ON ROBUST THEOREMS DUE TO BOLZANO, WEIERSTRASS, JORDAN, AND CANTOR

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Abstract. Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics where the aim is to identify the *minimal* axioms needed to prove a given theorem from ordinary, i.e. non-set theoretic, mathematics. This program has unveiled surprising regularities: the minimal axioms are very often *equivalent* to the theorem over the *base theory*, a weak system of 'computable mathematics', while most theorems are either provable in this base theory, or equivalent to one of only *four* logical systems. The latter plus the base theory are called the 'Big Five' and the associated equivalences are *robust* following Montalbán, i.e. stable under small variations of the theorems at hand. Working in Kohlenbach's *higher-order* RM, we obtain two new and long series of equivalences based on theorems due to Bolzano, Weierstrass, Jordan, and Cantor; these equivalences are extremely robust and have no counterpart among the Big Five systems. Thus, higher-order RM is much richer than its second-order cousin, boasting at least two extra 'Big' systems.

1. Introduction

1.1. **Motivation and caveat.** Like Hilbert ([34]), we believe the infinite to be a central object of study in mathematics. That the infinite comes in 'different sizes' is a relatively new insight, due to Cantor around 1874 ([9]), in the guise of the *uncountability of the real numbers*, also known simply as *Cantor's theorem*.

With the notion 'countable versus uncountable' in place, it is an empirical observation, witnessed by many textbooks, that to show that a set is countable one often constructs an injection (or bijection) to $\mathbb{N}$. When *given* a countable set, one (additionally) assumes that this set can be *enumerated*, i.e. represented by some sequence. In this light, implicit in much of mathematical practise is the following most basic principle about countable sets:

a set that can be mapped to $\mathbb{N}$ *via an injection* (*or bijection*) *can be enumerated.*

This principle was studied in [73, 86, 88] as part of the study of the uncountability of $\mathbb{R}$. In this paper, we continue the study of this principle in *Reverse Mathematics* (RM hereafter) and connect it to well-known 'household name' theorems due to

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Attributions. The direct contributions of the first author are mostly in Section 2, esp. the intricate proofs. The results in Section 3 are however partially-but-essentialy based on conceptual ideas due to the first author, the structure functional Ω pioneered in [75] in particular. The second author made partial contributions to both sections.

Bolzano-Weierstrass, Cantor, Jordan, and Heine-Borel, as discussed in detail in Section 1.2. We assume basic familiarity with RM, also sketched in Section 1.3.1. In particular, working in Kohlenbach's *higher-order* RM, we obtain two new long series of extremely robust equivalences involving the aforementioned theorems. In this concrete way, third-order arithmetic is *much* richer than its second-order cousin in that the former boasts (at least) two extra 'Big' systems[1] compared to the latter.

For all the aforementioned reasons, our results provide new answers to one of the driving questions behind RM, formulated as follows by Montalbán.

> The way I view it, gaining a greater understanding of [the big five] phenomenon is currently one of the driving questions behind reverse mathematics. To study [this] phenomenon, one distinction that I think is worth making is the one between robust systems and non-robust systems. A system is *robust* if it is equivalent to small perturbations of itself. This is not a precise notion yet, but we can still recognize some robust systems. All the big five systems are very robust. For example, most theorems about ordinals, stated in different possible ways, are all equivalent to each other and to ATR_0. Apart from those systems, weak weak Kőnig's Lemma WWKL_0 is also robust, and we know no more than one or two other systems that may be robust. ([59, p. 432], emphasis in original)

Finally, the uncountability of $\mathbb{R}$ deals with arbitrary mappings with domain $\mathbb{R}$ and is therefore best studied in a language that has such objects as first-class citizens. Obviousness, much more than beauty, is however in the eye of the beholder. Lest we be misunderstood, we formulate a blanket caveat: all notions (computation, continuity, function, open set, et cetera) used in this paper are to be interpreted via their higher-order definitions, also listed below, *unless explicitly stated otherwise.*

1.2. **From Bolzano-Weierstrass to Heine-Borel and Jordan.** In this section, we provide an overview of our results; in a nutshell, we obtain a large number of robust equivalences involving the *Bolzano-Weierstrass theorem* for countable sets and many theorems concerned with countable sets and related notions. We also obtain equivalences for theorems that *do not* involve countable sets in any obvious or direct way at all, namely the *Jordan decomposition theorem* and similar results on functions of bounded variation and related notions.

First of all, the *Bolzano-Weierstrass theorem* comes in different formulations. Weierstrass formulates this theorem around 1860 in [105, p. 77] as follows, while Bolzano [81, p. 174] states the existence of suprema rather than just limit points.

> If a function has a definite property infinitely often within a finite domain, then there is a point such that in any neighbourhood of this point there are infinitely many points with the property.

We start by studying the Bolzano-Weierstrass theorem for countable sets as in Principle 1.1. Precise definitions of all notions involved can be found in Section 1.3.2 while motivation for our choice of definitions is provided in Section 3.3.3.

Principle 1.1 (BWC)**.** *For a countable set $A \subset 2^{\mathbb{N}}$, the supremum* $\sup A$ *exists.*

[1]A logical system is called 'Big' if it boasts many equivalences involving robust principles.

Unless explicitly stated otherwise, the supremum is taken relative to the lexicographic ordering. A number of variations BWC_i^j of Principle 1.1 are possible, which we shall express via the indicated super- and sub-scripts as follows.

- For $i = 0$, *countable sets* are defined via **injections** to $\mathbb{N}$ (Definition 1.4).
- For $i = 1$, we restrict to *strongly countable sets*, which are defined via **bijections** to $\mathbb{N}$ (Definition 1.4).
- For j including seq, we additionally have that a **sequence** $(f_n)_{n\in\mathbb{N}}$ in A is given with $\lim_{n\to\infty} f_n = \sup A$.
- For j including fun, we additionally have that $\sup_{f\in A} F(f)$ exists for aribitrary **functionals** $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.
- For j including pwo, the supremum is relative to the pointwise[2] ordering.

Since Cantor space with the lexicographic ordering and $[0, 1]$ with its usual ordering are intimately connected, we take the former ordering to be fundamental. We have shown in [73] that $\mathsf{BWC}_0^{\mathsf{fun}}$ is 'explosive' in that it yields the much stronger $\Pi^1_2\text{-}\mathsf{CA}_0$ when combined with the Suslin functional, i.e. higher-order $\Pi^1_1\text{-}\mathsf{CA}_0$. Previously, metrisation theorems from topology were needed to reach $\Pi^1_2\text{-}\mathsf{CA}_0$ via $\Pi^1_1\text{-}\mathsf{CA}_0$ ([62–64]), while Rathjen states in [76] that $\Pi^1_2\text{-}\mathsf{CA}_0$ *dwarfs* $\Pi^1_1\text{-}\mathsf{CA}_0$ and Martin-Löf talks of a *chasm* and *abyss* between these two systems in [56]. Analogous results hold at the level of computability theory, in the sense of Kleene's S1-S9 ([45]), while we even obtain $\exists^3$, and hence full second-order arithmetic, if we assume $\mathsf{V{=}L}$, by [73, Theorem 4.6]. Thus, the following natural questions arise.

(Q0) Is the 'extra information' as in 'fun' or 'seq' necessary for explosions?
(Q1) Is it possible to 'split' e.g. BWC_0 in 'less explosive' components?
(Q2) Since BWC_0 is formulated using injections, is there an equivalent formulation only based on *bijections*?
(Q3) Is the explosive nature of BWC_0 caused by the use of injections or bijections?
(Q4) Are there equivalences involving BWC_0 from ordinary mathematics, especially involving theorems not related to countability in any obvious way?

Secondly, to answer (Q0), we connect BWC_0 to the other variations BWC_i^j, as part of Kohlenbach's *higher-order Reverse Mathematics*, briefly introduced in Section 1.3.1. We assume basic familiarity with Reverse Mathematics (RM hereafter), to which [99] provides an introduction. We establish the series of equivalences in (EQ) in Section 2.2, where IND_i are fragments of the induction axiom.

$$\begin{gathered}\mathsf{BWC}_0^{\mathsf{fun}} \leftrightarrow \mathsf{BWC}_0^{\mathsf{seq}} \leftrightarrow \mathsf{BWC}_0^{\mathsf{pwo}} \leftrightarrow [\mathsf{BWC}_0 + \mathsf{IND}_0] \leftrightarrow \mathsf{BWC}_0^{\mathsf{fun,pwo}}. \\ \mathsf{cocode}_1 \leftrightarrow \Delta\text{-}\mathsf{CA}_C^- \leftrightarrow \mathsf{BW}_1^{\mathsf{seq}} \leftrightarrow [\mathsf{BW}_1 + \mathsf{IND}_1] \leftrightarrow \mathsf{BW}_1^{\mathsf{pwo}}.\end{gathered} \tag{EQ}$$

Here, $\Delta\text{-}\mathsf{CA}_C^-$ is a peculiar axiom inspired by Δ^0_1-comprehension while cocode_1 expresses that *strongly countable* sets, i.e. boasting bijections to $\mathbb{N}$, can be enumerated. We point out that $\mathsf{BWC}_0 \leftrightarrow \mathsf{BWC}_0^{\mathsf{seq}}$ is interesting as follows: to obtain the extra sequence in the latter, the only method[3] seems to use countable choice, while the equivalence is provable *without* the latter. Thus, the extra sequence from $\mathsf{BWC}_0^{\mathsf{seq}}$, while seemingly a choice function, can be defined explicitly in terms of the other

[2]The pointwise ordering '$\leq_1$' is defined as $f \leq_1 g \equiv (\forall n \in \mathbb{N})(f(n) \leq_{\mathbb{N}} g(n))$ for any $f, g \in \mathbb{N}^{\mathbb{N}}$. The sequence $\sup A$ is the supremum of $A \subset 2^{\mathbb{N}}$ for this ordering if $(\forall f \in A)(f \leq_1 \sup A)$ and $(\forall k \in \mathbb{N})((\sup A)(k) = 1 \to (\exists f \in A)(f(k) = 1))$.

[3]Apply countable choice to $(\forall n \in \mathbb{N})(\exists f \in A)(d(f, \sup A) < \frac{1}{2^n})$ which holds by definition.

data, i.e. without the Axiom of Choice. By Remark 2.8, the second line of (EQ) is connected to *hyperarithmetical analysis*.

Thirdly, in answer to (Q3), the principles from (EQ) are formulated using injections and bijections to $\mathbb{N}$, while items (a)-(c) below are basic theorems about the real line $\mathbb{R}$ based on *enumerable* sets, i.e. listed by (possibly infinite) sequences, which is essentially the notion of countable set used in second-order RM:

(a) accu: a non-enumerable **closed** set in $\mathbb{R}$ has a limit point,
(b) accu': a non-enumerable set in $\mathbb{R}$ contains a limit point,
(c) ccc: a collection of disjoint open intervals in $\mathbb{R}$ is enumerable.
(d) cloq: a countable linear ordering is order-isomorphic to a subset of $\mathbb{Q}$.

Closed sets are defined as in Definition 1.2, which generalises the second-order notion ([97, II.5.6]). The principles ccc_i and accu_i for $i = 0, 1$ are defined as for BWC_i above. We establish the following series of implications in Section 2.4.

$$\mathsf{accu} \leftrightarrow \mathsf{accu}' \leftrightarrow \mathsf{ccc} \leftrightarrow \mathsf{BW}_0 \leftrightarrow [\mathsf{CB}\mathbb{N} + \mathsf{BW}_1] \leftrightarrow [\mathsf{CWO}^\omega + \mathsf{IND}_0]. \tag{EQ2}$$

$$\mathsf{ccc}_1 \leftrightarrow \mathsf{CB}\mathbb{N} \leftrightarrow \mathsf{accu}_1, \text{ and } \mathsf{cocode}_0 \leftrightarrow [\mathsf{cloq} + \mathsf{IND}_0] \leftrightarrow [\mathsf{cloq}' + \mathsf{IND}_0].$$

Here, $\mathsf{CB}\mathbb{N}$ is the *Cantor-Bernstein theorem* for $\mathbb{N}$ as in Principle 2.14, which is *independent* of BWC_1 by Theorem 2.16, thus answering (Q2). The principle CWO^ω expresses that countable well-orderings are comparable, while cloq' is Cantor's theorem characterising the order type η of $\mathbb{Q}$. The notion of *limit point* goes back to Cantor ([14, p. 98]) in 1872; he also proved the first instance of the *countable chain condition* ccc in [14, p. 161] and introduced order types, including η, in [12, 13].

Fourth, following (Q4), we also study BWC_0 and BWC_1 in the grand(er) scheme of things, namely how they connect to set theory and ordinary mathematics. In Section 3.2, we obtain equivalences between BWC_0 and BWC_1, and fragments of the well-known *countable union theorem* from set theory (see e.g. [32, §3.1]). As to ordinary mathematics, in Section 3.1, we establish equivalences between BWC_0 and versions of the *Lindelöf lemma* and *Heine-Borel theorem* as studied in [68, 73]. In Section 3.3, we establish equivalences between BWC_0, the *Jordan decomposition theorem*, and related results from [74, 75]. The latter theorem and its ilk have no obvious or direct connection to countability *at all*.

Finally, we discuss how these results provide detailed answers to (Q0)-(Q4) in the below sections. In light of all the aforementioned equivalences, we believe the following quote by Friedman to be apt:

> When a theorem is proved from the right axioms, the axioms can be proved from the theorem. ([26])

Next, Section 1.3 details the definitions used in this paper while a neat motivation for our choice of definitions is provided in Section 3.3.3, with the gift of hindsight.

1.3. **Preliminaries and definitions.** We briefly introduce *Reverse Mathematics* and *higher-order computability theory* in Section 1.3.1. We introduce some essential definitions in Section 1.3.2. A full introduction may be found in e.g. [73, §2]. In Section 3.3.3, we motivate our choice of definitions, Definition 1.2 in particular.

1.3.1. *Reverse Mathematics and higher-order computability theory.* Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics initiated around 1975 by Friedman ([26, 27]) and developed extensively by Simpson ([97]).

The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics.

We refer to [99] for a basic introduction to RM and to [96, 97] for an overview of RM. We expect basic familiarity with RM, in particular Kohlenbach's *higher-order* RM ([47]) essential to this paper, including the base theory RCA_0^ω. An extensive introduction can be found in e.g. [68, 71, 73]. We have chosen to include a brief introduction as a technical appendix, namely Section A. All undefined notions may be found in the latter.

Next, some of our main results will be proved using techniques from computability theory. Thus, we first make our notion of 'computability' precise as follows.

(I) We adopt ZFC, i.e. Zermelo-Fraenkel set theory with the Axiom of Choice, as the official metatheory for all results, unless explicitly stated otherwise.
(II) We adopt Kleene's notion of *higher-order computation* as given by his nine clauses S1-S9 (see [54, Ch. 5] or [45]) as our official notion of 'computable'.

We refer to [54] for a thorough overview of higher-order computability theory.

1.3.2. *Some definitions in higher-order arithmetic.* We introduce the standard definitions for countable set and related notions.

First of all, the main topic of [73] is the logical and computational properties of *the uncountability of* $\mathbb{R}$, established in 1874 by Cantor in his *first* set theory paper [9], in the guise of the following natural principles:

- NIN: *there is no injection from* $[0, 1]$ *to* $\mathbb{N}$,
- NBI: *there is no bijection from* $[0, 1]$ *to* $\mathbb{N}$.

As it happens, NIN and NBI are among the weakest principles that require a lot of comprehension for a proof. An overview may be found in [73, Figure 1].

Secondly, we shall make use of the following notion of (open) set, which was studied in detail in [70, 73, 88]. We motivate this choice in detail in Section 3.3.3.

Definition 1.2. [Sets in RCA_0^ω] We let $Y : \mathbb{R} \to \mathbb{R}$ represent subsets of $\mathbb{R}$ as follows: we write '$x \in Y$' for '$Y(x) >_{\mathbb{R}} 0$' and call a set $Y \subseteq \mathbb{R}$ 'open' if for every $x \in Y$, there is an open ball $B(x, r) \subset Y$ with $r^0 > 0$. A set Y is called 'closed' if the complement, denoted $Y^c = \{x \in \mathbb{R} : x \notin Y\}$, is open.

Note that for open Y as in the previous definition, the formula '$x \in Y$' has the same complexity (modulo higher types) as in second-order RM (see [97, II.5.6]), while given $(\exists^2)$ from Section A.1.4 the former becomes a 'proper' characteristic function, only taking values '0' and '1'. Hereafter, an '(open) set' refers to Definition 1.2, while 'RM-open set' refers to the second-order definition from RM.

The attentive reader has of course noted that e.g. the unit interval is only a set in the sense of Definition 1.2 in case we assume $\mathsf{ACA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\exists^2)$. For this reason, we shall adopt the latter as our base theory in our paper. We discuss how the reader may obtain equivalences over RCA_0^ω in Remark 2.1.

Thirdly, the notion of 'countable set' can be formalised in various ways, namely via Definitions 1.3 and 1.4.

Definition 1.3. [Enumerable sets of reals] A set $A \subset \mathbb{R}$ is *enumerable* if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $(\forall x \in \mathbb{R})(x \in A \to (\exists n \in \mathbb{N})(x =_{\mathbb{R}} x_n))$.

This definition reflects the RM-notion of 'countable set' from [97, V.4.2]. We note that given μ^2 from Section A.1.4, we may replace the final implication in Definition 1.3 by an equivalence.

The usual definition of 'countable set' is as follows in RCA_0^ω.

Definition 1.4. [Countable subset of $\mathbb{R}$] A set $A \subset \mathbb{R}$ is *countable* if there exists $Y : \mathbb{R} \to \mathbb{N}$ such that $(\forall x, y \in A)(Y(x) =_0 Y(y) \to x =_{\mathbb{R}} y)$. If $Y : \mathbb{R} \to \mathbb{N}$ is also *surjective*, i.e. $(\forall n \in \mathbb{N})(\exists x \in A)(Y(x) = n)$, we call A *strongly countable*.

The first part of Definition 1.4 is from Kunen's set theory textbook ([50, p. 63]) and the second part is taken from Hrbacek-Jech's set theory textbook [38] (where the term 'countable' is used instead of 'strongly countable'). For the rest of this paper, 'strongly countable' and 'countable' shall exclusively refer to Definition 1.4, *except when explicitly stated otherwise*.

Finally, we shall use the following definition of finite and infinite set.

Definition 1.5. [Finite and infinite sets] A set $A \subset \mathbb{R}$ is called *infinite* if

$$(\forall n \in \mathbb{N})(\exists w^{1^*})[|w| \geq n \wedge (\forall i, j < |w|)(w(i), w(j) \in A \wedge (i \neq j \to w(i) \neq w(j)))],$$

i.e. there are arbitrary long finite sequences of distinct elements in A. A set $A \subset \mathbb{R}$ is *finite* if it is not infinite.

The exact definition of (in)finite set plays a *minor* role in most of this paper, but a *major* role in the study of the Jordan decomposition theorem and related topics in Section 3.3. This observation is explained at length in Remark 3.31. In a nutshell, the notion of finite set as in Definition 1.5 is suitable for the RM-study of functions of bounded variation, whereas the 'usual' definitions of finite set, involving injections or bijections to $\mathbb{N}$, are not.

1.3.3. *Some axioms of higher-order arithmetic.* We introduce a number of axioms of higher-order arithmetic, including the 'higher-order counterparts' of WKL_0 and ACA_0. We motivate the latter term in detail based on Remark 1.12.

First of all, with Definitions 1.2 and 1.4 in place, the following principle has interesting properties, as studied in [73, 86, 88].

Principle 1.6 (cocode_0). *For any non-empty countable set $A \subseteq [0, 1]$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $(\forall x \in \mathbb{R})(x \in A \leftrightarrow (\exists n \in \mathbb{N})(x_n =_{\mathbb{R}} x))$.*

Indeed, as explored in [73, 88], we have $\mathsf{cocode}_0 \leftrightarrow \mathsf{BWC}_0^{\mathsf{fun}}$ over ACA_0^ω, while another interesting equivalence is based on the 'projection' axiom studied in [83]:

$$(\forall Y^2)(\exists X \subset \mathbb{N})\big(\forall n \in \mathbb{N})(n \in X \leftrightarrow (\exists f \in \mathbb{N}^{\mathbb{N}})(Y(f, n) = 0)\big). \tag{BOOT}$$

We mention that BOOT is equivalent to e.g. the monotone convergence theorem for nets indexed by Baire space (see [83, §3]), while it is essentially Feferman's $(\mathsf{Proj1})$ from [24] without set parameters. The axiom BOOT^- results from restricting BOOT to functionals Y with the following 'at most one' condition:

$$(\forall n \in \mathbb{N})(\exists \text{ at most one } f \in \mathbb{N}^{\mathbb{N}})(Y(f, n) = 0), \tag{1.1}$$

where similar constructs appear in the RM of ATR_0 by [97, V.5.2]. The weaker BOOT^- appears prominently in the RM-study of open sets given as (third-order) characteristic functions ([70]). In turn, BOOT_C^- is BOOT^- with '$\mathbb{N}^{\mathbb{N}}$' replaced by '$2^{\mathbb{N}}$' everywhere; BOOT_C^- was introduced in [73, §3.1] in the study of $\mathsf{BWC}_0^{\mathsf{fun}}$, and we have $\mathsf{BOOT}_C^- \leftrightarrow \mathsf{BWC}_0^{\mathsf{fun}}$ over RCA_0^ω by [88, Theorem 3.12]. In light of [97, V.5.2], $\mathsf{ACA}_0^\omega + \mathsf{BOOT}_C^-$ proves ATR_0.

Secondly, related to BOOT_C^- and cocode_0 is the following principle.

Principle 1.7 (range_0)**.** *For $Y : 2^{\mathbb{N}} \to \mathbb{N}$ an injection on $A \subset 2^{\mathbb{N}}$, we have*

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})\big[(\exists f \in A)(Y(f) = n) \leftrightarrow n \in X\big],$$

i.e. the range of Y restricted to A exists.

With the gift of hindsight[4] from [73, 86, 88], we see that cocode_0 is equivalent to:

$$\textit{a linear order } (A, \preceq_A) \textit{ for countable } A \subset \mathbb{R} \textit{ can be enumerated.} \tag{1.2}$$

In second-order RM, countable linear orders are represented by sequences (see [97, V.1.1]), i.e. the previous principle seems essential if one wants to interpret theorems about countable linear orders in higher-order arithmetic or set theory. Another useful fragment of BOOT is Δ-CA, which is central to 'lifting' second-order reversals to higher-order arithmetic (see [85, 87]).

Principle 1.8 (Δ-CA)**.** *For $i = 0, 1$, Y_i^2, and $A_i(n) \equiv (\exists f \in \mathbb{N}^{\mathbb{N}})(Y_i(f, n) = 0)$:*

$$(\forall n \in \mathbb{N})(A_0(n) \leftrightarrow \neg A_1(n)) \to (\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow A_0(n)).$$

This principle borrows its name from the fact that the ECF-translation (see Remark 1.12) converts Δ-CA into Δ_1^0-comprehension. As will become clear below, Δ-CA with the 'at most one' condition (1.1) plays an important role in the RM of the Bolzano-Weierstrass theorem.

Thirdly, the Heine-Borel theorem states the existence of a finite sub-covering for an open covering of certain spaces. Now, a functional $\Psi : \mathbb{R} \to \mathbb{R}^+$ gives rise to the *canonical covering* $\cup_{x \in I} I_x^\Psi$ for $I \equiv [0, 1]$, where I_x^Ψ is the open interval $(x - \Psi(x), x + \Psi(x))$. Hence, the uncountable covering $\cup_{x \in I} I_x^\Psi$ has a finite sub-covering by the Heine-Borel theorem, which yields the following principle.

Principle 1.9 (HBU)**.** $(\forall \Psi : \mathbb{R} \to \mathbb{R}^+)(\exists y_0, \dots, y_k \in I)(\forall x \in I)(\exists i \leq k)(x \in I_{y_i}^\Psi)$.

Note that HBU is essentially *Cousin's lemma* (see [18, p. 22]), i.e. the Heine-Borel theorem for canonical coverings. By [68, 71], Z_2^Ω proves HBU, but $\mathsf{Z}_2^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,1}$ cannot. Basic properties of the *gauge integral* ([61, 100]) are equivalent to HBU. By [68, Theorem 3.3], HBU is equivalent to the same compactness property for $2^{\mathbb{N}}$.

Principle 1.10 (HBU_c)**.** $(\forall G^2)(\exists f_1, \dots, f_k \in 2^{\mathbb{N}})(\forall f \in 2^{\mathbb{N}})(\exists i \leq k)(f \in [\overline{f_i}G(f_i)])$.

As studied in [84, §3.1], canonical coverings as in HBU are not suitable for the study of basic topological notions like paracompactness and dimension. This suggests the need for a more general notion of covering; the solution adopted in [84] it to allow $\psi : I \to \mathbb{R}$, i.e. I_x^ψ is empty in case $\psi(x) \leq 0$. In this way, we say that '$\cup_{x \in I} I_x^\psi$ covers $[0, 1]$' if $(\forall x \in [0, 1])(\exists y \in [0, 1])(x \in I_y^\psi)$. Thus, we obtain the Heine-Borel theorem as in HBT, going back to Lebesgue in 1898 (see [52, p. 133]).

Principle 1.11 (HBT)**.** *For $\psi : [0, 1] \to \mathbb{R}$, , if $\cup_{x \in I} I_x^\psi$ covers $[0, 1]$, then there are $y_1, \dots, y_k \in [0, 1]$ such that $\cup_{i \leq k} I_{y_i}^\psi$ covers $[0, 1]$.*

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[4] A countable $A \subset \mathbb{R}$ yields a linear order via $x \preceq y \equiv Y(x) \leq Y(y)$, where Y is injective on A.

As shown in [84, §3], we have $\mathsf{HBU} \leftrightarrow \mathsf{HBT}$ over various natural base theories, some of which we shall discuss and use in Section 3.1.4.

Finally, as discussed in detail in [47, §2], the base theories RCA_0^ω and RCA_0 prove the same L_2-sentences 'up to language' as the latter is set-based (the L_2-language) and the former function-based (the L_ω-language). Here, L_2 is the language of second-order arithmetic, while L_ω is the language of all finite types. This conservation result is obtained via the so-called ECF-interpretation, discussed next.

Remark 1.12 (The ECF-interpretation)**.** The (rather) technical definition of ECF may be found in [102, p. 138, §2.6]. Intuitively, the ECF-interpretation $[A]_{\mathsf{ECF}}$ of a formula $A \in \mathsf{L}_\omega$ is just A with all variables of type two and higher replaced by type one variables ranging over so-called 'associates' or 'RM-codes' (see [46, §4]); the latter are (countable) representations of continuous functionals. The ECF-interpretation connects RCA_0^ω and RCA_0 (see [47, Prop. 3.1]) in that if RCA_0^ω proves A, then RCA_0 proves $[A]_{\mathsf{ECF}}$, again 'up to language', as RCA_0 is formulated using sets, and $[A]_{\mathsf{ECF}}$ is formulated using types, i.e. using type zero and one objects.

In light of the widespread use of codes in RM and the common practise of identifying codes with the objects being coded, it is no exaggeration to refer to ECF as the *canonical* embedding of higher-order into second-order arithmetic. Moreover, $\mathsf{RCA}_0^\omega + \mathsf{BOOT}$ is called the 'higher-order counterpart' of ACA_0 as the former is a conservative extension of the latter, and ECF maps BOOT to ACA_0. Similarly, $\mathsf{RCA}_0^\omega + \mathsf{HBT}$ is the 'higher-order counterpart' of WKL_0.

As a neat application of the ECF-interpretation, Remark 3.28 establishes that the Jordan decomposition theorem (see Section 3.3.1) does not imply $(\exists^2)$, although the former theorem applies to discontinuous functions.

2. Equivalences for the Bolzano-Weierstrass theorem

2.1. **Introduction.**

We establish the results sketched in Section 1.2 and (EQ).

In Section 2.2.1, we establish the equivalence between cocode_1 and the Bolzano-Weierstrass theorem for *strongly* countable sets in Cantor space in various guises, including BWC_1. In Section 2.2.2, we do the same for cocode_0 and BWC_0 and variations. In Section 2.3, we study $\mathsf{CB}\mathbb{N}$, the Cantor-Berstein theorem for $\mathbb{N}$, and show that it is strictly weaker than BWC_0 in that $\mathsf{Z}_2^\omega + \mathsf{CB}\mathbb{N}$ cannot even prove NBI. In Section 2.4, we study items (a)-(d) from Section 1.2, which are basic theorems about limit points in $\mathbb{R}$ and related concepts, all going back to Cantor somehow. We establish equivalences between versions of these items on one hand, and $\mathsf{CB}\mathbb{N}$ and cocode_0 on the other hand; unlike the latter, items (a)-(c) do not mention 'injections' or 'bijections'.

As to technical machinery, we mention the 'excluded middle trick' pioneered in [71]. While we adopt ACA_0^ω as our base theory, the following trick can be used to replace the latter theory by RCA_0^ω *if the reader so desires*.

Remark 2.1 (Excluded middle trick)**.** The law of excluded middle as in $(\exists^2)\vee\neg(\exists^2)$ is quite useful as follows: suppose we are proving $T \to \mathsf{cocode}_0$ over RCA_0^ω. Now, in case $\neg(\exists^2)$, all functions on $\mathbb{R}$ are continuous by [47, §3] and cocode_0 trivially holds. Hence, what remains is to establish $T \to \mathsf{cocode}_0$ *in case we have* $(\exists^2)$. However, the latter axiom e.g. implies ACA_0 and can uniformly convert reals to their binary representations. In this way, finding a proof in $\mathsf{RCA}_0^\omega + (\exists^2)$ is 'much easier' than finding a proof in RCA_0^ω. In a nutshell, we may wlog assume $(\exists^2)$ when proving theorems that are trivial (or readily proved) when all functions (on $\mathbb{R}$ or $\mathbb{N}^\mathbb{N}$) are continuous, like cocode_0.

We stress that the previous trick should be used sparingly: the unit interval is not a set in the sense of Definition 1.2 in the absence of $(\exists^2)$.

In addition to the previous remark, we shall need a coding trick based on the well-known lexicographic ordering $<_{\mathsf{lex}}$, as described in Notation 2.2. For brevity, we sometimes abbreviate $\langle n\rangle * w^{0^*} * f^1$ as nwf if all types are clear from context.

Notation 2.2 (Sequences with information)**.** For a finite binary sequence s^{0^*}, define w_s by replacing 0 in s with the word 1001 and 1 in s with 101. Conversely, if w^{0^*} is a finite conjunction of words 1001 and 101, we let s_w be the finite binary sequence s such that $w_{s_w} =_{0^*} w$. This coding and decoding transfers directly to infinite binary sequences and infinite conjunctions of the words 1001 and 101. A *sequence with information* is any coded presentation $g = w_s 0 f$ of a pair (s, f) where s^{0^*} is a finite binary sequence and $f \in 2^\mathbb{N}$.

This notation is convenient when trying to define the set X of binary sequences s^{0^*} such that $(\exists f \in 2^\mathbb{N})[Y(s, f) = 0]$ for some fixed Y^2. Indeed, one point is that the coding as in Notation 2.2 preserves the lexicographic ordering of the sequences. Another point is that if s_1 is a strict subsequence of s_2, and $w_{s_1}0f_1$ and $w_{s_2}0f_2$ are two sequences with information, then $w_{s_1}0f_1 <_{\mathsf{lex}} w_{s_2}0f_2$. In this way, the above versions of the Bolzano-Weierstrass are applied to sets of sequences with information in such a way that the information parts do not show up in the supremum.

2.2. **Bolzano-Weierstrass theorem and (strongly) countable sets.** In this section, we study the RM of the Bolzano-Weierstrass theorem in the guise of BWC_i^j from Section 1.2. In particular, we provide a positive answer to question (Q0) from the latter by establishing the equivalences in (EQ).

2.2.1. *Strongly countable sets.* We connect the Bolzano-Weierstrass theorem for strongly countable sets to cocode_1, which is cocode_0 restricted to strongly countable sets. We discuss the connection to hyperarithmetical analysis in Remark 2.8.

First of all, we need a little bit of the induction axiom, formulated as in IND_1 in Principle 2.3. The equivalence between induction and bounded comprehension is well-known in second-order RM ([97, X.4.4]).

Principle 2.3 (IND_1)**.** *Let Y^2 satisfy $(\forall n \in \mathbb{N})(\exists! f \in 2^\mathbb{N})[Y(n, f) = 0]$. Then $(\forall n \in \mathbb{N})(\exists w^{1^*})\big[|w| = n \wedge (\forall i < n)(Y(i, w(i)) = 0)\big]$.*

Note that IND_1 is a special case of the axiom of finite choice, and is valid in all models considered in [66–73]. Moreover, IND_1 is trivial in case $\neg(\exists^2)$ since the condition on Y is then false.

Lemma 2.4. *The system ACA_0^ω proves $\mathsf{cocode}_1 \to \mathsf{IND}_1$.*

Proof. To show that $\mathsf{cocode}_1 \to \mathsf{IND}_1$, assume $(\forall n \in \mathbb{N})(\exists! f \in 2^\mathbb{N})A_0(n, f)$ where A_0 is quantifier-free. Let $\langle n\rangle * f \in A$ if $A_0(n, f)$ and define $F(g) := g(0)$, i.e. $F(\langle n\rangle * f) = n$. Modulo coding, we may view A as a subset of $2^\mathbb{N}$. By assumption, F is a bijection from A to $\mathbb{N}$, and by cocode_1, A is enumerable as $\{g_i\}_{i\in\mathbb{N}}$. From this enumeration, we can (Turing) compute $n \mapsto f_n$ where f_n is the unique f with $A_0(n, f)$ for any $n \in \mathbb{N}$, and in particular an object as claimed to exist by IND_1. $\square$

Secondly, the following theorem completes most of the results in (EQ) for BWC_1.

Theorem 2.5 (ACA_0^ω). $[\mathsf{BWC}_1 + \mathsf{IND}_1] \leftrightarrow \mathsf{BWC}_1^{\mathsf{pwo}} \leftrightarrow \mathsf{cocode}_1$.

Proof. We have already established that $\mathsf{cocode}_1 \rightarrow \mathsf{IND}_1$ in Lemma 2.4. Moreover, it is straightforward to prove both $\mathsf{BWC}_1^{\mathsf{pwo}}$ and BWC_1 from cocode_1. We first prove that $\mathsf{BWC}_1^{\mathsf{pwo}} \rightarrow \mathsf{cocode}_1$ in RCA_0^ω. To this end, let $F : 2^{\mathbb{N}} \rightarrow \mathbb{N}$ be a bijection on $A \subseteq 2^{\mathbb{N}}$. Define the set $B \subset 2^{\mathbb{N}}$ as follows: $g \in B$ if the following items are satisfied:

- for all $n, m, a, b \in \mathbb{N}$, $g(\langle n, a\rangle) = g(\langle m, b\rangle) = 1 \rightarrow n = m$,
- for a unique $n_0 \in \mathbb{N}$, $g(\langle n_0, 0\rangle) = 1$,
- for this n_0, the function $\lambda a.g(\langle n_0, a + 1\rangle)$ is in A and maps to n_0 under F.

Clearly, B is strongly countable and $\mathsf{BWC}_1^{\mathsf{pwo}}$ yields a pointwise least upper bound for B. This is essentially the characteristic function of the disjoint union of the sets (with characteristic functions) in A, and we can recover an enumeration of A.

Next, we prove that $\mathsf{BWC}_1 \rightarrow \mathsf{cocode}_1$, using IND_1. Let F be bijective on $A \subset 2^{\mathbb{N}}$. We will construct a strongly countable set B such that $g(\langle i, j\rangle) = F^{-1}(i)(j)$ is coded as the lexicographic supremum of B. Let $w0f \in B$ if $f = g_0 \oplus g_1 \oplus \cdots \oplus g_{k-1}$ where k is the length of s_w, where $F(g_i) = i$ for $i < k$, and where $s_w(\langle i, j\rangle) = g_i(j)$ whenever $\langle i, j\rangle < k$. We let $G(w0f)$ be the length of s_w. Then G is a bijection on B. We need IND_1 to establish the unique existence of $g_0 \oplus g_1 \oplus \cdots \oplus g_{k-1}$ for each k for this otherwise trivial fact. The supremum of B in the lexicographic ordering now codes the enumeration of A via the inverse of F and the $0 \mapsto 1001$ and $1 \mapsto 101$ translation from Notation 2.2. □

Thirdly, by the following, $\mathsf{ACA}_0^\omega + \mathsf{BWC}_1^{\mathsf{pwo}}$ and $\mathsf{ACA}_0^\omega + \mathsf{BWC}_1 + \mathsf{IND}_1$ are connected to *hyperarithmetical analysis*. We discuss this connection in Remark 2.8

Corollary 2.6. *The system* $\mathsf{ACA}_0^\omega + \mathsf{BWC}_1^{\mathsf{pwo}}$ *proves* $\mathsf{weak}\text{-}\Sigma_1^1\text{-}\mathsf{AC}_0$*; the former yields a conservative extension when added to* Σ_1^1-AC_0.

Proof. By [88, Theorem 3.17], $\mathsf{QF}\text{-}\mathsf{AC}^{0,1} \rightarrow \mathsf{cocode}_1 \rightarrow !\mathsf{QF}\text{-}\mathsf{AC}^{0,1}$, where the final principle is the first principle with a uniqueness condition. Now, $\mathsf{ACA}_0^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,1}$ is a conservative extension of Σ_1^1-AC_0 by [39, Cor. 2.7], while $\mathsf{ACA}_0^\omega + !\mathsf{QF}\text{-}\mathsf{AC}^{0,1}$ clearly proves $\mathsf{weak}\text{-}\Sigma_1^1\text{-}\mathsf{AC}_0$ □

We note that the monotone convergence theorem for nets with *strongly countable* index set, called $\mathsf{MCT}_1^{\mathsf{net}}$ in [88], is equivalent to cocode_1 over ACA_0^ω by [88, Theorem 3.10]. Hence, this theorem has the same status as e.g. $\mathsf{BWC}_1 + \mathsf{IND}_1$.

Finally, the previous results suggest a connection between cocode_1 and hyperarithmetical analysis. A well-known system here is Δ_1^1-comprehension (see [97, Table 4, p. 54]) and we now connect the latter to cocode_1. To this end, let $\Delta\text{-}\mathsf{CA}_C^-$ be Δ-CA restricted to formulas $A_i(n) \equiv (\exists f \in 2^{\mathbb{N}})(Y_i(f, n) = 0)$ also satisfying $(\forall n \in \mathbb{N})(\exists \text{ at most one } f \in 2^{\mathbb{N}})(Y_i(f, n) = 0)$ for $i = 0, 1$. In this way, $\Delta\text{-}\mathsf{CA}_C^-$ is similar in role and form to BOOT_C^-. We have the following surprising result.

Theorem 2.7. *The system* ACA_0^ω *proves that the following are equivalent:*

(a) cocode_1*: any strongly countable set can be enumerated,*
(b) *For strongly countable* $A \subset [0, 1]$*, any subset of* A *can be enumerated,*
(c) $\Delta\text{-}\mathsf{CA}_C^-$*: the axiom* Δ-CA *with an 'at most one' condition for* $2^{\mathbb{N}}$.

Proof. For the implication (a) → (b), let $A \subset [0,1]$ be strongly countable and use cocode_1 to obtain a sequence listing all elements of A. For $B \subset A$, use μ^2 to remove all elements in $A \setminus B$ from this sequence. For (b) → (c), fix Y_i^2 for $i = 0,1$ as in $\Delta\text{-}\mathsf{CA}_C^-$ and define the following subsets of Cantor space:

$$A := \{f \in 2^{\mathbb{N}} : (\exists n \in \mathbb{N})(Y_0(f,n) = 0)\} \text{ and } B := \{g \in 2^{\mathbb{N}} : (\exists m \in \mathbb{N})(Y_1(g,m) = 0)\}.$$

Define $Z, W : 2^{\mathbb{N}} \to \mathbb{N}$ as $Z(f) := (\mu n)(Y_0(f,n) = 0)$ and $W(g) := (\mu m)(Y_1(g,m) = 0)$. By the assumption on Y_0 (resp. Y_1), Z (resp. W) is injective on A (resp. B). Now let $A \,\dot\cup\, B$ be the disjoint[5] union of A and B and define the following:

$$V(h) := \begin{cases} Z(h(1) * h(2) * \dots) & h(0) = 0 \wedge h(1) * h(2) * \dots \in A \\ W(h(1) * h(2) * \dots) & h(0) = 1 \wedge h(1) * h(2) * \dots \in B \\ 0 & \text{otherwise} \end{cases}. \tag{2.1}$$

Now, $V : 2^{\mathbb{N}} \to \mathbb{N}$ defined as in (2.1) is *bijective* on $A \,\dot\cup\, B$, which is readily verified via a tedious-but-straightforward case distinction. Hence, $A \,\dot\cup\, B$ is strongly countable and applying item (b) yields an enumeration $(f_n)_{n\in\mathbb{N}}$ of A. By the definition of A, we have $(\exists f \in 2^{\mathbb{N}})(Y_0(f,n) = 0) \leftrightarrow (\exists m \in \mathbb{N})(Y_0(f_m, n) = 0)$, for any $n \in \mathbb{N}$. Now define $X \subset \mathbb{N}$ as follows: $n \in X \leftrightarrow (\exists m \in \mathbb{N})(Y_0(f_m, n) = 0)$. This set is exactly as needed for $\Delta\text{-}\mathsf{CA}_C^-$, and we are done.

For the implication $\Delta\text{-}\mathsf{CA}_C^- \to \mathsf{cocode}_1$, let $Y : 2^{\mathbb{N}} \to \mathbb{N}$ be bijective on $A \subset 2^{\mathbb{N}}$. Now consider, for any $n, m \in \mathbb{N}$ and $i = 0, 1$, the following:

$$(\exists g \in A)(g(m) = i \wedge Y(g) = n) \leftrightarrow (\forall f \in A)(f(m) \neq i \to Y(f) \neq n),$$

which follows by definition and satisfies the required 'at most one' conditions. Then $\Delta\text{-}\mathsf{CA}_C^-$ provides $X \subset \mathbb{N}^3$ such that

$$(n, m, i) \in X \leftrightarrow (\exists g \in A)(g(m) = i \wedge Y(g) = n)$$

for any $n, m \in \mathbb{N}$ and $i = 0, 1$. The enumeration of A is given by $f_n(m) = i$ for the unique i such that $(n, m, i) \in X$, and we are done. □

The 'at most one' conditions in $\Delta\text{-}\mathsf{CA}_C^-$ may seem strange, but similar constructs exist in second-order RM: as discussed in [97, p. 181], a version of Suslin's classical result that the Borel sets are exactly the Δ_1^1-sets can be proved in ATR_0. However, Borel sets in second-order RM are in fact given by Δ_1^1-formulas that satisfy an 'at most one' condition, in light of [97, V.3.3-4].

We finish this section with a remark on hyperarithmetical analysis.

Remark 2.8. The notion of *hyperarithmetical set* ([97, VIII.3]) gives rise to the (second-order) definition of *system/statement of hyperarithmetical analyis* (see e.g. [58] for the exact definition), which includes systems like $\Sigma_1^1\text{-}\mathsf{CA}_0$ (see [97, VII.6.1]). Montalbán claims in [58] that INDEC, a special case of [44, IV.3.3], is the first 'mathematical' statement of hyperarithmetical analysis. The latter theorem by Jullien can be found in [25, 6.3.4.(3)] and [78, Lemma 10.3].

The monographs [25, 44, 78] are all 'rather logical' in nature and INDEC is the *restriction* of a higher-order statement to countable linear orders in the sense of RM ([97, V.1.1]), i.e. such orders are given by sequences. In our opinion, the statements $\mathsf{MCT}_1^{\mathsf{net}}$ and BWC_1 introduced above are (much) more natural than INDEC as they are obtained from theorems of mainstream mathematics by a (similar to the case

[5]The disjoint union $A \,\dot\cup\, B$ can be defined as $\{(\langle n\rangle * f) \in 2^{\mathbb{N}} : (n = 0 \wedge f \in A) \vee (n = 1 \wedge f \in B)\}$.

of INDEC) restriction, namely to strongly countable sets. Now consider, $\mathsf{ACA}_0^\omega + \mathsf{X}$ where X is either $\mathsf{MCT}_1^{[0,1]}$, cocode_1, $\Delta\text{-}\mathsf{CA}_C^-$, or $\mathsf{BWC}_1 + \mathsf{IND}_1$. By the above, $\mathsf{ACA}_0^\omega + \mathsf{X}$ is a rather natural system *in the range of hyperarithmetical analysis*, namely sitting between $\mathsf{RCA}_0^\omega + \mathsf{weak}\text{-}\Sigma_1^1\text{-}\mathsf{CA}_0$ and $\mathsf{ACA}_0^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,1} \equiv_{\mathsf{L}_2} \Sigma_1^1\text{-}\mathsf{CA}_0$.

2.2.2. *Countable sets.* We study the Bolzano-Weierstrass for countable sets in its various guises and connect it to cocode_0.

Firstly, as perhaps expected in light of the use of IND_1 above, we also need a fragment of the induction axiom, as follows.

Definition 2.9. [IND_0] Let Y^2 satisfy $(\forall n \in \mathbb{N})(\exists \text{ at most one } f \in 2^{\mathbb{N}})(Y(f,n) = 0)$. For $k \in \mathbb{N}$, there is w^{1^*} with $|w| = k$ such that for $m \leq k$, we have:

$$(w(m) \in 2^{\mathbb{N}} \wedge Y(w(m),m) = 0) \leftrightarrow (\exists f \in 2^{\mathbb{N}})(Y(f,m) = 0).$$

Note that $\mathsf{IND}_0 \to \mathsf{IND}_1$ by definition. The following theorem is a first approximation of the results in (EQ).

Theorem 2.10. *The system* ACA_0^ω *proves* $\mathsf{BWC}_0^{\mathsf{pwo}} \leftrightarrow \mathsf{cocode}_0$

Proof. The reverse implication is immediate as cocode_0 converts A into a sequence. Of course, $(\exists^2)$ implies ACA_0 and hence the second-order Bolzano-Weierstrass theorem by [97, III.2]. For the forward implication, the construction in the proof of Theorem 2.5 is readily adapted. □

Secondly, what remains to establish (EQ) is the following.

Theorem 2.11. *The system* ACA_0^ω *proves*

$$\mathsf{cocode}_0 \leftrightarrow [\mathsf{BWC}_0 + \mathsf{IND}_0] \leftrightarrow \mathsf{range}_0 \leftrightarrow \mathsf{BWC}_0^{\mathsf{pwo}}. \tag{2.2}$$

Proof. The implication $\mathsf{BWC}_0^{\mathsf{pwo}} \to [\mathsf{BWC}_0 + \mathsf{IND}_0]$ follows in the same way as for $\mathsf{BWC}_1^{\mathsf{pwo}} \to [\mathsf{BWC}_1 + \mathsf{IND}_1]$ in the proof of Theorem 2.5, i.e. via cocode_0. To prove $[\mathsf{BWC}_0 + \mathsf{IND}_0] \to \mathsf{range}_0$, let $F : 2^{\mathbb{N}} \to \mathbb{N}$ be injective on $A \subset 2^{\mathbb{N}}$. Define the set B of sequences with information $w0g$ such that $w0g \in B$ if g is of the form $g_0 \oplus \cdots \oplus g_{k-1}$, where k is the length of s_w, and such that $F(g_i) = i$ whenever $s_w(i) = 1$. Then B is clearly countable since A is countable. Using IND_0 we see that for each k there is a w_k such that s_{w_k} has length k and approximates the characteristic function of the range of F. Using IND_0 again, there is $g = g_0 \oplus \cdots \oplus g_{k-1}$ such that $w_k 0 g \in B$. This object is the lexicographicly largest object $w'0g' \in B$ such that the length of $s_{w'} \leq k$. It follows that $\sup B$ will approximate a coded representation of the characteristic function of the range of F, and range_0 follows.

To prove $\mathsf{range}_0 \to \mathsf{BWC}_0^{\mathsf{pwo}}$, let $F : 2^{\mathbb{N}} \to \mathbb{N}$ be injective on $A \subset 2^{\mathbb{N}}$. Let $nf \in B$ if $f \in A$ and $f(n) = 1$ and let $G(nf) = \langle n, F(f)\rangle$. Then G is injective on B, so let X be the range of B under G. The pointwise least upper bound f of A is then definable from X and $\exists^2$ by $f(n) = 1 \leftrightarrow (\exists k)(\langle n,k\rangle \in X)$. □

Thirdly, we also obtain some nice equivalences for IND_0, which can be proved as well for IND_1 and *strongly* countable sets. Note that the third item uses the 'set theoretic' definition of finite sets of reals, also discussed in Section 3.3.3.

Theorem 2.12 ($\mathsf{ACA}_0^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,1}$)**.** *The following are equivalent.*

- IND_0.
- *A countable and finite set can be enumerated (by a finite sequence).*

- *A set $A \subset [0,1]$ with $Y : [0,1] \to \mathbb{N}$ injective and bounded on A, can be enumerated (by a finite sequence).*

We only need $\mathsf{QF\text{-}AC}^{0,1}$ *to obtain the second item.*

Proof. The second item readily implies the third one. We now prove the second item from the first one. To this end, assume A, Y are as in the second item and suppose $(\forall n \in \mathbb{N})(\exists x \in A)(Y(x) > n)$. Apply $\mathsf{QF\text{-}AC}^{0,1}$ and let $(x_n)_{n\in\mathbb{N}}$ be the resulting sequence. Define $g : \mathbb{N} \to \mathbb{N}$ as follows:

$$g(0) := 0 \text{ and } g(n+1) := Y(x_{g(n)}).$$

for which we use the primitive recursion scheme in RCA_0^ω. Now note that $(x_{g(n)})_{n\in\mathbb{N}}$ is a sequence of distinct reals in A, contradicting the assumption that it is finite (as in Definition 1.5). The previous contradiction implies that there is $N \in \mathbb{N}$ such that $(\forall x \in A)(Y(x) \leq N)$. Since Y is injective on A, we also have $(\forall n \leq N)(\exists \text{ at most one } x \in A)(Y(x) = n)$. Now apply IND_0 to obtain the desired enumeration of A. To prove IND_0 from the third item, let Y be as in the former and fix $k \in \mathbb{N}$. Define the set $A := \{f \in 2^\mathbb{N} : (\exists n \leq k)(Y(f,n) = 0)\}$ and define $Z(f) := (\mu n \leq k)(Y(f,n) = 0)$, if such there is, and 0 otherwise. Clearly, Z is injective and bounded (by k) on A. Applying the third item, we can enumerate A, yielding w^{1^*} as required by IND_0. □

The previous theorem suggests that IND_0 (and even cocode_0) cannot prove that a finite set is enumerable, due to the absence of an injection. However, finite sets (that come without any obvious injection) do occur 'in the wild', namely in the study of functions of bounded variation, as discussed in detail in Section 3.3.2.

Finally, we discuss equivalences for cocode_0 from other parts of mathematics.

Remark 2.13 (Lifting results)**.** Firstly, consider the following algebra statement:

any countable sub-field of $\mathbb{R}$ is isomorphic to a sub-field of an algebraically closed countable field.

The second-order version of the latter is equivalent to ACA_0 by [97, III.3.2]. Now, the centred statement with 'countable' removed everywhere is (equivalent to) ALCL from [87, §3.6.1]; it is shown in [87, Theorem 3.31] that

$$\mathsf{ACA}_0^\omega + \Delta\text{-}\mathsf{CA} \text{ proves } \mathsf{ALCL} \to \mathsf{BOOT}. \tag{2.3}$$

without any essential modification to the proof of [97, III.3.2], i.e. the proof of the latter is 'lifted' to the proof in (2.3) by 'bumping up' all the relevant types by one. Now let ALCL_0 be the above centred statement in italics with 'countable' interpreted as in Definition 1.4. One readily modifies the proof from (2.3) to yield:

$$\mathsf{ACA}_0^\omega + \Delta\text{-}\mathsf{CA}_C^- \text{ proves } \mathsf{ALCL}_0 \to \mathsf{BOOT}_C^-, \tag{2.4}$$

therewith yielding $[\mathsf{ALCL}_0 + \mathsf{cocode}_1] \leftrightarrow \mathsf{cocode}_0$ over RCA_0^ω by Theorem 2.7. One can obtain similar results for the other proofs in [85, 87], and most likely for any second-order reversal involving countable algebra (and beyond).

2.3. **The Cantor-Bernstein theorem.** We connect $\mathsf{BWC}_0^{\mathsf{pwo}}$ to the *Cantor-Bernstein theorem* for $\mathbb{N}$ as studied in [88] and defined as in Principle 2.14. As it happens, this theorem is studied in second-order RM as [15, Problem 1] and was studied by Cantor already in 1878 in [10]. Our results provide an answer to (Q1).

Principle 2.14 ($\mathsf{CB}\mathbb{N}$)**.** *A countable set $A \subset \mathbb{R}$ is strongly countable if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of pairwise distinct reals such that $(\forall n \in \mathbb{N})(x_n \in A)$.*

First of all, the equivalence $[\mathsf{CB}\mathbb{N}+\mathsf{cocode}_1] \leftrightarrow \mathsf{cocode}_0$ is proved[6] in [88, §3]. We have the following corollary to Theorems 2.5 and 2.10, which provides an answer to (Q1), as $\mathsf{BW}_0^{\mathsf{pwo}}$ can be split further.

Corollary 2.15. *The system ACA_0^ω proves $\mathsf{BWC}_0^{\mathsf{pwo}} \leftrightarrow [\mathsf{CB}\mathbb{N} + \mathsf{BWC}_1^{\mathsf{pwo}}]$.*

Proof. Immediate from $[\mathsf{CB}\mathbb{N}+\mathsf{cocode}_1] \leftrightarrow \mathsf{cocode}_0$ and Theorems 2.5 and 2.10. □

Secondly, we show that $\mathsf{CB}\mathbb{N}$ does not imply cocode_1, based on the proof of [73, Theorem 3.26]. This establishes that the statements inside the same square brackets in (2.5) are independent, even relative to Z_2^ω:

$$\mathsf{cocode}_0 \leftrightarrow [\mathsf{BWC}_1^{\mathsf{pwo}} + \mathsf{CB}\mathbb{N}] \leftrightarrow [\mathsf{cocode}_1 + \mathsf{CB}\mathbb{N}] \leftrightarrow [\Delta\text{-}\mathsf{CA}_C^- + \mathsf{CB}\mathbb{N}]. \tag{2.5}$$

Note that (2.5) follows from Corollary 2.15, while trivially $\mathsf{cocode}_1 \rightarrow \mathsf{NBI}$.

Theorem 2.16. *The system $\mathsf{Z}_2^\omega + \mathsf{CB}\mathbb{N} + \mathsf{IND}_0$ cannot prove NBI.*

Proof. The proof of [73, Theorem 3.28] discusses a model $\mathbf{Q}^*$ of $\mathsf{Z}_2^\omega + \neg\mathsf{NBI}$, which implies that Z_2^ω cannot prove NBI (or cocode_1). This model is defined in [73, Definition 2.28] and its properties are based on [73, Lemma 2.16 and Theorem 2.17]. Here, we will explain the properties of $\mathbf{Q}^*$ essential for the proof of our theorem, namely that this model satisfies $\mathsf{CB}\mathbb{N} + \mathsf{IND}_0$. For the proofs of (most of) these properties, we refer to [73].

First of all, the construction of the model is based on Kleene-computability relative to the functionals S_k^2, where S_k^2 is the characteristic function of some complete Π_k^1-subset of $\mathbb{N}^\mathbb{N}$. Using the Löwenheim-Skolem theorem, we let $A \subseteq \mathbb{N}^\mathbb{N}$ be a countable set such that all Π_k^1-formulas are absolute for A for all k. We let $(g_k)_{k\in\mathbb{N}}$ be an enumeration of A and we let A_k be the set of functions computable in S_k^2 and $\{g_0, \dots, g_{k-1}\}$. The key properties are that $A_k \subsetneq A_{k+1} \subsetneq A$ and that A_{k+1} contains an enumeration of A_k for each k.

Secondly, we let $\mathbf{Q}[1] = A$ be the elements of the model of pure type 1. The definition of $\mathbf{Q}[2]$ is as follows: If $F : A \rightarrow \mathbb{N}$ we let $F \in \mathbf{Q}[2]$ if there is a k_0 such that for all $k \geq k_0$, the restriction of F to A_k is partially computable in S_k^2 and $\{g_0, \dots, g_{k-1}\}$. No uniformity is required. On top of this, we close $\mathbf{Q}[1]$ and $\mathbf{Q}[2]$ under Kleene computability hereditarily for each pure type. As proved in [73], this will not add new elements of type 1 or type 2 to the structure. Finally, we use a canonical extension to interpretations of all finite types. The resulting type structure, named $\mathbf{Q}^*$ in [73], is a model of Z_2^ω and satisfies our weak induction axioms IND_0 and IND_1. Indeed, the models are constructed as computational closures, implying that for any sequence $f_0, \dots, f_n$ of elements in the model, the coded sequence $(f_0, \dots, f_n)$ is also in the model, and the two induction axioms IND_0 and IND_1 readily follow.

Thirdly, having witnessed the construction of the model $\mathbf{Q}^*$, we now show that it satisfies that all infinite subsets of $\mathbb{N}^\mathbb{N}$ are strongly countable. In particular, we have that $\mathsf{CB}\mathbb{N}$ holds in $\mathbf{Q}^*$. To this end, fix some arbitrary $B \in \mathbf{Q}[2]$ that is (the

[6]The reverse implication is trivial, assuming $\exists^2$. For the forward implication: if A is countable, consider a set B isomorphic to $\mathbb{N} \oplus A$. Apply $\mathsf{CB}\mathbb{N}$ to this set to show that it is strongly countable, and then cocode_1 to show that it is enumerable. Thus, A is enumerable.

characteristic function of) an infinite subset of $A = \mathbf{Q}[1]$. We have established in the proof of [73, Theorem 3.26] that $\mathbf{Q}[2]$ contains a bijection $\phi : \mathbf{Q}[1] \to \mathbb{N}$ with the extra property that ϕ_k, the restriction of ϕ to A_k, is partially computable in S_k^2 and $g_0, \dots, g_{k-1}$. We do not need the explicit construction of ϕ: it suffices to split the argument for finding a bijection from B to $\mathbb{N}$ in two cases, as follows.

- If $B \subseteq A_k$ for some k, then B is enumerable in $A_{k'}$ for some $k' > k$ (property of the model $\mathbf{Q}^*$), and the inverse can be found directly.
- In the 'otherwise' case, we construct an increasing sequence of functionals $\psi_k : (B \cap A_k) \to \mathbb{N}$ as being equal to the restriction of ϕ to $B \cap A_k$ except at finitely many points; we use the finite set of exceptions to make $\psi := \lim_{k\to\infty} \psi_k$ surjective. Now, for infinitely many k we have that $B \cap (A_{k+1} \setminus A_k) \neq \emptyset$. At each stage where this is the case, and where the range of $A_k \cap B$ under ψ_k is a proper subset of the range of A_k under ϕ, we define ψ_{k+1} as follows.
  - Choose one element f in $B \cap (A_{k+1} \setminus A_k)$. Let n be the least element in the range of A_k under ϕ that is not in the range of $B \cap A_k$ under ψ_k, and define $\psi_{k+1}(f) := n$.
  - We let ψ_{k+1} be equal to ϕ on the rest of $B \cap (A_{k+1} \setminus A_k)$, noticing that the injectivity of ϕ ensures that the value n used above will not be in the range of $A_{k+1} \setminus A_k$ under ϕ, so injectivity is preserved.

  Since B and ϕ are elements in $\mathbf{Q}[2]$ and ψ differs from the restriction of ϕ to B at only finitely many points in each A_k, it follows that $\psi \in \mathbf{Q}[2]$.

The previous case distinction finishes the proof. □

We now list some other interesting properties of the model $\mathbf{Q}^*$ constructed above. If A and A_k are as in the construction, and $B \subset A$ is such that $B \cap A_k$ is finite for each k, then automatically $B \in \mathbf{Q}[2]$. Since each set $A_{k+1} \setminus A_k$ is dense in $\mathbb{N}^{\mathbb{N}}$, this opens up numerous possibilities for counter-intuitive properties consistent with Z_2^ω. A few examples are as follows.

- There is a strongly countable set such that all enumerable subsets are finite.
- There is an infinite subset of $[0, 1]$ with no cluster-point.
- There is an infinite subset of $[0, 1]$ with one cluster-point 0, but with no sequence from the set converging to 0.

By the above, $\mathsf{CB}\mathbb{N}$ is weaker than BWC_0 and we also conjecture that the former is 'less explosive' than the latter as follows.

Conjecture 2.17. *The system* $\Pi_1^1\text{-}\mathsf{CA}_0^\omega + \mathsf{CB}\mathbb{N}$ *cannot prove* $\Pi_2^1\text{-}\mathsf{CA}_0$.

Proving the previous conjecture may be difficult, as Theorem 2.19 suggests that $\mathsf{CB}\mathbb{N}$ is 'very close' to BWC_0 in explosive power.

Now, the Cantor-Bernstein theorem is a standard exercise in axiomatic set theory (see e.g. [38, p. 69]). Experience bears out that when the students are asked to construct a bijection $H : A \to B$ from given injections $F : A \to B$ and $G : B \to A$, the successful solutions will all have the property that for each $a \in A$, either $H(a) = F(a)$ or $a = G(H(a))$. Let H with this property be called a *canonical witness* to the Cantor-Bernstein theorem. We define $\mathsf{CB}\mathbb{N}^+$ as $\mathsf{CB}\mathbb{N}$ augmented with the existence of such a canonical witness, namely as follows.

Principle 2.18 ($\mathsf{CB}\mathbb{N}^+$)**.** *Assume* $A \subset [0, 1]$ *satisfies the following:*

- *there is* $F : [0, 1] \to \mathbb{N}$ *injective on* A*,*

- *there is* $(x_n)_{n\in\mathbb{N}}$ *consisting of pairwise distinct reals with* $(\forall n \in \mathbb{N})(x_n \in A)$.

Then there is $H : [0,1] \to \mathbb{N}$ *which is bijective on* A *and such that for* $x \in A$ *either* $H(x) = F(x)$ *or* $x = x_{H(x)}$.

Note that canonical witnesses occur in [15, Problem 1] as part of the study of the Cantor-Bernstein theorem in second-order RM.

Theorem 2.19. *The system* $\Pi^1_1\text{-}\mathsf{CA}^\omega_0 + \mathsf{CB}\mathbb{N}^+$ *proves* $\Pi^1_2\text{-}\mathsf{CA}_0$.

Proof. We prove $\mathsf{CB}\mathbb{N}^+ \to \mathsf{BOOT}^-_C$ and note that [73, Theorem 3.23] yield $\Pi^1_2\text{-}\mathsf{CA}_0$ via $\Pi^1_1\text{-}\mathsf{CA}^\omega_0$. Let Y^2 be such that $(\forall n \in \mathbb{N})(\exists \text{ at most one } f \in 2^{\mathbb{N}})(Y(f,n) = 0)$. Let f_0 be the constant zero function and define $A \subset \mathbb{N} \times 2^{\mathbb{N}}$ as follows:

$$(m,f) \in A \leftrightarrow (\exists n \in \mathbb{N})[(m = 2n+1 \wedge Y(n,f) = 0) \vee (m = 2n \wedge f = f_0)].$$

Modulo coding, we can view A as a subset of $2^{\mathbb{N}}$. Define $F : A \to \mathbb{N}$ and $G : \mathbb{N} \to A$ as follows: $F\big((k,f)\big) := k$ and let $G(n) := (2n, f_0)$. Both functions are injective, so let $H : A \to \mathbb{N}$ be a canonical witness as in $\mathsf{CB}\mathbb{N}^+$. Now consider the following:

$$(\exists f \in 2^{\mathbb{N}})(Y(f,n) = 0) \leftrightarrow [H\big(2(2n+1), f_0\big) = 2(2n+1)]. \tag{2.6}$$

To prove (2.6), assume the left-hand side of (2.6) for fixed $n \in \mathbb{N}$. Then there is $f \in 2^{\mathbb{N}}$ such that $(2n+1, f) \in A$. Since this $(2n+1, f)$ is not in the range of G, we must have that $H(2n+1, f) = F(2n+1, f) = 2n+1$ and $G(2n+1) = (2(2n+1), f_0)$. Since the case that $H(2(2n+1), f_0) = 2n+1$ (the inverse of G) violates that H is injective, we must have that $H(2(2n+1), f_0) = 2(2n+1)$. Now assume the left-hand side of (2.6) is false for fixed $n \in \mathbb{N}$. Then there is no $f \in 2^{\mathbb{N}}$ such that $F(f) = 2n+1$. Since there is an $m \in \mathbb{N}$ and a $g \in 2^{\mathbb{N}}$ such that $(m,g) \in A$ and $H(m,g) = 2n+1$, we must have used the G^{-1}-part of H and have that $m = 2(2n+1)$ and $g = f_0$. This contradicts that $H(2(2n+1), f_0) = 2(2n+1)$. □

Corollary 2.20. *The system* ACA^ω_0 *proves* $\mathsf{BOOT}^-_C \leftrightarrow \mathsf{CB}\mathbb{N}^+$.

Proof. The proof of Theorem 2.19 establishes the reverse direction over ACA^ω_0. For the forward direction, we have $\mathsf{BOOT}^-_C \leftrightarrow \mathsf{cocode}_0$ over RCA^ω_0 by [88, Theorem 3.6]. Now let A, F, and $(x_n)_{n\in\mathbb{N}}$ be as in the antecedent of Principle 2.18. The enumeration of A provided by cocode_0 reduces the existence of a canonical witness for F and $(x_n)_{n\in\mathbb{N}}$ to the existence of a canonical witness for certain injections $f, g : \mathbb{N} \to \mathbb{N}$. For the latter, the standard proofs of the Cantor-Bernstein theorem can be formalised in ACA_0, yielding the required canonical witness. □

2.4. **Theorems going back to Cantor.** In this section, we establish ($\mathsf{EQ2}$) from Section 1.2. In particular, we extend Theorem 2.11 via a number of equivalences involving basic theorems about the real line or limit points, all going back to Cantor one way or the other. While interesting in their own right, our results also provide (positive) answers to questions (Q2)-(Q3) from Section 1.2. On a conceptual note, the order type η of $\mathbb{Q}$ appears throughout the second-order RM, but Cantor's characterisation of η as in cloq' below is *quite* explosive by Corollary 2.36.

First of all, the *perfect set theorem* or the *Cantor-Bendixson theorem* (see [97, V and VI] for the RM-study) imply that a nonempty *uncountable* and *closed* set has a perfect subset, and therefore *the original set has at least one limit point*. We shall study the latter for closed sets as in Definition 1.2. We note that the modern notion of limit/accumulation point was first articulated by Cantor in [14, p. 98].

Principle 2.21. *A non-enumerable and closed set in $\mathbb{R}$ has a limit point.*

Theorem 2.31 shows that $\mathsf{BW}_0^{\mathsf{fun}}$ is equivalent to a version of Principle 2.21, which is interesting as the latter does not mention bijections or injections. In particular, Principle 2.21 is a sentence of second-order arithmetic[7] *with one single modification*, namely the use of Definition 1.2 rather than RM-closed sets.

Secondly, Cantor shows in [14, p. 161, Hilfsatz II] that a collection of disjoint open intervals in $\mathbb{R}$ is countable; this is the first instance of the well-known *countable chain condition*. The following principle ccc expresses the former property *without* mentioning the words 'injection' or 'bijection'.

Principle 2.22 (ccc)**.** *Let $A \subset \mathbb{R}^2$ be such that for any non-identical intervals (a, b) and (c, d) in A, the intersection is empty. Then A can be enumerated.*

Let ccc_0 be ccc with the conclusion 'A is countable'. As will become clear in the proof of Theorem 2.31, ccc_0 is provable in RCA_0^ω, akin to how Cantor's theorem is provable in RCA_0 by [97, II.4.7].

Thirdly, the countable chain condition is found in the original version of *Suslin's hypothesis*, first formulated by Suslin in [101]. In this context, Cantor contributed the following theorem (for any countable set), as discussed in [77, p. 122-123].

Principle 2.23 (cloq)**.** *A countable linear ordering $(X, \preceq_X)$ for $X \subset \mathbb{R}$ is order-isomorphic to a subset of $\mathbb{Q}$.*

Moreover, Cantor introduces the notion of *order type* in [12] and characterises the order type η of $\mathbb{Q}$ in [13] based on the following (for any countable set).

Principle 2.24 (cloq')**.** *A countable and dense linear ordering without endpoints $(X, \preceq_X)$ for $X \subset \mathbb{R}$ is order-isomorphic to $\mathbb{Q}$.*

We use the usual[8] definition of *linear ordering* where '$\preceq_X$' is given by a characteristic function $F_X : \mathbb{R}^2 \to \mathbb{N}$, i.e. $x \preceq_X y \equiv F_X(x, y) =_0 1$, while $(X, \preceq_X)$ is called *countable* if $X \subset \mathbb{R}$ is. Similarly, an order-isomorphism from $(X, \preceq_X)$ to $(Y, \preceq_Y)$ is a surjective[9] $Y : X \to Y$ that respects the order relation (see [97, Def. V.2.7]), i.e.

$$(\forall x, x' \in X)(x \preceq_X x' \leftrightarrow Y(x) \preceq_Y Y(x')), \tag{2.7}$$

while a *well-founded linear order* (well-order) has no strictly descending sequences. The reader should verify that using a stronger definition of order-isomorphism does not change the below equivalences.

As to well-orders, Simpson calls the comparability of countable well-orders 'indispensable' for a decent theory of ordinals, pioneered by Cantor in [11]. We agree that it would be very indecent indeed to have incomparable countable well-orders, suggesting the following principle, which is just the second-order CWO from [97, V.6] formulated for linear orders over $\mathbb{R}$ that are countable.

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[7]Let $\mathsf{accu}_{\mathsf{RM}}$ be Principle 2.21 formulated with RM-closed sets. Since ATR_0 implies the perfect set theorem ([97, I.11.5]), we have the first implication in $\mathsf{ATR}_0 \to \mathsf{accu}_{\mathsf{RM}} \to \mathsf{ACA}_0$, while the second one follows via the proof of [97, III.2.2]. We believe that the final implication reverses.

[8]Namely that the relation $\preceq_X$ is transitive, anti-symmetric, and connex, just like in [97, V.1.1].

[9]Note that (2.7) implies that $(\forall x, x' \in X)(Y(x) =_Y Y(x') \to x =_X x')$, i.e. Y is injective relative to the equalities '$=_X$' and '$=_Y$', i.e. 'surjective' may be replaced by 'bijective'.

Principle 2.25 (CWO^ω)**.** *For countable well-orders $(X, \preceq_X)$ and $(Y, \preceq_Y)$ where $X, Y \subset \mathbb{R}$, the former order is order-isomorphic to the latter order or an initial segment of the latter order, or vice versa.*

Thirdly, we present a preliminary result that got everything started.

Theorem 2.26 (ACA_0^ω)**.** *Principle 2.21 implies the uncountability of $\mathbb{R}$ as in* NIN*.*

Proof. Let $Y : [0,1) \to \mathbb{N}$ be an injection and use $\exists^2$ to define $A \subset \mathbb{R}$ as follows:

$$x \in A \leftrightarrow (\exists n \in \mathbb{N})(n \leq x < n+1 \wedge Y(x-n) = n). \tag{2.8}$$

Intuitively, A is the set $\{z+Y(z) : z \in [0,1)\}$, although the latter need not exist (as a set) in ACA_0^ω. Each $[m, m+1)\cap A$ has at most one element as $x, y \in \big([m, m+1)\cap A\big)$ implies $Y(x-m) = Y(y-m)$ by (2.8) and hence $x =_{\mathbb{R}} y$ by the injectivity of Y. In this light, A does not have a limit point, while this set is trivially closed.

Towards a contradiction, we now show that A is non-enumerable. Suppose $(x_n)_{n\in\mathbb{N}}$ lists all elements of A, i.e. $(\forall x \in A)(\exists n \in \mathbb{N})(x =_{\mathbb{R}} x_n)$. Since we have $x \in A \leftrightarrow (Y(x - \lfloor x\rfloor) = \lfloor x\rfloor)$ for non-negative $x \in \mathbb{R}$, the sequence $(x_n - \lfloor x_n\rfloor)_{n\in\mathbb{N}}$ lists all elements of $[0, 1]$. Indeed, for $y_0 \in [0, 1]$, $(y_0 + Y(y_0)) \in A$ by definition and suppose $x_{n_0} =_{\mathbb{R}} y_0 + Y(y_0)$. Hence, $\lfloor x_{n_0}\rfloor = Y(y_0)$, and hence $y_0 = x_{n_0} - \lfloor x_{n_0}\rfloor$. A sequence listing the reals in $[0, 1)$ yields a contradiction by [97, II.4.7]. □

As is often the case (see e.g. [73,86,87]), the previous proof can be generalised to yield cocode_0. As noted above, there is however a fundamental difference between NIN and cocode_0: the latter combined with Π^1_1-CA_0^ω proves Π^1_2-CA_0, while the former does not (seem to go beyond Π^1_1-CA_0).

Corollary 2.27 (ACA_0^ω)**.** *Principle 2.21 implies* cocode_0*.*

Proof. Let $B \subset [0, 1]$ be a countable set, i.e. there exists $Y : [0, 1] \to \mathbb{N}$ such that Y is injective on B. Without loss of generality, we may assume $\mathbb{Q} \cap B = \emptyset$ as Feferman's μ^2 can enumerate the rationals in B. Similar to (2.8), we define the following set using $\exists^2$:

$$x \in A \leftrightarrow (\exists n \in \mathbb{N})(n \leq x < n+1 \wedge Y(x-n) = n \wedge (x-n) \in B). \tag{2.9}$$

Since Y is an injection on B, $[m, m+1)\cap A$ has at most one element for $m \in \mathbb{N}$. Thus, A is a closed subset with no limit point. By the contraposition of Principle 2.21, there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x \in A \leftrightarrow (\exists n \in \mathbb{N})(x = x_n)$. Clearly, $(x_n - \lfloor x_n\rfloor))_{n\in\mathbb{N}}$ yields an enumeration of B, and we are done. □

The previous corollary is interesting as follows: let PST and CBT be the *perfect set theorem* and the *Cantor-Bendixson theorem* formulated as in [70], i.e. for closed sets as in Definition 1.2 that are not enumerable. Note that Π^1_1-CA_0 proves these theorems formulated for RM-closed sets (and in L_2) by [97, V and VI].

Corollary 2.28. *The system Π^1_1-CA_0^ω cannot prove* PST *or* CBT*.*

Proof. By Theorem 2.27, both PST and CBT imply Principle 2.21. If Π^1_1-CA_0^ω could prove e.g. PST, we would obtain Π^1_2-CA_0 by [73, Theorem 4.22]. However, Π^1_1-CA_0^ω is Π^1_3-conservative over Π^1_1-CA_0 by [82, Theorem 2.2]. □

By the previous proof, Π^1_1-$\mathsf{CA}_0^\omega + \mathsf{PST}$ proves Π^1_2-CA_0 (and the same for CBT), i.e. Definition 1.2 makes these theorems quite explosive.

Unfortunately, we could not find a way to obtain the reversal of Corollary 2.27. On the other hand, *assuming* Principle 2.21, in case $A \subset \mathbb{R}$ is closed and has no limit points, one readily defines $G : \mathbb{R} \to \mathbb{N}$ (using $\exists^2$) such that

$$(\forall x \in A)(\exists n \leq G(x))\big(B(x, \tfrac{1}{2^n}) \cap A = \{x\}\big), \tag{2.10}$$

where (2.10) expresses that G is a witnessing functional for 'A has no limit points'. In other words, Principle 2.21 'enriches itself' with a witnessing functional G, while the set A from (2.9) has an almost trivial such witnessing functional (again using $\exists^2$). All this suggests the latter witnessing construct merits further study.

Fourth, to obtain the equivalences in Theorem 2.31, we seem to need the following slight constructive enrichment of Principle 2.21, as provided by (2.10).

Principle 2.29 (accu). *For any closed $A \subseteq \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{N}$ such that* (2.10)*, there is $(x_n)_{n\in\mathbb{N}}$ in A with $(\forall x \in \mathbb{R})\big(x \in A \leftrightarrow (\exists n \in \mathbb{N})(x = x_n)\big)$.*

We also study the following, apparently stronger, variation in Theorem 2.31.

Principle 2.30 (accu'). *For any $A \subseteq \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{N}$ such that* (2.10)*, there is $(x_n)_{n\in\mathbb{N}}$ in A with $(\forall x \in \mathbb{R})\big(x \in A \leftrightarrow (\exists n \in \mathbb{N})(x = x_n)\big)$.*

We note that Theorem 2.31 provides a positive answer to (Q3) from Section 1.2 as accu and ccc do not involve the notions 'injection' or 'bijection'. Moreover, $\mathsf{accu}' \leftrightarrow \mathsf{accu}$ is a nice robustness result, showing that quantifying over all sub-sets of $\mathbb{R}$ (rather than just the closed ones) need not be problematic.

Theorem 2.31. *The following are equivalent over ACA_0^ω:*

(a) cocode_0,
(b) $\mathsf{BWC}_0^{\mathsf{pwo}}$ (*Bolzano-Weierstrass*),
(c) accu',
(d) accu,
(e) ccc.

Proof. The equivalence (a) $\leftrightarrow$ (b) can be found in Theorem 2.11. We note that $\frac{1}{1+e^x}$ defines an injection from $\mathbb{R}$ to $(0,1)$. Hence, using $\exists^2$, one readily extends cocode_0 to subsets $A \subset \mathbb{R}$.

The implication $\mathsf{accu} \to \mathsf{cocode}_0$ is (essentially) proved in Corollary 2.27, as the functional G as in (2.10) is readily defined in this case. To prove $\mathsf{ccc} \to \mathsf{accu}$, let A, G be as in the latter, i.e. satisfying (2.10). By the latter, we have that for $x, y \in A$, the intersection $B(x, \frac{1}{2^{G(x)+1}}) \cap B(y, \frac{1}{2^{G(y)+1}})$ is empty in case $x \neq y$. We shall now define the set consisting of $B(x, \frac{1}{2^{G(x)+1}})$ for $x \in A$. To this end, define the set $B \subset \mathbb{R}^2$ as follows:

$$(a,b) \in B \leftrightarrow \big[\tfrac{a+b}{2} \in A \wedge a =_{\mathbb{R}} \tfrac{a+b}{2} - \tfrac{1}{2^{G(\frac{a+b}{2})+1}} \wedge b =_{\mathbb{R}} \tfrac{a+b}{2} + \tfrac{1}{2^{G(\frac{a+b}{2})+1}}\big]. \tag{2.11}$$

Now apply ccc to the set B, which yield sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ such that:

$$(\forall a, b \in \mathbb{R})\big((a,b) \in B \leftrightarrow (\exists n \in \mathbb{N})(a_n = a \wedge b_n = b)\big).$$

Then the sequence $(\frac{a_n+b_n}{2})_{n\in\mathbb{N}}$ enumerates A, and this implication is done.

For $\mathsf{cocode}_0 \to \mathsf{ccc}$, we first prove ccc_0 in ACA_0^ω. To the latter end, let $A \subset \mathbb{R}^2$ be as in ccc_0 and fix some enumeration $(q_n)_{n\in\mathbb{N}}$ of $\mathbb{Q}$. Define $Y((a,b))$ as the least $n \in \mathbb{N}$ such that $q_n \in (a,b)$ if such there is, and 0 otherwise. Clearly, Y is injective on A and the latter is countable, i.e. ccc_0 follows inside ACA_0^ω. Clearly, the combination $\mathsf{cocode}_0 + \mathsf{ccc}_0$ implies ccc. Thus, $\mathsf{cocode}_0 \to \mathsf{ccc}$ over ACA_0^ω follows.

Finally, the reverse implication in $\textsf{accu} \leftrightarrow \textsf{accu}'$ is trivial. Now fix $A \subset \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{N}$ satisfying (2.10). Then (2.11) yields a collection of open disjoint intervals in $\mathbb{R}$. Since $\textsf{accu} \to \textsf{ccc}$, this collection can be enumerated, yielding $\textsf{accu}'$. □

We shall obtain an equivalence between Principle 2.21 and $\textsf{cocode}_0$ over an elegant base theory in Section 3.3.2

Fifth, the theorem has some interesting corollaries as follows. Let $\textsf{accu}_0$ be $\textsf{accu}$ with the consequent weakened to stating that A is countable. In contrast to $\textsf{BWC}_0^j$, the former principles for countable sets are weak, as follows.

Corollary 2.32. *The system* $\textsf{ACA}_0^\omega$ *proves* $\textsf{accu}_0$ *and* $\textsf{ccc}_0$.

Proof. Note that $\textsf{ccc}_0$ was proved in $\textsf{ACA}_0^\omega$ in the proof of the theorem. To prove $\textsf{accu}_0$, note that (2.10) yields an injection to $\mathbb{Q}$. □

Let $\textsf{accu}_1$ be the restriction of $\textsf{accu}$ to infinite sets $A \subset \mathbb{R}$ and with conclusion weakened to: there is a bijection from A to $\mathbb{N}$. Similarly, let $\textsf{ccc}_1$ be $\textsf{ccc}$ with the weaker conclusion 'A is strongly countable' for infinite $A \subset \mathbb{R}$.

Corollary 2.33. *The system* $\textsf{ACA}_0^\omega$ *proves* $\textsf{cocode}_0 \leftrightarrow [\textsf{accu}_1 + \textsf{cocode}_1]$ *and* $\textsf{ccc}_1 \leftrightarrow \textsf{CB}\mathbb{N} \leftrightarrow \textsf{accu}_1$.

Proof. For the second part, $\textsf{RCA}_0^\omega$ proves $\textsf{ccc}_0$ by the proof of the theorem. Applying $\textsf{CB}\mathbb{N}$ to the conclusion of the former for infinite sets, one obtains $\textsf{ccc}_1$. The proof of $\textsf{ccc}_1 \to \textsf{accu}_1$ follows from the proof of $\textsf{ccc} \to \textsf{accu}$. Finally, the proof of Corollary 2.27 is readily adapted to $\textsf{accu}_1 \to \textsf{CB}\mathbb{N}$. For the first part, we note that $\textsf{accu}_1$ only deals with infinite sets. To obtain the same results for finite sets $A \subset \mathbb{R}$, consider the infinite set $B := A \cup \mathbb{Q}$ and note that μ^2 can enumerate all elements in $A \cap \mathbb{Q}$. Given an enumeration of B, one similarly obtains an enumeration of A. □

The first equivalence is interesting as the left-hand side (only) deals with injections, while the right-hand side (only) deals with bijections. Similarly, we have $\textsf{BWC}_0 \leftrightarrow [\textsf{ccc}_1 + \textsf{BW}_1]$ by Corollary 2.15. Thus, we have provided an answer to question (Q2) from Section 1.2. Next, we consider $\textsf{CWO}^\omega$ as follows.

Theorem 2.34. *The system* $\textsf{ACA}_0^\omega$ *proves* $\textsf{cocode}_0 \leftrightarrow [\textsf{CWO}^\omega + \textsf{IND}_0]$.

Proof. For $\textsf{cocode}_0 \to \textsf{CWO}^\omega$, use the proof that $\textsf{ATR}_0 \to \textsf{CWO}$ over $\textsf{RCA}_0$ from [97, V.6.8]. Note that $\textsf{ACA}_0^\omega + \textsf{BOOT}_C^-$ proves $\textsf{ATR}_0$ by [97, V.5.2]. Recall that $\textsf{cocode}_0 \to \textsf{IND}_0$ is proved in Theorem 2.7.

For $[\textsf{CWO}^\omega + \textsf{IND}_0] \to \textsf{cocode}_0$, let $Y : \mathbb{R} \to \mathbb{N}$ be injective on $A \subset [0, 1]$. In case $(\exists m \in \mathbb{N})(\forall x \in A)(Y(x) \leq m)$, $\textsf{IND}_0$ provides an enumeration of A as we have

$$(\forall n \in \mathbb{N})(\exists \text{ at most one } x \in [0, 1])(x \in A \wedge Y(x) = n).$$

Hence, we may assume $(\forall m \in \mathbb{N})(\exists x \in A)(Y(x) \geq m)$. Now define the linear order $(A, \preceq_A)$ via the following formula:

$$x \preceq_A y \equiv \big[Y(y) = n_0 \vee [Y(x) \neq n_0 \wedge Y(x) \leq_{\mathbb{N}} Y(y)]\big],$$

where $n_0 \in \mathbb{N}$ is the least $n \in \mathbb{N}$ such that $(\exists x \in A)(Y(x) = n)$; this number is readily defined using $\textsf{IND}_0$. Let $y_0 \in A$ be such that $Y(y_0) = n_0$. Intuitively, $(A, \preceq_A)$ has order type $\omega + 1$, i.e. the order of $\mathbb{N}$ followed by one element. Hence, of the four different possibilities provided by the consequent of $\textsf{CWO}^\omega$, three lead to contradiction. Indeed, a finite initial segment of either $(\mathbb{N}, \leq_{\mathbb{N}})$ or $(A, \preceq_A)$ has only got finitely many elements (since Y is an injection), while $\mathbb{N}$ is infinite and A satisfies $(\forall m \in \mathbb{N})(\exists x \in A)(Y(x) \geq m)$. Similarly, an order-isomorphism $W : A \to \mathbb{N}$ leads to contradiction as follows: since there is $y_0 \in A$ such that $Y(y_0) = n_0$, there cannot be a injection from $A \setminus \{y_0\}$ to $\{0, 1, \dots, W(y_0)\}$, as the latter set is finite, while the former is not. Similarly, an order-isomorphism $Z : \mathbb{N} \to A$ yields a contradiction as any $n \geq n_0$ is mapped below $Z(n_0) \in A$ (relative to $\preceq_A$), which is not possible as Y is an injection. The only remaining possibility is that CWO^ω provides an order-isomorphism $Z : \mathbb{N} \to A \setminus \{y_0\}$, where $A \setminus \{y_0\} = \{y \in A : y \prec y_0\}$ is an initial segment of A. The morphism Z is then a sequence satisfying $(\forall x \in A \setminus \{y_0\})(\exists n \in \mathbb{N})(Z(n) =_{\mathbb{R}} x)$, i.e. we obtain an enumeration of A. □

Theorem 2.35. *The system* ACA_0^ω *proves* $\mathsf{cocode}_0 \leftrightarrow \mathsf{cloq}$.

Proof. To prove $\mathsf{cocode}_0 \to \mathsf{cloq}$, use the well-known 'back-and-forth' proof based on the enumeration of A (see [77, p. 123]). By Theorem 2.11, we only need to prove $\mathsf{cloq} \to \mathsf{range}_0$ in ACA_0^ω. To this end, fix $A \subset [0, 1]$ and let $Y : [0, 1] \to \mathbb{N}$ be countable in A. Wlog we may assume that $0, 1 \notin A$. Now define the set $R \subset \mathbb{R}$ as follows: $y \in R$ if and only we have either $(\exists n \in \mathbb{N})(y =_{\mathbb{R}} n)$, or the following holds

$$(\exists q \in \mathbb{Q})(|y - q| \in A) \wedge (\forall m \in \mathbb{N})(m < |y| < m + 1 \to Y(|y - q|) = m).$$

Clearly, the set R is countable and $(R, \leq_{\mathbb{R}})$ is a linear order. Apply cloq to obtain $Q \subset \mathbb{Q}$ and $Z : R \to \mathbb{Q}$ such that Z is an order-isomorphism from $(R, \leq_{\mathbb{R}})$ to $(Q, \leq_{\mathbb{Q}})$. Now consider the following formula where $n \in \mathbb{N}$:

$$\begin{aligned}(\exists x \in A)(Y(x) = n) &\leftrightarrow (\exists y \in (n, n + 1))(y \in R) \\ &\leftrightarrow (\exists q \in Q)(Z(n) <_{\mathbb{Q}} q <_{\mathbb{Q}} Z(n + 1)). \end{aligned} \tag{2.12}$$

The first equivalence holds by the definition of R, while the second equivalence follows from the fact that Z is an order-isomorphism. Since (2.12) is decidable given $(\exists^2)$, range_0 is now immediate. □

Inspired by the previous proof, a version of Hausdorf's decomposition theorem for countable linear orders (see [16, Theorem 12] for the second-order RM version) should imply cocode_0. In turn, the previous proof inspires the following corollary.

Corollary 2.36. *The system* ACA_0^ω *proves* $\mathsf{cocode}_0 \leftrightarrow [\mathsf{cloq}' + \mathsf{IND}_0]$.

Proof. To prove $\mathsf{cocode}_0 \to \mathsf{cloq}'$, use the well-known 'back-and-forth' proof based on the enumeration of A (see [77, p. 123]). To prove $\mathsf{cloq}' \to \mathsf{range}_0$, fix $A \subset [0, 1]$ and let $Y : [0, 1] \to \mathbb{N}$ be countable in A. Wlog we may assume that $A \cap \mathbb{Q} = \emptyset$ as Feferman's μ^2 allows us to list the rationals in A. Now define the set $R' \subset \mathbb{R}$ as follows: $y \in R'$ if and only we have either $(\exists q \in \mathbb{Q})(y =_{\mathbb{R}} q)$, or the following holds

$$(\exists q \in \mathbb{Q})(|y - q| \in A) \wedge (\forall m \in \mathbb{N})(m <_{\mathbb{R}} |y| <_{\mathbb{R}} m + 1 \to Y(|y - q|) = m).$$

Clearly, the set R' is countable and $(R', \leq_{\mathbb{R}})$ is a dense linear order without end points. Apply cloq' to obtain an order-isomorphism Z from $(R', \leq_{\mathbb{R}})$ to $(\mathbb{Q}, \leq_{\mathbb{Q}})$. Now consider the following formula where $n \in \mathbb{N}$:

$$\begin{aligned}(\exists x \in A)(Y(x) = n) &\leftrightarrow (\exists y \in (n, n + 1))(y \in R' \wedge y \text{ is irrational}) \\ &\leftrightarrow (\exists q \in \mathbb{Q} \cap (Z(n), Z(n + 1)))(\forall r \in \mathbb{Q} \cap (n, n + 1))(Z(r) \neq_{\mathbb{Q}} q). \end{aligned} \tag{2.13}$$

The first equivalence holds by definition while the second equivalence follows from the fact that Z is an order-isomorphism. As for the theorem, range_0 follows. □

Restricting cloq' to strongly countable sets, one readily obtains an equivalence to $\mathsf{cocode}_1 + \mathsf{IND}_1$ by introducing an extra condition '$x > p$' in (2.13) with $p \in \mathbb{Q}$.

Finally, as to related research, Mal'tsev's theorem on countable ordered groups ([55]) is studied in second-order RM ([98]), and seems to imply cocode_0.

3. The bigger picture

Section 2 yields many (robust) equivalences for the Bolzano-Weierstrass theorem as in BW_0 and BWC_1. With these in place, it is time to connect the latter to the bigger picture, namely ordinary mathematics and set theory, as follows.

- In Section 3.1, we connect the Bolzano-Weierstrass theorem as in BWC_0 to the Heine-Borel theorem and the Lindelöf lemma as studied in [68, 69].
- We connect BWC_0 to the *countable union theorem* from set theory (Section 3.2); a natural restriction of the latter is equivalent to the former.
- In Section 3.3, we show that BWC_0 is equivalent to the *Jordan decomposition theorem* and similar results on functions of *bounded variation*. We also consider theorems on *regulated* functions.

Regarding the final item, the Jordan decomposition theorem and its ilk have no obvious or direct connection to countability at all, and have been studied in second-order RM ([49, 65]).

3.1. **Heine-Borel and Lindelöf.**

3.1.1. *Introduction.* In this section, we connect the Bolzano-Weierstrass theorem as in BWC_0 to the Heine-Borel theorem and the Lindelöf lemma. An overview of our results is as follows.

In Section 3.1.2, we identify weak/countable versions of the Heine-Borel theorem and Lindelöf lemma that are equivalent to BW_0. In Section 3.1.3, we show that for LIN, a most general version of the Lindelöf lemma for $\mathbb{N}^{\mathbb{N}}$, we have $\mathsf{BOOT} + \mathsf{QF}\text{-}\mathsf{AC}^{0,1} \to \mathsf{LIN} \to \mathsf{BWC}_0$, working over ACA_0^ω. In Section 3.1.4, assuming a fragment of the induction axiom, we similarly establish:

$$\mathsf{BOOT} \to \mathsf{HBT} \to \mathsf{BWC}_0 \to \mathsf{BWC}_1. \tag{3.1}$$

Recall that BOOT and HBT are the higher-order counterparts of ACA_0 and WKL_0 (see Remark 1.12). In this light, higher-order RM yields a much richer picture than its second-order counterpart, in that there are at least two extra 'Big' systems.

Next, the following series of implications is also established in Section 3.1.4, *without* the use of extra induction:

$$\mathsf{BOOT} \to \Sigma\text{-}\mathsf{SEP} \to \mathsf{BWC}_0 \to \mathsf{BWC}_1, \tag{3.2}$$

where $\Sigma\text{-}\mathsf{SEP}$ is the higher-order counterpart of Σ_1^0-separation. The latter is equivalent to WKL_0 by [97, IV.4.4] and ECF maps $\Sigma\text{-}\mathsf{SEP}$ to Σ_1^0-separation; we believe that HBU 'speaks more to the imagination' than $\Sigma\text{-}\mathsf{SEP}$. Moreover, $\mathsf{HBU} \leftrightarrow \mathsf{HBT} \leftrightarrow \Sigma\text{-}\mathsf{SEP}$ is established in Section 3.1.4, assuming extra axioms discussed next.

Finally, we should say a few words on the *neighbourhood function principle* NFP from [103, p. 215]. Restricted to the L_2-langauge, NFP is equivalent to the usual comprehension principle of Z_2. Now, the higher-order generalisation of comprehension, in the form of the functionals S_k^2, does not provide a satisfactory classification of e.g. HBU. Indeed, we know that Z_2^Ω proves $\mathsf{BOOT}, \mathsf{HBU}, \mathsf{BWC}_0, \mathsf{BWC}_1$ and Z_2^ω does not, while of course $\mathsf{Z}_2 \equiv_{\mathsf{L}_2} \mathsf{Z}_2^\omega \equiv_{\mathsf{L}_2} \mathsf{Z}_2^\Omega$. As explored in [73, 83, 84], the higher-order generalisation of NFP provides a more satisfactory classification of these principles: there are natural fragments of NFP equivalent to BOOT, HBU, and the Lindelöf lemma, *assuming* a fragment of NFP called A_0, discussed in Section 3.1.4. By Theorem 3.10 and Corollary 3.11, we have $\mathsf{HBU} \leftrightarrow \Sigma\text{-}\mathsf{SEP} \leftrightarrow \mathsf{HBT}$ working over $\mathsf{ACA}_0^\omega + \mathsf{A}_0$.

We should not have to point out that second-order RM assumes/needs Δ_1^0-comprehension in the base theory. Thus, it stands to reason that the development of RM based on NFP requires a fragment of the latter, like the A_0 axiom, in the base theory. This argument is explored at length in [83, 84]. Moreover, by Corollary 3.12, there is even a fragment of NFP, similar to A_0, that is equivalent to BWC_0.

3.1.2. *Countable coverings.* We connect BWC_0 to versions of the Heine-Borel theorem and Lindelöf lemma for coverings that are countable as in Definition 1.4.

First of all, the following version of the 'countable' Heine-Borel theorem implies NIN by [73, Cor. 3.20], but no reversal is known.

Principle 3.1 (HBC_0)**.** *For countable $A \subset \mathbb{R}^2$ with $(\forall x \in [0,1])(\exists (a,b) \in A)(x \in (a,b))$, there are $(a_0,b_0), \dots (a_k,b_k) \in A$ with $(\forall x \in [0,1])(\exists i \leq k)(x \in (a_i,b_i))$.*

The Heine-Borel theorem for different representations of open coverings is studied in RM ([94]), i.e. the motivation for HBC_0 is already present in second-order RM. Moreover, Borel in [6, p. 42] uses 'countable infinity of intervals' and **not** 'sequence of intervals' in his formulation[10] of the Heine-Borel theorem. He also mentions in [6, p. 42, Footnote (1)] a 'theoretical method' for 'effectively determining' the finite sub-covering at hand. In this light, we may assume that the finite sub-covering in HBC_0 is given by a finite sequence of reals without fear of adding 'extra data'.

We shall study a 'sequential' version of HBC_0 involving sequences of (sub-)coverings. Such sequential theorems are well-studied in RM, starting with [97, IV.2.12], and also in [21, 22, 29, 30, 36, 37, 106].

Principle 3.2 ($\mathsf{HBC}_0^{\mathsf{seq}}$)**.** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathbb{R}^2$ with countable union. Then there is $(b_n)_{n\in\mathbb{N}}$ such that for $n \in \mathbb{N}$, b_n is a finite sequence of elements of A_n and if the intervals in A_n cover $[0,1]$, then so do the intervals in b_n.*

On a related note, let LIN_0 be the Lindelöf lemma for countable sets $A \subset \mathbb{R}$, i.e. for $\Psi : \mathbb{R} \to \mathbb{R}^+$, there is $(x_n)_{n\in\mathbb{N}}$ in A such that $\cup_{n\in\mathbb{N}} B(x_n, \Psi(x_n))$ covers A. We have the following theorem connecting the aforementioned principles.

Theorem 3.3. *The system ACA_0^ω proves $\mathsf{LIN}_0 \leftrightarrow \mathsf{cocode}_0 \leftrightarrow \mathsf{HBC}_0^{\mathsf{seq}}$.*

Proof. The implication $\mathsf{LIN}_0 \leftarrow \mathsf{cocode}_0$ is trivial, while the reversal follows from $\mathsf{LIN}_0 \to \mathsf{accu}$ which in turn follows from applying LIN_0 to the covering provided by $\frac{1}{2^{G(x)+1}}$ as in (2.10). The implication $\mathsf{cocode}_0 \to \mathsf{HBC}_0^{\mathsf{seq}}$ is also straightforward as cocode_0 allows us to convert countable sets into sequences. The usual second-order proof from [97, IV.1] in WKL now yields HBC_0, while [106, Theorem 2.7] yields the sequential version, also working in WKL_0.

Finally, to obtain $\mathsf{HBC}_0^{\mathsf{seq}} \to \mathsf{cocode}_0$, fix a set $A \subset [0,1]$ with $Y : [0,1] \to \mathbb{N}$ injective on A. Now define the sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ as follows:

$$(a,b) \in A_n \leftrightarrow \left[\tfrac{a+b}{2} \in A \wedge Y(\tfrac{a+b}{2}) = n \wedge b - a = 4\max\left(|1 - \tfrac{a+b}{2}|, \tfrac{a+b}{2}\right)\right].$$

[10] In fact, Borel's explicitly mentions a version of cocode_1 in [6, p. 6] while the proof of the Heine-Borel theorem in [6, p. 42] starts with an application of cocode_1 and then proceeds with the usual 'interval-halving' proof, similar to Cousin's proof in [18].

By definition, each A_n has at most one element and the union is countable as $\cup_{n\in\mathbb{N}}A_n$ is a variation of A. Let $(b_n)_{n\in\mathbb{N}}$ be as provided by $\mathsf{HBC}_0^{\mathsf{seq}}$ and note that

$$(\exists x \in A)(Y(x) = n) \leftrightarrow (\exists (a,b) \in b_n)(Y(\tfrac{a+b}{2}) = n),$$

which immediately yields cocode_0, and we are done. □

Note that the formulation of $\mathsf{HBC}_0^{\mathsf{seq}}$ avoids the countable union theorem, which happens to be the topic of Section 3.2. Theorem 3.3 also has a certain robustness: the second equivalence still goes through if we let $(b_n)_{n\in\mathbb{N}}$ be a sequence of non-empty finite sets, while assuming cocode_1. Moreover, we believe that many sequential versions of theorems are equivalent to cocode_0, like e.g. ADS and RT_2^2 from the RM zoo (see [35]). An exception is cloq', as shown in Section 3.2.

Finally, by Theorem 3.3, the general Lindelöf lemma for *any* set $A \subset \mathbb{R}$ is quite explosive, yielding Π_2^1-CA_0 when combined with Π_1^1-CA_0^ω. Nonetheless, we show in the next section that this general version is still provable from $\mathsf{BOOT} + \mathsf{QF}\text{-}\mathsf{AC}^{0,1}$.

3.1.3. *A general Lindelöf lemma.* We show that a most general formulation of the Lindelöf lemma still follows from BOOT. We have established a similar result for the Heine-Borel theorem for uncountable coverings of closed sets in [70, Theorem 4.5]. We note that Lindelöf proves his eponymous lemma for *any* set in $\mathbb{R}^n$ in [53].

Principle 3.4 (LIN)**.** *For any G^2 and $D \subseteq \mathbb{N}^\mathbb{N}$, there is $(f_n)_{n\in\mathbb{N}}$ in D such that $\cup_{n\in\mathbb{N}}[\overline{f_n}G(f_n)]$ covers D.*

The following theorem is the main result of this section.

Theorem 3.5. *The system $\mathsf{ACA}_0^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,1} + \mathsf{BOOT}$ proves LIN.*

Proof. Fix a non-empty set $D \subset \mathbb{N}^\mathbb{N}$ and G^2 and let $(\sigma_n)_{n\in\mathbb{N}}$ be a list of all finite sequences. Use BOOT to define $X \subset \mathbb{N}$ such that

$$n \in X \leftrightarrow (\exists f \in D)\big(f \in [\sigma_n] \wedge \sigma_n =_{0^*} \overline{f}G(f)\big). \tag{3.3}$$

Define τ_0 as σ_{n_0} where $n_0 := (\mu n)(n \in X)$, and define τ_{n+1} as σ_{n+1} if $n+1 \in X$, and τ_n otherwise. Then $\cup_{n\in\mathbb{N}}[\tau_n]$ also covers D, but we still need to 'identify' the associated $f \in D$ from (3.3). To this end, apply $\mathsf{QF}\text{-}\mathsf{AC}^{0,1}$ to

$$(\forall n \in X)(\exists f \in D)(f \in [\sigma_n] \wedge \sigma_n =_{0^*} \overline{f}G(f)).$$

The resulting sequence provides the countable sub-covering as required by the conclusion of Principle 3.4. □

As shown in [68, 71], the Lindelöf lemma for the full Baire space yields Π_1^1-CA_0 when combined with $(\exists^2)$. Moreover, by Theorem 3.3, $\mathsf{LIN} \rightarrow \mathsf{cocode}_0$ is immediate, implying that that Π_1^1-$\mathsf{CA}_0^\omega + \mathsf{LIN}$ proves Π_2^1-CA_0. We also have

$$[\mathsf{BOOT} + \mathsf{QF}\text{-}\mathsf{AC}^{0,1}] \rightarrow \mathsf{LIN} \rightarrow \mathsf{BWC}_0 \rightarrow \mathsf{BWC}_1.$$

Finally, since Baire space is not σ-compact, we believe the use of countable choice in the previous proof to be essential.

3.1.4. *Uncountable coverings.* In this section, we connect HBT and related principles to BWC_0 as sketched in Section 3.1.1

First of all, in more detail, our main result is $\mathsf{HBU} \to \mathsf{BWC}_0$ assuming an extra axiom A_0 introduced in [83, 84] and discussed below. This implication is established using the intermediate principle Σ-SEP as in Principle 3.6. The latter is the third-order counterpart of the Σ^0_1-separation principle, which is equivalent to WKL_0 by [97, IV.4.4]. Since HBU is the higher-order counterpart of WKL_0, one expects $\mathsf{HBU} \leftrightarrow \Sigma$-$\mathsf{SEP}$, which is indeed proved in Theorem 3.10, also assuming A_0. Regarding (3.1), weakening A_0 is possible as in (3.12). We note that ECF maps both HBU and Σ-SEP to WKL_0, while $\mathsf{A}_0, \mathsf{BWC}_0, \mathsf{BWC}_1$ are trivial under ECF. Moreover, a version of A_0 turns out to be equivalent to cocode_0 by Corollary 3.12.

Secondly, we have previously considered a separation principle in connection to HBU in [83], namely as follows.

Principle 3.6 (Σ-SEP)**.** *For $i = 0, 1$, Y_i^2, and $\varphi_i(n) \equiv (\exists f_i \in \mathbb{N}^\mathbb{N})(Y_i(f_i, n) = 0)$,*

$$(\forall n \in \mathbb{N})(\neg\varphi_0(n) \vee \neg\varphi_1(n)) \to (\exists Z \subset \mathbb{N})(\forall n \in \mathbb{N})\big[\varphi_0(n) \to n \in Z \wedge \varphi_1(n) \to n \notin Z\big].$$

The following theorem implies that Π^1_1-$\mathsf{CA}^\omega_0 + \Sigma$-$\mathsf{SEP}$ proves Π^1_2-CA_0, which also follows immediately from [97, VII.6.14].

Theorem 3.7. *The system ACA^ω_0 proves Σ-$\mathsf{SEP} \to \mathsf{cocode}_0$.*

Proof. Let $Y : \mathbb{R} \to \mathbb{N}$ be injective on the non-empty set $A \subset [0, 1]$. Define the formula $\varphi_i(n, q)$ as follows where $n \in \mathbb{N}$ and $q \in \mathbb{Q} \cap (0, 1)$:

$$\varphi_0(n, q) \equiv (\exists x \in A)(Y(x) = n \wedge x >_\mathbb{R} q) \tag{3.4}$$

$$\varphi_1(n, q) \equiv (\exists x \in A)(Y(x) = n \wedge x \leq_\mathbb{R} q). \tag{3.5}$$

Since Y is injective on A, we have $(\forall n \in \mathbb{N}, q \in \mathbb{Q} \cap (0, 1))(\neg\varphi_0(n, q) \vee \neg\varphi_1(n, q))$. Let $Z \subset \mathbb{N} \times \mathbb{Q}$ be as in Σ-SEP and note that for $n \in \mathbb{N}, q \in \mathbb{Q} \cap (0, 1)$, we have

$$(n, q) \in Z \to (\forall x \in A)(Y(x) = n \to x >_\mathbb{R} q), \tag{3.6}$$

$$(n, q) \notin Z \to (\forall x \in A)(Y(x) = n \to x \leq_\mathbb{R} q). \tag{3.7}$$

Based on (3.6) and (3.7), define a sequence $(x_n)_{n\in\mathbb{N}}$ of reals in $[0, 1]$ as follows: $[x_n](0)$ is $\frac{1}{2}$ if $(n, \frac{1}{2}) \in Z$, and 0 otherwise; $[x_n](k+1)$ is $[x_n](k)+\frac{1}{2^{k+1}}$ if $(n, [x_n](k)+\frac{1}{2^{k+1}}) \in Z$, and $[x_n](k)$ otherwise. Using Feferman's μ^2, define $(y_n)_{n\in\mathbb{N}}$ as a subsequence (possibly with repetitions) of $(x_n)_{n\in\mathbb{N}}$ such that $(\forall n \in \mathbb{N})(y_n \in A)$. Then $(y_n)_{n\in\mathbb{N}}$ is an enumeration of A such that for all $k \in \mathbb{N}$:

$$(\exists x \in A)(Y(x) = k) \leftrightarrow (\exists m \in \mathbb{N})(Y(y_m) = k). \tag{3.8}$$

Indeed, the reverse implication in (3.8) is immediate by the definition of $(y_n)_{n\in\mathbb{N}}$. For the forward implication if $(\exists x \in A)(Y(x) = k)$ for fixed $k \in \mathbb{N}$, then $Y(x_k) = k$ and $x_k \in A$, by the definition of $(x_n)_{n\in\mathbb{N}}$. Hence, the right-hand side of (3.8) follows, and we observe that $(y_n)_{n\in\mathbb{N}}$ enumerates A. □

We can obtain an equivalence via the following 'at most one' condition:

$$(\forall i \in \{0, 1\})(\forall n \in \mathbb{N})(\exists \text{ at most one } f \in 2^\mathbb{N})(Y_i(f, n) = 0). \tag{3.9}$$

Let Σ-SEP^-_C be Σ-SEP with all type 1 quantifiers restricted to $2^\mathbb{N}$ and (3.9).

Corollary 3.8. *The system ACA^ω_0 proves Σ-$\mathsf{SEP}^-_C \leftrightarrow \mathsf{cocode}_0$.*

Proof. The forward implication is immediate from the proof of the theorem as (3.4) and (3.5) satisfy the required 'at most one' conditions. For the reverse implication, let Y_i^2 be as in $\Sigma\text{-}\mathsf{SEP}_C^-$ and define $A_i := \{f \in 2^{\mathbb{N}} : (\exists n \in \mathbb{N})(Y_i(f,n) = 0)\}$. Clearly, this set is countable as $Z_i(f) := (\mu n)(Y_i(f,n) = 0)$ yields an injection on A_i. Hence, cocode_0 provides an enumeration $(f_m)_{m\in\mathbb{N}}$ of A_0, implying

$$\varphi_0(n) \leftrightarrow (\exists f \in 2^{\mathbb{N}})(Y_0(f,n) = 0) \leftrightarrow (\exists m \in \mathbb{N})(Y(f_m,n) = 0),$$

i.e. $\varphi_0(n)$ is decidable modulo $\exists^2$. The same holds for $\varphi_1(n)$ and we are done. $\square$

Next, as shown in [83, §5] and [84], HBU, BOOT, and the Lindelöf lemma are equivalent to elegant fragments of the *neighbourhood function principle* NFP from [103]. In the same way as Δ_1^0-comprehension is included in RCA_0, the RM of NFP warrants a base theory that includes the following fragment of NFP, as discussed *at length and in minute detail* in [83, §5] and [84, §3.5].

Definition 3.9. $[\mathsf{A}_0]$ For Y^2 and $A(\sigma^{0^*}) \equiv (\exists f \in 2^{\mathbb{N}})(Y(f,\sigma) = 0)$, we have

$$(\forall f \in \mathbb{N}^{\mathbb{N}})(\exists n \in \mathbb{N})A(\overline{f}n) \to (\exists \Phi^2)(\forall f \in \mathbb{N}^{\mathbb{N}})A(\overline{f}\Phi(f)).$$

Recall the equivalence from [97, X.4.4] between Σ_1^0-induction and bounded Σ_1^0-comprehension. As noted above, IND_0 occupies the same category as the latter axiom, while an equivalence between HBU and $\Sigma\text{-}\mathsf{SEP}$ needs *bounded separation*, as follows. The axiom '$\mathsf{bounded}\text{-}\Sigma\text{-}\mathsf{SEP}$' is $\Sigma\text{-}\mathsf{SEP}$ weakened such that for any $k \in \mathbb{N}$:

$$(\forall n \leq k)(\neg\varphi_0(n)\vee\neg\varphi_1(n)) \to (\exists Z \subset \mathbb{N})(\forall n \leq k)\big[\varphi_0(n) \to n \in Z\wedge\varphi_1(n) \to n \notin Z\big].$$

Clearly, $\mathsf{bounded}\text{-}\Sigma\text{-}\mathsf{SEP}$ only provides a finite/bounded fragment of the separating set from $\Sigma\text{-}\mathsf{SEP}$, and the former follows from the induction axiom. We now have the following theorem which establishes (3.1).

Theorem 3.10. *The system* $\mathsf{ACA}_0^\omega + \mathsf{A}_0$ *proves* $[\mathsf{HBU} + \mathsf{bounded}\text{-}\Sigma\text{-}\mathsf{SEP}] \leftrightarrow \Sigma\text{-}\mathsf{SEP}$*; the reverse implication holds over* ACA_0^ω*.*

Proof. Assume HBU and suppose $\neg\Sigma\text{-}\mathsf{SEP}$. Fix Y_0, Y_1 as in the latter and let $A(\overline{Z}n)$ be the following, i.e. the formula in square brackets in $\Sigma\text{-}\mathsf{SEP}$:

$$(\varphi_0(n) \to n \in Z) \wedge (\varphi_1(n) \to n \notin Z), \tag{3.10}$$

where the notation '$\overline{Z}n$' in $A(\overline{Z}n)$ is justified by noting that the set Z is only invoked in (3.10) in the form '$n \in Z$'. By assumption, we have $(\forall Z \subset \mathbb{N})(\exists n \in \mathbb{N})\neg A(\overline{Z}n)$, which has the right form to apply A_0. Hence, there is $G : 2^{\mathbb{N}} \to \mathbb{N}$ such that $(\forall Z \subset \mathbb{N})\neg A(\overline{Z}G(Z))$. Apply HBU to obtain $f_1, \dots, f_k \in 2^{\mathbb{N}}$, a finite sub-covering of the canonical covering $\cup_{f\in 2^{\mathbb{N}}}[\overline{f}G(f)]$. Define $n_0 := \max_{i\leq k} G(f_i)$ and note that $(\forall Z \subset \mathbb{N})(\exists n \leq n_0)\neg A(\overline{Z}n)$. However, $\mathsf{bounded}\text{-}\Sigma\text{-}\mathsf{SEP}$ provides a set $Z_0 \subset \mathbb{N}$ such that for $m \leq n_0 + 1$, we have $A(\overline{Z_0}m)$, a contradiction, and we are done.

For the reverse implication, $\Sigma\text{-}\mathsf{SEP}$ implies $\mathsf{bounded}\text{-}\Sigma\text{-}\mathsf{SEP}$. Now assume $\Sigma\text{-}\mathsf{SEP}$ and suppose HBU fails for $\Psi_0 : [0,1] \to \mathbb{R}^+$. Consider the following for $q \in \mathbb{Q}\cap(0,1)$:

$$\varphi_0(q) \equiv (\exists w^{1^*})\big[(\forall i < |w|)(w(i) \in [0,1]) \wedge [0,q] \subset \cup_{i<|w|} I_{w(i)}^{\Psi_0}\big],$$

$$\varphi_1(q) \equiv (\exists v^{1^*})\big[(\forall j < |v|)(v(j) \in [0,1]) \wedge [q,1] \subset \cup_{j<|v|} I_{v(j)}^{\Psi_0}\big],$$

where $(\forall q \in \mathbb{Q}\cap(0,1))(\neg\varphi_0(q)\vee\neg\varphi_1(q))$ by assumption. Let $Z_0 \subset \mathbb{N}$ be as provided by $\Sigma\text{-}\mathsf{SEP}$ and define a real $x_0 \in [0,1]$ as follows. Define $[x_0](0)$ as $\frac{1}{2}$ if $\frac{1}{2} \in Z_0$, and

0 otherwise; define $[x_0](k+1)$ as $[x_0](k) + \frac{1}{2^{k+1}}$ if $[x_0](k) + \frac{1}{2^{k+1}} \in Z$, and $[x_0](k)$ otherwise. By definition, the real x_0 satisfies the following:

$$(\forall w^{1^*})\big[(\forall i < |w|)(w(i) \in [0,1]) \to \big([x_0](k), [x_0](k) + \tfrac{1}{2^{k+1}}\big) \not\subset \cup_{i<|w|} I^{\Psi_0}_{w(i)}\big], \quad (3.11)$$

which immediately yields a contradiction as $\big([x_0](k), [x_0](k) + \frac{1}{2^{k+1}}\big) \subset I^{\Psi_0}_{x_0}$ for k large enough, and we are done. □

Corollary 3.11. *The system* ACA_0^ω *proves* $[\mathsf{HBT} + \text{bounded-}\Sigma\text{-}\mathsf{SEP}] \leftrightarrow \Sigma\text{-}\mathsf{SEP}$.

Proof. The reverse implication readily follows from the second part of the proof of the theorem. For the forward implication, consider $(\forall Z \subset \mathbb{N})(\exists n \in \mathbb{N})\neg A(\overline{Z}n)$ as in the proof of the theorem. As noted above, we may use $\exists^2$ to code $\mathbb{N} \to \mathbb{N}$ sequences as binary sequences. Let Y be the characteristic function of the formula obtained by omitting the leading existential quantifiers (over $2^{\mathbb{N}}$) of $\neg A(\sigma)$. Define the function $\psi : [0,1] \to \mathbb{R}$ as follows: $\psi(x) := 0$ if there is no initial segment σ^{0^*} of the binary expansion $\sigma * f$ of x such that $Y(f,\sigma) = 0$; otherwise $\psi(x) := \frac{1}{2^k}$ where k is the length of the shortest such initial segment. Then ψ yields a covering of $[0,1]$ to which HBT applies. In the same was as in the proof of the theorem, one obtains a contradiction using bounded-Σ-SEP. □

It is straightforward to show that HBT implies the fragment of A_0 needed to prove $\mathsf{HBU} \to \mathsf{HBT}$. Another interesting exercise is to consider A_0^- which is A_0 with the extra condition $(\forall \sigma^{0^*} \leq_{0^*} 1)(\exists$ at most one $f \in 2^{\mathbb{N}})(Y(f,\sigma) = 0)$. Using the above results, one readily shows that over ACA_0^ω:

$$\mathsf{BOOT} \to [\mathsf{HBU} + \mathsf{A}_0^-] \to \mathsf{cocode}_0 \to \mathsf{A}_0^-, \quad (3.12)$$

$$[\mathsf{HBU} + \text{ bounded-}\Sigma\text{-}\mathsf{SEP} + \mathsf{A}_0^-] \leftrightarrow \Sigma\text{-}\mathsf{SEP}. \quad (3.13)$$

What is more important is the following corollary to Theorem 3.10 related to A_0^-. Let Σ-NFP_C^- be A_0^- with the conclusion strengthened as in NFP, i.e. $(\exists \gamma \in K_0)(\forall f \in 2^{\mathbb{N}})A(\overline{f}\gamma(f))$. Note that '$\gamma \in K_0$' is the notation used in NFP from [103] for γ^1 being a total RM-code/associate. Let bounded-Σ-SEP_C^- be bounded-Σ-SEP with the same restrictions as Σ-SEP_C^-.

Corollary 3.12. *The system* ACA_0^ω *proves* $\mathsf{cocode}_0 \leftrightarrow [\Sigma\text{-}\mathsf{NFP}_C^- + \text{ bounded-}\Sigma\text{-}\mathsf{SEP}_C^-]$.

Proof. The forward implication is straightforward: BOOT_C^- makes $A(\sigma)$ from Σ-NFP_C^- decidable, i.e. there is X, up to coding a subset of $\mathbb{N}$, such that

$$(\forall \sigma^{0^*} \leq 1)\big[\sigma \in X \leftrightarrow (\exists f \in 2^{\mathbb{N}})(Y(f,\sigma) = 0)\big].$$

Using QF-$\mathsf{AC}^{1,0}$ (and induction), we obtain G^2 such that $(\forall f \in 2^{\mathbb{N}})A(\overline{f}G(f))$, where $G(f)$ is the least such number. Clearly, G^2 has an RM-code, and NFP_C^- follows.

For the reverse implication, we prove $[\Sigma\text{-}\mathsf{NFP}_C^- + \text{ bounded-}\Sigma\text{-}\mathsf{SEP}_C^-] \to \Sigma\text{-}\mathsf{SEP}_C^-$ and Corollary 3.8 finishes the proof. To obtain Σ-SEP_C^-, consider $A(\sigma)$ as in (3.10). Note that $(\forall Z \subset \mathbb{N})(\exists n \in \mathbb{N})\neg A(\overline{Z}n)$ has the right form to apply Σ-NFP_C^-. The resulting function $\gamma \in K_0$ has an upper bound given WKL by [97, IV.2.2]. Now use bounded-Σ-SEP_C^- to obtain a contradiction in the same way as in the proof of Theorem 3.10. Note that Corollary 3.8 yields cocode_0. □

Finally, A_1 is A_0 but for formulas $A(\sigma^{0^*}) \equiv (\forall f \in 2^{\mathbb{N}})(Y(f,\sigma) = 0)$ and proves the equivalence between accu and Principle 2.21. The axiom A_1 implies that any continuous function on $\mathbb{N}^{\mathbb{N}}$ has an associate/RM-code, as explored in [83, §5].

3.1.5. *More on separation.* In this section, we show that Π-SEP, a separation principle much weaker than Σ-SEP, implies cocode_1. We also obtain an equivalence based on a weakening of Π-SEP.

First of all, note that the following principle is readily proved by applying QF-$\mathsf{AC}^{0,1}$ to the antecedent (see also [97, V.5.7]). Theorem 3.14 is reminiscent of the fact that Π^1_1-separation implies Δ^1_1-comprehension.

Principle 3.13 (Π-SEP). *For $i = 0, 1$, Y_i^2, and $\varphi_i(n) \equiv (\forall f_i \in \mathbb{N}^{\mathbb{N}})(Y_i(f_i, n) = 0)$,*

$$(\forall n \in \mathbb{N})(\neg\varphi_0(n) \vee \neg\varphi_1(n)) \to (\exists Z \subset \mathbb{N})(\forall n \in \mathbb{N})\big[\varphi_0(n) \to n \in Z \wedge \varphi_1(n) \to n \notin Z\big].$$

Theorem 3.14. *The system ACA_0^ω proves Π-$\mathsf{SEP} \to \mathsf{cocode}_1$.*

Proof. Let $Y : \mathbb{R} \to \mathbb{N}$ be bijective on the non-empty set $A \subset [0, 1]$. Define the formula $\varphi_i(n, q)$ as follows where $n \in \mathbb{N}$ and $q \in \mathbb{Q} \cap (0, 1)$:

$$\varphi_0(n, q) \equiv (\forall x \in A)(Y(x) = n \to x >_{\mathbb{R}} q) \tag{3.14}$$

$$\varphi_1(n, q) \equiv (\forall x \in A)(Y(x) = n \to x \leq_{\mathbb{R}} q). \tag{3.15}$$

Since Y is bijective on A, we have $(\forall n \in \mathbb{N}, q \in \mathbb{Q} \cap (0, 1))(\neg\varphi_0(n, q) \vee \neg\varphi_1(n, q))$. Let $Z \subset \mathbb{N} \times \mathbb{Q}$ be as in Π-SEP and note that for $n \in \mathbb{N}, q \in \mathbb{Q} \cap (0, 1)$, we have

$$(n, q) \in Z \to (\exists x \in A)(Y(x) = n \wedge x >_{\mathbb{R}} q), \tag{3.16}$$

$$(n, q) \notin Z \to (\exists x \in A)(Y(x) = n \wedge x \leq_{\mathbb{R}} q). \tag{3.17}$$

Now proceed as in the proof of Theorem 3.7 to define an enumeration of A. □

Finally, let Π-$\mathsf{SEP}!$ be Π-SEP restricted to Y_i^2 such that

$$(\forall n \in \mathbb{N})(\exists! f \in 2^{\mathbb{N}})\big[Y_0(f, n) \neq 0 \vee Y_1(f, n) \neq 0\big], \tag{3.18}$$

and all type 1 quantifiers restricted to $2^{\mathbb{N}}$. We have the following corollary.

Corollary 3.15. *The system ACA_0^ω proves Π-$\mathsf{SEP}! \leftrightarrow \mathsf{cocode}_1$.*

Proof. The forward direction is immediate from the proof of the theorem as (3.18) is satisfied by the formulas (3.14) and (3.15). For the reverse implication, the set $\{f \in 2^{\mathbb{N}} : Y_0(f, n) \neq 0 \vee Y_1(f, n) \neq 0\}$ is strongly countable. The enumeration provided by cocode_1 readily provides the set Z from Π-$\mathsf{SEP}!$ and we are done. □

3.2. **Countable unions and the Axiom of Choice.** In this section, we study the connection between the Bolzano-Weierstrass theorem, the *countable union theorem* for $\mathbb{R}$, and the existence of sets not in the class $\mathbf{F}_\sigma$. By Corollary 3.20, there are natural versions of the countable union theorem equivalent to BWC_i for $i = 0, 1$.

First of all, the Axiom of Choice (AC for short) is perhaps the most (in)famous axiom of the usual foundations of mathematics, i.e. ZFC set theory. It is known that very weak fragments of AC are independent of ZF, like the *countable union theorem* which expresses that a countable union of countable (or even 2-element) sets is again countable. We refer to [32] for an overview of this kind of results on AC, while we note that Cantor already considered the countable union theorem in 1878, namely in [10, p. 243]. The countable union theorem involving enumerations and (codes of) analytic sets may be found in second-order RM as [97, V.4.10], i.e. the following principle is a quite natural object of study in higher-order RM. We discuss the naturalness and generality of CUC in Remark 3.24.

Principle 3.16 (CUC)**.** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathbb{R}$ such that for all $n \in \mathbb{N}$, there is an enumeration of A_n. Then there is an enumeration of $\cup_{n\in\mathbb{N}}A_n$.*

Note that we need $(\exists^2)$ to guarantee that the union in CUC exists. As noted above, the countable union theorem *for 2-element sets* is still unprovable in ZF. In this light, define $\mathsf{CUC}(2)$ as CUC where each A_n has exactly two elements, i.e.

$$(\forall x, y, z \in A_n)(x =_{\mathbb{R}} y \vee x =_{\mathbb{R}} z) \wedge (\exists w, v \in A_n)(w \neq_{\mathbb{R}} v). \tag{3.19}$$

The following principle is (possibly) weaker than the countable union theorem according to [32, Diagram 3.4, p. 23]: $\mathbb{R}$ *is not a countable union of countable sets.* We distill the following principle from the latter.

Principle 3.17 (RUC)**.** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathbb{R}$ such that for all $n \in \mathbb{N}$, there exists an enumeration of A_n. Then there is $y \in \mathbb{R}$ not in $\cup_{n\in\mathbb{N}}A_n$.*

Note that RUC fails in the model $\mathbf{Q}^*$ constructed in the proof of Theorem 2.16, i.e. $\neg\mathsf{RUC}$ is consistent with Z_2^ω. By [97, II.4.7], Cantor's theorem (that the reals cannot be enumerated) is provable in RCA_0, and hence $\mathsf{CUC} \to \mathsf{RUC}$ over ACA_0^ω. The connection between RUC and the following principle is however more interesting.

Principle 3.18 ($\mathbf{NF}_\sigma$)**.** *There exists a subset of $\mathbb{R}$ that is not $\mathbf{F}_\sigma$.*

To be precise, we let $\mathbf{F}_\sigma$ be the class of sets obtained by closing the class of closed sets under unions of countable subclasses, always assuming that the unions exist. The following theorem connects CUC and RUC to BWC_0 and BWC_1.

Theorem 3.19. *The system ACA_0^ω proves $\mathsf{CUC} \to \mathsf{cocode}_0 \to \mathsf{CUC}(2) \to \mathsf{cocode}_1$ and $\mathbf{NF}_\sigma \to \mathsf{RUC} \to \mathsf{NIN}$.*

Proof. For the first part, fix non-empty $A \subseteq [0,1]$ and $Y : [0,1] \to \mathbb{N}$ such that the latter is injective on the former. Let $x_0 \in A$ be some element in A and define the sequence of sets $(A_n)_{n\in\mathbb{N}}$ as follows:

$$x \in A_n \equiv \big[[x \in A \wedge Y(x) = n] \vee x =_{\mathbb{R}} x_0\big]. \tag{3.20}$$

Clearly, for each $n \in \mathbb{N}$, there *exists* an enumeration of A_n, namely either the sequence $x_0, x_0, \dots$ or the sequence $x_0, y, x_0, y, \dots$ where $y \in [0,1]$ satisfies $Y(y) = n$, if such there is. By CUC, there is an enumeration of $A = \cup_{n\in\mathbb{N}}A_n$, yielding cocode_0. Now assume the latter and fix a sequence $(A_n)_{n\in\mathbb{N}}$ satisfying (3.19). By the latter, we have the following

$$(\forall n \in \mathbb{N})(\exists! x \in [0,1])(\exists! y \in [0,1])(x, y \in A_n \wedge x <_{\mathbb{R}} y), \tag{3.21}$$

as A_n has exactly two elements. Recall that $\mathsf{cocode}_1 \leftrightarrow !\mathsf{QF}\text{-}\mathsf{AC}^{0,1}$ by [88, Theorem 3.17]. Modulo some coding $!\mathsf{QF}\text{-}\mathsf{AC}^{0,1}$ applies to (3.21), and let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be the resulting sequences. Use $\exists^2$ to remove any reals from $(y_n)_{n\in\mathbb{N}}$ already in $(x_n)_{n\in\mathbb{N}}$. Then $Y : \mathbb{R} \to \mathbb{N}$ is injective on $\cup_{n\in\mathbb{N}}A_n$:

$$Y(x) := \begin{cases} 0 & x \notin \cup_{n\in\mathbb{N}}A_n \\ 2P(x) & (\exists n \in \mathbb{N})(x =_{\mathbb{R}} x_n) \\ 2P(x)+1 & (\exists n \in \mathbb{N})(x =_{\mathbb{R}} y_n) \end{cases}, \tag{3.22}$$

where $P(x) := (\mu n)(x \in A_n)$. Then cocode_0 yields $\mathsf{CUC}(2)$, as required. For the implication $\mathsf{CUC}(2) \to \mathsf{cocode}_1$, fix $A \subset [0,1]$ such that $Y : [0,1] \to \mathbb{N}$ is bijective on A. Define the set $A_n := \{x \in A : Y(x) = n \vee Y(x) = n+1\}$ and note that (3.19) is satisfied. Applying $\mathsf{CUC}(2)$ yields an enumeration of $A = \cup_{n\in\mathbb{N}}A_n$, as required.

For the second part, suppose $\mathbb{R} = \cup_n A_n$, where for each $n \in \mathbb{N}$ there *exists* an enumeration of A_n. Then all subsets of $\mathbb{R}$ are $\mathbf{F}_\sigma$ as follows: for $E \subset \mathbb{R}$, one defines an enumeration of $E \cap A_n$ by checking each element in the enumeration of A_n for elementhood in E. Hence, $E = \cup_{n\in\mathbb{N}}[A_n \cap E]$ is a countable union of enumerable sets, and therefore $\mathbf{F}_\sigma$. For $\mathsf{RUC} \rightarrow \mathsf{NIN}$, suppose $Y : \mathbb{R} \rightarrow \mathbb{N}$ is an injection. Define a sequence $(A_n)_{n\in\mathbb{N}}$ as follows $x \in A_n \equiv \big[Y(x) =_0 n \vee x =_{\mathbb{R}} 0\big]$. Clearly, for each $n \in \mathbb{N}$, there *exists* an enumeration of A_n. By RUC, there is $y \in \mathbb{R}$ not in $\cup_{n\in\mathbb{N}}A_n$. However, $\mathbb{R} = \cup_{n\in\mathbb{N}}A_n$ by definition, yielding $\mathsf{RUC} \rightarrow \mathsf{NIN}$. □

Assuming $\mathsf{ACA}_0^\omega + \neg\mathsf{RUC}$, the previous proof implies that all subsets of $\mathbb{R}$ are $\mathbf{F}_\sigma$, and considering complements implies that all subsets are also $\mathbf{G}_\delta$. In stronger systems, the class $\mathbf{F}_\sigma \cap \mathbf{G}_\delta$ corresponds to Δ^0_2-formulas with function parameters.

Let $\mathsf{CUC}_0(2)$ be $\mathsf{CUC}(2)$ without the second conjunct of (3.19) and let $\mathsf{CUC}_1(2)$ be $\mathsf{CUC}(2)$ where we additionally assume the sets A_n to be pairwise disjoint.

Corollary 3.20 (ACA_0^ω)**.** *We have* $\mathsf{cocode}_0 \leftrightarrow \mathsf{CUC}_0(2)$ *and* $\mathsf{cocode}_1 \leftrightarrow \mathsf{CUC}_1(2)$.

Proof. The proof of $\mathsf{CUC}(2) \rightarrow \mathsf{cocode}_1$ from the theorem yields $\mathsf{CUC}_1(2) \rightarrow \mathsf{cocode}_1$ as $A_n := \{x \in A : Y(x) = 2n \vee Y(x) = 2n + 1\}$ are indeed pairwise disjoint. The proof of $\mathsf{cocode}_0 \rightarrow \mathsf{CUC}(2)$ yields $\mathsf{cocode}_1 \rightarrow \mathsf{CUC}_1(2)$ as the extra 'pairwise disjoint' condition in $\mathsf{CUC}_1(2)$ guarantees that Y defined in (3.22) is bijective on $\cup_{n\in\mathbb{N}}A_n$. The proof of $\mathsf{CUC} \rightarrow \mathsf{cocode}_0$ from the theorem yields a proof of $\mathsf{CUC}_0(2) \rightarrow \mathsf{cocode}_0$ as the sets from (3.20) have at most two elements. The proof of $\mathsf{cocode}_0 \rightarrow \mathsf{CUC}(2)$ from the theorem can be adapted as follows: consider the following formula, where the boldface text is different from (3.21):

$$(\forall n \in \mathbb{N})(\exists\ \textbf{at most one}\ (x, y) \in \mathbb{R}^2)(x, y \in A_n \wedge x <_{\mathbb{R}} y), \tag{3.23}$$

to which BOOT_C^- applies modulo coding. For the resulting set $X \subset \mathbb{N}$ we have

$$(\forall n \in X)(\exists! x \in [0, 1])(\exists! y \in [0, 1])(x, y \in A_n \wedge x <_{\mathbb{R}} y).$$

One now readily modifies (3.22) to the case at hand, which yields an enumeration of all A_n that have exactly two elements. To enumerate the A_n that are singletons, consider the following:

$$(\forall n \in \mathbb{N} \setminus X)(\exists \text{ at most one } x \in \mathbb{R})(x \in A_n), \tag{3.24}$$

to which BOOT_C^- applies modulo coding. For the resulting set $Z \subset \mathbb{N}$ we have

$$(\forall n \in Z)(\exists! x \in [0, 1])(x \in A_n),$$

which readily yields the required enumeration. □

By the previous, one can view CUC as the sequential version of cocode_0. However, the sequential version of e.g. BWC_0 is readily proved in Z_2^Ω (and hence ZF). By contrast, the sequential version of cloq' is equivalent to CUC by Corollary 3.22,

Principle 3.21 ($\mathsf{cloq}'_{\mathsf{seq}}$)**.** *Let* $(X_n, \preceq_n)_{n\in\mathbb{N}}$ *be a sequence of dense linear orderings without endpoints, with each* $X_n \subset \mathbb{R}$ *countable. Then there is a sequence* $(Z_n)_{n\in\mathbb{N}}$ *with* $Z_n : \mathbb{R} \rightarrow \mathbb{Q}$ *an order-isomorphism from* $(X_n, \preceq_n)$ *to* $\mathbb{Q}$ *for each* $n \in \mathbb{N}$.

Corollary 3.22. *The system* ACA_0^ω *proves* $[\mathsf{cloq}'_{\mathsf{seq}} + \mathsf{IND}_0] \leftrightarrow \mathsf{CUC}$.

Proof. For the reverse implication, CUC yields cocode_0 by Theorem 3.19. Hence, if $(X_n, \preceq_n)_{n\in\mathbb{N}}$ is as in the antecedent of $\mathsf{cloq}'_{\mathsf{seq}}$, cocode_0 implies that for each X_n, there is an enumeration. By CUC, there is a 'master' enumeration of $\cup_{n\in\mathbb{N}}X_n$. Use the well-known 'back-and-forth' proof (see [77, p. 123]) for each $(X_n, \preceq_n)$, uniformly in $\mathbb{N}$ and based on the master enumeration, to yield a sequence as in the consequence of $\mathsf{cloq}'_{\mathsf{seq}}$.

For the forward implication, we have access to cocode_0 by Corollary 2.36. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence as in CUC and define $A := \cup_{n\in\mathbb{N}}A_n$. Note that $(\exists^2)$ shows that each A_n is countable via an obvious injection. Without loss of generality, we may assume that $\mathbb{Q}\cap A$ is $\emptyset$, since Feferman's μ^2 can list all the rationals in a given set of reals. Now define $X_n := \mathbb{Q}\cup A_n$ and $\preceq_n$ the usual ordering of the reals. Let $(Z_n)_{n\in\mathbb{N}}$ be as provided by $\mathsf{cloq}'_{\mathsf{seq}}$, let $(p_n)_{n\in\mathbb{N}}$ be the usual list of primes, and let $G : \mathbb{Q} \to (\mathbb{N}\setminus\{0\})$ be an injection. Define $H(x)$ as $(\mu n)(x \in A_n)$ and define $Y : \mathbb{R} \to \mathbb{N}$ as $Y(x) := p_{H(x)}^{G(Z_{H(x)}(x))}$. By definition, Y is an injection on A; the latter is therefore countable, and enumerable by Corollary 2.36. □

We note in passing that the weak choice principle WCC from [5] is intermediate between cocode_0 and cocode_1 by the previous. We also have the following corollary.

Corollary 3.23. *$\mathsf{Z}_2^\Omega + \mathsf{QF\text{-}AC}^{0,1}$ proves CUC; $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ cannot prove RUC.*

Proof. For the negative result, NIN is not provable in $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ by [73, Theorem 3.2], while $\mathsf{RUC} \to \mathsf{NIN}$ over ACA_0^ω by Theorem 3.19. For the positive result, the antecedent of CUC expresses the following:

$$(\forall n \in \mathbb{N})(\exists (x_m)_{m\in\mathbb{N}})(\forall y \in \mathbb{R})\big[y \in A_n \leftrightarrow (\exists k \in \mathbb{N})(x_k =_{\mathbb{R}} y)\big].$$

Using $\exists^3$ and $\mathsf{QF\text{-}AC}^{0,1}$, there is a 'master' sequence, yielding CUC. □

We finish this section with a remark on the naturalness and generality of CUC.

Remark 3.24 (CUC, old and new)**.** First of all, an L_2-version of CUC for sets represented by analytic codes is proved in [97, V.4.10], inside ATR_0. Note that enumerable sets are automatically Borel, and therefore analytic. Similarly, (codes for) Borel sets are closed under countable unions in second-order RM by [97, V.3.3], also working in ATR_0. Modulo coding, there is thus antecedent for the study of CUC in second-order RM.

Secondly, in contrast to the second-order principles from the previous paragraph, CUC does (seem to) quantify over all enumerable subsets of $\mathbb{R}$. This apparent generality of CUC should not be overstated: an enumerated set is of course measurable (provably having measure zero in ACA_0^ω), and the class of (codes for) measurable sets is closed under countable unions in second-order RM, as mentioned in [97, X.1.17]. Similarly, enumerated sets are clearly Borel sets (of low level) in ACA_0^ω. Hence, CUC is of a level of generality comparable to what one studies in RM, but formulated with third-order characteristic functions rather than second-order codes.

Thirdly, in Section 3.3, we connect cocode_0 to theorems pertaining to bounded variation (and related concepts), like the *Jordan decomposition theorem* as in Theorem 3.27. On one hand, this theorem readily implies cocode_0, while the reversal *should* go through, seeing as though functions of bounded variation only have countably many points of discontinuity. Indeed, an enumeration of the latter set even guarantees that Jordan's original proof ([42]) of the Jordan decomposition theorem goes through. Try as we might, the aforementioned reversal only goes through assuming the following (seemingly trivial) fragment of the countable union theorem, which however does not[11] even imply NIN over $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$.

Principle 3.25 ($\mathsf{CUC}_{\mathsf{fin}}$)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be subsets of $\mathbb{R}$ such that $\cup_{n\in\mathbb{N}}X_n$ is not countable. Then X_m is not finite for some $m \in \mathbb{N}$.*

Recall our notion of 'finite set' from Definition 1.5, to be discussed in detail in Section 3.3.2. In the below, we even obtain equivalences involving $\mathsf{CUC}_{\mathsf{fin}}$, i.e. the countable union theorem is a natural/useful object of study in this context.

3.3. **Bounded variation and related concepts.** In this section, we establish an equivalence between BWC_0 and the well-known *Jordan decomposition theorem* as in Theorem 3.27. We also obtain other equivalences involving theorems about *bounded variation* and *regulated* functions. We introduce definitions for the previous italicised notions in Section 3.3.1, while our main results are in Section 3.3.2. The latter results provide some non-trivial motivation for our choice of definition of closed and finite set, as discussed in Section 3.3.3.

3.3.1. *Definitions: bounded variation and related notions.* We formulate the definitions of bounded variation and regulated functions, as well as some background.

Firstly, the notion of *bounded variation* (often abbreviated BV below) was first explicitly[12] introduced by Jordan around 1881 ([42]) yielding a generalisation of Dirichlet's convergence theorems for Fourier series. Indeed, Dirichlet's convergence results are restricted to functions that are continuous except at a finite number of points, while BV-functions can have infinitely many points of discontinuity, as already studied by Jordan, namely in [42, p. 230]. Nowadays, the *total variation* of a function $f : [a, b] \to \mathbb{R}$ is defined as follows:

$$V_a^b(f) := \sup_{a\leq x_0<\cdots<x_n\leq b} \textstyle\sum_{i=0}^n |f(x_i) - f(x_{i+1})|. \tag{3.25}$$

If this quantity exists and is finite, one says that f has bounded variation on $[a, b]$. Now, the notion of bounded variation is defined in [65] *without* mentioning the supremum in (3.25); this approach can also be found in [3, 4, 49]. Hence, we shall distinguish between the two notions in Definition 3.26. As it happens, Jordan seems to use item (a) of Definition 3.26 in [42, p. 228-229]. This definition suggests a two-fold variation for any result on functions of bounded variation, namely depending on whether the supremum (3.25) is given, or only an upper bound on the latter.

Definition 3.26. [Variations on variation]

(a) The function $f : [a, b] \to \mathbb{R}$ *has bounded variation* on $[a, b]$ if there is $k_0 \in \mathbb{N}$ such that $k_0 \geq \sum_{i=0}^n |f(x_i) - f(x_{i+1})|$ for any partition $x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b$.

(b) The function $f : [a, b] \to \mathbb{R}$ *has a variation* on $[a, b]$ if the supremum in (3.25) exists and is finite.

Secondly, the fundamental theorem about BV-functions is formulated as follows.

Theorem 3.27 (Jordan decomposition theorem, [42, p. 229])**.** *A BV-function $f : [0, 1] \to \mathbb{R}$ is the difference of two non-decreasing functions $g, h : [0, 1] \to \mathbb{R}$.*

[11] Note that $\neg\mathsf{NIN}$ implies $\mathsf{CUC}_{\mathsf{fin}}$, while $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ does not prove NIN by [73, §3].

[12] Lakatos in [51, p. 148] claims that Jordan did not invent or introduce the notion of bounded variation in [42], but rather discovered it in Dirichlet's 1829 paper [19].

Theorem 3.27 has been studied via second-order representations in [31, 49, 65, 107]. The same holds for constructive analysis by [3, 4, 33, 79], involving different (but related) constructive enrichments. Now, ACA_0 suffices to derive Theorem 3.27 for various kinds of second-order *representations* of BV-functions in [49, 65]. By contrast, our results imply that $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ cannot prove the third-order version of Theorem 3.27, as the latter is equivalent to BWC_0 over a suitable base theory (see Theorem 3.34). Nonetheless, the third-order Jordan decomposition theorem does not imply much comprehension, by the following remark.

Remark 3.28 (Comprehension and Jordan decompositions)**.** The third-order version of the Jordan decomposition theorem (Theorem 3.27) implies neither $(\exists^2)$ nor any theorem of Z_2 not provable in ACA_0, working over RCA_0^ω. Indeed, the ECF-translation (Remark 1.12) of the former is implied by $\mathsf{Jordan}_{\mathsf{cont}}$, the second-order version of Theorem 3.27 from [65] and provable in ACA_0. By contrast, ECF translates $(\exists^2)$ to '$0 = 1$' while second-order sentences are translated to themselves.

Thirdly, Jordan proves in [43, §105] that BV-functions are exactly those for which the notion of 'length of the graph of the function' makes sense. In particular, $f \in BV$ if and only if the 'length of the graph of f', defined as follows:

$$L(f, [0,1]) := \sup_{0=t_0<t_1<\dots<t_m=1} \sum_{i=0}^{m-1} \sqrt{(t_i - t_{i+1})^2 + (f(t_i) - f(t_{i+1}))^2} \quad (3.26)$$

exists and is finite by [1, Thm. 3.28.(c)]. In case the supremum in (3.26) exists (and is finite), f is also called *rectifiable*. Rectifiable curves predate BV-functions: in [93, §1-2], it is claimed that (3.26) is essentially equivalent to Duhamel's 1866 approach from [23, Ch. VI]. Around 1833, Dirksen, the PhD supervisor of Jacobi and Heine, already provides a definition of arc length that is (very) similar to (3.26) (see [20, §2, p. 128]), but with some conceptual problems as discussed in [17, §3].

Fourth, a function is *regulated* (called 'regular' in [1]) if for every x_0 in the domain, the 'left' and 'right' limit $f(x_0-) = \lim_{x\to x_0-} f(x)$ and $f(x_0+) = \lim_{x\to x_0+} f(x)$ exist. Scheeffer studies discontinuous regulated functions in [93] (without using the term 'regulated'), while Bourbaki develops Riemann integration based on regulated functions in [7]. Now, BV-functions are regulated (see Theorem 3.33), while Weierstrass' 'monster' function is a natural example of a regulated function not in BV. An interesting observation about regular functions and continuity is as follows.

Remark 3.29 (Continuity and the Axiom of Choice)**.** As discussed in [47, §3], the *local* equivalence for functions on Baire space between sequential and 'epsilon-delta' continuity can be proved in $\mathsf{RCA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$, but not in ZF. By the final item in Theorem 3.33, this equivalence for *regulated* functions is provable in ACA_0^ω.

Finally, the Jordan decomposition theorem as in Theorem 3.27 shows that a BV-function can be 'decomposed' as the difference of monotone functions. This is however not the only result of its kind: Sierpiński e.g. establishes in [95] that for regulated $f : [0,1] \to \mathbb{R}$, there are g, h such that $f = g \circ h$ with g continuous and h strictly increasing on their respective domains.

3.3.2. *Bounded variation and Reverse Mathematics.* In this section, we develop the RM of the Jordan decomposition theorem and related results on bounded variation and regulated functions. As will become clear, the principle $\mathsf{CUC}_{\mathsf{fin}}$ from Remark 3.24 is central to this enterprise.

First of all, we recall our particular notion of 'finite set' to be used in $\mathsf{CUC}_{\mathsf{fin}}$ and provide some motivation in Remark 3.31 right below. On a historical note, the study of various definitions of finite set (in set theory) was the topic of Mostowski's dissertation, as suggested by Tarski ([60, p. 18-19]).

Definition 3.30 (Finite)**.** Any $X \subset \mathbb{R}$ is *finite* if there is $N \in \mathbb{N}$ such that for any finite sequence $(x_0, \dots, x_N)$ of distinct reals, there is $i \leq N$ such that $x_i \notin X$.

The number $N \in \mathbb{N}$ from the previous definition is called an *upper bound* on the size of the finite set $X \subset \mathbb{R}$, and we use '$|X| \leq N$' as purely symbolic notation for this. Note that Definition 3.30 is not circular as 'finite sequences of reals' are just objects of type 1, modulo coding using $\exists^2$. We now motivate Definition 3.30.

Remark 3.31 (Finite sets by any other name)**.** First of all, working in set theory, the various definitions[13] of 'finite set' are not equivalent over ZF, while countable choice suffices to establish the equivalence ([41]). Hence, it should not be a surprise that studying finite sets in weak systems requires one to choose a specific definition.

Secondly, consider the following set where f is a function of bounded variation:

$$A_n := \big\{x \in (0,1) : |f(x+) - f(x)| > \tfrac{1}{2^n} \vee |f(x-) - f(x)| > \tfrac{1}{2^n}\big\} \tag{3.27}$$

This set is finite as each element of A_n contributes at least $\frac{1}{2^n}$ to the total variation. Finite as A_n may be, we are unable to exhibit an injection from A_n to $\{0, 1, \dots, k\}$ for some $k \in \mathbb{N}$, say computable in some S_m^k (see Remark 3.37 for details). By contrast, A_n is trivially finite in the sense of Definition 3.30 in ACA_0^ω.

In conclusion, *if* one wants to work in a weak logical system, *then* (certain) finite sets that 'appear in the wild' are best studied via Definition 3.30, and not the definition from Footnote 13 involving bijections or injections. Moreover, Theorem 2.12 suggests that IND_0 (and cocode_0) does not suffice to study finite sets as in Definition 3.30; as noted in Remark 3.24, we indeed seem to need $\mathsf{CUC}_{\mathsf{fin}}$.

Secondly, we need Theorem 3.33 to establish basic properties of BV and regulated functions. We shall make (seemingly essential) use of the following fragment of the induction axiom, which also follows from $\mathsf{QF\text{-}AC}^{0,1}$.

Definition 3.32. [IND_2] Let Y^2, k^0 satisfy $(\forall n \leq k)(\exists f \in 2^{\mathbb{N}})(Y(f, n) = 0)$. There is w^{1^*} such that $(\forall n \leq k)(\exists i < |w|)(Y(w(i), n) = 0)$.

Note that we use the 'standard' definition of left and right limits, i.e. as in (3.29).

Theorem 3.33 (ACA_0^ω)**.**

- *Assuming* IND_2, *any* BV*-function* $f : [0,1] \to \mathbb{R}$ *is regulated.*
- *Any monotone function* $f : [0,1] \to \mathbb{R}$ *has bounded variation.*
- *For any monotone function* $f : \mathbb{R} \to \mathbb{R}$, *there is a sequence* $(x_n)_{n \in \mathbb{N}}$ *that enumerates all* $x \in [0,1]$ *such that* f *is discontinuous at* x.
- *For regulated* $f : [0,1] \to \mathbb{R}$ *and* $x \in [0,1]$, f *is sequentially continuous at* x *if and only if* f *is epsilon-delta continuous at* x.
- *For finite* $X \subset [0,1]$, *the function* $\mathbb{1}_X$ *has bounded variation.*

[13]In ZF, a set A is 'finite' if there is a bijection to $\{0, 1, \dots, n\}$ for some $n \in \mathbb{N}$; a set A is 'Dedekind finite' if any injective mapping from A to A is also surjective.

Proof. For the first item, assume $f(c-)$ does not exist for $c \in (0,1]$. We obtain a contradiction using $\mathsf{QF\text{-}AC}^{0,1}$ and then using IND_2. Hence, there is $\varepsilon > 0$ with

$$(\forall k \in \mathbb{N})(\exists x, y \in (c - \tfrac{1}{2^k}, c))(x < y \wedge |f(x) - f(y)| > \varepsilon). \tag{3.28}$$

Apply $\mathsf{QF\text{-}AC}^{0,1}$ to (3.28); modify the resulting sequence $(x_n, y_n)_{n\in\mathbb{N}}$ to guarantee

$$x_m < y_m < c - \tfrac{1}{2^{m+1}} < x_{m+1} < y_{m+1} < c - \tfrac{1}{2^{m+1}}$$

for large enough $m \in \mathbb{N}$. By definition, $|f(x_k) - f(y_k)| > \varepsilon$ for large enough $k \in \mathbb{N}$, i.e. collecting enough such points in a partition, the associated variation is arbitrary large. We now observe how the previous proof is readily modified: apply IND_2 to (3.28) after choosing large enough (relative to the variation of f) upper bound on k in (3.28). Hence, $f(c-)$ must exist and the other cases follow in the same way.

For the second part, assume $f : [0,1] \to \mathbb{R}$ is monotone. Then the usual telescoping sum trick implies that the total variation of f as in (3.25) exists and equals $|f(0) - f(1)|$. The third part is follows from [74, Lemma 7], which applies to $[0,1]$ but trivially generalises to $\mathbb{R}$.

For the fourth item, let $f : [0,1] \to \mathbb{R}$ be regulated and fix $x_0 \in [0,1]$. We only need to prove the forward implication, i.e. assume f is sequentially continuous at x_0. To show that $f(x_0-) = f(x_0)$, consider $y_n := x_0 - \frac{1}{2^{n+1}}$ and note that $(y_n)_{n\in\mathbb{N}}$ converges to x_0, implying that $(f(y_n))_{n\in\mathbb{N}}$ converges to $f(x_0)$. Now consider the definition of 'the left limit $f(x_0-)$ exists' as follows:

$$(\exists y \in \mathbb{R})(\forall k \in \mathbb{N})(\exists N \in \mathbb{N})(\forall z \in (x_0 - \tfrac{1}{2^N}, x_0))(|f(z) - y| < \tfrac{1}{2^k}) \tag{3.29}$$

Since $(y_n)_{n\in\mathbb{N}}$ converges to x_0 and $(f(y_n))_{n\in\mathbb{N}}$ converges to $f(x_0)$, we have $y = f(x_0)$ in (3.29). In the same way, one shows that $f(x_0+) = f(x_0)$. Then (3.29) and the associated 'right limit' version imply that f is epsilon-delta continuous at x_0.

For the fifth item, fix finite $X \subset [0,1]$ with $N \in \mathbb{N}$ as in Definition 3.30. To show that $f(x) := \mathbb{1}_X(x)$ has bounded variation, note that for a partition $x_0 = 0, x_1, \ldots, x_k, x_{k+1} = 1$ of $[0,1]$ we have $\sum_{i=0}^{k} |f(x_{i+1}) - f(x_i)| \leq J$, where $J \leq N$ is the number of elements of X among the partition points. □

Thirdly, we can now connect the Jordan decomposition theorem and cocode_0. Note that 'bounded variation' refers to item (a) in Definition 3.26.

Theorem 3.34 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_2 + \mathsf{CUC}_{\mathsf{fin}}$)**.** *The following are equivalent.*

(i) *The principle* cocode_0.
(ii) *The Jordan decomposition theorem* (*Theorem 3.27*).
(iii) *For a BV-function* $f : [0,1] \to \mathbb{R}$*, there is a sequence enumerating all points where* f *is discontinuous.*

The previous upward implications are provable over ACA_0^ω*. Assuming* $\mathsf{QF\text{-}AC}^{0,1}$*, the above are equivalent to the following.*

(iv) *For regulated* $f : [0,1] \to \mathbb{R}$*, there is a sequence enumerating all points where* f *is discontinuous.*
(v) (*Sierpiński*) *For regulated* $f : [0,1] \to \mathbb{R}$*, there are* g, h *such that* $f = g \circ h$ *with* g *continuous and* h *strictly increasing on their interval domains.*

The previous upward implications are provable over $\mathsf{ACA}_0^\omega + \mathsf{IND}_2$.

Proof. The equivalence (ii) $\leftrightarrow$ (iii) follows from Theorem 3.33 and the usual proof of the Jordan decomposition theorem. Indeed, we can 'imitate' the supremum in (3.25) as follows: use μ^2 to define, for any $x \in [0,1]$, the following:

$$V(x) := \sup_{0\leq y_0 < \dots < y_n \leq x} \sum_{i=0}^{n} |f(y_i) - f(y_{i+1})|, \tag{3.30}$$

where $(y_i)_{i\in\mathbb{N}}$ is the sequence consisting of $\mathbb{Q} \cap [0,1]$ together with the sequence provided by item (iii). Trivially, $g(x) := \lambda x.V(x)$ is increasing on $[0,1]$ and the same holds for $h(x) := V(x) - f(x)$. Indeed, for $0 \leq y < z \leq 1$, we have

$$h(z) - h(y) = V(z) - f(z) - V(y) + f(y) = (V(z) - V(y)) - (f(z) - f(y)) \geq 0,$$

where the final inequality follows from the definition of V. We now have $f(x) - g(x) = h(x)$ for all $x \in [0,1]$, yielding the Jordan decomposition theorem.

For the implication (iii) $\rightarrow$ (i), fix $A \subset [0,1]$ and $Y : [0,1] \rightarrow \mathbb{N}$ injective on A. Define $f(x)$ as $\frac{1}{2^{Y(x)+1}}$ in case $x \in A$, and 0 otherwise. Clearly, $f \in BV$ as any sum $\sum_{i=0}^{n} |f(x_i) - f(x_{i+1})|$ is at most $\sum_{i=0}^{n+1} \frac{1}{2^{i+1}}$, which is bounded by 1 for any $n \in \mathbb{N}$. The points of discontinuity for f are exactly the points of A, and cocode_0 follows.

For the implication (i) $\rightarrow$ (iii), fix a BV-function $f : [0,1] \rightarrow \mathbb{R}$ and $n \in \mathbb{N}$. We may assume that the upper bound as in item (a) in Def. 3.26 is 1. The first item of Theorem 3.33 guarantees that f is regular. Now define the following set

$$A_n := \left\{x \in (0,1) : |f(x+) - f(x)| > \tfrac{1}{2^n} \vee |f(x-) - f(x)| > \tfrac{1}{2^n}\right\} \tag{3.31}$$

which is finite (in the sense of Definition 3.30). Indeed, assuming A_n were not finite, there are arbitrary long finite sequences of elements of A_n. However, each element of A_n contributes at least $\frac{1}{2^n}$ to the variation of f, a contradiction. Hence, A_n is finite (and has at most 2^n elements). Using the contraposition of $\mathsf{CUC}_{\mathsf{fin}}$, the union $A := \cup_{n\in\mathbb{N}} A_n$ is countable. This union can now be enumerated thanks to cocode_0, yielding a sequence listing all points of discontinuity of f.

The implications (v)$\rightarrow$(iv)$\rightarrow$(iii) are immediate by Theorem 3.33. For (iv) $\rightarrow$ (v), fix regulated $f : [0,1] \rightarrow \mathbb{R}$ and consider the proof of [1, Theorem 0.36, p. 28], going back to [95]. This proof establishes the existence of g, h such that $f = g \circ h$ with g continuous and h strictly increasing. Moreover, one finds an *explicit construction* (modulo $\exists^2$) of the function h required, *assuming* a sequence listing all points of discontinuity of f on $[0,1]$. The function g is then defined as $\lambda y.f(h^{-1}(y))$ where h^{-1} is the inverse of h, definable using $\exists^2$.

Finally, we shall make use of $\mathsf{QF\text{-}AC}^{0,1}$ to prove (i)$\rightarrow$(iv); fix regulated $f : [0,1] \rightarrow \mathbb{R}$ and $n \in \mathbb{N}$ and note that A_n as in (3.31) is again finite. Indeed, assuming A_n were not finite, $\mathsf{QF\text{-}AC}^{0,1}$ provides a sequence $(x_j)_{j\in\mathbb{N}}$ of elements of A_n. By the Bolzano-Weierstrass theorem, this sequence has a convergent sub-sequence, say with limit $c \in [0,1]$. However, $f(c+)$ and $f(c-)$ do not exist by the definition of A_n (via the usual epsilon-delta argument), a contradiction. In conclusion, the union $A := \cup_{n\in\mathbb{N}} A_n$ can now be enumerated, thanks to item (i) and $\mathsf{CUC}_{\mathsf{fin}}$. □

The use of $\mathsf{QF\text{-}AC}^{0,1}$ in the theorem can be avoided in various ways, one of which is the principle NCC from [73]. We will explore this in a follow-up paper.

Fourth, we establish a (more) elegant result as in Theorem 3.35. In the latter, the *uniform finite union theorem* expresses the existence of $h : \mathbb{N} \rightarrow \mathbb{N}$ such that $|X_n| \leq h(n)$ for a sequence of finite sets $(X_n)_{n\in\mathbb{N}}$ in $[0,1]$. The *finite union theorem* expresses (only) that for such a sequence, each $\cup_{n\leq k} X_n$ is finite for $k \in \mathbb{N}$.

Regarding item (h), Principle 2.21 was studied in Corollary 2.27 and we can now obtain an equivalence involving the former and $\textsf{cocode}_0$.

Theorem 3.35 ($\textsf{ACA}_0^\omega + \textsf{QF-AC}^{0,1}$)**.** *The following are equivalent.*

(a) *The combination* $\textsf{CUC}_{\textsf{fin}} + \textsf{cocode}_0$*.*
(b) *For regulated* $f : \mathbb{R} \to \mathbb{R}$*, there is a sequence enumerating the points of discontinuity.*
(c) *For regulated* $f : [0,1] \to \mathbb{R}$*, there is a sequence enumerating the points of discontinuity.*
(d) *The* uniform finite union theorem *plus the Jordan decomposition theorem.*
(e) *The* uniform finite union theorem *plus: for* $f : [0,1] \to \mathbb{R}$ *in* BV*, there is a sequence enumerating the points of discontinuity.*
(f) *The* finite union theorem *plus the Jordan decomposition theorem on the half-line: for* $f : \mathbb{R} \to \mathbb{R}$ *with bounded variation on* $[0, y]$ *for any* $y \in \mathbb{R}^+$*, there are monotone* g, h *such that* $f(x) = g(x) - h(x)$ *for any* $x \geq 0$*.*
(g) *The* finite union theorem *plus: for* $f : \mathbb{R} \to \mathbb{R}$ *with bounded variation on* $[0, y]$ *for any* $y \in \mathbb{R}^+$*, there is a sequence enumerating the points of discontinuity of* f *on* $[0, +\infty)$*.*
(h) *A non-enumerable and closed set in* $\mathbb{R}$ *has a limit point* (*Principle 2.21*)*.*

Proof. First of all, we derive the following basic properties concerning finite sets, working in our base theory $\textsf{ACA}_0^\omega + \textsf{QF-AC}^{0,1}$.

(x1) Any item (a)-(h) implies that *a finite set of reals can be enumerated.*
(x2) Item (c) implies the *finite union theorem.*
(x3) Item (b) implies the *uniform finite union theorem* and $\textsf{CUC}_{\textsf{fin}}$.

For item (x1), a finite set has characteristic function that is in BV and regulated by Theorem 3.33, assuming $\textsf{IND}_2$ which follows from $\textsf{QF-AC}^{0,1}$. Hence, over our base theory, items (a)-(g) imply that a finite set can be enumerated (as a finite sequence, using μ^2), where we note the third item of Theorem 3.33. Since finite sets do not have limit points, item (x1) also holds for item (h).

For item (x2), let $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite sets. We may assume $0, 1 \notin \cup_{n\in\mathbb{N}} X_n$. Now consider $f_k : [0,1] \to \mathbb{N}$ for $k \geq 2$ defined as follows: define $Y_i := \{y \in (\frac{i}{k}, \frac{i+1}{k}) : k(y - \frac{i}{k}) \in X_i\}$ and $f_k(x) := \sum_{i=0}^{k} \mathbb{1}_{Y_i}(x)$. By definition, Y_i is the set X_i for $i \leq k$, but shrunk by a factor $\frac{1}{k}$ and moved to $(\frac{i}{k}, \frac{i+1}{k})$. Hence, Y_i is finite for $i \leq k$ and since $f_k(x)$ equals $\mathbb{1}_{Y_i}(x)$ for $x \in [\frac{i}{k}, \frac{i+1}{k}]$, the function f_k is regulated by Theorem 3.33. Thus, item (c) implies that the points of discontinuity of f_k can be enumerated, which means $\cup_{i\leq k} Y_i$ can be enumerated. Using μ^2 and the latter enumeration, one finds an upper bound $N_i \in \mathbb{N}$ for each Y_i. Taking the sum, $\cup_{i\leq k} Y_i$ (and hence $\cup_{i\leq k} X_i$) is finite. One obtains item (x3) in the same way: let Z_i be the set X_i moved to $(i+1, i+2)$ without shrinking for $i \in \mathbb{N}$. Then the function $\mathbb{1}_{\cup_{n\in\mathbb{N}} Z_n}$ is regular on $\mathbb{R}$ and item (b) provides an enumeration of $\cup_{n\in\mathbb{N}} Z_n$, which readily yields $\textsf{CUC}_{\textsf{fin}}$. Using this enumeration and μ^2, one obtains the function $h : \mathbb{N} \to \mathbb{N}$ as in the *uniform* finite union theorem.

Secondly, we establish (a) $\to$ (b) $\to$ (c) $\to$ (a). Now, (a) $\to$ (b) follows from the proof of (i)$\to$(iv) in Theorem 3.34 by replacing $[0,1]$ by $\mathbb{R}$. In turn, (b) $\to$ (c) is trivial while (c) $\to$ (a) is proved as follows: let $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite sets in $[0,1]$ and define the following function:

$$g(x) := \begin{cases} \frac{1}{2^n} & x \in X_n \text{ and } n \text{ is the least such number} \\ 0 & \text{otherwise} \end{cases}. \tag{3.32}$$

To show that $g : [0,1] \to \mathbb{R}$ is regulated, fix $x \in [0,1]$ and $k \in \mathbb{N}$. Then $\cup_{i\leq k} X_i$ is finite by the finite union theorem, which is available due to item (x2) from the first paragraph of this proof. Then $(\exists m^0)(\forall y \in B(x, \frac{1}{2^m}) \setminus \{x\})(y \notin \cup_{i\leq k} X_i)$ readily[14] follows by contradiction. By definition, $g(x) \leq \frac{1}{2^{k+1}}$ on this punctured disc, i.e. g becomes arbitrarily small near x, implying $g(x+) = 0 = g(x-)$. Item (c) now provides a list $(x_n)_{n\in\mathbb{N}}$ with all points where g is discontinuous; this sequence also enumerates $\cup_{n\in\mathbb{N}} X_n$. Indeed, $g(x_m-) = 0 = g(x_m+)$ implies that $g(x_m) > 0$ as g must be discontinuous at x_m; by (3.32), x_m is in $\cup_{n\in\mathbb{N}} X_n$. Similarly, if y is in the latter union, we have $g(y) > 0$ by (3.32); hence g is discontinuous at y, implying there is $m \in \mathbb{N}$ with $y = x_m$. Hence, $\cup_{n\in\mathbb{N}} X_n$ can be enumerated, which immediately implies cocode_0 and $\mathsf{CUC}_{\mathsf{fin}}$. We have established (a) $\leftrightarrow$ (b) $\leftrightarrow$ (c).

Thirdly, we show that (b) $\to$ (d) $\to$ (e) $\to$ (a). The implication (b) $\to$ (d) follows from item (x3) and Theorem 3.34. The implication (d) $\to$ (e) follows by the third item of Theorem 3.33. For the implication (e)$\to$(a), modify (3.32) as follows:

$$g(x) := \begin{cases} \frac{1}{2^n}\frac{1}{h(n)+1} & x \in X_n \text{ and } n \text{ is the least such number} \\ 0 & \text{otherwise} \end{cases}, \tag{3.33}$$

where h is as provided by the uniform finite union theorem. Since $|X_n| \leq h(n)$ for all $n \in \mathbb{N}$, g as in (3.33) is in BV, with variation bounded by 1. Applying item (e), one obtains an enumeration of $\cup_{n\in\mathbb{N}} X_n$, as required for item (a). By the previous paragraph, we obtain (a) $\leftrightarrow$ (b) $\leftrightarrow$ (c) $\leftrightarrow$ (d) $\leftrightarrow$ (e).

Fourth, we show that (b) $\to$ (f) $\to$ (g) $\to$ (a). The implication (b) $\to$ (f) follows from item (x3) and the generalisation of (3.30) to arbitrary intervals $[0, y]$ for $y > 0$; the second part is essentially the same as the proof of (iii) $\to$ (ii) in Theorem 3.34. The implication (f) $\to$ (g) follows by the third item of Theorem 3.33. To prove that item (g) implies item (a), let $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite sets in $(0,1)$. Let Z_i be the set X_i moved to $(i+1, i+2)$ without shrinking for $i \in \mathbb{N}$. As above, the function $\mathbb{1}_{\cup_{n\in\mathbb{N}} Z_n}$ satisfies the conditions of item (g). Indeed, on the interval $[0, y]$ with $0 < y \leq m \in \mathbb{N}$, the function $\mathbb{1}_{\cup_{n\in\mathbb{N}} Z_n}$ reduces to $\mathbb{1}_{\cup_{n\leq m} Z_n}$, and the latter has bounded variation by the finite union theorem and the final item in Theorem 3.33. An enumeration of the points of discontinuity of $\mathbb{1}_{\cup_{n\in\mathbb{N}} Z_n}$ readily yields an enumeration of $\cup_{n\in\mathbb{N}} X_n$, as required for item (a). By the previous paragraph, we obtain (a) $\leftrightarrow$ (b) $\leftrightarrow \dots \leftrightarrow$ (g), i.e. all that remains is item (h),

Finally, we prove (a) $\to$ (h) $\to$ (e), finishing the theorem. Hence, assume (h) and fix $f : [0,1] \to \mathbb{R}$ in BV and consider A_n a in (3.31), which is well-defined thanks to Theorem 3.33. The set A_n is also finite as in the proof of Theorem 3.34 and we may assume $0, 1 \notin A_n$ for $n \in \mathbb{N}$. Now let B_n be a copy of A_n translated from $[0,1]$ to $[n+1, n+2]$ for $n \in \mathbb{N}$. Then $B := \cup_{n\in\mathbb{N}} B_n$ has no limit points, which one proves (by contradiction) using $\mathsf{QF\text{-}AC}^{0,1}$ and the Bolzano-Weierstrass theorem. Hence, item (h) yields an enumeration of B and hence a sequence listing all points where f is

[14] Suppose $(\forall m^0)(\exists y \in B(x, \frac{1}{2^m}) \setminus \{x\})(y \in \cup_{i\leq k} X_i)$ and apply $\mathsf{QF\text{-}AC}^{0,1}$ to obtain a sequence in $\cup_{i\leq k} X_i$ converging to x. Using μ^2, one modifies this sequence to guarantee it consists of pairwise disjoint reals. This however contradicts the finiteness of $\cup_{i\leq k} X_i$.

discontinuous. Similarly, item (h) implies the uniform finite union theorem. For the implication (a)→(h), fix closed $A \subset \mathbb{R}$ with no limit points. Then $A_n := A \cap [-n, n]$ is finite for any $n \in \mathbb{N}$, which one proves (by contradiction) using $\textsf{QF-AC}^{0,1}$ and the Bolzano-Weierstrass theorem. By $\textsf{CUC}_{\textsf{fin}}$, $\cup_{n\in\mathbb{N}} A_n$ is countable, and can be enumerated using $\textsf{cocode}_0$, and we are done. □

By the proof of Theorem 3.35, item (c) implies the finite union theorem, while the same does not seem to hold for the Jordan decomposition theorem. We believe this is due to fact that 'regulated' is a local property while 'bounded variation' is a global property (of the domain). Moreover, there are *many and very different* intermediate spaces (see [1] or [74, Remark 4.13]) between the space of regulated and of BV-functions; each of these intermediate spaces yields an equivalent generalisation of e.g. item (e) in Theorem 3.35, also showcasing a certain robustness.

Next, by the following theorem, we may replace the finite union theorem in Theorem 3.35 by 'more mathematical' principles.

Theorem 3.36 ($\textsf{ACA}_0^\omega + \textsf{QF-AC}^{0,1}$)**.** *The higher items imply the lower items.*

- *The combination* $\textsf{CUC}_{\textsf{fin}} + \textsf{cocode}_0$.
- *For* $f_1, \dots, f_k : [0, 1] \to \mathbb{R}$ *in* BV*, the sum* $\sum_{i=1}^k f_i$ *is in* BV.
- *The finite union theorem.*

Proof. For the first downward implication, the following set

$$A_{n,i} := \{x \in [0, 1] : |f_i(x+) - f_i(x)| > \tfrac{1}{2^n} \vee |f_i(x-) - f_i(x)| > \tfrac{1}{2^n}\}$$

is finite for all $n \in \mathbb{N}$ and $i \leq k$ if $f_1, \dots, f_k \in BV$. Clearly, the set $B_n := \cup_{i\leq k} A_{n,i}$ is finite. Hence, there is an enumeration of $\cup_{n\in\mathbb{N}} B_n$, yielding a sequence $(x_n)_{n\in\mathbb{N}}$ that lists all points of discontinuity of the functions f_i for $i \leq k$. Using (μ^2), we can compute $V_0^1(f_i)$ for $i \leq q$ as we can replace the usual supremum by one over $\mathbb{N}$ (and $\mathbb{Q}$). The proof that $V_0^1(f + g) \leq V_0^1(f) + V_0^1(g)$ in [1, p. 57] essentially amounts to the triangle inequality over $\mathbb{R}$, i.e. that $\sum_{i=1}^k f_i$ is in BV now follows.

The second downward implication is straightforward as a characteristic function $\mathbb{1}_X$ is in BV if $X \subset [0, 1]$ is finite by Theorem 3.33. □

We could replace the second item in Theorem 3.36 by the following statement:

for f *in* BV *and* $0 = x_0 < x_1 < \cdots < x_k < x_{k+1} = 1$, $V_0^1(f) = \sum_{i=0}^k V_{x_i}^{x_{i+1}}(f)$,

but this would entail a number of technical details. The same division property for the arc length of rectifiable functions would of course be rather natural.

3.3.3. *On the choice of definitions.* In this section, we discuss our choice of definitions and provide some motivation.

First of all, the following remark provides some motivation for the use of our definitions of finite and closed set as in Definitions 1.2 and 3.30.

Remark 3.37. As discussed above, the sets A_n from (3.27) are finite and hence closed. In particular, working in $\textsf{ZF}$ (or even $\textsf{Z}_2^\Omega$ from Section A.1.4), the following objects can be constructed:

- for $n \in \mathbb{N}$, an injection Y_n from A_n to some $\{0, 1, \dots, k\}$ with $k \in \mathbb{N}$,
- for $m \in \mathbb{N}$, an RM-code C_m (see [97, II.5.6]) for the closed sets A_m.

However, it is shown in [91,92] that neither Y_n nor C_n are computable (in the sense of Kleene S1-S9) in terms of any S^2_m and the other data. Hence, it seems Z_2^ω cannot prove the general existence of Y_n and C_n as in the previous items. By contrast, the system ACA_0^ω (and even fragments) suffice to show that A_n from (3.27) is finite in the sense of Definition 3.30, and closed in the sense of Definition 1.2.

In conclusion, the study of BV-functions readily yields finite (resp. closed) sets for which there is no reasonable injection to some fragment of $\mathbb{N}$ (resp. RM-code). This observation justifies our choice of definitions of closed and finite set as in Definitions 1.2 and 3.30

Secondly, Remark 3.37 has some ramifications for our choice of the definition of 'countable set', as follows. Indeed, one could reformulate $\mathsf{CUC}_{\mathsf{fin}} + \mathsf{cocode}_0$ as:

*a **height countable** set in the unit interval can be enumerated*,

where the boldface notion is defined as follows.

Definition 3.38. [Height countable] A set $A \subset \mathbb{R}$ is *height countable* if there is a *height* $H : \mathbb{R} \to \mathbb{N}$ for A, i.e. for all $n \in \mathbb{N}$, $A_n := \{x \in A : H(x) < n\}$ is finite.

The notion of 'height' is mentioned in e.g. [40, 48, 57, 80, 104] in connection to countability. Now, as to the naturalness of Definition 3.38, consider the set of discontinuities of a function $f \in BV$ (or even regular), definable in ACA_0^ω:

$$A := \{x \in [0,1] : f(x+) \neq f(x-)\}. \tag{3.34}$$

The set A is trivially height countable and central to many proofs in [1]. As discussed in [89], no S^2_m suffices to compute an injection from A to $\mathbb{N}$ in general.

In conclusion, the textbook study of BV-functions yields height countable sets occuring 'in the wild' but with no 'reasonable' injection (or bijection) to $\mathbb{N}$. Hence, it seems we have a choice between using $\mathsf{CUC}_{\mathsf{fin}}$ or adopting Definition 3.38 as our definition of countable set. We choose the former option as e.g. Theorem 3.35 is still quite elegant. By contrast, Definition 3.38 is used in [89, 90], as this seems to be the **only** way of obtaining elegant equivalences for the uncountability of $\mathbb{R}$. To be absolutely clear, as documented in [89, 90], the statement *the unit interval is not height countable* readily gives rise to many interesting equivalences while NIN does not (seem to), say working over $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ or fragments.

Finally, our notion of 'finite set' as in Definition 3.30 is different from the mainstream set theory definition (see Footnote 13), for reasons discussed in Remark 3.31. Nonetheless, the reader may desire an equivalence in Theorem 3.35 involving a (more) mainstream definition of finite set. To this end, let $\mathsf{CUC}'_{\mathsf{fin}}$ be $\mathsf{CUC}_{\mathsf{fin}}$ formulated with the following finiteness notion.

Definition 3.39. [Set theory finite] A set $X \subset \mathbb{R}$ is *set theory finite* if there are $k \in \mathbb{N}$ and $Y : [0,1] \to \mathbb{N}$ such that on X, Y is bounded by k and injective.

One readily shows that the following are equivalent, say over $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$.

- (Bolzano-Weierstrass) For $X \subset [0,1]$ which is not set theory finite, there is a limit point $y \in [0,1]$, i.e. $(\forall k \in \mathbb{N})(\exists x \in X)(|x - y| < \frac{1}{2^k})$.
- A finite set (in the sense of Definition 3.30) is set theory finite.

Letting BW be the first item, we note that item (a) from Theorem 3.35 is equivalent to $\mathsf{BW} + \mathsf{CUC}'_{\mathsf{fin}} + \mathsf{cocode}_0$, and where the latter uses Definition 3.39 exclusively. Jordan mentions BW in e.g. [43, p. 23, §27]. We intend to explore the content of the previous remark in a future paper. In conclusion, we note that the insights in this section (esp. regarding Definition 3.30) came about after a recent FOM-discussion initiated by Friedman ([28]).

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Appendix A. Reverse Mathematics: introduction and definitions

A.1. **Reverse Mathematics.** We discuss Reverse Mathematics (Section A.1.1) and introduce -in full detail- Kohlenbach's base theory of *higher-order* Reverse Mathematics (Section A.1.2). Some essential axioms, functionals, and notations may be found in Sections A.1.3 and A.1.4.

A.1.1. *Introduction.* Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics initiated around 1975 by Friedman ([26, 27]) and developed extensively by Simpson ([97]). The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics.

We refer to [99] for a basic introduction to RM and to [96, 97] for an overview of RM. We expect basic familiarity with RM, but do sketch some aspects of Kohlenbach's *higher-order* RM ([47]) essential to this paper, including the base theory RCA_0^ω (Definition A.1).

First of all, in contrast to 'classical' RM based on *second-order arithmetic* Z_2, higher-order RM uses L_ω, the richer language of *higher-order arithmetic*. Indeed, while the former is restricted to natural numbers and sets of natural numbers, higher-order arithmetic can accommodate sets of sets of natural numbers, sets of sets of sets of natural numbers, et cetera. To formalise this idea, we introduce the collection of *all finite types* $\mathbf{T}$, defined by the two clauses:

$$\text{(i) } 0 \in \mathbf{T} \text{ and (ii) If } \sigma, \tau \in \mathbf{T} \text{ then } (\sigma \to \tau) \in \mathbf{T},$$

where 0 is the type of natural numbers, and $\sigma \to \tau$ is the type of mappings from objects of type σ to objects of type τ. In this way, $1 \equiv 0 \to 0$ is the type of functions from numbers to numbers, and $n + 1 \equiv n \to 0$. Viewing sets as given by characteristic functions, we note that Z_2 only includes objects of type 0 and 1.

Secondly, the language L_ω includes variables $x^\rho, y^\rho, z^\rho, \ldots$ of any finite type $\rho \in \mathbf{T}$. Types may be omitted when they can be inferred from context. The constants of L_ω include the type 0 objects $0, 1$ and $<_0, +_0, \times_0, =_0$ which are intended to have their usual meaning as operations on $\mathbb{N}$. Equality at higher types is defined in terms of '$=_0$' as follows: for any objects x^τ, y^τ, we have

$$[x =_\tau y] \equiv (\forall z_1^{\tau_1} \ldots z_k^{\tau_k})[xz_1 \ldots z_k =_0 yz_1 \ldots z_k], \tag{A.1}$$

if the type τ is composed as $\tau \equiv (\tau_1 \to \ldots \to \tau_k \to 0)$. Furthermore, L_ω also includes the *recursor constant* $\mathbf{R}_\sigma$ for any $\sigma \in \mathbf{T}$, which allows for iteration on type σ-objects as in the special case (A.2). Formulas and terms are defined as usual. One obtains the sub-language L_{n+2} by restricting the above type formation rule to produce only type $n + 1$ objects (and related types of similar complexity).

A.1.2. *The base theory of higher-order Reverse Mathematics.* We introduce Kohlenbach's base theory RCA_0^ω, first introduced in [47, §2].

Definition A.1. The base theory RCA_0^ω consists of the following axioms.

(a) Basic axioms expressing that $0, 1, <_0, +_0, \times_0$ form an ordered semi-ring with equality $=_0$.
(b) Basic axioms defining the well-known Π and Σ combinators (aka K and S in [2]), which allow for the definition of λ-*abstraction*.
(c) The defining axiom of the recursor constant $\mathbf{R}_0$: for m^0 and f^1:

$$\mathbf{R}_0(f, m, 0) := m \text{ and } \mathbf{R}_0(f, m, n+1) := f(n, \mathbf{R}_0(f, m, n)). \tag{A.2}$$

(d) The *axiom of extensionality*: for all $\rho, \tau \in \mathbf{T}$, we have:

$$(\forall x^\rho, y^\rho, \varphi^{\rho\to\tau})\big[x =_\rho y \to \varphi(x) =_\tau \varphi(y)\big]. \tag{$\mathsf{E}_{\rho,\tau}$}$$

(e) The induction axiom for quantifier-free formulas of L_ω.
(f) $\mathsf{QF\text{-}AC}^{1,0}$: the quantifier-free Axiom of Choice as in Definition A.2.

Note that variables (of any finite type) are allowed in quantifier-free formulas of the language L_ω: only quantifiers are banned. Recursion as in (A.2) is called *primitive recursion*; the class of functionals obtained from $\mathbf{R}_\rho$ for all $\rho \in \mathbf{T}$ is called *Gödel's system T* of all (higher-order) primitive recursive functionals.

Definition A.2. The axiom $\mathsf{QF\text{-}AC}$ consists of the following for all $\sigma, \tau \in \mathbf{T}$:

$$(\forall x^\sigma)(\exists y^\tau)A(x, y) \to (\exists Y^{\sigma\to\tau})(\forall x^\sigma)A(x, Y(x)), \tag{$\mathsf{QF\text{-}AC}^{\sigma,\tau}$}$$

for any quantifier-free formula A in the language of L_ω.

As discussed in [47, §2], RCA_0^ω and RCA_0 prove the same sentences 'up to language' as the latter is set-based and the former function-based. This conservation result is obtained via the so-called ECF-interpretation discussed in Remark 1.12.

A.1.3. *Notations and the like.* We introduce the usual notations for common mathematical notions, like real numbers, as also introduced in [47].

Definition A.3 (Real numbers and related notions in RCA_0^ω)**.**

(a) Natural numbers correspond to type zero objects, and we use 'n^0' and '$n \in \mathbb{N}$' interchangeably. Rational numbers are defined as signed quotients of natural numbers, and '$q \in \mathbb{Q}$' and '$<_\mathbb{Q}$' have their usual meaning.
(b) Real numbers are coded by fast-converging Cauchy sequences $q_{(\cdot)} : \mathbb{N} \to \mathbb{Q}$, i.e. such that $(\forall n^0, i^0)(|q_n - q_{n+i}| <_\mathbb{Q} \frac{1}{2^n})$. We use Kohlenbach's 'hat function' from [47, p. 289] to guarantee that every q^1 defines a real number.
(c) We write '$x \in \mathbb{R}$' to express that $x^1 := (q^1_{(\cdot)})$ represents a real as in the previous item and write $[x](k) := q_k$ for the k-th approximation of x.
(d) Two reals x, y represented by $q_{(\cdot)}$ and $r_{(\cdot)}$ are *equal*, denoted $x =_\mathbb{R} y$, if $(\forall n^0)(|q_n - r_n| \le 2^{-n+1})$. Inequality '$<_\mathbb{R}$' is defined similarly. We sometimes omit the subscript '$\mathbb{R}$' if it is clear from context.
(e) Functions $F : \mathbb{R} \to \mathbb{R}$ are represented by $\Phi^{1\to 1}$ mapping equal reals to equal reals, i.e. extensionality as in $(\forall x, y \in \mathbb{R})(x =_\mathbb{R} y \to \Phi(x) =_\mathbb{R} \Phi(y))$.
(f) The relation '$x \le_\tau y$' is defined as in (A.1) but with '$\le_0$' instead of '$=_0$'. Binary sequences are denoted '$f^1, g^1 \le_1 1$', but also '$f, g \in C$' or '$f, g \in 2^\mathbb{N}$'. Elements of Baire space are given by f^1, g^1, but also denoted '$f, g \in \mathbb{N}^\mathbb{N}$'.
(g) For a binary sequence f^1, the associated real in $[0, 1]$ is $\mathbb{r}(f) := \sum_{n=0}^\infty \frac{f(n)}{2^{n+1}}$.

(h) Sets of type ρ objects $X^{\rho\to 0}, Y^{\rho\to 0}, \dots$ are given by their characteristic functions $F_X^{\rho\to 0} \leq_{\rho\to 0} 1$, i.e. we write '$x \in X$' for $F_X(x) =_0 1$.

For completeness, we list the following notational convention for finite sequences.

Notation A.4 (Finite sequences)**.** The type for 'finite sequences of objects of type ρ' is denoted ρ^*, which we shall only use for $\rho = 0, 1$. Since the usual coding of pairs of numbers goes through in RCA_0^ω, we shall not always distinguish between 0 and 0^*. Similarly, we assume a fixed coding for finite sequences of type 1 and shall make use of the type '1^*'. In general, we do not always distinguish between 's^ρ' and '$\langle s^\rho\rangle$', where the former is 'the object s of type ρ', and the latter is 'the sequence of type ρ^* with only element s^ρ'. The empty sequence for the type ρ^* is denoted by '$\langle\rangle_\rho$', usually with the typing omitted.

Furthermore, we denote by '$|s| = n$' the length of the finite sequence $s^{\rho^*} = \langle s_0^\rho, s_1^\rho, \dots, s_{n-1}^\rho\rangle$, where $|\langle\rangle| = 0$, i.e. the empty sequence has length zero. For sequences s^{ρ^*}, t^{ρ^*}, we denote by '$s*t$' the concatenation of s and t, i.e. $(s*t)(i) = s(i)$ for $i < |s|$ and $(s*t)(j) = t(|s|-j)$ for $|s| \leq j < |s|+|t|$. For a sequence s^{ρ^*}, we define $\overline{s}N := \langle s(0), s(1), \dots, s(N-1)\rangle$ for $N^0 < |s|$. For a sequence $\alpha^{0\to\rho}$, we also write $\overline{\alpha}N = \langle \alpha(0), \alpha(1), \dots, \alpha(N-1)\rangle$ for *any* N^0. By way of shorthand, $(\forall q^\rho \in Q^{\rho^*})A(q)$ abbreviates $(\forall i^0 < |Q|)A(Q(i))$, which is (equivalent to) quantifier-free if A is.

A.1.4. *Some comprehension functionals.* As noted in Section 1.2, the logical hardness of a theorem is measured via what fragment of the comprehension axiom is needed for a proof. For this reason, we introduce some axioms and functionals related to *higher-order comprehension* in this section. We are mostly dealing with *conventional* comprehension here, i.e. only parameters over $\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$ are allowed in formula classes like Π_k^1 and Σ_k^1.

First of all, the following functional is clearly discontinuous at $f = 11\dots$; in fact, $(\exists^2)$ is equivalent to the existence of $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = 1$ if $x >_\mathbb{R} 0$, and 0 otherwise ([47, §3]). This fact shall be repeated often.

$$(\exists\varphi^2 \leq_2 1)(\forall f^1)\big[(\exists n)(f(n) = 0) \leftrightarrow \varphi(f) = 0\big]. \tag{$\exists^2$}$$

Related to $(\exists^2)$, the functional μ^2 in (μ^2) is also called *Feferman's* μ ([2]).

$$(\exists\mu^2)(\forall f^1)\big[(\exists n)(f(n) = 0) \to [f(\mu(f)) = 0 \wedge (\forall i < \mu(f))(f(i) \neq 0)] \\ \wedge [(\forall n)(f(n) \neq 0) \to \mu(f) = 0]\big], \tag{μ^2}$$

We have $(\exists^2) \leftrightarrow (\mu^2)$ over RCA_0^ω and $\mathsf{ACA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\exists^2)$ proves the same sentences as ACA_0 by [39, Theorem 2.5].

Secondly, the functional S^2 in (S^2) is called *the Suslin functional* ([47]).

$$(\exists\mathsf{S}^2 \leq_2 1)(\forall f^1)\big[(\exists g^1)(\forall n^0)(f(\overline{g}n) = 0) \leftrightarrow \mathsf{S}(f) = 0\big], \tag{S^2}$$

The system $\Pi_1^1\text{-}\mathsf{CA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\mathsf{S}^2)$ proves the same Π_3^1-sentences as $\Pi_1^1\text{-}\mathsf{CA}_0$ by [82, Theorem 2.2]. By definition, the Suslin functional S^2 can decide whether a Σ_1^1-formula as in the left-hand side of (S^2) is true or false. We similarly define the functional S_k^2 which decides the truth or falsity of Σ_k^1-formulas from L_2; we also define the system $\Pi_k^1\text{-}\mathsf{CA}_0^\omega$ as $\mathsf{RCA}_0^\omega + (\mathsf{S}_k^2)$, where (S_k^2) expresses that S_k^2 exists. We note that the operators ν_n from [8, p. 129] are essentially S_n^2 strengthened to return a witness (if existant) to the Σ_n^1-formula at hand.

Thirdly, full second-order arithmetic Z_2 is readily derived from $\cup_k \Pi^1_k\text{-}\mathsf{CA}_0^\omega$, or from:

$$(\exists E^3 \leq_3 1)(\forall Y^2)\big[(\exists f^1)(Y(f)=0) \leftrightarrow E(Y)=0\big], \qquad (\exists^3)$$

and we therefore define $\mathsf{Z}_2^\Omega \equiv \mathsf{RCA}_0^\omega + (\exists^3)$ and $\mathsf{Z}_2^\omega \equiv \cup_k \Pi^1_k\text{-}\mathsf{CA}_0^\omega$, which are conservative over Z_2 by [39, Cor. 2.6]. Despite this close connection, Z_2^ω and Z_2^Ω can behave quite differently, as discussed in e.g. [68, §2.2]. The functional from $(\exists^3)$ is also called '$\exists^3$', and we use the same convention for other functionals.

References

[1] Jürgen Appell, Józef Banaś, and Nelson Merentes, *Bounded variation and around*, De Gruyter Series in Nonlinear Analysis and Applications, vol. 17, De Gruyter, Berlin, 2014.

[2] Jeremy Avigad and Solomon Feferman, *Gödel's functional ("Dialectica") interpretation*, Handbook of proof theory, Stud. Logic Found. Math., vol. 137, 1998, pp. 337–405.

[3] Douglas Bridges, *A constructive look at functions of bounded variation*, Bull. London Math. Soc. **32** (2000), no. 3, 316–324.

[4] Douglas Bridges and Ayan Mahalanobis, *Bounded variation implies regulated: a constructive proof*, J. Symbolic Logic **66** (2001), no. 4, 1695–1700.

[5] Douglas Bridges, Fred Richman, and Peter Schuster, *A weak countable choice principle*, Proc. Amer. Math. Soc. **128** (2000), no. 9, 2749–2752.

[6] E. Borel, *Leçons sur la théorie des fonctions*, Gauthier-Villars, Paris, 1898.

[7] N. Bourbaki, *Élements de mathématique, Livre IV: Fonctions d'une variable réelle. (Théorie élémentaire)*, Actualités Sci. Ind., no. 1132, Hermann et Cie., Paris, 1951 (French).

[8] Wilfried Buchholz, Solomon Feferman, Wolfram Pohlers, and Wilfried Sieg, *Iterated inductive definitions and subsystems of analysis*, LNM 897, Springer, 1981.

[9] Georg Cantor, *Ueber eine Eigenschaft des Inbegriffs aller reellen algebraischen Zahlen*, J. Reine Angew. Math. **77** (1874), 258–262.

[10] ———, *Ein Beitrag zur Mannigfaltigkeitslehre.*, Journal für die reine und angewandte Mathematik **84** (1878), 242-258.

[11] ———, *Ueber unendliche, lineare Punktmannichfaltigkeiten*, Math. Ann. **21** (1883), no. 4, 545–591.

[12] ———, *Zur Lehre vom Transfiniten: gesammelte Abhandlungen aus der Zeitschrift für Philosophie und Philosophische Kritik, vom Jahre 1887*, Pfeffer, Halle, 1890.

[13] ———, *Beiträge zur Begründung der transfiniten Mengenlehre*, Mathematische Annalen **46** (1895), 481-512.

[14] ———, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, Springer, 1980. Reprint of the 1932 original, vii+489.

[15] Douglas Cenzer and Jeffrey B. Remmel, *Proof-theoretic strength of the stable marriage theorem and other problems*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, ASL, 2005, pp. 67–103.

[16] Peter Clote, *The metamathematics of scattered linear orderings*, Arch. Math. Logic **29** (1989), no. 1, 9–20.

[17] J. L. Coolidge, *The lengths of curves*, Amer. Math. Monthly **60** (1953), 89–93.

[18] Pierre Cousin, *Sur les fonctions de n variables complexes*, Acta Math. **19** (1895), 1–61.

[19] Lejeune P. G. Dirichlet, *Über die Darstellung ganz willkürlicher Funktionen durch Sinus- und Cosinusreihen*, Repertorium der physik, von H.W. Dove und L. Moser, bd. 1, 1837.

[20] Enno Dirksen, *Ueber die Anwendung der Analysis auf die Rectification der Curven*, Akademie der Wissenschaften zu Berlin (1833), 123-168.

[21] François G. Dorais, *Classical consequences of continuous choice principles from intuitionistic analysis*, Notre Dame J. Form. Log. **55** (2014), no. 1, 25–39.

[22] François G. Dorais, Damir D. Dzhafarov, Jeffry L. Hirst, Joseph R. Mileti, and Paul Shafer, *On uniform relationships between combinatorial problems*, Trans. Amer. Math. Soc. **368** (2016), no. 2, 1321–1359.

[23] J. M. C. Dunham, *Des méthodes dans les sciences de raisonnement. Application des méthodes générales à la science des nombres et à la science de l'étendue*, Vol II, Gauthier-Villars, 1866.

[24] Solomon Feferman, *How a Little Bit goes a Long Way: Predicative Foundations of Analysis*, 2013. unpublished notes from 1977-1981 with updated introduction, `https://math.stanford.edu/~feferman/papers/pfa(1).pdf`.

[25] Roland Fraïssé, *Theory of relations*, Studies in Logic and the Foundations of Mathematics, vol. 145, North-Holland, 2000. With an appendix by Norbert Sauer.

[26] Harvey Friedman, *Some systems of second order arithmetic and their use*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, 1975, pp. 235–242.

[27] ______, *Systems of second order arithmetic with restricted induction, I & II (Abstracts)*, Journal of Symbolic Logic **41** (1976), 557–559.

[28] ______, *Remarks on Reverse Mathematics /1*, FOM mailing list (Sept. 21st, 2021). `https://cs.nyu.edu/pipermail/fom/2021-September/022875.html`.

[29] Makoto Fujiwara and Keita Yokoyama, *A note on the sequential version of Π^1_2 statements*, Lecture Notes in Comput. Sci., vol. 7921, Springer, Heidelberg, 2013, pp. 171–180.

[30] Makoto Fujiwara, Kojiro Higuchi, and Takayuki Kihara, *On the strength of marriage theorems and uniformity*, MLQ Math. Log. Q. **60** (2014), no. 3.

[31] Noam Greenberg, Joseph S. Miller, and André Nies, *Highness properties close to PA-completeness*, To appear in Israel Journal of Mathematics (2019).

[32] Horst Herrlich, *Axiom of choice*, Lecture Notes in Mathematics, vol. 1876, Springer, 2006.

[33] Arend Heyting, *Recent progress in intuitionistic analysis*, Intuitionism and Proof Theory (Proc. Conf., Buffalo, N.Y., 1968), North-Holland, Amsterdam, 1970, pp. 95–100.

[34] David Hilbert, *Über das Unendliche*, Math. Ann. **95** (1926), no. 1, 161–190 (German).

[35] Denis R. Hirschfeldt, *Slicing the truth*, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, vol. 28, World Scientific Publishing, 2015.

[36] Jeffry L. Hirst and Carl Mummert, *Reverse mathematics and uniformity in proofs without excluded middle*, Notre Dame J. Form. Log. **52** (2011), no. 2, 149–162.

[37] Jeffry L. Hirst, *Representations of reals in reverse mathematics*, Bull. Pol. Acad. Sci. Math. **55** (2007), no. 4, 303–316.

[38] Karel Hrbacek and Thomas Jech, *Introduction to set theory*, 3rd ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 220, Marcel Dekker, Inc., New York, 1999.

[39] James Hunter, *Higher-order reverse topology*, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–The University of Wisconsin - Madison.

[40] M. N. Huxley, *Area, lattice points, and exponential sums*, London Mathematical Society Monographs. New Series, vol. 13, The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.

[41] Thomas J. Jech, *The axiom of choice*, Studies in Logic and the Foundations of Mathematics, Vol. 75, North-Holland, 1973.

[42] Camillie Jordan, *Sur la série de Fourier*, Comptes rendus de l'Académie des Sciences, Paris, Gauthier-Villars **92** (1881), 228–230.

[43] Camille Jordan, *Cours d'analyse de l'École polytechnique. Tome I*, Les Grands Classiques Gauthier-Villars, Éditions Jacques Gabay, 1991. Reprint of the third (1909) edition; first edition: 1883.

[44] Pierre Jullien, *Contribution à l'étude des types d'ordres dispersés*, PhD thesis, University of Marseilles, 1969.

[45] Stephen C. Kleene, *Recursive functionals and quantifiers of finite types. I*, Trans. Amer. Math. Soc. **91** (1959), 1–52.

[46] Ulrich Kohlenbach, *Foundational and mathematical uses of higher types*, Reflections on the foundations of mathematics, Lect. Notes Log., vol. 15, ASL, 2002, pp. 92–116.

[47] ______, *Higher order reverse mathematics*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, ASL, 2005, pp. 281–295.

[48] A. N. Kolmogorov and S. V. Fomin, *Elements of the theory of functions and functional analysis. Vol. 1. Metric and normed spaces*, Graylock Press, Rochester, N.Y., 1957. Translated from the first Russian edition by Leo F. Boron.

[49] Alexander P. Kreuzer, *Bounded variation and the strength of Helly's selection theorem*, Log. Methods Comput. Sci. **10** (2014), no. 4, 4:16, 15.

[50] Kenneth Kunen, *Set theory*, Studies in Logic, vol. 34, College Publications, London, 2011.

[51] Imre Lakatos, *Proofs and refutations*, Cambridge Philosophy Classics, Cambridge University Press, 2015. The logic of mathematical discovery; Edited by John Worrall and Elie Zahar; With a new preface by Paolo Mancosu; Originally published in 1976.

[52] Henri Lebesgue, *Comptes rendus et analyses: Review of Young and Young, The theory of sets of points*, Bulletin des sciences mathématiques **31** (1907), no. 2, 132–134.

[53] Ernst Lindelöf, *Sur Quelques Points De La Théorie Des Ensembles*, Comptes Rendus (1903), 697–700.

[54] John Longley and Dag Normann, *Higher-order Computability*, Theory and Applications of Computability, Springer, 2015.

[55] A. I. Mal′cev, *On ordered groups*, Izvestiya Akad. Nauk SSSR. Ser. Mat. **13** (1949), 473–482 (Russian).

[56] Per Martin-Löf, *The Hilbert-Brouwer controversy resolved?*, in: *One Hundred Years of Intuitionism (1907-2007)*, 1967, pp. 243–256.

[57] Victor H. Moll, *Numbers and functions*, Student Mathematical Library, vol. 65, American Mathematical Society, 2012.

[58] Antonio Montalbán, *Indecomposable linear orderings and hyperarithmetic analysis*, J. Math. Log. **6** (2006), no. 1, 89–120.

[59] Antonio Montalbán, *Open questions in reverse mathematics*, Bull. Sym. Logic **17** (2011), no. 3, 431–454.

[60] Andrzej Mostowski, *Foundational studies. Selected works. Vol. I*, Studies in Logic and the Foundations of Mathematics, vol. 93, North-Holland; PWN—Polish Scientific Publishers, 1979.

[61] P. Muldowney, *A general theory of integration in function spaces, including Wiener and Feynman integration*, Vol. 153, Longman Scientific & Technical, Harlow; John Wiley, 1987.

[62] Carl Mummert and Stephen G. Simpson, *Reverse mathematics and* Π^1_2 *comprehension*, Bull. Symbolic Logic **11** (2005), no. 4, 526–533.

[63] Carl Mummert, *On the reverse mathematics of general topology*, ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)–The Pennsylvania State University.

[64] ———, *Reverse mathematics of MF spaces*, J. Math. Log. **6** (2006), no. 2, 203–232.

[65] André Nies, Marcus A. Triplett, and Keita Yokoyama, *The reverse mathematics of theorems of Jordan and Lebesgue*, The Journal of Symbolic Logic (2021), 1–18.

[66] Dag Normann and Sam Sanders, *Nonstandard Analysis, Computability Theory, and their connections*, Journal of Symbolic Logic **84** (2019), no. 4, 1422–1465.

[67] ———, *The strength of compactness in Computability Theory and Nonstandard Analysis*, Annals of Pure and Applied Logic, Article 102710 **170** (2019), no. 11.

[68] ———, *On the mathematical and foundational significance of the uncountable*, Journal of Mathematical Logic, `https://doi.org/10.1142/S0219061319500016` (2019).

[69] ———, *Representations in measure theory*, Submitted, arXiv: `https://arxiv.org/abs/1902.02756` (2019).

[70] ———, *Open sets in Reverse Mathematics and Computability Theory*, Journal of Logic and Computation **30** (2020), no. 8, pp. 40.

[71] ———, *Pincherle's theorem in reverse mathematics and computability theory*, Ann. Pure Appl. Logic **171** (2020), no. 5, 102788, 41.

[72] ———, *The Axiom of Choice in Computability Theory and Reverse Mathematics*, Journal of logic and computation **31** (2021), no. 1, 297–325.

[73] ———, *On the uncountability of* $\mathbb{R}$, Journal of Symbolic Logic, doi:10.1017/jsl.2022.27 (2022), 1-45.

[74] ———, *Betwixt Turing and Kleene*, LNCS 13137, proceedings of LFCS22 (2022), pp. 18.

[75] ———, *On the computational properties of basic mathematical notions*, Submitted, arxiv: `https://arxiv.org/abs/2203.05250` (2022), pp. 43.

[76] Michael Rathjen, *The art of ordinal analysis*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006.

[77] Judith Roitman, *Introduction to modern set theory*, Pure and Applied Mathematics (New York), John Wiley, 1990.

[78] Joseph G. Rosenstein, *Linear orderings*, Pure and Applied Mathematics, vol. 98, Academic Press, 1982.

[79] Fred Richman, *Omniscience principles and functions of bounded variation*, MLQ **48** (2002), 111–116.

[80] Halsey L. Royden, *Real Analysis*, Lecture Notes in Mathematics, Pearson Education, 1989.
[81] S.B. Russ, *A translation of Bolzano's paper on the intermediate value theorem.*, Hist. Math. **7** (1980), 156–185.
[82] Nobuyuki Sakamoto and Takeshi Yamazaki, *Uniform versions of some axioms of second order arithmetic*, MLQ Math. Log. Q. **50** (2004), no. 6, 587–593.
[83] Sam Sanders, *Plato and the foundations of mathematics*, Submitted, arxiv: `https://arxiv.org/abs/1908.05676` (2019), pp. 40.
[84] ______, *Reverse Mathematics of topology: dimension, paracompactness, and splittings*, Notre Dame Journal for Formal Logic **61** (2020), no. 4, 537-559.
[85] ______, *Lifting recursive counterexamples to higher-order arithmetic*, Proceedings of LFCS2020, Lecture Notes in Computer Science 11972, Springer (2020), 249-267.
[86] ______, *Reverse Mathematics of the uncountability of $\mathbb{R}$: Baire classes, metric spaces, and unordered sums*, Submitted, arXiv: `https://arxiv.org/abs/2011.02915` (2020), pp. 15.
[87] ______, *Lifting countable to uncountable mathematics*, Information and Computation, Springer, paper 104762 **287** (2021), pp. 25.
[88] ______, *Countable sets versus sets that are countable in Reverse Mathematics*, Computability, vol. 11, no. 1, pp. 9-39 (2022).
[89] ______, *On the computational properties of the uncountability of the reals*, LNCS 13468, Proceedings of WoLLIC22, Springer (2022).
[90] ______, *Big in Reverse Mathematics: the uncountability of the reals*, Submitted, arxiv: `https://arxiv.org/abs/2208.03027` (2022), pp. 21.
[91] ______, *Finding points of continuity in real analysis*, Submitted (2022).
[92] ______, *On the computational properties of the Baire category theorem*, Submitted (2022).
[93] Ludwig Scheeffer, *Allgemeine Untersuchungen über Rectification der Curven*, Acta Math. **5** (1884), no. 1, 49–82 (German).
[94] Paul Shafer, *The strength of compactness for countable complete linear orders*, Computability **9** (2020), no. 1, 25–36.
[95] Waclaw Sierpiński, *Sur une propriété des fonctions qui n'ont que des discontinuités de première espèce*, Bull. Acad. Sci. Roumaine **16** (1933), 1–4 (French).
[96] Stephen G. Simpson (ed.), *Reverse mathematics 2001*, Lecture Notes in Logic, vol. 21, ASL, La Jolla, CA, 2005.
[97] ______, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, CUP, 2009.
[98] Reed Solomon, Π^1_1-CA_0 *and order types of countable ordered groups*, J. Symbolic Logic **66** (2001), no. 1, 192–206.
[99] J. Stillwell, *Reverse mathematics, proofs from the inside out*, Princeton Univ. Press, 2018.
[100] Charles Swartz, *Introduction to gauge integrals*, World Scientific, 2001.
[101] Michael Suslin, *Problème 3*, Fundamenta Mathematicae **1** (1920), 223.
[102] Anne Sjerp Troelstra, *Metamathematical investigation of intuitionistic arithmetic and analysis*, Springer Berlin, 1973. Lecture Notes in Mathematics, Vol. 344.
[103] Anne Sjerp Troelstra and Dirk van Dalen, *Constructivism in mathematics. Vol. I*, Studies in Logic and the Foundations of Mathematics, vol. 121, North-Holland, 1988.
[104] B.S. Vatssa, *Discrete Mathematics (4th edition)*, New Age International, 1993.
[105] Karl Weierstrass, *Einleiting in die Theorie der analytischen Funktionen*, Schriftenr. Math. Inst. Univ. Münster, 2. Ser. 38, 108 S., 1986.
[106] Keita Yokoyama, *Standard and non-standard analysis in second order arithmetic*, Tohoku Mathematical Publications, vol. 34, Sendai, 2009. PhD Thesis, Tohoku University, 2007.
[107] Xizhong Zheng and Robert Rettinger, *Effective Jordan decomposition*, Theory Comput. Syst. **38** (2005), no. 2, 189–209.