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Decomposition of the Kähler differentials

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

This thesis is concerned with the module of Kähler differentials of an affine scheme and its primary decomposition. For a smooth scheme, these are the local building blocks of the cotangent bundle. Each inclusion of a closed subscheme induces a reversed map of Kähler differentials. We aim to study the kernel of this map, as well as its primary decomposition. First we review Kähler differentials, then we review primary decompositions. Next we relate the primary decomposition of the closed subscheme ideal to the kernel of the map of differentials. Lastly, we look at the geometric aspect of the theory of differentials. We will be working in the case of curves in the affine plane, but we expect that many of our findings remain true for more general schemes and their closed subschemes.

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Contents

Introduction

Kähler differentials were introduced by the German mathematician Erich Kähler in the 1930s. They provide an algebraic analog to differential forms in Differential Geometry. These essentially allow us to do calculus on schemes, the object of interest in modern algebraic geometry.

Let A be a ring, B an A-algebra and M a B-module. An A-*derivation* is an A-linear map

$$B \to M$$

that satisfies the *Leibniz rule*. The set of such A-derivations is a B-module and gives rise to a covariant functor of B-modules. This functor is representable in the sense that

$$\mathrm{Der}_A(B, \_) \simeq \mathrm{Hom}_B(\Omega_B, \_),$$

where Ω_B is exactly the Kähler differentials. When globalizing the Kähler differentials, one gets a sheaf, called the *cotangent sheaf*, which is dual to the *tangent sheaf*. As the name indicates, the tangent sheaf contains information about all tangent spaces on a given scheme. For example, a scheme is smooth if and only if the tangent sheaf is locally free, or equivalently, a vector bundle.

The cotangent sheaf also gives rise to the *canonical sheaf*, an object we can attach to any scheme and is important when studying its geometry. For example, when a scheme X is non-singular, then the canonical sheaf is the *dualizing sheaf*. The sheaf of differentials is also used when computing algebraic De Rham cohomology. Grothendieck [Gro66] showed that this can be used to compute singular cohomology of smooth schemes over the complex numbers. Later, it was shown by Liebermann and Herrera [HL71] that one can also do this for singular schemes over $\mathbb{C}$. In this thesis, we are concerned with local properties of the cotangent sheaf, so we consider an affine scheme $X = \mathrm{Spec}(A)$ for some ring A. The problem we are concerned with, is the following.

Problem 1. *Given a closed subscheme* $\mathrm{Spec}(A/\mathfrak{a}) \to \mathrm{Spec}(A)$*, what is a primary decomposition of* $\Omega_{A/\mathfrak{a}}$*, and how does it relate to a primary decomposition of* $\mathfrak{a}$*?*

In some sense, a primary decomposition of ideals abstracts prime factorization of integers. On the other hand one can view it as an algebraic phenomenon, that, in fact, gives a decomposition of schemes into irreducible components. One could also look at this problem in a more global setting. In [DG67, Chapter 4, Section 3], Grothendieck introduces primary decompositions of $\mathcal{O}_X$-modules for a scheme X. Thus, one can decompose the cotangent bundle Ω_X as an $\mathcal{O}_X$-module. Such decompositions give us information about the geometric representations of the cotangent sheaf, for example the tangents in a point on the underlying scheme. The cotangent sheaf contains more information than just its geometric representations, and the primary decomposition also gives information about this algebraic structure. Primary decompositions of $\mathcal{O}_X$-modules requires that one can do this locally, which is why we focus on the affine situation in this thesis.

Outline

In Chapter 1 we introduce the module of Kähler differentials and some of its properties. Our interest is the kernel of the differential map induced by the quotient map $A \to A/\mathfrak{a}$, which we would like to decompose. Therefore, we establish some properties of the differentials on ideals which we can use in the decomposition. We also compute the kernel of such a map for a general ring and ideal, inspired by [Sta22].

In Chapter 2 we recall the fundamentals of the theory on primary decompositions of ideals. We then generalize much of this to modules and prove many of the results we will need when decomposing the differentials. To illustrate the situation, we give an example of the associated primes to the Kähler differentials of a quotient $A/\mathfrak{a}$. Furthermore, we examine the relation between the primary components of the kernel of

$$\Omega_A \to \Omega_{A/\mathfrak{a}},$$

and the primary components of $\mathfrak{a}$.

Chapter 3 is devoted to computing the decomposition of $\Omega_{A/\mathfrak{a}}$. First, we conjecture a statement about when the kernel of the induced quotient map is primary. Then, we conjecture that there are more components in the differentials than in the ideal, and finally, we find the missing component and conjecture that it is primary.

Finally, Chapter 4 is meant to give some motivation for computing this decomposition, and examines the geometry of the differentials. As the differentials are dual to tangent spaces, they are geometric by nature, although, as we have said, there are different uses for the differentials.

In the Appendix A, we compute several examples in [Macaulay2].

CHAPTER 1

The differentials of closed subschemes

We want to look at how differentials behave relative to ideals of a commutative ring, and we are especially interested in the components of these ideals. All rings in this thesis will be assumed to be commutative. We begin by giving an introduction to differentials and some examples of those. Much of these preliminaries are based on Hartshorne [Har77], Ellingsrud and Ottem [EO], and Eisenbud[Eis13].

1.1 Kähler differentials

The Kähler differentials are, as the name conveys, a module that contains information about the 'differentiated' elements of a ring. They are the analog of covectors in the study of differential geometry and are named after the German mathematician Erich Kähler. To define the differentials, we must first define what it means for a map to be a derivation.

Definition 1.1.1 (Derivation)**.** Let A be a ring, B an A-algebra and M a B-module. Let $d : B \to M$ be an A-linear map satisfying the *Leibniz rule*:

$$d(b_1 b_2) = b_1 \cdot d(b_2) + b_2 \cdot d(b_1)$$

We call such a map a *derivation*.

For our purposes, we will almost always assume that $A = k$ and B will usually be a polynomial ring or a quotient of one. Let us look at an easy example to demystify this definition.

Example 1.1.2. Let $A = k$ where k is an algebraically closed field, $B = M = k[x]$. Then the map that sends a polynomial $f(x) \in B$ to its formal derivative with respect to x is a derivation, more specifically a k-derivation. For instance,

$$f(x) = 3x^2 \mapsto 6x.$$

The set of derivations is denoted by $\mathrm{Der}_A(B, M)$. This is in fact a B-module: Let d_1 and d_2 be two different derivations from B into M. The sum of these is defined to be

$$(d_1 + d_2)(b) = d_1(b) + d_2(b),$$

where $b \in B$. This is a derivation from B to M. Linearity also follows from the linearity of each of d_1 and d_2.

Definition 1.1.3. Let $\Omega_{B|A}$ denote the B-module that represents the set of derivations, i.e. the module such that

$$\mathrm{Der}_A(B, M) \simeq \mathrm{Hom}_B(\Omega_{B|A}, M).$$

We call this module the *Kähler differentials*, or the *differentials* for short.

If we let $M = \Omega_{B|A}$ in the definition, then the derivation

$$d \in \mathrm{Der}_A(B, \Omega_{B|A})$$

that corresponds to the identity in the homomorphism-module

$$1 \in \mathrm{Hom}_B(\Omega_{B|A}, \Omega_{B|A})$$

is called the *universal derivation.* This is a derivation

$$d : B \to \Omega_{B|A}$$

such that for each derivation from B to M, there exists a unique B-module homomorphism

$$\alpha : \Omega_B \to M,$$

such that the derivation factors through this map.

Clearly Ω_B contains a lot of information about how the derivations for that ring will be. Naturally we want to know more about Ω_B, which is referred to as the Kähler differentials. We sometimes also refer to the Kähler differentials of a ring A as 'applying omega' to that ring.

The Kähler differentials for a ring B over A can be constructed by taking the direct sum

$$\bigoplus_{b \in B} Bdb.$$

Here d is the universal derivation, and we quotient out by the relations of the derivation map which are

$$d(bb') = bd(b') + b'd(b) \tag{1.1}$$
$$d(b + b') = (d(b) + d(b')) \tag{1.2}$$
$$d(a) = 0 \tag{1.3}$$

for $b, b' \in B$ and $a \in A$. Note that Equation (1.1) stems from the Leibniz rule, Equation (1.2) from linearity, and Equation (1.3) from the fact that d is A-linear and the Leibniz rule combined. To see this, let $a \in A$, then

$$d(a \cdot 1) = a \cdot d(1).$$

Furthermore, because of the Leibniz rule

$$d(a \cdot 1) = a \cdot d(1) + 1 \cdot d(a),$$

which means $a \cdot d(1) = 0$ for all $a \in A$. These relations are in perfect correspondence with the way we are used to differentiate polynomials with respect to some variable. The universal derivation is simply defined by

$$B \ \to \Omega_B$$

$$b \mapsto db.$$

We will in general assume that all rings we take the differentials of are over k unless we specify otherwise. In the same spirit we will by the notation Ω_A mean $\Omega_{A|k}$. Let us see an example of the Kähler differentials of a ring.

Example 1.1.4. Let $R = k[x, y]$. We claim that Ω_R will be generated by dx and dy as a $k[x, y]$-module, i.e. any element can be written as $f dx + g dy$ for $f, g \in k[x, y]$. Consider for example $f = 2x^3 + 4xy^2$ and let us see what happens when we apply d to this element. Since we have the relation $d(b+b') = d(b)+d(b')$ in this module, we get

$$df = d(2x^3) + d(4xy^2).$$

Furthermore, since d is k-linear we get

$$df = 2d(x^3) + 4d(xy^2),$$

and using the Leibniz rule we get

$$d(x^3) = x(dx^2) + x^2(dx) = (x^2dx + x^2dx) + x^2dx = 3x^2dx,$$

which is what we are used to from normal differentiation. Furthermore,

$$4d(xy^2) = 4y^2dx + 8xydy,$$

and so we have

$$df = (6x^2 + 4y^2)dx + 8xydy,$$

which are the partial derivatives of f with respect to x and y respectively. Let $g \in R$ be an arbitrary polynomial. We want to show that one can write dg as $g_1dx + g_2dy$ for $g_1, g_2 \in k[x, y]$. Now, g will be of the form

$$g = a_0 + a_1x + b_1y + c_2xy + a_1x^2 + b_2y^2 \cdots + a_nx^n + b_ny^n,$$

where $a_i, b_i, c_i \in k$. Let $c_ix^iy^j$ be an arbitrary term in g, then

$$d(c_ix^iy^j) = c_i(i \cdot y^jx^{i-1}dx + j \cdot x^idy),$$

so any term in g can be written in this way. Then, by linearity, dg can be written as

$$dg = g_1dx + g_2dy.$$

Thus, any element of Ω_R is written as a sum of dx and dy with coefficients from R.

We can state this generally:

Proposition 1.1.5 ([Eis13, Proposition 16.1])**.** *If $R = A[x_1, \ldots, x_n]$ is the polynomial ring in n variables, then*

$$\Omega_{R|A} = Rdx_1 \oplus \cdots \oplus Rdx_n.$$

For the proof we refer to [Eis13]. So essentially for any polynomial ring, the corresponding Kähler differentials are generated by the dx_i's as a module over the polynomial ring.

Before we move on to studying the Kähler differentials of closed subschemes, let us establish some basic properties of the universal derivation as an operator.

The d-operator and its properties

For our purposes, it will be fruitful to have some notion of what the universal derivation d or the 'd-operator' does to ideals and which properties are preserved and which are not. We will only be concerned with rings that are Noetherian.

First of all, we must clarify what we mean by applying the d-operator to an ideal. Let $\mathfrak{b} \subset B$ be some ideal in a ring B. By the notation $d\mathfrak{b}$ we will mean the module generated by the image of the ideal through the universal derivation

$$d : B \to \Omega_{B|A},$$

in other words

$$d\mathfrak{b} = \{r \cdot d(f) \mid f \in \mathfrak{b}, r \in B\} \subset \Omega_B.$$

So $d\mathfrak{b}$ is the submodule of Ω_B generated by df for all generators f of $\mathfrak{b}$. Note that it is a B-module, therefore it is closed under multiplication by elements from B. Let us show that inclusions are preserved.

Proposition 1.1.6. *Let $\mathfrak{a}, \mathfrak{b} \subset B$ be two ideals of a Noetherian ring B such that $\mathfrak{a} \subset \mathfrak{b}$. Then*

$$d\mathfrak{a} \subset d\mathfrak{b}.$$

Proof. Since B is Noetherian, any ideal in this ring is finitely generated, so we can write

$$\mathfrak{a} = (f_1, \ldots, f_n)$$

and

$$\mathfrak{b} = (g_1, \ldots, g_m).$$

Let $\omega_{\mathfrak{a}} \in d\mathfrak{a}$. We want to show that $\omega_{\mathfrak{a}} \in d\mathfrak{b}$. We can write $\omega_{\mathfrak{a}}$ as

$$\omega_{\mathfrak{a}} = d\left(\sum_{i=1}^{n} a_i f_i\right)$$

for some $a_i \in B$. Since $\mathfrak{a} \subset \mathfrak{b}$, any generator of $\mathfrak{a}$ can be written as

$$f_i = \sum_{j=1}^{m} b_j g_j,$$

where again $b_j \in B$. But then we can rewrite $\omega_{\mathfrak{a}}$ in the following way:

$$\begin{aligned}\omega_{\mathfrak{a}} = d\left(\sum_{i=1}^{n} a_i f_i\right) &= d\left(\sum_{i=1}^{n} a_i \left(\sum_{j=1}^{m} b_j g_j\right)\right) \\ &= d\left(\sum_{j=1}^{m} (a_1 + a_2 + \cdots + a_n) b_j g_j\right).\end{aligned}$$

If we set $c_j = (a_1 + a_2 + \cdots + a_n) b_j$, then we can write $\omega_{\mathfrak{a}}$ as

$$\omega_{\mathfrak{a}} = d\left(\sum_{j=1}^{n} c_j g_j\right),$$

where c_j is an element of B. Now we have written $\omega_{\mathfrak{a}}$ as an element of $d\mathfrak{b}$, hence $d\mathfrak{a} \subset d\mathfrak{b}$, which was what we wanted to show. ∎

Let us see what applying d does to a sum of ideals.

Proposition 1.1.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a Noetherian ring B. Then*

$$d(\mathfrak{a}+\mathfrak{b}) = d\mathfrak{a} + d\mathfrak{b}.$$

Proof. As before we assume that

$$\mathfrak{a} = (f_1, \dots, f_n)$$

and

$$\mathfrak{b} = (g_1, \dots, g_m).$$

Then, we know from basic abstract algebra that $\mathfrak{a}+\mathfrak{b}$ is generated by all of these, i.e

$$\mathfrak{a}+\mathfrak{b} = (f_1, \dots, f_n, g_1, \dots, g_m).$$

Now, let $\omega \in d(\mathfrak{a}+\mathfrak{b})$, which means

$$\omega = d\left(\sum_{i=1}^{n} a_i f_i + \sum_{j=1}^{m} b_j g_j\right)$$

for $a_i, b_j \in B$. But d is linear, so we can rewrite this as

$$\omega = d\left(\sum_{i=1}^{n} a_i f_i\right) + d\left(\sum_{j=1}^{m} b_j g_j\right),$$

but clearly

$$d\left(\sum_{i=1}^{n} a_i f_i\right) \in d\mathfrak{a}$$

and

$$d\left(\sum_{j=1}^{m} b_j g_j\right) \in d\mathfrak{b},$$

hence

$$\omega \in d\mathfrak{a} + d\mathfrak{b}.$$

This means $d(\mathfrak{a}+\mathfrak{b}) \subset d\mathfrak{a} + d\mathfrak{b}$.

To show the converse let $\omega_1 + \omega_2 \in d\mathfrak{a} + d\mathfrak{b}$, thus

$$\begin{aligned} \omega_1 + \omega_2 &= d\left(\sum_{i=1}^{n} a_i f_i\right) + d\left(\sum_{j=1}^{m} b_j g_j\right) \\ &= d\left(\sum_{i=1}^{n} a_i f_i + \sum_{j=1}^{m} b_j g_j\right), \end{aligned}$$

which is an element of $d(\mathfrak{a}+\mathfrak{b})$. This finishes the proof. ■

Another natural property to check is how the d-operator behaves with maps of rings and more specifically maps of ideals. The case with rings is a known result where the arguments are based on those of Eisenbud in [Eis13].

Proposition 1.1.8 ([Eis13, p. 386]). *The d-operator is functorial on the category of Noetherian rings in the following sense. For any map of rings over an algebraically closed field k*

$$\phi : A \to B,$$

there is an induced map of A-modules

$$\Omega_\phi : \Omega_A \to \Omega_B.$$

Proof. We have

$$\begin{array}{ccc} \Omega_A & & \Omega_B \\ {\scriptstyle d}\uparrow & & \uparrow{\scriptstyle d} \\ A & \xrightarrow{\phi} & B \end{array}$$

and we would like to find a map between Ω_A and Ω_B. Observe that Ω_B is an A-module through the composition $d \circ \phi$. Furthermore, this composition is a k-derivation since

$$\begin{aligned} d \circ \phi(aa') &= d(\phi(aa')) && (1.4) \\ &= d(\phi(a) \cdot \phi(a')) && (1.5) \\ &= \phi(a') \cdot d(\phi(a)) + \phi(a) \cdot d(\phi(a')), && (1.6) \end{aligned}$$

where Equation (1.5) follows from linearity of ϕ and Equation (1.6) follows from the fact that d is a derivation. Hence, we see that Leibniz rule is satisfied. It is also k-linear by a similar argument:

$$\begin{aligned} d \circ \phi(r \cdot a) &= d(\phi(r) \cdot \phi(a)) \\ &= \phi(r) \cdot d(\phi(a)), \end{aligned}$$

where we have used A-linearity of ϕ and k-linearity of d. Thus, the composition $d \circ \phi$ is a k-derivation

$$d \circ \phi : A \to \Omega_B,$$

and then by the universal property of Ω_A, there is a unique map

$$\Omega_\phi : \Omega_A \to \Omega_B$$

such that the diagram in Figure 1.1 commutes. This finishes the proof. ■

$$\begin{array}{ccc} \Omega_A & \xrightarrow{\Omega_\phi} & \Omega_B \\ {\scriptstyle d}\uparrow & & \uparrow{\scriptstyle d} \\ A & \xrightarrow{\phi} & B \end{array}$$

Figure 1.1: Commutative diagram

Let us see that the same is true for ideals of rings. In other words, that after applying d to the rings, $d\mathfrak{a}$ will be mapped into $d\mathfrak{b}$ through this map.

Proposition 1.1.9. *Let A, B and ϕ be as in Proposition 1.1.8 and assume further that $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ are two ideals such that*

$$\phi(\mathfrak{a}) \subset \mathfrak{b}.$$

Then, applying the d-operator to both the ideals and rings we get

$$\Omega_\phi(d\mathfrak{a}) \subset d\mathfrak{b}.$$

Proof. Let $\omega \in d\mathfrak{a}$. As earlier, we write it as

$$\omega = d\left(\sum_{i=1}^{n} a_i f_i\right),$$

where $\sum_{i=1}^{n} a_i f_i = a \in A$. Since $\omega \in \Omega_A$ we can map it through Ω_ϕ to Ω_B. We need to show that the image is contained in $d\mathfrak{b}$. Now, the element ω will be mapped to

$$\omega \mapsto \Omega_\phi(\omega).$$

Note that this means that a is mapped to $\Omega_\phi(\omega)$ through the composition $\Omega_\phi \circ d$. However, the diagram in Figure 1.1 is commutative, thus we have

$$d \circ \phi(a) \ = \Omega_\phi \circ d(a).$$

Recall that $\phi(\mathfrak{a}) \subset \mathfrak{b}$, hence $\phi(a) \in \mathfrak{b}$ and then

$$\Omega_\phi \circ d(a) = d \circ \phi(a) \in d\mathfrak{b}.$$

Thus, $d\mathfrak{a}$ is mapped into $d\mathfrak{b}$. This finishes the proof. ∎

Now that we have a good grasp on how the universal derivation behaves with some basic operations, we can move on to examining how the ‘omega-operator’ behaves with polynomial rings, and specifically quotient rings of polynomial rings. By the ‘omega-operator’ we mean the action of applying omega to some ring A to get Ω_A.

1.2 The induced quotient map

We are interested in comparing the Kähler differentials of a subscheme with those of the ambient scheme. We will limit ourselves to studying subschemes of affine schemes $X = \mathrm{Spec}(A)$, for some ring A. Our examples of closed subschemes will primarily be curves in $\mathbb{A}^2_k$, the affine plane over some algebraically closed field k.

Explicitly we have a polynomial ring $A = k[x, y]$, where k is an algebraically closed field. Assume further we have an ideal $\mathfrak{a} \subset A$, which will define a subscheme of $\mathbb{A}^2$

$$\mathrm{Spec}(A/\mathfrak{a}) \hookrightarrow \mathrm{Spec}(A).$$

We want to consider the map

$$\Omega_A \to \Omega_{(A/I)},$$

which we saw in Proposition 1.1.8 is naturally induced by the quotient map

$$A \to A/\mathfrak{a}.$$

To understand the map of differentials we should first familiarize ourselves with how the module $\Omega_{A/I}$ looks.

Theorem 1.2.1 ([EO, Theorem 17.13]). *Let A be a ring and let $B = A[x_1, \dots, x_n]/(f_1, \dots, f_n)$. Then*

$$\Omega_{B|A} = \frac{\bigoplus_i Bdx_i}{\sum_j B\left(\sum_i (\partial f_j/\partial x_i)\, dx_i\right)}$$

and the universal A-derivation is given as

$$d_{B/A}(f) = \sum_{i=1}^{n} (\partial f/\partial x_i)\, dx_i$$

for some $f \in B$.

This means that when taking the differentials of a quotient ring, we must also quotient out by the image of the ideal through d. Let us see an example of such a map between the differentials.

Example 1.2.2. Consider the polynomial ring $A = k[x, y]$ and $\mathfrak{a} = (y - x^2)$. Then the corresponding differential modules become

$$\Omega_A = k[x, y]dx \oplus k[x, y]dy,$$

$$\Omega_{A/\mathfrak{a}} = \frac{k[x, y]/(y - x^2)dx \oplus k[x, y]/(y - x^2)dy}{dy - 2xdx}.$$

We can simplify the situation by observing that $k[x, y]/(y - x^2) \simeq k[x]$. Then, the quotient map

$$k[x, y] \to k[x, y]/(y - x^2) \simeq k[x]$$

is defined by

$$x \mapsto x$$
$$y \mapsto x^2.$$

The induced map between differentials will then be defined by

$$dx \mapsto dx$$

$$dy \mapsto dx^2 = xdx + xdx = 2xdx$$

and the coefficients are naturally induced by the A-module structure.

Our goal is to find the kernel of the map between the differential modules. In this simple example this is quite easy since we can just ‘kill’ one of the generators, namely dy. We then end up with a free module with just one generator, dx, modulo the relation $\mathfrak{a} = (y - x^2)$. Accordingly, we get

$$\Omega_{A/\mathfrak{a}} = \frac{k[x, y]/(y - x^2)dx \oplus k[x, y]/(y - x^2)dy}{dy - 2xdx} \simeq k[x, y]/(y - x^2)dx, 0$$

where this isomorphism is induced by mapping $dy \mapsto 2xdx$ and $dx \mapsto dx$. In this case it is clear that the kernel is equal to the set

$$\mathfrak{a} \cdot \Omega_A + A \cdot d\mathfrak{a}$$

since $\mathfrak{a}$ is the zero ideal in $(A/\mathfrak{a}) \cdot dx$ and

$$d\mathfrak{a} = (dy - 2xdx)$$

is sent to zero because we map $dy \mapsto 2xdx$. Then, the sum of these two submodules must be the kernel since these are the only relations we have.

As we will see, the submodule

$$\mathfrak{a} \cdot \Omega_A + A \cdot d\mathfrak{a}$$

is equal to the kernel also in a general setting. For a general ring, the computation is not that easy since we cannot always end up with an isomorphism to a module generated by one element as above. This is because we might have relations both between the coefficients and the basis elements of the module. Let us look at one such situation.

Example 1.2.3. Consider

$$\begin{aligned} A &= k[x, y] \\ \mathfrak{a} &= (y^2 - x^2). \end{aligned}$$

We have from Theorem 1.2.1 that the Kähler differentials of the quotient ring becomes

$$\Omega_{(A/\mathfrak{a})} = \frac{k[x, y]/(y^2 - x^2)dx \oplus k[x, y]/(y^2 - x^2)dy}{2ydy - 2xdx}.$$

Now we cannot 'kill' one of the basis elements of the module since we have a y-coefficient and a x-coefficient in the relation between dx and dy. The elements x and y are not invertible in A, which means we cannot 'ignore' these terms in the relation. So, assume we have an element in the kernel

$$fdx + gdy \mapsto 0$$

for some $f, g \in k[x, y]$. Determining the kernel is equivalent to finding all the different ways to write zero in the module. Assume g has a factor of y which means it can be written

$$g = y \cdot g'$$

for some $g' \in A$. Assuming this we are in an equivalent situation as in Example 1.2.2. Now, the relation

$$(2ydy - 2xdx) = 0$$

gives us

$$fdx + gdy = fdx + g'ydy = fdx + g'xdx = (f + g'x)dx$$

in $\Omega_{A/\mathfrak{a}}$, and for this to be zero we need $(f + g'x) \in (y^2 - x^2)$. That means

$$f + g'x = h(y^2 - x^2)$$

for some $h \in A$. To summarize

$$fdx + gdy - g'(ydy - xdx) + g'(ydy - xdx) = fdx + g'xdx + g'(ydy - xdx)$$

and

$$f + g'xdx = h(y^2 - x^2)dx$$

which shows that $fdx + gdy$ is mapped to zero,0 since it is equal to $h(y^2 - x^2)dx$ modulo $(ydy - xdx)$. However, when g does not have a factor of y, the situation is much more complicated. This is because we cannot replace all elements of the form fdy with something of the form gdx.

As stated above, the kernel of the canonical map

$$\Omega_A \to \Omega_{A/\mathfrak{a}}$$

for an ideal $\mathfrak{a}$ of a Noetherian ring A will be the A-module

$$\mathfrak{a} \cdot \Omega_A + d\mathfrak{a}$$

where the f_i's are the generators of $\mathfrak{a}$. We will from now on denote this submodule as

$$\Omega_{\mathfrak{a}} = \mathfrak{a} \cdot \Omega_A + d\mathfrak{a}. \tag{1.7}$$

For this to make sense one must first establish which ring this ideal $\mathfrak{a}$ is in. If this is not clear from the context we denote this by

$$\Omega_{\mathfrak{a} \subset A},$$

to make it clear that we are regarding $\mathfrak{a}$ as an ideal of A and not some other ring. For most of our purposes the ring in question will be understood from the context.

However, before we show that this submodule is indeed the kernel we are looking for, let us establish some basic properties of the omega-operator for ideals as we did for the d-operator. By the omega-operator we just mean applying omega to an ideal as in Equation (1.7).

1.3 The omega-operator and its properties

We establish that omega preserves inclusions, just as the d-operator does.

Proposition 1.3.1. *Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of a Noetherian ring A such that $\mathfrak{a} \subset \mathfrak{b}$. Then the modules $\Omega_{\mathfrak{a}}$ and $\Omega_{\mathfrak{b}}$ are such that*

$$\Omega_{\mathfrak{a}} \subset \Omega_{\mathfrak{b}}.$$

Proof. Recall that we defined $\Omega_{\mathfrak{a}}$ as

$$\Omega_{\mathfrak{a}} = \mathfrak{a} \cdot \Omega_A + A \cdot d\mathfrak{a}.$$

From Proposition 1.1.6 we have that

$$d\mathfrak{a} \subset d\mathfrak{b},$$

so it remains to show that

$$\mathfrak{a} \cdot \Omega_A \subset \mathfrak{b} \cdot \Omega_A,$$

but this is clear since $\mathfrak{a} \subset \mathfrak{b}$. ■

The omega-operator also preserves sums of ideals.

Proposition 1.3.2. *Assume that* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are as in Proposition 1.3.1. Then*

$$\Omega_{\mathfrak{a}+\mathfrak{b}} = \Omega_{\mathfrak{a}} + \Omega_{\mathfrak{b}}.$$

Proof. Since A is Noetherian, we can assume

$$\begin{aligned} \mathfrak{a} &= (f_1, \ldots, f_n) \\ \mathfrak{b} &= (g_1, \ldots, g_m). \end{aligned}$$

Now, let $\omega \in \Omega_{\mathfrak{a}+\mathfrak{b}}$. We can write it on the form

$$\omega = (a+b) \cdot \gamma + \sum_{i=1}^{n} c_i df_i + \sum_{j=1}^{m} c_{n+j} dg_j,$$

where $a \in \mathfrak{a}$, $b \in \mathfrak{b}$, $c_i \in A$ and $\gamma \in \Omega_A$. We want to show that ω is in $\Omega_{\mathfrak{a}} + \Omega_{\mathfrak{b}}$. This follows readily from factoring terms by whether they come from $\mathfrak{a}$ or $\mathfrak{b}$:

$$\begin{aligned} \omega &= (a+b)dg + \sum_{i=1}^{n} c_i df_i + \sum_{j=1}^{m} c_{i+j} dg_j \\ &= \left(adg + \sum_{i=1}^{n} c_i df_i \right) + \left(bdg + \sum_{j=1}^{m} c_{i+j} dg_j \right). \end{aligned}$$

Observe that now the first term is an element of $\Omega_{\mathfrak{a}}$ and the second of $\Omega_{\mathfrak{b}}$, so it is exactly an element of $\Omega_{\mathfrak{a}} + \Omega_{\mathfrak{b}}$. This means we have showed

$$\Omega_{\mathfrak{a}+\mathfrak{b}} \subset \Omega_{\mathfrak{a}} + \Omega_{\mathfrak{b}}. \tag{1.8}$$

The converse is easily checked using the fact that the omega-operator preserves inclusions. Since $\mathfrak{a} \subset \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}$ we have that

$$\Omega_{\mathfrak{a}} \subset \Omega_{\mathfrak{a}+\mathfrak{b}},$$

and

$$\Omega_{\mathfrak{b}} \subset \Omega_{\mathfrak{a}+\mathfrak{b}}.$$

Now, the sum of two sets contained in a third set is also contained in that set, hence we have

$$\Omega_{\mathfrak{a}} + \Omega_{\mathfrak{b}} \subset \Omega_{\mathfrak{a}+\mathfrak{b}}.$$

Combining this and Equation (1.8) we get the desired equality

$$\Omega_{\mathfrak{a}} + \Omega_{\mathfrak{b}} = \Omega_{\mathfrak{a}+\mathfrak{b}}.$$

■

We also show that omega behaves well with ring homomorphisms.

Proposition 1.3.3. *Assume A and B are rings, and that $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of these rings, respectively. Further, assume that we have a ring homomorphism*

$$\phi : A \to B$$

such that ϕ maps $\mathfrak{a}$ into $\mathfrak{b}$, i.e.

$$\phi(\mathfrak{a}) \subset \mathfrak{b}.$$

Then, applying the omega-operator, we get

$$\Omega_\phi(\Omega_\mathfrak{a}) \subset \Omega_\mathfrak{b}.$$

Proof. We know that applying omega to

$$\phi : A \to B$$

we just get

$$\Omega_\phi : \Omega_A \to \Omega_B.$$

We would like to show that an element of $\Omega_\mathfrak{a}$ is sent to $\Omega_\mathfrak{b}$ through the induced map Ω_ϕ. Now, let $\omega \in \mathfrak{a}$, which means it is of the form

$$\omega = a \cdot \gamma + \omega_\mathfrak{a},$$

where $a \in \mathfrak{a}$, $\gamma \in \Omega_A$ and $\omega_\mathfrak{a} \in d\mathfrak{a}$. From Proposition 1.1.9 we have that

$$\Omega_\phi(\omega_\mathfrak{a}) \in d\mathfrak{b},$$

and since Ω_ϕ is a module homomorphism, we need only check that

$$\Omega_\phi(a \cdot \gamma) \in \Omega_\mathfrak{b}.$$

Well, Ω_ϕ is an A-module homomorphism, which means

$$\Omega_\phi(a \cdot \gamma) = \phi(a) \cdot \Omega_\phi(\gamma).$$

Remember that ϕ maps $\mathfrak{a}$ into $\mathfrak{b}$, hence $\phi(a) \in \mathfrak{b}$. Thus,

$$\phi(a) \cdot \Omega_\phi(\gamma) \in \mathfrak{b} \cdot \Omega_B \subset \Omega_\mathfrak{b},$$

which finishes the proof. ∎

We state what omega does to maximal ideals.

Proposition 1.3.4. *Let A be a finitely generated k-algebra, where k is an algebraically closed field. If $\mathfrak{m} \subset A$ is some maximal ideal, then*

$$\Omega_\mathfrak{m} = \Omega_{A|k}.$$

In other words, applying the 'omega-operator' to a maximal ideal gives us all Kähler differentials of the ring.

Proof. This is just because the module $\Omega_{A/\mathfrak{m}} = 0$, since

$$A/\mathfrak{m} \simeq k,$$

and clearly $\Omega_{k|k} = 0$. ∎

Alternate proof. We can also show this by considering $\Omega_{\mathfrak{m}}$. Since A is a finitely generated k-algebra we know from Noether's normalization lemma that we can find $x_1, \ldots, x_n$ such that

$$A \simeq k[x_1, \ldots, x_n].$$

Then, we know by Hilberts Nullstellensatz that any maximal ideal in such a ring is of the form

$$\mathfrak{m} = (x_1 - \alpha_1, \ldots, x_n - \alpha_n),$$

where $\alpha_i \in k$. Applying d to this ideal we get for each generator

$$d(x_i - \alpha_i) = d(x_i) - d(\alpha_i) \tag{1.9}$$

$$= d(x_i), \tag{1.10}$$

where we have used linearity of d and that α_i is a constant. But from Proposition 1.1.5 we know that these dx_i's generate Ω_A, and

$$d\mathfrak{m} \subset \Omega_{\mathfrak{m}},$$

hence

$$\Omega_{\mathfrak{m}} = \Omega_A.$$

■

1.4 The kernel of the quotient map of differentials

We are now ready to prove that the kernel of the map

$$\Omega_A \to \Omega_{A/\mathfrak{a}},$$

for a ring A and ideal $\mathfrak{a}$, is exactly the submodule we introduced earlier,

$$\Omega_{\mathfrak{a}} = \mathfrak{a} \cdot \Omega_A + d\mathfrak{a}.$$

The proof is heavily inspired by [Sta22, Lemma 10.131.6].

Proposition 1.4.1 ([Sta22, Lemma 10.131.6])**.** *Let A be a ring over some algebraically closed field k and let $\mathfrak{a} \subset A$ be some ideal of A. The quotient map*

$$A \to A/\mathfrak{a}$$

induces a map between the corresponding differential modules

$$\Omega_A \to \Omega_{A/\mathfrak{a}}.$$

Then, the kernel of this map is

$$\mathfrak{a} \cdot \Omega_A + A \cdot d\mathfrak{a}.$$

If $(f_1, f_2, \ldots, f_n)$ generates $\mathfrak{a}$, then $d\mathfrak{a}$ is the module generated by

$$(df_1, df_2, \ldots, df_n).$$

Proof. To find the kernel for a general ring it can be fruitful to find a resolution through free modules. To this end, consider the module

$$\bigoplus_{a \in A} A[a],$$

which is a direct sum of polynomial rings where each element of the ring is a variable in one part of the sum. Now, define D to be the map from this graded ring into Ω_A by

$$\bigoplus_{\alpha} r_\alpha[a_\alpha] \overset{D}{\longmapsto} \sum_{\alpha} r_\alpha d(a_\alpha),$$

where $r_\alpha \in A$ are the coefficients in the graded ring, the $[a_\alpha]$'s are the variables, and d is the universal derivation. Then, because d is a derivation, all elements of the form $[ab] - a[b] - b[a]$, $[a+b] - [a] - [b]$, and $[r]$ are in the kernel, where $r \in k$ and $a, b \in A$. Next, we create an exact sequence by mapping a free module with these relations as variables into $\bigoplus_{a \in A} A[a]$. We can write this module as a direct sum of polynomial rings

$$\bigoplus_{i} A[i],$$

where the i's are all the relations mapped to zero through the map D. For example, one relation is $i = [xy] - x[y] - y[x]$ for $x, y \in A$, which stems from the Leibniz rule. We map these relations to their corresponding element in $\bigoplus_{a \in A} A[a]$. In other words,

$$[xy] - x[y] - y[x] \mapsto [xy] - x[y] - y[x].$$

So, these relations are the elements mapped to zero by D. This yields the exact sequence

$$\bigoplus_{i} A[i] \longrightarrow \bigoplus_{a \in A} A[a] \overset{D}{\longrightarrow} \Omega_A \longrightarrow 0.$$

It is exact on the right since Ω_A is generated by $d(a)$ for all $a \in A$, and for any $da \in \Omega_A$, we have $[a] \in \bigoplus_{a \in A} A[a]$ which is sent to da. So the map of the generators of Ω_A is surjective. The sequence has also been defined to be exact in the middle. We can make a commutative diagram by mapping this sequence into the corresponding sequence quotiented by the ideal $\mathfrak{a}$. This gives the diagram in Figure 1.2.

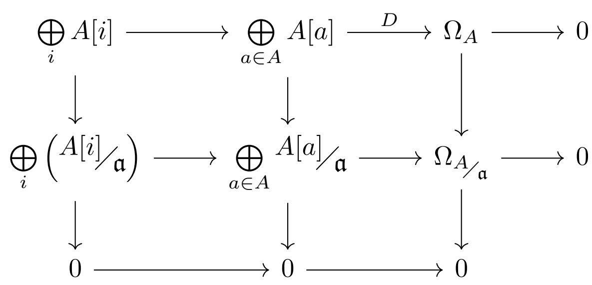

Figure 1.2: Free resolution of the kernel

The vertical sequences are exact since the maps are just quotient maps which are automatically surjective. The elements are sent to their equivalence class modulo $\mathfrak{a}$ in the two leftmost vertical maps. The map

$$\Phi : \Omega_A \to \Omega_{A/\mathfrak{a}}$$

is induced by the quotient map of the rings and is also surjective. Recall that we are interested in finding the kernel of this map, and we will use this diagram to determine it. So assume an element $\omega \in \Omega_A$ is in the kernel so

$$\Phi(\omega) = 0 \in \Omega_{A/\mathfrak{a}}.$$

Since D is surjective we can find an element of the form

$$f = (r_\alpha \cdot [a_\alpha]) \in \bigoplus_{a \in A} A[a]$$

such that

$$D((r_\alpha \cdot [a_\alpha])) = \sum_\alpha r_\alpha d(a_\alpha) = \omega,$$

where the r_α's are elements in A that act like coefficients in this polynomial ring, and the $[a_\alpha]$'s are the variables. Now that we have an element in $\bigoplus_{a \in A} A[a]$, we can map it through the quotient map

$$\bigoplus_{a \in A} A[a] \to \bigoplus_{a \in A} A[a]/\mathfrak{a}$$
$$f \mapsto \bar{f}.$$

Consider the rightmost part of the diagram in Figure 1.2, which is depicted in Figure 1.3. Recall that ω is mapped to zero through Φ, and since this diagram

$$\begin{array}{ccc} \bigoplus_{a \in A} A[a] & \xrightarrow{D} & \Omega_A \\ \downarrow & & \downarrow \Phi \\ \bigoplus_{a \in A} A[a]/\mathfrak{a} & \xrightarrow{\bar{D}} & \Omega_{A/\mathfrak{a}} \end{array}$$

Figure 1.3: Right side of the diagram in Figure 1.2.

commutes, $\bar{f}$ must also be mapped to zero through $\bar{D}$. But then $\bar{f}$ is in the kernel of $\bar{D}$, so it comes from an element in the relations module

$$\bigoplus_i A/\mathfrak{a}[i].$$

Let us call this element $\bar{g} \in \bigoplus_i A[i]/\mathfrak{a}$. We have

$$\bigoplus_i A[i]/\mathfrak{a} \to \bigoplus_{a \in A} A[a]/\mathfrak{a}$$
$$\bar{g} \mapsto \bar{f}.$$

Now, since the quotient map is surjective we can find an element g of

$$\bigoplus_i A[i]$$

that is mapped to $\bar{g}$. Consider the leftmost part of our diagram, depicted in Figure 1.4. Here we have that $\psi(g) = \bar{g}$ and $\gamma(f) = \bar{f} = \bar{r}(\bar{g})$ since this diagram

$$\begin{array}{ccc} \bigoplus_i A[i] & \xrightarrow{r} & \bigoplus_{a\in A} A[a] \\ \downarrow \psi & & \downarrow \gamma \\ \bigoplus_i \left(A[i]/\mathfrak{a} \right) & \xrightarrow{\bar{r}} & \bigoplus_{a\in A} A[a]/\mathfrak{a} \end{array}$$

Figure 1.4: Left part of our diagram

also commutes. In particular that means that $f = r(g)$ modulo $\mathfrak{a}$. If we now consider the element $h = f - r(g)$, then we see

$$\gamma(h) = \gamma(f - r(g)) = \bar{f} - \bar{r}(\bar{g}) = 0$$

by linearity, so h is in the kernel of γ. The salient point is that h also maps to the element ω we started with through D. This is true by noting that

$$g \in \bigoplus_i A[i],$$

and by exactness we have

$$D(r(g)) = 0.$$

The map D is linear by the linearity of the universal derivation, hence

$$D(h) = D(f - r(g)) = D(r(g)) - D(f) = 0 - D(f).$$

That means we have found an element of $\bigoplus_{a\in A} A[a]$ that is mapped to ω and also mapped to zero through γ. By this diagram chase, for each element $\omega \in \Omega_A$ in the kernel of Φ, we can find an element $h \in \bigoplus_{a\in A} A[a]$ that maps to ω, and is also contained in the kernel of γ. So for any element in the kernel of

$$\Phi : \Omega_A \to \Omega_{A/\mathfrak{a}}$$

we can find an element in the kernel of

$$\bigoplus_{a\in A} A[a] \xrightarrow{\gamma} \bigoplus_{a\in A} A[a]/\mathfrak{a},$$

so we determine $\mathrm{Ker}(\gamma)$ to determine the $\mathrm{Ker}(\Phi)$. Let $x = \oplus_\alpha(r_\alpha \cdot a_\alpha) \in \bigoplus_{a\in A} A[a]$ be an element in the kernel of γ. The image of this element through γ is

$$\oplus_\alpha(\overline{r_\alpha}\,\overline{a_\alpha}).$$

This is zero if and only if it is zero for all α, so let α be some given index. We have

$$\overline{r_\alpha}\,\overline{a_\alpha} = 0,$$

which means that either $\overline{r_\alpha} = 0$ or $\overline{a_\alpha} = 0$. So, either $\overline{r_\alpha} \in \mathfrak{a}$ or we have $\overline{a_\alpha} \in \mathfrak{a}$. Thus, the kernel is

$$\operatorname{Ker}(\gamma) = \mathfrak{a} \cdot [A] + A \cdot [\mathfrak{a}].$$

But, now we have a characterization of the kernel of

$$\Omega_A \xrightarrow{\Phi} \Omega_{A/\mathfrak{a}}$$

by the fact that each element corresponds to at least one element of $\bigoplus_{a \in A} A[a]$ that is in the kernel of γ. But that means

$$\operatorname{Ker}\Phi = D(\operatorname{Ker}\gamma) = D(\mathfrak{a} \cdot [A] + A \cdot [\mathfrak{a}]) = \mathfrak{a} \cdot \Omega_A + A \cdot d\mathfrak{a},$$

so now we have showed that

$$\operatorname{Ker}\Phi \subset \mathfrak{a} \cdot \Omega_A + A \cdot d\mathfrak{a}.$$

It is clear that the converse holds true, hence

$$\operatorname{Ker}\Phi = \mathfrak{a} \cdot \Omega_A + A \cdot d\mathfrak{a}.$$

■

Now that we have determined the kernel of the map

$$\Omega_A \xrightarrow{\Phi} \Omega_{A/\mathfrak{a}}$$

we would like to compute the primary decomposition of this, and see if there are relations between the primary components of $\Omega_\mathfrak{a}$ and the primary components of the ideal $\mathfrak{a}$. To do this we need some theory from commutative algebra on primary decomposition of ideals and modules.

CHAPTER 2

Primary decomposition of ideals and modules

We want to find a primary decomposition of the omega module. In this chapter we will establish some preliminaries on primary decompositions. First, we establish what it means for a module to be primary. Then, we study what a decomposition into primary modules is.

We will introduce the theory for ideals in rings, and then generalize this to modules. The results on ideals of rings are already heavily covered in the reference literature, therefore we omit the proofs here. Much of the theory on ideals is based on the theory in [AM69] and [Ell].

2.1 Primary ideals and primary modules

Definition 2.1.1. Let $\mathfrak{a}$ be some ideal of a ring R. The ideal is said to be *primary* if the following condition is met:

$$x \cdot y \in \mathfrak{a} \implies x \in \mathfrak{a} \text{ or } y^n \in \mathfrak{a}, \text{ for some } n \in \mathbb{N}.$$

If $\sqrt{a} = \mathfrak{p}$ for a prime ideal $\mathfrak{p}$, we say that $\mathfrak{a}$ is $\mathfrak{p}$-*primary*

Note that this means that all prime ideals are primary since the definition for prime ideals is the same by letting $n = 1$. This leads us to the next proposition.

Proposition 2.1.2 ([Ell, Proposition 10.3])**.** *If $\mathfrak{a}$ is a primary ideal of a ring R then the radical of $\mathfrak{a}$*

$$\sqrt{\mathfrak{a}} = \{x \in R \mid x^n \in \mathfrak{a}\}$$

is a prime ideal.

Now that we have established what it means for ideals of rings to be primary, we generalize this notion to modules of rings. First, recall that the annihilator of an R-module is defined as

$$\mathrm{Ann}(M) = \ \{x \in R | \ x \cdot M \subset 0\},$$

where we write 0 in the sense of the zero submodule of M. In other words, it is the set of elements in R that multiplies the entire R-module M into the zero module. Now we can define a primary submodule.

Definition 2.1.3. Let M be a module of some ring R and N be a submodule of M. We say that N is a *primary submodule* if it satisfies the following condition:

$$x \cdot y \in N \implies y \in N \text{ or } x^n \in \mathrm{Ann}(M/N) \text{ for some } n \in \mathbb{N}, 0$$

where $x \in R$ and $y \in M$.

If $y \in N$, then any ring element will multiply y into N. Note that in the case of modules we have to define a primary module relative to some ambient module. When specializing this to the case of ideals of rings we just consider the ring as a module over itself and the ideal as the submodule. Further, note that $x^n \in \mathrm{Ann}(M/N)$ is equivalent to $x \in \sqrt{\mathrm{Ann}(M/N)}$. We will say that a primary module $N \subset M$ is $\mathfrak{p}$-primary if

$$\sqrt{\mathrm{Ann}(M/N)} = \mathfrak{p}.$$

Consider the following example.

Example 2.1.4. Let $M = k[x,y]$ and $N = (x^2)$. Then, $N \subset M$ are both naturally $k[x,y]$-modules, and we are in fact just checking if N is a primary ideal. If

$$f \cdot g \in (x^2)$$

for some $g \in k[x,y]/(x^2)$ and $f \in k[x,y]$, we have

$$f \cdot g = hx^2$$

for some $h \in k[x,y]$. If $g \in (x^2)$ we are done, so assume $g \notin (x^2)$. However, then we must have that f has a factor of x, since x is irreducible in this ring. Then we have $f \in (x)$, and then clearly

$$f^2 \in \mathrm{Ann}(k[x,y]/(x^2)),$$

which shows that N is (x)-primary.

Let us look at an example of a primary submodule of the differentials.

Example 2.1.5. Assume that $R = k[x,y]$ is a polynomial ring of two variables and consider the ideal $(x) \subset R$. This ideal is prime, so it is certainly primary. We know that

$$\Omega_{(x)} = (x) \cdot \Omega_R + (dx) \cdot R,$$

and we are interested in whether $\Omega_{(x)}$ is (x)-primary. Assume that $\omega \in \Omega_{k[x,y]}/\Omega_{(x)}$ is nonzero and $a \in k[x,y]$ such that

$$a \cdot \omega \in \Omega_{(x)}.$$

We want to show that this means that $a^n \in \mathrm{Ann}(\Omega_{k[x,y]}/\Omega_{(x)})$ for some $n \in \mathbb{N}$. Since $\omega \in \Omega_R/\Omega_{(x)}$ we have that $\omega = (\omega_1 dx + \omega_2 dy)$ for some $\omega_i \in k[x,y]$. Further,

$$a \cdot \omega \in \Omega_{(x)}$$

means that

$$a \cdot \omega = x(udx + vdy) + w \cdot dx$$

for some $u, v, w \in R$. We can split this equation by the generators of $\Omega_{k[x,y]}$ and get

$$a \cdot \omega_1 = xu + w \tag{2.1}$$

$$a \cdot \omega_2 = xv. \tag{2.2}$$

Note that ω_1 and ω_2 cannot both be multiples of (x) since then $\omega \in \Omega_{(x)}$. From Equation (2.2) we get that either $\omega_2 \in (x)$ or $a \in (x)$. If the latter is true, then we are done since then certainly $a = a'x$ multiplies all of Ω_R into $\Omega_{(x)}$. Assume then that $\omega_2 = \omega_2' x$. However, now we have

$$\omega = \omega_1 dx + \omega_2' x dy$$

which is an element of $\Omega_{(x)}$, which is a contradiction. Therefore, we must have $a \in (x)$, so we have that $\Omega_{(x)}$ is primary. We can also show that $\Omega_{(x)}$ is in fact (x)-primary by showing that

$$\sqrt{\operatorname{Ann}(\Omega_{k[x,y]}/\Omega_{(x)})} = \sqrt{\{r \in R : r \cdot \Omega_{k[x,y]} \subset \Omega_{(x)}\}}$$

is equal to (x) when considering $\Omega_{(x)}$ as a submodule of $\Omega_{k[x,y]}/\Omega_{(x)}$. So we are interested in polynomials of x and y over k such that they multiply all differentials into $\Omega_{(x)}$. But clearly this is just (x) since all differentials of the form (dx) are already contained so what remains are those of the form (dy). But then these must be multiplied by something in (x) to be in $\Omega_{(x)}$, because there are no relations between dx and dy. So

$$\operatorname{Ann}\left(\Omega_{k[x,y]}/\Omega_{(x)}\right) = (x),$$

and $\Omega_{(x)}$ is (x)-primary.

If possible we want to figure out what the primary components of a given ideal is because it simplifies the situation and breaks the ideal into smaller parts. This is done by primary decomposition, where we write a given ideal as an intersection of primary ideals. Geometrically this corresponds to considering a scheme or subscheme as a union of smaller schemes which have nicer properties. We start by introducing primary decomposition for ideals of rings, and then, as before, generalize this to modules of rings.

2.2 Existence of primary decompositions of ideals and their uniqueness

Proposition 2.2.1 ([Ell, Proposition 10.18])**.** *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Then there exists a decomposition of $\mathfrak{a}$ into finitely many primary ideals.*

Such a decomposition is not necessarily unique, but there are things we can do to make them as unique as possible. To this end we let a minimal primary decomposition $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ denote a decomposition such that all the radicals of the components $\sqrt{\mathfrak{a}_i}$ are unique, and the intersection is irredundant in the sense that no two components are contained in one another.

Note that when intersecting two sets where one contains the other, one can always remove the larger set and the resulting set will not change. It

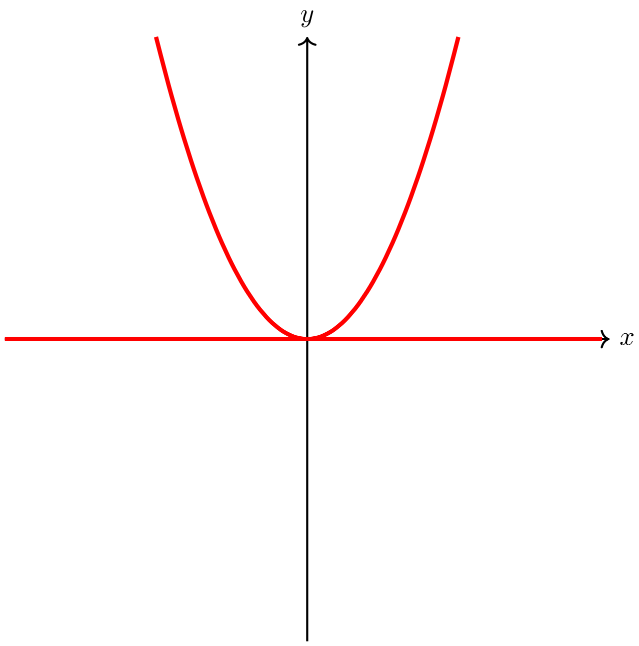

Figure 2.1: Graph of $(y^2 - yx^2)$

turns out that one can always find a minimal decomposition given a primary decomposition of an ideal.

Lemma 2.2.2 ([Ell, Theorem 10.19])**.** *Let R and $\mathfrak{a}$ be as in Proposition 2.2.1 and assume we have a primary decomposition $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n$. Then, there exists a subset of these primary components $\mathfrak{q}_i$ such that their radicals are unique, and the intersection is irredundant.*

Let us look at an example where we get a minimal primary decomposition from one that is not minimal.

Example 2.2.3. Let $R = k[x, y]$, where k is an algebraically closed field. Consider the ideal $\mathfrak{a} = (y^2 - yx^2) \subset R$. Geometrically this corresponds to the union of the x-axis and the parabola $y = x^2$ in $\mathbb{A}^2$, illustrated in Figure 2.1. It is clear that

$$(y^2 - yx^2) = y(y - x^2)$$

One primary decomposition of this ideal is

$$\mathfrak{a} = (y) \cap (y - x^2) \cap (x, y).$$

However, note that

$$(y) \subset (x, y),$$

so the component (x, y) does not contribute anything to the intersection, and we can omit it. This means that the primary decomposition above is not minimal because the intersection is redundant. Removing the last component we get:

$$\mathfrak{a} = (y) \cap (y - x^2),$$

where both components are prime ideals and clearly not contained in one another, which means we have a minimal primary decomposition.

Let us see an example of non-unique radicals.

Example 2.2.4. Let $R = k[x, y]$, where k is an algebraically closed field. Consider the ideal

$$(y^2) \cap (y) \cap (x).$$

Taking radicals of each component, we get $(y), (y)$, and (x), so clearly the radicals are not unique. We see that we can remove the second component

$$(y^2) \cap (y) \cap (x) = (y^2) \cap (x).$$

This is also clear from the fact that $(y^2) \subset (y)$.

We state the uniqueness of the radicals.

Theorem 2.2.5 ([Ell, Theorem 10.23])**.** *Let $\mathfrak{a} \subset R$ be an ideal of a Noetherian ring. Then, the radicals of the components in a minimal primary decomposition of $\mathfrak{a}$ are uniquely determined by the ideal.*

Essentially, in a minimal primary decomposition the radicals of the components are always the same. One cannot have two minimal primary decompositions of the same ideal with different radicals.

In a primary decomposition, we categorize the components as isolated or embedded. The isolated components are the ones with minimal radicals, which means their radicals do not contain radicals of other components, so in Example 2.2.3 above

$$(y) \text{ and } (y - x^2)$$

are isolated components. Embedded components are ones which do not have minimal radical, so in Example 2.2.3

$$(x, y)$$

is an embedded component. The next theorem states that the isolated components are unique. However, no such statement exists for the embedded components, in fact, we will see examples of the embedded components not being unique.

The name embedded comes from the geometrical viewpoint, which is why it is inverse to our algebraic intuition, since it is an ideal that contains other ideals. But, as we know from algebraic geometry when we go from ideals to closed sets of $\operatorname{Spec}(R)$ we reverse this inclusion. The ideal (x, y) becomes $V(x, y)$ which is a point and (y) becomes $V(y)$ which is the x-axis.

Theorem 2.2.6 ([Ell, Theorem 10.30])**.** *The isolated primary components of an ideal $\mathfrak{a}$ in a Noetherian ring R are uniquely determined by $\mathfrak{a}$.*

This means that the isolated components in a primary decomposition will always be the same for a given ideal. However, there are no restraints on the embedded components. Thus, we can have infinitely many primary decompositions.

Now let us go back to the world of modules and generalize what we have learned for ideals.

2.3 Primary decompositions of Noetherian submodules and their uniqueness

An essential part we must establish is that modules have primary decompositions and the uniqueness of the radicals. Many of these statements are from the Commutative Algebra lecture book of Robert B. Ash ([Ash]), and the proofs are also heavily inspired by the proofs in his book. We include the proofs to increase our intuition about the theory. First, we state existence.

Theorem 2.3.1 ([Ash, Theorem 1.2.5]). *Let R be a Noetherian ring and M be a finitely generated R-module. Further, let N be a submodule of M. Then there exists a primary decomposition of N, in other words it can be written as*

$$N = \bigcap_{i=1}^{n} N_i$$

where each N_i is $\mathfrak{p}_i$-primary for prime ideals $\mathfrak{p}_i$.

Before we prove this theorem, we need to establish some statements that will help us. First, we need to define an irreducible submodule.

Definition 2.3.2. Let M be a module and let N be a submodule of M. If there are no submodules $N_1 \subset M$ and $N_2 \subset M$ such that

$$N = N_1 \cap N_2,$$

where N is properly contained in both N_1 and N_2, then we say that N is *irreducible.*

In the Noetherian case, irreducible modules correspond to primary modules.

Proposition 2.3.3 ([Ash, Proposition 1.2.4]). *Let N be an irreducible submodule of M, a Noetherian module. Then N is a primary module.*

Proof. Assume N is an irreducible submodule of a Noetherian R-module M, and assume for contradiction that N is not primary. This means that there is some $a \in R$ such that the map

$$\begin{aligned} \phi_a : M/N &\to M/N \\ m &\mapsto a \cdot m \end{aligned}$$

is neither injective nor nilpotent. The chain of submodules

$$\operatorname{Ker}(\phi_a) \subset \operatorname{Ker}(\phi_a^2) \subset \dots$$

must terminate since M is Noetherian. Assume that it terminates at $\operatorname{Ker}(\phi_a^k)$ for $k \in \mathbb{N}$, hence

$$\operatorname{Ker}(\phi_a^k) = \operatorname{Ker}(\phi_a^{k+1}).$$

For simplicity, let $\psi = \phi_a^k$, then of course $\operatorname{Ker}(\psi) = \operatorname{Ker}(\psi^2)$. We claim that

$$\operatorname{im}(\psi) \cap \operatorname{Ker}(\psi) = 0,$$

where 0 is the zero-module of M. To see this, assume $x \in \mathrm{im}(\psi) \cap \mathrm{Ker}(\psi)$. This means $\psi(x) = 0$, and that there exists some $y \in M/N$ such that $\psi(y) = x$, moreover $\psi^2(y) = \psi(x) = 0$. However, $\mathrm{Ker}(\psi) = \mathrm{Ker}(\psi^2)$, thus

$$y \in \mathrm{Ker}(\psi^2) \implies y \in \mathrm{Ker}(\psi).$$

This means that $\psi(y) = 0$, but $\psi(y) = x$, so $x = 0$, which proves the claim. The map ϕ_a is not injective, i.e. $\mathrm{Ker}(\phi_a) \neq 0$, but

$$\mathrm{Ker}(\phi_a) \subset \mathrm{Ker}(\psi).$$

Hence, $\mathrm{Ker}(\psi) \neq 0$. Also, since ϕ_a is not nilpotent we have $\phi^m a \neq 0$ for all $m \in \mathbb{N}$. Therefore, there are elements not mapped to zero through ψ, i.e $\mathrm{im}(\psi) \neq 0$. Now, let $p : M \to M/N$ be the naturally induced quotient homomorphism of modules. Further, let $N_1 = p^{-1}(\mathrm{Ker}(\psi))$, and $N_2 = p^{-1}(\mathrm{im}(\psi))$. Now, since we have shown that the sets $\mathrm{Ker}(\psi)$ and $\mathrm{im}(\psi)$ are not equal to zero, the sets N_1 and N_2 are not equal to the kernel of p, which we know is just N.

We will show that $N = N_1 \cap N_2$ with $N \subsetneq N_1$ and $N \subsetneq N_2$, which means N is reducible, contradicting our assumption. Let $x \in N_1 \cap N_2$, then

$$x \in p^{-1}(\mathrm{Ker}(\psi)) \cap p^{-1}(\mathrm{im}(\psi)) \implies$$
$$p(x) \in \mathrm{Ker}(\psi) \cap \mathrm{im}(\psi)$$

since inverse images distribute over intersections. Recall that we showed $\mathrm{Ker}(\psi) \cap \mathrm{im}(\psi) = 0$, hence $p(x) = 0$, which means $x \in \mathrm{Ker}(p) = N$. For the converse assume that $x \in N$, which means

$$p(x) = 0 \in \mathrm{Ker}(\psi) \cap \mathrm{im}(\psi).$$

So x is mapped to $\mathrm{Ker}(\psi) \cap \mathrm{im}(\psi)$, which means it is in

$$p^{-1}(\mathrm{Ker}(\psi) \cap \mathrm{im}(\psi)),$$

which is just $N_1 \cap N_2$, so $x \in N_1 \cap N_2$.

The next part is showing N_1 and N_2 properly contain N. Therefore, let $y \in \mathrm{Ker}(\psi)$ be a nonzero element. We know such a choice is possible since we earlier showed that $\mathrm{Ker}(\psi) \neq 0$. Now, since p is a quotient map it is surjective, and therefore we can find $x \in M$ such that $p(x) = y$. This means that $x \in p^{-1}\,\mathrm{Ker}(\psi) = N_1$, but y is nonzero, hence

$$y \notin \mathrm{Ker}(p) = N,$$

so N_1 contains elements that are not in N. Similarly choose $y \in \mathrm{im}(\psi)$ such that $y \neq 0$, possible by earlier arguments. Again, there exists $x \in M$ such that $p(x) = y$, which means $x \in p^{-1}(\mathrm{im}(\psi)) = N_2$. However, $y \neq 0$, hence $x \notin N$. This means we have showed that N is reducible, a contradiction, so N must be primary.

■

Now we are ready to prove Theorem 2.3.1, existence of primary decompositions of finitely generated modules. Using Proposition 2.3.3, our proof becomes quite simple and beautiful indeed.

Proof. Assume that there exists submodules of M that cannot be decomposed into primary components. We let the set of such submodules be denoted

$$S = \{N \subset M \mid N \text{ does not have a primary decomposition}\}.$$

We claim that $S = \emptyset$. Since M is a Noetherian module we know such a set has a maximal element, assume this is $N \in S$. This means that N cannot itself be primary, hence it cannot be irreducible, so there exists $N_1, N_2 \subset M$ such that

$$N = N_1 \cap N_2,$$

and N is properly contained in both of N_1 and N_2. However, N is maximal in S, so N_1 and N_2 cannot be in S, which means they each have primary decompositions. But, then of course N can be decomposed using the decompositions of N_1 and N_2, hence N is decomposable after all, so $N \notin S$ and S is empty. This finishes the proof. ■

The radical of each primary component is called an associated prime to the ideal one is decomposing. They are, in loose terms, the information giving the geometry of the module.

Definition 2.3.4 (Associated prime)**.** Let M be an A-module, where A is some ring. If there exists some $m \in M$ such that

$$\mathrm{Ann}(m) = \mathfrak{p},$$

and $\mathfrak{p}$ is a prime ideal, then we say that $\mathfrak{p}$ is an *associated prime* to M. We denote the set of associated primes to M by $\mathrm{Ass}(M)$.

Example 2.3.5. Let $R = k[x, y, z]$ and consider the ideal $(xyz) \subset R$. The minimal primary decomposition of this ideal is

$$(xyz) = (x) \cap (y) \cap (z).$$

In fact, these three ideals are also the associated primes of (xyz). To see this, observe that

$$\mathrm{Ann}(xy) = (z),$$

where we are taking the annihilator in the module

$$k[x, y, z]/(xyz).$$

Similar computations can be made for the other two ideals. One interpretation of the associated primes are as the building blocks of the ideal. This makes more sense geometrically when we observe that (xyz) corresponds to the union of the $xy-$, $yz-$, and xz-planes in three-dimensional space. Now, the ideal (z) corresponds to the xy-plane, (x) corresponds to the yz-plane, and (y) to the xz-plane. So each associated prime makes up one geometric component of the ideal we are decomposing. Observe that the ideal (xyz^2) can be decomposed

$$(xyz^2) = (x) \cap (y) \cap (z^2),$$

but the associated primes are the same. The geometry is also the same because

$$V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a})$$

from a fundamental result about the Zariski topology [EO, Lemma 2.5].

The next result tells us that this connection between the associated primes and components of the primary decomposition is a general pattern. However, before we state it, we need a helpful lemma from commutative algebra. The proof is inspired by [htt].

Lemma 2.3.6. *Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal, and $\{\mathfrak{a}_i\}$ a set of ideals $\mathfrak{a}_i \subset A$ such that*

$$\bigcap_{i=1}^{n} \mathfrak{a}_i^{n_i} \subset \mathfrak{p}.$$

Then, $\mathfrak{a}_i \subset \mathfrak{p}$ for some $\mathfrak{a}_i$.

Proof. Assume that this is not the case, so we have for each i, some $r_i \in \mathfrak{a}_i^{n_i} \setminus \mathfrak{p}$. The product

$$r_1 r_2 \cdots r_j \in \left(\bigcap_{i=1}^{j} \mathfrak{a}_i^{n_i} \right),$$

so it must be in $\mathfrak{p}$. However, because $\mathfrak{p}$ is prime we must have $r_1 \in \mathfrak{p}$ or $r_2 \cdots r_n \in \mathfrak{p}$. If the former is true we have a contradiction, if the latter is true we do this inductively until we end up with $r_i \in \mathfrak{p}$ for at least one i, which is a contradiction. Hence, we have

$$\mathfrak{a}_i^{n_i} \subset \mathfrak{p},$$

but taking radicals on both sides we get the desired inclusion,

$$\mathfrak{a}_i \subset \mathfrak{p}.$$

■

Theorem 2.3.7 ([Ash, Theorem 1.3.9])**.** *Let M be a nonzero finitely generated module over a Noetherian ring A. Assume N has a primary decomposition*

$$N = \bigcap_i N_i,$$

where each N_i is $\mathfrak{p}_i$-primary for some prime ideals $\mathfrak{p}_i$ of A. Then

$$\operatorname{Ass}(M/N) = \{\mathfrak{p}_1, \ldots \mathfrak{p}_n\}.$$

Proof. We start by showing that each associated prime is one of the radicals in the decomposition. Let $\mathfrak{p} \in \operatorname{Ass}(M/N)$, which means that there exists some nonzero $m \in M/N$ such that

$$\mathfrak{p} \cdot m \subset N.$$

Now, we want to show that there is some N_i that is $\mathfrak{p}$-primary. We renumber the N_i's of the primary decomposition such that $m \notin N_i$ for $1 \leq i \leq j$, and $m \in N_i$ for $j+1 \leq i \leq r$. Since N_i is $\mathfrak{p}_i$-primary, we know that

$$\mathfrak{p}_i = \sqrt{\operatorname{Ann}(M/N_i)}.$$

Further, we know that N_i is finitely generated which means that there exists some $n_i \in \mathbb{N}$, such that

$$\mathfrak{p}_i^{n_i} M \subset N_i.$$

Now, this means
$$\left(\bigcap_{i=1}^{j} \mathfrak{p}_i^{n_i}\right) m \subseteq \bigcap_{i=1}^{r} N_i = N.$$
This is because for each i, all elements in $\mathfrak{p}_i^{n_i}$ multiplies m into N_i. So, if

$$r \in \left(\bigcap_{i=1}^{j} \mathfrak{p}_i^{n_i}\right),$$

then r multiplies m into all N_i's for $i = 1, \ldots, j$. For the rest of the $i = j+1, \ldots r$, we have that $m \in N_i$ so then of course $\mathfrak{p}_i^{n_i} \in N_i$, hence the inclusion holds. This gives us
$$\left(\bigcap_{i=1}^{j} \mathfrak{p}_i^{n_i}\right) \subset \mathrm{Ann}_{M/N}(m) = \mathfrak{p}.$$
Now we can apply our helpful Lemma 2.3.6 to get $\mathfrak{p}_i \subset \mathfrak{p}$ for some $i \in \{1, \ldots, j\}$. We want to show that in fact $\mathfrak{p} = \mathfrak{p}_i$ for this i. Let $a \in \mathfrak{p}$. Then, we know that $a \cdot m \in N$. Further, since $i \leq j$, we have that $m \notin N_i$. This means the map

$$\phi_a : M/N_i \to M/N_i$$

is not injective, and as N_i is primary, ϕ_a must be nilpotent. That means that $a^{n_i} M \subset N_i$ for some $n_i \in \mathbb{N}$. In other words,

$$a \in \sqrt{\mathrm{Ann}(M/N_i)} = \mathfrak{p}_i.$$

This means $\mathfrak{p} = \mathfrak{p}_i$, so any associated prime to N must be one of the $\mathfrak{p}_i$'s.

For the converse we want to show that any of the $\mathfrak{p}_i$'s is an associated prime. Since the numbering does not matter, we can just choose $i = 1$. Recall that our decomposition is minimal, which means N_1 is not contained in the intersection of the other N_i's. This means we can choose $m \in (N_2 \cap \cdots \cap N_r) \setminus N_1$. Since N_1 is $\mathfrak{p}_1$-primary we know that

$$\mathfrak{p}_1^n \cdot m \ \subset N_1$$

and $\mathfrak{p}_1^{n-1} \cdot m \not\subset N_1$ for some $n \geq 1$. If $n = 1$, we take $\mathfrak{p}_1^0$ to be the entire ring R. Now, if we take
$$y \in (\mathfrak{p}_1^{n-1} \cdot m) \setminus N_1,$$
then $y \notin N$. If we can show that $\mathfrak{p}_1 = \mathrm{Ann}_{M/N}(y)$, the proof is done, since this is exactly what it means to be an associated prime. We have

$$\mathfrak{p}_1 \cdot y \subset \mathfrak{p}_1^n \cdot m \subset N_1,$$

and, further we chose $m \in \bigcap_{i=2}^{r} N_i$, which of course means $\mathfrak{p}_i^n \cdot m \subset \bigcap_{i=2}^{r} N_i$. However, this gives us
$$\mathfrak{p}_1^n \cdot m \subset \bigcap_{i=1}^{r} N_i = N,$$

hence $\mathfrak{p}_1 \ \subset \mathrm{Ann}_{M/N}(y)$.

For the converse we must show that $\mathrm{Ann}_{M/N}(y) \subset \mathfrak{p}_1$, so assume $a \in \mathrm{Ann}_{M/N}(y)$, thus $a \cdot y \in N$. However, then $a \cdot y \in N_1$, and $y \notin N_1$, which means the map

$$\phi_a : M/N_1 \to M/N_1$$

is not injective. Then, ϕ_a must be nilpotent, hence $a \in \sqrt{\mathrm{Ann}(M/N_1)} = \mathfrak{p}_1$, moreover $\mathfrak{p}_1 = \mathrm{Ann}_{M/N}(y)$. This finishes the proof. ■

This means that the associated primes of M/N are exactly the ones that make up the prime ideals that the components of a primary decomposition has to be primary to. Determining a primary decomposition can then be simplified by first computing the associated primes and vice versa. This lets us show the analogue of the first uniqueness theorem for modules.

Theorem 2.3.8 ([Ash, Theorem 1.3.10])**.** *Let M be a finitely generated module over a Noetherian ring R. If N is a submodule with the primary decomposition*

$$N = \bigcap_{i=1}^{r} N_i,$$

where N_i is $\mathfrak{p}_i$ primary for a prime ideal $\mathfrak{p}_i$ for $i = 1, \dots, r$, then these prime ideals $\mathfrak{p}_i$ are uniquely determined by the submodule N.

Proof. A reduced primary decomposition of N is given by $\bigcap_{i=1}^{r} N_i/N$ where N_i/N is $\mathfrak{p}_i$-primary for $1 \leq i \leq r$. Then, by Theorem 2.3.7 we have that

$$\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \dots \mathfrak{p}_r\}.$$

However, the associated primes of M/N are determined by N, which of course by the above equation means that the $\mathfrak{p}_i$'s are determined by N. ■

The following result is quite useful when determining whether submodules are $\mathfrak{p}$-primary to some prime ideal $\mathfrak{p}$.

Corollary 2.3.9 ([Ash, Corollary 1.3.11])**.** *Let N be a submodule of M, both finitely generated over a Noetherian ring R. Then, N is $\mathfrak{p}$-primary for a prime ideal $\mathfrak{p}$ if and only if*

$$\mathrm{Ass}(M/N) = \{\mathfrak{p}\}.$$

Proof. Assume N is primary, then a minimal primary decomposition of N is just N itself. We also know $\mathfrak{p} = \sqrt{\mathrm{Ann}(M/N)}$ for some prime ideal $\mathfrak{p}$. From Theorem 2.3.7 we know that $\mathfrak{p}$ is in fact the only associated prime of M/N. For the converse assume that $\mathfrak{p}$ is the only associated prime of M/N, then again by Theorem 2.3.7 we get that N is $\mathfrak{p}$-primary. ■

We include the second uniqueness theorem for good measure. Proving this will lead us quite far astray, so we omit the proof and refer to [Ash].

Theorem 2.3.10 ([Ash, Theorem 1.4.5])**.** *Let M be a finitely generated R-module where R is a Noetherian ring. Suppose that*

$$N = \bigcap_{i} N_i$$

is a minimal primary decomposition of the submodule N, and that N_i is $\mathfrak{p}_i$-primary for $i = 1, \dots, r$. If $\mathfrak{p}_i$ is minimal, then N_i is uniquely determined by N.

The following result is often very useful to determine the associated primes of a module. For the proof we refer to [Ash].

Proposition 2.3.11 ([Ash, Proposition 1.5.6]). *If M is a finitely generated module of a Noetherian ring R, then*

$$\bigcap_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p} = \sqrt{\mathrm{Ann}(M)}.$$

2.4 The associated primes of a differential module

Our conjecture is that the isolated components of $\mathfrak{a}$ are one-to-one with the isolated components of $\Omega_{\mathfrak{a}}$ for an ideal $\mathfrak{a} \subset A$. We have not been able to prove that they are the same, but we can at least show that each primary component of $\mathfrak{a}$ will be included in one of the components of $\Omega_{\mathfrak{a}}$.

Proposition 2.4.1. *Let A be a Noetherian ring and assume $\mathfrak{a} \subset A$ is an ideal. Assume $\mathfrak{a}$ has a primary decomposition*

$$\mathfrak{a} = \bigcap_i \mathfrak{q}_i,$$

such that each $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary for a prime ideal $\mathfrak{p}_i$. Further, assume that we have a primary decomposition of $\Omega_{\mathfrak{a}}$

$$\Omega_{\mathfrak{a}} = \bigcap_j N_j,$$

where each N_j is $\mathfrak{b}_j$-primary for a prime ideal $\mathfrak{b}_j$. Then each $\mathfrak{p}_i$ is contained in one of the $\mathfrak{b}_j$'s.

Proof. Assume we are in the situation above. Since there is a nonzero element $r \in A/\mathfrak{a}$ such that

$$\mathfrak{p}_i \cdot r \in \mathfrak{a},$$

then we should certainly also have

$$\mathfrak{p}_i \cdot rdg \in \Omega_{\mathfrak{a}}$$

for any $g \in A$, since then $\mathfrak{p}_i \cdot r \subset \mathfrak{a}$. But then

$$\mathfrak{p}_i \cdot rdg \subset \mathfrak{a} \cdot \Omega_A.$$

This means that

$$\mathfrak{p}_i \subset \mathrm{Ann}_{\Omega_A/\Omega_{\mathfrak{a}}}(rdg),$$

so $\mathfrak{p}$ is at least contained in a prime associated to $\Omega_A/\Omega_{\mathfrak{a}}$, in other words, one of the $\mathfrak{b}_j$'s. ∎

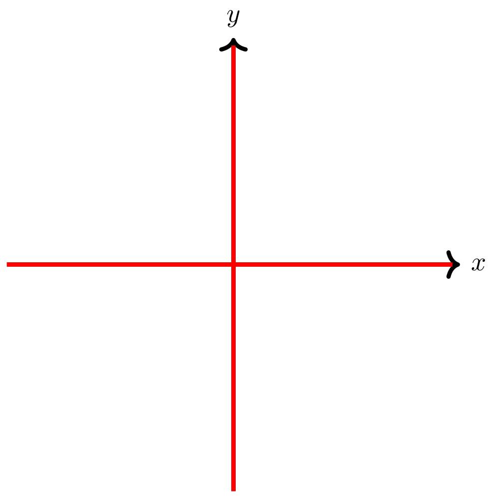

Figure 2.2: The axes in $\mathbb{A}^2$.

We conjecture in general that the isolated components of each decomposition are equal, but that the omega module can contain more embedded components that the ideal does not have. Let us look at an example where we can show this.

Example 2.4.2. Let $A = k[x, y]$, where k is an algebraically closed field, and consider the ideal

$$\mathfrak{a} = (xy).$$

Geometrically we know that this corresponds to the union of the x- and y-axis in the plane, $\mathbb{A}^2$. From Proposition 1.4.1 we know that the kernel of

$$\Omega_{k[x,y]} \to \Omega_{k[x,y]/(xy)}$$

is

$$\begin{aligned}\Omega_{(xy)} &= (xy) \cdot \Omega_A + d(xy) \cdot A \\ &= (xy) \cdot \Omega_A + (xdy + ydx) \cdot A.\end{aligned}$$

We regard this as a $k[x, y]$-module in the canonical way, so we can multiply with elements from A. Now, the primary decomposition of (xy) is simply

$$(xy) = (x) \cap (y),$$

and these components clearly have different radicals that do not include each other, so they are both isolated components. The ideals are both prime, so their radicals equal the ideals themselves([AM69, Proposition 1.14]). Thus, the associated primes are

$$\text{Ass}(k[x, y]/(xy)) = \{(x), (y)\}.$$

From Proposition 2.4.1, we know that these two primes should also be associated to the corresponding omega module, $\Omega_A/\Omega_{(xy)}$. This is not hard to see, let us show this first for (y).

Consider the element

$$\omega = xdx \in \Omega_A. \tag{2.3}$$

From arguments made earlier ω cannot be in $\Omega_{\mathfrak{a}}$. When considering

$$\operatorname{Ass}(\Omega_A/\Omega_{\mathfrak{a}}),$$

we have the relation

$$(xdy + ydx). \tag{2.4}$$

We will make a choice to 'disregard' all elements on the form

$$ydx,$$

and replace these with

$$-xdy.$$

This is legal since we are working in

$$\Omega_{k[x,y]}/\Omega_{(xy)}.$$

It is clear that the only way to multiply Equation (2.3) into $\Omega_{\mathfrak{a}}$ is by pushing it into

$$(xy) \cdot \Omega_A,$$

since it can never become something of the form $(xdy + ydx)$ with just multiplication from $k[x, y]$. Then, we have

$$\operatorname{Ann}_{\Omega_A/\Omega_{\mathfrak{a}}}(xdx) = (y),$$

which means that

$$(y) \in \operatorname{Ass}(\Omega_A/\Omega_{\mathfrak{a}}).$$

To show that

$$(x) \in \operatorname{Ass}(\Omega_A/\Omega_{\mathfrak{a}}),$$

we consider the nonzero element $ydy \in \Omega_A/\Omega_{\mathfrak{a}}$. Now, by the same arguments as above the only way to multiply this into $\Omega_{\mathfrak{a}}$ is by some element in (x), which means

$$\operatorname{Ann}_{\Omega_A/\Omega_{\mathfrak{a}}}(ydy) = (x).$$

So (x) is also an associated prime. However, there should also be a third component, which is (x, y)-primary. In other words, we should have that

$$\operatorname{Ass}(\Omega_A/\Omega_{\mathfrak{a}}) = \{(x), (y), (x, y)\}.$$

This is where our choice of xdy in the relation Equation (2.4) comes into play. Consider the element

$$xdy \in \Omega_A.$$

Now this is clearly 'killed' by all of (y), but by observing that

$$xdy = -ydx,$$

we also see that it is killed by all of (x), which means

$$\operatorname{Ann}_{\Omega_A/\Omega_{\mathfrak{a}}}(xdy) = (x, y).$$

To conclude, now we have shown that

$$\mathrm{Ass}(\Omega_A/\Omega_{\mathfrak{a}}) \supset \{(x), (y), (x, y)\}.$$

We show that this is in fact an equality, so assume that $\mathfrak{p}$ is some associated prime of $\Omega_A/\Omega_{\mathfrak{a}}$ which is not in

$$\{(x), (y), (x, y)\}.$$

This means that

$$\mathfrak{p} = \mathrm{Ann}_{\Omega_A/\Omega_{\mathfrak{a}}}(\omega)$$

for some $\omega \in \Omega_A$. We also have that $(x), (y), (x, y)$ annihilates some elements in $\Omega_A/\Omega_{(xy)}$, which means

$$(xy) = (x) \cap (y) \cap (x, y) \supset \mathrm{Ann}(\Omega_A/\Omega_{(xy)}).$$

The converse, that $(xy) \ \in \mathrm{Ann}(\Omega_A/\Omega_{(xy)})$ is clear, so

$$(xy) = \mathrm{Ann}(\Omega_A/\Omega_{(xy)}).$$

Of course, we must have $\mathfrak{p} \supset \mathrm{Ann}(\Omega_A/\Omega_{(xy)})$, thus

$$\mathfrak{p} \supset (xy).$$

So the only possibility is that $\mathfrak{p}$ contains (xy). Since we are in $k[x, y]$ we need only check primes of the form (f) for some irreducible f, and maximal ideals $(x - \alpha, y - \beta)$ for $\alpha, \beta \in k$. Assume first that $\mathfrak{p} = (f)$ for some irreducible $f \in k[x, y]$. We have $(xy) \subset (f)$, in other words, $xy = f \cdot h$ for some $h \in A$. However, this implies either $f \in (x)$ or $f \in (y)$, which contradicts our assumption that

$$\mathfrak{p} \not\subset \{(x), (y), (x, y)\}.$$

Therefore, assume that $\mathfrak{p} = (x - \alpha, y - \beta)$ for some $\alpha, \beta \in k$. We can assume that at least one of α and β is nonzero. Further, if both of them are nonzero, then it is impossible that

$$(xy) \subset (x - \alpha, y - \beta).$$

Hence, we assume that $\alpha = 0$. So, we have some element ω such that $(x, y - \beta) \cdot \omega \in \Omega_{(xy)}$. We assume that $\omega \notin \Omega_{(xy)}$. Then, modulo $\Omega_{(xy)}$, we can write it on the form

$$\omega = (a_0 + a_1 x + \cdots + a^n x^n)dx + (b_0 + b_1 y + \cdots + b^n y^n)dy + (c_1 y dx + c_2 x dy),$$

for some $a_i, b_i, c_i \in k$. This is possible because each time we have terms on the form $f \cdot (xy)dx$ or $f \cdot (xy)dy$ for some $f \in A$, we can remove them since they are already in $\Omega_{(xy)}$. We can also remove terms on the form y^2dx, x^2dy, since

$$y^2dx = y \cdot (ydx + xdy) - (xy)dx$$
$$x^2dy = x \cdot (ydx + xdy) - (xy)dy,$$

so these are both in $\Omega_{(xy)}$. Now, we have that $(x, y - \beta) \cdot \omega \in \Omega_{(xy)}$, so $(y - \beta)$ should multiply this element into $\Omega_{(xy)}$. We have

$$-\beta\omega = -\beta(a_0 + a_1 x + \cdots + a_n x^n)dx - \beta(c_1 y dx)$$

$$- \beta(b_0 + b_1 y + \dots b_n y^n)dy - \beta(c_2 x dy)$$

which should be in $\Omega_{(xy)}$. Looking at the degree zero part of this, there is no way to choose a_0 and b_0 such that $-\beta(a_0 dx + b_0 dy) \in \Omega_{(xy)}$ without choosing $a_0 = b_0 = 0$, since $\Omega_{(xy)}$ has no elements of degree zero. There are also no terms of degree zero coming from $y \cdot \omega$ because y is of degree 1, and only the constants are units in A. Then, we must let $a_0 = b_0 = 0$, which means there are also no terms of degree 1 in $y \cdot \omega$. We see that there is no way to choose a_1, b_1 nonzero such that

$$-\beta(a_1 x dx + b_1 y dy) + c_1 y dx + c_2 x dy = (xy)(u dx + v dy) + w(x dy + y dx).$$

This is because there is no way to get terms of degree 1 on the right side of the form xdx or ydy, so we must have $a_1 = b_1 = 0$. Using this argument inductively on i in a_i, b_i, we get that $a_i = b_i = 0$ for all i, which leaves us with

$$-\beta\omega = -\beta(c_2 x dy + c_1 y dx) \in \Omega_{(xy)}.$$

However, this implies either $c_2 \in (y)$ and $c_1 \in (x)$ or $c_1 = c_2$, and both of these cases gives us that

$$\omega = (c_2 x dy + c_1 y dx) \in \Omega_{(xy)},$$

which is a contradiction. Now we have proven that the claim holds when $\alpha = 0$, however, the case when $\beta = 0$ is completely symmetrical. Thus, there are no more associated primes to $\Omega_{k[x,y]}/\Omega_{(xy)}$, i.e.

$$\mathrm{Ass}(\Omega_A/\Omega_{\mathfrak{a}}) = \{(x), (y), (x, y)\}.$$

This equality is confirmed by the computations in Example A.2.1. Intuitively it makes sense that these are the only components, because we have only one point of intersection and two components. Each isolated component, (x) and (y), corresponds to a component in the omega module, and there is one embedded component, which corresponds to the intersection of these.

CHAPTER 3

Computing the decomposition of the omega module

Now that we have a good grasp on primary decompositions of modules we would like to decompose the kernel of the naturally induced map

$$\Omega_A \ \to \Omega_{A/\mathfrak{a}},$$

where A is a ring and $\mathfrak{a}$ an ideal of this ring. The goal of this thesis is examining any relations between the primary decomposition of the ideal

$$\mathfrak{a} = \bigcap_i^n \mathfrak{q}_i,$$

and the primary decomposition of

$$\Omega_{\mathfrak{a}} = \bigcap_i^m N_i.$$

From Theorem 2.3.1, we know that such decompositions exist.

In this chapter we conjecture that the isolated components of the two are the same. We can check this by computing the associated primes of each and compare these. For this thesis we will focus on curves in $\mathbb{A}^2$, but we expect that our findings can be extended to the general case. We will then propose a general primary decomposition for a specific type of curve.

3.1 An embedded component

Let $\mathfrak{p} = (f)$ and $\mathfrak{q} = (g)$ be two ideals of the polynomial ring of two variables $R = k[x, y]$. If we assume these to be prime and to not have common components, then the primary decomposition of $(f)(g)$ is just

$$(fg) = (f)(g) = \mathfrak{p} \cap \mathfrak{q} = (f) \cap (g).$$

We would like to get a grasp of how the differential modules behave with such products of ideals. Note that geometrically this corresponds to union of curves in $\mathbb{A}^2$. Therefore, understanding this will give us information about how the tangents of two curves in $\mathbb{A}^2$ behave when taking the union of them. We also get more information about the cotangent sheaf, which is a very important structure in algebraic geometry. The cotangent sheaf, among other things, gives rise to the canonical sheaf. One might think that 'applying omega' to the product of the ideals we would get

$$\Omega_{(fg)} = \Omega_{(f)} \cap \Omega_{(g)},$$

but this is not the case in general. When we say 'applying omega' to an ideal we will mean the submodule

$$\Omega_{\mathfrak{a}} = \mathfrak{a} \cdot \Omega_A + d\mathfrak{a}.$$

If the two curves intersect in a point we get a third embedded component corresponding to the intersection. So we would like to prove that

$$\Omega_{(fg)} \neq \Omega_{(f)} \cap \Omega_{(g)}$$

when (f) and (g) intersect at a point. For simplicity we will always assume that this intersection point is the origin, but one could make a linear change of coordinates to move this point anywhere in the plane.

Example 3.1.1. A good example to keep in mind is $f = x$ and $g = y$ in $R = k[x, y]$. Then, geometrically $(fg) = (xy)$ corresponds to the axes in the plane. By definition, we have that

$$\Omega_{(xy)} = (xy) \cdot \Omega_R + (xdy + ydx) \cdot R.$$

Consider now the element

$$xdy \in \Omega_R.$$

Note that this is in $(x) \cdot \Omega_R$ and $(dy) \cdot R$, which are subsets of $\Omega_{(x)}$ and $\Omega_{(y)}$, respectively. This means that $xdy \in \Omega_{(x)} \cap \Omega_{(y)}$. We see that $xdy \notin \Omega_{(xy)}$ because the only way to get an element of degree one is to take

$$r \cdot (xdy + ydx)$$

for $r \in k$. However, then we have the element $xdy + ydx$ and have to rid ourselves of ydx by adding an element from $(xy) \cdot \Omega_R$, but it is impossible to write

$$ydx = (xy)(udx + vdy)$$

for any $u, v \in R$. This is because the right side has at least degree 2 and the left side has degree equal to 1. Hence,

$$\Omega_{(xy)} \neq \Omega_{(x)} \cap \Omega_{(y)}.$$

Let us prove this generally.

Lemma 3.1.2. *Let $f, g \in k[x, y]$ be two polynomials that do not share common components. Then,*

$$\Omega_{(fg)} \subseteq \Omega_{(f)} \cap \Omega_{(g)}.$$

Furthermore, if f and g are irreducible and their corresponding curves intersect in at least one point, then we have

$$\Omega_{(fg)} \subsetneq \Omega_{(f)} \cap \Omega_{(g)}.$$

Proof. We begin by showing that

$$\Omega_{(fg)} = (fg) \cdot \Omega_R + R \cdot (fdg + gdf)$$

is contained in the intersection, i.e. contained in both $\Omega_{(f)}$ and $\Omega_{(g)}$. Note that $(fg) \subset (f)$ and $(fg) \subset (g)$. We have from Proposition 1.3.1 that

$$\Omega_{(fg)} \subset \Omega_{(f)}$$
$$\Omega_{(fg)} \subset \Omega_{(g)},$$

which of course implies

$$\Omega_{(fg)} \subset \Omega_{(f)} \cap \Omega_{(g)}.$$

Now, assume that f and g are irreducible, and assume their curves intersect in a point. We want to show that the inclusion above is proper. To do this, we find an element in the intersection $\Omega_{(f)} \cap \Omega_{(g)}$ that is not in $\Omega_{(fg)}$. One candidate is $f \cdot dg$, which is clearly in the intersection since $fdg \in (f) \cdot \Omega_R$ and $fdg \in R \cdot (dg)$. We show that it is not in $\Omega_{(fg)}$. To be in $\Omega_{(fg)}$ it must be of the form

$$fdg = afgdx + bfgdy + c(fdg + gdf)$$

for some polynomials $a, b, c \in R$. We have

$$fdg = f(agdx + bgdy + cdg) + cgdf$$

where the left side is divisible by f, so the right side must also be in (f), hence $c \in (f)$, since f and g do not have common components. Then, we get

$$dg = (agdx + bgdy + c'fdg + c'gdf)$$

$$g_x dx + g_y dy = g(adx + bdy + c'f_x dx + c'f_y dy) + c'fdg$$

for some $c' \in k[x, y]$. Observe that evaluating this at $(0, 0)$, we get zero on the right side, since we have assumed that f and g intersect in the origin. However, we have assumed g to be smooth, so both g_x and g_y cannot be zero in the origin, which means we have a contradiction. ■

Let us see an example of the proper inclusion and an example of equality.

Example 3.1.3. Consider $f = y$ and $g = y - x^2$ in $R = k[x, y]$, which as we have seen corresponds to the graph in Figure 2.1. It is clear that these two curves only intersect in the origin, which means Lemma 3.1.2 applies. We have the element

$$\omega = yd(y - x^2) = ydy - 2xydx \in \Omega_R.$$

Note that $d((y - x^2)) = (dy - 2xdx) \cdot k[x, y]$, and that we can write

$$\omega = ydy - 2xydx = y(dy - 2xdx),$$

which means that ω is in $\Omega_{(y-x^2)}$, because

$$\Omega_{(y-x^2)} = (y - x^2) \cdot \Omega_R + R \cdot (dy - 2xdx).$$

Also, ω is trivially in $\Omega_{(y)}$, hence it is in the intersection. However, it is not in

$$\Omega_{(y^2-yx^2)} = (y^2 - yx^2) \cdot \Omega_{k[x,y]} + (-2xdx + (2y - x^2)dy) \cdot k[x, y].$$

We can see this by observing that the only way to get the term ydy in $\Omega_{(y^2-yx^2)}$ is by

$$\frac{1}{2}\cdot(-2xdx+(2y-x^2)dy) = xdx + \left(y - \frac{x^2}{2}\right)dy.$$

However, now we have to remove $xdx - \frac{x^2}{2}dy$, which is impossible by any choice of $u, v \in k[x, y]$ in

$$(y^2 - yx^2)(udx + vdy),$$

since all terms here have degree at least 2, and we have a term of degree 1. We can also see this by the argument in our proof of this lemma, assuming

$$y(dy - 2xdx) = (y^2 - yx^2)(udx + vdy) + w(-2xdx + (2y - x^2)dy),$$

for some $u, v, w \in k[x, y]$. But, $w \in (y)$ by the same argument as in the proof, and then

$$(dy - 2xdx) = (y - x^2)(udx + vdy) + w'(-2xdx + (2y - x^2)dy).$$

Evaluating this equation in the origin we get

$$dy = 0,$$

a contradiction.

Example 3.1.4. Let $f = y$ and $g = y - 1$, which in $\mathbb{A}^2$ are two parallel lines that never intersect. The curves of these polynomials are depicted in Figure 3.1. Our claim is that $\Omega_{(y^2-y)} = \Omega_{(y)} \cap \Omega_{(y-1)}$. We have already shown

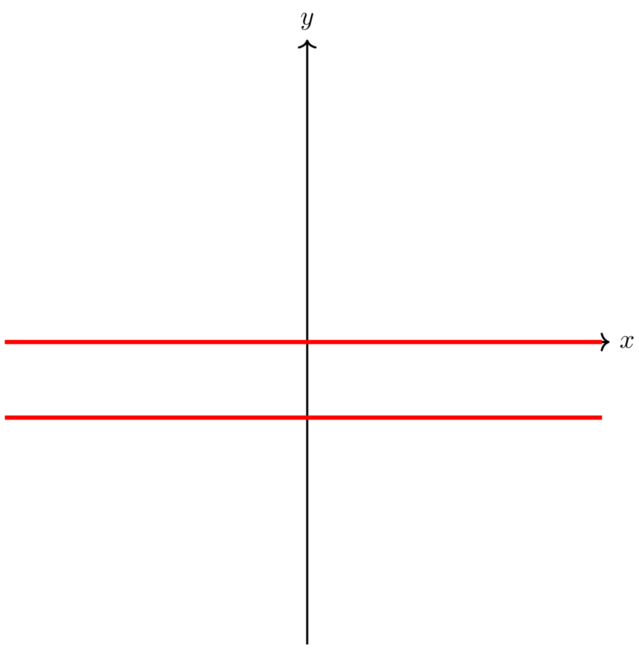

Figure 3.1: The two lines defined by y and $y - 1$.

$$\Omega_{(y^2-y)} \subset \Omega_{(y)} \cap \Omega_{(y-1)}$$

generally in Lemma 3.1.2. The other inclusion is computed in Macaulay2, in Example A.2.3.

The example showcases why we need to assume that we have intersecting curves when stating proper inclusion in Lemma 3.1.2.

This was the first step in showing that applying omega to a product of ideals has an embedded component corresponding to the intersection of the two subschemes. Next we show that $\Omega_{(f)}$ is in fact (f)-primary. Then, we know that $\Omega_{(g)}$ is also (g)-primary, and that $\Omega_{(fg)}$ is contained in the intersection of two primary components. This in turn will mean that $\Omega_{(fg)}$ must have at least a third associated prime, hence a third primary component. The number of additional components is completely determined by the amount of intersections between the two closed subschemes in $\mathbb{A}^2$.

3.2 Curves that give rise to primary modules

For a given ideal $\mathfrak{a}$, if we have that $\Omega_{\mathfrak{a}}$ is primary, we say that $\mathfrak{a}$ is a *ω-primary ideal.* We conjecture that irreducible ideals are ω-primary ideals. In fact, from our computations in Macaulay2 we are led to believe that even primary ideals are ω-primary, but proving this is outside the scope of this thesis.

Conjecture 3.2.1. *Let $f \in k[x, y]$ be an irreducible and smooth polynomial. Then,*

$$\Omega_{(f)} = (f) \cdot \Omega_{k[x,y]} + (df) \cdot k[x, y]$$

is a (f)-primary module, in other words (f) is a ω-primary ideal.

We will show that (f) is at least contained in all associated primes. The claim is that

$$\sqrt{\operatorname{Ann}\left({}^{\Omega_R}\!\diagup_{\Omega_{(f)}}\right)} = (f).$$

Because of Proposition 2.3.11, this means that the intersection of all associated primes is equal to (f), which of course means they all contain (f). It is clear that (f) is contained in the left side since $(f) \cdot \Omega_R \subset \Omega_{(f)}$, so (f) kills all elements of Ω_R. To show the converse, assume that

$$r \in \sqrt{\operatorname{Ann}\left({}^{\Omega_R}\!\diagup_{\Omega_{(f)}}\right)}$$

which means that r^n multiplies all of Ω_R into $\Omega_{(f)}$. Let $adx+bdy$ be an arbitrary element of Ω_R. Since r^n kills all elements of Ω_R, we can let $b = 0$ and let a be free for the moment. We get

$$\begin{aligned} r^n(adx) &= ufdx + vfdy + w\left(f_x dx + f_y dy\right) \\ r^n a &= uf + w \cdot f_x \\ 0 &= vf + w \cdot f_y \end{aligned}$$

for some $u, v, w \in k[x, y]$. Since f is irreducible we know that f_y can not divide f which means that w must divide f, in other words $w = fw'$ for some $w' \in R$. Then we are left with

$$r^n a = uf + w'f \cdot f_x$$

$$r^n a = f\left(u + w' \cdot f_x\right).$$

However, the right side is a multiple of f, hence the left side must also be a multiple of f. This means that in general r^n must be a multiple of f since we can just choose $a \notin (f)$. So, we have $r^n \in (f)$, hence

$$\sqrt{\operatorname{Ann}\left(\Omega_R/\Omega_{(f)}\right)} = (f).$$

Even so, this does not mean that $\Omega_{(f)}$ is primary. For it to be primary we must show that every zero-divisor in

$$\Omega_R/\Omega_{(f)}$$

is nilpotent. For now let us show that (f) is in fact an associated prime. From Proposition 2.3.11 we see that if there are other associated primes, these must contain (f). However, by basic commutative algebra (f) can only be contained in primes that are maximal. Consider the element

$$dx \in \Omega_{k[x,y]}/\Omega_{(f)}.$$

Now, we know that the only relations on this module is

$$(f) \cdot (adx + bdy) + c(f_x dx + f_y dy) = 0,$$

for all $a, b, c \in k[x, y]$. Assume $f_x \notin k$, which means that there are no relations on dx, because there are no other ways to get dx from our aforementioned relations. Then, it is clear that the only way to multiply dx into $\Omega_{(f)}$ is by multiplying by (f), which means

$$\operatorname{Ann}_{\Omega_{k[x,y]}/\Omega_{(f)}}(dx) = (f),$$

which means that

$$(f) \ \in \operatorname{Ass}(\Omega_{k[x,y]}/\Omega_{(f)}).$$

If however, $f_x \in k$, then we have a relation

$$dx = -\frac{f_y}{f_x} dy$$

modulo (df). Now if $f_y \in k$, then we still get

$$\operatorname{Ann}_{\Omega_{k[x,y]}/\Omega_{(f)}}(dx) = (f),$$

but of course we get

$$\operatorname{Ann}_{\Omega_{k[x,y]}/\Omega_{(f)}}(dy) = (f)$$

as well, since dx and dy are equal up to multiplication by a constant in this case. If $f_y \notin k$, we just consider dy instead and by the same argument as for dx we get

$$\operatorname{Ann}_{\Omega_{k[x,y]}/\Omega_{(f)}}(dy) = (f).$$

This means that (f) is some prime associated to $\Omega_{(f)}$. To show that it is the only associated prime, we must somehow use that f is a smooth polynomial, because the statement does not hold for singular curves. We will see why we need smoothness in Example 3.2.3. Let us see some examples.

Example 3.2.2. Let $f = x \in k[x,y]$, which geometrically just corresponds to the y-axis. We show that

$$\mathrm{Ass}\left(\Omega_{k[x,y]}/\Omega_{(x)}\right) = \{(x)\},$$

in other words that $\Omega_{(x)}$ is (x)-primary. As we argued earlier, by results in commutative algebra and Proposition 2.3.11, if there are other associated primes, then (x) is contained in these. Thus, any other associated prime is of the form

$$\mathfrak{m}_{(x,y-\beta)} = (x, y-\beta)$$

for some $\beta \in k$. Assume that this is the case, so there is some element

$$\omega = \omega_1 dx + \omega_2 dy \in \Omega_{k[x,y]}$$

such that

$$(x, y-\beta) = \mathrm{Ann}_{\Omega_{k[x,y]}/\Omega_{(f)}}(\omega).$$

So, for all $a, b \in k[x,y]$,

$$(ax + b(y-\beta))(\omega_1 dx + \omega_2 dy) = x(udx + vdy) + w(dx)$$

for some $u, v, w \in k[x,y]$. Splitting this equation, we get

$$\begin{aligned}(ax + b(y-\beta))\omega_1 &= xu + w \\ (ax + b(y-\beta))\omega_2 &= xv.\end{aligned} \tag{3.1}$$

Now, let $a = 0$. Then Equation (3.1) becomes

$$b(y-\beta)\omega_2 = xv,$$

hence $\omega_2 \in (x)$. Then, we can rewrite our element

$$\omega_1 dx + \omega_2 dy = \omega_1 dx + \omega_2' x dy$$

for some $\omega_2' \in k[x,y]$. Now it is clear that $\omega \in \Omega_{(x)}$. This shows that the only elements in

$$\Omega_{k[x,y]}/\Omega_{(x)}$$

that are multiplied into $\Omega_{(x)}$ by all of

$$\mathfrak{m}_{(x,y-\beta)} = (x, y-\beta)$$

are the ones already in $\Omega_{(x)}$, which means that there are no nonzero elements in

$$\Omega_{k[x,y]}/\Omega_{(x)}$$

that gets multiplied to zero by any other prime ideal than (x), hence $\Omega_{(x)}$ is (x)-primary, i.e. (x) is ω-primary.

Example 3.2.3. A non-example is the singular nodal curve defined by

$$f = y^2 - x^3 - x^2.$$

We know that (f) is a prime ideal, but it is not smooth, which should mean it is not ω-primary. In Example A.2.2 we show that $\Omega_{(f)}$ has two components, one $(y^2 - x^3 - x^2)$-primary and one (x,y)-primary, since the curve intersects itself in the origin. This shows why we must assume (f) is smooth when we claim that (f) is ω-primary.

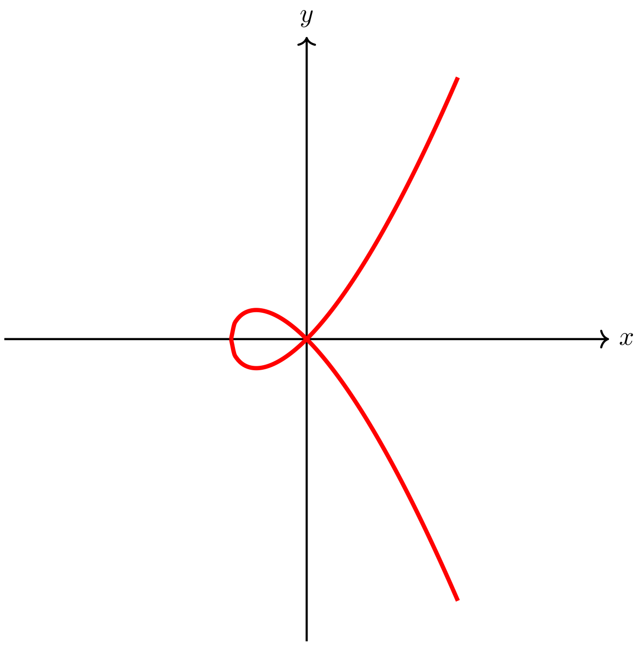

Figure 3.2: The nodal cubic defined by $(y^2 - x^2(x+1))$.

3.3 The last component of the decomposition

We have seen that there must be a third component in the decomposition of

$$\Omega_{(fg)}.$$

The third component we propose is

$$N_{(fg)} = \Omega_{(fg)} + (x^d dx, y^d dx), \tag{3.2}$$

where $d = deg(fg)$. The last generators are included to make the module (x, y)-primary. Many calculations of different examples of f and g in Macaulay2 has given us an idea of how $N_{(fg)}$ should look, which led us to Equation (3.2). Examples of such calculations can be found in Appendix A. We claim that raising x and y to the d-th power ensures that x^d and y^d can be written as

$$x^d = a_1 \cdot f + b_1 \cdot g$$
$$y^d = a_2 \cdot f + b_2 \cdot g$$

for some $a_i, b_i \in k[x, y]$. Since f and g only intersect in the origin, we know that

$$\sqrt{(f, g)} = (x, y).$$

This means that $x^{n_1} \in (f, g)$ and $y^{n_2} \in (f, g)$ for some $n_1, n_2 \in \mathbb{N}$. By Bezout's theorem, we know that f and g intersect $\deg(f)\deg(g)$ times, counting multiplicities. In our case, we have assumed they only intersect in the origin, therefore the multiplicity of this point is $\deg(f)\deg(g)$. This means that the $k[x, y]$-module

$$k[x, y]/(f, g)$$

has maximum length $\deg(f)\deg(g)$. Hence, $x^n \in (f,g)$ when $n = \deg(f)\deg(g)+1$. However, we know that $\deg(fg) > \deg(f)\deg(g)$ for all non-constant f and g, hence choosing $d = \deg(fg)$ is enough to get that $x^d \in (f,g)$ and $y^d \in (f,g)$.

As mentioned earlier, we assume that our curves only intersect in the origin. Therefore, we can just consider the cases where we have (x,y)-primary components, which means the module geometrically is in the origin.

Note that $N_{(fg)}$ is equal to the module we want to decompose, $\Omega_{(fg)}$, but with some extra generators. We must therefore show that adding these generators does not make the intersection contain other elements than $\Omega_{(fg)}$. The following lemma will be useful.

Lemma 3.3.1. *Let A, B and N be modules of the same ring R. Assume that $B \subset A$ and $N \cap A \subset B$. Then,*

$$A \cap (N+B) = B.$$

Proof. We begin by showing the right to left inclusion. Let $b \in B$. Since $B \subset A$, we have $b \in A$. Further, $B \subset N+B$, so $b \in N+B$, hence

$$b \in A \cap (N+B),$$

which proves the right to left inclusion. For the converse, let

$$a \in A \cap (N+B).$$

Then, we have $a \in N+B$, which means $a = n+b$ for some $n \in N$ and $b \in B$. Since A is a module and $b \in A$, we can subtract b from a and still get an element in A. We get

$$a - b = n \in A.$$

This is in N and A, i.e.

$$n \in A \cap N.$$

By assumption, we have $A \cap N \subset B$, so $n \in B$. Then, $a = n+b \in B$, hence $A \cap (N+B) \subset B$. This means we have equality between the two modules. ■

Having this lemma means that to show

$$\Omega_{(fg)} = \Omega_{(f)} \cap \Omega_{(g)} \cap N_{(fg)},$$

we need only show that

$$\Omega_{(f)} \cap \Omega_{(g)} \cap (x^d dx, y^d dy) \subset \Omega_{(fg)}.$$

To see this, let

$$\begin{aligned} R &= k[x,y], \\ B &= \Omega_{(fg)}, \\ A &= \Omega_{(f)} \cap \Omega_{(g)}, \\ N &= (x^d dx, y^d dy) \end{aligned}$$

in Lemma 3.3.1. Then we see that by adding

$$N = (x^d dx, y^d dy),$$

we still get the desired equality. For this to be the desired last component, it remains to show that

$$\Omega_{(f)} \cap \Omega_{(g)} \cap (x^d dx, y^d dy) \subset \Omega_{(fg)}.$$

We state the conjecture.

Conjecture 3.3.2. *Let f and g be two irreducible, smooth polynomials of $k[x, y]$ that do not share components. Then,*

$$\Omega_{(f)} \cap \Omega_{(g)} \cap (x^d dx, y^d dy) \subset \Omega_{(fg)},$$

where $d = \deg(fg)$.

We look at an example.

Example 3.3.3. Let $f = y$ and $g = x$, then

$$(fg) = (xy),$$

which corresponds to the axes in the plane. We need to show that

$$\Omega_{(y)} \cap \Omega_{(x)} \cap (x^2 dx, y^2 dy) \subset \Omega_{(xy)}.$$

Consider $\omega = \omega_1 x^2 dx + \omega_2 y^2 dy \in (x^2 dx, y^2 dy)$ and assume that

$$\omega \in \Omega_{(y)} \cap \Omega_{(x)}.$$

First, since $\omega \in \Omega_{(x)}$, we get the following:

$$\omega_1 x^2 dx + \omega_2 y^2 dy = (x)(u_1 dx + v_1 dy) + w_1 \cdot dx \tag{3.3}$$

for some $u_1, v_1, w_1 \in k[x, y]$. An equivalent argument for $\Omega_{(y)}$ gives us that

$$\omega_1 x^2 dx + \omega_2 y^2 dy = (y)(u_2 dx + v_2 dy) + w_2 \cdot dx \tag{3.4}$$

for $u_2, v_2, w_2 \in k[x, y]$. Splitting Equation (3.3) by the two generators of $\Omega_{k[x,y]}$ gives us

$$\omega_1 x^2 = x u_1 + w_1 \tag{3.5}$$
$$\omega_2 y^2 = x v_1. \tag{3.6}$$

Doing the same for Equation (3.4) gives

$$\omega_1 x^2 = y u_2 \tag{3.7}$$
$$\omega_2 y^2 = y v_2 + w_2. \tag{3.8}$$

Given these equations, we want to show that the element we started with is in

$$\Omega_{(xy)} = (xy) \cdot \Omega_{k[x,y]} + (xdy + ydx) \cdot k[x, y].$$

From Equation (3.6), we get that

$$\omega_2 \in (x) \text{ and } v_1 \in (y^2).$$

Furthermore, from Equation (3.7), we get that

$$\omega_1 \in (y) \text{ and } v_2 \in (x^2).$$

However, now we can rewrite the element we started with as

$$\begin{aligned}\omega_1 x^2 dx + \omega_2 y^2 &= h_1 y x^2 dx + h_2 x y^2 dy && (3.9)\\ &= xy(h_1 x dx + h_2 y dy) \in \Omega_{(xy)} && (3.10)\end{aligned}$$

for some $h_1, h_2 \in k[x, y]$. Thus,

$$\Omega_{(y)} \cap \Omega_{(x)} \cap (x^2dx, y^2dy) \subset \Omega_{(xy)}.$$

Now, using Lemma 3.3.1, we get that

$$\Omega_{(y)} \cap \Omega_{(x)} \cap (x^2dx, y^2dy) = \Omega_{(xy)},$$

which means we have a candidate for a primary decomposition of $\Omega_{(xy)}$. We only need to check that $N_{(fg)}$ is (x, y)-primary. A computation of this example can also be found in Example A.2.1.

Let us look at a more interesting example.

Example 3.3.4. Let $f = y - x^2$ and $g = x$ be two polynomials in $A = k[x, y]$. We claim that

$$\Omega_{(yx-x^3)} = \Omega_{(y-x^2)} \cap \Omega_{(x)} \cap N_{(yx-x^3)},$$

where $N_{(yx-x^3)} = \Omega_{(yx-x^3)} + (x^3dx, y^3dy)$, is a primary decomposition. We want to show that these two modules are equal, but we already know that $\Omega_{(yx-x^3)}$ is contained in the right side by Lemma 3.1.2, so it remains to show the converse. We can write the module

$$\Omega_{(y-x^2)} = A \cdot (y - x^2)dx + A \cdot (y - x^2)dy + A \cdot (dy - 2xdx),$$

where $A = k[x, y]$. However, the generator $A \cdot (y - x^2)dy$ is superfluous to generate $\Omega_{(fg)}$, since we can write

$$(y - x^2)dy = (y - x^2) \cdot (dy - 2xdx) + 2x \cdot (y - x^2)dx.$$

So, we can write the middle generator as a linear combination of the other two over A. This means we have

$$\Omega_{(y-x^2)} = A \cdot (y - x^2)dx + A \cdot (dy - 2xdx).$$

We push $\Omega_{(y-x^2)}$ into $\Omega_{(x)}$. Observe that the first generator, $A \cdot (y-x^2)dx \in \Omega_{(x)}$, but the generator $A \cdot (dy - 2xdx)$ is not in $\Omega_{(x)}$ in general, so we must multiply by x to push the dy part into $(x) \cdot \Omega_A$. So, we get

$$\Omega_{(y-x^2)} \cap \Omega_{(x)} = A \cdot (y - x^2)dx + A \cdot (xdy - 2x^2dx).$$

What remains is to intersect this module with (x^3dx, y^3dy). To do this, we must multiply with elements from A to make the coefficients of each dx in the generators be in (x^3), and analogously each dy coefficient be in (y^3). We see that to get $A \cdot (y-x^2)dx$ in $\Omega_{(x)}$ we must multiply with x^3, so this generator becomes

$A \cdot (x^3y - x^5)dx$. Similarly, for the other generator we get $A \cdot (xy^3)(xdy - 2x^2dx)$. To summarize, we now have

$$\Omega_{(y-x^2)} \cap \Omega_{(x)} \cap N_{(yx-x^3)} = A \cdot x^3(y-x^2)dx + A \cdot (xy^3)(xdy - 2x^2dx).$$

If we show that each of these generators are in $\Omega_{(xy-x^3)}$ we are done. Recall

$$\Omega_{(xy-x^3)} = A \cdot (xy-x^3)dx + A \cdot (xy-x^3)dy + A \cdot ((y-x^2)dx + x(dy - 2xdx)).$$

Clearly, $x^3(y-x^2)dx \in A \cdot (xy - x^3)dx$, so this generator is in $\Omega_{(xy-x^3)}$. For the other one, we can write

$$(xy^3)(xdy - 2x^2dx) = (xy^3)((y-x^2)dx + x(dy - 2xdx)) - y^3(xy - x^3)dx,$$

where we have written $(xy^3)(xdy - 2x^2dx)$ as an element of the generators of $\Omega_{(xy-x^3)}$. This means that

$$\Omega_{(yx-x^3)} \supset \Omega_{(y-x^2)} \cap \Omega_{(x)} \cap N_{(yx-x^3)},$$

moreover

$$\Omega_{(yx-x^3)} = \Omega_{(y-x^2)} \cap \Omega_{(x)} \cap N_{(yx-x^3)}.$$

We have computed more examples of this in Appendix A.3.

3.4 Showing the last component is primary

The remaining part of establishing that

$$\Omega_{(fg)} = \Omega_{(f)} \cap \Omega_{(g)} \cap N_{(fg)}$$

is a primary decomposition, is to show that $N_{(fg)}$ is primary. As we have stated before, we conjecture it to be (x, y)-primary. Let us see an example of this.

Example 3.4.1. Let $f = y$ and $g = x + y$. Geometrically the principal ideals these generate correspond to the x-axis and the line $y = -x$, respectively. An illustration is depicted in Figure 3.3.

In order to show that

$$N_{(x(x+y))} = \Omega_{(x^2+xy)} + (x^2dx, y^2dy)$$

is primary, let us first show that

$$x^d dy, y^d dx \in N_{(fg)}.$$

We first show that $x^d dy$ is in $N_{(fg)}$, so we want to take a general element of $N_{(x^2+xy)}$,

$$fg(udx + vdy) + w((fg_x + gf_x)dx + (fg_y + gf_y)dy) + \alpha \cdot x^d dx + \beta y^d dy$$

and choose u, v, w, α and β such that this element is equal to

$$x^d dy.$$

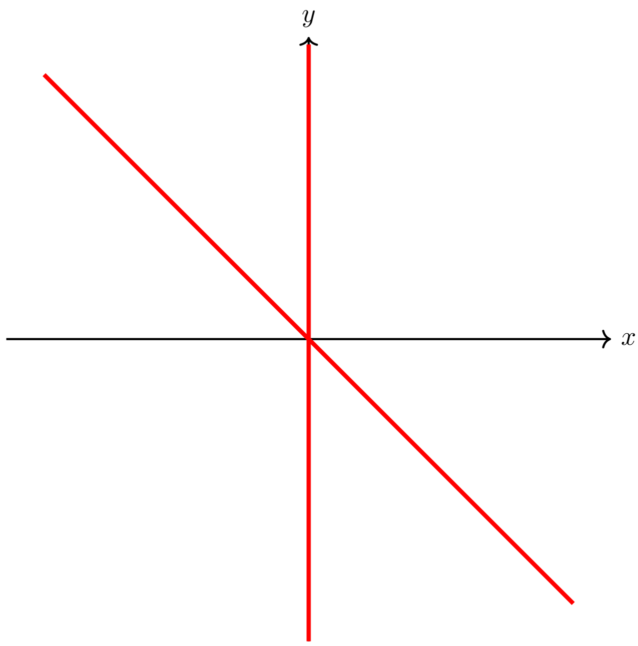

Figure 3.3: Zero set of $(yx + y^2)$.

We can split the computation into one computation for each generator of $\Omega_{k[x,y]}$. For dx, we want

$$fgu + w(fg_x + gf_x) + \alpha x^d = 0. \tag{3.11}$$

And for dy, we want

$$fgv + w(fg_y + gf_y) + \beta y^d = x^d. \tag{3.12}$$

This is for general f and g, so let us insert our example here. Equation (3.11) becomes

$$(yx + y^2)u + w(y) + \alpha x^2 = 0, \tag{3.13}$$

and Equation (3.12) becomes

$$\begin{aligned}(yx + y^2)v + w(y + (x + y)) + \beta y^2 &= x^2\\ (yx + y^2)v + w(x + 2y) + \beta y^2 &= x^2,\end{aligned} \tag{3.14}$$

since $d = \deg(yx + y^2) = 2$. However, these two curves only intersect in the origin, thus we know that

$$\sqrt{(y, x + y)} = (x, y), \tag{3.15}$$

which means that there exists some $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned}x^{N_1} &= a_1 y + b_1(x + y)\\ y^{N_2} &= a_2 y + b_2(x + y)\end{aligned}$$

for some $a_1, a_2, b_1, b_2 \in k[x, y]$. It will suffice to use $d = \deg(fg)$, by the arguments made before Lemma 3.3.1. In this simple example we actually do not need to use this fact though, we see from Equation (3.14) that the only way to get a term of the form x^2 is to choose

$$w = x + w',$$

for some $w' \in k[x, y]$, because all other terms have a factor of y. Inserting this fact into Equation (3.13), we get

$$(xy + y^2)u + (x + w')y + \alpha x^2 = 0,$$

and we see in order to remove the xy term here, we must choose $u = -1$, which gives us

$$(xy + y^2)(-1) + (x + w')y + \alpha x^2 = -y^2 + w'y + \alpha x^2.$$

Now, the only way to get rid of the term $-y^2$ is to choose

$$w' = y,$$

and then we can just let

$$\alpha = 0$$

to satisfy Equation (3.13). Now, back to Equation (3.14), the choice of

$$w = x + y$$

gives us

$$\begin{aligned} x^2 &= (xy + y^2)v + y(x + 2y) + x(x + 2y) + \beta y^2 \\ &= (xy + y^2)v + 3xy + 2y^2 + x^2 + \beta y^2, \end{aligned}$$

and we want to choose $v, \beta \in k[x, y]$ such that this holds. We want to remove the terms

$$3xy + 2y^2,$$

and we begin by choosing

$$u = -3$$

in Equation (3.14). Then we end up with the equality

$$(yx + y^2)(-3) + x^2 + 3xy + 2y^2 + \beta y^2 = x^2 - y^2,$$

and now we just need to remove the last term, $-y^2$, but this is simply done by choosing

$$\beta = -1,$$

and we have the desired equality Equation (3.14). We have now written

$$x^2 dy$$

as an element of $N_{(fg)}$, which means

$$x^2 dy \in N_{(fg)}.$$

Next we show that also

$$y^2 dx \in N_{(fg)}$$

in a similar fashion. So, we want

$$y^d dx = (xy + y^2)(udx + vdy) + w(ydx + (x + 2y)dy) + \alpha x^2 dx + \beta y^2 dy.$$

Splitting as we did before we get

$$y^2 = (xy + y^2)u + wy + \alpha x^2 \tag{3.16}$$

$$0 = (xy + y^2)v + w(x + 2y) + \beta y^2. \tag{3.17}$$

We would like to find some choice of $u, v, w, \alpha, \beta \in k[x, y]$ such that these equations hold. Consider Equation (3.16) first. There are two ways to get a term of y^2 on the right side. We try one of them and see if it works. The first is choosing

$$u = 1 + u'$$

for some $u' \in k[x, y]$. This gives us

$$(xy + y^2)(1 + u') + wy + \alpha x^2 = xy + y^2 + u'(xy + y^2) + wy + \alpha x^2,$$

and we want to remove the term xy, which we do by choosing

$$w = -x + w'$$

for some $w' \in k[x, y]$. Let us see how this fits into Equation (3.17). We get

$$(xy+y^2)v+(-x+w')(x+2y)+\beta y^2 = (xy+y^2)v+w'(x+2y)-x^2-2xy+\beta y^2,$$

but there is now no way to choose v or β to remove x^2. Therefore, this cannot be the right choice of w, and we have to try the other option. Again let all the coefficients be arbitrary and consider Equation (3.16) where we now let

$$w = y + w'.$$

Let us check how this fits into Equation (3.17):

$$(xy+y^2)v+(y+w')(x+2y)+\beta y^2 = (xy+y^2)v+xy+2y^2+w'(x+2y)+\beta y^2.$$

We can now let

$$v = -1,$$

and we get

$$\begin{aligned}&(xy + y^2)(-1) + xy + 2y^2 + w'(x + 2y) + \beta y^2\\ &= -y^2 + 2y^2 + w'(x + 2y) + \beta y^2\\ &= y^2 + \beta y^2 + w'(x + 2y),\end{aligned}$$

where we see that if we let

$$\beta = -1 \tag{3.18}$$

$$w' = 0. \tag{3.19}$$

Then, we get the desired equality

$$(xy + y^2)(-1) + y(x + 2y) + (-1)y^2 = 0.$$

Now, let us go back to Equation (3.16) and see if we can solve this. Recall that we chose $w = y$, thus we get

$$(xy + y^2)u + y^2 + \alpha x^2,$$

but now we can just let

$$u = 0 \quad (3.20)$$
$$\alpha = 0. \quad (3.21)$$

We have now written

$$y^2 dx,$$

of the form of an element of

$$N_{(fg)} = \Omega_{(xy+y^2)} + (x^2dx, y^2dy),$$

which means that

$$y^2 dx \in N_{(fg)}.$$

Now that we have both

$$y^2 dx \in N_{(fg)}$$
$$x^2 dy \in N_{(fg)}$$

we can use this to conclude that $N_{(fg)}$ is (x, y)-primary. Any element in $\Omega_{k[x,y]}$ is of the form

$$adx + bdy,$$

where $a, b \in k[x, y]$. Well, using that we have

$$\sqrt{\operatorname{Ann}\left(\Omega_{k[x,y]}/N_{(fg)}\right)} = (x, y)$$

and Proposition 2.3.11, we know that

$$\bigcap_{\mathfrak{p} \in \operatorname{Ass}(M/N_{(fg)})} \mathfrak{p} = (x, y),$$

where we have denoted $M = \Omega_{k[x,y]}$. However, now we have an intersection of prime ideals equal to a maximal ideal, which can only mean that the set of intersection is just the maximal ideal itself, i.e

$$\operatorname{Ass}(M/N_{(fg)}) = \{(x, y)\}.$$

From Corollary 2.3.9 we have that this means precisely that $N_{(fg)}$ is (x, y)-primary, which is what we wanted to show.

Proving this fact in general has shown to be much harder, and one would need to use the fact that the two curves $V(f)$ and $V(g)$ only intersect in the origin, in other words that

$$\sqrt{(f, g)} = (x, y).$$

We have not been able to prove this in this thesis, but have checked many cases in Macaulay2, see Appendix A.1 and Appendix A.3.

Conjecture 3.4.2. *Let $f, g \in k[x, y]$ be two polynomials such that their corresponding curves in $\mathbb{A}^2$ are smooth, irreducible and only intersect in the origin. Then it is our contention that the submodule of $\Omega_{k[x,y]}$;*

$$N_{(fg)} = \Omega_{(fg)} + (x^d dx, y^d dy)$$

where $d = \deg(fg)$, is (x, y)-primary.

Let us show another simple example.

Example 3.4.3. Let $f = x$ and $g = y$, which means we are geometrically considering the x- and y-axes. We want to show that

$$N_{(fg)} = \Omega_{(xy)} + (x^2 dx, y^2 dy)$$

is (x, y)-primary. Again we do this by showing that

$$(x^2 dy, y^2 dx) \in N_{(fg)}.$$

Recall that

$$\Omega_{(xy)} = (xy) \cdot \Omega_{k[x,y]} + (xdy + ydx) \cdot k[x, y].$$

Consider the equations

$$(xy)u + wy + \alpha x^2 = 0 \tag{3.22}$$

$$(xy)v + wx + \beta y^2 = x^2. \tag{3.23}$$

Now, if we can choose $u, v, w, \alpha, \beta \in k[x, y]$ such that both of these equations hold, then we are done. Observing the latter we need

$$w = x + w'$$

for some $w' \in k[x, y]$. Let us now insert this fact into Equation (3.22), which gives us

$$(xy)u + (x + w')y + \alpha x^2,$$

so if we choose $u = -1$, then we can let

$$w' = 0$$

$$\alpha = 0$$

which gives

$$(xy)(-1) + xy + 0 \cdot x^2 = 0$$

as we wanted. Now we have chosen

$$w = x.$$

Let us insert this into Equation (3.23). We get

$$(xy)v + x^2 + \beta y^2,$$

and we want this to be equal to x^2 for some choice of $v, \beta \in k[x, y]$. However, this is easy, just let

$$v = 0$$

$$\beta = 0,$$

and we get the desired equality. This means that

$$x^2 dy \in N_{(fg)}.$$

Let us show that we also have $y^2 dx \in N_{(fg)}$. To be clear, we want to find $u, v, w, \alpha, \beta \in k[x, y]$ such that

$$y^2 = (xy)u + wy + \alpha x^2 \tag{3.24}$$
$$0 = (xy)v + wx + \beta y^2. \tag{3.25}$$

From Equation (3.24) we see that we need $w = y + w'$ for some $w' \in k[x, y]$. Let us insert this fact into Equation (3.25) and see what we get.

$$0 = (xy)v + (y + w')x + \beta y^2.$$

Now we need to choose v, w', β such that this equation holds, and we see that the only term we have no control over is xy, which we need to remove. We can do this by letting $v = -1$. Then the equation becomes

$$-xy + xy + w'x + \beta y^2 = w'x + \beta y^2,$$

where we can just choose

$$w' = \beta = 0,$$

and we get the that the desired equality holds. Now, we go back to Equation (3.24), inserting that $w' = 0$, and get

$$y^2 = (xy)u + y^2 + \alpha x^2,$$

which one sees easily holds if we choose

$$u = \alpha = 0.$$

This means that

$$y^2 dx \in N_{(fg)},$$

so now we have that

$$(x^2 dy, y^2 dx) \subset N_{(fg)}.$$

Again, by the fact that

$$\sqrt{\operatorname{Ann}\left(\Omega_{k[x,y]}/N_{(fg)}\right)} = (x, y),$$

and Proposition 2.3.11, we know that

$$\operatorname{Ass}(\Omega_{k[x,y]}/N_{(fg)}) = \{(x, y)\},$$

which is equivalent to $N_{(fg)}$ being (x, y)-primary.

Let us look at a more interesting example.

Example 3.4.4. Consider the polynomials $f = y - x^2$ and $g = x$, which geometrically define the y-axis and the parabola $y = x^2$. We claim that $N_{(yx-x^3)}$ is (x, y)-primary, and we start by showing that $(x^3dy, y^3dx) \subset N_{(fg)}$. So, we want to show that $x^3dy \in N_{(yx-x^3)}$, where

$$N_{(yx-x^3)} = ((yx - x^3) \cdot \Omega_{k[x,y]} + (xdy - (y - 3x^2)dx)) + (x^3dx, y^3dy).$$

So we want

$$x^3dy = (yx - x^3)(udx + vdy) + w(xdy - (y - 3x^2)dx) + \alpha x^3dx + \beta y^3dy,$$

for some $u, v, w, \alpha, \beta \in k[x, y]$. Splitting this by generators we get

$$0 = (yx - x^3)u + w(3x^2 - y) + \alpha x^3 \tag{3.26}$$

$$x^3 = (yx - x^3)v + wx + \beta y^3. \tag{3.27}$$

One possibility is to choose $w = x^2 + w'$ for some $w' \in k[x, y]$, and insert this into Equation (3.26). We get

$$0 = (yx - x^3)u + (x^2 + w')(3x^2 - y) + \alpha x^3,$$

where we need to choose u, w' and α such that this holds. Let $u = x + u'$ for some $u' \in k[x, y]$, then we are left with

$$(yx - x^3)u' + w'(3x^2 - y) + 2x^4 + \alpha x^3.$$

Now, we can just choose $u' = w' = 0$ and $\alpha = -2x$, which makes Equation (3.26) hold. Back to Equation (3.27), we get

$$x^3 = (yx - x^3)v + x^3 + \beta y^3,$$

but now we can just choose $v = \beta = 0$. Hence, $x^3dy \in N_{(yx-x^3)}$.

Next we want to do the same for y^3dx. This gives the equations

$$y^3 = (yx - x^3)u + w(3x^2 - y) + \alpha x^3 \tag{3.28}$$

$$0 = (yx - x^3)v + wx + \beta y^3. \tag{3.29}$$

The only way to get a term y^3 in Equation (3.28) is by choosing $w = -y^2 + w'$ for some $w' \in k[x, y]$. Inserting this into Equation (3.29), we get

$$(yx - x^3)v + (w' - y^2)x + \beta y^3,$$

so again we want to choose $v, w', \beta \in k[x, y]$ such that it is zero. To cancel the $-y^2x$ term we choose $v = y$, and get

$$-yx^3 + w'x + \beta y^3.$$

Now, choosing $w' = yx^2$ and $\beta = 0$ this becomes zero, so Equation (3.29) holds. We chose $w = -y^2 + yx^2$, so inserting this into Equation (3.28) we get

$$(yx - x^3)u + (-y^2 + yx^2)(3x^2 - y) + \alpha x^3 \tag{3.30}$$

$$= (yx - x^3)u - 3x^2y^2 + y^3 - 3yx^4 + x^2y^2 + \alpha x^3 \tag{3.31}$$

$$= (yx - x^3)u + xy(-3xy - 3x^3 + xy) + \alpha x^3 + y^3 \tag{3.32}$$

$$= (yx - x^3)u + xy(xy - 3xy - 3x^3) + \alpha x^3 + y^3. \tag{3.33}$$

Let $u = (-3xy^2 - 3yx^3 + xy^2)$, then we get

$$(-3xy^2 - 3yx^3 + xy^2)(-x^3) + \alpha x^3 + y^3,$$

and if we choose $\alpha = -u$ we get the desired equality Equation (3.29). We have showed that $(x^3dy, y^3dx) \subset N_{(yx-x^3)}$, and by the same arguments as in the last example, that $N_{(yx-x^3)}$ is (x, y)-primary.

In Example 3.4.1, Example 3.4.3 and Example 3.4.4 we showed that $N_{(fg)}$ is (x, y)-primary by showing that

$$(x^d dy, y^d dx) \subset N_{(fg)}.$$

Because of this, we could just as well have defined $N_{(fg)}$ as

$$N_{(fg)} = \Omega_{(fg)} + (x^d dx, x^d dy, y^d dx, y^d dy),$$

where d is the degree of fg. This has the advantage we get that $N_{(fg)}$ is (x, y)-primary. The downside is that showing the equality

$$\Omega_{(fg)} = \Omega_{(f)} \cap \Omega_{(g)} \cap N_{(fg)}$$

gets much harder, since we have a larger $N_{(fg)}$ module.

CHAPTER 4

Examining the geometry of the Kähler differentials

The differentials are directly connected to the tangent space of a scheme. For a general scheme the tangent space is the dual of the differential module. The differentials, or globally the cotangent sheaf, are invariant to the underlying scheme and contains information about its geometry. To understand more about the geometry of the Kähler differentials, we examine some geometric representations of them in this chapter.

First we examine homomorphisms from the differentials into k, where $k = k(p)$ is the residue field at a point p. In other words, the fiber of the global sections of the tangent sheaf at a point p. We start by considering a polynomial ring $R = k[x, y]$ and the ideal we have already worked with

$$\mathfrak{a} = (y^2 - yx^2).$$

Recall that this ideal corresponds to the union of the x-axis and the parabola $y = x^2$ in $\mathbb{A}^2$.

In general, we conjecture that the omega module $\Omega_{\mathfrak{a}}$ of an ideal $\mathfrak{a}$ does not behave like the ideal. By this we mean that the decomposition of $\Omega_{(y^2-yx^2)}$ is not simply applying omega to each of

$$(y) \text{ and } (y - x^2)$$

and intersecting them, instead we get a third component that gives the information about the point of intersection, namely the origin. For this specific example we can even show that this is true, which we have done in Appendix A. As we have seen earlier, the corresponding omega module is

$$\Omega_{\mathfrak{a}} = (y^2 - yx^2) \cdot \Omega_R + (-2xydx + (2y - x^2)dy) \cdot R.$$

Consider the homomorphisms of Ω_R into R. This is a set of vector fields of $\mathbb{A}^2$ where the fiber at a point is a tangent space. We know from [EO, Definition 17.2] that the definition of the tangent bundle is

Definition 4.0.1. Let X be a smooth scheme over an algebraically closed field k. Then we define the *tangent bundle* to be the sheaf

$$\mathscr{T}_X = \mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X).$$

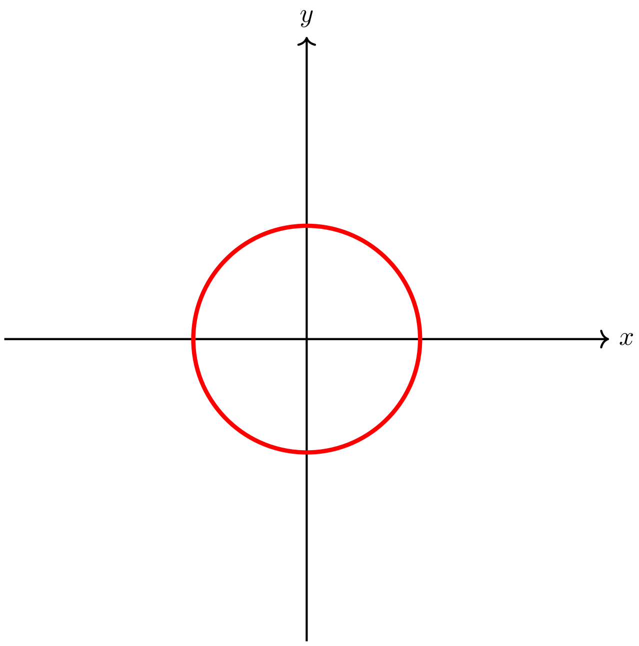

Figure 4.1: The unit circle in $\mathbb{A}^2$.

Example 4.0.2. Let us look at a simple affine example. Assume that

$$X = \mathrm{Spec}(A),$$

where

$$A = k[x, y]/(y^2 + x^2 - 1).$$

This is the unit circle in $\mathbb{A}^2$, which we know is smooth everywhere.

We know that the tangent space at any point of X is just the line in $\mathbb{A}^2$ that is tangent to the circle in this point. This fits into our definition of the tangent bundle since choosing a point on the unit circle and computing the tangent space is exactly equivalent to taking the fiber of the tangent bundle in this point. So the tangent space at a point (α_0, β_0) on the unit circle is

$$\mathrm{Hom}_A(\Omega_A, A)_{(x-\alpha_0, y-\beta_0)} \otimes_{A_{(x_0,y_0)}} k(p),$$

where p is the point (α_0, β_0). If we choose a point (α_0, β_0) not on the unit circle, i.e.

$$\alpha_0^2 + \beta_0^2 \neq 1,$$

then

$$\mathrm{Hom}_A(\Omega_A, A)_{(x-\alpha_0, y-\beta_0)} = 0,$$

since localizing $A = k[x, y]/(x^2 + y^2 - 1)$ outside the unit circle will invert the ideal $(x^2 + y^2 - 1)$. In other words, the zero ideal will be inverted, hence the whole ring becomes the zero ring. This means we have the module of homomorphisms into the trivial ring, which is just the zero homomorphism.

This makes sense with our intuition of tangent spaces since a scheme should not have tangents defined outside its topological space. Let us assume $(\alpha_0, \beta_0) \in k^2$ are such that

$$\alpha_2^2 + \beta_0^2 = 1.$$

In other words, (α_0, β_0) corresponds to a point on the unit circle. As we saw earlier, the tangent bundle is defined by

$$dx \mapsto \alpha_1$$
$$dy \mapsto \beta_1$$

such that $2x \cdot \alpha_1 + 2y \cdot \beta_1 = 0$. We are looking at the stalk in the point corresponding to the maximal ideal

$$\mathfrak{m}_{\alpha_0,\beta_0} = (x - \alpha_0, y - \beta_0),$$

and everything not in this ideal is inverted. Now, if $\alpha_0 \neq 0$ we can invert x and therefore let β_1 be free and write

$$\alpha_1 = -\frac{y\beta_1}{x},$$

which means we have a one-dimensional tangent space in such a point when taking the fiber. If $\alpha_0 = 0$, then we know $\beta_0 \neq 0$ since otherwise we are not on the unit circle. Then we can invert y, so we let α_1 be free and write β_1 as

$$\beta_1 = -\frac{x\alpha_1}{y}.$$

Then, taking the fiber at any point on the circle gives us a one-dimensional tangent space. This makes sense since for any point on the circle we know the tangents are just the lines through the point. In Figure 4.2 we see the line tangent to the circle in the point

$$(\alpha_0, \beta_0) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),$$

and the tangent space are all the vectors on this line.

Here we considered the tangents globally as a sheaf, and to look at the tangents at a point, we take the fiber. This fits with the definition of the tangent bundle in differential geometry, where a tangent bundle is a map

$$E \xrightarrow{\pi} M,$$

where M is a manifold and $E \simeq M \times \mathbb{R}^n$ for some $n \in \mathbb{N}$. Computing the tangent space in this case is taking the fiber of this map at a point $p \in M$. One could also just look at the point from the start by considering the Zariski tangent space.

Definition 4.0.3. The *Zariski tangent space* T_xX to X at the point $x \in X$ is the dual vector space of $\mathfrak{m}_x/\mathfrak{m}_x^2$. That is,

$$T_xX = \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)).$$

The space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is called the *Zariski cotangent space* of X at x. An element of T_xX is called a tangent vector; it is a linear functional

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \to k(x).$$

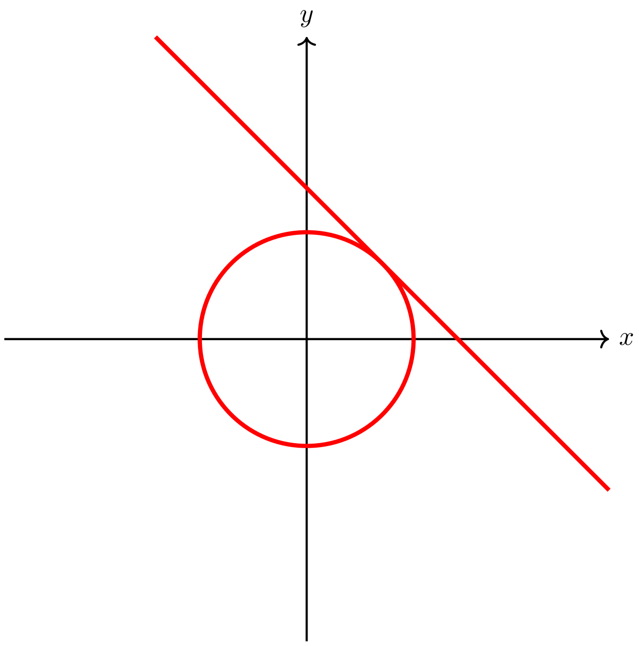

Figure 4.2: Tangent space to a point on unit circle

We prefer however, to work with the Kähler differentials rather than the maximal ideals at the point one is considering. This has the advantage that of being a global structure, which corresponds to the structure at the point when one takes the fiber. Luckily, there is a relation we can use in certain situations for this exact purpose. For more general settings one can use the statement [EO, Proposition 17.35], but for our case we can make the situation simpler.

Proposition 4.0.4. *Let R be a Noetherian ring over an algebraically closed field k. Further, let $x \in \operatorname{Spec}(R)$ be a point, so it corresponds to some maximal ideal $\mathfrak{m}_x$. Then $\mathfrak{m}_x/\mathfrak{m}_x^2$ represents the derivations*

$$\operatorname{Der}_{k(x)}(R, k(x)).$$

In other words,

$$\operatorname{Der}_{k(x)}(R, k(x)) \simeq \operatorname{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)).$$

Proof. We begin by noting that since R is a ring over k, an algebraically closed field, we have that

$$k(x) = R/\mathfrak{m}_x \simeq k.$$

Now, we want to show that

$$\operatorname{Der}_{k(x)}(R, k(x)) \simeq \operatorname{Hom}_{k(x)}((\mathfrak{m}_x/\mathfrak{m}_x^2, k(x))),$$

so we define a map between them and show it is an isomorphism. Define the map by

$$\begin{aligned} \operatorname{Der}_k(R, k) &\to \operatorname{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k) \\ D &\mapsto T_D, \end{aligned}$$

where $T_D(m) = D(m)$ for $m \in \mathfrak{m}_x/\mathfrak{m}_x^2$, and remember that k is an R-module. To check that it is well-defined, let $m \cdot n \in \mathfrak{m}_x/\mathfrak{m}_x^2$. Then we have that

$$T_D(mn) = D(mn) = mD(n) + nD(m)$$

by the Leibniz rule. However, m and n is zero in k through the exactness of

$$0 \to \mathfrak{m}_x \to R \to R/\mathfrak{m}_x \simeq k,$$

hence

$$T_D(mn) = mD(n) + nD(m) = 0D(n) + 0D(m) = 0,$$

which means the map is well-defined. To see that it is surjective, let $T_D \in \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$. Any element in R can be written as $a + m$, where $a \in R$ is a unit and $m \in \mathfrak{m}_x$. But, $D(a + m) = D(m)$, since $a \in k$, and then $D(a+m) = T_D(m)$. To show the map is injective, assume $T_D = 0$, which means all of $\mathfrak{m}_x$ is mapped to zero in k. Then, the corresponding D also maps all of $\mathfrak{m}_x$ to zero, but as we said,

$$a + \mathfrak{m}_x = R$$

and

$$D(a + m) = D(a) + D(m) = 0,$$

so D is the zero map. This finishes the proof. ■

Now that we have this fact, we can use Ω_R instead of $\mathfrak{m}_x/\mathfrak{m}_x^2$ when computing the Zariski tangent space at a point. Let us consider the situation we had in Example 4.0.2. We consider the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and compute the Zariski tangent space at this point. Choosing this point we map

$$\begin{aligned} k[x, y]/(y^2 + x^2 - 1) &\to k \\ x &\to \frac{1}{\sqrt{2}} \\ y &\to \frac{1}{\sqrt{2}} \end{aligned}$$

which gives the $R = k[x, y]/(y^2 + x^2 - 1)$-module structure to k. Now we want to map Ω_R into k given this structure. Recall that

$$\Omega_R \simeq \Omega_{k[x,y]}/\Omega_{(y^2+x^2-1)}$$

where

$$\Omega_{(y^2+x^2-1)} = (y^2 + x^2 - 1) \cdot \Omega_{k[x,y]} + (ydy + xdx) \cdot R.$$

Through our definitions above $(y^2 + x^2 - 1)$ is already mapped to zero through our R-module structure, so what remains is to map

$$\begin{aligned} dx &\to \alpha \\ dy &\to \beta \end{aligned}$$

such that $(ydy + xdx)$ is mapped to zero. In other words, $\frac{1}{\sqrt{2}}\alpha = -\frac{1}{\sqrt{2}}\beta$. But, this just means

$$\alpha = -\beta,$$

so we can let β be free and let α depend on it. This gives a one-dimensional tangent space, which is exactly the same we found in Example 4.0.2 when we computed the fiber of the stalk of the tangent bundle in this point.

However, these two notions are not always equivalent. Whenever we have a smooth scheme they are equivalent, but when we look at singular points they are different. In rough terms; the Zariski tangent space is only interested in the point, while the tangent sheaf is a local structure which always keeps some information about the scheme around the point one localizes at. Note that when we have a non-smooth scheme we refer to the sheaf known as the tangent bundle as the tangent sheaf. This is because when the scheme is singular, then the sheaf is not a bundle.

Let us see an example of the tangent sheaf being different from the Zariski tangent space at a singular point.

Example 4.0.5. Let $f = y$ and $g = x$, so (fg) gives rise to the scheme in Figure 3.3. We choose the point $(0, 0)$, which is certainly on this graph and a singular point, so the Zariski tangent space and the fiber of the tangent bundle should not be equivalent here. The tangent bundle is not defined for singular schemes, but let us observe what happens when we look at the tangent sheaf. Choosing this point is equivalent with choosing the module structure

$$\begin{aligned} k[x, y]/(xy) &\to k \\ x &\mapsto 0 \\ y &\mapsto 0. \end{aligned}$$

Now, computing the Zariski tangent space we need to map $\Omega_{k[x,y]/(xy)}$ into k with the structure above. We need to choose $\alpha, \beta \in k$ such that

$$\begin{aligned} \Omega_{k[x,y]/(xy)} \simeq \Omega_{k[x,y]}/\Omega_{(xy)} &\to k \\ dx &\mapsto \alpha \\ dy &\mapsto \beta \end{aligned}$$

is a homomorphism. In other words, we need $(xdy + ydx)$ to be mapped to zero. Recall that through the module structure we gave above, $(xdy + ydx)$ becomes $0 \cdot dy + 0 \cdot dx$, so we need $0 \cdot \beta + 0 \cdot \alpha = 0$. This is true for any $\alpha, \beta \in k$, so we can choose these freely. Hence, the tangent space is the set of all vectors in $\mathbb{A}^2$ located in the origin. The situation is illustrated in Figure 4.3, the red vector is just one example, we can let this vector be anywhere in the plane as long as it is located in the origin. This is quite strange, since this vector is certainly not tangent to our scheme in the geometric sense as we are used to. Let us compare this to the tangent sheaf. We want to compute the fiber of the tangent sheaf at this point, which is

$$\mathscr{T}_{X,(x,y)} \otimes_{\mathcal{O}_{X,(x,y)}} k(0) = \mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)_{(x,y)} \otimes_{\mathcal{O}_{X,(x,y)}} k(0)$$

where $X = \operatorname{Spec}(k[x, y](xy))$. First taking the stalk is just

$$S^{-1} \operatorname{Hom}_B(\Omega_B, B)$$

where S is the localizing set. So we must choose where to send

$$\begin{aligned} dx &\mapsto f \\ dy &\mapsto g, \end{aligned}$$

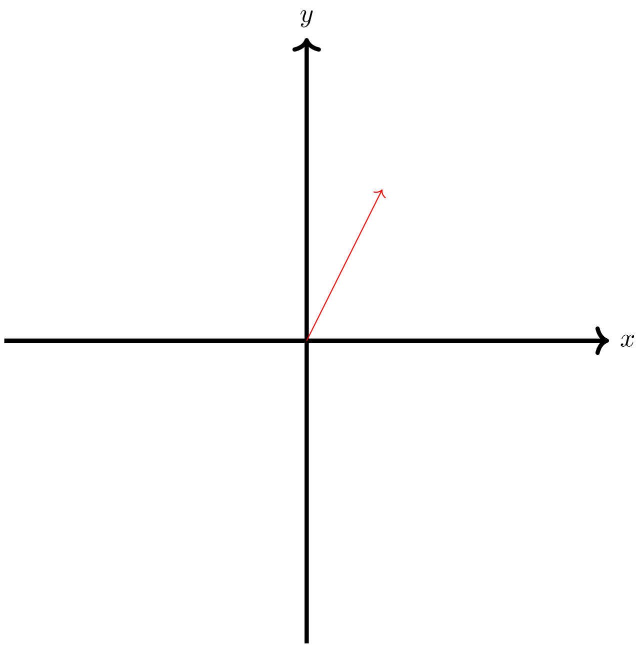

Figure 4.3: The tangent space in $(0, 0)$ of Figure 3.3.

for $f, g \in B$. We must choose f and g such that $xg + yf = 0$, i.e. $f \in (x)$ and $g \in (y)$. Now, for each choice of f and g such that this holds, we get a vector field, and evaluating in the origin we see that $f(0, 0) = g(0, 0) = 0$, so at the origin this vector field is always zero. Thus, we see that the tangent sheaf in the origin, which is a singular point, does not correspond to the Zariski tangent space, which was a two-dimensional space. This is why we assume that X is a smooth scheme when defining the tangent bundle, because in the singular case the sheaf is no longer a bundle.

4.1 What the embedded component means for the cotangent sheaf

Let $B = k[x, y]$, where k is an algebraically closed field. We have computed that

$$\Omega_{k[x,y]/(xy)} \simeq \Omega_{k[x,y]}/\Omega_{(xy)},$$

and that $\Omega_{(xy)}$ contains more than a (x)-primary component and a (y)-primary component. As we have shown, a primary decomposition is

$$\Omega_{(xy)} = \Omega_{(x)} \cap \Omega_{(y)} \cap (\Omega_{(x,y)} + (x^2dx, y^2dy),$$

whereas the ideal is just $(xy) = (x) \cap (y)$. This begs the question, what this last component means geometrically. We compute the Zariski tangent space of (xy). Consider the origin in $\text{Spec}(k[x, y]/(xy))$.

$$\Omega_{k[x,y](xy)}(k[x, y]/(xy), k) \to k$$

$$dx \mapsto \alpha$$
$$dy \mapsto \beta.$$

For this map to be well-defined we need elements that are in $\Omega_{(x)} \cap \Omega_{(y)}$ to be mapped to zero. An element in $\Omega_{(x)}$ looks like

$$\omega = x(udx + vdy) + w(dx)$$

for some $u, v, w \in k$, and for ω to be in $\Omega_{(y)}$, we need $xu + w \in (y)$. Hence, an element in the intersection looks like

$$x(vdy) + yh(dx),$$

where $yh = xu + w$ for some $h \in k[x, y]$. Well, by the module structure this will be mapped to

$$0 \cdot (v(0,0) \cdot \beta) + 0 \cdot h(0,0) \cdot \alpha,$$

which is always zero, so α and β are free, hence we have a two-dimensional tangent space in the origin. This is the same as we get computing

$$\mathrm{Hom}_B(\Omega_B/\Omega_{(xy)}, k).$$

This is in fact true for every evaluation at a point on (xy), so geometrically there is no difference. However, let us consider the stalk of each of them at the (y)-axis. The one corresponding to $\Omega_{(xy)}$:

$$\left(\Omega_B/\Omega_{(xy)}\right)_{(x)}.$$

Here (y) is inverted, and the relation

$$xdy + ydx = 0$$

becomes

$$\frac{x}{y}dy + dx = 0.$$

However, $\Omega_{(x)} \cap \Omega_{(y)}$ is generated by

$$xdy \text{ and } ydx$$

and so the relations on

$$\left(\Omega_B/\Omega_{(x)} \cap \Omega_{(y)}\right)_{(x)}$$

becomes

$$xdy = 0 \text{ and } dx = 0,$$

since y is a unit. So, we have

$$\left(\Omega_B/\Omega_{(xy)}\right)_{(x)} \simeq \frac{\Omega_{S^{-1}B}}{\left(dx - \frac{x}{y}dy\right)}$$

and

$$\left(\Omega_B/\Omega_{(x)} \cap \Omega_{(y)}\right)_{(x)} \simeq \Omega_{S^{-1}B}/(dx, xdy).$$

Note that once we evaluate at any point on (xy) these are both equal. They are however subtly different before looking at just a point. Thus, this third component does change the algebraic structure of the Kähler differentials, but the geometric representation does not change.

APPENDIX A

Computations in Macaulay2

A.1 The script used to compute decompositions

This is the program we use to check if our proposed decomposition of $\Omega_{\mathfrak{a}}$ is right. We give the program an ideal $I \subset k[x, y]$, and it computes the actual decomposition, and then we check if our decomposition has the same associated primes and intersects to the correct module Ω_I. Note that

```
decompomeg
```

can be used to calculate the decomposition of any $\Omega_{\mathfrak{a}}$, not just the ones we work with in this thesis.

One can also calculate that our proposed decomposition generalizes to curves f and g intersecting in several points by adding

$$N_i = \Omega_{(fg)} + ((x - \alpha)^d dx, (y - \beta)^d dy)$$

for each point (α, β) they intersect in, to the intersection.

```
-- Program to calculate primary decomposition of omega module
restart
omegagens = (I) -> (
    -- Creates the matrix containing the generators of omega(I).
    -- Applying image to the output will give us the module omega(I).
    pols = entries gens I;
    -- Makes a nonempty matrix
    mat = matrix{{1,0}};
    for i in pols_0 do (
        -- The (I)\cdot omega(k[x,y]) part
        mat = mat || matrix{{i, 0}};
        mat = mat || matrix{{0, i}};
        -- The d(I)\cdot k[x,y] part
        mat = mat || matrix{{diff(x,i), diff(y,i)}};
    );
    -- Remove the 0 entry
    mat = submatrix'(mat, {0},{})
)

decompomeg = (I) -> (
    -- Computes an accurate primary decomposition of omega(I)
    -- Get the generators of omega(I)
    mat = transpose matrix omegagens(I);
    N = image mat; -- omega(I)
    M = ambient N;
    T = M/N;
```

```
    -- This gives us the primary decomposition as subquotients
    decomp = primaryDecomposition T;
    comps = {1};
    -- We resolve the subquotients as k[x,y]-modules
    for i from 0 to #decomp-1 do (
        test = image gens(decomp_i);
        T2 = T/test;

        phi = inducedMap(T2,M);
        comps = append(comps, trim image matrix entries gens trim ker phi);
    );
    comps = delete(1, comps);
    comps
    -- List of components in decomposition
    -- The intersection will be omega(I)
)

computeEmbedded = (I, prim) -> (
    -- I is the ideal we are taking omega of.
    -- prim is the associated prime we are creating
    -- a component for.
    -- prim will usually be (y,x)
    pols = entries gens I;
    mat = matrix{{1,0}};
    -- Create the module omega(I)
    for i in pols_0 do (
        mat = mat || matrix{{i, 0}};
        mat = mat || matrix{{0, i}};
        mat = mat || matrix{{diff(x,i), diff(y,i)}};
    );
    d = degree I;
    -- Add the generators of (x^ddx, y^ddy)
    mat = mat || matrix{{prim_1^d, 0}, {0, prim_0^d}};
    -- Remove unnecessary element
    mat = submatrix'(mat, {0},{})
)

compareDecomps = (I) -> (
    -- Computes our proposed analytic decomposition for I
    -- and compares it with the one macaulay gives.
    -- The comparison checks whether they intersect to the
    -- same ideal and it checks whether
    -- our proposed decomposition consists of primary components.
    -- If the proposed decomposition is right this will print nothing,
    -- if, however, the decomposition is not equal to omega(I) it will
    -- print "not equal sets".
    -- If the proposed decomp has non-primary components it will print
    -- "not primary".

    actualcomps = (decompomeg(I));
    lng = #actualcomps;
    N = intersect(actualcomps);
    par = primaryDecomposition I;
    propComps = {1};
    count = 0;
    for i in par do (
        count = count + 1;
        isoComp = trim image transpose omegagens(i);
        -- The isolated components
        propComps = append(propComps, isoComp);
    );
    rest = lng - count;
```

```
    -- Compute all embedded components of our proposed
    for i from 1 to rest do (
         -- Checks what the remaining components are primary to,
         -- in our examples this will just be (x,y).
        curr_comp = actualcomps_(lng-i);
        prim = associatedPrimes (ambient curr_comp/curr_comp);
        -- The prime ideal the (lng-i)th component is primary to
        prim2 = entries gens prim_0;
        -- Compute the module N_2 = omega(I) + (x^d, y^d)
        propLast = image transpose computeEmbedded(I, prim2_0);
        propComps = append(propComps, propLast);
    );
    -- This will be the proposed analytic primary decomposition of omega(I)
    propComps = delete(1, propComps);
    for i from 0 to (lng-1) do (
        -- Check if the analytic components are in fact primary
        if not isPrimary(ambient propComps_i, propComps_i) then (
            print(i | "not primary");
            );
        );
    -- Intersecting the components we have calculated,
    -- if everything is right this should be omega(I)
    N2 = intersect(propComps);
    if not (N2==N) then (
        -- Check if the decompositions are equal sets
        -- when you intersect the components.
        print("Not equal sets");
    );
    -- The two decompositions (proposed, macaulay2 computed)
    {propComps, actualcomps}
)

R = QQ[x,y]
p = ideal(x)
decomp = compareDecomps(p)
```

A.2 Computing examples

Example A.2.1. The computation of Example 3.3.3 and Example 2.4.2. We show that N_2 is (x, y)-primary, that our proposed decomposition is correct, and that $\Omega_{(xy)}$ has the associated primes

$$\{(x), (y), (x, y)\}.$$

```
R = QQ[x,y]
p = ideal(x*y)
decomp = compareDecomps(p)
-- This prints only the two decompositions;
-- the one macaulay calculates, and the one we propose.
-- Nothing else is printed, which means the proposed decomposition
-- is correct.

-- Checking whether N_2 is primary:
associatedPrimes(ambient decomp_2/decomp_2)
-- This prints out (y, x), so the N_2 component is (x,y)-primary.

-- Writing omega(xy) analytically and checking whether is has
-- the proposed associated primes.
-- omegafg will be omega(xy), the matrix entries correspond to the
```

```
-- generators (xydx), (xydy) and (ydx+xdy), respectively.
omegafg = image matrix entries transpose matrix{{x*y,0},{0,x*y},{y,x}}
associatedPrimes(ambient omegafg/omegafg)
-- Prints list of (x), (y) and (x,y), the ideals we conjectured.
```

The following example shows why we must assume (f) defines a smooth curve when claiming $\Omega_{(f)}$ is (f)-primary.

Example A.2.2. We let $f = y^2 - x^2(x+1)$, so we are working with the nodal curve in Example 3.2.3.

```
-- We check that omega of the nodal cubic is not primary.
-- gensnod will be the generators of omega of the nodal cubic.
gensnod = matrix{{(y^2-x^3-x^2),0},{0,(y^2-x^3-x^2)},{-3*x^2-2*x,2*y}}
omegafg = image matrix entries transpose gensnodal
-- The module omega(f) for f=(y^2-x^3-x^2)
associatedPrimes(ambient omegafg/omegafg)
-- Prints out {(x,y), (y^2-x^3-x^2)}, in other words,
-- it is not primary.
```

We compute Example 3.1.4.

Example A.2.3.

```
-- Computing omega(y)\cap omega(y-1) = omega(y^2-y)
omegaf = intersect(decompomeg(ideal(y)));
omegag = intersect(decompomeg(ideal(y-1)));
omegafg = intersect(decompomeg(ideal(y^2-y)));
intsct = intersect(omegaf, omegag);
intsct == omegafg
-- prints true.
-- This means omega(fg) = omega(f)\cap omega(g)
-- in this specific example.
```

A.3 Computing more examples of the conjectures

In the last section we verified some statements for specific examples in the thesis. Here we compute examples of the conjectures made in Chapter 3 on a larger scale in an attempt to verify the statements made.

The first conjecture claimed that for any smooth and irreducible $f \in k[x, y]$, the module

$$\Omega_{(f)} = (f) \cdot \Omega_{k[x,y]} + (df) \cdot k[x, y]$$

will be primary. To generate the module $\Omega_{(f)}$ we can use the

```
omegagens
```

from Appendix A.

Computing primary omega modules

```
checkPrimary = (I) -> (
    -- Checks if omega(I) is primary, and then
    -- if it is rad(I)-primary.
    -- If it is rad(I)-primary nothing gets printed.
    omegafg = trim image transpose omegagens(I);
    asscomps = associatedPrimes(ambient omegafg/omegafg);
```

```
    if not (#asscomps==1) then (
        print("Not primary!!");
        print(asscomps);
    );
    if not ((radical I) == asscomps_0) then (
        print("Not rad(I)-primary!!");
        print(asscomps);
    );
)

checkPrimary(ideal(x))
-- Prints nothing, so omega(x) is (x)-primary
checkPrimary(ideal(x^2))
-- Same, omega(x^2) is (x)-primary
-- Note that this means that not only irreducible
-- polynomials can give rise to primary omega modules.
checkPrimary(ideal(y-x^2))
-- parabola, true
checkPrimary(ideal(y-x))
-- line, true
checkPrimary(ideal(y+x^2))
-- inverse parabola, true
checkPrimary(ideal(y^2-x^2*(x+1)))
-- The nodal cubic, a primary ideal, but not smooth
-- so it is not a sound ideal, hence we get two
-- associated primes.
checkPrimary(ideal(y-x^3))
-- true

-- One can also calculate this for large n
n = 1000
for i from 4 to n do (
    checkPrimary(ideal(y-x^n))
)
-- this prints nothing, so ideals of this form
-- are sound up to at least n=1000.

-- We can change the situation a little
n = 1000
for i from 4 to n do (
    if (i%25==0) then (
        print(i);
    );
    checkPrimary(ideal(y-x^n+x^(n-1)))
)
-- Also true, takes a while so we print a counter
-- to see how far in the computation we are.
```

Computing the equality of the sets

We check the statements Conjecture 3.3.2 and Conjecture 3.4.2. Checking this for an ideal I is just running the function

```
compareDecomps(I)
```

The script tells whether the sets are not equal or if any of the components in the intersection are not primary. Assume we have the script in Appendix A.1. The script is explained in more detail in that section.

```
compareDecomps(ideal(x*y))
```

```
-- true
compareDecomps(ideal(x))
-- true
compareDecomps(ideal(y-x^2))
-- true
compareDecomps(intersect(ideal(x+y), ideal(x-y)))
-- true

-- Doing the same as when computing primary modules
n = 1000
for i from 4 to n do (
    if (i%25==0) then (
    -- We do this to see how far
    -- in the computation we are.
        print(i);
    );
    compareDecomps(ideal(y-x^n))
)
-- true
```

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