

REGULARIZATION OF MULTIPLICATIVE SDES THROUGH ADDITIVE NOISE

BY LUCIO GALEATI^{1,a} AND FABIAN A. HARANG^{2,b}

¹*Institute of Applied Mathematics, University of Bonn, lucio.galeati@iam.uni-bonn.de*

²*Department of Mathematics, University of Oslo, fabian.a.harang@bi.no*

We investigate the regularizing effect of certain additive continuous perturbations on SDEs with multiplicative fractional Brownian motion (fBm). Traditionally, a Lipschitz requirement on the drift and diffusion coefficients is imposed to ensure existence and uniqueness of the SDE. We show that suitable perturbations restore existence, uniqueness and regularity of the flow for the resulting equation, even when both the drift and the diffusion coefficients are distributional, thus extending the program of regularization by noise to the case of multiplicative SDEs. Our method relies on a combination of the nonlinear Young formalism developed by Catellier and Gubinelli (*Stochastic Process. Appl.* **126** (2016) 2323–2366), and stochastic averaging estimates recently obtained by Hairer and Li (*Ann. Probab.* **48** (2020) 1826–1860).

1. Introduction. In this paper we deal with multidimensional stochastic differential equations of the form

$$(1.1) \quad dx_t = b_1(t, x_t) dt + b_2(t, x_t) d\beta_t + dw_t, \quad x_0 \in \mathbb{R}^d,$$

where β is a fractional Brownian motion with Hurst parameter $H > 1/2$ and w is a deterministic continuous path. Specifically, we are interested in understanding how the additive perturbation affects the SDE, by identifying *analytic* conditions on w which ensure well-posedness for (1.1) even when it fails for $w \equiv 0$, in the style of *regularisation by noise* phenomena.

Let us first provide a short account of the main known results for (1.1) with $w \equiv 0$. Since $H > 1/2$, the SDE is pathwise meaningful either in the sense of Young integrals or fractional calculus; for b_1 and b_2 sufficiently smooth, existence of a unique solution is classical; see, for example, [15, 30], as well as [5], Appendix D, for a general survey. Sharp conditions for well-posedness, in the form of Osgood-type regularity for b_1 and b_2 , are given in [35], generalizing to the case $H > 1/2$ the results from [33, 36] for $H = 1/2$; this includes the case of b_1 and b_2 Lipschitz. If $d = 1$ and $b_2 \equiv 1$, the authors in [29] establish pathwise uniqueness for b_1 satisfying suitable Hölder regularity. This result can be extended to a broader class of nondegenerate diffusion coefficients b_2 by means of a Doss–Sussman transformation, in the style of [2]. Recently, [23] investigated the case $b_1 \equiv 0$ and b_2 nondegenerate of bounded variation; however, the conditions included therein for well-posedness are fairly specific and require verification for each choice of b_2 .

None of the results mentioned above includes the case of general Hölder continuous diffusion b_2 and smooth drift b_1 . This is not due to technical limitations of the proofs; in fact, uniqueness does in general not hold. To see this, let $d = 1$, let y be a solution to the ODE $\dot{y}_t = f(y_t)$ with $y_0 = 0$, and define the process $x_t := y(\beta_t)$. Under the assumption that f is α -Hölder with $H(1 + \alpha) > 1$, the Young chain rule shows that x satisfies the SDE

$$dx_t = f(x_t) d\beta_t, \quad x_0 = 0.$$

Received August 2020; revised May 2021.

MSC2020 subject classifications. Primary 34F05, 60H10; secondary 37H10.

Key words and phrases. Stochastic differential equations, regularization by noise, multiplicative noise, Young integration, rough path theory.

As a consequence, to any solution of the ODE we can associate a solution of the SDE; if uniqueness fails for the first, it will also fail for latter. For instance, we can take

$$f(z) = \frac{1}{1-\alpha}|z|^\alpha, \quad y_t^1 = 0, \quad y_t^2 = t^{\frac{1}{1-\alpha}},$$

which implies that $x_t^1 = 0$ and $x_t^2 = (\beta_t)^{1/(1-\alpha)}$ are two different solutions starting from 0 to the same SDE; the above procedure actually allows to construct infinitely many of them.

Therefore, the well-posedness theory for SDEs driven by fBm with $H > 1/2$ can not be better than the one for classical ODEs. At the same time, since existence of solutions is granted by compactness arguments under mild regularity assumptions on b_1 and b_2 , it is reasonable to ask whether, among the many mathematical solutions, some are more meaningful than others. If the SDE models a physically observed phenomenon, then its solutions intuitively should be stable under very small perturbations. In this sense, establishing uniqueness for (1.1) with very small, nontrivial w , can be seen as the first step in this context of the more general program on vanishing noise selection of solutions outlined in [13].

Investigations on well-posedness of the SDE (1.1) with w sampled as a stochastic process date back to the pioneering work of Zvonkin [37] and the literature on the topic has grown extensively; see, for example, [4, 14, 25, 28, 34] and the review [13]. However, to the best of our knowledge, only the case $b_2 \equiv 0$ has been treated so far; the presence of a diffusion term, combined with the fact that in the regime $H > 1/2$ many classical probabilistic tools (martingale problems, Markov processes and generators) are not available, creates new difficulties and different sets of idea must be introduced.

Our approach to the problem follows the ideas introduced in [8], where analytic conditions on w which imply well-posedness for (1.1) with $b_2 \equiv 0$ and possibly distributional drift b_1 are identified. In recent years, this analytic approach to regularization by noise phenomena has been considerably expanded; see [16, 21, 22].

From now on, in order not to hinder the main contributions of this work with technical details, we will focus for simplicity on the additively perturbed SDE (in integral form)

$$(1.2) \quad x_t = x_0 + \int_0^t b(x_s) d\beta_s + w_t$$

namely, with $b_1 \equiv 0$ and b_2 not depending on time, but being possibly distributional. Indeed (1.2) presents the same main difficulties and, once they are properly understood, generalising the results to (1.1) is almost straightforward, as will be shown in Section 5.

Our main strategy is based on readapting the nonlinear Young formalism introduced in [8] in this setting. Given a solution x to (1.2), $\theta := x - w$ formally solves

$$(1.3) \quad \theta_t = \theta_0 + \int_0^t b(\theta_s + w_s) d\beta_s.$$

If both b and w are sufficiently regular, then equation (1.3) can be reinterpreted as a nonlinear Young differential equation (nonlinear YDE for short) of the form

$$(1.4) \quad \theta_t = \theta_0 + \int_0^t \Gamma^w b(ds, \theta_s),$$

where we denote by $\Gamma^w b$ the *multiplicative averaged field*, formally defined as

$$(1.5) \quad \Gamma^w b(t, y) = \int_0^t b(y + w_r) d\beta_r, \quad t \in [0, T], y \in \mathbb{R}^d.$$

It plays in this context the same role as the *classical averaged field* $T^w b$ from [8], given by

$$T^w b(t, y) = \int_0^t b(y + w_r) dr, \quad t \in [0, T], y \in \mathbb{R}^d.$$

We can then *define* x to be a solution to (1.2) by imposing the ansatz $x = w + \theta$, with θ solution to (1.4); in this way we can give meaning to (1.2) for less regular choices of b and w , assuming we are able to prove the required regularity for $\Gamma^w b$. Existence and uniqueness of x then reduces to that of θ , which in turn follows from the abstract theory of nonlinear YDEs (see Section 2.2 for a recap) applied to the random field $\Gamma^w b$.

There are, however, some major problems in achieving the program outlined above, compared to the case of perturbed ODEs treated in [8]. Indeed, the classical averaged field $T^w b$ is by now a well-understood object, which is always analytically well defined as a distribution. Moreover, many stochastic estimates are available for $T^w b$ when w is sampled as suitable stochastic processes; see Section 2.1 for an overview. In contrast, in order to define the integral appearing in (1.5) as a Young integral, we need at least to require w to be δ -Hölder continuous with $H + \delta > 1$; without this assumption, it is unclear how to interpret neither (1.2) nor (1.5), even when b is a smooth function. At the same time, it is now clear from [8, 16, 22] that a strong regularisation effect is expected to hold for especially rough w , that is, for very small values of δ , thus making the requirement $H + \delta > 1$ too restrictive.

In order to overcome this difficulty, we must invoke recently developed stochastic estimates by Hairer and Li [20], regarding Wiener integrals of the form

$$\int_0^t f_s d\beta_s$$

with β fBm with $H > 1/2$ and $f : [0, T] \rightarrow \mathbb{R}$ possibly distributional. Remarkably, this not only allows to define $\Gamma^w b$ as a random field, but also relates its space-time Hölder regularity to that of $T^w b$, with no restrictions on the value $\delta \in (0, 1)$. With this tool at hand, we can then apply the already existing results for $T^w b$ in order to define $\Gamma^w b$ and solve the associated equation (1.4).

Our approach presents several nice features: it identifies sufficient analytic conditions for w to regularise the SDE, in the form of regularity requirements for $T^w b$; it provides a path-wise solution concept for (1.2) in terms of equation (1.4), which should be regarded as a random nonlinear YDE rather than an SDE; no adaptedness requirements are needed to guarantee uniqueness; finally, the existence of an associated Lipschitz flow is a direct consequence of the nonlinear YDE theory.

1.1. Main results. In all the next statements, whenever referring to a fractional Brownian motion β of parameter H , we will consider it to be the canonical process on $(\Omega, \mathcal{F}, \mu^H)$, where $\Omega = C([0, T]; \mathbb{R}^m)$, μ^H is the fBm law on Ω and \mathcal{F} is the completion of the $\mathcal{B}(C([0, T]; \mathbb{R}^m))$ w.r.t. μ^H ; the process $\beta = \{\beta_t\}_{t \in [0, T]}$ is given by $\beta_t(\omega) = \omega(t)$. However, as will be discussed, the concept of path-by-path well-posedness only depends on the law μ^H , therefore, the results automatically carry over to any other probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which an fBm of parameter $H > 1/2$ is defined. We will frequently refer to the averaged fields $T^w b$ and $\Gamma^w b$, formally given in (1.5) and rigorously defined in Sections 2 and 3 respectively.

The following statement summarizes our main findings.

THEOREM 1. *Let $H \in (1/2, 1)$, $b \in \mathcal{D}(\mathbb{R}^d)$ and w be a deterministic path such that*

$$(1.6) \quad T^w b \in C_t^\gamma C_x^2 \quad \text{for some } \gamma \in \left(\frac{3}{2} - H, 1\right);$$

then path-by-path well-posedness holds for the SDE

$$dx_t = b(x_t) d\beta_t + dw_t.$$

In particular, for any $x_0 \in \mathbb{R}^d$, any two pathwise solutions defined on $(\Omega, \mathcal{F}, \mathbb{P})$ starting from x_0 are indistinguishable. Moreover, solutions are adapted to the filtration generated by β and they form a random $C^1_{x,\text{loc}}$ flow; specifically, the unique solution starting at x_0 is given by

$$(1.7) \quad x_t(\omega) = w_t + \mathcal{I}(\Gamma^w b(\omega))(t, x_0 - w_0),$$

where $\mathcal{I}(\Gamma^w b)$ is another random $C^1_{x,\text{loc}}$ flow.

For the definitions of pathwise solution and path-by-path well-posedness, we refer to Section 4.3. Let us mention that pathwise solutions need not be adapted, which is instead a consequence of Theorem 1; this is a nontrivial fact, as there are SDEs for which path-by-path uniqueness holds but there exist no adapted solutions; see [32].

A rigorous construction of the random field $\omega \mapsto \Gamma^w(\omega)$, together with its space-time regularity, is presented in Section 3. The notation $\mathcal{I}(\Gamma^w b(\omega))$ is not by chance: as shown in Corollary 20, it is possible to define a continuous function $\mathcal{I}(\cdot)$ which maps drifts of prescribed regularity into flows. Therefore, equation (1.7) implies that the solution map admits the following decomposition:

$$\omega \mapsto \Gamma^w b(\omega) \mapsto \mathcal{I}(\Gamma^w b(\omega)) \mapsto x(\omega),$$

where the first map is measurable, but the other ones are continuous; this is in a nice analogy with the classical decomposition of the Itô–Lyons map from rough path theory.

A justification of our interpretation of the SDE, in terms of a nonlinear YDE related to $\Gamma^w b$, comes from the next result. It also serves the purpose of helping the intuition of the reader, who is not yet familiar with the intrinsic concept of solution based on nonlinear YDEs (which will be given in full detail later on in Section 4.3): the solution x (resp. the flow $\mathcal{I}(\Gamma^w b)$) may also be defined as the unique limit point of the solutions x^n (resp. flows $\mathcal{I}(\Gamma^{w^n} b^n)$) associated to regular pair (b^n, w^n) satisfying point (iii) below.

PROPOSITION 2. *Let H, b, w, β as in Theorem 1. Then:*

(i) *If b and w are regular, then any pathwise solution to the SDE*

$$x_t(\omega) = x_0 + \int_0^t b(x_s(\omega)) \, d\beta_s(\omega) + w_t,$$

where the integral is interpreted in the Young sense, is also a pathwise solution in the sense of Definition 39.

(ii) *If condition (1.6) holds, then it is possible to find sequences (b^n, w^n) of regular coefficients such that $(b^n, w^n) \rightarrow (b, w)$ and the associated pathwise solutions x^n converge in probability to the unique pathwise solution x given by Theorem 1.*

(iii) *More generally, if condition (1.6) holds, for any sequence of regular coefficients $(b^n, w^n) \rightarrow (b, w)$ such that*

$$T^{w^n} b^n \text{ is Cauchy in } C_t^\gamma C_x^2 \quad \text{for some } \gamma \in \left(\frac{3}{2} - H, 1\right)$$

the associated pathwise solutions x^n converge in probability to x .

We have left some of the details of Proposition 2 (the exact regularity, the notions of convergence, etc.) vague on purpose, as it should be regarded as some kind of meta theorem or general principle; more details will be given in the proof in Section 4.4.

Let us stress that condition $(b^n, w^n) \rightarrow (b, w)$ alone is *not* enough to deduce $x^n \rightarrow x$! Indeed, if we mollify the path w first, then its irregularity and its regularising effect on the

equation (measured by the regularity of $T^w b$) are completely lost; in order to build approximations schemes, one needs to first approximate b by a more regular version b^n and only then approximate $T^w b^n$ by $T^{w^n} b^n$, so that at each step the regularity of the averaged field is preserved.

Direct-to-check conditions on the regularity of $T^w b$, as well as higher regularity for the flow, are given by the next statement.

THEOREM 3. *Let $b \in C_x^\alpha$, $\alpha \in \mathbb{R}$, w be such that $T^w b \in C_t^{1/2} C_x^{\alpha+\nu}$ for $\nu > 0$ satisfying*

$$(1.8) \quad \alpha + \nu(2H - 1) > 2.$$

Then the hypothesis of Theorem 1 are met. If in addition $T^w b \in C_t^{1/2} C_x^{\alpha+\nu}$ with

$$(1.9) \quad \alpha + \nu(2H - 1) > n + 1,$$

then the random flow associated to the SDE is $C_{x,\text{loc}}^n$.

If both the diffusion coefficient b and the perturbation w are sufficiently regular to give meaning to the SDE as a classical Young differential equation, but not to establish its uniqueness, we can exploit the double formulation of the problem, as a Young SDE and a nonlinear YDE, to establish uniqueness under weaker regularity for $T^w b$ than that of Theorem 3. However, this comes at the price of prescribing some Hölder regularity for w , which might limit its regularising effect.

THEOREM 4. *Let β be a fractional Brownian motion with parameter $H \in (\frac{1}{2}, 1)$, $b \in C_x^\alpha$ for some $\alpha \in (0, 1)$ and $w \in C_t^\delta$ a deterministic path with $H + \alpha\delta > 1$; suppose that $T^w b \in C_t^{1/2} C_x^{\alpha+\nu}$ for some $\nu > 0$ satisfying*

$$(1.10) \quad \alpha + \nu(2H - 1) > 1 + \frac{1}{2H}.$$

Then for μ^H -a.e. ω the following holds: for every $x_0 \in \mathbb{R}^d$ there exists a unique solution to

$$x_t = x_0 + \int_0^t b(x_s) d\beta_s(\omega) + w_t$$

in the class $x \in (w + C_t^{H-}) \cap C_t^\delta$, where the above integral is meaningful in the Young sense.

The proofs of Theorems 1–4 will be presented in Section 4.4; observe that they only rely on the analytical regularity of $T^w b$, where w is a deterministic continuous path. There is plenty of choice for w , as the next statements show.

COROLLARY 5. *Let w be sampled as an fBm of parameter $\delta \in (0, 1)$, b be a compactly supported distribution of regularity C_x^α , $\alpha \in \mathbb{R}$, such that*

$$(1.11) \quad \alpha > 2 - \frac{1}{\delta} \left(H - \frac{1}{2} \right).$$

Then almost every realisation of w satisfies condition (1.8). If in addition

$$(1.12) \quad \alpha > n + 1 - \frac{1}{\delta} \left(H - \frac{1}{2} \right),$$

then almost every realisation satisfies condition (1.9). Moreover, under (1.11) (resp. (1.12)), generic $w \in C_t^\delta$ satisfy (1.8) (resp. (1.9)), genericity being understood in the sense of prevalence. Finally, if w is sampled as either a $p - \log$ -Brownian motion or an infinite series of fBMs (see Section 4 from [22]), then any choice of $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ is allowed and we can drop the assumption of compact support on $b \in C_x^\alpha$.

PROOF. The case of w sampled as an fBm follows from the results from [16] (see, for instance, Remark 7 or Section 3.3 more in general); indeed for b as in the assumptions, almost every realisation of w satisfies

$$T^w b \in C_t^{\frac{1}{2}} C_x^{\alpha+\nu} \quad \forall \nu < \frac{1}{2\delta}.$$

Under condition (1.11), it is possible to find $\varepsilon > 0$ small enough such that $\nu = 1/(2\delta) - \varepsilon$ satisfies (1.8); similarly under condition (1.12), we can choose $\nu = 1/(2\delta) - \varepsilon$ so that (1.9) holds. The conclusion follows from an application of Theorem 3. The statement for generic $w \in C_t^\delta$ follows from the exact same reasoning, only applying Theorem 2 from [16] instead. The last statement follows from the fact that these processes are infinitely regularising (see Section 4 from [22] for more details), so that $T^w b \in C_t^\alpha C_x^n$ for all $\alpha \in (0, 1)$ and $n \in \mathbb{N}$. \square

REMARK 6. The result shows that the introduction of a suitable perturbation w allows to give meaning and solve the SDE with arbitrarily irregular distributional drift b ; moreover, the associated flow of solutions can become arbitrarily regular in space.

COROLLARY 7. *Let w be sampled as an fBm of parameter $\delta \in (0, 1)$ such that $\delta + H < 1$ and b be a compactly supported distribution of regularity C_x^α such that*

$$(1.13) \quad \alpha > \max \left\{ \frac{1-H}{\delta}, 1 + \frac{1}{2H} - \frac{1}{\delta} \left(H - \frac{1}{2} \right) \right\}.$$

Then almost every realisation of w satisfies the assumptions of Theorem 4. Moreover, under (1.13), generic $w \in C_t^\delta$ satisfy (1.10), genericity being understood in the sense of prevalence.

PROOF. The proof is analogue to that of Corollary 5, only relying on Theorem 4 instead. Under condition (1.13), $H + \alpha\delta > 1$ and we can find $\nu = 1/(2\delta) - \varepsilon$ with $\varepsilon > 0$ sufficiently small such that (1.10) holds. The conclusion then follows from the results from [16] and Theorem 4. \square

REMARK 8. It can be checked that, in order for condition (1.13) to be satisfied for some $\alpha < 1$, it must be imposed $H > \sqrt{2}/2$. With a slight abuse, we can consider the fBm of parameter $H = 1$ to be given by $\beta_t = Nt$, where N is a standard normal (this is the only possible 1-self-similar centered Gaussian process); observe that in the limit $H \uparrow 1$ conditions (1.11), (1.13) become respectively

$$\alpha > 2 - \frac{1}{2\delta}, \quad \alpha > \max \left\{ 0, \frac{3}{2} - \frac{1}{2\delta} \right\}$$

which is consistent with the results from [8] with $d\beta_t$ replaced by dt .

1.2. *Outline of the paper.* In Section 2 we give a short overview of the existing theory on classical averaged fields and nonlinear Young integration. In Section 3 we investigate the multiplicative averaged field, both from an analytic and probabilistic point of view, and establish its space-time regularity. Section 4 deals with regularisation of SDEs by additive perturbations; several theorems regarding existence and uniqueness are given, as well as a discussion of the meaning of well-posedness of these random equations. Proofs of the main results from Section 1.1 are given here. In Section 5, some elementary extensions of the previous results are provided. We conclude in Section 6 with a discussion on open problems and future directions.

1.3. *Notation.* Below is a list of frequently used notation and conventions:

- Throughout the paper we will consider a finite time horizon $T > 0$.
- We write $a \lesssim b$ whenever there exists a constant $C > 0$ such that $a \leq Cb$. If the constant C depends on a parameter p of interest, we write $a \lesssim_p b$.
- We denote by $C_c^\infty(\mathbb{R}^d)$ the space of smooth compactly supported functions and by $\mathcal{D}(\mathbb{R}^d)$ its dual.
- Similarly, $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing functions on \mathbb{R}^d , $\mathcal{S}'(\mathbb{R}^d)$ its dual.
- $\mathcal{B}_{p,q}^\alpha$ denotes the classical inhomogeneous Besov spaces, for $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$.
- We write $C_x^\alpha := \mathcal{B}_{\infty,\infty}^\alpha(\mathbb{R}^d)$; $C_b^n(\mathbb{R}^d; \mathbb{R}^n)$ is the space of bounded, n -times differentiable maps with bounded derivatives up to order n . Their norms are denoted respectively by $\|\cdot\|_\alpha, \|\cdot\|_{C_b^n}$.
- Given a Banach space E , $\gamma \in (0, 1)$, $C_t^\gamma E = C^\gamma([0, T]; E)$ denotes the classical Hölder space of E -valued functions; we equip it with the Hölder seminorm and norm

$$\llbracket f \rrbracket_{\gamma,E} := \sup_{s \neq t \in [0,T]} \frac{\|f_{s,t}\|_E}{|t-s|^\gamma}, \quad \|f\|_{\gamma,E} = \|f(0)\|_E + \llbracket f \rrbracket_{\gamma,E},$$

where we use the increment notation $f_{s,t} := f(t) - f(s)$.

- Of particular interest will be the choices $E = \mathbb{R}^d$, $E = C_x^\eta$ and $E = C_x^{\eta,\lambda}$, where $C_x^{\eta,\lambda}$ denotes a weighted Hölder space (see Definition 15); they define the spaces $C_t^\gamma = C_t^\gamma \mathbb{R}^d$, $C_t^\gamma C_x^\eta$ and $C_t^\gamma C_x^{\eta,\lambda}$. Their norms will be denoted respectively by $\|\cdot\|_\gamma, \|\cdot\|_{\gamma,\eta}, \|\cdot\|_{\gamma,\eta,\lambda}$.
- Whenever there is no possible ambiguity, we will keep using the shorthand notation $\|b\|_\alpha, \|\beta\|_H, \llbracket w \rrbracket_\delta, \|T^w b\|_{\gamma,\eta}, \|\Gamma^w b\|_{\gamma,\eta,\lambda}$, etc.
- For $z \in \mathbb{R}^d$, we define the translation operator τ acting on fields $b : \mathbb{R}^d \rightarrow \mathbb{R}^n$ by $\tau^z b = b(\cdot + z)$.
- Given a continuous path w , for any $\gamma \in (0, 1)$, we set $w + C_t^\gamma := \{w + g, g \in C_t^\gamma\}$.
- We denote by B_R the open ball in \mathbb{R}^d centered at 0 with radius $R > 0$.
- Whenever a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ appears, it is always assumed that \mathcal{F} is \mathbb{P} -complete and that $\{\mathcal{F}_t\}$ satisfies the usual assumptions. We denote by \mathbb{E} expectation with respect to \mathbb{P} .

2. Preliminaries on averaging and nonlinear Young integration.

2.1. *Properties of classical averaged fields.* The averaged field $T^w b$ is by now a well-studied object (see, e.g., [8, 16, 17, 22]); there is, however, not a unique way to define it and, depending on the situations, some definitions might be more practical than others. For self-containedness, we provide here to the reader a brief overview of the topic, together with some of its properties which will be handy for later analysis. We start with an analytical definition of $T^w b$.

DEFINITION 9 (Averaging operator and averaged field). Let $w : [0, T] \rightarrow \mathbb{R}^d$ be a measurable path and E be a separable Banach space, continuously embedded in $\mathcal{S}'(\mathbb{R}^d)$, on which translations act isometrically, that is, $\|\tau^v b\|_E = \|b\|_E$. We define the averaging operator T^w as the continuous linear map from E to $\text{Lip}([0, T], E)$ given by

$$T_t^w b = \int_0^t \tau^{w_s} b \, ds \quad \forall t \in [0, T],$$

where the integral is meaningful in the Bochner sense. We will refer to $T^w b$ as an averaged field.

If $E \hookrightarrow C(\mathbb{R}^d)$, then Definition 9 corresponds to the pointwise one given by

$$T_t^w b(x) = \int_0^t b(x + w_s) ds.$$

If in addition w is a continuous path, then it is easy to check that T^w maps $C_c^\infty(\mathbb{R}^d)$ continuously into itself, allowing to define by duality T^w on $\mathcal{D}(\mathbb{R}^d)$ by setting

$$\langle T^w \varphi, \psi \rangle := \langle \varphi, T^{-w} \psi \rangle \quad \forall \varphi \in \mathcal{D}, \psi \in C_c^\infty.$$

The main advantage of this definition is that it requires no underlying probability space and already allows to deduce some basic properties of the operators T^w .

LEMMA 10. *Let w and b be as in Definition 9. Then the following properties hold:*

- (i) *Averaging and spatial differentiation commute, that is, $\partial_i T^w b = T^w \partial_i b$ for all $i = 1, \dots, d$.*
- (ii) *Averaging and spatial convolution commutes, that is, for any $K \in C_c^\infty(\mathbb{R}^d)$, the following relation hold*

$$K * (T^w b) = T^w (K * b) = (T^w K) * b.$$

We omit the proof, which can be found in Section 3.1 from [16]. Let us mention that Definition 9 is fairly elastic and allows to consider also time-dependent b ; at the same time, its main drawback is that it does not allow to quantify the spatial regularity improvement of $T^w b$, compared to the original b , as an effect of the averaging procedure and the oscillatory nature of w . Nevertheless, if $T^w b$ is known to be regular, it provides efficient ways to approximate it.

LEMMA 11. *Let $b \in E$ for some E as in Definition 9 be such that $T^w b \in C_t^\gamma C_x^\alpha$ for some $\gamma \in (0, 1]$ and $\alpha > 0$, $(\rho^\varepsilon)_{\varepsilon>0}$ be a family of standard mollifiers and define $b^\varepsilon := \rho^\varepsilon * b$. Then for any $\delta > 0$, $T^w b^\varepsilon \rightarrow T^w b$ in $C_t^{\gamma-\delta} C_x^{\alpha-\delta}$ as $\varepsilon \rightarrow 0$.*

PROOF. The lemma is a slight improvement of Lemma 4 from [16], the only difference being the claim that $T^w b^\varepsilon \rightarrow T^w b$ in $C_t^{\gamma-\delta} C_x^{\alpha-\delta}$ globally instead of just locally. As in [16], thanks to the properties of averaging it holds

$$\|T^w b^\varepsilon\|_{\gamma,\alpha} = \|\rho^\varepsilon * T^w b\|_{\gamma,\alpha} \leq \|T^w b\|_{\gamma,\alpha} \quad \forall \varepsilon > 0.$$

Moreover, by properties of convolution, we have

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |(\rho^\varepsilon * T^w b)(t,x) - T^w b(t,x)| \lesssim \varepsilon^\alpha \|T^w b\|_{\gamma,\alpha} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

that is, uniform convergence holds. Standard interpolation estimates between the convergence in $C([0, T] \times \mathbb{R}^d)$ and the uniform bound in $C_t^\gamma C_x^\alpha$ imply the conclusion. \square

Another more probabilistic way to construct an averaged field is to consider a given distribution $b \in \mathcal{S}'(\mathbb{R}^d)$ and a continuous \mathbb{R}^d -valued stochastic process $(w_t)_{t \in [0,T]}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Typically in this setting the goal is to show that \mathbb{P} -a.s. $T^w b$ is a well-defined, continuous random field, even if the original b was not. We say that the process w is ρ -regularising the distribution $b \in C_x^\alpha$ if \mathbb{P} -a.s. $T^w b \in C_t^\gamma C_{x,\text{loc}}^{\alpha+\rho}$ for some $\gamma > 1/2$ and $\rho > 0$.

In this sense, Gubinelli and Catellier proved in [8] that if $b \in C_x^\alpha$ and w is an fBm of parameter $H \in (0, 1)$, then w is ρ -regularising for any $\rho < 1/(2H)$ (the results in [8] actually

also establish global estimates for $T^w b$, which require the introduction of suitable weighted Hölder norms similar to those in (3.9)). Their results have then been extended to other classes of fields b , possibly of the form $b \in L_t^p C_x^\alpha$, in Section 7 from [26] and Section 3.3 from [16].

Thus choosing a fBm with H very small, the regularity of the associated averaged field $T^w b$ gets better. As the techniques used to prove the regularity of $T^w b$ are of probabilistic nature, the set of $\omega \in \Omega$ for which $T^{w(\omega)} b$ has the desired regularity depends on the given b and cannot in general be chosen to be the same for all possible $b \in C_x^\alpha$. At the same time, it provides sharp estimates, which remarkably do not depend on the dimension of the ambient space \mathbb{R}^d .

A third approach, which combines analytic and probabilistic techniques, is based on the following observation: for any continuous path w , we have

$$(2.1) \quad T_{s,t}^w b(x) = b * \bar{\mu}_{s,t}^w(x),$$

where the measure $\bar{\mu}^w$ denotes the reflection of the occupation measure μ^w , that is, $\bar{\mu}_{s,t}^w(A) := \mu_{s,t}(-A)$ for any $A \in \mathcal{B}(\mathbb{R}^d)$. The occupation measure μ^w associated to w is defined as

$$\mu_t(A) = \lambda\{s \leq t | w_s \in A\}$$

for any Borel set $A \subset \mathbb{R}^d$, where λ denotes the Lebesgue measure on $[0, T]$. We say that w admits a local time if μ^w is absolutely continuous w.r.t. the Lebesgue measure on \mathbb{R}^d , in which case the local time L^w is exactly the density of μ^w . Namely, it is the only nonnegative element of $L^1(\mathbb{R}^d)$ such that

$$\mu_t(A) = \int_A L_t(z) dz \quad \forall A \in \mathcal{B}(\mathbb{R}^d).$$

In this case $T^w b = b * \bar{L}_t^w$ where $\bar{L}_t(x) := L_t(-x)$ and in order to show its regularity improvement, it suffices to establish the joint space-time regularity of the map $(t, x) \mapsto L_t^w(x)$. This line of approach was first explored in [8], via the notion of ρ -irregularity; the study of the joint space-time regularity of L^w is, however, a topic of independent interest which has received a lot of attention; see [19] for a review.

It is shown in [22], Theorem 17, that if a Gaussian process $w : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ satisfies the following local nondeterminism condition for some $\zeta \in (0, 2)$

$$\inf_{t>0} \inf_{s \in [0,t]} \inf_{z \in \mathbb{R}^d; |z|=1} \frac{z^\top \text{Var}(w_t | \mathcal{F}_s) z}{(t-s)^\zeta} > 0,$$

where z^\top denotes the transpose of the vector z . Then, \mathbb{P} -a.s. the local time L^w is contained in the space $C_t^\gamma H^k$ for some

$$\gamma > \frac{1}{2}, \quad k < \frac{1}{\zeta} - \frac{d}{2},$$

where H^k denotes the L^2 -based Sobolev space. This result, combined with the relation (2.1), allows to establish a regularising effect for all possible b in a suitable class. Namely, if we denote by $\Omega' \subset \Omega$ the set of full measure where L^w has the desired regularity, then by an application of Young's convolution inequality, we obtain that

$$\|T^{w(\omega)} b\|_{C_t^\gamma C_x^{\beta+k}} \lesssim \|b\|_{H^\beta} \|L^{w(\omega)}\|_{C_t^\gamma H_x^k} \quad \forall b \in H_x^\beta$$

for all $\omega \in \Omega'$. In this case the regularity improvement holds on a set of full probability which is independent of the choice of $b \in H^\beta$. We can view T^w as a (random) continuous linear operator from H^β to $C_t^\gamma C_x^{k+\beta}$; in this sense we can call it an averaging operator.

The main drawback of this approach is that in general the regularity improvement will depend heavily on the dimension d of the ambient space \mathbb{R}^d ; for instance, if w is sampled as a Brownian motion, then its local time L^w exists only for $d = 1$, making the reasoning not applicable for $d \geq 2$. On the other hand, the aforementioned results for the averaged field $T^w b$ still provide a regularisation effect of order $\rho \sim 1$. For this reason in this article we will mostly refrain from considering the operator T^w , but rather only assume to be working with an averaged field $T^w b$ of suitable regularity.

Let us finally mention that in the papers [16, 17], Gubinelli and one of the authors showed that the regularity properties of $T^w b$ (resp. L^w) in fact hold for almost all continuous paths (in the sense of prevalence); see Theorem 1 from [16]. This largely speaks to the generality that is obtained through considerations of averaged fields in connection with ODEs, as in principle one does not impose any statistical assumption on the perturbation w . For instance, the results from [16] can be combined with our results, Theorems 3 and 4, to deduce that generic perturbations w regularise multiplicative SDEs driven by fBm.

2.2. Nonlinear Young integration and equations. We recall in this section some of the main results on the theory of abstract nonlinear Young differential equations, which is by now a well-understood topic; see [8, 16, 22, 24].

We start by introducing the class of vector fields $A : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ we will work with; from now on, whenever A appears, it will be implicitly assumed that $A(0, x) = 0$ for all x . We also adopt the incremental notation $A_{s,t}(x) = A(t, x) - A(s, x)$.

DEFINITION 12. We say that $f \in C(\mathbb{R}^d; \mathbb{R}^d)$ belongs to $C_{x,\text{loc}}^\eta$ for $\eta \in (0, 1)$ if the following quantities are finite for any $R > 0$:

$$\llbracket f \rrbracket_{\eta,R} := \sup_{x,y \in B_R; x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta}, \quad \|f\|_{\eta,R} := \llbracket f \rrbracket_{\eta,R} + \sup_{x \in B_R} |f(x)|.$$

Given $A \in C([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$, we say that $A \in C_t^\gamma C_{x,\text{loc}}^\eta$ for $\gamma, \eta \in (0, 1)$ if similarly, for any $R > 0$, it holds

$$\llbracket A \rrbracket_{\gamma,\eta,R} := \sup_{0 \leq s < t \leq T} \frac{\llbracket A_{s,t} \rrbracket_{\eta,R}}{|t - s|^\gamma} < \infty, \quad \|A\|_{\gamma,\eta,R} := \sup_{0 \leq s < t \leq T} \frac{\|A_{s,t}\|_{\eta,R}}{|t - s|^\gamma} < \infty.$$

$A^n \rightarrow A$ in $C_t^\gamma C_{x,\text{loc}}^\eta$ if $\|A^n - A\|_{\gamma,\eta,R} \rightarrow 0$ as $n \rightarrow \infty$ for any $R \geq 0$; $A \in C_t^\gamma C_{x,\text{loc}}^{n+\eta}$ if A admits spatial derivatives up to order n and $D_x^k A \in C_t^\gamma C_{x,\text{loc}}^\eta$ for any $k \leq n$.

Given A as described above Definition 12, we can define the nonlinear Young integral of A along a curve θ .

THEOREM 13. Let $A \in C_t^\gamma C_{x,\text{loc}}^\eta$ and $\theta \in C_t^\nu$ with $\gamma + \eta\nu > 1$. Then along any sequence of partitions $\{\mathcal{P}^n\}_n$ of $[0, T]$ with infinitesimal meshes $|\mathcal{P}^n| \rightarrow 0$, the following limit exists and is independent of the chosen sequence $\{\mathcal{P}^n\}_n$:

$$\int_0^T A(\text{d}u, \theta_u) = \lim_{n \rightarrow \infty} \sum_{[t_i, t_{i+1}] \in \mathcal{P}^n} A_{t_i, t_{i+1}}(\theta_{t_i}).$$

We say that $\int_0^T A(\text{d}u, \theta_u)$ is a nonlinear Young integral. More generally, the construction holds for any subinterval $[s, t] \subset [0, T]$ and allows to define a map $t \mapsto \int_0^t A(\text{d}u, \theta_u)$ with the following properties:

- (i) $\int_0^s A(\text{d}u, \theta_u) + \int_s^t A(\text{d}u, \theta_u) = \int_0^t A(\text{d}u, \theta_u)$ for all $0 \leq s \leq t \leq T$.

(ii) $\int_0^\cdot A(du, \theta_u) \in C_t^\gamma$ and there exists a constant $C = C(\gamma, \gamma + \eta\nu, T)$ such that, taking $R = \|\theta\|_\infty$, it holds

$$\left| \int_s^t A(du, \theta_u) - A_{s,t}(\theta_s) \right| \leq C|t - s|^{\gamma + \eta\nu} \llbracket A \rrbracket_{\gamma, \beta, R} \llbracket \theta \rrbracket_\nu^\eta,$$

$$\left\| \int_0^\cdot A(du, \theta_u) \right\|_\gamma \leq C \|A\|_{\gamma, \eta, R} (1 + \llbracket \theta \rrbracket_\nu^\eta).$$

(iii) If in addition $\partial_t A$ exists and is continuous, then $\int_0^\cdot A(du, \theta_u) = \int_0^\cdot \partial_u A(u, \theta_u) du$.

(iv) The map from $C_t^\gamma C_{x, \text{loc}}^\eta \times C_t^\nu \rightarrow C_t^\gamma$ given by $(A, \theta) \mapsto \int_0^\cdot A(du, \theta_u)$ is linear in A and continuous in both variables (in the respective topologies).

We can then pass to define the nonlinear Young differential equation (YDE) associated to a drift $A \in C_t^\gamma C_{x, \text{loc}}^\eta$.

DEFINITION 14. Let A be given as in Theorem 13. We say that $\theta \in C_t^\nu$ is a solution starting at $\theta_0 \in \mathbb{R}^d$ to the nonlinear YDE

$$(2.2) \quad d\theta_t = A(dt, \theta_t)$$

if $\gamma + \eta\nu > 1$ and θ satisfies

$$(2.3) \quad \theta_t = \theta_0 + \int_0^t A(du, \theta_u) \quad \forall t \in [0, T].$$

In order to provide a global solution theory, local bounds on A are not enough and suitable growth conditions must be introduced.

DEFINITION 15. For $\eta, \lambda \in (0, 1)$, we define the weighted Hölder space $C_x^{\eta, \lambda} = C^{\eta, \lambda}(\mathbb{R}^d; \mathbb{R}^d)$ as the collection of all fields $f \in C_{x, \text{loc}}^\eta$ such that

$$\|f\|_{\eta, \lambda} := |f(0)| + \sup_{R \geq 1} R^{-\lambda} \llbracket f \rrbracket_{\eta, R} < \infty.$$

$C_x^{\eta, \lambda}$ is a Banach space with the norm $\|\cdot\|_{\eta, \lambda}$; similar definitions hold for $C_x^{n+\eta, \lambda}$, $n \in \mathbb{N}$.

DEFINITION 16. We say that $A \in C_t^\gamma C_x^\eta$ if it satisfies global bounds, namely, if

$$\llbracket A \rrbracket_{\gamma, \eta} := \sup_{0 \leq s < t \leq T} \frac{\llbracket A_{s,t} \rrbracket_\eta}{|t - s|^\gamma} < \infty, \quad \|A\|_{\gamma, \eta} := \sup_{0 \leq s < t \leq T} \frac{\|A_{s,t}\|_\eta}{|t - s|^\gamma} < \infty,$$

where $\llbracket \cdot \rrbracket_\eta$, $\|\cdot\|_\eta$ denote the classical Besov–Hölder seminorm and norm of $C^\eta(\mathbb{R}^d; \mathbb{R}^d)$ respectively. Similarly, $A \in C_t^\gamma C_x^{\eta, \lambda}$ for $\gamma, \eta, \lambda \in (0, 1)$ if

$$\|A\|_{\gamma, \eta, \lambda} := \sup_{0 \leq s < t \leq T} \frac{\|A_{s,t}\|_{\eta, \lambda}}{|t - s|^\gamma} < \infty.$$

Observe that $C_t^\gamma C_x^{\eta, \lambda}$ is a Banach space endowed with the norm $\|\cdot\|_{\gamma, \eta, \lambda}$. The definitions for $C_t^\gamma C_x^{n+\eta}$ and $C_t^\gamma C_x^{n+\eta, \lambda}$ are analogous.

REMARK 17. Although the quantities $\|\cdot\|_{\gamma, \eta, R}$ and $\|\cdot\|_{\gamma, \eta, \lambda}$ are related, since the latter measures how the first grows as a function of R , we ask the reader to keep in mind that they represents two different quantities. Throughout the text $R \geq 0$ will always denote the radius of a ball $B(0, R) \subset \mathbb{R}^d$ centered at zero, and so $\|\cdot\|_{\gamma, \eta, R}$ denotes the Hölder norm restricted to $[0, T] \times B(0, R)$; instead the parameter $\lambda \in (0, 1)$ will be consistently used in relation to the weighted Hölder space $C_x^{\eta, \lambda}$. We believe that the exact meaning of the norm will always be clear from the context.

Observe that for $A \in C_t^\gamma C_x^{\eta,\lambda}$ we have an upper bound on the growth of $A_{s,t}$ at infinity. Indeed, for any $x \in \mathbb{R}^d$ such that $|x| \geq 1$, it holds that

$$|A_{s,t}(x)| \leq |A_{s,t}(x) - A_{s,t}(0)| + |A_{s,t}(0)| \leq \llbracket A \rrbracket_{\gamma,\eta,\lambda} |t - s|^\gamma |x|^{\eta+\lambda} + \|A\|_{\gamma,\eta,\lambda} |t - s|^\gamma.$$

In particular, if $\eta + \lambda \leq 1$, then $A_{s,t}$ has at most linear growth.

Throughout this article, we will often work with parameters γ, η, λ , satisfying the condition

$$(A) \quad \gamma, \eta, \lambda \in (0, 1), \quad \gamma > 1/2, \quad \gamma(1 + \eta) > 1, \quad \eta + \lambda \leq 1.$$

The following theorem gives sufficient conditions for well-posedness of the YDE associated to A , as well as existence and regularity of the associated flow.

THEOREM 18. *Suppose $A \in C_t^\gamma C_x^{\eta,\lambda}$ for some γ, η, λ satisfying (A). Then for any $\theta_0 \in \mathbb{R}^d$ there exists a solution $\theta \in C_t^\gamma$ to the YDE (2.2) starting from θ_0 , as well as a constant $C = C(\gamma, \eta, T)$ such that*

$$(2.4) \quad \|\theta\|_\gamma \leq C \exp(C \|A\|_{\gamma,\eta,\lambda}^2) (1 + |\theta_0|).$$

If $A \in C_t^\gamma C_x^{\eta,\lambda} \cap C_t^\gamma C_{x,\text{loc}}^{1+\eta}$, such solution is unique and the YDE admits a $C_t^\gamma C_{x,\text{loc}}^1$ flow. Finally, if $A \in C_t^\gamma C_{x,\text{loc}}^{n+\eta}$, then the flow belongs to $C_t^\gamma C_{x,\text{loc}}^n$.

PROOF. The existence of a global solution under the condition $C_t^\gamma C_x^{\eta,\lambda}$, together with the a priori estimate (2.4), follows from Theorem 3.1 from [24] (see also Theorem 2.9 from [8]). Since estimate (2.4) is uniform over all possible θ_0 in a bounded ball, we can apply localization arguments (see Remark 2.10 and Section 2.3 from [8], as well as Remark 14 from [16]) and assume w.l.o.g. $A \in C_t^\gamma C_x^{1+\eta}$ (resp. $C_t^\gamma C_x^{n+\eta}$); uniqueness and $C_t^\gamma C_x^1$ -regularity of the flow are then consequences of Theorem 3.5 from [24] (see also Theorems 16 and 17 from [16] or Proposition 28 from [22]). Finally, higher regularity follows from Theorem 2 from [22] or equivalently Theorem 18 from [16]. \square

In order to compare solutions associated to different data (θ_0, A) , a general methodology was introduced in [8], based on what are therein called comparison principles. The version given here is based on Theorem 9 from [16].

THEOREM 19. *Let $R, M > 0$, $A^i \in C_t^\gamma C_x^{1+\eta,\lambda}$ for γ, η, λ satisfying (A). Suppose $\|A^i\|_{\gamma,1+\eta,\lambda} \leq M$, $|\theta_0^i| \leq R$ for $i = 1, 2$, and denote by θ^i the unique solution associated to (A^i, θ_0^i) . Then there exists a constant $C = C(\gamma, \eta, T, R, M)$, increasing in the last two variables, such that*

$$(2.5) \quad \|\theta^1 - \theta^2\|_\gamma \leq C (|\theta_0^1 - \theta_0^2| + \|A^1 - A^2\|_{\gamma,1+\eta,\lambda}).$$

PROOF. We only sketch the proof as it is almost identical to the one of Theorem 9 from [16]. Thanks to the a priori bound (2.4), we can localize everything and assume $A^i \in C_t^\gamma C_x^{1+\eta}$ (the localization will produce constants depending on R and M which are incorporated in the final C). It follows from Lemma 6 in [16] that $v := \theta^1 - \theta^2$ satisfies an affine classical YDE of the form

$$v_t = v_0 + \int_0^t v_s \cdot dV_s + \psi_t,$$

where

$$V_t = \int_0^1 \int_0^t \nabla_x A^1(ds, \theta_s^2 + \lambda(\theta_s^1 - \theta_s^2)) d\lambda, \quad \psi_t = \int_0^t (A^1 - A^2)(ds, \theta_s^2).$$

Standard estimates for solutions to affine Young equations are known (see, for instance, Lemma 19 from [16] or Section 6.2 from [27]); by points (i) and (ii) of Theorem 13, we can estimate ψ by

$$\|\psi\|_\gamma \lesssim \|A^1 - A^2\|_{\gamma,\eta} (1 + \|\theta^2\|_\gamma) \lesssim \|A^1 - A^2\|_{\gamma,\eta}$$

and the conclusion follows. \square

As a nice corollary, we deduce continuous dependence of the flow Φ on the drift A .

COROLLARY 20. *Define a map \mathcal{I} on $C_t^\gamma C_x^{1+\eta,\lambda}$ by $A \mapsto \mathcal{I}(A)$, where $\mathcal{I}(A)$ is the flow associated to A . Then \mathcal{I} is a continuous map from $C_t^\gamma C_x^{1+\eta,\lambda}$ to $C([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$, the latter being endowed with the topology of uniform convergence on compact sets. As a consequence, to any random field $A \in C_t^\gamma C_x^{1+\eta,\lambda}$, we can associate a unique random flow $\Phi = \mathcal{I}(A)$.*

PROOF. The statement is an immediate consequence of estimate (2.5). Indeed, given $A^i \in C_t^\gamma C_x^{1+\eta,\lambda}$ with $\|A^i\|_{\gamma,1+\eta,\lambda} \leq M$, the solutions θ^i associated to (A^i, θ_0, \cdot) correspond to $\theta^i_t = \mathcal{I}(A^i)(t, \theta_0)$ and, therefore, from (2.5) we deduce that

$$\begin{aligned} \sup_{\theta_0 \in B_R, t \in [0, T]} |\mathcal{I}(A^1)(t, \theta_0) - \mathcal{I}(A^2)(t, \theta_0)| &\leq \sup_{\theta_0 \in B_R} \|\mathcal{I}(A^1)(\cdot, \theta_0) - \mathcal{I}(A^2)(\cdot, \theta_0)\|_\gamma \\ &\leq C \|A^1 - A^2\|_{\gamma,1+\eta,\lambda}. \end{aligned}$$

Given a sequence $A^n \rightarrow A$ in $C_t^\gamma C_x^{1+\eta,\lambda}$, it must be bounded in $C_t^\gamma C_x^{1+\eta,\lambda}$ and, therefore, for any $R > 0$ we can find $C_R > 0$ such that

$$\sup_{\theta_0 \in B_R, t \in [0, T]} |\mathcal{I}(A^n)(t, \theta_0) - \mathcal{I}(A)(t, \theta_0)| \leq C_R \|A^n - A\|_{\gamma,1+\eta,\lambda} \rightarrow 0$$

which shows uniform convergence on compact sets of $\mathcal{I}(A^n)$ to $\mathcal{I}(A)$. The last statement follows from the fact that continuous image of measurable functions is still measurable. \square

REMARK 21. The results from [8, 16, 22, 24] actually show that, given a bounded family $\{A_n\}_n$ in $C_t^\gamma C_x^{1+\eta,\lambda}$, the associated flows $\mathcal{I}(A_n)$ are bounded in $C_t^\gamma C_x^{1,\text{loc}}$ (in the sense that all seminorms $\|\mathcal{I}(A_n)\|_{\gamma,1,R}$ are controlled). Thus interpolation estimates allow to improve the previous result by showing that, if $A_n \rightarrow A$ in $C_t^\gamma C_x^{1+\eta,\lambda}$, then $\mathcal{I}(A_n) \rightarrow \mathcal{I}(A)$ in $C_t^{\gamma-\varepsilon} C_x^{1-\varepsilon,\text{loc}}$ for any $\varepsilon > 0$.

3. Averaged fields with multiplicative noise. An averaged field with multiplicative noise is formally given by

$$(3.1) \quad \Gamma_{s,t}^w b(x) = \int_s^t b(x + w_r) dh_r, \quad x \in \mathbb{R}^d, [s, t] \subset [0, T],$$

where we consider in general $w \in C([0, T]; \mathbb{R}^d)$, $b \in \mathcal{D}(\mathbb{R}^d; \mathbb{R}^{d \times m})$ and $h \in C^H([0, T]; \mathbb{R}^m)$ to be a Hölder continuous path with $H > 1/2$.

The main goal of this section is to prove the following result, which allows to rigorously construct $\Gamma^w b$ as a random field and to relate its space-time regularity to that of the classical averaged field $T^w b$.

THEOREM 22. *Let β be a fBm of Hurst parameter $H > 1/2$, with values in \mathbb{R}^m , defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then for any deterministic $b \in \mathcal{S}(\mathbb{R}^d; \mathbb{R}^{d \times m})$ and $w \in C^\delta([0, T]; \mathbb{R}^d)$ with $H + \delta > 1$, it is possible to define the averaged field $\Gamma^w b$ in (3.1) pathwise as a Young integral with $h = \beta(\omega)$; $\Gamma^w b$ can be regarded as a random field from $[0, T] \times \mathbb{R}^d$ to \mathbb{R}^d .*

The definition extends continuously in a unique way to any pair (b, w) with $b \in \mathcal{D}(\mathbb{R}^d; \mathbb{R}^{d \times m})$, $w \in C([0, T]; \mathbb{R}^d)$ such that $T^w b \in C_t^\gamma C_x^\eta$ for some $\gamma > \frac{3}{2} - H$, $\eta \in (0, 1)$. In that case

$$\Gamma^w b \in L^p(\Omega; C_t^{\gamma'} C_x^{\eta', \lambda}) \quad \forall p < \infty, \gamma' < \gamma + H - 1, \eta' < \eta, \lambda > 0$$

and there exists $C > 0$ (depending on all the above parameters) such that for any (b^i, w^i) satisfying $T^{w^i} b^i \in C_t^\gamma C_x^\eta$ it holds

$$(3.2) \quad \mathbb{E}[\|\Gamma^{w^1} b^1 - \Gamma^{w^2} b^2\|_{\gamma', \eta', \lambda}^p] \leq C \|T^{w^1} b^1 - T^{w^2} b^2\|_{\gamma, \eta}^p.$$

More generally, estimate (3.2) holds replacing η', η with $n + \eta', n + \eta$ respectively, for any $n \in \mathbb{N}$; namely, $\Gamma^w b$ inherits higher space regularity from $T^w b$.

REMARK 23. Observe that in the statement of Theorem 22, if $T^w b \in C_t^\gamma C_x^2$ for some $\gamma > \frac{3}{2} - H$, it is always possible to choose the parameters γ', η', λ so that $\gamma' > 1/2$, $\gamma'(1 + \eta') > 1$ and $\eta' + \lambda < 1$, namely, satisfying condition (A).

The proof of Theorem 22 is presented throughout the section, which is structured as follows.

We first consider the more regular case in which $w \in C_t^\delta$ with $\delta + H > 1$. Here we can give a rigorous analytical construction of the operator $b \mapsto \Gamma^w b$, as a map from $\mathcal{D}(\mathbb{R}^d)$ into itself; in this case, the definition is purely analytical and holds for any given H -Hölder continuous path h .

Next we restrict our attention to the fBm case $h = \beta(\omega)$, in which by more probabilistic techniques we can extend the definition of $\Gamma^w b$ to a larger class of (w, b) ; this class is defined only in terms of the regularity of the classical averaged field $T^w b$. A key point will be the use of a lemma from [20] to obtain suitable $L^p(\Omega)$ bounds for $\Gamma^w b$, combined with a modified version of the Garsia–Rodemich–Rumsey lemma.

3.1. Definition of averaging operator. The purpose of this section is to analytically define the multiplicative averaging operator Γ^w as a map from $\mathcal{D}(\mathbb{R}^d)$ to itself; to this end, we need to impose some regularity on w and h , namely, require $H + \delta > 1$.

We will see in the next sections that in the fBm case, we can drop the condition $H + \delta > 1$, by defining $\Gamma^w b$ as a random field.

Recall that for any $v \in \mathbb{R}^d$, τ^v denotes the translation operator by v , that is, $\tau^v b(\cdot) = b(\cdot + v)$.

LEMMA 24. *Let $\alpha \in \mathbb{R}$, $w \in C_t^\delta$, $h \in C_t^H$ and $\eta \in (0, 1]$ such that*

$$H + \eta\delta > 1.$$

Then for any $b \in C_x^{\alpha+\eta}$ there exists a unique element of $C_t^H C_x^\alpha$, which we denote by $\Gamma^w b$ and which we will refer to as a multiplicative averaged field, such that for any $s < t \in [0, T]$

$$\|\Gamma_{s,t}^w b - b(\cdot + w_s)h_{s,t}\|_\alpha \lesssim |t - s|^{H+\eta\delta}.$$

Moreover, there exists a constant $C = C(H + \eta\delta, T)$ such that for any $b \in C_x^{\alpha+\eta}$ it holds

$$(3.3) \quad \|\Gamma^w b\|_{H, \alpha} \leq C \|b\|_{\alpha+\eta} \|h\|_H (1 + \|w\|_\delta).$$

In particular, the map $\Gamma^w : b \mapsto \Gamma^w b$ is an element of $\mathcal{L}(C_x^{\alpha+\eta}; C_t^H C_x^\alpha)$. If $\alpha > 0$, then $\Gamma^w b$ defined as above coincides with the pointwise map defined by the Young integral

$$(3.4) \quad (\Gamma_{s,t}^w b)(x) = \int_s^t b(x + w_r) dh_r.$$

PROOF. All the statements easily follow from an application of the sewing lemma (e.g., [15], Lemma 4.2). Set, for any $s \leq t$, $\Xi_{s,t} := (\tau^{w_s} b)h_{s,t} \in C_x^\alpha$; it holds $\delta \Xi_{s,u,t} = (\tau^{w_s} b - \tau^{w_u} b)h_{s,t}$ with the estimates

$$\begin{aligned} \|\delta \Xi_{s,u,t}\|_\alpha &= \|\tau^{w_s} b - \tau^{w_u} b\|_\alpha |h_{s,t}| \lesssim \|b\|_{\alpha+\eta} |w_{s,u}|^\eta |h_{s,t}| \\ &\leq \|b\|_{\alpha+\eta} [w]_\delta^\eta [h]_H |t - s|^{H+\delta\eta}, \end{aligned}$$

where we used the basic estimate

$$(3.5) \quad \|\tau^y b - \tau^z b\|_\alpha \lesssim |y - z|^\eta \|b\|_{\alpha+\eta}.$$

To see (3.5), observe that by Bernstein estimates, for any Littlewood–Paley block of b it holds

$$\|\tau^y \Delta_n b - \tau^z \Delta_n b\|_\infty \lesssim \|\Delta_n b\|_\infty, \quad \|\tau^y \Delta_n b - \tau^z \Delta_n b\|_\infty \lesssim 2^n |y - z| \|\Delta_n b\|_\infty,$$

which interpolated together provide, for any $\eta \in [0, 1]$,

$$\begin{aligned} \|\tau^y b - \tau^z b\|_\alpha &= \sup_n \{2^{n\alpha} \|\tau^y \Delta_n b - \tau^z \Delta_n b\|_\infty\} \\ &\lesssim |y - z|^\eta \sup_n \{2^{n(\alpha+\eta)} \|\Delta_n b\|_\infty\} = |y - z|^\eta \|b\|_{\alpha+\eta}. \end{aligned}$$

The sewing lemma thus implies the existence and uniqueness of $\Gamma^w b$, as well as the bound

$$\|\Gamma_{s,t}^w b - b(\cdot + w_s)h_{s,t}\|_\alpha \lesssim \|b\|_{\alpha+\eta} [w]_\delta^\eta [h]_H.$$

We then have

$$\begin{aligned} \|\Gamma_{s,t}^w b\|_\alpha &\leq \|\tau^{w_s} b\|_\alpha |h_{s,t}| + C \|b\|_{\alpha+\eta} [w]_\delta^\eta [h]_H |t - s|^{H+\eta\delta} \\ &\lesssim_T |t - s|^H \|b\|_{\alpha+\eta} [h]_H (1 + [w]_\delta), \end{aligned}$$

which implies bound (3.3). The last claim follows from the fact that the Young integral in (3.4) corresponds to the sewing of $\langle \Xi_{s,t}, \delta_x \rangle$ and thus must coincide with $\langle \Gamma_{s,t}^w b, \delta_x \rangle$. \square

The operator Γ^w behaves similar to the classical averaging operator T^w ; we summarize some of its properties in the following two lemmas.

LEMMA 25. *Let $\Gamma^w b$ be given as in Lemma 24. Then the following properties hold:*

(i) *Averaging and space differentiation (in the distributional sense) commute:*

$$\partial_i \Gamma^w b = \Gamma^w \partial_i b \quad \forall b \in C_x^\alpha, i = 1, \dots, d.$$

(ii) *Averaging and spatial convolution commute: for any $\varphi \in C_c^\infty$ it holds*

$$\varphi * (\Gamma^w b) = \Gamma^w (\varphi * b) \quad \forall b \in C_x^\alpha.$$

(iii) *If b is compactly supported, then so is $\Gamma^w b$, with $\text{supp } \Gamma_{s,t}^w b \subset \text{supp } b + B(0, \|w\|_\infty)$ for all s, t . Similarly, if b^1 and b^2 coincide on $B(0, R)$, then $\Gamma^w b^1$ and $\Gamma^w b^2$ coincide on $B(0, R - \|w\|_\infty)$.*

(iv) *The operator Γ^w can be extended to an operator from $\mathcal{D}(\mathbb{R}^d)$ to itself by the duality formula*

$$\langle \Gamma_{s,t}^w \psi, \varphi \rangle := \langle \psi, \Gamma_{s,t}^{-w} \varphi \rangle \quad \forall \psi \in \mathcal{D}(\mathbb{R}^d), \varphi \in C_c^\infty(\mathbb{R}^d).$$

PROOF. The proof is analogue to that of Lemma 24. Indeed, by setting $\Xi[b]_{s,t} := (\tau^{w_s} b)h_{s,t}$, it is immediate to check that

$$\partial_{x_i} \Xi[b] = \Xi[\partial_{x_i} b], \quad \varphi * \Xi_{s,t}[b] = \Xi_{s,t}[\varphi * b]$$

and so the same relations must hold between the respective sewings, proving points (i) and (ii). The first part of point (iii) follows from the fact that, for any $s < t$, $\Xi_{s,t}[b]$ is supported on $\text{supp } b + B(0, w_s) \subset \text{supp } b + B(0, \|w\|_\infty)$ and the second part by applying a similar reasoning to $b^1 - b^2$. Finally, it follows from Lemma 24 and point (iii) that $\Gamma_{s,t}^w$ continuously maps C_c^∞ into itself; therefore, also the dual definition from $\mathcal{D}(\mathbb{R}^d)$ to itself is meaningful. Whenever ψ and φ are both smooth, we have the relation

$$\langle (\tau^{w_s} \psi)h_{s,t}, \varphi \rangle = \langle \psi, (\tau^{-w_s} \varphi)h_{s,t} \rangle$$

which implies the same relation for the respective sewings, that is, $\langle \Gamma_{s,t}^w \psi, \varphi \rangle = \langle \psi, \Gamma_{s,t}^{-w} \varphi \rangle$. □

LEMMA 26. *Let $b \in \mathcal{D}(\mathbb{R}^d)$ be such that $\Gamma^w b \in C_t^\gamma C_x^{\alpha,\lambda}$ for some $\gamma, \lambda \in (0, 1)$ and $\alpha \in (0, \infty)$. Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers and set $b^\varepsilon = \rho^\varepsilon * b$. Then for any $\varepsilon > 0$ it holds $\Gamma^w b^\varepsilon \in C_t^\gamma C_x^{\alpha,\lambda}$ with*

$$(3.6) \quad \|\Gamma^w b^\varepsilon\|_{\gamma,\alpha,\lambda} \lesssim \|\Gamma^w b\|_{\gamma,\alpha,\lambda};$$

moreover, $\Gamma^w b^\varepsilon \rightarrow \Gamma^w b$ as $\varepsilon \rightarrow 0$ in $C_t^{\gamma'} C_x^{\alpha',\lambda}$ for any $\gamma' < \gamma$ and $\alpha' < \alpha$.

PROOF. It is enough to prove the claim for $\alpha \in (0, 1)$, as the other cases follow by repeating the same argument for $D^k \Gamma^w b = \Gamma^w D^k b$. The bound (3.6) follows from point (iii) of Lemma 25, since we have

$$\|\Gamma^w b^\varepsilon\|_{\gamma,\alpha,R} = \|\rho^\varepsilon * \Gamma^w b\|_{\gamma,\alpha,R} \lesssim \|\Gamma^w b\|_{\gamma,\alpha,R+\varepsilon} \lesssim R^\lambda \|\Gamma^w b\|_{\gamma,\alpha,\lambda},$$

where we used the fact that ρ^ε is supported in B_ε and $(R + \varepsilon)^\lambda \sim R^\lambda$ since $R \geq 1$ and $\varepsilon \in (0, 1)$. By properties of convolutions, it holds

$$\sup_{(t,x) \in [0,T] \times B_R} |\Gamma^w b^\varepsilon(t,x) - \Gamma^w b(t,x)| \lesssim \varepsilon^\alpha \|\Gamma^w b\|_{\gamma,\alpha,R+\varepsilon} \lesssim \varepsilon^\alpha R^\lambda \|\Gamma^w b\|_{\gamma,\alpha,\lambda}.$$

Interpolating this estimate with the uniform bound (3.6), we obtain that for any $\theta \in (0, 1)$ it holds

$$\begin{aligned} \|\Gamma^w b^\varepsilon - \Gamma^w b\|_{\theta\gamma,\theta\alpha,\lambda} &= \sup_{R \geq 1} \{R^{-\lambda} \|\Gamma^w b^\varepsilon - \Gamma^w b\|_{\theta\gamma,\theta\alpha,\lambda}\} \\ &\lesssim \varepsilon^{(1-\theta)\alpha} \|\Gamma^w b\|_{\gamma,\alpha,\lambda} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

By the arbitrariness of $\theta \in (0, 1)$ we can conclude. □

3.2. L^p bounds for averaging operators with multiplicative fBm in the smooth case. We will now assume that β is sampled as a fractional Brownian motion with $H > 1/2$, with trajectories in C_t^{H-} ; observe that all the results from the previous section still apply with H replaced by $H - \varepsilon$, ε sufficiently small. Through probabilistic techniques, we will show that we can extend the definition of $\Gamma^w b$ to other choices of b and w and that $\Gamma^w b$ inherits the spatial regularity of $T^w b$ (at least locally). To this end, we will use a probabilistic inequality for integration with respect to a fractional Brownian motion with $H > \frac{1}{2}$ proven by Hairer and Li [20], which we recall first.

PROPOSITION 27. *Let $\beta : [0, T] \times \Omega \rightarrow \mathbb{R}^m$ be a fractional Brownian motion with Hurst parameter $H > 1/2$, $f : [0, T] \rightarrow \mathbb{R}$ be a smooth deterministic function. Then for any $\gamma > 3/2 - H$ and any finite $p \geq 2$ there exists a constant $C = C(p, \gamma, H, T)$ such that*

$$\left\| \int_s^t f_r \, d\beta_r \right\|_{L^p(\Omega)} \leq C \left\| \int_0^\cdot f_r \, dr \right\|_\gamma |t - s|^{H+\gamma-1} \quad \forall [s, t] \subset [0, T].$$

By density and linearity, this immediately allows to extend the definition of $\int_s^t f_r \, d\beta_r$ as a random variable to any distribution f such that $\int_0^\cdot f_r \, dr$ belongs to C_t^γ for some $\gamma > 3/2 - H$.

PROOF. The statement is a particular subcase of [20], Prop. 3.4, in the case of deterministic f . Therein it is required that $f \in C_t^{-\kappa}$, in the sense of $\int_0^\cdot f \in C_t^{1-\kappa}$, for some $\kappa < H - 1/2$, corresponding to $\gamma = 1 - \kappa > 3/2 - H$. \square

REMARK 28. The rather elegant point of Proposition 27 is that it extends the class of integrands with respect to fBm to distributions $f \in \mathcal{S}'(\mathbb{R})$ such that $\int_0^\cdot f_r \, dr \in C_t^\gamma$ for some $\gamma > 3/2 - H$. It immediately extends to the case $f \in \mathcal{S}'(\mathbb{R}; \mathbb{R}^{d \times m})$ by reasoning component-wise. Keeping in mind that our interest is in averaging operators, by setting $f_r = \tau^{w_r} b(x)$ for some continuous path w , $\int_s^t f_r \, d\beta_r$ is a well-defined random variable in $L^p(\Omega)$ as long as $\int_0^\cdot \tau^{w_r} b(x) \, dr = \int_0^\cdot b(x + w_r) \, dr = T^w b(\cdot, x)$ belongs to C_t^γ .

LEMMA 29. *Let $b \in C_b^2(\mathbb{R}^d; \mathbb{R}^{d \times m})$, β be an fBm of parameter $H > 1/2$ and $w \in C_t^\delta$ a deterministic path such that $H + \delta > 1$. Define the multiplicative averaged field $\Gamma^w b$ pathwise as in Lemma 24; namely, for any $\omega \in \Omega$ such that $\beta(\omega) \in C_t^{H-}$, set*

$$(3.7) \quad \Gamma_{s,t}^w b(x)(\omega) := \int_s^t b(x + w_r) \, d\beta_r(\omega).$$

Then for any $p \geq 2$, $\eta \in (0, 1)$ and $\gamma > 3/2 - H$ we have the following estimates:

- (i) $\|\Gamma_{s,t}^w b(x)\|_{L^p(\Omega)} \lesssim \|T^w b\|_{\gamma,\eta} |t - s|^{H+\gamma-1}$,
- (ii) $\|\Gamma_{s,t}^w b(x) - \Gamma_{s,t}^w b(y)\|_{L^p(\Omega)} \lesssim \|T^w b\|_{\gamma,\eta} |x - y|^\eta |t - s|^{H+\gamma-1}$,
- (iii) $\|\nabla \Gamma_{s,t}^w b(x) - \nabla \Gamma_{s,t}^w b(y)\|_{L^p(\Omega)} \lesssim \|T^w b\|_{\gamma,1+\eta} |x - y|^\eta |t - s|^{H+\gamma-1}$.

PROOF. The results are a direct application of Proposition 27. It follows from Lemma 24, for the choice $\alpha = \eta = 1$, that $\Gamma^w b \in C_t^{H-} C_b^1$, as well as $T^w b \in C_t^1 C_b^2$; for any $p \geq 2$ it holds

$$\begin{aligned} \|\Gamma_{s,t}^w b(x)\|_{L^p(\Omega)} &= \left\| \int_s^t b(x + w_r) \, d\beta_r \right\|_{L^p(\Omega)} \\ &\lesssim \left\| \int_0^\cdot b(x + w_r) \, dr \right\|_\gamma |t - s|^{H+\gamma-1} \\ &\sim \|T^w b(\cdot, x)\|_\gamma |t - s|^{H+\gamma-1}, \end{aligned}$$

which implies that point (i) holds. Similarly, for any $x, y \in \mathbb{R}^d$ we have

$$\begin{aligned} \|\Gamma_{s,t}^w b(x) - \Gamma_{s,t}^w b(y)\|_{L^p(\Omega)} &\lesssim \|T^w b(\cdot, x) - T^w b(\cdot, y)\|_\gamma |t - s|^{H+\gamma-1} \\ &\lesssim \|T^w b\|_{\gamma,\eta} |x - y|^\eta |t - s|^{H+\gamma-1}. \end{aligned}$$

Point (iii) follows from the fact that $\nabla \Gamma^w b = \Gamma^w \nabla b$ and an application of points (i) and (ii) with b replaced by ∇b . \square

In order to provide a control on the joint space-time regularity of $\Gamma^w b$ in terms of that of $T^w b$, we need to combine Lemma 29 with a suitable modification of the classical Garsia–Rodemich–Rumsey (GRR) lemma; a direct application of the results from [18] is not enough, as it only provides local estimates, while the theory outlined in Section 2.2 requires the additional growth condition $\Gamma^w b \in C_t^\gamma C_x^{1+\eta,\lambda}$.

Recall that for general $A : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ it holds

$$(3.8) \quad \|A\|_{\gamma,\eta,\lambda} \lesssim \llbracket A \rrbracket_{\gamma,\eta,\lambda} + \|A(\cdot, 0)\|_\gamma,$$

where by definition

$$\llbracket A \rrbracket_{\gamma,\eta,\lambda} = \sup_{0 \leq s < t \leq T} \frac{\llbracket A_{s,t} \rrbracket_{\eta,\lambda}}{|t - s|^\gamma},$$

and we recall that for $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, the weighted Hölder seminorm is given by

$$(3.9) \quad \llbracket f \rrbracket_{\eta,\lambda} := \sup_{R \geq 1} R^{-\lambda} \llbracket f \rrbracket_{\eta,\lambda,R} = \sup_{R \geq 1} \sup_{x,y \in B_R; x \neq y} \frac{|f(x) - f(y)|}{R^\lambda |x - y|^\eta}.$$

In order to establish $C_t^\gamma C_x^{\eta,\lambda}$ -regularity of random fields, we need the following lemma.

LEMMA 30. *Let $\{A(t, x) : t \in [0, T], x \in \mathbb{R}^d\}$ be a family of \mathbb{R}^d -valued random variables satisfying the following condition for some $\kappa > 0$ and $p \geq 1$:*

$$(3.10) \quad \mathbb{E}[|A_{s,t}(x) - A_{s,t}(y)|^p] \leq \kappa |t - s|^{1+\beta_1} |x - y|^{d+\beta_2} \quad \forall 0 \leq s \leq t \leq T, x, y \in \mathbb{R}^d.$$

Then for any $\gamma, \eta, \lambda \in (0, 1)$ such that

$$\gamma < \frac{\beta_1}{p}, \quad \eta < \frac{\beta_2}{p}, \quad \lambda > \frac{\beta_2 + d}{p} - \eta,$$

there exists a constant $C = C(\eta, \gamma, \lambda, \beta_1, \beta_2, p, d)$ and a continuous modification of A such that

$$(3.11) \quad \mathbb{E}[\llbracket A \rrbracket_{\gamma,\eta,\lambda}^p] \leq C\kappa.$$

PROOF. Existence of a jointly continuous modification of A which is locally Hölder continuous follows from classical application of GRR lemma, so we only need to focus on estimate (3.11). We can assume A to take values in \mathbb{R} , as the general case follows reasoning componentwise. We will first prove the following claim: if b is a continuous random field such that

$$\mathbb{E}[|b(x) - b(y)|^p] \leq \kappa |x - y|^{d+\beta} \quad \forall x, y \in \mathbb{R}^d,$$

then for any $\eta < \beta/p$ and λ such that $\eta + \lambda < (\beta + d)/p$, then $b \in C_x^{\eta,\lambda}$ and there exists a constant $c_1 = c_1(d, p, \eta, \beta)$ such that

$$(3.12) \quad \mathbb{E}[\llbracket b \rrbracket_{\eta,\lambda}^p] \leq c_1\kappa.$$

Indeed by the classical GRR lemma, for any continuous function f , there exists a constant $c_2 = c_2(d, \eta, \beta, p)$ which is independent of R such that

$$\|f\|_{\eta,R}^p = \left(\sup_{x,y \in B_R; x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta} \right)^p \leq c_2 \int_{B_R \times B_R} \frac{|f(x) - f(y)|^p}{|x - y|^{2d+\eta p}} dx dy.$$

Applied to the field b , this implies that for any $R > 0$ it holds

$$(3.13) \quad \begin{aligned} \mathbb{E}[R^{-\lambda p} \|b\|_{\eta,R}^p] &\leq c_2 \kappa R^{-\lambda p} \int_{B_R \times B_R} |x - y|^{\beta - \alpha p - d} dx dy \\ &= c_1 \kappa R^{\beta + d - \eta p - \lambda p} \end{aligned}$$

for any $\eta < \beta/p$. Now consider the sequence $R = 2^n$ with $n \in \mathbb{N}$, then

$$(3.14) \quad \begin{aligned} \mathbb{E}\left[\left(\sup_{R=2^n, n \in \mathbb{N}} R^{-\lambda} \|b\|_{\eta,R}\right)^p\right] &\leq \mathbb{E}\left[\sum_{R=2^n} R^{-\lambda p} \|b\|_{\eta,R}^p\right] \\ &\leq c_1 \kappa \sum_n 2^{n(\beta + d - \eta p - \lambda p)} \leq c_3 \kappa \end{aligned}$$

for some $c_3 = c_3(d, \eta, \beta, \lambda, p)$, under the condition $\beta + d - \eta p - \lambda p < 0$. Finally, for any $R \geq 1$, choosing $n \in \mathbb{N}$ such that $2^n \leq R < 2^{n+1}$, it holds

$$\begin{aligned} R^{-\lambda} \|b\|_{\eta,R} &\leq R^{-\lambda} \|b\|_{\eta,2^{n+1}} \\ &\leq R^{-\lambda} 2^{\lambda(n+1)} \sup_{r=2^m, m \in \mathbb{N}} r^{-\lambda} \|b\|_{\eta,r} \leq 2^\lambda \sup_{r=2^m, m \in \mathbb{N}} r^{-\lambda} \|b\|_{\eta,r} \end{aligned}$$

which combined with estimates in (3.13) and (3.14) implies the claim (3.12). In order to conclude, observe that for any $s \leq t$, applying the above to $b = A_{s,t}$, by hypothesis (3.10) we obtain

$$\mathbb{E}[\|A_{s,t}\|_{\eta,\lambda}^p] \leq c_1 \kappa |t - s|^{1+\beta_1}$$

and the conclusion follows by applying classical Kolmogorov continuity criterion. \square

3.3. Proof of Theorem 22. We now have all the ingredients to complete the proof of the main result of this section. We start by showing that estimate (3.2) is true when b and w are taken sufficiently regular.

LEMMA 31. *Let $b^1, b^2, w^1, w^2, \beta$ be as in Lemma 29, $\gamma > \frac{3}{2} - H$ and $\eta \in (0, 1)$ fixed parameters. Then for any choice of $(p, \gamma', \eta', \lambda)$ such that*

$$p \geq 2, \quad \gamma' < \gamma + H - 1, \quad \eta' < \eta, \quad \lambda > 0,$$

there exists a constant C (which depends on d, T and the parameters above) such that

$$(3.15) \quad \mathbb{E}[\|\Gamma^{w^1} b^1 - \Gamma^{w^2} b^2\|_{\gamma', \eta', \lambda}^p] \leq C \|T^{w^1} b^1 - T^{w^2} b^2\|_{\gamma, \eta}^p.$$

PROOF. As the multiplicative averaging acts linearly, it suffices to show the statement for a single $T^w b$. By bound (ii) of Lemma 29, it holds

$$\|\Gamma_{s,t}^w b(x) - \Gamma_{s,t}^w b(y)\|_{L^p(\Omega)} \lesssim \|T^w b\|_{\gamma, \eta} |t - s|^{H+\gamma-1} |x - y|^\eta \quad \forall p \geq 2, x, y \in \mathbb{R}^d.$$

Therefore, $\Gamma_{s,t}^w b$ satisfies condition (3.10) for the choice $\beta_1 = p(H + \gamma - 1) - 1$ and $\beta_2 = p\eta - d$; since p can be chosen arbitrarily large, we conclude by Lemma 30 that for any

$$\gamma' < H + \gamma - 1, \quad \eta' < \eta, \quad \lambda > 0,$$

it holds

$$\mathbb{E}[\|\Gamma^w b\|_{\gamma', \eta', \lambda}^p] \leq C \|T^w b\|_{\gamma, \eta}^p.$$

For the same choice of p and γ' , we may also apply bound (i) of Lemma 29 to $\Gamma_{s,t}^w(0)$, together with Kolmogorov’s continuity theorem, to deduce that

$$\mathbb{E}[\|\Gamma^w(\cdot, 0)\|_{\gamma'}^p] \leq C' \|T^w b\|_{\gamma, \eta}^p;$$

the conclusion then follows by virtue of inequality (3.8). \square

PROOF OF THEOREM 22. The proof is divided in two natural steps: we will first show that, thanks to Lemma 31, we can extend the definition of $\Gamma^w b$ to the case of regular b and continuous (but not necessarily Hölder regular) w ; then we will show that, under the assumption that $T^w b$ is sufficiently regular, the definition further extends to the case of distributional b .

Step 1. Let $b \in C_b^2$, $\{w^n\}_n$ be a sequence in C_t^δ , with $\delta + H > 1$, such that $w^n \rightarrow w$ uniformly on $[0, T]$. Our aim is to show that the sequence $\Gamma^{w^n} b$ is Cauchy in a suitable weighted Hölder space and thus admits a unique limit, which we define to be $\Gamma^w b$. In particular, while we cannot define anymore the field $\Gamma^w b$ analytically as done in Section 3.1, it is still well defined as a random variable.

Since $b \in C_b^2$, for any $n, m \in \mathbb{N}$ we have the estimates

$$\begin{aligned} & \left| \int_s^t b(x + w_r^n) dr - \int_s^t b(x + w_r^m) dr \right| \\ & \leq \int_s^t \|b\|_{C_b^1} |w_r^n - w_r^m| dr \leq \|w^n - w^m\|_\infty \|b\|_{C_b^1} |t - s| \end{aligned}$$

and similarly, for fixed n and any $x, y \in \mathbb{R}^d$,

$$\left| \int_s^t b(x + w_r^n) dr - \int_s^t b(y + w_r^n) dr \right| \leq |x - y| \|b\|_{C_x^1} |t - s|.$$

One can then apply triangular inequality and interpolate the two inequalities above to deduce that, for any $\eta \in (0, 1)$, it holds

$$|T_{s,t}^{w^n} b(x) - T_{s,t}^{w^m} b(y)| \lesssim \|b\|_{C_b^1} |x - y|^\eta \|w^n - w^m\|_\infty^{1-\eta} |t - s|.$$

Since $w^n \rightarrow w$ uniformly in $[0, T]$, the sequence $\{w^n\}_n$ is Cauchy, and by the above estimate so is $\{T^{w^n} b\}_n$ in $C_t^1 C_x^\eta$, for any $\eta < 1$. Combined with (3.15), this implies that for any $\gamma' < H$, $\eta' < \eta$, $\lambda > 0$ and $p \in [2, \infty)$ it holds

$$\mathbb{E}[\|\Gamma^{w^n} b - \Gamma^{w^m} b\|_{\gamma', \eta', \lambda}^p] \lesssim \|T^{w^n} b - T^{w^m} b\|_{1, \eta'+\varepsilon}^p \lesssim \|b\|_{C_b^1} \|w^n - w^m\|_\infty^{1-\eta'-\varepsilon},$$

where we chose $\varepsilon > 0$ s.t. $\eta' + \varepsilon < 1$. Therefore, the sequence $\{\Gamma^{w^n} b\}_n$ is Cauchy in $L^p(\Omega; C_t^{\gamma'} C_x^{\eta', \lambda})$ and it admits a unique limit, which we define to be $\Gamma^w b$. It follows from the arguments above that this is a good definition, as it does not depend on the chosen sequence $\{w_n\}_n$ such that $w_n \rightarrow w$.

More generally, by iterating the reasoning to $D^k b$ for $k \leq n$, the above procedure shows that if $b \in C_b^{n+1}$ and w is a continuous path, then $\Gamma^w b$ belongs to $C_t^{\gamma'} C_x^{n+\eta', \lambda}$. By construction, inequality (3.15) still holds for any pairs (w^i, b^i) with $w^i \in C_t^0$ and $b^i \in C_b^2$.

Step 2. We now want to pass to the case in which b is distributional, w is continuous and $T^w b \in C_t^\gamma C_x^\eta$ (resp. $C_t^\gamma C_x^{n+\eta}$) for some $\gamma > 3/2 - H$.

By Lemma 11 we can choose a family of mollifiers $\{\rho_\varepsilon\}_{\varepsilon>0}$, a parameter $\delta > 0$ arbitrarily small and a sequence $\varepsilon_n \rightarrow 0$ such that setting $b_n = b^{\varepsilon_n} = \rho_{\varepsilon_n} * b$, it holds that $T^w b_n \rightarrow T^w b$ in $C_t^{\gamma-\delta} C_x^{\eta-\delta}$. In particular, $\{T^w b_n\}_n$ is a Cauchy sequence in $C_t^{\gamma-\delta} C_x^{\eta-\delta}$ and choosing δ such that $\gamma - \delta > 3/2 - H$, by the previous step $\Gamma^w b_n$ are well-defined random fields; moreover, for any $\gamma' < \gamma + H - \delta - 1$, $\eta' < \eta - \delta$, $\lambda > 0$ and $p \in [2, \infty)$ it follows from Lemma 31 that they satisfy

$$(3.16) \quad \mathbb{E}[\| \Gamma^w b_n - \Gamma^w b_m \|_{\gamma', \eta', \lambda}^p] \lesssim \| T^w b_n - T^w b_m \|_{\gamma-\delta, \eta-\delta}^p.$$

This implies that $\{\Gamma^w b_n\}_n$ is a Cauchy sequence in $L^p(\Omega; C_t^{\gamma'} C_x^{1+\eta', \lambda})$ and thus admits a unique limit, which we define to be $\Gamma^w b$. It follows from Lemma 11 that $\Gamma^w b$ does not depend on the chosen family of mollifiers. Indeed, given another sequence \tilde{b}_n , it holds $T^w \tilde{b}_n \rightarrow T^w b$ in $C_t^{\gamma-\delta} C_x^{\eta-\delta}$ as well, so that $T^w b_n - T^w \tilde{b}_n$ converge to 0 in $C_t^{\gamma-\delta} C_x^{\eta-\delta}$; therefore, by Lemma 31, the random fields $\Gamma^w b_n - \Gamma^w \tilde{b}_n$ converge to 0 in $L^p(\Omega; C_t^{\gamma'} C_x^{1+\eta', \lambda})$, implying they both converge to $\Gamma^w b$. More generally, the reasoning shows that for any sequence of smooth functions b_n s.t. $T^w b_n \rightarrow T^w b$ in $C_t^{\gamma-\delta} C_x^{\eta-\delta}$, the associated multiplicative averaged fields $\Gamma^w b_n$ must converge to $\Gamma^w b$. Moreover, for any pair of random fields $\Gamma^{w_1} b_1, \Gamma^{w_2} b_2$ defined in this way, for w^i continuous paths and b_i possibly distributional fields, we have the inequality

$$\mathbb{E}[\| \Gamma^{w_1} b_1 - \Gamma^{w_2} b_2 \|_{\gamma', \eta', \lambda}^p] \lesssim \| T^{w_1} b_1 - T^{w_2} b_2 \|_{\gamma, \eta}^p,$$

which can be rephrased as the fact that the multiplicative averaging, seen as a map $T^w b \mapsto \Gamma^w b$ from $C_t^\gamma C_x^\eta$ to $L^p(\Omega; C_t^{\gamma'} C_x^{\eta', \lambda})$, is linear and continuous.

The general case of $T^w b \in C_t^\gamma C_x^{n+\eta}$ follows as before by iterating the reasoning to the derivatives $D^k T^w b = T^w D^k b$. \square

REMARK 32. If $w \in C_t^\delta$ with $\delta + H > 1$, the procedure from Theorem 22 is consistent with the one from Section 3.1, namely, the random field $\Gamma^w b$ is a regular representative of the random distribution defined pathwise by means of Lemma 24.

REMARK 33. Several properties satisfied by the analytical definition of $\Gamma^w b$ from Lemma 25 extend by the approximation procedure to the more general definition of Theorem 22, once they are interpreted as equalities between random variables. For instance, it is still true that, for $K \in C_c^\infty$, $K * \Gamma^w b = \Gamma^w(K * b)$; similarly, if both $T^w b$ and $T^w \nabla b$ are regular enough, then $\Gamma^w \nabla b = \nabla \Gamma^w b$.

REMARK 34. The proof of Theorem 22 also contains the following fact: if $T^w b \in C_t^\gamma C_x^{n+\eta}$, then it is possible to find a sequence (b^n, w^n) with $b^n \in C_x^\infty$, $w^n \in C_t^1$ such that $b^n \rightarrow b$ in the sense of distributions, $w^n \rightarrow w$ in the uniform convergence and $\Gamma^{w^n} b^n \rightarrow \Gamma^w b$ in $L^p(\Omega; C_t^{\gamma'} C_t^{n+\eta', \lambda})$ for any $\gamma' < \gamma + H - 1$, $\eta' < \eta$ and $\lambda > 0$.

4. Regularisation of SDEs by additive perturbations. We are now ready to prove the regularizing effect of certain paths on SDEs with multiplicative noise. Towards this aim, we begin to motivate this section by showing that when b is a smooth vector field, $w \in C_t^\delta$, and $t \mapsto \beta_t$ is a sample path of a fractional Brownian motion with $H \in (\frac{1}{2}, 1)$ such that $\delta + H > 1$, then multiplicative SDEs formally given by

$$(4.1) \quad dx_t = b(x_t) d\beta_t + dw_t, \quad x_0 \in \mathbb{R}^d$$

can be solved in the nonlinear Young equations framework, outlined in Section 2.2. Just as in the nonmultiplicative case, these results can then be generalised to allow for distributional

drifts b , still under the assumption that $\delta + H > 1$. These solutions preserve the natural notion of a pathwise solution, in the sense that if $\{b^n\}_n$ is a sequence of smooth functions approximating the distribution b in a suitable distribution space, then the corresponding solutions $x^n \rightarrow x$ in C_t^δ .

4.1. *Classical YDEs as averaged equations.* The content of this section, similar to that of Section 3.1, is entirely analytic and holds also when β is replaced by a deterministic path $h \in C_t^H$, similar to Section 3.1. All the statements generalize to the case $h \in C_t^{H-}$, as the conditions on H are always in the form of a strict inequality, thus they can be applied to typical realizations $\beta(\omega)$ of fBm.

Let us briefly recall the setting: here $b \in \mathcal{D}(\mathbb{R}^d; \mathbb{R}^{d \times m})$ (mostly regular for the moment), $w \in C^\delta([0, T]; \mathbb{R}^d)$ and $h \in C^H([0, T]; \mathbb{R}^m)$; we look for a solution $x \in C([0, T]; \mathbb{R}^d)$.

We start by showing that the nonlinear YDE formulation of the problem is a natural generalisation of the original one, whenever b and w are sufficiently regular.

PROPOSITION 35. *Let $b \in C_b^2$, $w \in C_t^\delta$ and $h \in C_t^H$ with $H > 1/2$, $H + \delta > 1$. Then for any $x_0 \in \mathbb{R}^d$ there exists a unique solution $x \in C_t^\delta$ to the perturbed Young differential equation*

$$(4.2) \quad x_t = x_0 + \int_0^t b(x_s) dh_s + w_t \quad \forall t \in [0, T];$$

in particular, $x = \theta + w$, where $\theta \in C_t^H$ is the unique solution to the nonlinear YDE

$$(4.3) \quad \theta_t = \theta_0 + \int_0^t \Gamma^w b(ds, \theta_s).$$

For any $\alpha \in (0, 1)$ satisfying $H + \alpha\delta > 1$ there exists a constant $C = C(\alpha, \delta, H, T)$ such that θ satisfies the a priori estimate

$$(4.4) \quad \llbracket \theta \rrbracket_H \leq C(1 + \|b\|_\alpha^2 \llbracket h \rrbracket_H^2)(1 + \llbracket w \rrbracket_\delta).$$

PROOF. It is easy to check that $x \in C_t^\delta$ solves (4.2) iff $\theta = x - w \in C_t^\delta$ satisfies

$$\theta_t = \theta_0 + \int_0^t b(\theta_s + w_s) dh_s = \theta_0 + \int_0^t \tilde{b}(s, \theta_s) dh_s \quad \forall t \in [0, T],$$

where $\tilde{b}(t, z) := b(z + w_t)$; by properties of Young integrals, any such θ must also belong to C_t^H . The drift \tilde{b} satisfies

$$|\tilde{b}(t, z_1) - \tilde{b}(s, z_2)| + |\nabla \tilde{b}(t, z_1) - \nabla \tilde{b}(s, z_2)| \lesssim \|b\|_{C_b^2} |z_1 - z_2| + \|b\|_{C_b^2} \llbracket w \rrbracket_\delta |t - s|^\delta$$

which by classical results implies existence and uniqueness of solutions to the YDE associated to \tilde{b} in the class C_t^H ; see, for instance, Theorem 2.1 from [30] or Section 3 from [10].

In order to show that θ solves (4.3), it is enough to prove that $\int_0^t b(w_s + \theta_s) dh_s = \int_0^t \Gamma^w b(ds, \theta_s)$. Since $b \in C_b^2$ and $H + \delta > 1$, by Lemma 24 we have $\Gamma^w b \in C_t^H C_x^1$ and the nonlinear Young integral $\int_0^t \Gamma^w b(ds, \theta_s)$ is well defined (because $\theta \in C_t^H$ and $H > 1/2$). By the respective definition of the two integrals, it holds

$$\begin{aligned} & \left| \int_s^t b(w_r + \theta_r) dh_r - \int_s^t \Gamma^w b(dr, \theta_s) \right| \\ &= \left| \int_s^t b(w_r + \theta_r) dh_r \pm b(w_s + \theta_s) h_{s,t} \pm \Gamma_{s,t}^w b(\theta_s) - \int_s^t \Gamma^w b(dr, \theta_s) \right| \\ &\lesssim |t - s|^{H+\delta} + \left| b(w_s + \theta_s) h_{s,t} - \int_s^t b(\theta_s + w_r) dh_r \right| \lesssim |t - s|^{H+\delta} \end{aligned}$$

which implies that they must coincide.

We now move on to prove (4.4). For any $0 < \Delta < T$, denote by $[\theta]_{H,\Delta}$ (resp. $[\theta]_{\delta,\Delta}$) the quantity

$$[\theta]_{H,\Delta} = \sup_{|t-s| \leq \Delta} \frac{|\theta_{s,t}|}{|t-s|^H}.$$

By properties of Young integrals, for any $s < t$ such that $|t-s| < \Delta$ it holds

$$\begin{aligned} |\theta_{s,t}| &= \left| \int_s^t b(w_r + \theta_r) dh_r \right| \\ &\lesssim |b(w_s + \theta_s)h_{s,t}| + |t-s|^{H+\alpha\delta} [b]_\alpha [h]_H [\theta + w]_{\delta,\Delta}^\alpha \\ &\lesssim |t-s|^H \|b\|_\alpha [h]_H + |t-s|^H \Delta^{\alpha\delta} [b]_\alpha [h]_H (1 + [w]_\delta + [\theta]_{\delta,\Delta}) \\ &\lesssim |t-s|^H \|b\|_\alpha [h]_H (1 + \Delta^{\alpha\delta} + \Delta^{\alpha\delta} [w]_\delta) + |t-s|^H \Delta^{\alpha\delta} \|b\|_\alpha [h]_H [\theta]_{H,\Delta}. \end{aligned}$$

Dividing by $|t-s|^H$, taking the supremum over $|t-s| \leq \Delta$, we find $\kappa = \kappa(\alpha, \delta, H, T)$ s.t.

$$[\theta]_{H,\Delta} \leq \kappa \|b\|_\alpha [h]_H (1 + \Delta^{\alpha\delta} + \Delta^{\alpha\delta} [w]_\delta) + \kappa \Delta^{\alpha\delta} \|b\|_\alpha [h]_H [\theta]_{H,\Delta};$$

choosing Δ such that $\kappa \Delta^{\alpha\delta} \|b\|_\alpha [h]_H \leq 1/2$, $\kappa \Delta^{\alpha\delta} \|b\|_\alpha [h]_H \sim 1$ we obtain

$$[\theta]_{H,\Delta} \lesssim 1 + \|b\|_\alpha [h]_H + [w]_\delta.$$

Applying Exercise 4.24 from [15] we deduce

$$[\theta]_H \lesssim \Delta^{H-1} (1 + \|b\|_\alpha [h]_H + [w]_\delta) \lesssim (\|b\|_\alpha [h]_H)^{\frac{1-H}{\alpha\delta}} (1 + \|b\|_\alpha [h]_H + [w]_\delta)$$

and the conclusion follows from the fact that $(1-H)/(\alpha\delta) < 1$ by hypothesis. \square

4.2. *General YDEs as averaged equations.* In the case b is regular enough for the classical YDE (4.2) to be meaningful, the nonlinear Young formalism still gives nontrivial criteria in order to establish uniqueness of solutions, as the next proposition shows.

PROPOSITION 36. *Let $b \in C_x^\alpha$ for some $\alpha \in (0, 1)$ such that $H + \alpha\delta > 1$. Then for any $x_0 \in \mathbb{R}^d$ there exists at least one solution $x \in C_t^\delta$, $x \in w + C_t^H$ to the YDE (4.2). If $\Gamma^w b \in C_t^\gamma C_{x,\text{loc}}^{1+\eta}$ for some $\gamma, \eta \in (0, 1)$ satisfying*

$$\gamma + \eta H > 1,$$

then such solution x is unique in the class $w + C_t^H$.

PROOF. The proof follows a similar reasoning to those from Section 4.1 of [16], so we will mostly sketch it.

Step 1: Existence. Let b^ε be a sequence of mollifications of b and denote by x^ε the unique solution of the YDE (4.2) associated to b^ε with initial data x_0 . Then $x^\varepsilon = \theta^\varepsilon + w$ satisfy the a priori bound (4.4), uniformly in $\varepsilon > 0$ and so by Ascoli–Arzelà we can extract a subsequence θ^{ε_n} such that $\theta^{\varepsilon_n} \rightarrow \theta$ in $C_t^{H'}$ for any $H' < H$. Combining this fact with $b^{\varepsilon_n} \rightarrow b$ in $C_x^{\alpha'}$ for any $\alpha' < \alpha$, it is easy to check by the continuity properties of Young integrals that $x := \theta + w$ must be a solution to the YDE associated to b , with initial data x_0 .

Step 2: Averaging formulation. Reasoning as in the proof of Proposition 35, it can be shown that θ is also a solution of (4.3).

Step 3: Separation property. Given any two solutions x^1, x^2 for the same initial data x_0 , $x^i = \theta^i + w$ with $\theta^i \in C_t^H$, we claim that their difference $v = x^1 - x^2 = \theta^1 - \theta^2$ satisfies a linear YDE of the form

$$(4.5) \quad dv_t = v_t \cdot dV_t, \quad V_t = \int_0^t \int_0^1 \nabla \Gamma^w b(ds, \lambda \theta_s^1 + (1-\lambda)\theta_s^2) d\lambda.$$

This follows from the general fact that for any θ^i as above, and any $A \in C_t^\gamma C_{x,\text{loc}}^{1+\eta}$, it holds

$$\int_0^t A(ds, \theta_s^1) - \int_0^t A(ds, \theta_s^2) = \int_0^t (\theta_s^1 - \theta_s^2) \cdot dV[A]_s,$$

$$V[A]_t := \int_0^1 \int_0^t \nabla A(ds, \lambda\theta_s^1 + (1-\lambda)\theta_s^2) d\lambda$$

which can be shown by going through the same proof as in Lemma 6 from [16].

Step 4: Conclusion. The difference $v = x^1 - x^2$ satisfies a linear YDE with initial data $v_0 = 0$. Uniqueness for such equations is well known, thus necessarily $v \equiv 0$. \square

Our general aim is to show that the introduction of suitable perturbations w allows to restore existence and uniqueness for the SDE and provides a consistent solution theory even when b is merely distributional; the next lemmas show that, when it is possible to carry out this program, we can also recover our generalised solutions as limits of those associated to more classical YDEs of the form (4.2) with regular coefficients.

LEMMA 37. Consider sequences b^n of regular functions (e.g., in C_b^2), $x_0^n \in \mathbb{R}^d$ and $w^n \in C_t^\delta$ with $\delta + H > 1$; denote by x^n the unique solution starting from x_0^n to the classical YDE

$$dx^n = b^n(x^n) dh + dw^n.$$

Suppose that

$$x_0^n \rightarrow x_0 \text{ in } \mathbb{R}^d, \quad w^n \rightarrow w \text{ in } C_t^0, \quad \Gamma^{w^n} b^n \rightarrow A \text{ in } C_t^\gamma C_x^{1+\eta,\lambda},$$

where γ, η, λ are parameters satisfying $\gamma > 1/2, \gamma(1 + \eta) > 1$ and $\eta + \lambda \leq 1$. Then x^n converge uniformly to $w + \theta$, where θ is the unique solution starting from $\theta_0 := x_0 - w_0$ to the nonlinear YDE associated to A .

PROOF. We know from Proposition 35 that in the smooth case, $\theta^n := x^n - w^n$ is a solution to the nonlinear YDE associated to $(\Gamma^{w^n} b^n, x_0^n - w_0^n)$, where the multiplicative averaging operator $\Gamma^{w^n} b^n$ is classically defined pointwise and by hypothesis $(\Gamma^{w^n} b^n, x_0^n - w_0^n) \rightarrow (A, \theta_0)$ in $C_t^\gamma C_x^{1+\eta,\lambda} \times \mathbb{R}^d$. It then follows from Theorem 19 that $\theta^n \rightarrow \theta$ in C_t^γ ; since $w^n \rightarrow w$, it follows that $x^n = w^n + \theta^n \rightarrow w + \theta$. \square

We stated Lemma 37 in a general fashion, so that it can be applied even in situations in which after the limit w does not belong to C_t^δ with $\delta > 1 - H$. In this case the analytic definition of $\Gamma^w b$ breaks down, even in the distributional sense, regardless the regularity of b ; therefore, we must invoke the stochastic construction of $\Gamma^w b$ from Section 3.2, which truly relies on $h = \beta(\omega)$ being a typical realization of fBm. However, in the regime $H + \delta > 1$, if the regularity of $\Gamma^w b$ is known, the approximating sequence can be constructed explicitly and we obtain the following result, which holds for any given continuous path $h \in C_t^H$, not necessarily sampled as a stochastic process.

PROPOSITION 38. Let $b \in \mathcal{D}(\mathbb{R}^d)$ be such that $\Gamma^w b \in C_t^\gamma C_x^{1+\eta,\lambda}$ for some γ, η, λ satisfying (A). Then for any $\theta_0 \in \mathbb{R}^d$ there exists a unique solution $\theta \in C_t^\gamma$ to the nonlinear YDE

$$(4.6) \quad \theta_t = \theta_0 + \int_0^t \Gamma^w b(ds, \theta_s).$$

Moreover, denoting by b^ε a sequence of mollifications of b and by x^ε the solutions associated to

$$x_t^\varepsilon = \theta_0 + \int_0^t b^\varepsilon(x_s^\varepsilon) dh_s + w_t,$$

then setting $\theta^\varepsilon = x^\varepsilon - w$, it holds $\theta^\varepsilon \rightarrow \theta$ in C_t^γ as $\varepsilon \rightarrow 0$.

PROOF. The first claim follows from Theorem 18. By Lemma 26, $\Gamma^w b^\varepsilon$ are uniformly bounded in $C_t^\gamma C_x^{1+\eta,\lambda}$ and they are converging to $\Gamma^w b$ in $C_t^{\gamma'} C_x^{1+\eta',\lambda}$ for any $\gamma' < \gamma$ and $\eta' < \eta$; we can choose them so that $\gamma' > 1/2$, $\gamma'(1 + \eta') > 1$, $\eta' + \lambda \geq 1$. The conclusion then follows from Lemma 37. \square

4.3. Concepts of existence and uniqueness.

DEFINITION 39. Let $\{\beta_t\}_{t \in [0, T]}$ be a fBm of Hurst parameter $H > 1/2$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, w a continuous deterministic path and b a distributional field. We say that a process x is a *pathwise solution* starting at $x_0 \in \mathbb{R}^d$ to the SDE

$$dx_t = b(x_t) d\beta_t + dw_t$$

if there exist parameters γ, η, λ satisfying (A) and a set $\Omega' \subset \Omega$ of full probability such that, for all $\omega \in \Omega'$, the following hold:

- (i) $\Gamma^w b$ is well defined in the sense of Theorem 22 and $\Gamma^w b(\omega) \in C_t^\gamma C_x^{\eta,\lambda}$.
- (ii) $x(\omega)_0 = x_0$ and $x(\omega) \in w + C_t^\gamma$.
- (iii) $\theta(\omega) := x(\omega) - w$ satisfies the nonlinear YDE

$$\theta_t(\omega) = \theta_0 + \int_0^t \Gamma^w b(\omega)(ds, \theta_s(\omega)).$$

Let us comment on the above definition. First of all observe that no filtration on the space $(\Omega, \mathcal{F}, \mathcal{P})$ is considered and no adaptability is required on the process x . Second, the equation satisfied by $\theta_t(\omega)$ is analytically meaningful, once $\Gamma^w b(\omega)$ has the prescribed regularity. In this sense, it is a *random solution* to a *random YDE* rather than a solution to an SDE; in other terms, differently from classical SDEs driven by Brownian motion, all integrals appearing are pathwise defined, which is why we chose the terminology of *pathwise solution*.

Our definition is in some sense closer in spirit to the concept of superposition solution considered in [12] (which is itself inspired by the one from [1]) than to classical concepts of solutions for SDEs. Another way to see it is to define, for γ, η, λ satisfying (A) and for any $A \in C_t^\gamma C_x^{\eta,\lambda}$, $\theta_0 \in \mathbb{R}^d$ the set

$$(4.7) \quad C(\theta_0, A) := \left\{ \theta \in C_t^\gamma : \theta_t = \theta_0 + \int_0^t A(ds, \theta_s) \forall t \in [0, T] \right\}.$$

Then conditions (i) and (iii) from Definition 39 may be written as

$$\mathbb{P}(\omega \in \Omega : \Gamma^w b(\omega) \in C_t^\gamma C_x^{\eta,\lambda}, \theta(\omega) \in C(\theta_0, \Gamma^w b(\omega))) = 1$$

which can be interpreted as the fact that θ , as a random variable on C_t^γ , is concentrated on the random set $\omega \mapsto C(\theta_0, \Gamma^w b(\omega))$; we will soon rigorously show that this defines a random set, but let us proceed in the discussion for the moment. As a consequence, if $C(\theta_0, \Gamma^w b(\omega))$ is a singleton for \mathbb{P} -a.e. ω , then θ is uniquely determined. This motivates the following definition.

DEFINITION 40. Let β, w, b and the parameters γ, η, λ be as in Definition 39. We say that *path-by-path well-posedness* holds for the SDE if

$$(4.8) \quad \mathbb{P}(\omega \in \Omega : \Gamma^w b(\omega) \in C_t^\gamma C_x^{\eta, \lambda}, \text{Card}(C(\theta_0, \Gamma^w b(\omega))) = 1 \text{ for all } \theta_0 \in \mathbb{R}^d) = 1.$$

We adopt this terminology, instead of the more classical *path-by-path uniqueness*, to stress the fact that the “good set” of full probability on which uniqueness holds is the same for all $\theta_0 \in \mathbb{R}^d$, differently from the original result by Davie from [11].

REMARK 41. By the construction from Theorem 22, the random field $\Gamma^w b$ is adapted to the filtration generated by $\beta, \Gamma^w b = \Gamma^w b(\beta)$; therefore, (4.8) is exclusively a requirement on the law of β and does not depend on the specific probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in consideration.

As a consequence of the theory outlined in Section 2.2, in particular Theorem 18, we immediately deduce the following.

LEMMA 42. Let β, w, b and the parameters γ, η, λ be as in Definition 39 and suppose that

$$\mathbb{P}(\omega \in \Omega : \Gamma^w b(\omega) \in C_t^\gamma C_x^{1+\eta, \lambda}) = 1.$$

Then *path-by-path well-posedness* holds for the SDE.

The rest of the section is dedicated to the proof that $\omega \mapsto C(\theta_0, \Gamma^w b(\omega))$ is a random set, as well as some of its properties. Thus, we believe that it contains results of independent interest regarding nonlinear YDEs.

Before proceeding further, we need to recall a few things on random sets; for a more detailed exposition we refer to [7]. Given a complete vector space (E, d) , the distance between $a \in E$ and a compact $K \subset E$ is given by

$$d(a, K) = \inf_{b \in K} d(a, b) = \min_{b \in K} d(a, b),$$

where the infimum is realised since K is compact. Given K_1, K_2 compact subsets of E , their Hausdorff distance d_H is defined as

$$d_H(K_1, K_2) = \max \left\{ \sup_{a \in K_1} d(a, K_2), \sup_{b \in K_2} d(b, K_1) \right\}.$$

Setting $\mathcal{K}(E) = \{K \subset E : K \text{ compact}\}$, $(\mathcal{K}(E), d_H)$ is a complete metric space and, moreover, we have the identity

$$d_H(K_1, K_2) = \sup_{a \in E} |d(a, K_1) - d(a, K_2)| = \max_{a \in K_1 \cup K_2} |d(a, K_1) - d(a, K_2)|.$$

Consider $(\mathcal{K}(E), d_H)$ endowed with its Borel σ -algebra, and let (F, \mathcal{A}) be another measurable space; then it can be shown that a map $X : (F, \mathcal{A}) \rightarrow (\mathcal{K}(E), d_H)$ is measurable if and only if the map $d(a, X(\cdot))$ is measurable from (F, \mathcal{A}) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, for all $a \in E$. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a *random compact set* is a measurable map $X : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathcal{K}(E), d_H)$.

PROPOSITION 43. Let γ, η, λ be parameters satisfying (A). Then for any $\theta_0 \in \mathbb{R}^d$ and $A \in C_t^\gamma C_x^{\eta, \lambda}$, the set $C(\theta_0, A)$ is a nonempty, compact subset of C_t^γ . Moreover, the map

$$(\theta_0, A) \mapsto C(\theta_0, A)$$

is measurable from $\mathbb{R}^d \times C_t^\gamma C_x^{\eta, \lambda}$ to $\mathcal{K}(C_t^\gamma)$.

PROOF. The fact that $C(\theta_0, A)$ is nonempty follows from Theorem 3.1 from [24]. By the a priori estimate (2.4), $C(\theta_0, A)$ is bounded in C_t^γ ; therefore, given a sequence $\{\theta^n\} \subset C(\theta_0, A)$, by Ascoli–Arzelà we can extract a subsequence (not relabelled for simplicity) such that $\theta^n \rightarrow \theta$ in $C_t^{\gamma-\varepsilon}$ for any $\varepsilon > 0$. Choosing ε sufficiently small such that $\gamma + \eta(\gamma - \varepsilon) > 1$, it follows from the continuity of Young integrals that

$$\theta^n = \theta_0 + \int_0^\cdot A(ds, \theta_s^n) \rightarrow \theta_0 + \int_0^\cdot A(ds, \theta_s) = \theta \quad \text{in } C_t^\gamma.$$

Namely, θ^n converge in C_t^γ to an element of $C(\theta_0, A)$, which shows compactness.

In order to prove the second claim, it is enough to show that for any $y \in C_t^\gamma$, the map

$$\mathbb{R}^d \times C_t^\gamma C_x^{\eta,\lambda} \ni (\theta_0, A) \mapsto d(y, C(\theta_0, A)) \in \mathbb{R}$$

is measurable. We will actually show that it is lower semicontinuous, implying that it has closed sublevel sets and thus its measurability. Fix $y \in C_t^\gamma$ and let $(\theta_0^n, A^n) \rightarrow (\theta_0, A)$; by compactness of $C(\theta_0^n, A^n)$, for each n there exists $\theta^n \in C(\theta_0^n, A^n)$ such that $d(y, \theta^n) = d(y, C(\theta_0^n, A^n))$. Up to extracting a subsequence which realizes the liminf, we can assume without loss of generality that $\lim d(y, C(\theta_0^n, A^n))$ exists; as the sequence (θ_0^n, A^n) is convergent, it must also be bounded, which implies by (2.4) that $\{\theta^n\}_n$ is bounded in C_t^γ . Invoking Ascoli–Arzelà and reasoning as in the previous point, using the continuity of nonlinear Young integrals, we can find a (not relabelled) subsequence such that $\theta^n \rightarrow \theta \in C(\theta_0, A)$ in C_t^γ . As a consequence

$$d(y, C(\theta_0, A)) \leq d(y, \theta) = \lim_{n \rightarrow \infty} d(y, \theta_n) = \liminf_{n \rightarrow \infty} d(y, C(\theta_0^n, A^n))$$

which implies lower semicontinuity, and thus concludes the proof. \square

The fact that $C(\theta_0, \Gamma^w b(\omega))$ is a random set follows from the following more general result.

COROLLARY 44. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a random field $A = A(\omega) \in C_t^\gamma C_x^{\eta,\lambda}$ and a random vector $\xi = \xi(\omega) \in \mathbb{R}^d$ is defined. Then the map*

$$\omega \mapsto C(\xi(\omega), A(\omega))$$

defines a random compact subset of C_t^γ .

PROOF. It is an immediate consequence of Proposition 43 and the fact that composition of measurable functions is measurable. \square

4.4. *Proofs of the main results.* The goal is to find specific conditions on the parameters H, δ and the regularity of b in order to obtain existence and uniqueness of (4.2). To this end, we will distinguish our analysis into to different cases: when $b \in C_x^\alpha$ with $\alpha \in (0, 1)$, we will find conditions for δ, H and α through application of Proposition 36 to ensure well-posedness for (4.2). For the general case of $b \in \mathcal{D}(\mathbb{R}^d)$, we will consult Proposition 38 to find conditions for δ, H and α such that existence and uniqueness holds.

We are now ready to give the proofs of our main results.

PROOF OF THEOREM 1. It follows from Corollary 31 and Remark 23 that, under the regularity assumption $T^w b \in C_t^{\gamma'} C_x^2$, the multiplicative averaged field $\Gamma^w b$ is a well-defined random field and we can find γ', η, λ satisfying (A) such that

$$\Gamma^w b \in C_t^{\gamma'} C_x^{1+\eta,\lambda} \quad \mathbb{P}\text{-a.s.}$$

Therefore, path-by-path well-posedness follows from Lemma 42. Given two pathwise solutions $x^i = \theta^i + w$ starting at x_0 , setting $\theta_0 = x_0 - w_0$, it holds

$$\begin{aligned} \mathbb{P}(x^1 = x^2 \text{ in } C_t^0) &= \mathbb{P}(\theta^1 = \theta^2 \text{ in } C_t^0) \\ &\geq \mathbb{P}(\theta^i \in C_t^{\gamma'}, \Gamma^w b \in C_t^{\gamma'} C_x^{\eta, \lambda}, \theta^i \in C(\theta_0, \Gamma^w b)) \\ &\geq \mathbb{P}(\Gamma^w b \in C_t^{\gamma'} C_x^{\eta, \lambda}, C(\theta_0, \Gamma^w b) \text{ is a singleton}) = 1 \end{aligned}$$

which shows indistinguishability. Adaptedness follows from the formula $\theta(\omega) = \mathcal{I}(\Gamma^w b(\omega))(\cdot, \theta_0)$ and the fact that by construction the field $\Gamma^w b$ is adapted to β , in the sense that $\{\Gamma_s^w b, s \in [0, t]\} \subset \sigma\{\beta_s : s \in [0, t]\}$. Finally, formula (1.7) follows from the one for θ and the change of variables $x = \theta + w$. \square

PROOF OF PROPOSITION 2. Part (i) is just a consequence of Proposition 35; in particular, it is enough to require $b \in C_b^2, w \in C_t^\delta$ with $\delta + H > 1$.

Under condition (1.6), by Remark 34 we can find a sequence (b^n, w^n) (for instance, in $C_b^2 \times C_t^\delta$) such that $b^n \rightarrow b$ in the sense of distributions, $w^n \rightarrow w$ uniformly and $\Gamma^{w^n} b^n(\omega) \rightarrow \Gamma^w b(\omega)$ in $C_t^{\gamma'} C_x^{1+\eta, \lambda}$ for \mathbb{P} -a.e. ω ; moreover, we can choose the parameters γ', η, λ satisfying condition (A). Therefore, point (ii) follows from an application of Lemma 37.

Suppose now (b^n, w^n) is a sequence in $C_b^2 \times C_t^\delta$ satisfying the assumptions of point (iii); by properties of classical averaged fields, $T^{w^n} b^n \rightarrow T^w b$ in the sense of distributions, which implies that $T^w b \in C_t^\gamma C_x^2$ and $T^{w^n} b^n \rightarrow T^w b$ in $C_t^\gamma C_x^2$. But then by Theorem 22 and Remark 23, we can find γ', η, λ satisfying (A) such that $\Gamma^{w^n} b^n \rightarrow \Gamma^w b$ in $L^p(\Omega; C_t^{\gamma'} C_x^{\eta, \lambda})$. The conclusion then follows again from an application of Lemma 37. \square

In order to specialize the above criterion to cases of practical interest, we need the following lemma.

LEMMA 45. *Let $b \in C_x^\alpha$ for some $\alpha \in \mathbb{R}$, w a continuous path s.t. $T^w b \in C_t^{1/2} C_x^{\alpha+\nu}$. Then*

$$T^w b \in C_t^\gamma C_x^{\alpha+2\nu(1-\gamma)} \quad \forall \gamma \in [1/2, 1].$$

PROOF. Since $b \in C_x^\alpha, T^w b \in C_t^1 C_x^\alpha$; the claim then follows from interpolation estimates. Indeed, by Besov interpolation inequality (see [3], Theorem 2.80), for any $\theta \in [0, 1]$ it holds

$$\|T_{s,t}^w b\|_{\alpha+(1-\theta)\nu} \lesssim \|T_{s,t}^w b\|_\alpha^\theta \|T_{s,t}^w b\|_{\alpha+\nu}^{1-\theta} \lesssim |t-s|^{\theta+(1-\theta)/2} \|T_{s,t}^w b\|_{1,\alpha}^\theta \|T_{s,t}^w b\|_{1/2,\alpha+\nu}^{1-\theta}$$

and the conclusion follows by taking $\gamma = (1 + \theta)/2$. \square

PROOF OF THEOREM 3. To show the first statement, we need to verify that under condition (1.8), $T^w b \in C_t^\gamma C_x^2$ for some $\gamma > 3/2 - H$; by the assumption $T^w b \in C_t^{1/2} C_x^{\alpha+\nu}$ and Lemma 45, it is enough to verify that

$$(4.9) \quad \begin{cases} \gamma > 3/2 - H, \\ \alpha + 2\nu(1 - \gamma) > 2. \end{cases}$$

It is easy to check that one can find $\gamma \in (0, 1)$ satisfying (4.9) if and only if (1.8) holds. Similar computations show that, under (1.9), $T^w b \in C_t^{\gamma'} C_x^{n+1}$, which implies that we can find γ', η, λ satisfying (A) such that $\Gamma^w b \in C_t^{\gamma'} C_x^{n+\eta, \lambda}$; the regularity of the flow then follows from the last part of Theorem 18. \square

PROOF OF THEOREM 4. The proof follows the same lines as that of Theorem 3, only this time we want to check that the conditions of Proposition 36 are met. By the assumptions and Lemma 45, $T^w b \in C_t^\gamma C_x^{\alpha+2\nu(1-\gamma)}$ for any $\gamma > 1/2$; taking $\gamma > 3/2 - H$ and applying Corollary 31, we deduce that \mathbb{P} -a.s. $\Gamma^w b \in C_t^{\gamma'} C_x^{1+\eta,\lambda}$ for any $\gamma' < \gamma + H - 1$, $1 + \eta < \alpha + 2\nu(1 - \gamma)$ and λ sufficiently small. In order to find γ' , η such that $\gamma' + H\eta > 1$ it is, therefore, enough to verify that there exists $\gamma > 1/2$ such that

$$(4.10) \quad \begin{cases} \gamma > 3/2 - H, \\ \frac{1}{2} + H(\alpha + 2\nu(1 - \gamma) - 1) > 1 \end{cases}$$

or equivalently

$$(4.11) \quad \begin{cases} \gamma > 3/2 - H, \\ \alpha + 2\nu(1 - \gamma) > 1 + \frac{1}{2H}. \end{cases}$$

Taking γ of the form $\gamma = 3/2 - H + \varepsilon$ with $\varepsilon > 0$ sufficiently small, it is easy to check that the conditions (4.10) or (4.11) are satisfied under assumption (1.10). \square

5. Further extensions.

5.1. *Time inhomogeneous diffusion coefficient.* So far we assumed the diffusion coefficient b to be homogeneous, in the sense that $b(t, x) = b(x)$. However, our method can be easily extended to the general case of time inhomogeneous b . We will outline here the necessary conditions in order to obtain well-posedness of equations with time homogeneous coefficients of the form

$$dx_t = b(t, x_t) d\beta_t + dw_t.$$

The first step in this direction is to define the multiplicative averaged field $\Gamma^w b$. To this end, it is readily seen that if $(t, x) \mapsto b(t, x)$ is smooth in both variables and $w \in C_t^\delta$ with not too small δ , the analytical definition of $\Gamma^w b$ from Lemma 24 still holds. In fact, if $b \in C_t^\rho C_x^{\alpha+\eta}$ with $\rho > 1 - H$, $\alpha \in \mathbb{R}$ and $\eta \in (0, 1]$, under the assumption $H + \eta\delta > 1$, there exists a unique distribution $\Gamma^w b \in C_t^H C_x^\alpha$ such that

$$(5.1) \quad \|\Gamma_{s,t}^w b - b(s, \cdot + w_s)\beta_{s,t}\|_{C_x^\alpha} \lesssim |t - s|^{H+\eta\delta}.$$

Indeed, setting $\Xi_{s,t} = \tau^{w_s} b(s, \cdot)\beta_{s,t}$, we observe that

$$\begin{aligned} \|\delta \Xi_{s,u,t}\|_{C_x^\alpha} &\lesssim [\|b(s, \cdot + w_u) - b(u, \cdot + w_u)\|_{C_x^\alpha} \\ &\quad + \|b(s, \cdot + w_u) - b(s, \cdot + w_s)\|_{C_x^\alpha}] |\beta_{u,t}|. \end{aligned}$$

Invoking the assumptions of Hölder regularity in $t \mapsto b(t, \cdot)$, w and β , we obtain

$$\|\delta \Xi_{s,u,t}\|_{C_x^\alpha} \lesssim \|b\|_{C_t^\rho C_x^{\alpha+\eta}} \| \beta \|_{C_t^H} (1 + \|w\|_{C_t^\delta}) |t - s|^{H+\eta\delta\wedge\rho},$$

where we have employed estimates similar to those of Lemma 24. An application of the sewing lemma then implies (5.1). Thus, from an analytical perspective it is readily seen that the multiplicative averaged field is well defined. In order to obtain the regularizing effect from w , we then need to use the stochastic construction of $\Gamma^w b$ by application of Proposition 27. Lemma 29 is thus readily extended to the time inhomogeneous case, under the assumption that the classical averaged field $T^w b \in C_t^\gamma C_x^{1+\eta}$. For example, in [16] it is shown that $T^w b \in C_t^\gamma C_x^{1+\eta}$ for $b \in L^q([0, T]; C_x^\alpha)$ with $q > 2$ and $\alpha \in \mathbb{R}$ under suitable conditions on w . For a

more detailed analytical construction of the classical averaged field with time inhomogeneous b ; see [16]. In a similar spirit, one can then readily apply the modified GRR lemma 30 in order to obtain almost sure space-time Hölder regularity of $\Gamma^w b$.

With the time inhomogeneous multiplicative averaged field at hand, one can then go through the same abstract procedure for existence and uniqueness of nonlinear young equations as shown in Section 2.2 by setting $A_{s,t}(x) = \Gamma_{s,t}^w b_s(x)$ in Theorem 13 and Theorem 18. These theorems can then be used to extend the results in Section 4 to allow for time inhomogeneous diffusion coefficients b with possibly distributional spatial dependence.

5.2. *Including a non-Lipschitz drift term.* So far, we have only considered (1.1) in the case when $b_1 \equiv 0$ and $b_2 = b$. However, our results immediately extend to equations with both nontrivial drift and diffusion, of the form

$$x_t = x_0 + \int_0^t b_1(x_s) ds + \int_0^t b_2(x_s) d\beta_s + w_t, \quad x_0 \in \mathbb{R}^d.$$

Again, by the change of variables $\theta = x - w$, we see that θ formally solves the equation

$$\theta_t = x_0 + \int_0^t b_1(\theta_s + w_s) ds + \int_0^t b_2(\theta_s + w_s) d\beta_s.$$

Setting

$$A_{s,t}(x) := T_{s,t}^w b_1(x) + \Gamma_{s,t}^w b_2(x),$$

we can interpret the equation in the Young integral sense as

$$\theta_t = x_0 + \int_0^t A(ds, \theta_s).$$

Under the condition that A is sufficiently regular, existence and uniqueness for the YDE holds by Theorem 18. It is, therefore, enough to require $T^w b_1$ and $\Gamma^w b_2$ to belong to $C_t^\gamma C_x^{1+\beta,\lambda}$ for suitable γ, β, λ ; then the results in Section 4 can be extended directly. In this case one can also consider time inhomogeneous drift and diffusion b_1 and b_2 by following the steps outlined in Section 5.1.

5.3. *Random initial condition.* So far we have only considered deterministic initial data $x_0 \in \mathbb{R}^d$ (resp. $\theta_0 = x_0 - w_0 \in \mathbb{R}^d$). However, especially in view of applications to optimal transport and fluid dynamics equations, it is often interesting to allow random initial data for the SDE. This extension can be easily implemented in the framework of Section 4.3, as we are now going to show.

DEFINITION 46. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which an fBm $\{\beta_t\}_{t \in [0, T]}$ of Hurst parameter $H > 1/2$, as well as an independent \mathbb{R}^d -valued random variable ξ , are defined; consider also a continuous deterministic path w and a distributional field b . We say that a process x is a *pathwise solution* to the SDE

$$dx_t = b(x_t) d\beta_t + dw_t, \quad x_0 = \xi$$

if there exist parameters γ, η, λ satisfying (A) such that $\Gamma^w b$ is well defined in the sense of Theorem 22 and, setting $\theta = x - w, \zeta = \xi - w_0$, it holds

$$\mathbb{P}(\omega \in \Omega : \Gamma^w b(\omega) \in C_t^\gamma C_x^{\eta,\lambda}, \theta(\omega) = C_t^\gamma, \theta(\omega) \in C(\zeta(\omega), \Gamma^w b(\omega))) = 1.$$

As a consequence of the theory from Section 2.2, in particular Theorem 18 and Corollary 20, we deduce the following result.

COROLLARY 47. *Let β, b, w, ξ, ζ be as in Definition 46 and such that the assumptions of Lemma 42 are satisfied. Then any pathwise solution x to the SDE with initial condition $\xi, x = \theta + w$, satisfies*

$$\mathbb{P}(w \in \Omega : \theta(\omega)_t = \mathcal{I}(\Gamma^w b(\omega))(t, \zeta(\omega)) \text{ for all } t \in [0, T]) = 1,$$

where \mathcal{I} is the map defined in Corollary 20, that is, $\mathcal{I}(\Gamma^w b(\omega))$ is the flow associated to $\Gamma^w b(\omega)$. In particular, all the conclusions follow if the assumptions of Theorem 1 are satisfied.

6. Concluding remarks. We have shown that through a suitable perturbation of a continuous but irregular path w , the SDE

$$(6.1) \quad dx_t = b(x_t) d\beta_t + dw_t, \quad x_0 \in \mathbb{R}^d$$

is well posed and admits a unique solution even for distributional coefficients b in terms of Definitions 39 and 40, in the case $\{\beta_t\}_{t \in [0, T]}$ is a fBm with $H \in (\frac{1}{2}, 1)$. This can be seen as a first step in a more general program of proving regularization of multiplicative SDEs through perturbation by irregular/rough paths. The first question one could ask is whether it is possible to require less restrictive conditions on b given a certain regularizing path w . For example, in [8, 16] (and partially related [6]), sharper results are obtained for SDEs with additive drift (nonmultiplicative case) by exploiting the Girsanov transform. If w is sampled as an fBm of parameter δ , another possible way to solve the SDE in (6.1) (say w.l.o.g. for $x_0 = 0$) would be to check that the process

$$\tilde{w}_t = w_t - \int_0^t b(w_s) d\beta_s$$

is again an fBm of parameter δ under a new probability law \mathbb{Q} ; if that is the case, then w itself is a solution to the equation w.r.t. \tilde{w} . However, the estimates from Proposition 27 are not enough to establish exponential integrability and thus to check if Novikov holds. Another possibility to obtain sharper results could be to apply the recently developed stochastic sewing lemma [26], in combination with a more direct application of the results obtained by Hairer and Li in [20]. Probably in that case, existence and uniqueness in the class of adapted processes is more straightforward. Our results, on the other hand, have the advantages that: (i) uniqueness also holds without adaptability requirements (although a posteriori the unique solution will be adapted); (ii) existence and uniqueness of solutions immediately comes with a regular flow (which is quite difficult to establish by means of stochastic techniques); (iii) the resulting equation has a pathwise analytical meaning, its randomness being in the random field $\Gamma^w b$ but not the YDE itself.

A possibly more challenging extension of our results, is to consider the case of multiplicative fBm with $0 < H \leq \frac{1}{2}$. As seen through our analysis, such an extension would be highly dependent on showing the relation between the multiplicative averaged field $\Gamma^w b$ with the classical averaged field $T^w b$ when Γ^w is driven by a fBm with $H \leq \frac{1}{2}$. In this case, Proposition 27 breaks down, and thus a similar statement in the rough case would be needed. Furthermore, if one can prove that $\Gamma^w b \in C_t^\gamma C_{x, \text{loc}}^\eta$ for general distributions b , one can not hope for a $\gamma > \frac{1}{2}$, which is required to apply the nonlinear Young formalism employed in this article. To this end, one could hope to use techniques developed on nonlinear rough paths (see, e.g., [9, 31]), but the exact formulation of the equation in this context is not completely clear.

Observe that for smooth functions b , under the assumption $H + \delta > 1$ (recall that $\delta \in (0, 1)$) is the Hölder regularity of w), it holds

$$(6.2) \quad \Gamma^w b = \Gamma^w (b * \delta_0) = b * \Gamma^w \delta_0 = b * \bar{v}^w,$$

where $\tilde{\nu}^w$ is the reflection of ν^w formally given by

$$\nu_{s,t}^w = \int_s^t \delta_{w_r} d\beta_r,$$

and for $y \in \mathbb{R}^d$, δ_y denotes the Dirac delta centered at y . It is tempting to think of ν^w as being a form of “weighted occupation measure”. However, in general ν^w will NOT be a measure. Anyway, applying the approximation procedure from Section 3, identity (6.2) is preserved also in the case $H + \delta \leq 1$, once interpreted as random variables: for fixed b ,

$$\Gamma^w b(\omega) = b * \tilde{\nu}^w(\omega) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Now on the r.h.s. the random variable appearing does not depend on b anymore, so it can be regarded as a regular version of the family of random variables $\{\Gamma^w b\}_{b \in E}$: once we fix the set $\Omega' \subset \Omega$ on which ν^w is defined and regular, so are $\Gamma^w b$. In this sense, in many considerations we could also make the full probability set independent of b , deriving the regularity of $\Gamma^w b$ from that of ν^w and Young’s convolution inequality, which can then be seen analogously to constructing the classical averaged field as a convolution between a function b and the reflected local time associated to w .

One could also readapt the concept of ρ -irregularity (see, e.g., [17]) in this setting. Indeed at least formally, convolution with ν^w coincides at the Fourier level to a Fourier multiplier of the form

$$\hat{\nu}^w(\xi) = \int_s^t e^{i\xi \cdot w_r} d\beta_r,$$

where for any fixed ξ , $\hat{\nu}^w(\xi)$ is a well-defined random variable (random path actually, once we apply Kolmogorov) by the lemma from [20]. Combining this with the classical ρ -irregularity property, one should obtain that if w is (γ, ρ) -irregular, then for any $\gamma' < \gamma + H - 1$, $\rho' < \rho$ it holds

$$\mathbb{E}[\|\hat{\nu}^w(\xi)\|_{\gamma'}^{\rho'}]^{1/\rho'} \lesssim |\xi|^{-\rho'}.$$

One could then ask the more difficult question of whether it is possible to establish that

$$\mathbb{P}\left(\sup_{\xi \in \mathbb{R}^d} |\xi|^{\rho'} \|\hat{\nu}^w(\xi)\|_{\gamma'} < \infty\right) = 1$$

which would be a true analogue of the ρ -irregularity property.

Acknowledgments. The authors would like to thank the anonymous referees for their constructive comments that improved the quality of this paper.

Funding. L. Galeati is funded by the DFG under Germany’s Excellence Strategy—GZ 2047/1, project-id 390685813.

F. Harang is gratefully acknowledging the financial support from the Research Council of Norway (RCN). Project STORM, project number: 274410.

REFERENCES

[1] AMBROSIO, L. (2004). Transport equation and Cauchy problem for BV vector fields. *Invent. Math.* **158** 227–260. MR2096794 <https://doi.org/10.1007/s00222-004-0367-2>

[2] ATHREYA, S., BHAR, S. and SHEKHAR, A. (2017). Smoothness of flow and path-by-path uniqueness in stochastic differential equations. arXiv preprint. Available at arXiv:1709.02115.

[3] BAHOURI, H., CHEMIN, J.-Y. and DANCHIN, R. (2011). *Fourier Analysis and Nonlinear Partial Differential Equations. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **343**. Springer, Heidelberg. MR2768550 <https://doi.org/10.1007/978-3-642-16830-7>

- [4] BECK, L., FLANDOLI, F., GUBINELLI, M. and MAURELLI, M. (2019). Stochastic ODEs and stochastic linear PDEs with critical drift: Regularity, duality and uniqueness. *Electron. J. Probab.* **24** Paper No. 136, 72. MR4040996 <https://doi.org/10.1214/19-ejp379>
- [5] BIAGINI, F., HU, Y., ØKSENDAL, B. and ZHANG, T. (2008). *Stochastic Calculus for Fractional Brownian Motion and Applications. Probability and Its Applications (New York)*. Springer London, Ltd., London. MR2387368 <https://doi.org/10.1007/978-1-84628-797-8>
- [6] BUTKOVSKY, O., DAREIOTIS, K. and GERENCSÉR, M. (2021). Approximation of SDEs: A stochastic sewing approach. *Probab. Theory Related Fields* **181** 975–1034. MR4344136 <https://doi.org/10.1007/s00440-021-01080-2>
- [7] CASTAING, C. and VALADIER, M. (1977). *Convex Analysis and Measurable Multifunctions. Lecture Notes in Mathematics* **580**. Springer, Berlin. MR0467310
- [8] CATELLIER, R. and GUBINELLI, M. (2016). Averaging along irregular curves and regularisation of ODEs. *Stochastic Process. Appl.* **126** 2323–2366. MR3505229 <https://doi.org/10.1016/j.spa.2016.02.002>
- [9] COGHI, M. and NILSSEN, T. (2021). Rough nonlocal diffusions. *Stochastic Process. Appl.* **141** 1–56. MR4293767 <https://doi.org/10.1016/j.spa.2021.07.002>
- [10] CONG, N. D., DUC, L. H. and HONG, P. T. (2018). Nonautonomous Young differential equations revisited. *J. Dynam. Differential Equations* **30** 1921–1943. MR3871629 <https://doi.org/10.1007/s10884-017-9634-y>
- [11] DAVIE, A. M. (2007). Uniqueness of solutions of stochastic differential equations. *Int. Math. Res. Not. IMRN* **24** Art. ID rnm124, 26. MR2377011 <https://doi.org/10.1093/imrn/rnm124>
- [12] FLANDOLI, F. (2009). Remarks on uniqueness and strong solutions to deterministic and stochastic differential equations. *Metrika* **69** 101–123. MR2481917 <https://doi.org/10.1007/s00184-008-0210-7>
- [13] FLANDOLI, F. (2011). *Random Perturbation of PDEs and Fluid Dynamic Models: Lectures from the 40th Probability Summer School Held in Saint-Flour, 2010, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School]. Lecture Notes in Math.* **2015**. Springer, Heidelberg. MR2796837 <https://doi.org/10.1007/978-3-642-18231-0>
- [14] FLANDOLI, F., GUBINELLI, M. and PRIOLA, E. (2010). Well-posedness of the transport equation by stochastic perturbation. *Invent. Math.* **180** 1–53. MR2593276 <https://doi.org/10.1007/s00222-009-0224-4>
- [15] FRIZ, P. K. and HAIRER, M. (2014). *A Course on Rough Paths: With an Introduction to Regularity Structures. Universitext*. Springer, Cham. MR3289027 <https://doi.org/10.1007/978-3-319-08332-2>
- [16] GALEATI, L. and GUBINELLI, M. (2021). Noiseless regularisation by noise. *Rev. Mat. Iberoam.* <https://doi.org/10.4171/RMI/1280>
- [17] GALEATI, L. and GUBINELLI, M. (2020). Prevalence of ρ -irregularity and related properties. arXiv preprint. Available at arXiv:2004.00872.
- [18] GARSIA, A. M., RODEMICH, E. and RUMSEY, H. JR. (1970/71). A real variable lemma and the continuity of paths of some Gaussian processes. *Indiana Univ. Math. J.* **20** 565–578. MR0267632 <https://doi.org/10.1512/iumj.1970.20.20046>
- [19] GEMAN, D. and HOROWITZ, J. (1973). Occupation times for smooth stationary processes. *Ann. Probab.* **1** 131–137. MR0350833 <https://doi.org/10.1214/aop/1176997029>
- [20] HAIRER, M. and LI, X.-M. (2020). Averaging dynamics driven by fractional Brownian motion. *Ann. Probab.* **48** 1826–1860. MR4124526 <https://doi.org/10.1214/19-AOP1408>
- [21] HARANG, F. A. and LING, C. (2021). Regularity of local times associated to Volterra–Lévy processes and path-wise regularization of stochastic differential equations. *J. Theor. Probab.* <https://doi.org/10.1007/s10959-021-01114-4>
- [22] HARANG, F. A. and PERKOWSKI, N. (2021). C^∞ -regularization of ODEs perturbed by noise. *Stoch. Dyn.* **21** Paper No. 2140010, 29. MR4342752 <https://doi.org/10.1142/S0219493721400104>
- [23] HINZ, M., TÖLLE, J. M. and VIITASAARI, L. (2020). Variability of paths and differential equations with BV coefficients. arXiv preprint. Available at arXiv:2003.11698.
- [24] HU, Y. and LÊ, K. (2017). Nonlinear Young integrals and differential systems in Hölder media. *Trans. Amer. Math. Soc.* **369** 1935–2002. MR3581224 <https://doi.org/10.1090/tran/6774>
- [25] KRYLOV, N. V. and RÖCKNER, M. (2005). Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields* **131** 154–196. MR2117951 <https://doi.org/10.1007/s00440-004-0361-z>
- [26] LÊ, K. (2020). A stochastic sewing lemma and applications. *Electron. J. Probab.* **25** Paper No. 38, 55. MR4089788 <https://doi.org/10.1214/20-ejp442>
- [27] LEJAY, A. (2010). Controlled differential equations as Young integrals: A simple approach. *J. Differential Equations* **249** 1777–1798. MR2679003 <https://doi.org/10.1016/j.jde.2010.05.006>

- [28] MOHAMMED, S.-E. A., NILSSEN, T. K. and PROSKE, F. N. (2015). Sobolev differentiable stochastic flows for SDEs with singular coefficients: Applications to the transport equation. *Ann. Probab.* **43** 1535–1576. MR3342670 <https://doi.org/10.1214/14-AOP909>
- [29] NUALART, D. and OUKNINE, Y. (2002). Regularization of differential equations by fractional noise. *Stochastic Process. Appl.* **102** 103–116. MR1934157 [https://doi.org/10.1016/S0304-4149\(02\)00155-2](https://doi.org/10.1016/S0304-4149(02)00155-2)
- [30] NUALART, D. and RĂȘCANU, A. (2002). Differential equations driven by fractional Brownian motion. *Collect. Math.* **53** 55–81. MR1893308
- [31] NUALART, D. and XIA, P. (2020). On nonlinear rough paths. *ALEA Lat. Am. J. Probab. Math. Stat.* **17** 545–587. MR4105930 <https://doi.org/10.30757/alea.v17-22>
- [32] SHAPOSHNIKOV, A. and WRESCH, L. (2020). Pathwise vs. path-by-path uniqueness. arXiv preprint. Available at [arXiv:2001.02869](https://arxiv.org/abs/2001.02869).
- [33] TANIGUCHI, T. (1992). Successive approximations to solutions of stochastic differential equations. *J. Differential Equations* **96** 152–169. MR1153313 [https://doi.org/10.1016/0022-0396\(92\)90148-G](https://doi.org/10.1016/0022-0396(92)90148-G)
- [34] VERETENNIKOV, A. Y. (1981). On strong solution and explicit formulas for solutions of stochastic integral equations. *Math. USSR, Sb.* **39** 387–403.
- [35] XU, Y., PEI, B. and WU, J.-L. (2017). Stochastic averaging principle for differential equations with non-Lipschitz coefficients driven by fractional Brownian motion. *Stoch. Dyn.* **17** 1750013, 16. MR3626744 <https://doi.org/10.1142/S0219493717500137>
- [36] YAMADA, T. (1981). On the successive approximation of solutions of stochastic differential equations. *J. Math. Kyoto Univ.* **21** 501–515. MR0629781 <https://doi.org/10.1215/kjm/1250521975>
- [37] ZVONKIN, A. K. (1974). A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)* **93** 129–149, 152. MR0336813