, -, 0)$, also in this case, is not the actual optimal strategy space. Again, numerous iterations within the scope of conceivable price changes showed that none of the naive feasible strategy spaces were available in this market – so neither $S_2 = (+, +, 0)$, $S_3 = (+, 0, 0)$, $SH_4 = (+, -, -)$ and $SH_5 = (+, -, 0)$.

Our model shows that in a market of 10% decline displayed in Figure 6.34, the investor manages to salvage only 62.14% of the portfolio value with the three-step strategy space $S_5 = (+, +, +)$, which is the actual optimal strategy space. Even in a declining market of 50% shown in Figure 6.33 the investor only manages to escape with 48.62% of the portfolio value, incurring a 37.86% liquidity cost in the 10% decline and a liquidity cost of 51.38% in the 50% decline. Furthermore, in both cases the investor is losing towards the market, which declines 10% and 50%, respectively. Hence, a big decrease in the market only yields a minor reduced sales value (13.52 percentage points) compared to the 10% market decline. Consequently, the bounce-back function ψ is weaker in a big market drop than in a small market drop.

Figure 6.32 shows that the investment strategy space $S_5 = (+, +, +)$ is the actual optimal strategy space until the market declines 74.65%. At this point the model suggests $S_7 = (+, 0, +)$, which is the boundary between $S_5 = (+, +, +)$ and $T_4 = (+, -, +)$, as the optimal strategy space yielding a 42.89% sales value and thus incurring a 57.11% liquidity cost. It is interesting that the investor in this case beats the market with 17.54 percentage points.

Even the 90% market decline yields a positive result on the market decline as the investor's sales value is 40.98%. This is a markup on the market of 30.98 percentage points. In this case the model suggest the investment strategy space $T_4 = (+, -, +)$. See Figure 6.31.

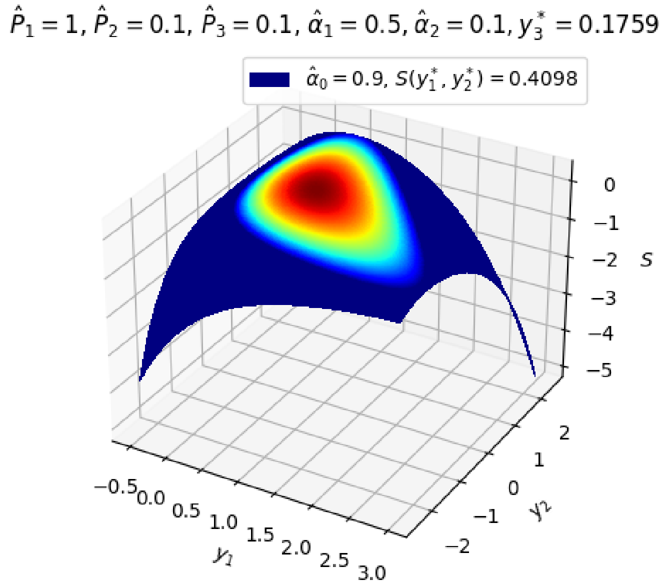
Based on numerical iterations of $\hat{P}_2 = \hat{P}_3$ we see that the first and third coordinate of our 3-tuple (y_1, y_2, y_3) seem to stay positive independent of whether we make $\hat{P}_2$ go towards 0 or 1.

In general, it seems like a small market drop results in the investor losing towards the market, as the sales value of the portfolio is much smaller than the market drop, see Figures 6.33 and 6.34. Contrary, as the market experiences a dramatic drop the investor, in these cases, beat the market as the sales value of the portfolio is bigger than the market drop, see Figures 6.31 and 6.32.

6.3.2.9 Bear: $\hat{P}_1 > \hat{P}_2 > \hat{P}_3$

In Section 6.2.2.9 on page 74 we found the naive optimal strategy space to be the short selling strategy space $SH_3 = (+, 0, -)$, where the investor sells the entire portfolio, including borrowed stock at t_1, does nothing at t_2, for then to buy back the borrowed stock at t_3.

Our numerical analysis unravels that our naive optimal strategy space $SH_3 = (+, 0, -)$ can be an actual optimal strategy space in a Bear market, as it is a boundary strategy space for $D_1 = (+, +, -)$. It may thus, under the given circumstances, be the optimal strategy space the investor is searching for.



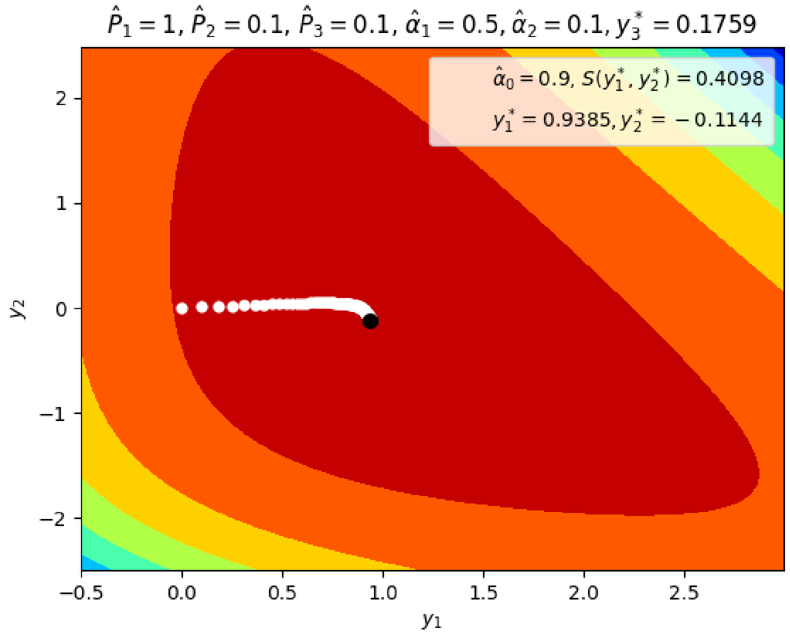


Figure 6.31: An example of $T_4 = (+, -, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$ and $\hat{P}_2 = \hat{P}_3 = 0.1$ in a BearFlat market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

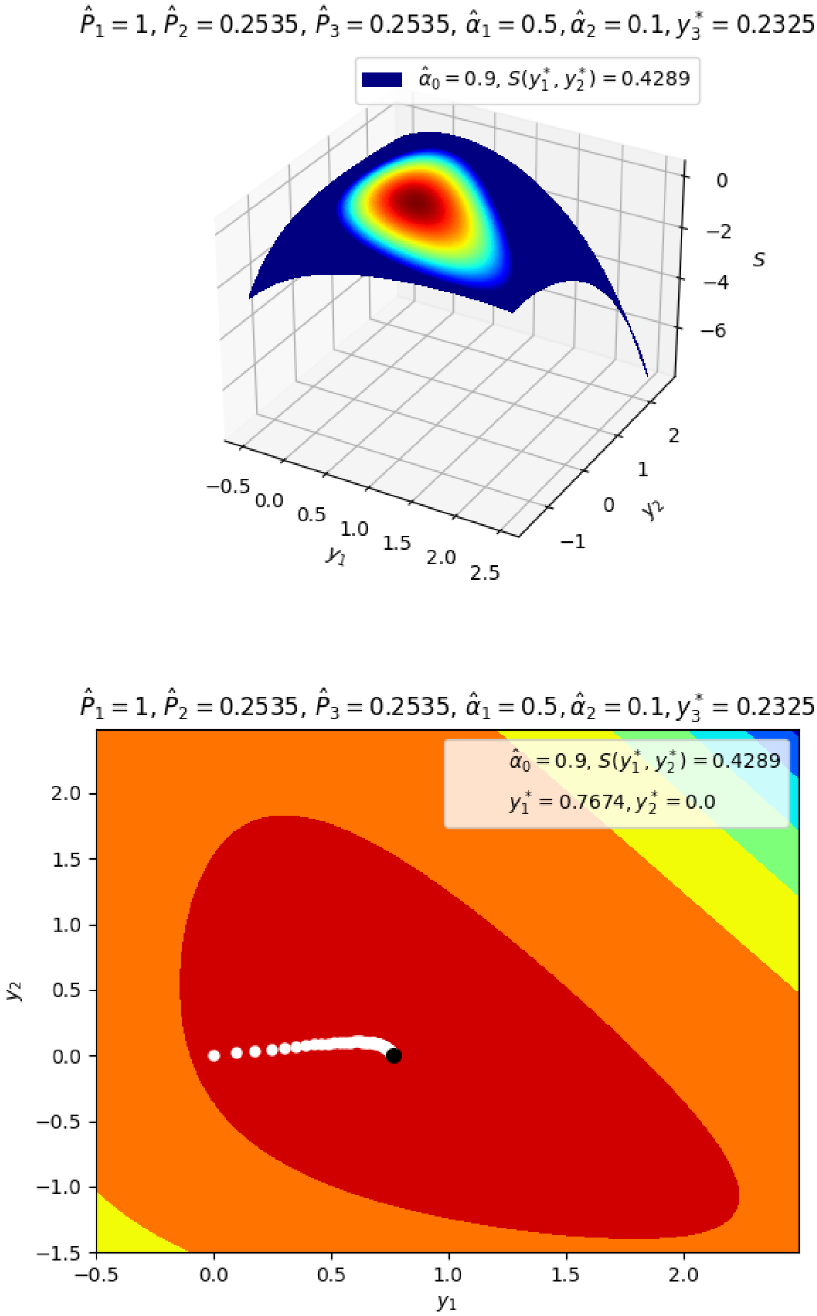
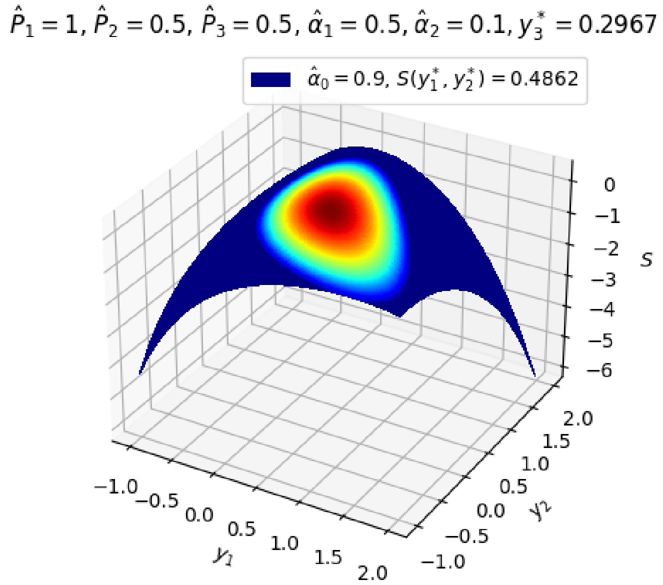


Figure 6.32: An example of $S_7 = (+, 0, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$ and $\hat{P}_2 = \hat{P}_3 = 0.2535$ in a BearFlat market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.



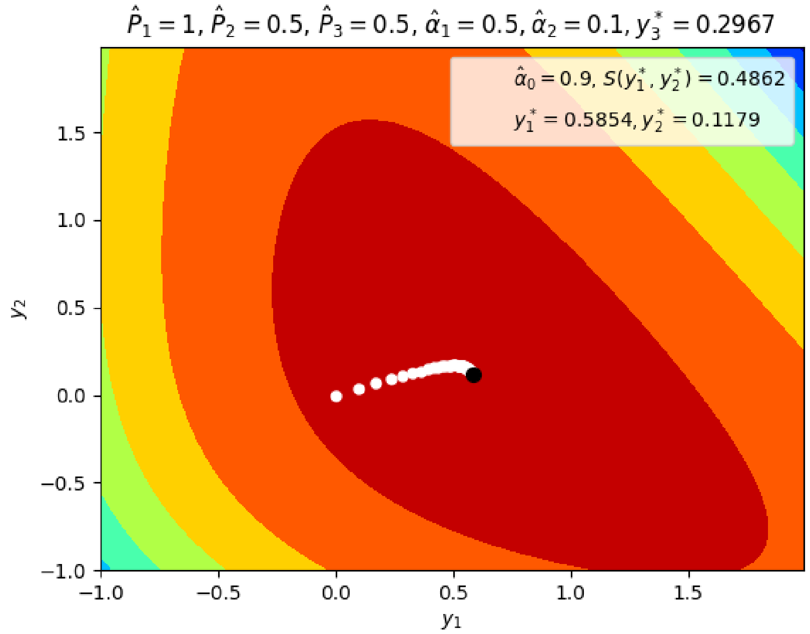


Figure 6.33: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$ and $\hat{P}_2 = \hat{P}_3 = 0.5$ in a BearFlat market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

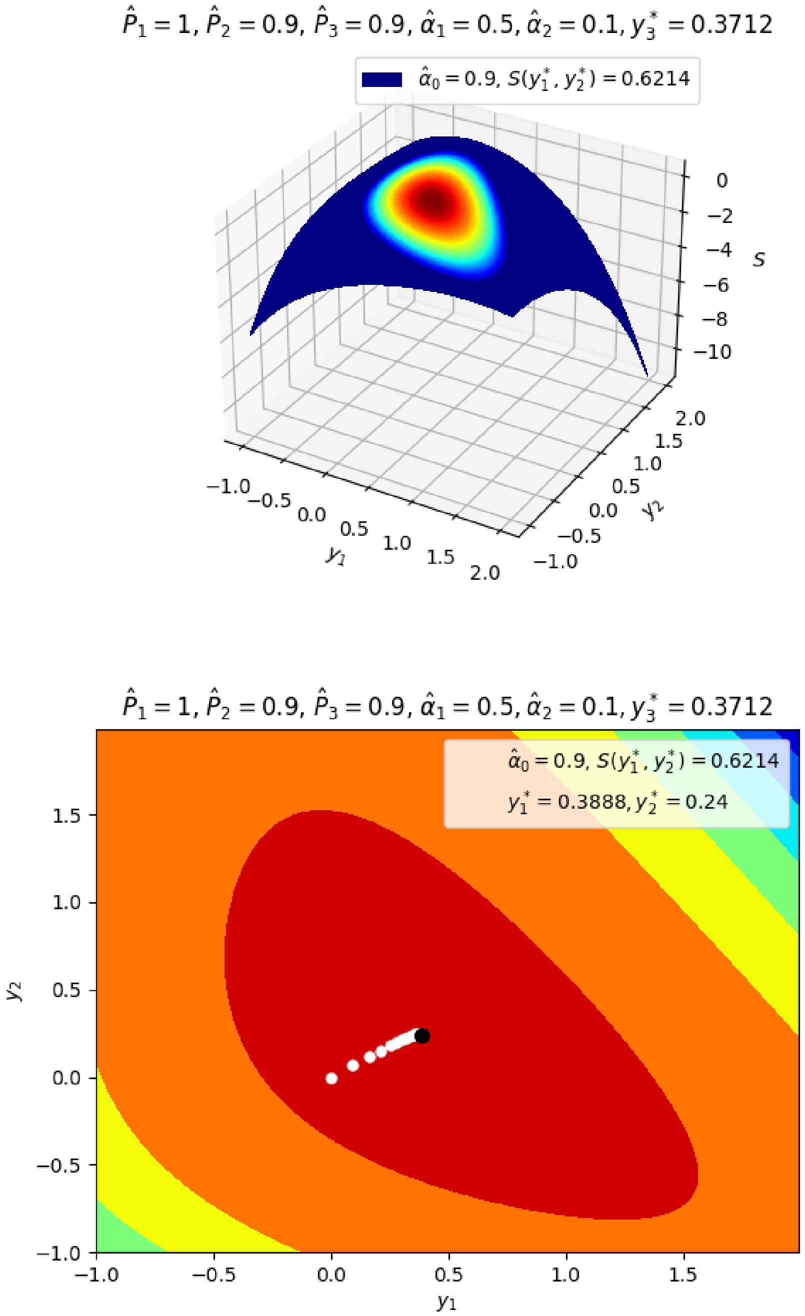


Figure 6.34: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$ and $\hat{P}_2 = \hat{P}_3 = 0.9$ in a BearFlat market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

Another set of iterations within the scope of conceivable prices found one incident where a naive feasible strategy space was the actual optimal strategy space, namely $D_1 = (+,+,-)$ shown in Figure 6.35. In addition, we found that the naive strategy spaces $T_4 = (+,-,+)$ and $S_5 = (+,+,+)$ shown in Figures 6.36 and 6.37 to yield actual optimal strategy spaces. Consequently, we see that the boundary strategy space between $S_5 = (+,+,+)$ and $T_4 = (+,-,+)$, $S_7 = (+,0,+)$, also must be an actual optimal strategy space. Furthermore, the boundary strategy space between $S_5 = (+,+,+)$ and $D_1 = (+,+,-)$, $S_2 = (+,+,0)$, is also an optimal strategy space. The naive feasible spaces S_7 and S_2 were not found by iteration, however the Intermediate Value Theorem 2.2.11 ensures their existence.

Our model shows that in a moderately declining market displayed in Figure 6.38, the investor manages to salvage only 62.97% of the portfolio value with this three-step strategy space, incurring a liquidity cost of 37.03%. Even in the extreme market shown in Figure 6.37, the investor manages to escape with 52.11% of the portfolio value, incurring a 47.89% liquidity cost due to the size of the portfolio. In the latter case, it is interesting to see that the investor beats the market by 2.11 percentage points, while the investor in the first case is beaten by the market by 27.03 percentage points. This follows the pattern in the BearFlat where a smaller market drop yields a bigger liquidity cost C, and a bigger market drop yields a smaller liquidity cost C than the market decline.

So, why is it that $SH_3 = (+,0,-)$ is not an actual optimal strategy space at all times? It seems that the bounce-back function ψ in the last step t_1 does not have enough bounce. Hence, the price drop of flooding the market at t_1 will outperform the profit from the extra stocks bought back in step t_3. Again, it may be argued that an insignificant portfolio could benefit from the SH_3 strategy space, as the bounce-back function ψ will be negligible.

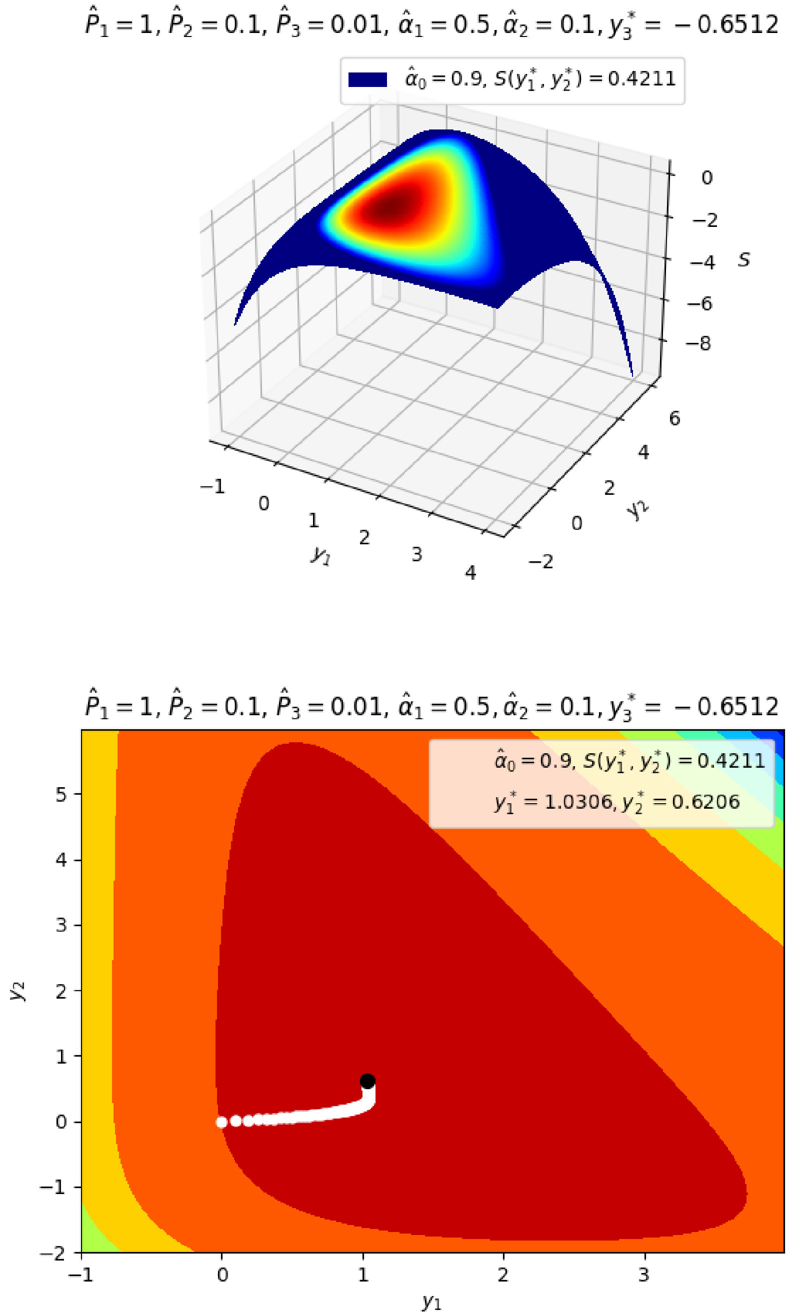
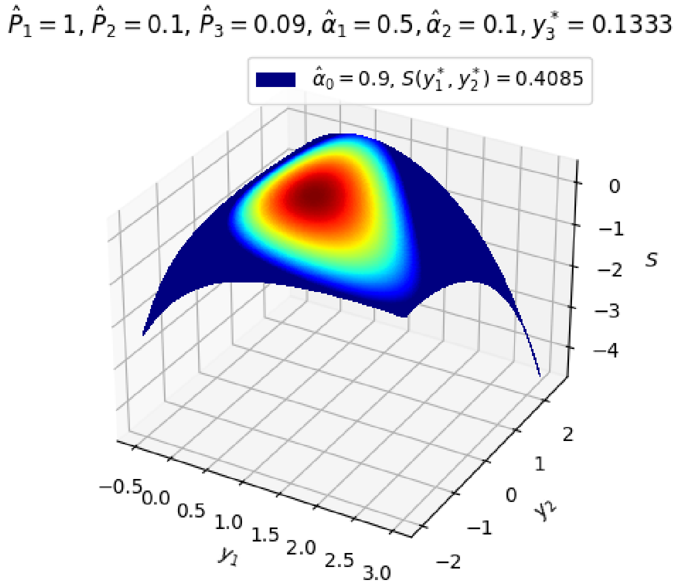


Figure 6.35: An example of $D_1 = (+, +, -)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 0.1$ and $\hat{P}_3 = 0.01$ in a Bear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.



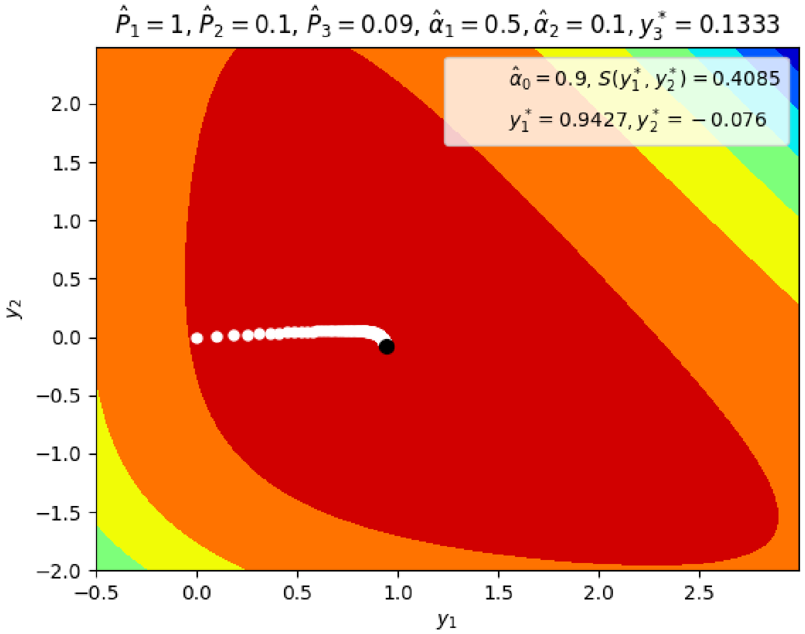


Figure 6.36: An example of $T_4 = (+, -, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 0.1$ and $\hat{P}_3 = 0.09$ in a Bear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

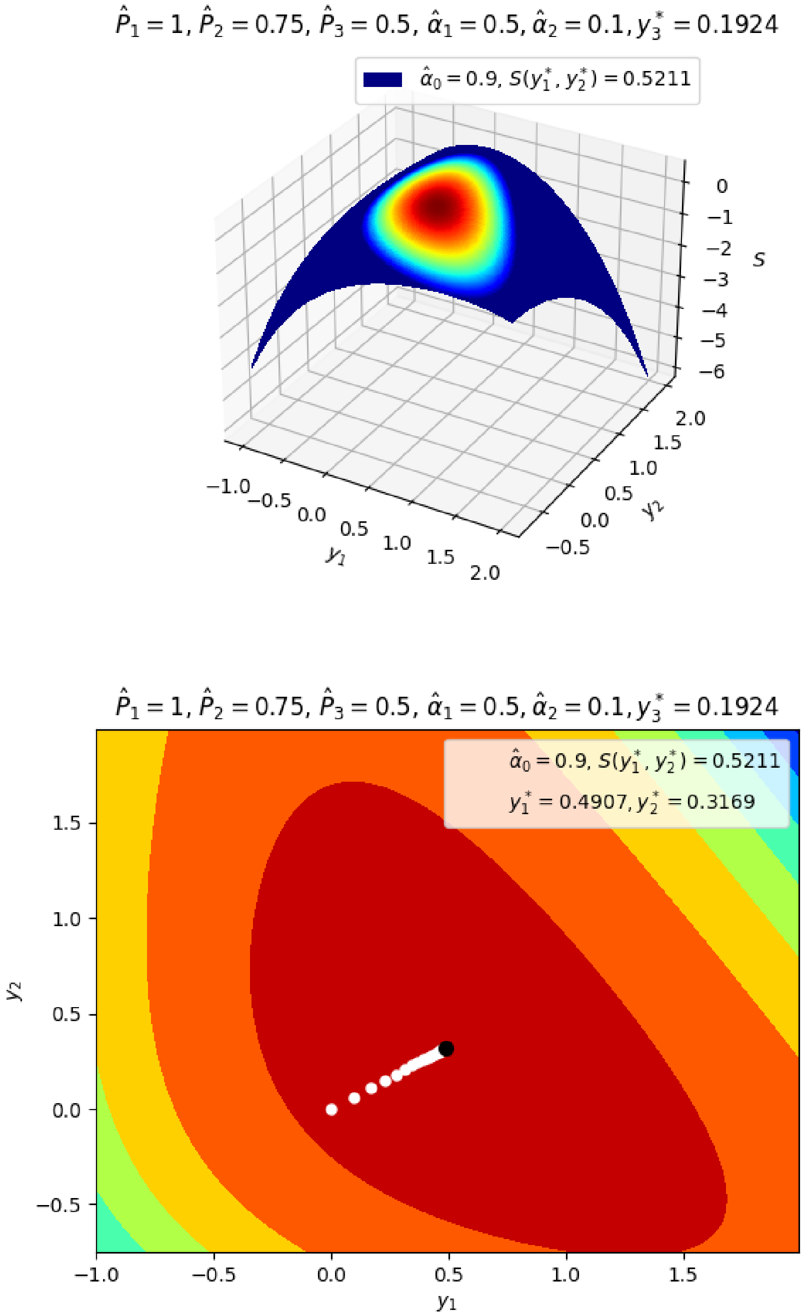
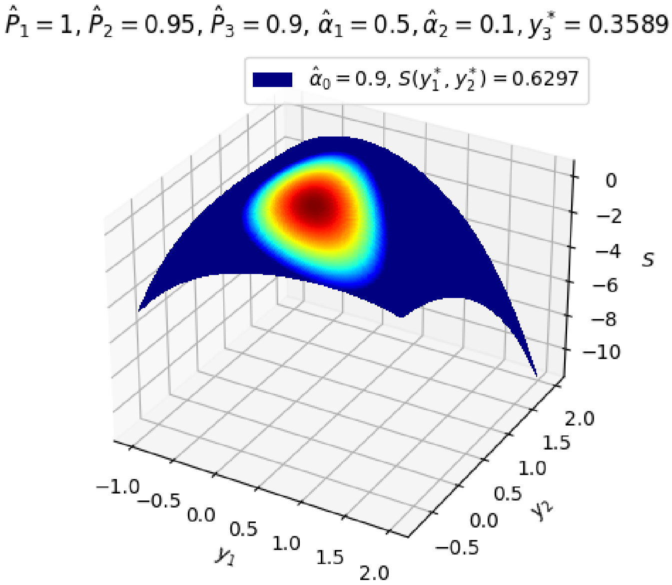


Figure 6.37: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 0.75$ and $\hat{P}_3 = 0.5$ in a Bear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.



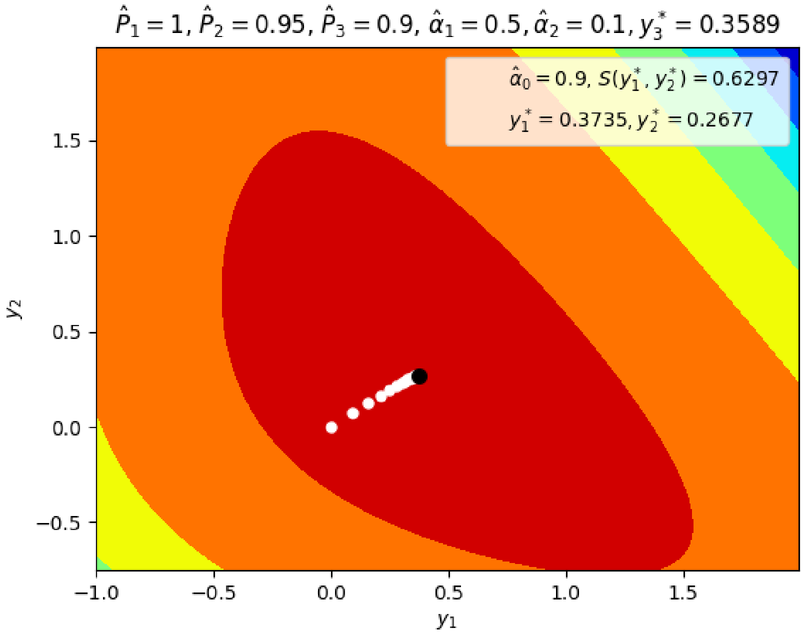


Figure 6.38: An example of $S_5 = (+, +, +)$. The upper plot shows the complete sales value surface for $\hat{P}_1 = 1$, $\hat{P}_2 = 0.95$ and $\hat{P}_3 = 0.9$ in a Bear market. The contour plot below shows the sales value surface and the iterations of the Gradient Ascent to the maximum.

6.4 Comparative analysis of different market types

It is natural to wonder if a BullFlat market behaves equal to a FlatBull market. At first glance, that may even be expected. The same goes for market types that just seem to have changed the order of movement – do they actually commute? Hence, this section seeks to shed some light on how market types that naively may be viewed as equal actually move, based on our model S.

6.4.1 Bull vs. Flat vs. Bear

The only strategy space the three pure markets Bull, Flat and Bear have in common is $S_5 = (+, +, +)$. While the naive strategy spaces for both Bull and Bear markets were actual optimal strategy spaces in the model, this was not the case in the Flat market.

Also, Bull and Bear markets have several strategy spaces as actual optimal, while the Flat market just has $S_5 = (+, +, +)$. The Bull market has actual optimal strategy spaces in $T_4 = (+, -, +)$ and $T_2 = (-, +, +)$ in addition to $PU_1 = (-, -, +)$ and $S_5 = (+, +, +)$, and the Bear market has actual optimal strategy spaces in $SH_3 = (+, 0, -)$, $D_1 = (+, +, -)$, $S_2 = (+, +, 0)$, $T_4 = (+, -, +)$, $S_5 = (+, +, +)$ and $S_7 = (+, 0, +)$.

The bounce-back function ψ displays different behaviours in the three pure markets. While the bounce-back function ψ in a Flat market has a 33.91% liquidity cost, the bounce-back function in a Bull market influences the sales value S less by a bigger market increase than a smaller market increase. Moreover, in the Bear market a smaller decrease yields a bigger liquidity cost than a bigger decrease.

6.4.2 FlatBull vs. BullFlat

Naively it may seem intuitive that these two should have the same result, as the only difference on the surface is the order of the markets. This is not the case. The naive optimal strategy space in both the BullFlat and the FlatBull markets ends up as non-feasible strategy spaces in our model. Again, both have $S_5 = (+, +, +)$ as an actual optimal strategy space. However, The BullFlat market includes $T_2 = (-, +, +)$ and $S_4 = (0, +, +)$ as optimal strategy spaces, while the FlatBull market includes $S_7 = (+, 0, +)$ and $T_4 = (+, -, +)$ as well.

Furthermore, the FlatBull market performs worse than the BullFlat market with respectively a sales value of 68.49% and 70.24% in the 10% market increase. These sales values are 79.50% and 88.43% with the 50% market increase.

Also, in a BullFlat market, a bigger market increase only helps reduce the liquidity cost by 18.19 percentage points, while in a FlatBull market a bigger market increase only helps reduce the liquidity cost by 11.01 percentage points, both while the market moves 40 percentage points. Hence, the bigger the market increase the bigger the effect from the bounce-back function ψ in both cases.

Interestingly, in the FlatBull market, the difference in sales value between a 10% and a 50% market increase is only 11.01 percentage points, while in a BullFlat market this delta is 18.19 percentage points.

6.4.3 FlatBear vs. BearFlat

Also here, the naive optimal strategy space in both the BearFlat and the FlatBear markets ends up as non-feasible strategy spaces in our model. Yet again, both have $S_5 = (+,+,+)$ as an actual optimal strategy space. However, The BearFlat market includes $T_4 = (+,-,+)$ and $S_7 = (+,0,+)$ as actual optimal strategy spaces, while the FlatBear market includes $D_1 = (+,+,-)$ and $S_2 = (+,+,0)$ as well.

We see the same pattern of significant differences between BearFlat and FlatBear as we did with BullFlat and FlatBull. Also in this case, starting with a Flat market yields a smaller drop in sales value compared to ending with a Flat market. In BearFlat we see a sales value of 62.14% in the 10% market drop and 48.62% in the 50% market drop, compared to a sales value of 60.20% in the 10% market drop and 57.54% in the 50% market drop. This gives a sales value differences within the same market drops of 13.53 and 2.66 percentage points, respectively. Hence, in the BearFlat market a big decrease in the market yields a smaller reduction in sales value (13.52 percentage points) than the 10% market decline. While in a FlatBear market, an additional 40 percentage point drop in the market yields only a marginally increased sales value (2.66 percentage points). Consequently, the bounce-back function ψ is stronger the bigger the market drop.

6.4.4 BullBear vs. BearBull

The BullBear market does not have its naive optimal strategy space $SH_1 = (-,+,-)$ as an actual optimal strategy space, while the BearBull market does. The naive optimal strategy space is one of the actual optimal strategy spaces, $T_4 = (+,-,+)$. Both market types have other actual optimal investment strategy spaces as well. The BullBear market includes $D_1 = (+,+,-)$, $T_2 = (-,+,+)$, $S_2 = (+,+,0)$, $S_4 = (0,+,+)$, as well as the all-rounder $S_5 = (+,+,+)$. The optimal strategy space $S_5 = (+,+,+)$ is also an actual optimal strategy space in the BearBull market, accompanied by $S_7 = (+,0,+)$.

The BullBear market also offers the investor a negative liquidity cost. This is not the case in the BearBull market. Now, in the BullBear market where the market first rise 10%, for then to return to its original state, the sales value is 67.97% and the liquidity cost is 32.03%. In the BearBull market with a 90% decrease before returning to its original state, the sales value is 87.28% with a liquidity cost of 12.72%. Again we see these markets behaving differently.

This asymmetric movement is expected in this case as the base for percentage change is not the same. Interestingly, the change in sales value accounts for close to 20 percentage points (18.19) which is the same as the 20 percentage point change (22.78) in the $\hat{P}_2$ for the two market types.

CHAPTER 7

Closing remark

On a grander scale, the business world is primarily about mitigating unnecessary risk, and at the same time optimise return in an attempt to find a fulcrum where it all balances. In order to converge towards these points the field is in need of more accurate risk measures, for then to set more accurate anchors, which in return will give a better expected return estimate. It would have been interesting to work on a continuation of this aspect, however the scope of this project is limited to an exploration of a model in discrete time, which includes risk as a part of all the other market parameters in the model.

This Master's thesis investigates the normalised sales value model S in one, two and three dimensions. The model describes how an investor can optimise market moving portfolio liquidation in certain markets defined by $\hat{P}_n$, with respect to liquidity, market value, sales value, liquidity cost and liquidity cost ratio. Possible investment strategies are sorted into strategy spaces which are clustered under the investment opportunities buy, sell, hold, pumping, dumping and short selling. In order to decide which strategy space and numerical strategies that optimise the liquidation we used naive, analytical and numerical methods. The thesis begins with the Fixed Point method, however changes to Gradient Ascent method as the first one becomes unstable in $N = 2$.

Across the different dimensions $N = 1, 2, 3$, we discover the following: The investigation in $N = 1$ finds that the liquidity cost ratio LCR is constant across all price developments as long as y_1 can be bigger than 1, and that the closer to 1 $\hat{\alpha}_0$ moves the bigger the liquidity cost C.

In $N = 2$ we find that the model S has a unique maximum, and there is no available optimal pumping strategy for $S(y_1)$ under the constraint $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$ and given $S'(0) \geq 0$. However, there may be pumping strategies available under certain market sentiments as long as $S'(0) < 0$. There is also no dumping strategy for $S(y_1)$ under the constraint $0 < \hat{\alpha}_1 < \hat{\alpha}_0 < 1$ and given $S'(1) \leq 0$. Moreover, there may exist dumping strategies under certain market sentiments as long as $S'(1) > 0$. Also, there are levels where the estimated price $\hat{P}_2$ triggers a certain investment strategy, and these investment strategies are found in distinct types of markets.

For $N = 3$ the naive analysis seems to be severely insufficient compared to an analysis based on numerical method, as most of the optimal feasible strategy spaces are non-feasible. Furthermore, the sequence of market types does not commute, i.e., a BearFlat market does not behave equal to a FlatBear market.

Also, it seems more risky to choose strategy spaces with a hold action, i.e., a 0 in the 3-tuple, as they are boundary spaces, and thus more difficult to target.

Moreover, this thesis has its clear limitations. The vast opportunity space in $N = 3$ reduces the analysis in three dimensions to a game with numbers rather than a complete rigorous analysis. As a consequence, this analysis also lacks an investigation of an arbitrary N. Also, our model is discrete in time while financial markets are modelled continuously. Furthermore, these models are often based on time series and our model is independent of time. Besides, there are numerous factors that drive the stock markets, while this model consolidates them all into the $\hat{\alpha}_n$ parameters.

Finally, the analysis enlightens how difficult it may be to read the financial markets based on naive intuition and reasoning, and that mathematical modelling has its clear advantages. Someone who has had a great deal of success with this approach is the previous mathematics professor Jim Simons and his investment company Renaissance Technologies, who have delivered impressive returns for decades.

APPENDIX A

Python code

A.1 The case $N = 1$

This Python program runs the model S for $N = 1$ and prints all plots in Chapter 4. The different plots are a result of different values for α_0. Short descriptions of each block in the program are written in the script.

```
from math import exp
import numpy as np
import matplotlib.pyplot as plt

# Defining the functions
def S(alpha, P, y):
    return y*P*np.exp(-alpha*y)
def M(P,y):
    return P*y
def E(a0):
    return (-a0)

# Some parameters and y-axis
y = np.linspace(0,2.5,100)
P = 1
alphas = [0.99, 0.75, 0.50, 0.25, 0.1]
fig = plt.figure()
ax = fig.add_subplot(1, 1, 1)
a0 = np.linspace(0,1,100)

# Making the plots
for alpha in alphas:
    y0 = 1/alpha
    d = (M(P,y0)-S(alpha,P,y0))/M(P,y0)
    print(d)
    top_point = (P/alpha)*np.exp(-1)
    f = S(alpha = alpha, P = P, y = y)
    plt.plot(y,f, linewidth=2, label = 'a0={0} Total sales value'.format(alpha
     ))
    if (y0 <= 2):
        plt.plot([y0],[top_point],'o', color='k')
        ax.axvline(x=y0, linewidth=0.5, linestyle = '--')
plt.plot(y, M(P = P, y=y), linewidth=2, color='magenta' ,label='Estimated
     value')

# Eliminate upper and right axes
ax.spines['right'].set_color('none')
ax.spines['top'].set_color('none')

```

```
# Show ticks in the left and lower axes only
ax.xaxis.set_ticks_position('bottom')
ax.yaxis.set_ticks_position('left')
plt.xlabel('Block size, $y$')
plt.ylabel('Total sales value, $S$')
plt.legend(loc='best')
plt.show()

# Show ticks in the left and lower axes only of elasticity plot
plt.plot(a0,E(a0), linewidth=2, label = 'Elasticity')
ax.xaxis.set_ticks_position('bottom')
ax.yaxis.set_ticks_position('left')
plt.xlabel('Market, $\hat{\\alpha}_0$')
plt.ylabel('Elasticity, $E_{\psi}$')
plt.legend(loc='best')
plt.show()
```

A.2 The case $N = 2$

A.2.1 $\hat{\alpha}_0$, Fixed Point Method

This Python program runs the model S for $N = 2$ with the Fixed Point method. It prints all plots with a singular $\hat{\alpha}_0$ and several $\hat{\alpha}_1$s in Chapter 5. The different plots are a result of different carefully chosen values for α_0 and $\hat{\alpha}_1$s, as well as $\hat{P}_2$. Short descriptions of each block in the program are written in the script. The script is used for the Flat and the Bear type market.

```
from math import exp
from mpl_toolkits import mplot3d
import matplotlib.pyplot as plt
import numpy as np

# Defining the functions
def dL(y, params):
    ah_0 = params["ah_0"]
    ah_1 = params["ah_1"]
    return Eh_1(y, params)*(1-y*ah_0)-(1-y)*ah_1*Eh_2(y,1-y,params)\
    -(Eh_2(y,1-y, params)-ah_0*(1-y)*Eh_2(y,1-y, params))
def ph_next1(ph, params):
    ah_0 = params["ah_0"]
    ah_1 = params["ah_1"]
    return (Eh_2(ph,1-ph, params)*(ah_1-ah_0)+Eh_2(ph,1-ph, params)-Eh_1(ph,
     params))/ \
        (Eh_2(ph,1-ph, params)*(ah_1-ah_0)-ah_0*Eh_1(ph, params))
def S(y_1, y_2, params):
    return y_1*Eh_1(y_1,params)+y_2*Eh_2(y_1,y_2,params)
def Eh_1(y_1, params):
    ah_0 = params["ah_0"]
    return np.exp(-ah_0*y_1)
def Eh_2(y_1,y_2, params):
    ah_0 = params["ah_0"]
    ah_1 = params["ah_1"]
    Ph_2 = params["Ph_2"]
    return Ph_2*np.exp(-(ah_0*y_2+ah_1*y_1))
def S2(y, params):
    return y*Eh_1(y, params)+(1-y)*Eh_2(y,1-y, params)

# Deciding the market types
P2 = 0.9

# Fixed point alogrithm
def fixed_point(ah_0, ah_1, Ph_2, ph=0, tol=1e-9, n=int(1e5)):
    Ph_2 = P2
    svar_y = []
    svar_S = []
    print(ah_0, ah_1)
    params = {"ah_0": ah_0, "ah_1": ah_1, "Ph_2": Ph_2}
    for _ in range(n):
        Sval = S2(ph, params)
        if abs(dL(ph, params))<tol:
            print('jippi')
            break
        ph=ph_next1(ph, params)
        svar_y.append(ph)
        svar_S.append(S2(ph, params))
    return svar_y, svar_S, ph, Sval

# Producing plots and Latex-tables
```

```
if __name__ == "__main__":
    tableline = "${alpha_0}$ & ${alpha_1}$ & {alpha_diff:0.4f} & {y_1:0.4f} & {y_2:0.4f} & {S:0.4f} & {Smin:0.4f} & {diff:0.4f} \\\\ \n"
    a0list = [0.4]
    a1list = [0.3999, 0.35, 0.25, 0.1, 0.01, 0.0001]
    latexstring = ""
    latexstringshort = ""
    color = ['orange', 'cyan' ,'purple', 'lightblue', 'yellow', 'green', 'grey']
    for a_0 in a0list:
        legends = []
        if a_0 != 0:
            latexstring += "\\midrule \n"
        a1_values = list(filter(lambda x: x <= a_0, a1list))
        for index,a_1 in enumerate(a1_values):
            ah_0 = a_0
            ah_1 = a_1
            da = ah_0-ah_1
            svar_y, svar_S, ph, S = fixed_point(ah_0, ah_1, P2, ph=5, n=int(1e5))
            y_1 = ph
            y_2 = 1-ph
            params = {"ah_0":ah_0, "ah_1":ah_1, "Ph_2":P2}
            Smin = S2(1,params)
            line = tableline.format(alpha_0=a_0,\
                 alpha_1=a_1, alpha_diff=ah_0-ah_1,\
                      y_1=y_1, y_2=y_2, S=S, Smin=Smin, diff=S-Smin)
            latexstring += line
            if index == 0 or index == len(list(a1_values))-1:
                latexstringshort += line
# Making plots
            xmin = -1
            xmax = 4
            y=np.linspace(xmin,xmax,1000)
            plt.xlim(xmin,xmax)
            f_1=S2(y, params)
            plt.ylim(0, np.max(f_1)*1.15)
            plt.plot(y,f_1,label="Total sales value as a quotient of initial portfolio value")
            plt.plot(svar_y,svar_S,'*',color=color[index], label='_nolegend_')
            plt.plot(svar_y[-1],svar_S[-1],f'*r')
            plt.xlabel("Volume of shares sold $y_1$")
            plt.ylabel("Sales value $S$")
            plt.title(f"$\\alpha_0 = {ah_0}$, $\\hatP_2 = {P2}$")
            legends += [f"$\\alpha_1 = {ah_1}$, $\\Delta\\alpha = {round(da,3)}$",\
            f"$S(y^*) = {round(svar_S[-1],3)}$, $y^* = {round(svar_y[-1],3)}$"]
            plt.legend(legends)
        plt.savefig(f"a0_{a_0}a1{a_1}P2{P2}.png")
        plt.show()
    latexstring += "\\bottomrule"
    print(latexstring)
    with open(f"a0_{a_0}_a1_{a_1}_P2{P2}.tex", "w") as file:
        file.write(latexstring)
```

A.2.2 $\hat{\alpha}_1$, Fixed Point Method

This Python program runs the model S for $N = 2$ with the Fixed Point method. It prints all plots with a several $\hat{\alpha}_0$s and one $\hat{\alpha}_1$ in Chapter 5. The different plots are a result of different carefully chosen values for α_0s and $\hat{\alpha}_1$, as well as $\hat{P}_2$. Short descriptions of each block in the program are written in the script. The script is used for the Flat and the Bear type market.

```
from math import exp
from mpl_toolkits import mplot3d
import matplotlib.pyplot as plt
import numpy as np

# Defining the functions
def dL(y, params):
    ah_0 = params["ah_0"]
    ah_1 = params["ah_1"]
    return Eh_1(y, params)*(1-y*ah_0)-(1-y)*ah_1*Eh_2(y,1-y,params)\
    -(Eh_2(y,1-y, params)-ah_0*(1-y)*Eh_2(y,1-y, params))
def ph_next1(ph, params):
    ah_0 = params["ah_0"]
    ah_1 = params["ah_1"]
    return (Eh_2(ph,1-ph, params)*(ah_1-ah_0)+Eh_2(ph,1-ph, params)-Eh_1(ph, params))/ \
        (Eh_2(ph,1-ph, params)*(ah_1-ah_0)-ah_0*Eh_1(ph, params))
def S(y_1, y_2, params):
    return y_1*Eh_1(y_1,params)+y_2*Eh_2(y_1,y_2,params)
def Eh_1(y_1, params):
    ah_0 = params["ah_0"]
    return np.exp(-ah_0*y_1)
def Eh_2(y_1,y_2, params):
    ah_0 = params["ah_0"]
    ah_1 = params["ah_1"]
    Ph_2 = params["Ph_2"]
    return Ph_2*np.exp(-(ah_0*y_2+ah_1*y_1))
def S2(y, params):
    return y*Eh_1(y, params)+(1-y)*Eh_2(y,1-y, params)

# Estimated price
P2 = 0.5

# Fixed point alogrithm
def fixed_point(ah_0, ah_1, Ph_2, ph=0.3, tol=1e-9, n=int(1e5)):
    Ph_2 = P2
    svar_y = []
    svar_S = []
    print(ah_0, ah_1)
    params = {"ah_0": ah_0, "ah_1": ah_1, "Ph_2": Ph_2}
    for _ in range(n):
        Sval = S2(ph, params)
        if abs(dL(ph, params))<tol*ah_0:
            print('jippi')
            break
        ph=ph_next1(ph, params)
        svar_y.append(ph)
        svar_S.append(S2(ph, params))
    return svar_y, svar_S, ph, Sval

# Making labels nicer
def prettify(n):
    if n == 0:
        return "0"
```

```python
    else:
        return "n"

# Producing plots and Latex-tables
if __name__ == "__main__":
    tableline = "${alpha_0}$ & ${alpha_1}$ & {alpha_diff:0.4f} & {y_1:0.4f} & {y_2:0.4f} & {S:0.4f} \\\\ \n"
    a0list = [0.99]
    a1list = [0.99]
    latexstring = ""
    latexstringshort = ""
    color = ['orange', 'cyan' ,'purple', 'lightblue', 'yellow', 'green', 'grey']
    legends = []
    for a_1 in a1list:
        if a_1 != 0:
            latexstring += "\\midrule \n"
        a0_values = list(filter(lambda x: x > a_1, a0list))
        for index,a_0 in enumerate(a0_values):
            ah_0 = a_0
            ah_1 = a_1
            da = ah_0-ah_1
            svar_y, svar_S, ph, S = fixed_point(ah_0, ah_1, P2, ph=-20, n=int(1e5))
            y_1 = ph
            y_2 = 1-ph
            params = {"ah_0":ah_0, "ah_1":ah_1, "Ph_2":P2}
            Smin = S2(1,params)
            line = tableline.format(alpha_0=prettify(a_0),\
                 alpha_1=prettify(a_1), alpha_diff=ah_0-ah_1,\
                      y_1=y_1, y_2=y_2, S=S)
            latexstring += line
            if index == 0 or index == len(list(a0_values))-1:
                latexstringshort += line
# Making plots
            xmin = -0.5
            xmax = 3
            y=np.linspace(xmin,xmax,1000)
            plt.xlim(xmin,xmax)
            plt.ylim(0, S*1.15)
            f_1=S2(y, params)
            plt.plot(y,f_1,label="Total sales value as a quotient of initial portfolio value")
            plt.plot(svar_y,svar_S,'*',color=color[index], label='_nolegend_')
            plt.plot(svar_y[-1],svar_S[-1],f'*r')
            plt.xlabel("Volume of shares sold $y_1$")
            plt.ylabel("Sales value $S$")
            plt.title(f"$\\alpha_1 = {ah_1}$, $\\hatP_2 = {P2}$")
            legends += [f"$\\alpha_0 = {ah_0}$, $\\Delta\\alpha = {round(da,3)}$",\
            f"$S(y^*) = {round(svar_S[-1],3)}$, $y^* = {round(svar_y[-1],3)}$"]
            plt.legend(legends)
    plt.savefig(f"a1_{a_1}P2{P2}.png")
    plt.show()
    latexstring += "\\bottomrule"
    with open("a0_099_P2_05.tex", "w") as file:
        file.write(latexstring)
    with open("a0_099_P2_05short.tex", "w") as file:
        file.write(latexstringshort)
```

A.2.3 $\hat{\alpha}_0$, Gradient Ascent Method

This Python program runs the model S for $N = 2$ with the Gradient Ascent method. It prints all plots with a one $\hat{\alpha}_0$ and several $\hat{\alpha}_1$s in Chapter 5. The different plots are a result of different carefully chosen values for α_0 and $\hat{\alpha}_1$s, as well as $\hat{P}_2$. Short descriptions of each block in the program are written in the script. The script is used for the Bull type market.

```
from math import exp
from scipy import misc
from mpl_toolkits import mplot3d
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.pyplot as plt
from matplotlib import cm
import matplotlib as mpl
from matplotlib.ticker import LinearLocator, FormatStrFormatter
import random
import numpy as np

def dL(y, params):
    ah_0 = params["ah_0"]
    ah_1 = params["ah_1"]
    return Eh_1(y, params)*(1-y*ah_0)-(1-y)*ah_1*Eh_2(y,1-y,params)\
    -(Eh_2(y,1-y, params)-ah_0*(1-y)*Eh_2(y,1-y, params))
def ph_next1(ph, params):
    ah_0 = params["ah_0"]
    ah_1 = params["ah_1"]
    return (Eh_2(ph,1-ph, params)*(ah_1-ah_0)+Eh_2(ph,1-ph, params)-Eh_1(ph,
     params))/ \
        (Eh_2(ph,1-ph, params)*(ah_1-ah_0)-ah_0*Eh_1(ph, params))
def S(y_1, y_2, params):
    return y_1*Eh_1(y_1,params)+y_2*Eh_2(y_1,y_2,params)
def Eh_1(y_1, params):
    ah_0 = params["ah_0"]
    return np.exp(-ah_0*y_1)
def Eh_2(y_1,y_2, params):
    ah_0 = params["ah_0"]
    ah_1 = params["ah_1"]
    Ph_2 = params["Ph_2"]
    return Ph_2*np.exp(-(ah_0*y_2+ah_1*y_1))
def S2(y, params):
    return y*Eh_1(y, params)+(1-y)*Eh_2(y,1-y, params)

# Finding the derivative for the Gradient Ascent
def derivative(params, var=0, point=[]):
    args = point[:]
    def wraps(x):
        args[var] = x
        args[1] = 1-x
        return S(*args,params)
    return misc.derivative(wraps, point[var], dx = 1e-10)

def gradient_ascent(params):  # iteration
    lr = 0.1 # learning rate
    nb_max_iter = 1000 # Nb max iteration
    eps = 1e-9 # stop condition

    svar_y1 = []
    svar_y2 = []
    svar_S = []

    y_1 = -1 # start point
```

```python
    y_2 = -1
    z0 = S(y_1,y_2,params)
    svar_y1.append(y_1)
    svar_S.append(z0)

    cond = eps + 10.0 # start with cond greater than eps (assumption)
    nb_iter = 0
    tmp_z0 = z0
    while cond > eps and nb_iter < nb_max_iter:
        dydt = derivative(params, 0, [y_1,y_2])
        tmp_y1 = y_1 + lr * dydt
        y_1 = tmp_y1
        y_2 = 1-y_1
        z0 = S(y_1,y_2,params)
        svar_S.append(z0)
        nb_iter = nb_iter + 1
        cond = abs( tmp_z0 - z0 )
        tmp_z0 = z0
        svar_y1.append(y_1)
        svar_y2.append(y_2)
    return np.array(svar_y1), np.array(svar_y2), np.array(svar_S), y_1, y_2, z0

# Making 2D
if __name__ == "__main__":
    tableline = "${alpha_0}$ & ${alpha_1}$ & {alpha_diff:0.4f} & {y_1:0.4f} & {y_2:0.4f} & {S:0.4f} & {Smin:0.4f} & {diff:0.4f} \\\\ \n"
    a0list = [0.99]
    a1list = [0.98, 0.00001]
    latexstring = ""
    latexstringshort = ""

    color = ['orange', 'cyan' ,'purple', 'lightblue', 'yellow', 'green', 'grey']

# Estimated price
    P1 = 1
    P2 = 1.1

    color_index = 0
    color = ['orange', 'green', 'black',
             'purple', 'yellow', 'cyan', 'orange']
    mycmap = plt.get_cmap('gist_earth')
    legends=[]
    for index, ah_1 in enumerate(a1list):
        a0_values = list(filter(lambda x: x >= ah_1, a0list))

        if (len(a0_values) > 0):
            latexstring += "\\midrule \n"
        for ah_0 in a0_values:
            params = {"ah_0": ah_0, "ah_1": ah_1, "P1": P1, "Ph_2": P2}
            svar_y1, svar_y2, svar_S, y1val, y2val, Sval = gradient_ascent(params)
            da = ah_0 - ah_1
            xmin = -2
            xmax = 2
            y=np.linspace(xmin,xmax,1000)
            plt.xlim(xmin,xmax)
            f_1=S2(y, params)
            plt.ylim(0, np.max(f_1)*1.15)
            plt.plot(y,f_1,label="Total sales value as a quotient of initial portfolio value")
```

```
            plt.plot(svar_y1,svar_S,'*',color=color[index], label='_nolegend_')
            plt.plot(svar_y1[-1],svar_S[-1],f'*r')
            plt.xlabel("Volume of shares sold $y^*_1$")
            plt.ylabel("Sales value $S$")
            plt.title(f"$\\alpha_0 = {ah_0}$, $\\hatP_2 = {P2}$")
            Smin = S2(xmin,params)
            line = tableline.format(alpha_0=ah_0,\
                alpha_1=ah_1, alpha_diff=ah_0-ah_1,\
                    y_1=y1val, y_2=y2val, S=Sval, Smin=Smin, diff=Sval-Smin)
            latexstring += line
            legends += [f"$\\alpha_1 = {ah_1}$, $\\Delta\\alpha = {round(da,3)}$",\
            f"$S(y^*) = {round(svar_S[-1],3)}$, $y^* = {round(svar_y1[-1],3)}$"]
    plt.legend(legends)
    plt.savefig(f"ah_0{ah_0}ah_1{ah_1}P2{P2}.png")
    plt.show()
    latexstring += "\\bottomrule"
    with open(f"ah_0_{ah_0}_ah_1_{ah_1}_P2{P2}.tex", "w") as file:
        file.write(latexstring)
```

A.2.4 $\hat{\alpha}_1$, Gradient Ascent Method

This Python program runs the model S for $N = 2$ with the Gradient Ascent method. It prints all plots with a several $\hat{\alpha}_0$s and one $\hat{\alpha}_1$ in Chapter 5. The different plots are a result of different carefully chosen values for α_0s and $\hat{\alpha}_1$, as well as $\hat{P}_2$. Short descriptions of each block in the program are written in the script. The script is used for the Bull type market.

```
from math import exp
from scipy import misc
from mpl_toolkits import mplot3d
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.pyplot as plt
from matplotlib import cm
import matplotlib as mpl
from matplotlib.ticker import LinearLocator, FormatStrFormatter
import random
import numpy as np

def dL(y, params):
    ah_0 = params["ah_0"]
    ah_1 = params["ah_1"]
    return Eh_1(y, params)*(1-y*ah_0)-(1-y)*ah_1*Eh_2(y,1-y,params)\
    -(Eh_2(y,1-y, params)-ah_0*(1-y)*Eh_2(y,1-y, params))
def ph_next1(ph, params):
    ah_0 = params["ah_0"]
    ah_1 = params["ah_1"]
    return (Eh_2(ph,1-ph, params)*(ah_1-ah_0)+Eh_2(ph,1-ph, params)-Eh_1(ph, params))/ \
        (Eh_2(ph,1-ph, params)*(ah_1-ah_0)-ah_0*Eh_1(ph, params))
def S(y_1, y_2, params):
    return y_1*Eh_1(y_1,params)+y_2*Eh_2(y_1,y_2,params)
def Eh_1(y_1, params):
    ah_0 = params["ah_0"]
    return np.exp(-ah_0*y_1)
def Eh_2(y_1,y_2, params):
    ah_0 = params["ah_0"]
    ah_1 = params["ah_1"]
    Ph_2 = params["Ph_2"]
    return Ph_2*np.exp(-(ah_0*y_2+ah_1*y_1))
def S2(y, params):
    return y*Eh_1(y, params)+(1-y)*Eh_2(y,1-y, params)

# Finding the derivative for the Gradient Ascent
def derivative(params, var=0, point=[]):
    args = point[:]
    def wraps(x):
        args[var] = x
        args[1] = 1-x
        return S(*args,params)
    return misc.derivative(wraps, point[var], dx = 1e-10)

def gradient_ascent(params):  # iteration
    lr = 0.1 # learning rate
    nb_max_iter = 1000 # Nb max iteration
    eps = 1e-9 # stop condition

    svar_y1 = []
    svar_y2 = []
    svar_S = []

    y_1 = -1 # start point
```

```
    y_2 = -1
    z0 = S(y_1,y_2,params)
    svar_y1.append(y_1)
    svar_S.append(z0)

    cond = eps + 10.0 # start with cond greater than eps (assumption)
    nb_iter = 0
    tmp_z0 = z0
    while cond > eps and nb_iter < nb_max_iter:
        dydt = derivative(params, 0, [y_1,y_2])
        tmp_y1 = y_1 + lr * dydt
        y_1 = tmp_y1
        y_2 = 1-y_1
        z0 = S(y_1,y_2,params)
        svar_S.append(z0)
        nb_iter = nb_iter + 1
        cond = abs( tmp_z0 - z0 )
        tmp_z0 = z0
        svar_y1.append(y_1)
        svar_y2.append(y_2)
    return np.array(svar_y1), np.array(svar_y2), np.array(svar_S), y_1, y_2, z0

# Making 2D
if __name__ == "__main__":
    tableline = "${alpha_0}$ & ${alpha_1}$ & {alpha_diff:0.4f} & {y_1:0.4f} & {y_2:0.4f} & {S:0.4f} & {Smin:0.4f} & {diff:0.4f} \\\\ \n"
    a0list = [0.99, 0.497, 0.192, 0.115, 0.0471, 0.03]
    a1list = [0.00001]
    latexstring = ""
    latexstringshort = ""
    color = ['orange', 'cyan' ,'purple', 'lightblue', 'yellow', 'green', 'grey']

# Estimated price
    P1 = 1
    P2 = 1.5

    color_index = 0
    color = ['orange', 'green', 'black',
             'purple', 'yellow', 'cyan', 'orange']
    mycmap = plt.get_cmap('gist_earth')
    legends=[]
    for index, ah_0 in enumerate(a0list):
        a1_values = list(filter(lambda x: x < ah_0, a1list))

        if (len(a1_values) > 0):
            latexstring += "\\midrule \n"
        for ah_1 in a1_values:
            params = {"ah_0": ah_0, "ah_1": ah_1, "P1": P1, "Ph_2": P2}
            svar_y1, svar_y2, svar_S, y1val, y2val, Sval = gradient_ascent(params)
            da = ah_0 - ah_1
            xmin = -9
            xmax = 4
            y=np.linspace(xmin,xmax,1000)
            plt.xlim(xmin,xmax)
            f_1=S2(y, params)
            plt.ylim(0, np.max(f_1)*1.15)
            plt.plot(y,f_1,label="Total sales value as a quotient of initial portfolio value")
            plt.plot(svar_y1,svar_S,'*',color=color[index], label='_nolegend_'
```

```
)
            plt.plot(svar_y1[-1],svar_S[-1],f'*r')
            plt.xlabel("Volume of shares sold $y^*_1$")
            plt.ylabel("Sales value $S$")
            plt.title(f"$\\alpha_1 = {ah_1}$, $\\hatP_2 = {P2}$")
            Smin = S2(xmin,params)
            line = tableline.format(alpha_0=ah_0,\
                 alpha_1=ah_1, alpha_diff=ah_0-ah_1,\
                      y_1=y1val, y_2=y2val, S=Sval, Smin=Smin, diff=Sval-Smin)
            latexstring += line
            legends += [f"$\\alpha_0 = {ah_0}$, $\\Delta\\alpha = {round(da,3)}$",\
            f"$S(y^*) = {round(svar_S[-1],3)}$, $y^* = {round(svar_y1[-1],3)}$"]
    plt.legend(legends)
    plt.savefig(f"ah_1{ah_1}ah_0{ah_0}P2{P2}.png")
    plt.show()
    latexstring += "\\bottomrule"
    with open(f"ah_1_{ah_1}_ah_0_{ah_0}_P2{P2}.tex", "w") as file:
        file.write(latexstring)
```

A.3 The case $N = 3$

This Python program runs the model S for $N = 3$ with the Gradient Ascent method. It prints all plots with a several $\hat{\alpha}_0$s and several $\hat{\alpha}_1$s in Chapter 6. The different plots are a result of different carefully chosen values for α_0, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ as well as $\hat{P}_1$, $\hat{P}_2$ and $\hat{P}_3$. Short descriptions of each block in the program are written in the script. The script is used for all market types.

```
from math import exp
from scipy import misc
from mpl_toolkits import mplot3d
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.pyplot as plt
from matplotlib import cm
import matplotlib as mpl
from matplotlib.ticker import LinearLocator, FormatStrFormatter
import random
import numpy as np

# Defining the funcitons
def S(y1,y2,params):
    a0 = params["a0"]
    a1 = params["a1"]
    a2 = params["a2"]
    P1 = params["P1"]
    P2 = params["P2"]
    P3 = params["P3"]
    A = y1*P1*np.exp(-a0*y1)
    B = y2*P2*np.exp(-a0*y2-a1*y1)
    C = (1-y1-y2)*P3*np.exp((a0-a1)*y2+(a0-a2)*y1-a0)
    S = A+B+C
    return S

# Finding the partial derivatives for the Gradient Ascent
def partial_derivative(params, var=0, point=[]):
    args = point[:]
    def wraps(x):
        args[var] = x
        return S(*args,params)
    return misc.derivative(wraps, point[var], dx = 1e-10)

def gradient_ascent(params):  # iteration
    alpha = 0.1 # learning rate
    nb_max_iter = 1000 # Nb max iteration
    eps = 1e-10 # stop condition

    svar_y1 = []
    svar_y2 = []
    svar_S = []

    y1_0 = 0 # start point
    y2_0 = 0
    z0 = S(y1_0,y2_0,params)
    svar_y1.append(y1_0)
    svar_y2.append(y2_0)
    svar_S.append(z0)

    cond = eps + 10.0 # start with cond greater than eps (assumption)
    nb_iter = 0
    tmp_z0 = z0
    while cond > eps and nb_iter < nb_max_iter:
```

```
        tmp_y1_0 = y1_0 + alpha * partial_derivative(params, 0, [y1_0,y2_0])
        tmp_y2_0 = y2_0 + alpha * partial_derivative(params, 1, [y1_0,y2_0])
        y1_0 = tmp_y1_0
        y2_0 = tmp_y2_0
        y3_0 = 1.0 - y1_0 - y2_0
        z0 = S(y1_0,y2_0,params)
        svar_S.append(z0)
        nb_iter = nb_iter + 1
        cond = abs( tmp_z0 - z0 )
        tmp_z0 = z0
        svar_y1.append(y1_0)
        svar_y2.append(y2_0)
    return np.array(svar_y1), np.array(svar_y2), np.array(svar_S), y1_0, y2_0, z0

# Making 2D and 3D plots
if __name__ == "__main__":
# sentiment
    a0list = [0.9]
    a1list = [0.5]
    a2list = [0.1]
# Estimated prices
    P1 = 1
    P2 = 10
    P3 = 4
    color_index = 0
    color = ['orange', 'green', 'black',
             'purple', 'yellow', 'cyan', 'orange']
    mycmap = plt.get_cmap('gist_earth')
    for index, a2 in enumerate(a2list):
        a1_values = list(filter(lambda x: x >= a2, a1list))
        for a1 in a1_values:
            legends=[]
            fig = plt.figure()
            fig2 = plt.figure()
            ax = fig.add_subplot(111, projection='3d')
            ax.set_xlabel('$y_1$')
            ax.set_ylabel('$y_2$')
            ax.set_zlabel('$S$')
            ax2 = fig2.add_subplot()
            ax2.set_xlabel('$y_1$')
            ax2.set_ylabel('$y_2$')
            a0_values = list(filter(lambda x: x >= a1, a0list))
            y3 = 0
            y1 = np.arange(-2.5, 1.5, 0.01)
            y2 = np.arange(-0.25, 2.75, 0.01)
            for a0 in a0_values:
                params = {"a0": a0, "a1": a1, "a2": a2, "P1": P1, "P2": P2, "P3": P3}
                svar_y1, svar_y2, svar_S, y1val, y2val, Sval = gradient_ascent(params)
                print(params)
                X, Y = np.meshgrid(y1, y2)
                Z = S(X, Y, params)
                my_col = cm.jet(Z/np.amax(Z))
                # In parallell
                h = ax2.contourf(X,Y,Z,cmap='jet')
                this_color = color[color_index % 2]
                other_color = color[2+(color_index % 2)]
                ax2.scatter(svar_y1, svar_y2, s=25,  label='_nolegend_',color='white')
```

```
            ax2.scatter(y1val, y2val, s=50,color='black')
            surf = ax.plot_surface(X, Y, Z,cmap=mycmap,rstride=1, cstride=1, facecolors = my_col,
               linewidth=0, antialiased=False)
            surf._edgecolors2d = surf._edgecolor3d
            surf._facecolors2d = surf._facecolor3d

            legends += [f"$\\hat \\alpha_0 = {a0}$, $S(y_1^*,y_2^*) = {round(svar_S[-1],4)}$",
                       f"$y_1^* = {round(svar_y1[-1],4)}, y_2^* = {round(svar_y2[-1],4)}$"]
            color_index+=1
            y3 =     1 - y1val - y2val
        title = f"$\\hat P_1 = {P1}$, $\\hat P_2 = {P2}$, $\\hat P_3 = {P3}$, $\\hat \\alpha_1 = {a1}, \\hat \\alpha_2 = {a2}, y_3^* = {round(y3,4)}$"
        window_title = f"P2 = {P2}, P3 = {P3}, a1 = {a1}, a2 = {a2}"
        ax.set_title(title)
        ax2.set_title(title)
        figure_name = f"{P2}-{P3}-{a1}-{a2}.png"
        figure_name_contour = f"{P2}-{P3}-{a1}-{a2}-contour.png"
        fig.canvas.manager.set_window_title(window_title)
        fig2.canvas.manager.set_window_title(window_title)

        ax.legend(legends)
        ax2.legend(legends)
        fig.savefig(figure_name)
        fig2.savefig(figure_name_contour)
plt.show()
```

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