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On the DT-PT Correspondence

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

Donaldson-Thomas (DT) theory and Pandharipande-Thomas (PT) theory represent two ways of counting curves on complex threefolds by integration against virtual classes of Hilbert schemes and stable pair spaces, respectively. The DT-PT correspondence, proven by Bridgeland and Toda, equates these two theories via an equality of generating series of invariants. In this thesis we review the theory and techniques required to investigate this correspondence and its analogues in specific examples.

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Contents

Introduction

Classically, *enumerative geometry* deals with problems involving counting geometric objects (lines, circles, conics, etc.), satisfying some incidence conditions. These incidence conditions could be requiring objects to pass through a number of points, to be contained in a specific surface, or perhaps to satisfy certain tangency conditions. Consider the *problem of Apollonius*:

> *Given three circles in the plane, how many other circles are tangent to all three?*

Often, the way to approach an enumerative problem is to find a "space" *parametrizing* the objects to be counted, and study how imposing the incidence conditions reduces that space. Understanding the reduction is part of the subject of *intersection theory*. A circle in the plane is determined by three independent parameters (or *moduli*): the two coordinates of the center point, and the radius. From this observation, we can associate a three-dimensional space, denoted by $\mathbb{P}^3$, where each point corresponds to a unique circle in the plane. Using Bezout's theorem, it can be shown that after reducing this space of circles three times by imposing the condition that we only consider circles which are tangent to the given ones, what we should end up with is 8 points.

Even when the *actual* number of tangent circles is more or less than 8, the number 8 is the "correct" answer–it conveys deeper information about the problem, as it does not depend on the specific choice of the three circles. In this thesis we discuss two frameworks for defining enumerative invariants associated to curves embedded in threefolds.

Modern enumerative geometry In the last 60 years, there has been significant advancement in the study of moduli spaces and intersection theory. Work of Grothendieck related to representable functors, such as Hilb and Quot, and Mumford's *Geometric Invariant Theory* (GIT) are foundational to the topics in this thesis. The theory of stacks, also built on these ideas, plays an important role, though goes beyond the scope of this thesis. A robust intersection theory, as set down by Fulton, is a central tool for studying enumerative invariants based on moduli spaces.

Unlike $\mathbb{P}^3$ in the example problem above, moduli spaces in the modern theory can be extremely difficult to work with. Indeed, before the 1990s, most "curve-counting" theories were only equipped to handle lines, (e.g., Schubert calculus) or other genus 0 curves [PT14a].

With a maturing theory of moduli spaces and intersection theory in place, Li and Tian [LT98] and Behrend and Fantechi [BF97] introduced notions of *virtual classes* which could be used as "corrected" moduli spaces. Based on virtual classes, several modern curve-counting theories have been developed, [PT14a]. The first of these theories was Gromov-Witten theory, an enumerative theory based on the moduli space of *stable maps* from fixed curves into a space.

Donaldson-Thomas theory Donaldson-Thomas (DT) theory, as presented in this thesis, is based on work of [Tho00]. This is considered a *sheaf*-counting theory, applying to threefolds, and originally defined as a holomorphic analogue to the Casson invariant. A major contribution of [Tho00], was the introduction of a deformation-obstruction theory for *stable sheaves* defining a virtual class; it serves as a model for now several invariants defined from moduli of objects in the derived category with some notion of stability.

Donaldson-Thomas theory may also be viewed as a *curve*-counting theory by the equivalence stable rank one sheaves with trivial determinant and *ideal* sheaves. In other words, DT moduli spaces are represented by Hilbert schemes. As an enumerative theory, Hilbert schemes have the disadvantage that curves degenerate into curves of higher genus plus "free points". An example of this is the twisted cubic in $\mathbb{P}^3$ deforming in a flat family to a nodal plane cubic with an embedded point that may further deform into a smooth genus one curve and a free point (see [Har77, III §9] for the first degeneration).

Seminal work of [Mau+06a] relates Donaldson-Thomas invariants to Gromov-Witten invariants, via their generating series. The conjecture (now theorem in most cases) was that after a change of variable, the two generating series differ only by the contribution of the free points on the DT side.

Pandharipande-Thomas theory At first conjecturally, Pandharipande-Thomas (PT), or stable pairs, invariants are the geometric realization of quotienting DT invariants by the contribution of free points. In fact, that they are related this way *is* the DT-PT correspondence. Like the Hilbert scheme, the moduli space underlying the PT invariants parametrizes embedded curves, but in a way such that zero dimensional contributions are, in some sense, confined to the curve. As explained in [PT09a], the stable pairs moduli space can be viewed as a modification of the Hønsen compactification of Cohen-Macaulay curves [Høn04], which admits a virtual fundamental class.

Scope of this thesis The DT-PT correspondence is an expression of wall-crossing for counting invariants in the derived category. However, we will not address the correspondence at this level. Instead, we will develop the relevant theory, examples and computational tools to explore the correspondence empirically.

Outline

We begin Chapter 1 by recalling the Hilbert scheme, Hilbert polynomials and how they define stability for coherent sheaves. We associate the Hilbert scheme with stable rank one sheaves (referring to [Tho00]) in order to pass freely between the two notions in later sections. The Hilbert scheme of points on affine space is also discussed, as it will be the model for degree zero DT invariants. Stable pairs are then derived from the general notion of a coherent system. We discuss some of their geometric properties and briefly reflect on the relation to the Hilbert scheme. The next section introduces moduli spaces of quiver representations. The reason for this is that such schemes are naturally the critical locus of a regular function inside a smooth ambient variety; in this situation, DT and PT invariants are computable by the Behrend function. Finally, we define torus actions and localization in the non-virtual case as a primer for virtual localization. We compute the 27 lines on the cubic surface as an example.

In Chapter 2 we develop the main technical tools in DT theory, both to define and compute invariants. These are virtual fundamental classes, virtual localization and Behrend functions. We use virtual localization to compute low degree DT invariants for $\mathbb{A}^3$ and see that our result agrees with the partition function computed by [Mau+06b]. We briefly discuss the other MNOP conjectures.

Finally, in Chapter 3, we introduce the PT invariants and compute generating series for the local $\mathbb{P}^1$. We discuss the general form of DT-PT correspondences, before exploring the *unweighted* correspondence. In particular, we show that it fails in the simplest case when $\dim X \neq 3$. We end by discussing the original DT-PT correspondence, in general and in the case of the local $\mathbb{P}^1$, as well as its K-theoretic refinement.

Conventions

By a Calabi-Yau threefold X, we mean a nonsingular variety over $\mathbb{C}$ with $\dim X = 3$, trivial canonical bundle $\omega_X \simeq \mathcal{O}_X$ and vanishing h^1, $H^1(X, \mathcal{O}_X) = 0$. The names “resolved conifold” and “local $\mathbb{P}^1$” are used interchangebly for the total space of the bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

CHAPTER 1

Preliminaries

1.1 Hilbert schemes

Moduli functors and Hilb

Moduli spaces parametrize isomorphism classes of a given set of, typically geometric, objects. A *coarse moduli space* realizes a one-to-one correspondence between the (closed) points of some scheme and isomorphism classes of desired objects, e.g., line bundles or space curves. The correspondence between closed points of a scheme X over k and maps from $\operatorname{Spec} k \to X$ hints at a way to upgrade the problem. A scheme X with the association k-points to isomorphism classes extended into a contravariant functor $\mathsf{Sch} \to \mathsf{Set}$, contains much more information about the objects parametrized. Maps from more general schemes should correspond to *families*, and varying a point continuously in the source scheme should correspond to a continuous deformation.

Definition 1.1.1. Suppose $F : \mathsf{Sch} \to \mathsf{Set}$ is a moduli functor, a functor mapping schemes to "families." If there exists a scheme M with functor of points $h_M = \operatorname{Hom}(-, M)$ naturally equivalent to F, then we say that F is representable (by M) and that M is a *fine moduli space*.

Universal families Fine moduli spaces are especially desirable because of the presence of *universal families*. If $\mathcal{M}$ represents a moduli functor F, then there is a canonical family $U \to M$ corresponding to the identity in $\operatorname{Hom}(\mathcal{M}, \mathcal{M})$ called the *universal* or *tautological* family. It typically has the property that the schematic fiber over a point is isomorphic to the object that point corresponds to.

One of the most important moduli functors is the *Hilbert functor* denoted Hilb. In this case, families are *flat families* [Har77, III, §9], which, roughly speaking, exclude undesirable behavior such as sudden increase in dimension.

Definition 1.1.2 ([Str96, Def. 3.1])**.** An *algebraic family of closed subschemes of X, parametrized by T*, is a closed subscheme $Z \subseteq X_T = X \times_k T$. The family is *flat* if the induced morphism $Z \to T$ is flat.

That the flatness condition above is the right one seems mysterious from the definition. For families over integral noetherian schemes, perseveration of the *Hilbert polynomial* is a necessary and sufficient condition for flatness [Har77, III, Theorem 9.9].

Hilbert polynomials Let $S = k[x_0, \ldots, x_n]$. Declaring $\deg x_i = 1$, gives S the structure of a $\mathbb{Z}$-graded ring

$$S = \bigoplus_{k \in \mathbb{Z}} S_n,$$

where $S_k = 0$ if $k < 0$. Recall that closed subschemes of projective space $\mathbb{P}^n = \operatorname{Proj} S$ are given by homogenous ideals $I \subset k[x_0, \ldots, x_n]$. The Hilbert function of I_Z is defined to be the integer function mapping d to the dimension over k of the d-th graded piece of S/I with its induced grading. Note that while several homogenous ideals I may define the same Z, their Hilbert functions agree for $d \gg 0$. By [Har77, I, Theorem 7.5 (Hilbert-Serre)] these Hilbert functions are given by unique numerical polynomials for $d \gg 0$, and therefore any $Z \subseteq \mathbb{P}^n$ has a well-defined *Hilbert polynomial*, P_Z. Recall that for projective space $\mathbb{P}^n$ itself (given by the zero ideal) we have

$$P_{\mathbb{P}^n}(d) = \binom{n+d}{d}.$$

The definition of the Hilbert polynomial above may be extended to any projective scheme Z with ample line bundle $\mathcal{O}(1)$ defining an embedding into some projective space, note that P_Z depends on the projective embedding, i.e., choice of $\mathcal{O}(1)$. Nevertheless, some aspects of the Hilbert polynomial *are* independent of this choice, for example, the dimension of Z is equal to the degree of P_Z and the arithmetic genus $p_a(Z)$ is equal to $(-1)^r P_Z(0) - 1$, [Har77, I, §7].

We will return to the Hilbert polynomial in a more general form, in order to define *stability*. We now define the Hilbert functor and the scheme that represents it:

Definition 1.1.3 (Hilbert functor/scheme, [Str96, Def. 3.2])**.** Let $\operatorname{Hilb}_X(T)$ be the set of flat algebraic families of a closed subschemes Z of X parametrized by T. If $T' \to T$ is any morphism, $Z \to Z \times_T T'$ gives a map $\operatorname{Hilb}_X(T) \to \operatorname{Hilb}_X(T')$, which makes Hilb_X a contravariant functor on the category of k-schemes. If Hilb_X is representable, the k-scheme $\operatorname{Hilb} X$ representing it is called the *Hilbert scheme of X*.

Remark 1.1.4. If such a scheme exists, then $\operatorname{Hilb}_X = \bigsqcup \operatorname{Hilb}_X^P$; and the Hilbert functor defined above is covered by the open and closed subfunctors given by specifying the Hilbert polynomial P. The subtlety in the difference of notation between the Hilbert functor Hilb_X and the Hilbert scheme $\operatorname{Hilb} X$ is only a slight abuse by the Yoneda lemma.

Theorem 1.1.5 (Grothendieck, [Gro60])**.** *Assume that X is projective and P is a numerical polynomial. Then* $\operatorname{Hilb}^P X$ *exists and is projective.*

Example 1.1.6 (Curves in $\mathbb{P}^2$, [Har10, Ex. 1.1])**.** Consider $\mathbb{P}^2 = \operatorname{Proj} \mathbb{C}[x, y, z]$. A curve of degree d in $\mathbb{P}^2$ is defined (up to scaling) by a homogenous polynomial $a_0 x^d + \cdots + a_n z^d$, with $a_i \in k$ and $n = \binom{d+2}{2} - 1$, i.e., the number of degree

d monomials in $n+1$ variables. This gives a bijection between plane curves and $\mathbb{P}^n$ (the a_i's become the projective coordinates). Define a family $\mathcal{C}$ by the polynomial $a_0x^d + \cdots + a_nz^d$, where now the a_i are coordinates of $\mathbb{P}^n$. We see that under the projection $\mathcal{C} \to \mathbb{P}^n$, the fiber over a point $a \in \mathbb{P}^n$ is the curve a corresponds to. Using $\mathcal{C}$, one shows that $\mathbb{P}^n$ represents $\mathrm{Hilb}^{P_d}_{\mathbb{P}^2}$, where $P_d(z) = -\frac{1}{2}d(d-2z-3)$.

Hilb for quasi-projective varieties There are several variations and generalizations to the Hilbert functor and scheme defined above. Without modification, the Hilbert functor written above (1.1.3) is not representable in the quasi-projective case. In affine space, for example, any subvariety is contained in a flat family containing an empty fiber. The solution is to consider the Hilbert scheme of an ambient projective scheme Y, and to consider the open locus in $\mathrm{Hilb}\, Y$ of closed subschemes of Y not meeting the boundary $Y \setminus X$.

The Hilbert polynomial and stable sheaves

The realization of Hilbert schemes of curves as moduli spaces of *stable* sheaves is important for defining invariants of such spaces in the next chapter. It also indicates that there is more general framework for moduli spaces that are *like* the Hilbert scheme. We start by introducing some preliminary definitions from [HL10].

Definition 1.1.7 (Dimension of a coherent sheaf)**.** Let $\mathcal{F}$ be a coherent sheaf on a scheme X; Recall that the support of $\mathcal{F}$, $\mathrm{Supp}\,\mathcal{F}$ is defined to be $\{x \in X \mid \mathcal{F}_x \neq 0\}$. If X is Noetherian, $\mathrm{Supp}\,\mathcal{F}$ is closed [Har77, Exercise II.5.6], and we define the *dimension of a coherent sheaf*, $\dim \mathcal{F}$, to be the dimension of the support of $\mathcal{F}$.

Definition 1.1.8 (Hilbert polynomial for coherent sheaves, [Har77, Exercise III.5.2])**.** Let X be a projective scheme over a field k with very ample line bundle $\mathcal{O}(1)$ on X. For a coherent sheaf $\mathcal{F}$ on X, we define the *Hilbert polynomial* of $\mathcal{F}$ (with respect to $\mathcal{O}_X(1)$) to be the numerical polynomial $P_{\mathcal{F}}$ such that

$$P_{\mathcal{F}}(n) = \chi(\mathcal{F}(n))$$

for $n \gg 0$ and $\chi(\mathcal{F}(n))$ the sheaf-theoretic Euler characteristic: $\chi(\mathcal{F}(n)) = \sum_i (-1)^i H^i(X, \mathcal{F} \otimes \mathcal{O}(n))$. This agrees with the previous definition via $P_Z = P_{\mathcal{O}_Z}$.

Remark 1.1.9. Since the Euler characteristic is additive on exact sequences, so too is the Hilbert polynomial. It can be shown, for example in [HL10, Lemma 1.2.1], that for any coherent sheaf $\mathcal{F}$, $P_{\mathcal{F}}$ can be written uniquely as,

$$P_{\mathcal{F}}(n) = \sum_{i=0}^{\dim \mathcal{F}} \alpha_i(\mathcal{F}) \frac{m^i}{i!}$$

where $\alpha_i \in \mathbb{Q}$ and $\alpha_{\dim \mathcal{F}} \neq 0$ (this is called the *multiplicity* of $\mathcal{F}$). This form of $P_{\mathcal{F}}$ is used in the following definitions.

Definition 1.1.10 (Rank of a coherent sheaf, [HL10])**.** Let $\mathcal{F}$ be a coherent sheaf on a projective scheme X with $\dim \mathcal{F} = \dim X = d$. The *rank* of $\mathcal{F}$ is defined as

$$\mathrm{rk}(\mathcal{F}) = \frac{\alpha_d(\mathcal{F})}{\alpha_d(\mathcal{O}_X)}.$$

Remark 1.1.11. Note that if X is integral then it contains a dense open U such that $\mathcal{F}|_U$ is locally free and, by additivity of the Hilbert polynomial, has the same rank as $\mathcal{F}$. In the locally free case, the rank of a sheaf agrees with the rank of the associated vector bundle.

Definition 1.1.12 (Reduced Hilbert polynomial, [HL10])**.** The *reduced Hilbert polynomial* of a coherent sheaf $\mathcal{F}$ is of dimension d is defined as

$$p_{\mathcal{F}}(n) = \frac{P_{\mathcal{F}}(n)}{\alpha_d(\mathcal{F})}.$$

Definition 1.1.13 (semistable (stable) sheaf, [HL10])**.** A coherent sheaf $\mathcal{F}$ is *semistable* if

(i) $\mathcal{F}$ is *pure* (i.e., has no nontrivial subsheaves of lower dimension), and

(ii) for all proper subsheaves $\mathcal{G} \subseteq \mathcal{F}$, we have $p_{\mathcal{G}}(n) \leq p_{\mathcal{F}}(n)$, $n \gg 0$ (often written simply as $p_{\mathcal{G}} < p_{\mathcal{F}}$).

We say that $\mathcal{F}$ is *stable* if this inequality is strict.

Proposition 1.1.14. *Suppose that X is smooth projective variety of dimension $d \geq 3$. Let $C \subset X$ be a curve in X; the associated ideal sheaf $\mathcal{I}_C$ is stable, of rank one, and has trivial determinant.*

Proof. By additivity of the Hilbert polynomial on the sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

we have that $P_{\mathcal{O}_X} = P_{\mathcal{I}_C} + P_{\mathcal{O}_C}$. Since the degree of $P_{\mathcal{O}_C}$ is 1, we must have $\alpha_d(\mathcal{I}_C) = \alpha_d(\mathcal{O}_C)$ and thus $\mathrm{rk}(\mathcal{I}_C) = 1$. For stability, first note that $\mathcal{I}_C$ is pure: any subsheaf of lower dimension must be torsion, but $\mathcal{I}_C \subseteq \mathcal{O}_X$. Now by [HL10, p. 1.2.6] stability is equivalent to $p(\mathcal{I}_C) < p(\mathcal{Q})$ for all proper quotient sheaves $\mathcal{I}_C \to \mathcal{Q}$ with $\alpha_{\dim X}(\mathcal{Q}) > 0$. This is trivially satisfied, since the last condition would imply that $\mathcal{I}_C$ has a nontrivial subsheaf of lower rank, i.e., rank zero since X is integral, which contradicts purity. □

Remark 1.1.15. This is the easy direction of an equivalence (see [Mau+06a, p. 1.4] or [Tho00, p. 3.40] for the other direction). Furthermore, that equivalence is only the bijection on points between the Hilbert functor, and the relevant stable sheaf functor. Later we will take this equivalence as granted and freely pass between the Hilbert scheme and the moduli space of stable sheaves with fixed determinant.

Hilbert schemes of points

Using the definition above, we may define *the Hilbert scheme of n points* to be the component of Hilb indexed by the constant polynomial n. The subschemes parametrized by this component will be zero dimensional of *length n*. In the case of affine space, these subschemes are exactly those corresponding to ideals

$I \subset \mathbb{C}[x_1, \ldots, x_d]$ of *colength n*. Hilbert schemes of points underlie the *degree zero* DT invariants and will be important in the following sections. We recall some important properties.

The Hilbert scheme of points is well-behaved in dimension ≤ 3: If X is nonsingular then $\mathrm{Hilb}^n X$ is nonsingular when $\dim X \leq 2$ or $n \leq 3$; the bounds are strict for any X [Fan06, p. 7.2]. In the surface case $\mathrm{Hilb}^n S$ is a resolution of singularities over $\mathrm{Sym}^n S$ [Fog68].

Theorem 1.1.16 (Göttsche formula, [Göt90])**.** *Let S be a smooth projective surface over $\mathbb{C}$. The generating series for Euler characteristics of Hilbert schemes of points on S is given by the following:*

$$\sum_{n=0}^{\infty} \chi(\mathrm{Hilb}^n S) q^n = \prod_{m=1}^{\infty} (1 - q^m)^{-\chi(S)}. \tag{1.1}$$

Example 1.1.17 (K3 surfaces)**.** It's well known that Hilbert schemes points on $K3$ surfaces provide a large class of *hyperkähler manifolds*, those of so-called $K3^{[m]}$-type [Deb20]. Using the Göttsche formula, we may compute their Euler characteristics: The right side of (1.1) is approximated in *Mathematica*®by

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GF[u_, q_, x_] := Product[(1 - q^i)^-x, {i, u}]
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and since the topological Euler characteristic of any $K3$ surface S is 24, we compute via

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n = 6(*number of terms*);
Series[GF[n + 1, q, 24], {q, 0, n}];
CoefficientList[%, q]
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that $\chi(\mathrm{Hilb}^n S) = 24, 324, 3200, 25650, 176256, 1073720, \ldots$.

Theorem 1.1.18 (Cheah formula, [Che96])**.** *Let X be a smooth projective threefold over $\mathbb{C}$. The generating series for Euler characteristics of Hilbert schemes of points on X is given by the following:*

$$\sum_{n=0}^{\infty} \chi(\mathrm{Hilb}^n X) q^n = \prod_{m=1}^{\infty} (1 - q^m)^{-m\chi(X)} = M(q)^{\chi(X)} \tag{1.2}$$

where

$$M(q) = \prod_{m=1}^{\infty} (1 - q^m)^{-m}$$

is the MacMahon function [Mac16].

1.2 Stable pairs

From the perspective of counting curves, one of the main problems with Hilbert schemes is the presence of subschemes with "free points" in the component of any curve. Spaces of stable pairs give a different compactification of embedded curves. The main references for this section will be [Le 93; PT09a; PT14a; ST11].

Generalities

Definition 1.2.1 (Coherent System/Pair, [Le 93, p. 61])**.** Let X be a nonsingular projective variety of dimension n. A *coherent system* $(\mathcal{F}, \Gamma)$ of dimension d is a coherent sheaf $\mathcal{F}$ on X of dimension d and a subspace $\Gamma \subseteq H^0(\mathcal{F})$. A coherent system is a *pair* if $\dim \operatorname{Supp} \mathcal{F} = 1 = \dim \Gamma = \langle s \rangle$, which we often written simply as $(\mathcal{F}, s)$.

Remark 1.2.2. A coherent system is a generalization of a *linear system*, when $\mathcal{F}$ is a line bundle. Recall that $H^0(X, \mathcal{F}) \cong Hom(\mathcal{O}_X, \mathcal{F})$, therefore we will also often write a pair $(\mathcal{F}, s)$ as $s : \mathcal{O}_X \to \mathcal{F}$.

Definition 1.2.3 (Reduced Hilbert polynomial of a coherent system, [Le 93, p. 62])**.** Let $(\Gamma, \mathcal{F})$ be a coherent system on X. Let P_X denote the Hilbert polynomial of X, r the multiplicity of $\mathcal{F}$, and $p_{\mathcal{F}}$ the reduced Hilbert polynomial of $\mathcal{F}$. We define the *reduced Hilbert polynomial of* $(\Gamma, \mathcal{F})$ to be

$$p_{(\Gamma, \mathcal{F})} = \frac{\dim \Gamma}{r} P_X + p_{\mathcal{F}}.$$

We can now define the (semi)stability of coherent systems completely analogously to that of coherent sheaves:

Definition 1.2.4 (stability for coherent systems, [Le 93, p. 62])**.** A d-dimensional coherent system $(\Gamma, \mathcal{F})$ is *semistable* if,

(i) $\mathcal{F}$ is pure of dimension d, i.e., $\mathcal{F}$ has no nontrivial subsheaves of dimension $< d$, and

(ii) For any nontrivial coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$, the coherent system $(\Gamma', \mathcal{F}') = (\Gamma \cap H^0(\mathcal{F}'), \mathcal{F})$ we have

$$p_{(\Gamma', \mathcal{F}')} \leq p_{(\Gamma, \mathcal{F})}.$$

We say that $(\Gamma, \mathcal{F})$ is *stable* if this inequality is strict.

Example 1.2.5. Consider a coherent system $(0, \mathcal{F})$. We have $p_{(0,\mathcal{F})} = p_{\mathcal{F}}$, and therefore $\mathcal{F}$ is (semi)stable if and only if $(0, \mathcal{F})$ is.

Proposition 1.2.6 ([PT09a])**.** *A pair $(\mathcal{F}, s)$ is stable if and only if $\mathcal{F}$ is pure and $s : \mathcal{O}_X \to \mathcal{F}$ has zero dimensional cokernel.*

We begin to discuss moduli of stable pairs, and include the following definition of families of coherent systems for completeness.

Definition 1.2.7 (Families of coherent systems)**.** Let X be a smooth projective variety of dimension d. A *flat family of coherent systems* on X, parametrized by S consists of two sheaves $\mathscr{F}, \mathcal{Q}$, where

(i) $\mathscr{F}$ is coherent on $X \times S$ which is flat over S, and

(ii) $\mathcal{Q}$ is a locally free quotient of the relative Ext sheaf $\mathcal{E}xt^d_q(\mathscr{F}, p^*\omega_X)$.

Remark 1.2.8 (Fiber over a point)*.* Let us check that a family over a point is a pair. Le Potier shows [Le 93, Lemma 4.9] that over a closed point $s \in S$,

$$\mathcal{E}xt^d_q(\mathscr{F}, p^*\omega_X) = \operatorname{Ext}^d(\mathscr{F}_s, \omega_X).$$

By Serre duality, $\operatorname{Ext}^d(\mathcal{F}_s, \omega_X) = H^0(\mathcal{F}_s)$, and thus $\mathscr{F}_s = \mathcal{F}$ and the kernel of the quotient of $H^0(\mathcal{F}_s)$ defines a pair on X.

Theorem 1.2.9 ([Le 93, Théorème 4.13]). *Let X be a smooth projective variety, there is a projective scheme $\operatorname{Syst}^P X$ representing a functor Syst^P_X,*

$$\begin{aligned} \operatorname{Syst}^P_X : \mathsf{Sch} &\longrightarrow \mathsf{Set} \\ T &\longmapsto \left\{ \begin{matrix} \textit{flat families of semistable coherent systems} \\ (\mathscr{F}, \mathscr{Q}) \textit{with } P_{\mathscr{F}_t} = P \textit{ for all } t \in T \end{matrix} \right\}. \end{aligned}$$

Remark 1.2.10. Note that we may write this scheme as a disjoint union

$$\operatorname{Syst}^P X = \bigsqcup_{n \in \mathbb{N}} \operatorname{Syst}^{P,n} X$$

where $\operatorname{Syst}^{P,n} X$ parametrizes coherent systems with $\dim \Gamma = 1$. Thus, choosing a Hilbert polynomial P of degree one, we see that the space $\operatorname{Syst}^{P,1} X$ parametrizes stable pairs on X. From now on we will denote this space by $\operatorname{Pairs}^P X$.

Geometric aspects

In specializing from coherent systems to stable pairs, several geometric aspects emerge. In the best case, i.e., when a pair has smooth supporting curve, it is equivalent data to a Weil divisor on that curve.

Proposition 1.2.11 ([PT09a, p. 412]). *Let $(\mathcal{F}, s)$ be a stable pair on X, and let $C_{\mathcal{F}}$ denote the schematic support of $\mathcal{F}$. Then $C_{\mathcal{F}}$ is a Cohen-Macaulay curve on X with structure sheaf isomorphic to* $\operatorname{im}(s)$*.*

Definition 1.2.12 (Zero locus of a stable pair, [PT09a]). Given a stable pair $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$, from 1.2.6 we had that the cokernel of s has zero-dimensional support. Define the *zero locus* of the pair as the reduced support scheme of $\operatorname{coker} s$.

Proposition 1.2.13 (Stable pairs on smooth curves, [PT09a]). *Stable pairs supported on a smooth curve C correspond to Weil divisors on C.*

Proof. Let $(\mathcal{F}, s)$ be a stable pair supported on C. $\mathcal{F}$ may be viewed as a pure rank one sheaf on C, which implies $\mathcal{F}|_C$ is locally free of rank one. Since C is nonsingular, $\mathcal{F}$ is then given by $\mathcal{O}_C(D)$ for a Weil divisor D. □

Even when C is singular, we always have stable pairs with any given zero locus:

Proposition 1.2.14. *For any Cohen-Macaulay curve $C \subset X$ and any finite collection of closed points $D = \bigsqcup p_i$ on C, there exists a stable pair $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ with $C_{\mathcal{F}} = C$ and zero locus D.*

Relation to the Hilbert scheme In some sense, the goal of this thesis is to explore the relation between the Hilbert scheme and the Pairs space. Here however, we simply remark the following similarity: As we saw in the last section the Hilbert scheme may be thought of as parametrizing ideal sheaves $\mathcal{I}_Z \subset \mathcal{O}_X$, which are equivalent to quotients of the structure sheaf, $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$.

Stable pairs $\mathcal{O}_X \to \mathcal{F}$ are in one sense a relaxation of this, in that the map need not be surjective, but with zero dimensional cokernel, and in another sense a restriction, requiring that the sheaf $\mathcal{F}$ be pure. Both stable pairs schemes and Hilbert schemes parametrize stable objects in the derived category, with two different notations of stability.

1.3 Quiver representations

The main references for this section will be [Kin94; Ric21; Sze08; Tod21]. The aim is to define moduli spaces of quiver representations, and especially to realize certain examples, such as the Hilbert scheme of points $\mathrm{Hilb}^n \mathbb{A}^3$ and the local $\mathbb{P}^1$, relevant for this thesis as such spaces. More generally, quivers with superpotentials define varieties which are critical loci in smooth ambient varieties, and therefore carry a canonical symmetric perfect obstruction and zero dimensional virtual class. This makes them model spaces for defining and computing enumerative invariants.

Basic definitions

A *quiver* is simply a finite directed graph, allowing for multiple edges and loops. To fix our notation we make the following definition in accordance with [Sze08].

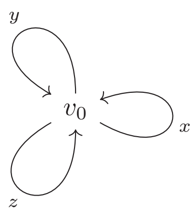
Definition 1.3.1. A *quiver* $Q = (V, E, h, t : E \to V)$ consists of a finite vertex set V, a finite edge set E, and maps h, t determining the orientation of a given edge. If the vertices are indexed by a set I, we will sometimes abuse notation and take values for h and t in I.

Example 1.3.2 (The A_2 quiver)**.** The A_2 quiver is $Q = (\{v_0, v_1\}, \{e\}, h : e \mapsto v_1, t : e \mapsto v_0)$, and can be drawn as,

$$v_0 \xrightarrow{e} v_1$$

Path algebra Given a quiver, it is natural to consider the paths contained therein. In the set of all paths, including the do-nothing path at each vertex, one may define a binary operation by concatenation if the two paths match up, and as zero otherwise. Taking formal linear sums over a given base field, k along with this operation defines an algebra known as the *path algebra*, kQ (as a k-vector space it has a basis consisting of all paths in Q).

Example 1.3.3 (The $\mathbb{A}^3$ quiver)**.** Below we have the quiver with $V = \{v\}$, $E = \{x, y, z\}$ and constant maps $h, t : e \mapsto v$.



The path algebra of Q is given by linear sums of monomials in x, y and z, i.e., $\mathbb{C}Q = \mathbb{C}\langle x, y, z\rangle$.

Superpotentials For any quiver Q, an element of the quotient by the commutator, $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ is called a *superpotential.* Since the product of non-adjacent paths is zero, we immediately see that superpotentials are represented by formal sums of *closed* paths in Q. Similarly, from the commutator relation, we get that cyclic permutations of a given path represent the same superpotential.

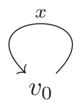
This last observation makes the following "formal differentiation" well-defined on superpotentials.

Definition 1.3.4 (Noncommutative derivative)**.** Given a cycle C in Q containing edges $e_1, \dots e_n$, define the formal derivative with respect to some edge $e \in Q$ as:

$$\partial_e C = \sum_{i=0}^{n-1} d_e(\sigma_i(C)), \quad d_e(\sigma_i(C)) = \begin{cases} w & \text{if } \sigma_i(C) = ew \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma_i(C)$ denotes the i-th cyclic permutation of C and w is some word in the e_i. This operation extends linearly to superpotentials, yielding a linear map $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q] \to \mathbb{C}Q$. Given a superpotential W we may thus define the *Jacobian ideal* just as in other contexts, $I_W = \langle \partial_e W \mid e \in E\rangle$.

Example 1.3.5 (One-loop quiver)**.** Consider the quiver $Q = (\{v_0\}, \{x\}, h, t)$.



with superpotential $W = x^{n+1}$. We see that noncommutative derivative agrees with the ordinary derivative, $\partial_x W = (n+1)x^n$.

Example 1.3.6 (Formal differentiation and Jacobian ideal)**.** Consider the $\mathbb{A}^3$ quiver Q from (1.3.3), $W = xyz - xzy$ defines a non-zero superpotential, and

$$\partial_x W = yz - zy, \quad \partial_y W = zx - xz, \quad \partial_z W = xy - yz.$$

Thus, we have $I_W = \langle yz - zy, zx - xz, xy - yz\rangle$, and therefore $\mathbb{C}Q/I_W \simeq \mathbb{C}[x, y, z]$.

Quiver representations and moduli

Definition 1.3.7 (Quiver representation)**.** A *representation* of a quiver $Q = (V, E, h, t)$ consists of the following data:

i. vector spaces V_i, one for each vertex v_i in V,

ii. linear maps $\phi_e : V_{t(e)} \to V_{h(e)}$, one for each $e \in E$.

Given a $|V|$-tuple, or *dimension vector*, $d = (d_1, \dots, d_{|V|}) \in \mathbb{N}^{|V|}$, we say that a Q-representation is *d-dimensional* if $\dim V_i = d_i$ for all $1 \le i \le |V|$. A morphism of Q-representations $f : (\{V_i\}, \{\phi_e\}) \to (\{U_i\}, \{\psi_e\})$ is a collection of linear maps $f_i : V_i \to U_i$ commuting with the maps ϕ_e, ψ_e.

Remark 1.3.8 (Path-algebra representations). *Representations* of a path algebra $\mathbb{C}Q$ are simply modules over $\mathbb{C}Q$. It can be easily shown that for any quiver Q, the category of quiver representations $\mathsf{Rep}(Q)$ is equivalent to the category of finitely generated $\mathbb{C}Q$ representations. This follows almost immediately from defining the functor, i.e., the map determining a $\mathbb{C}Q$ representation from a quiver representation, cf. [Cra08, Proposition 4.2]. Note that this implies that $\mathsf{Rep}(Q)$ is an abelian category. We will sometimes use the language of modules when describing quiver-representations and vice versa.

Before we consider moduli of quiver representations, we must first describe the isomorphism classes. From (1.3.7) it is clear that a morphism of quiver representations is an isomorphism if and only if the maps f_i are isomorphisms. Thus, from the perspective of determining isomorphism classes of quiver representations, the f_i might as well be *automorphisms*, i.e., elements $\mathrm{GL}(V_i)$. Indeed, with any given d-dimensional choice of vector spaces $\{V_i\}$, we can describe the isomorphism classes of d-dimensional Q-represention as the orbits of $G = \prod_{v\in V} \mathrm{GL}(V_i)$ acting on $\mathcal{R}(Q,d) = \bigoplus_{e\in E} \mathrm{Hom}(V_{t(e)}), V_{h(e)})$ by $g \cdot \phi_e \mapsto g_{h(e)}\phi_e g_{t(e)}^{-1}$. For example, if Q is the A_2 quiver (1.3.2), $v_1 \xrightarrow{e} v_1$, the action is given by

$$\begin{tikzcd} V_1 \arrow[r, "\phi_e"] \arrow[d, shift left, "g_1"] & V_2 \arrow[d, shift left, "g_2"] & \arrow[r, mapsto, "(g_1,g_2)\cdot"] & {} & V_1 \arrow[r, "\phi_e"] \arrow[d, shift left, "g_1"] & V_2 \arrow[d, shift left, "g_2"] \\ V_1 \arrow[u, shift left, "g_1^{-1}"] & V_2 \arrow[u, shift left, "g_2^{-1}"] & & & V_1 \arrow[u, shift left, "g_1^{-1}"] & V_2 \arrow[u, shift left, "g_2^{-1}"] \end{tikzcd}$$

Note that $\mathcal{R}(Q,d)$ is an affine space acted on by a linear reductive group. Thus, after determining the semi-stable locus, the moduli scheme can be described by the GIT quotient. By restricting to the view of *framed quivers* we achieve a nice description of this locus, namely the semi-stable representations will correspond to *cyclic* modules.

Definition 1.3.9 (Framed quiver)**.** A *(1-)framed* quiver is a quiver $\vec{Q}$ with a distinguished vertex (often denoted by ∞) and a single edge emanating from ∞ (the vertex it points to is often denoted by 0, or k). A framed quiver representation is a quiver representation with dimension vector d such that $\dim d_\infty = 1$.

Suppose that a framed quiver $\vec{Q}$ is defined by augmenting an ordinary quiver $Q = (V,E,h,t)$ by adding the vertex v_∞ to V, adding edge e_∞ pointing to some vertex v_k, and adjusting h and t accordingly. The space of framed representations of $\vec{Q}$ with dimension vector $\vec{d} = (1,d)$ (that is, $\vec{d}_\infty = 1$ and d is a dimension vector for Q), may be alternatively described as

$$\begin{aligned} \mathcal{R}(\vec{Q},\vec{d}) &= \prod_{e\in E\cup\{e_\infty\}} \mathrm{Hom}(V_{t(e)}, V_{h(e)}) \\ &= \prod_{e\in E} \mathrm{Hom}(V_{t(e)}, V_{h(e)}) \times \mathrm{Hom}(\mathbb{C}, V_k) \\ &= \prod_{e\in E} \mathrm{Hom}(V_{t(e)}, V_{h(e)}) \times V_k = \{(\phi_e)_{e\in E}, m \in V_k\} \end{aligned}$$

where the V_i are some starting choice of vector spaces satisfying d.

Now, following [Sze08], let S^0 be the open subvariety of $\mathcal{R}(\vec{Q}, \vec{d}) = \{(\phi_e)_{e \in E}, m \in V_k\}$ where the vector m generates the $\mathbb{C}Q$-module (V_i, ϕ_e), in other words, every V_i is spanned by vectors obtained by repeatedly applying the ϕ_e to m. If W is a superpotential for Q, let X denote the closed subscheme of S^0 of representations satisfying the relations $\partial_e W = 0$ for all $e \in E$ (1.3.4). Recalling the G action on $\mathcal{R}(\vec{Q}, \vec{d})$ from above, [Sze08] shows the following.

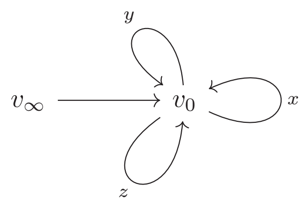
Theorem 1.3.10 ([Sze08, Proposition 1.2.2])**.**

(i) *There exists a smooth and quasi-projective geometric quotient N of S^0 by G, containing a closed subscheme $\mathcal{M}_{k,d} \subset N$ which is a quotient of X by G.*

(ii) *The space $\mathcal{M}_{k,d}$ carries a tautological family $(M_{k,d}, m_{k,d})$ of framed cyclic $\mathbb{C}Q/I_W$-modules, generated at the vertex k.*

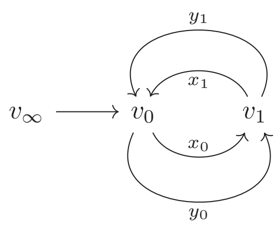
(iii) *The space $\mathcal{M}_{k,d}$ a fine moduli space representing a functor of certain $\mathbb{C}Q/I_W$-modules.*

Example 1.3.11 ($\mathrm{Hilb}^n \mathbb{A}^3$)**.** Consider the *framed* $\mathbb{A}^3$ quiver



with superpotential $W = xyz - xzy$ as in 1.3.6. From (1.3.10), the moduli space $\mathcal{M}_{v_0,n}$ parametrizes n-dimensional $\mathbb{C}[x, y, z]$-modules, with module structure induced by a linear surjection from $\mathbb{C}[x, y, z]$ (from the condition defining S^0, above). Such modules are identified with n-dimensional quotients of $\mathbb{C}[x, y, z]$, which we identify with the Hilbert scheme $\mathrm{Hilb}^n \mathbb{A}^3$.

Example 1.3.12 (Local $\mathbb{P}^1$)**.** Consider the following framed quiver



with superpotential $W = x_0x_1y_0y_1 - x_0y_1y_0x_1$ (the *Klebanov-Witten* superpotential). The algebra $A = \mathbb{C}Q/I_W$ is a non-commutative crepant resolution for the conifold singularity

$$Z = \mathrm{Spec}(\mathbb{C}[x, y, z, w]/(xy - zw)),$$

[Sze08, §2]. Most importantly, Van den Bergh showed in [Van04] that the bounded derived category $\mathrm{D^b}(X)$ where X is the local $\mathbb{P}^1$, is derived equivalent to $\mathrm{D^b}(A)$. Invariants based on $\mathrm{D^b}(X)$ may be calculated by studying the above quiver.

The condition defining S^0 used in 1.3.10 was a convenient way to describe the semistable locus with respect to the G-action. That the resulting moduli spaces in the cases above are also moduli of sheaves over varieties suggests a relation with notions of stability defined in previous sections. For quivers, we have the following more general definition.

Definition 1.3.13 (θ-stability, [Kin94])**.** Given a quiver Q, a dimension vector d, and a vector $\theta \in \mathbb{R}^{|V|}$ such that $\theta \cdot d = 0$. We say that a Q-representation (V_i, ϕ_e) of dimension d is *θ-semistable (stable)* if for any non-zero proper subrepresentation (V_i', ϕ_e') of dimension d' we have $\theta(v') \leq 0$ (resp. $\theta(v') < 0$).

Remark 1.3.14. Invariants defined for moduli of quiver representations $\mathcal{M}_{k,d}^{\theta}$ vary discretely with change of stability parameter θ, this is described as a *wall-chamber* structure.

Definition 1.3.15 (Trace of a superpotential)**.** Any superpotential W for a quiver Q, defines a regular function $\mathrm{Tr}(W) : \mathcal{R}(Q,d) \to \mathbb{C}$ called the *trace* of W. It suffices to define $\mathrm{Tr}(c)$ for an arbitrary cycle $c = e_{i_1} e_{i_2} \cdots e_{i_k}$ and extend linearly to any W.

$$\mathrm{Tr}(c) : \mathcal{R}(Q,d) \to \mathbb{C} \tag{1.3}$$

$$(V_i, \phi_i) \mapsto \mathrm{tr}(\phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_k}) \tag{1.4}$$

Remark 1.3.16. The trace of a superpotential $\mathrm{Tr}(W) = f$ descends to a regular function $\bar{f}$ on the moduli space $\mathcal{M}_{k,d}$ from 1.3.10. It can be shown that the vanishing of the 1-form *df* agrees with the vanishing of the Jacobian ideal I_W 1.3.4 (see [Sze08, Theorem 1.3.1] and [Seg08, Proposition 3.8]), and in fact, $\mathcal{M}_{k,d} = Z(d\bar{f})$ With (1.3.11), this shows that the Hilbert scheme $\mathrm{Hilb}^n \mathbb{A}^3$ is closed subscheme of smooth variety defined as a critical locus.

1.4 Torus actions and localization

Torus actions

Throughout this thesis, by tori we mean an *algebraic* torus, the scheme $\mathbb{C}^* = \mathrm{Spec}\, \mathbb{C}[t, t^{-1}]$ and higher products $(\mathbb{C}^*)^r = \mathrm{Spec}\, \mathbb{C}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$. They will often be denoted by $\mathbb{T}$. The ordinary group structure on $\mathbb{C}^*$ is algebraic, making tori into group schemes. Our main interest in this section will be to see how these groups act on a given space, and more importantly how that information can be leveraged to study the space being acted on. We require group actions to be *morphisms* $G \times X \to X$, commuting with the group operations. Likewise, a *representation* of a torus $\mathbb{T}$ is a morphism $\rho : \mathbb{T} \to \mathrm{GL}_n(\mathbb{C})$; the integer n is the *dimension* of the representation. One dimensional representations are called *characters*. It's well known that representations of algebraic tori decompose into sums of characters. As in the case of ordinary representations of groups, we will often refer to a representation as its underlying vector space when the action (the morphism ρ) is implicit.

Proposition 1.4.1. *The characters of* $\mathbb{C}^*$ *are given by* $\mathbb{C}$*-algebra maps* $\mathbb{C}[s^{\pm 1}] \to \mathbb{C}[t^{\pm 1}]$ *where* $t \mapsto s^w$ *for some* $w \in \mathbb{Z}$*. The integer* w *is then called the* weight *of the character.*

Proof. A character of $\mathbb{C}^*$ is a morphism $\operatorname{Spec}\mathbb{C}[t,t^{-1}] \to \operatorname{Spec}\mathbb{C}[s,s^{-1}]$, which is equivalent s to $\mathbb{C}$-algebra map $\varphi : \mathbb{C}[s^{\pm 1}] \to \mathbb{C}[t^{\pm 1}]$. This map is determined by the image of t, $\varphi(t)$ which must be invertible, i.e., $\varphi(t) = at^w$ for some integer w. Since the map must commute with multiplication by $\mathbb{C}$, we have $a = 1$. □

Remark 1.4.2. Knowing how representations of higher dimensional tori decompose and the above statement, we can identify the Grothendieck ring of representations of a torus $\mathbb{T} = (\mathbb{C}^*)^r$, i.e., the *representation ring* $R(\mathbb{T})$, with $\mathbb{Z}^r$. We also see that a given $\mathbb{T}$-representation V can be decomposed into *weight spaces* $V = \bigoplus_{\mathbb{Z}^r} V_\alpha$. Visually, this decomposition is the same as a $\mathbb{Z}^r$-grading of V. The correspondence between torus actions and graded modules is very general (see 1.4.3).

Proposition 1.4.3. *There is an equivalence of categories between* $\mathbb{T}$*-equivariant quasi-coherent sheaves and* $\mathbb{Z}^r$*-graded* A *modules:*

$$\mathsf{QCoh}^{\mathbb{T}}(\operatorname{Spec} A) \simeq \mathbb{Z}^r\text{-}\mathsf{GrMod}(R).$$

As a reference for this statement see [Gan19, Proposition 1.7].

Fixed loci

Definition 1.4.4 (Fixed-point scheme, [Mil17, §7.b])**.** Suppose $\mathbb{T}$ acts on a $\mathbb{C}$-scheme X by $a : \mathbb{T} \times X \to X$. We define the *fixed-point scheme* $X^{\mathbb{T}}$ as the largest subscheme of X on which $\mathbb{T}$ acts trivially. In other words, if $Z \subset X$ is any subscheme such that $a_t|_Z : Z \to X$ equals $\mathrm{id}_X|_Z : Z \to X$ (where a_t is the map $X \to X$ induced by fixing $t \in \mathbb{T}$), then inclusion $Z \hookrightarrow X$ factors uniquely through $X^{\mathbb{T}}$:

$$\begin{tikzcd} Z \arrow[d, dashed, "\exists ! g"'] \arrow[dr, hook] & \\ X^{\mathbb{T}} \arrow[r] & X \end{tikzcd}$$

Remark 1.4.5. Direct constructions of the fixed-point scheme are given in broad generality by [Fog73] and [Mil17]. In the case when X is smooth and projective, the fixed point locus is also smooth [Fog73; Ive72]. Iversen also proved that, in that case, the Euler characteristic (as the top Chern class of the tangent bundle) of X agrees with the Euler characteristic of the smooth fixed locus. More generally, Bialynicki-Birula proved the following, which we sometimes call “Euler localization”:

Theorem 1.4.6 ([Bia73, Corollary 2])**.** *Suppose that* X *is acted on by* $\mathbb{T} = \mathbb{C}^*$*, then*

$$\chi(X) = \chi(X^{\mathbb{T}}),$$

where χ *is the Euler-Poincaré characteristic of the underlying topological space.*

Example 1.4.7 ($\chi(\mathbb{P}^1) = 2.$)**.** Consider the action on $\mathbb{P}^1 = \operatorname{Proj}\mathbb{C}[x_0, x_1]$ given by $t \cdot (x_0, x_1) = (tx_0, t^2x_1)$. The only $\mathbb{C}^*$ fixed points are $[0;1]$ and $[1;0]$, so $|(\mathbb{P}^1)^{\mathbb{T}}| = 2 = \chi(\mathbb{P}^1)$.

Proposition 1.4.8 ([Ric21])**.** *Consider the standard action of* $\mathbb{T} = \mathbb{C}^{*d}$ *on* $\mathbb{A}^d$ *by scaling in the coordinate ring* $R = \mathbb{C}[x, y, z]$*. There are bijections between the following,*

(i) *fixed subschemes of length n,*

(ii) *monomial ideals in R of colength n,*

(iii) *d-dimensional partitions of size n.*

Corollary 1.4.9 (Euler)**.** *The generating series for the Hilbert scheme of points on $\mathbb{A}^2$ is:*

$$\sum_{n=1} \chi(\mathrm{Hilb}^n \mathbb{A}^2) q^n = \prod_{n=1} \frac{1}{1-q^n}.$$

Example 1.4.10. We can use Euler localization (1.4.6) to compute Euler characteristics of Hilbert schemes of points on projective spaces. The $\mathbb{P}^2$ and $\mathbb{P}^3$ columns can be checked by formulas of Göttsche [Göt90] and Cheah [Che96].

	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^3$	$\mathbb{P}^4$	$\mathbb{P}^5$	$\mathbb{P}^6$
$\chi(\mathrm{Hilb}^1(-))$	2	3	4	5	6	7
$\chi(\mathrm{Hilb}^2(-))$	3	9	18	30	45	63
$\chi(\mathrm{Hilb}^3(-))$	4	22	64	140	260	434
$\chi(\mathrm{Hilb}^4(-))$	5	51	215	615	1410	2800
$\chi(\mathrm{Hilb}^5(-))$	6	108	660	2476	7026	16632

Example 1.4.11 (Fixed loci of $\mathrm{Pairs}^P X$)**.** For X a nonsingular toric threefold, we have from [PT09b, Theorem 1] that any connected component of the $\mathbb{T}$-fixed locus is a product of $\mathbb{P}^1$'s, noting that the 0-th product is a point. Pandharipande and Thomas show this by characterizing the fixed points in terms of box configurations analogous to 1.4.8.

Atiyah-Bott localization

Here we will introduce Atiyah-Bott localization and review some applications. The statement is given in terms of equivariant cohomology/Euler classes, which we recall briefly. The main consequence of the theorem is that if a torus (here in the Lie group setting, which will denote by T, instead of $\mathbb{T}$) acts on a compact manifold, then the map on equivariant cohomology induced by the inclusion of the fixed locus becomes an isomorphism after a certain localization (in the commutative algebra sense). Moreover, the inverse map can be described explicitly, and can be used effectively for computations in the form of the *integration formula* 1.4.17.

Though we are primarily interested in applying localization in the algebraic setting, we will instead recall notions from equivariant (singular) cohomology and appeal to the analogy between homology and the Chow ring. This avoids the problem of dealing with the Borel construction algebraically. Equivariant (co)homology is a generalization of ordinary (co)homology that respects a given group action ($H^*_G = H^*$ when G acts trivially). In the best case, when the group action is free, the equivariant (co)homology is equal to the ordinary cohomology of the quotient.

In fact, the principles that the equivariant theory should behave like the cohomology of the quotient and also be homotopy invariant lead naturally to the construction 1.4.13, which first requires the following gadget.

Proposition 1.4.12 ([tDie08, §14.4])**.** *For any group G there exists a weakly contractible space EG on which G acts freely.*

It is the total space of the *universal bundle* $EG \to BG$, where BG is the *classifying space* of G and which we may take as the quotient of EG by the G action. We remark that this is also a universal object in the sense of the previous sections: BG represents the functor from the homotopy category to sets, mapping spaces Z to isomorphism classes of principal G-bundles over Z.

Definition 1.4.13 (Equivariant cohomology)**.** Let a group G act on a topological space X. By 1.4.12, the space $X \times EG$ has the same homotopy type as X and is acted on freely by G. Let $(X \times_G EG)$ denote $(X \times EG)/G$, this is known as the *Borel construction or Borel space.* Define

$$H^*_G(X) = H^*_G(X \times_G EG).$$

Note that when $X = \mathrm{pt}$, then $H^*_G(X) = H^*(BG)$.

Example 1.4.14 ($G = \mathbb{C}^*$, $X = \mathrm{pt}$)**.** G acts freely on $\mathbb{C}^n \setminus \{0\}$ by scaling, the homotopy groups $\pi_i(\mathbb{C}^n \setminus \{0\})$ are trivial for all $i < n$. Thus, by the universal property, we may take $EG = \mathbb{C}^\infty \setminus \{0\}$, whence $BG = \mathbb{P}^\infty$. Therefore,

$$H^*_G(X) = H^*_G(\mathbb{P}^\infty) = \mathbb{Z}[s_1].$$

Likewise, $H^*_{\mathbb{C}^{*r}}(X) = H^*_G((\mathbb{P}^\infty)^r) = \mathbb{Z}[s_1, \dots, s_r]$, with s_i of cohomological degree 2.

Definition 1.4.15 (Equivariant vector bundles)**.** Let a group G act on a topological space X. A *G-equivariant vector bundle* on X is a vector bundle $\pi : V \to X$ with a choice of G-action on V, $a_V : G \times V \to V$ commuting with projection and the action on X:

$$\begin{tikzcd} G \times V \arrow[r, "a_V"] \arrow[d, "\mathrm{id}\times\pi"'] & V \arrow[d, "\pi"] \\ G \times X \arrow[r, "a_X"] & X. \end{tikzcd} \tag{1.5}$$

Definition 1.4.16 (Equivariant Chern classes/Euler class)**.** Given a G-equivariant vector bundle V over X (1.4.15), we define the *G-equivariant Chern classes*, $c_i^G(V) \in H^*_G(X)$ to be the ordinary Chern classes (in cohomology) of the bundle induced by (1.5): $V \times_G EG \to X \times_G EG$. We define the *equivariant Euler class* as the top equivariant Chern class: $e^{\mathbb{T}}(V) = c^{\mathbb{T}}_{\mathrm{rk}\, V}(V)$, i.e., the Euler class of induced bundle.

Proposition 1.4.17 ([AB84], [BV83, Théorème 1.6])**.** *Let M be a compact manifold with the action of a torus T with an isolated set of fixed points M^T. For any $\phi \in H^*_T(M)$ we have*

$$\int_M \phi = \sum_{P \in M^T} \frac{i_P^* \phi}{e^T(N_{P|M})}$$

where $\int_M \phi$ denotes the pushforward of ϕ to a point, i_P is the inclusion of a point $P \hookrightarrow M$ and $e(N_{P|M})$ is the Euler class of the normal bundle at P.

Example 1.4.18 (Euler localization)**.** Using the integration formula (1.4.17) we can retroactively verify the principle of "Euler localization" above (in the smooth manifold case). Let T act on a manifold M with isolated fixed points

M^T. First, by the Poincaré-Hopf theorem, the Euler characteristic $\chi(M)$ is equal to the degree of top Chern class of the tangent bundle, from there we can apply localization:

$$\begin{aligned}\chi(M) &= \int_M e(\mathcal{T}M) \in H_0(\mathrm{pt}) \cong \mathbb{Z} \\ &= \sum_{P\in M^T} \frac{e(\mathcal{T}_P M)}{e^T(N_{P|M})} \\ &= \sum_{P\in M^T} \frac{e(\mathcal{T}_P M)}{e(\mathcal{T}_P M)} = |M^T|\end{aligned}$$

since the normal bundle to a point is the tangent space.

The analogous theory of equivariant Chow groups were defined in [EG98a], and, of greater utility for our purposes, the authors proved the following algebraic analogy of 1.4.17:

Proposition 1.4.19 ([EG98b])**.** *Let X be a smooth proper variety with the action of a torus $\mathbb{T}$ with an isolated set of fixed points $X^{\mathbb{T}}$, for any $\alpha \in A^T_{*,loc}$ we have*

$$\int_X \alpha = \sum_{Z\in X^T} \frac{i_Z^* \alpha}{e^{\mathbb{T}}(N_{Z|X})}. \tag{1.6}$$

Moreover, if $a \in A_0 X$ is the pullback of an element $\alpha \in A_0^{\mathbb{T}} X$, then $\deg a$ *may be evaluated by 1.6.*

Example 1.4.20 (27 lines on a cubic surface)**.** Consider the $\mathbb{T} = \mathbb{C}^*$ action on $\mathbb{P}^3 = \operatorname{Proj} \mathbb{C}[x_0, x_1, x_2, x_3]$ scaling by $t \cdot (x_0, x_1, x_2, x_3) \to (t^{w_0} x_0, t^{w_0} x_1, t^{w_0} x_2, t^{w_0} x_3)$ with unique weights $w_0, w_1, w_2, w_3 \in \mathbb{Z}$. This action lifts to the Grassmannian $G = \mathbb{G}(1,3)$ and has 6 fixed points corresponding to the coordinate axes: $\{\ell_{i,j} = Z(x_i, x_j)\}_{i<j}$. Recall the universal sequence ([EH16, §3.2]):

$$0 \to \mathcal{S} \to \mathcal{O}_G \otimes V \to \mathcal{Q} \to 0 \tag{1.7}$$

where $\mathcal{S}$ and $\mathcal{Q}$ are the universal sub and quotient bundles and $\mathbb{P}V = \mathbb{P}^3$, i.e., $V^* = H^0(\mathcal{O}_{\mathbb{P}^3}(1))$. It can be shown, say from the proof of [EH16, Proposition 6.4], that any cubic form f induces a global section σ_f of $\operatorname{Sym}^3 \mathcal{S}^*$ whose zero locus in G is exactly the lines contained in $X = Z(f)$. Thus, if this locus of lines is zero dimensional and reduced (which indeed it is, though we will omit this part) then the number of lines on a cubic surface is given by the degree of $c_4(\operatorname{Sym}^3 \mathcal{S}^*)$, the top Chern class ($\operatorname{Sym}^3 \mathcal{S}^*$ has rank 4 since $\operatorname{rk} \mathcal{S}^* = 2$). By

$$\begin{aligned}|\text{lines in } X| &= \int_{\mathbb{G}(1,3)} e(\operatorname{Sym}^3 \mathcal{S}^*) = \int_{\mathbb{G}(1,3)} e^{\mathbb{T}}(\operatorname{Sym}^3 \mathcal{S}^*) \\ &= \sum_{\ell_{i,j}} \frac{e^{\mathbb{T}}(\operatorname{Sym}^3 \mathcal{S}^*)|_{\ell_{i,j}}}{e^{\mathbb{T}}(N_{\ell_{i,j}|G})} \quad \text{(by localization)}\end{aligned}$$

Applying the Whitney sum formula to the normal bundle exact sequence (short, in this case) we find that $e^{\mathbb{T}}(N_{\ell|G}) = e^{\mathbb{T}}(T_\ell G)$. Recall also (again [EH16, §3.2]) that the tangent bundle of the Grassmannian is given by

$\mathcal{T}G = \mathcal{H}om(\mathcal{S}, \mathcal{Q}) = \mathcal{S}^* \otimes \mathcal{Q}$, so $T_\ell G = \mathcal{S}^*|_\ell \otimes \mathcal{Q}|_\ell$. By the tautological property of $\mathcal{S}$, we see $\mathcal{S}^*|_{\ell_{ij}} \cong \langle x_i, x_j \rangle$. It follows that

$$e^{\mathbb{T}}(T_{\ell_{i,j}} G) = \prod_{k \neq i,j} (w_i - w_k)(w_j - w_k),$$

and

$$\operatorname{Sym}^3 \mathcal{S}^*|_{\ell_{i,j}} \cong \operatorname{Sym}^3 \langle x_i, x_j \rangle \cong \langle x_i^3, x_i^2 x_j, x_i x_j^2, x_j^4 \rangle$$

so

$$e^{\mathbb{T}}(\operatorname{Sym}^3 \mathcal{S}^*)|_{\ell_{i,j}}) = (3w_i)(2w_i + w_j)(w_i + 2w_j)(3w_j).$$

We have computed,

$$|\text{lines in } X| = \sum_{\ell_{i,j}} \frac{(3w_i)(2w_i + w_j)(w_i + 2w_j)(3w_j)}{\prod_{k \neq i,j}(w_i - w_k)(w_j - w_k)} \tag{1.8}$$

which, for distinct weights w_0, w_1, w_2, w_3, simplifies to 27. This can be checked using various w_i using the following *Mathematica*®code:

```
lindices = {{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}};
loacs[w__] := Sum[
    ((3 w[[l[[1]]]]) * (2 w[[l[[1]]]] + w[[l[[2]]]]) *
    (w[[l[[1]]]] + 2 w[[l[[2]]]]) * (3 w[[l[[2]]]])) /
    Product[(w[[l[[1]]]] - w[[k]]) * (w[[l[[2]]]] - w[[k]]),
        {k, Complement[{1, 2, 3, 4}, l]}],
    {l , lindices}];

w = {0, 1, 2, 3};
loacs[w]
```

Remark 1.4.21 (Lines on a quintic threefold). It can be shown by the same technique that the general quintic hypersurface in $\mathbb{P}^4$ contains 2875 lines. Unlike the previous example, the locus in the Grassmannian of lines on such a threefold, i.e., the *Fano scheme* is not always reduced and zero dimensional. Indeed, the Fermat quintic, for example, contains 1-dimensional *families* of lines. Already from this example, we see that defining *invariants* based on counts of embedded curves in threefolds will require more sophisticated constructions.

CHAPTER 2

Donaldson-Thomas theory

In this chapter we introduce the main technical background and techniques for defining and computing Donaldson-Thomas invariants.

2.1 Virtual fundamental classes

In realizing spaces of ideal sheaves, stable pairs and quiver representations as schemes, we have crossed the first major hurdle in most modern enumerative problems. The language of schemes gives us several tools with which to define invariants and extract enumerative information. Nevertheless, some of these schemes are notoriously unwieldy (cf. [Vak06]). Indeed, the simplest moduli space relevant for enumerative invariants of threefolds is the Hilbert scheme of points $\mathrm{Hilb}^n \mathbb{A}^3$; this is singular already when $n = 3$ and reducible for $n \gg 0$.

The solution is to consider instead *virtual classes*, which represent a "corrected" version of these spaces. Any such notion should accord with the following principles:

1. A virtual class should be of expected dimension.

2. It should agree with the ordinary fundamental class in ideal settings, i.e., when it's representing something smooth and/or defined by transverse intersections.

3. It should, in some sense, be deformation invariant.

The following two examples are familiar situations invoking the principles of virtual cycles.

Example 2.1.1. The "virtual" dimension is a familiar concept: Let Y by a smooth variety of dimension d, and let E be a rank r vector bundle on Y. Consider $X = Z(s)$, the zero locus of some section of E. Locally, X is cut out by r equations and therefore the expected dimension or *virtual dimension* of X is $d - r$. It may happen that $\dim X \geq d - r$, yet there is a still a natural choice of cycle class associated to X of the "correct" dimension, the r-th Chern class of E. Indeed, if X is purely $d - r$ dimensional, then we have

$$c_r(E) = [X] \in A_{d-r}(X),$$

see [Ful98, Example 3.2.16].

Example 2.1.2 (Bezout)**.** Consider now $Y = \mathbb{P}^n$ and $\mathcal{E} = \mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \oplus \cdots \oplus \mathcal{O}(d_r)$. The vanishing $X = Z(s)$ of a section of $\mathcal{E}$ is then the intersection of r hypersurfaces of degrees d_i. By the Whitney sum formula we have

$$c_r(\mathcal{E}) = \prod_i c_1(\mathcal{O}(d_i)) = \prod_i d_i H$$

where H is the class of a hyperplane. When $r = n$ we are used to thinking of the right side of the equality as the "expected class" by Bezout's theorem.

Idealized virtual classes

Problems requiring virtual classes often arise in intersection theory. A well-defined intersection product has to handle degenerate cases such as non-transverse, "excess," and even self-intersection. The intersection theory of Fulton-MacPherson provides a robust framework for such problems, starting with idea of deformation to the normal cone. We apply similar ideas to define a virtual class $[X]^{\mathrm{vir}}$ in the case of a closed embedding of a scheme X into an ambient nonsingular variety Y, and call this the idealized setting. Later we will see that this, in fact, a local model for the far more general definition of a virtual fundamental class.

We first recall for a vector bundle $E \to X$ the flat pullback and the associated Gysin homomorphisms. This, along with the normal cone of an embedding $X \hookrightarrow Y$, will allow us to define a cycle, which at least is of the correct dimension.

Proposition 2.1.3 ([Ful98, Theorem 3.3])**.** *Let $\pi : E \to X$ be a vector bundle on a scheme X. The flat pullback*

$$\pi^* : A_{k-r}(X) \to A_k(E)$$

is an isomorphism for all k.

Remark 2.1.4 (Gysin maps). In the case of 2.1.3, we observe that $\mathrm{id}_X = \pi \circ 0_E$, where $0_E : X \to E$ denotes the zero section. Thus, the maps $0_E^* : A_k(E) \to A_{k-r}(X)$ are equal to the inverse of the π^*; these are *Gysin homomorphisms* [Ful98, p. 65].

Definition 2.1.5 (Normal cone, [Ful98, §4.2])**.** Let X be a closed subscheme of a scheme Y with ideal sheaf $\mathcal{I}_X$. We define the *normal cone* to X in Y to be

$$C_{X/Y} = \operatorname{Spec} \bigoplus_{n=0}^{\infty} \mathcal{I}_X^n / \mathcal{I}_X^{n+1}.$$

If Y is of pure dimension d, then $C_{X/Y}$ is of pure dimension d, [Ful98, B6.6].

Remark 2.1.6 (Normal sheaf). Recall that the normal sheaf is the dual of $\mathcal{I}_X/\mathcal{I}_X^2$ [Har77, II, §8]. In the smooth case, the normal *bundle* is then given by $\operatorname{Spec} \operatorname{Sym} \mathcal{I}_X/\mathcal{I}_X^2$. The natural surjection $\operatorname{Sym} \mathcal{I}_X/\mathcal{I}_X^2 \twoheadrightarrow \bigoplus_{n=0}^{\infty} \mathcal{I}_X^n/\mathcal{I}_X^{n+1}$ is an isomorphism when X is nonsingular.

Definition 2.1.7 (Idealized virtual classes, [BCM20])**.** Suppose a scheme X admits a closed embedding, $i : X \hookrightarrow Y$, into a smooth ambient variety Y of dimension d. An *obstruction bundle* is a vector bundle $E_{X/Y}$ over X with an embedding

$$C_{X/Y} \hookrightarrow E_{X/Y}$$

where $C_{X/Y}$ is normal cone. We define the *virtual fundamental class* with respect to $E_{X/Y}$ to be

$$[X]^{\text{vir}}_{E_{X/Y}} = 0_E^*[C_{X/Y}].$$

Example 2.1.8. Following [Ric21], we can easily describe a suitable embedding $C_{X/Y} \hookrightarrow E_{X/Y}$ in the further idealized case when X is the zero scheme of a section of vector bundle $E \to Y$, i.e., when we have the fiber diagram,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "s"] \\ Y \arrow[r, "0_E"] & E. \end{tikzcd}$$

Writing $\mathcal{E} = \operatorname{Spec}\operatorname{Sym} E^\vee$, the section s is a map of sheaves $\mathcal{O}_Y \to \mathcal{E}$ with kernel $\mathcal{I}_X$ the ideal sheaf of $X = Z(s)$. Dualizing, $\mathcal{I}_X$ is the *image* of $s^\vee : \mathcal{E}^* \to \mathcal{O}_Y$, and so pulling back to X gives a surjection onto the conormal sheaf

$$\mathcal{E}^*|_X \to \mathcal{I}/\mathcal{I}^2,$$

whence

$$N_{X/Y} \hookrightarrow E|_X$$

after applying Spec Sym. Composing with the inclusion $C_{X/Y} \hookrightarrow N_{X/Y}$ induced by the surjection $\operatorname{Sym}\mathcal{I}/\mathcal{I}^2 \twoheadrightarrow \bigoplus \mathcal{I}^n/\mathcal{I}^{n+1}$, we find that $E|_X$ is an appropriate obstruction bundle. Since $C_{X/Y}$ has pure dimension $d = \dim Y$, the induced virtual fundamental class is

$$[X]^{\text{vir}}_{E|_X} = 0^*_{E|_X}[C_{X/Y}] \in A_{d-r}X, \tag{2.1}$$

which is of expected dimension. Note that in this example, $[X]^{\text{vir}}$ is the *localized top Chern class* of E with respect to the s, [Ful98, §14.1].

Example 2.1.9 (Regular embedding, [BCM20, p. 4.7],[Tho00, p. 31])**.** When $X \subseteq Y$ is smooth, i.e., $X \hookrightarrow Y$ is a regular embedding, for any obstruction bundle $E_{X/Y}$ we have $C_{X/Y} = N_{X/Y}$ and

$$[X]^{\text{vir}}_{E_{X/Y}} = c_{\text{top}}(E)$$

where $E = \operatorname{coker}(N_{X/Y} \to E_{X/Y})$, by [Ful98, Ex. 4.1.8].

POTs and virtual fundamental classes

A *perfect obstruction theory* (POT) allows for the definition of a virtual fundamental class without requiring a global embedding, as opposed to the idealized case above. This more general notion is indeed an extension of the virtual class define above, however, as we will see in (2.1.19). The idealized model is still very useful; the construction is much easier to work with, applies to our main examples ($\mathbb{A}^3$ and the local $\mathbb{P}^1$), and may be viewed as the "local model" for the general construction (2.1.21).

We are most interested in a virtual fundamental class for ideal sheaves on threefolds; the relevant POT for projective threefolds is given by [Tho00], and we recall a few of its properties. Notably, in the Calabi-Yau case, the Thomas's construction is *symmetric* (2.1.23), and has a zero dimensional associated virtual fundamental class. Donaldson-Thomas invariants are defined in this most special case as the lengths of these virtual classes, as zero dimensional subschemes.

Definition 2.1.10 (Perfect obstruction theory, [BF97])**.** Let X be a scheme and let $\mathbb{L}_X^\bullet$ denote the truncated cotangent complex (described below). A *perfect obstruction theory* for X is (quasi-isomorphic to) a complex $\mathcal{E}^\bullet \in \mathrm{D^b}(X)$ of locally free sheaves supported in degrees $[-1, 0]$ and a morphism $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X^\bullet$ such that the induced maps:

i. $\phi|_{h^0} : h^0(\mathcal{E}^\bullet) \to h^0(\mathbb{L}_X^\bullet)$ is an isomorphism, and

ii. $\phi|_{h^{-1}} : h^{-1}(\mathcal{E}^\bullet) \to h^{-1}(\mathbb{L}_X^\bullet)$ is surjective.

Remark 2.1.11 (truncated cotangent complex)*.* Informally, for a morphism of k-schemes $f : X \to Y$, the cotangent complex generalizes the following exact sequences:

$$f^*\Omega_{X/Y} \to \Omega_X \to \Omega_{X/Y} \to 0, \tag{2.2}$$

and for f a closed embedding with ideal sheaf $\mathcal{I}$, the conormal sequence,

$$\mathcal{I}/\mathcal{I}^2 \to f^*\Omega_Y \to \Omega_X \to 0, \tag{2.3}$$

writing simply Ω_X for $\Omega_{X/k}$. The *truncated* cotangent complex for a scheme X has a nice local description [Tod21, p. 6]: For opens $U \subset X$ with closed embeddings $i : U \hookrightarrow Y$ with Y smooth, writing $\mathcal{I}_U$ for the ideal sheaf of $U \in Y$, the truncated cotangent complex $\mathbb{L}_X^\bullet$ restricted to U is given by,

$$\mathbb{L}_X^\bullet|_U = (\mathcal{I}_U/\mathcal{I}_U^2 \to \Omega_Y|_U),$$

coming from the conormal sequence (2.3), where $\Omega_Y|_U = i^*\Omega_Y$. Note that if X is quasi-projective then we may take $U = X$ and Y as an open subset of the ambient projective space to get a *global* description $\mathbb{L}_X^\bullet$.

Remark 2.1.12 (Virtual fundamental classes)*.* The construction of the virtual fundamental class with a given perfect obstruction theory is nontrivial [BF97], but analogous to the 2.1. It involves the construction of the *intrinsic normal cone*, which plays a similar role to the normal cone in the previous definition. As in the choice of obstruction bundle above, the virtual class, in general, depends on the choice of perfect obstruction theory. The virtual dimension in the [BF08] construction is the rank of $\mathcal{E}^\bullet$, i.e., the alternating sum of the ranks of the constituent sheaves.

Example 2.1.13 (Trivial perfect obstruction theory)**.** If X is smooth scheme then the cotangent complex of X is the cotangent bundle Ω_X. Taking $\mathcal{E}^\bullet = [0 \to \Omega_X]$ yields the ordinary fundamental class, $[X]^{\mathrm{vir}} = [X] \in A_{\dim X}X$.

We know include three critical results from [Tho00] that allow the construction of Donaldson-Thomas invariants (2.1.29), and realize them as a "sheaf-counting" theory.

Theorem 2.1.14 ([Tho00, Theorem 3.30])**.** *Let X be a smooth projective polarized variety. Let $\mathcal{M}$ denote the moduli space of stable sheaves on X with a* *fixed choice of Chern classes $c_i \in A^i(X)$ and let $\mathcal{M}_{\mathcal{L}}$ denote the subscheme of sheaves with some fixed determinant $\mathcal{L}$ with $c_1(\mathcal{L}) = c_1$. If the numbers*

$$\dim \operatorname{Ext}^i(\mathcal{E}, \mathcal{E}), \quad i \geq 3 \tag{2.4}$$

are constant over all $\mathcal{E}$ in $\mathcal{M}$, then for rank $r > 0$, $\mathcal{M}_{\mathcal{L}}$ admits a perfect obstruction theory.

The explicit construction of the obstruction theory is given in [Tho00]. Thomas also shows the following as a corollary:

Proposition 2.1.15 ([Tho00, Corollary 3.39])**.** *Let X be a smooth projective 3-fold with trivial or anti-effective canonical bundle, i.e., $H^0(X, \wedge^3 T_X) \neq 0$. If the* semi*stable sheaves with fixed data as in 2.1.14 are all stable then the $\mathcal{M}_{\mathcal{L}}$ of 2.1.14 admits a virtual fundamental class of dimension*

$$\mathrm{vdim} = \sum_{i=0}^{3} (-1)^{i+1} \dim \operatorname{Ext}_0^i(\mathcal{E}, \mathcal{E}).$$

Moreover, if $\mathcal{M}$ is smooth, then the virtual cycle is the top Chern class of the obstruction sheaf*,* $\mathrm{Ob} = h^1(\mathcal{E}^{\bullet\vee})$.

Remark 2.1.16. $\operatorname{Ext}_0^i(\mathcal{E}, \mathcal{E})$ denotes the *traceless* part of $\operatorname{Ext}^i(\mathcal{E}, \mathcal{E})$; recall that for $\operatorname{rk} \mathcal{E} > 0$, there is a splitting $\operatorname{Ext}^i(\mathcal{E}, \mathcal{E} \otimes \mathcal{I}) = \operatorname{Ext}_0^i(\mathcal{E}, \mathcal{E} \otimes \mathcal{I}) \oplus H^i(\mathcal{I})$, [Tho00, p. 382]. In our case, however, we will have $\operatorname{Ext}_0(\mathcal{I}, \mathcal{I}) = \operatorname{Ext}(\mathcal{I}, \mathcal{I})$ (2.1.18).

Corollary 2.1.17 (Main example, [Tho00, Corollary 3.40])**.** *By 2.1.15, Hilbert schemes of curves on a Calabi-Yau threefold X admit zero dimensional virtual fundamental classes.*

Remark 2.1.18. The previous proposition (2.1.15) applies to the Hilbert scheme by the identification of ideal sheaves and stable sheaves mentioned after 1.1.14. Following [Mau+06a], in the case that $H^i(\mathcal{O}_X) = 0$ when $i > 0$ we have $\operatorname{Ext}_0(\mathcal{I}, \mathcal{I}) = \operatorname{Ext}(\mathcal{I}, \mathcal{I})$, and thus $\mathrm{vdim} = 0$ follows from Serre duality [Huy06, Theorem 3.12]. More generally, later we will have $\mathrm{vdim} = 0$ by virtue of the obstruction theory being symmetric 2.1.23, 2.1.28.

Example 2.1.19 (Local model, [Tod21, p. 6])**.** Consider a locally free sheaf $\mathcal{E}$ over a smooth scheme Y and global section $s \in \Gamma(\mathcal{E})$. Let $\mathcal{I}$ be the image of the dual section $s^\vee : \mathcal{E}^\vee \to \mathcal{O}_Y$ and $X \subset Y$ the subscheme with ideal sheaf $\mathcal{I}$. We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{E}^\vee|_X \arrow[r] \arrow[d, "s^\vee|_X"'] & \Omega_Y|_X \arrow[d, shift right, "\mathrm{id}"'] \\ \mathcal{I}/\mathcal{I}^2 \arrow[r] & \Omega_Y|_X \arrow[u, shift right, "\mathrm{id}"'] \end{tikzcd} \tag{2.5}$$

where the top morphism is defined by composition. By setting $\mathcal{E}^\bullet = (\mathcal{E}^\vee|_X \to \Omega_Y|_X)$, the above diagram gives a perfect obstruction theory on X: First, $\Omega_Y|_X$ is locally free because Y is nonsingular, and similarly $\mathcal{E}_X^\vee$ is by assumption of $\mathcal{E}$. The requirements on the maps h^0 and h^{-1} hold by construction. Note the similarity with (2.1.8).

Remark 2.1.20. This example can also be found in [Ric21], [BCM20] and [PT14a], sometimes referred to as the “toy model.”

Proposition 2.1.21 ([Tod21, Remark 1.7]). *Any perfect obstruction theory is locally of the form in the diagram 2.5.*

Remark 2.1.22. Toda gives a short argument for this fact [Tod21, p.7], using the local description mentioned in 2.1.11. This claim justifies the term "local model" in 2.1.19, and lets us appeal to the idealized virtual class definition 2.1.7 when discussing invariants which are defined using the more general definition of virtual classes.

Definition 2.1.23 (Symmetic obstruction theory, [BF08]). Let $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X^\bullet$ be a perfect obstruction theory for a scheme X. We say that $(\mathcal{E}^\bullet, \phi)$ is *symmetric* if there exists an isomorphism

$$\theta : \mathcal{E}^\bullet \to \mathcal{E}^{\bullet\vee}[1] \quad \text{with} \quad \theta^\vee[1] = \theta.$$

Proposition 2.1.24 ([Beh09]). *All symmetric perfect obstruction theories on a scheme X induce the same zero-dimensional virtual fundamental class.*

Remark 2.1.25. That the associated virtual dimension is zero is easy to show:

$$\operatorname{vdim} = \operatorname{rk} \mathcal{E}^\bullet = \operatorname{rk} \mathcal{E}^{\bullet\vee}[1] = -\operatorname{rk} \mathcal{E}^\bullet$$

so $\operatorname{rk} \mathcal{E}^\bullet = 0$. That the virtual class of symmetric POT on X is intrinsic to X follows from Behrend's theorem (2.3.1). More generally, however, Siebert's result [Sie04, Theorem 4.6] says that the virtual fundamental class of a perfect obstruction theory depends only on its K-theory class.

Example 2.1.26. There is a large class of examples of symmetric POTs coming from the local model 2.1.19: when the section defining X is a closed 1-form a smooth ambient Y. Taking Ω_Y as the vector bundle $\mathcal{E}$ in 2.1.19 with section a closed 1-form $\omega = df$, defines a perfect obstruction theory.

$$\begin{tikzcd}
\mathcal{E}^\bullet \arrow[d, "\phi"'] & T_Y|_X \arrow[r, "\nabla\omega"] \arrow[d, "\omega^\vee|_X"'] & \Omega_Y|_X \arrow[d, shift right, "\mathrm{id}"'] \\
\mathbb{L}_X^\bullet & \mathcal{I}/\mathcal{I}^2 \arrow[r] & \Omega_Y|_X \arrow[u, shift right, "\mathrm{id}"']
\end{tikzcd} \tag{2.6}$$

Moreover $\nabla\omega$ is a symmetric bilinear form, and we may define θ in (2.1.23) as the identity. See [BF08, Example 1.4, 1.5, 1.19]. The result is that whenever a scheme X is defined as a *critical locus*, it has a canonical zero-dimensional virtual class.

Remark 2.1.27. The symmetric POT given above does *not* serve as the local model for an arbitrary symmetric POT in a way analogous to 2.1.19. In 2.1.26 it is sufficient that ω be an *almost closed* 1-form: $d\omega \in I\Omega_Y^2$, and in [PT14b] the authors construct such a form which is not, even locally, expressible as a critical locus in the sense of (2.1.26).

Proposition 2.1.28 ([BF08, Corollary 1.25]). *Let X be a smooth projective Calabi-Yau threefold. The perfect obstruction theory of [Tho00] (referenced in 2.1.14) is symmetric.*

Donaldson-Thomas invariants

Definition 2.1.29 (Donaldson-Thomas invariants). Let X be a smooth projective Calabi-Yau threefold. Denote by $[\mathrm{Hilb}^{n,\beta}]^{\mathrm{vir}}$ the virtual class of the Hilbert scheme, as in 2.1.17, determined by $[C] = \beta \in H_2(X)$ and $n = \chi(\mathcal{O}_C)$, Define the *Donaldson-Thomas invariant* $I_{n,\beta}(X)$ by

$$I_{n,\beta}(X) = \int_{[\mathrm{Hilb}^{n,\beta} X]^{\mathrm{vir}}} 1 \in \mathbb{Z}. \tag{2.7}$$

When X is implicit or arbitrary, we will often simply write $I_{n,\beta}$.

Remark 2.1.30. It follows from the construction in [BF97] that the virtual class is proper since $\mathrm{Hilb}^{n,\beta}$ is projective. Thus, the integral 2.7, i.e., the proper pushforward to a point, is well-defined, and can be associated with an integer.

Remark 2.1.31 ($\mathrm{Hilb}^{n,\beta}$). In flat families the homology class $[C] = \beta$ and the sheaf Euler characteristic $\chi(\mathcal{O}_C) = n$ are constant. This allows us to re-label our Hilbert schemes $\mathrm{Hilb}^{n,\beta}$ instead of Hilb^P, making it visually much clearer which curves we are considering. Note that when $\beta = 0$ this notation still agrees with the previous notation for Hilbert schemes of points. More generally, It is a consequence of the Grothendieck-Riemann-Roch theorem that the Chern character $(1, c_1, -\beta, -n)$ with $[C] = \beta \in H_2(X)$ and $n = \chi(\mathcal{O}_C)$, determines the Hilbert polynomial of C.

Remark 2.1.32 (Deformation invariance). The sense in which $I_{n,\beta}$ is really *invariant* is explained in [Tho00, Corollary 3.53], with the result being that DT invariants are constant in projective families.

Example 2.1.33 (Basic example [Tho00, p. 405]). Consider the Hilbert schemes of points $\mathrm{Hilb}^n X$, with $n = 1, 2$ on a Calabi-Yau threefold X. They are smooth and so, by (2.1.15), the virtual classes are the top Chern classes of the obstruction sheaf. Using Thomas' theory, or by 2.1.26 having realized $\mathrm{Hilb}^n \mathbb{A}^3$ as a critical locus in (1.3.16) and appealing to symmetry (1.3.16), the obstruction sheaf is then just the cotangent bundle to $\mathrm{Hilb}^n X$. Thus, we have that

$$I_{1,0} = -\chi(X) \quad I_{2,0} = -\chi(X^{[2]}).$$

2.2 Virtual localization

In [GP99], Graber and Pandharipande prove a localization formula analogous to Atiyah-Bott localization (1.4) in the context of virtual fundamental classes. Indeed, when X is nonsingular and equipped with a $\mathbb{C}^*$ action, using the trivial perfect obstruction theory (2.1.13), i.e., $[X]^{\mathrm{vir}} = [X]$, the *virtual localization formula* recovers 1.4.19. While a useful computational tool, the main application of the virtual localization for us will be to *define* Donaldson-Thomas invariants for toric Calabi-Yau threefolds.

Definition 2.2.1 (Equivariant perfect obstruction theory). Suppose $\mathbb{T} = (\mathbb{C}^*)^r$ acts on a $\mathbb{C}$-scheme X. A $\mathbb{T}$-*equivariant perfect obstruction theory* is a perfect obstruction theory (2.1.10), $(\mathcal{E}^\bullet, \phi)$, where $\mathcal{E}^\bullet$ is a $\mathbb{T}$-equivariant complex commuting with ϕ and the induced $\mathbb{T}$-action on $\mathbb{L}_X^\bullet$.

Proposition 2.2.2 ([GP99, Proposition 1]). *Let X be a $\mathbb{C}$-scheme acted on by $\mathbb{T} = \mathbb{C}^*$, and let $(\mathcal{E}^\bullet, \phi)$ be an equivariant perfect obstruction theory for X. The restriction of $(\mathcal{E}^\bullet, \phi)$ to any (possibly reducible) component of the fixed locus $X^{\mathbb{T}}$ defines a perfect obstruction theory for that component.*

Definition 2.2.3 (Virtual normal bundle, [GP99])**.** Given a $\mathbb{T}$-equivariant perfect obstruction theory, $\mathcal{E}^\bullet \to \mathbb{L}_X^\bullet$, we define the *virtual normal bundle* to be the moving part of $\mathcal{E}^\bullet$.

Remark 2.2.4. By "moving part" we mean the following: Analogous to the case of vector spaces, a $\mathbb{C}^*$ action on a coherent sheaf $\mathcal{F}$, determines a decomposition into $\mathbb{C}^*$-*eigensheaves*,

$$\mathcal{F} = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}_k,$$

indexed by the weight of the $\mathbb{C}^*$ action (cf. 1.4.3). We call $\mathcal{F}_0$ the *fixed part* of $\mathcal{F}$ and $\bigoplus_{k \neq 0} \mathcal{F}_k$ the *moving part* of $\mathcal{F}$. These notions extend naturally to complexes. Note also that in the non-virtual case, this coincides with the usual normal bundle.

Definition 2.2.5 (Euler class, [GP99])**.** The *Euler class* of a two term complex $[B_0 \to B_1]$ is the ratio $e(B_0)/e(B_1)$.

Theorem 2.2.6 (Virtual localizaiton formula [GP99])**.** *Let X be an algebraic scheme with a $\mathbb{C}^*$-action and a $\mathbb{C}^*$-equivariant perfect obstruction theory. Then*

$$[X]^{vir} = \iota_* \sum \frac{[X_i]^{vir}}{e^{\mathbb{C}^*}(N_i^{vir})}$$

in $A_*^{\mathbb{C}^*}(X) \otimes \mathbb{Q}[t, t^{-1}]$ where t is the generator of the $\mathbb{C}^*$-equivariant ring of a point.

Localization and DT invariants

One of the running assumptions of the previous section (2.1) was that the starting variety was *projective*, and we defined Donaldson-Thomas invariants (2.1.29) for projective and *Calabi-Yau* threefolds. We would like to use virtual localization to compute invariants, and one would naturally look to toric varieties for a class of examples. Note, however, that since toric varieties are rational they have geometric genus zero, whereas projective Calabi-Yau's have $p_g = 1$. Therefore, in order to *use* virtual localization in Donaldson-Thomas theory, we have to expand our definition to accommodate more general projective varieties, or make a new definition for toric Calabi-Yau's. In fact, both are possible. The specification of Calabi-Yau above was mainly for convenience, to ensure a zero dimensional virtual class; the integral (2.7) may be modified with "insertions" to produce integer invariants when the virtual dimension is positive. When X is toric, we make the following definition:

Definition 2.2.7 (Toric DT invariants)**.** Let X be a nonsingular toric threefold with proper $\mathbb{T}$-fixed point set $(\mathrm{Hilb}^{n,\beta} X)^{\mathbb{T}}$ under the induced action. Define

$$I_{n,\beta}(X) = \int_{[(\mathrm{Hilb}^{n,\beta} X)^{\mathbb{T}}]^{\mathrm{vir}}} \frac{1}{e^{\mathbb{T}}(N^{\mathrm{vir}})} \in \mathbb{Q}(s_1, s_2, s_3) \tag{2.8}$$

Remark 2.2.8. Agrees by localization theorem. Called in [BP08] "residue invariants." From that paper: the POT still well-behaved, or can be obtained by equivariant compactification and restriction.

Example 2.2.9 (Degree 0 invariants of $\mathbb{A}^3$)**.** We calculate the degree 0 DT-invariants $I_{n,0}(\mathbb{A}^3)$ for low n. The standard $(\mathbb{C}^*)^3$ action on $\mathbb{A}^3$ via the coordinate ring $\mathbb{C}[x,y,z]$, i.e., $(t_1,t_2,t_3)\cdot(x,y,z)\mapsto(t_1x,t_2y,t_3z)$, induces an action on $\mathrm{Hilb}^n\mathbb{A}^3$ with isolated fixed points. On a fixed component Z, the induced $\mathbb{T}$-equivariant POT gives virtual normal bundle $N_Z^{\mathrm{vir}}=\mathrm{Ext}^1(\mathcal{I}_Z,\mathcal{I}_Z)\to\mathrm{Ext}^2(\mathcal{I}_Z,\mathcal{I}_Z)$, [BB07; Mau+06a; Mau+06b; Ric21]. By virtual localization,

$$I_{n,0}(\mathbb{A}^3)=\sum_{Z\in(\mathrm{Hilb}^{n,\beta}X)^{\mathbb{T}}}\frac{e^{\mathbb{T}}(\mathrm{Ext}^2(I_Z,I_Z))}{e^{\mathbb{T}}(\mathrm{Ext}^1(I_Z,I_Z))}. \tag{2.9}$$

To compute $e^{\mathbb{T}}(\mathrm{Ext}^i(I_Z,I_Z))$, we use the correspondence between $\mathbb{T}$-actions and gradings 1.4.3. Thinking of the $\mathbb{T}$-representations $\mathrm{Ext}^i(I_Z,I_Z)$ as multigraded modules, the multigraded Hilbert series then determines the decomposition into weight spaces (see 1.4.2). This can be computed using Macaulay2 [GS], we may write $\mathrm{Ext}^i(I_Z,I_Z)$ as a sum of characters in $\mathbb{Z}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1}]$. Finally, with $s_i=e^{\mathbb{T}}(t_i)$, the Euler classes $e^{\mathbb{T}}(\mathrm{Ext}^i(I_Z,I_Z))$ can be determined using Chern class identities such as the Whitney sum formula, and $c_1(\mathcal{L}\otimes\mathcal{L}')=c_1(\mathcal{L})+c_1(\mathcal{L}')$ for line bundles $\mathcal{L},\mathcal{L}'$.

For $n=1$, the only $\mathbb{T}$-fixed ideal is $I_Z=(x,y,z)$, and

$$\mathrm{Ext}^1(I_Z,I_Z)=t_1^{-1}+t_2^{-1}+t_3^{-1} \tag{2.10}$$
$$\mathrm{Ext}^2(I_Z,I_Z)=t_1^{-1}t_2^{-1}+t_1^{-1}t_3^{-1}+t_2^{-1}t_3^{-1}. \tag{2.11}$$

Note that $\mathrm{Ext}^1(I_Z,I_Z)$ is indeed the tangent representation. It follows that

$$I_{1,0}(\mathbb{A}^3)=\frac{e^{\mathbb{T}}(\mathrm{Ext}^2(I_Z,I_Z))}{e^{\mathbb{T}}(\mathrm{Ext}^1(I_Z,I_Z))}=\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1s_2s_3}.$$

Note that from 2.10, specializing the action to the subtorus $\mathbb{T}_0\subset\mathbb{T}$ defined by $t_1t_2t_3=1$, we would have $I_{1,0}=1$.

For $n=2$, there are three fixed ideals: $(x^2,y,z),(x,y^2,z),(x,y,z^2)$. The analogous calculation yields:

$$I_{0,2}(\mathbb{A}^3)=I_{0,1}(\mathbb{A}^3)\cdot\frac{s_1^2(s_2+s_3)+s_1\left(s_2^2-3s_2s_3+s_3^2\right)+s_2s_3(s_2+s_3)}{2s_1s_2s_3}.$$

2.3 Behrend functions

Theorem 2.3.1 ([Beh09])**.** *For any scheme X over $\mathbb{C}$ there is a canonical constructible function $\nu_X : X\to\mathbb{Z}$ such that if X is proper and embeddable, then*

$$\int_{[X]^{vir}}1=\chi(X,\nu_X)=\sum_{n\in\mathbb{Z}}n\chi(\{\nu_X=n\})$$

with $[X]^{vir}$ given by any symmetric obstruction theory (2.1.23).

Remark 2.3.2. The function ν_X is often called the *Behrend function*. It comes from the *intrinsic normal cone* [BF97], which was used to define general virtual fundamental classes for a perfect obstruction theory. In fact, once the theory of Behrend and Fantechi for the intrinsic normal cone $\mathfrak{c}_X$ is in place, the function ν_X is relatively easy to *define*: Recall that MacPherson's *local Euler obstruction*, Eu, maps integral algebraic cycles on a $\mathbb{C}$-scheme X to constructible integer-valued functions on X, [Mac74], cf. [Ful98, p. 19.1.7]. The Behrend function is value of Eu for the intrinsic normal cone,

$$\nu_X = \mathrm{Eu}(\mathfrak{c}_X)$$

[Beh09, Definition 1.4]. Using the global index theorem of MacPherson and Behrend's construction in the smooth case recovers Gauss-Bonnet [Beh09].

Theorem 2.3.3 ([Beh09])**.** *Let M be a $\mathbb{C}$-scheme and ν_M its Behrend function.*

i. *At smooth points P of M we have $\nu_M(P) = (-1)^{\dim M}$.*

ii. *If M is the critical scheme of a regular function f for a smooth ambient scheme Y, i.e., $M = Z(df)$, then*

$$\nu_M(P) = (-1)^{\dim M}(1 - \chi(F_P)) \tag{2.12}$$

where F_P is the Milnor fiber, i.e., the intersection of a nearby fiber and small ball in Y around P.

DT invariant via Behrend functions Theorem 2.3.1 gives yet another way to compute Donadson-Thomas invariants, and a second way to *define* DT invariants when X is quasi-projective. It is known that this extension agrees with invariants defined via virtual localization 2.2.7 with a *Calabi-Yau* $\mathbb{T}$-equivariant obstruction theory in the toric case.

Example 2.3.4 (DT-invariants from quivers)**.** In 1.3 we saw how Hilbert schemes on $\mathbb{A}^3$ and the local $\mathbb{P}^1$, $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ are critical loci in smooth ambient varieties. [BF08] use the Behrend function to calculate the invariants

$$I_{0,n} = \int_{[\mathrm{Hilb}^n \mathbb{A}^3]^{\mathrm{vir}}} 1$$

localized with respect to a 1-dimensional subtorus of $\mathbb{T}_0 \subset (\mathbb{C}^*)^3$ given by $t_1 t_2 t_3 = 1$. By similar methods [Sze08] compute invariants for the resolved conifold.

Example 2.3.5 (Super-rigid curves)**.** [BB07] perform a similar calculation to [Sze08] in the case of the local $\mathbb{P}^1$. More generally, they compute the contribution to the DT series of the so-called "super-rigid" curves, i.e., embedded rational curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, (like that of the zero section in the local $\mathbb{P}^1$).

2.4 MNOP

The two highly influential papers [Mau+06a] and [Mau+06b] of Maulik et al. expand on work of Thomas, [Tho00] with a series of conjectures relating the sheaf-counting theory thenceforth known as *Donaldson-Thomas theory* with the more established field of Gromov-Witten invariants. The relation was given in terms of generating series or "partition functions" of invariants. The authors prove several of their conjectures in the toric projective case, starting a long series of papers building towards a GW/DT correspondence for all nonsingular threefolds.

We view the study of generating series' of invariants as the central influence of the MNOP papers. We start with the *DT-partition function*.

Definition 2.4.1 (DT-partition function)**.** Let X be a Calabi-Yau threefold. Using definitions 2.1.29 2.2.7 for the Donaldson-Thomas invariants $I_{n,\beta}$ of X, we define the *DT-partition function* of X (with respect to β)

$$\mathsf{DT}_\beta X(q) = \sum_{n\in\mathbb{Z}} I_{n,\beta}(X) q^n.$$

When $\beta = 0$, we call $\mathsf{DT}_0 X$ the *degree zero* series.

Remark 2.4.2. It can be shown that for any β, $I_{n,\beta} = 0$ for $n \ll 0$ (cf. [Tod09, Lemma 3.10]), thus $\mathsf{DT}_\beta X(q)$ is a formal Laurent series.

Note that degree zero partition function is the virtual version of the series described by the Cheah formula (1.1.18). That it admits a similarly elegant description is an interesting result on its own. Its form was conjectured and first proven in the projective toric case to be the following by [Mau+06a].

Theorem 2.4.3 ([LP09])**.** *Let X be a nonsingular projective, or quasi-projective with a torus action, threefold. The degree 0 Donaldson-Thomas partition function of X is determined by*

$$\mathsf{DT}_0 X(q) = M(-q)^{\int_X c_3(T_X \otimes K_X)}, \tag{2.13}$$

where M is the MacMahon function.

Remark 2.4.4. In the case when X is non-projective toric, the exponent $\int_X c_3(T_X \otimes K_X)$ is defined via localization 1.4.

Remark 2.4.5. Theorem 2.4.3 has been proven several times with varying levels of generality (cf. [Mau+06b], [Li06], [BF08]), before the most general proof was given in [LP09]. One technique of [LP09] involves degeneration to the toric case, further motivating the study of toric invariants.

Example 2.4.6 (Checking terms for $\mathbb{A}^3$)**.** If $X = \mathbb{A}^3$, we calculate the exponent in 2.13 by localization 1.4.19,

$$\int_X c_3(T_X \otimes K_X) = -\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3},$$

in $\mathbb{Q}(s_1, s_2, s_3)$,the equivariant cohomology ring of a point. By expanding the degree zero partition function 2.13, in this case using the *Mathematica*®script

```
MM[u_, q_] := Product[(1 - q^n)^-n, {n, u}];
DT0A3[u_, q_, s1_, s2_, s3_] :=
    MM[u, -q]^(-(s1 + s2) (s1 + s3) (s2 + s3)/(s1 s2 s3 ));
Series[DT0A3[5, q, s1, s2, s3], {q, 0, 3}];
CoefficientList[%, q] // FullSimplify
```

we observe that Theorem 2.4.3 agrees with our computations in 2.2.9 for $n = 1, 2$.

The equivariant DT vertex Another way to state the calculation of 2.13 in the case of $\mathbb{A}^3$ is as the *0-leg* equivariant DT vertex $\mathsf{W}_{DT}(\emptyset, \emptyset, \emptyset)$. For $\mathbb{A}^3$ recall the correspondence between $\mathbb{T}$-fixed subschemes, monomial ideals and 3d-partitions; the correspondence extends to partitions with infinite outgoing "legs", i.e., monomial ideals $I \subset \mathbb{C}[x, y, z] = R$ for which R/I is infinite dimensional over $\mathbb{C}$. We refer to [Mau+06b, §4] or [PT09a, §5.2] for the definition of the full equivariant DT vertex, $\mathsf{W}_{DT}(\lambda_1, \lambda_2, \lambda_3)$. Intuitively, they are generating series for DT invariants of $\mathbb{A}^3$ (or more generally, any $\mathbb{T}$-fixed point in toric X) with underlying curve specified by the outgoing 2-dimensional partitions $\lambda_1, \lambda_2, \lambda_3$.

Reduced series Roughly speaking, the degree zero invariants are the factors by which we over-count curves because of the presence of free-points. [Mau+06a] formally excise these degeneration contributions by forming the *reduced DT series*:

$$\mathsf{DT}'_\beta X(q) = \mathsf{DT}_\beta X(q)/\mathsf{DT}_0 X(q).$$

In the next chapter we discuss how the reduced series has its own geometric interpretation, but for now we recall some of its properties, conjectured in [Mau+06a, Conjecture 2].

Conjecture 2.4.7 ([Mau+06a, Conjecture 2])**.** *The reduced series* $\mathsf{DT}'_\beta X(q)$ *is a rational function of* q *symmetric under the transformation* $q \mapsto q^{-1}$*.*

Remark 2.4.8. This is now a theorem in the projective Calabi-Yau case by Bridgeland [Bri11] and [Tod10], via the DT-PT correspondence.

Gromov-Witten invariants Donaldson-Thomas invariants are to the Hilbert scheme what Gromov-Witten invariants are to the moduli space of *stable maps*. That space, $\overline{\mathcal{M}_g}(X, \beta)$, parametrizes maps $f : C \to X$ from at worst nodal genus g curves C with $\mathrm{Aut}(f)$ finite and $f_*[C] = \beta$. It also admits a virtual fundamental class, and the Gromov-Witten (GW) invariants are defined via integration against this class. Summing over the genus, a partition function $\mathsf{GW}_\beta X(q)$ is formed analogously to 2.4.1. It is related to the DT partition function by the third [Mau+06a] conjecture:

Conjecture 2.4.9 ([Mau+06a, Conjecture 3])**.** *Let* X *be a Calabi-Yau threefold, the change of variables* $q = -e^{iu}$ *equates the reduced DT and GW partition functions:*

$$\mathsf{DT}'_\beta X(-e^{iu}) = \mathsf{GW}_\beta X(u). \tag{2.14}$$

Remark 2.4.10. The toric case is proven by [Mau+11]. In the projective setting, [PP16] proves this correspondence for Calabi-Yau complete intersections in products of projective spaces, notably including the quintic in $\mathbb{P}^4$. Their result work factors through the DT-PT correspondence.

CHAPTER 3

DT-PT correspondences

3.1 Pandharipande-Thomas theory

The similarity between ideal sheaves and stable pairs extends to a construction of perfect obstruction theories by [PT09a], and we have:

Theorem 3.1.1 ([PT09a, Theorem 2.14])**.** *Let X be a smooth projective threefold, and let* $\mathrm{Pairs}^{n,\beta} X$ *denote the space of stable pairs determined by a curve class* $\beta \in H_2(X)$ *and Euler characteristic* $\chi(\mathcal{O}_C) = n$ *for* $[C] = \beta$*.* $\mathrm{Pairs}^{n,\beta} X$ *admits a virtual fundamental class*

$$[\mathrm{Pairs}^{n,\beta} X]^{vir} \in A_{c_\beta}(\mathrm{Pairs}^{n,\beta} X) \tag{3.1}$$

where

$$c_\beta = \mathrm{vdim} = \int_\beta c_1(X). \tag{3.2}$$

Remark 3.1.2. The relevant obstruction theory is inherited from one defined more generally for the moduli space of fixed determinant complexes in the bounded derived category. Indeed, before defining the relevant POT, Pandharipande and Thomas first show that stable pairs faithfully embed in $\mathrm{D^b}(X)$, i.e., that stable pairs $(\mathcal{F}, s)$ and $(\mathcal{F}', s')$ are isomorphic if and only if the associated complexes $\{\mathcal{O}_X \xrightarrow{s} \mathcal{F}\}$ and $\{\mathcal{O}_X \xrightarrow{s'} \mathcal{F}'\}$ are quasi-isomorphic. In some sense, this is the more modern perspective on these objects. Finally, from 3.2 we see that the Calabi-Yau condition ensures a zero dimensional virtual class.

Definition 3.1.3 (PT invariants)**.** Let X be a Calabi-Yau threefold. Fixing n and β as above, if X is projective we define

$$P_{n,\beta}(X) = \int_{[\mathrm{Pairs}^{n,\beta} X]^{\mathrm{vir}}} 1 \in \mathbb{Z} \tag{3.3}$$

and if X is toric,

$$P_{n,\beta}(X) = \int_{[(\mathrm{Pairs}^{n,\beta} X)^{\mathbb{T}}]^{\mathrm{vir}}} \frac{1}{e^{\mathbb{T}}(N^{\mathrm{vir}})} \in \mathbb{Q}(s_1, s_2, s_3) \tag{3.4}$$

where the s_i are generators for the equivariant cohomology of a point. We call $P_{n,\beta}$ the *stable pair* or *Pandharipande-Thomas (PT)* invariants.

Remark 3.1.4 (Generating series). We denote the generating series (or partition function) of PT invariants with specified curve class $\beta \in H_2(X)$ by

$$\mathsf{PT}_\beta X(q) = \sum_{n\in\mathbb{Z}} P_{n,\beta}(X) q^n \tag{3.5}$$

Remark 3.1.5. PT invariants may also be defined via Behrend functions. As in the DT case, this way of defining invariants in the toric Calabi-Yau case agrees with the definition above by specialization to a subtorus $\mathbb{T}_0 \subset \mathbb{T}$ such that the defining $\mathbb{T}$-equivariant perfect obstruction theory restricts to one which is $\mathbb{T}_0$-equivariantly Calabi-Yau.

Example 3.1.6 (The local $\mathbb{P}^1$)**.** Consider $X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The stable pairs $\mathrm{Pairs}^{n,[\mathbb{P}^1]}$, with underlying curve class the zero section $\mathbb{P}^1$, are parametrized by sections of $\mathcal{O}(n-1)$ up to scaling (cf. 1.2.13). Thus,

$$\mathrm{Pairs}^{n,[\mathbb{P}^1]} \cong \mathrm{Sym}^{n-1}\mathbb{P}^{n-1} \cong \mathbb{P}^{n-1}.$$

Since $\mathbb{P}^{n-1}$ is nonsingular, the virtual class is given by the top Chern class of the cotangent bundle and therefore,

$$P_{n,[\mathbb{P}^1]} = (-1)^{n-1}\chi(\mathbb{P}^{n-1}) = (-1)^{n-1} n.$$

The generating series is

$$Z_{P,[\mathbb{P}^1]} = q - 2q^2 + 3q^3 - 4q^4 + \cdots = \frac{q}{(1+q)^2}.$$

Note that this example may be generalized to compute the contribution of higher genus *isolated* nonsingular curves C, [PT09a, §4.2]:

$$Z^C_{P,[C]} = \sum_{d\geq 0} q^{1-g+d}(-1)^d \chi(\mathrm{Sym}^d(C)).$$

The PT vertex Torus fixed stable pairs correspond, locally, to *weighted* box-counts, similar to that of the DT case. These modified 3d-partitions always have one or more outgoing "legs", corresponding to the curve class of the support. The *PT equivariant vertex* $\mathsf{W}_{\mathsf{PT}}(\lambda_1, \lambda_2, \lambda_3)$ is a generating series which, roughly speaking, counts possible local contributions of type specified by outgoing legs λ_i of stable pairs, analogous to the DT equivariant vertex. This is defined rigorously in [PT09b], and the authors prove a form of the vertex in the case of one and two nonzero outgoing partitions λ_i.

DT-PT correspondences We now describe the general form of DT-PT correspondences. Following [JWY21] we make the distinction between the *geometric* correspondence between generating series for invariants (especially for the projective case), and the combinatorial correspondence between vertices. By a *geometric* DT-PT correspondence, we mean an equality of Laurent series

$$\sum_{n\in\mathbb{Z}} I^*_{n,\beta} q^n = \sum_{n\in\mathbb{Z}} P^*_{n,\beta} q^n \cdot \sum_{n\in\mathbb{Z}} I^*_{n,0} q^n \tag{3.6}$$

for invariants $I^*_{n,\beta}$ and $P^*_{n,\beta}$ of a nonsingular threefold, typically specified as projective, Calabi-Yau, and/or toric, defined based on the Hilbert scheme or the pairs space, respectively. Here, $I^*_{n,\beta}$ means any of $I_{n,\beta}$, $I^\chi_{n,\beta}$, or $I^K_{n,\beta}$, which refer to the classical, unweighted, and K-theoretic versions of the invariants, respectively. By the combinatorial correspondence, we mean a similar equality of the respective vertices.

3.2 Unweighted correspondence

Definition 3.2.1 (Unweighted invariants)**.** Let X be a nonsingular projective variety, consider the schemes $\mathrm{Hilb}^{n,\beta} X$ and $\mathrm{Pairs}^{n,\beta} X$, determined by a curve class $\beta \in H_2(X)$ and Euler characteristic n as in (2.1.29) and (3.1.3). We define the *unweighted DT and PT invariants*, $I^\chi_{n,\beta}(X)$ and $P^\chi_{n,\beta}(X)$ to be the Euler characteristics of these schemes,

$$I^\chi_{n,\beta}(X) = \chi(\mathrm{Hilb}^{n,\beta} X) \quad P^\chi_{n,\beta}(X) = \chi(\mathrm{Pairs}^{n,\beta} X).$$

Remark 3.2.2. Note that this definition does not involve virtual classes. By Euler localization 1.4.6, these invariants may be calculated by in simple cases by counting $\mathbb{T}$-fixed points, if isolated.

Theorem 3.2.3 (Unweighted DT-PT, [ST11])**.** *Let X be a nonsingular projective threefold. For any curve class $\beta \in H_2(X,\mathbb{Z})$ there is an equality of formal Laurent series:*

$$\sum_{n\in\mathbb{Z}} I^\chi_{n,\beta}(X)q^n = \sum_{n\in\mathbb{Z}} P^\chi_{n,\beta}(X)q^n \cdot \sum_{n\in\mathbb{Z}} I^\chi_{n,0}(X)q^n$$

Equivalently, for each $n \in \mathbb{Z}$,

$$I^\chi_{n,\beta}(X) = P^\chi_{n,\beta}(X) + \sum_{i=1} P^\chi_{n-i,\beta}(X) I^\chi_{i,0}(X).$$

Remark 3.2.4. This result in the projective *Calabi-Yau* case was also proven in Bridgeland [Bri11] and Toda [Tod10].

Theorem 3.2.5 ([ST11, Theorem 1.5])**.** *Let C be a Cohen-Macaulay curve in a smooth projective threefold. Define $I_{n,C}$ to be the Euler characteristic of the subscheme of* $\mathrm{Hilb}^{n+\chi(\mathcal{O}_C),\beta} X$ *consisting of ideal sheaves $\mathcal{I}_Z$ with underling Cohen-Macaulay curve C such that $\mathcal{I}_C/\mathcal{I}_Z$ is 0-dimensional of length n. Define $P_{n,C}$ to be the Euler characteristic of the subscheme of* $\mathrm{Pairs}^{n+\chi(\mathcal{O}_C),\beta} X$ *of pairs supported on C with cokernel of length n. We have*

$$I_{n,C} = P_{n,C} + \chi(X)P_{n-1,C} + \chi(\mathrm{Hilb}^2 X)P_{n-2,C} + \cdots + \chi(\mathrm{Hilb}^n X)P_{0,C}. \quad (3.7)$$

Remark 3.2.6. Stoppa and Thomas show that the subsets defining $I_{n,C}$ and $P_{n,C}$ are in fact schemes.

Proposition 3.2.7. *The unweighted correspondences* do not *hold in general, when dimension $d \neq 3$.*

Proof. We consider $X = \mathbb{P}^2 = \mathrm{Proj}\,\mathbb{C}[x_0, x_1, x_2]$. Let $C = Z(x_0) \cong \mathbb{P}^1$. In the simplest case, when $n = 1$, from 3.7, we check if

$$I_{1,C} = P_{1,C} + \chi(\mathbb{P}^2)P_{0,C}. \quad (3.8)$$

From Example 3.1.6, we have $P_{n,C} = \chi(\mathbb{P}^n)$, and thus (3.8)becomes $I_{1,C} = 2+3$. However, there are only three fixed points contributing to $I_{1,C}$, namely C and free point $Z(x_1, x_2)$, and C with an embedded point thickened once along the $x_1 = 0$ or $x_2 = 0$ lines. Applying this argument to each fixed line, provides a counter example to Theorem 3.2.3 in dimension two, when $X = \mathbb{P}^2$ and $\beta = [\mathbb{P}^1]$ □

3.3 The DT-PT correspondence

The following is the DT-PT correspondence, conjectured by Pandharipande and Thomas in [PT09a, Conjecture 3.3] and proven by Bridgeland, [Bri11, Theorem 1.1].

Theorem 3.3.1. *Let X be a smooth projective Calabi-Yau threefold. For each class $\beta \in H_2(X, \mathbb{Z})$ there is an equality of Laurent series:*

$$\mathsf{DT}_\beta(q) = \mathsf{PT}_\beta(q) \cdot \mathsf{DT}_0(q).$$

Theorem 3.3.2 (DT-PT vertex correspondence, [JWY21])**.** *(Calabi-Yau vertex)*

$$\mathsf{W}_{\mathsf{DT}}(\mu_1, \mu_2, \mu_3) = M(q)\mathsf{W}_{\mathsf{PT}}(\mu_1, \mu_2, \mu_3) \tag{3.9}$$

The local $\mathbb{P}^1$ With Theorem 2.4.3 and Example 3.1.6 we have two of the three components in the DT-PT correspondence for the local $\mathbb{P}^1$, with β as the class of the zero section. Using calculations of [BB07] or [Sze08] for the DT partition function, the correspondence can be verified directly, in this case.

The Hilbert scheme and the pairs space over the local $\mathbb{P}^1$ may be realized as moduli spaces of θ-stable (1.3.13) quiver representations for the conifold quiver and superpotential (Example 1.3.12), cf. [NN10; Sze08]. In fact, [NN10] classifies the entire wall-chamber structure.

K-theoretic correspondence Finally, we remark that a *K-theoretic* refinement of Donaldson-Thomas and stable pairs have recently been introduced [Oko17], and a DT-PT correspondence has been conjectured in [NO16]. A form of the K-theoretic analogue of the degree zero partition function, cf. Theorem 2.4.3, known as *Nekrasov's formula* has already been proven [Oko17], and directly generalizes the 0-leg equivariant DT vertex ([Oko17, Ex. 3.4.45]).

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