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Constant diagonal sums of doubly stochastic matrices

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This master's thesis is submitted under the master's programme *Lektor-programmet*, with programme option *Mathematics*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 30 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

The aim of this thesis is to investigate the diagonals of doubly stochastic matrices. More specifically, we have studied doubly stochastic matrices with constant diagonal sums, and how small modifications made to these matrices alter the diagonal sums. We have defined a specific operation in modifying these matrices, and found some results on how this operation alters the diagonals and diagonal sums of the matrices. Among the results, we found that the modified diagonal sums correspond to the modifications made to the matrices. In our work, we studied a specific class of doubly stochastic matrices with constant diagonal sums, in addition to randomly generated matrices using MATLAB.

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CHAPTER 1

Introduction

1.1 Introduction

The main topic of this thesis is diagonals of doubly stochastic matrices. More specifically, we have studied doubly stochastic matrices with restricted constant diagonal sums (RCDS). This entails that the sum of all diagonals which do not contain any zeros are constant. RCDS matrices have been studied by Brualdi & Dahl [3], and our work in this thesis is based on their article *Diagonal sums of doubly stochastic matrices* (2021). The patterns of RCDS matrices are complicated, and there does not yet exist a method to find them all. Brualdi & Dahl [3] have provided us with algorithms to construct some of them, in addition to certain characterizations of the matrices, which we have used in our work. Since the diagonals we are studying do not contain any zeros, the zero positions of matrices influence the possible diagonals that can be obtained. Lest we procure arbitrary results, we have restricted the matrices in our work to be fully indecomposable.

With the basis of RCDS matrices from Brualdi & Dahl [3], we have studied the diagonals and diagonal sums of these matrices. Moreover, we have made some modifications to the matrices, and studied how the modifications alter the diagonals and their sums. When exploring these matrices and their diagonals, we have used Matlab [10] to generate doubly stochastic and RCDS matrices. This work have produced some general results which will be presented in this thesis.

The current chapter will present preliminary theory relevant to our work. This entails theory on linear algebra, matrix theory and majorization, which is mainly based on the work of Brualdi [5], Dahl [7] and Marshall et al. [9]. Moreover, we will present theory on convexity and polyhedra, of which we have studied the work by Dahl [6]. There is a close relation between matrix theory and graph theory, and we will present some graph theory by Bondy & Murty [2], Brualdi & Ryser [4] and Dahl [7]. Matrix diagonals can be associated to perfect matchings in a bipartite graph and the optimal assignment problem, which will be discussed in this thesis. In the following chapters, we will go deeper into matrix diagonals and RCDS doubly stochastic matrices. In those chapters, several results from Brualdi & Dahl [3] will be presented.

1.2 Outline

The thesis is organized as follows:

Chapter 1 introduces the topic of the thesis and presents preliminaries, including theory on convexity, graphs, permutations and doubly stochastic matrices.

Chapter 2 introduces diagonals, fully indecomposable matrices, matrices with restricted constant diagonal sums (RCDS), and explores the diagonals of certain RCDS matrices.

Chapter 3 explores how modifications made to RCDS matrices alters the diagonals and diagonal sums of the matrices.

Chapter 4 explains the MATLAB codes that were used in the work of this thesis

Chapter 5 is a conclusion of the thesis.

Appendix A consists of MATLAB codes that were made and used for the work in this thesis.

1.3 Vector space

We begin by defining a vector space.

Definition 1.3.1. Let $M_{m,n}$ be the space consisting of real $m \times n$ matrices. $M_{m,n}$ is a vector space if the matrices are defined under addition and multiplication by scalars, i.e. the following conditions hold for all matrices $A = [a_{ij}]$, $B = [b_{ij}] \in M_{m,n}$ and scalars $c \in \mathbb{R}$,

(i) $A + B = C = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$

(ii) $cA = [c \cdot a_{ij}]$

In the case where $m = n$, we denote the vector space as M_n.

Definition 1.3.2. Let $\mathbb{R}^n$ be the Euclidean space consisting of real vectors of length n. $\mathbb{R}^n$ is a vector space if the vectors are defined under addition and multiplication by scalars, i.e. the following conditions hold for all vectors $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$,

(i) $u + v = w$ where $w = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$

(ii) $cu = (cu_1, cu_2, \ldots, cu_n)$

Consider the vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$. Following [11], a *linear combination* is a sum of scalar multiples of these vectors, i.e. $a_1x_1 + a_2x_2 + \cdots + a_nx_n$, where $a_1, a_2, \ldots, a_n \in \mathbb{R}$. The *span* of these vectors, denoted $\text{Span}(x_1, x_2, \ldots, x_n)$, is the set of all linear combinations of these vectors. If $\text{Span}(x_1, x_2, \ldots, x_n) = \mathbb{R}^n$, then the vectors $(x_1, x_2, \ldots, x_n)$ span $\mathbb{R}^n$.

1.4 Theory of convexity

In this section, we will introduce some basic theory of convex sets, convex functions and polyhedra. These concepts are closely related to linear optimization, and are relevant for further definitions and theory. All theory and definitions presented in this chapter is cited from Dahl [6].

Definition 1.4.1. [6] Let $C \in \mathbb{R}^n$ be a set and let $x_1, x_2 \in C$ be two points in the set. Then C is a *convex* set if the line between x_1 and x_2 lies in C, i.e. $\lambda x_1 + (1-\lambda)x_2 \in C$ for $0 \leq \lambda \leq 1$.

The definition of a convex set uses a certain type of linear combination of x_1, x_2, called a convex combination, see [6]. Consider the vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$ and nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\sum_{i \leq n} \lambda_i = 1$. Then $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$ is a *convex combination* of $x_1, x_2, \ldots, x_n$. The following is an example of convex combinations.

Example 1.4.2. Let $(0,0)$, $(1,0)$, $(1,1)$ be three points in R^2. The point $p = (2/3, 1/3)$ is a convex combination of the points, since $(2/3, 1/3) = 1/3\,(0,0) + 1/3\,(1,0) + 1/3\,(1,1)$ and $1/3 + 1/3 + 1/3 = 1$. Figure 1.1 illustrates this with a drawing in the plane. Notice that the points $(0,0)$, $(1,0)$, $(1,1)$ span the convex set $C = \{(x_1, x_2) : 0 \leq x_1, x_2 \leq 1,\ x_2 - x_1 \leq 0\}$, and the point $p = (2/3, 1/3)$ lies in C.

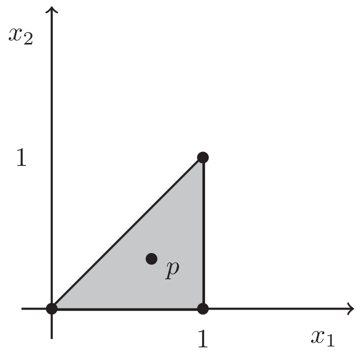


Figure 1.1: The convex set C and point p from Example 1.4.2

△

In Example 1.4.2 we saw a case where the convex set is closed under the operation of taking convex combinations. In fact, this is the case for all convex sets and convex combinations within them, which brings us to the following result.

Proposition 1.4.3. *[6, Proposition 2.1.1] A set $C \in \mathbb{R}^n$ is convex if and only if it contains all convex combinations of its points. A set $C \in \mathbb{R}^n$ is a convex cone if and only if it contains all nonnegative combinations of its points.*

We will now move on to the last part of this section, and introduce convex functions.

Definition 1.4.4. [6] Let $f : \mathbb{R} \to \mathbb{R}$. Then f is a convex function if the following inequality holds

$$f((1-\lambda)\,x + \lambda\,y) \leq (1-\lambda)\,f(x) + \lambda\,f(y) \quad \forall\, x, y \in \mathbb{R},\ 0 \leq \lambda \leq 1 \tag{1.1}$$

Figure 1.2 illustrates a convex function f with x, y, $z \in \mathbb{R}$. The line segment between $f(x)$ and $f(y)$ is expressed by $(1-\lambda)f(x) + \lambda f(y)$, and $f(z) = f((1-\lambda)x+\lambda y)$. We can see in the figure that $f(z) \leq (1-\lambda)f(x)+\lambda f(y)$ and thus the inequality condition 1.1 for the convex function f holds.

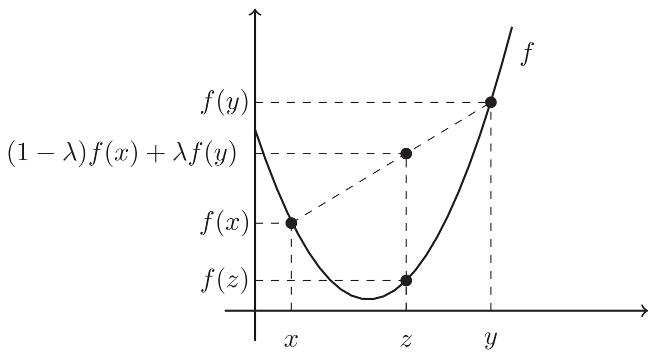


Figure 1.2: Example of a convex function f

We will now move on to introducing linear optimization and polyhedra theory, which is closely related to convexity theory.

1.5 Linear optimization and polyhedra theory

In this section, we will introduce theory of linear optimization and polyhedra, following Dahl [6]. This section is closely related to convexity, which we introduced in the previous section. In linear optimization, the goal is to maximize (or minimize) a linear function $c^T x$ (often called an *objective function*) of the vector $x = (x_1, x_2, \ldots, x_n)$ with respect to certain constraints. This is called a linear programming (LP) problem. A practical example is a company which produces n products and wishes to maximize the profit for each product $x = (x_1, x_2, \ldots, x_n)$. Some constraints may for example be the costs of production, transportation, raw material, or storage. Each constraint correspond to a linear equation or inequality, such that the objective function and set of constraints may be written as the following.

$$\begin{array}{lccccccccl}
\text{maximize} & c_1x_1 & + & c_2x_2 & + & \ldots & + & c_nx_n & & \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_1 & \leq b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_1 & \leq b_2 \\
 & & & & & \vdots & & & \\
 & a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & a_{nn}x_1 & \leq b_n \\
 & & & & & & & x_1, x_2, \ldots, x_n & \geq 0
\end{array} \tag{1.2}$$

The LP problem may be written in matrix form as the following.

$$\begin{array}{lrl} \max & c^T x & \\ \text{subject to} & Ax & \leq b, \\ & x & \geq 0 \end{array} \tag{1.3}$$

Each vector $x \in \mathbb{R}^n$ is *feasible* if it satisfies the constraints in the LP problem, see [6]. If an LP problem may be written on the form 1.3, the feasible solutions are $x \in \mathbb{R}^n$ such that they satisfy $Ax \leq b$, $x \geq 0$. Let P be the set of all feasible solutions to the LP problem, i.e. $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$. If one chooses any two distinct points x_1, x_2 in P, then the line segment $\lambda\, x_1 + (1-\lambda)\, x_2$ also lies in P. This can be shown as

$$A\,(\lambda\, x_1 + (1-\lambda)\, x_2) = \lambda\, A\, x_1 + (1-\lambda)\, A\, x_2 \leq \lambda\, b + (1-\lambda)\, b = b$$

Thus, P is a convex set. In fact, P is a polyhedron. A *polyhedron* in $\mathbb{R}^n$ is defined to be the set $\{x \in \mathbb{R}^n : Ax \leq b\}$ where $A \in M_{m,n}$, $b \in \mathbb{R}^m$. Moreover, from the previous argument we have the following result.

Proposition 1.5.1. *[6, Proposition 1.4.1]] The solution set of any linear system in the variable $x \in \mathbb{R}^n$ is a polyhedron. Every polyhedron is a convex set.*

We will now give an example of an LP problem, which also illustrates convexity and polyhedra.

Example 1.5.2. We will study the following linear programming problem.

$$\begin{array}{lrcrl} \text{maximize} & 3x_1 & + & x_2 & \\ \text{subject to} & 0.3x_1 & + & x_2 & \leq 5 \\ & x_1 & + & x_2 & \leq 8 \\ & x_1 & - & x_2 & \leq 4 \\ & & & x_1, x_2 & \geq 0 \end{array} \tag{1.4}$$

Figure 1.3 illustrates the LP problem in the plane. One can see that the feasible set P is a polyhedron. The objective function is represented as level lines, $3x_1 + x_2 = a$, where a is a constant, and which are drawn as stapled lines. In the feasible region, the objective function attains its maximum when $x = (6, 2)$ which give the optimal value $3x_1 + x_2 = 3 \cdot 6 + 2 = 20$.

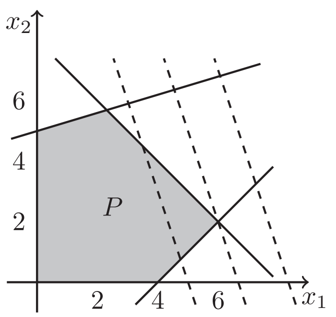


Figure 1.3: A drawing of the LP problem in Example 1.5.2

△

We will now introduce an important concept in polyhedra theory, namely the convex hull. The *convex hull* of a set S is the set of all convex combinations of points in S, following [6]. In fact, is the smallest convex set containing S, as we will see from the following proposition.

Proposition 1.5.3. *[6, Proposition 2.2.1] Let $S \subseteq \mathbb{R}^n$. Then* $\operatorname{conv}(S)$ *is equal to the intersection of all convex sets containing S. Thus,* $\operatorname{conv}(S)$ *is the smallest convex set containing S.*

An illustration of the convex hull can be seen in Figure 1.4. Notice that the feasible region in Figure 1.3 is the convex hull of the set of constraints corresponding to the previous LP problem. Actually, the convex hull of a finite set gives an important class of convex sets, namely polytopes. A set $P \in \mathbb{R}^n$ is called a *polytope* if it is the convex hull of a finite set of points in $\mathbb{R}^n$, see [6]. In fact, a set is a polytope if and only if it is a bounded polyhedron. This is an important result in polytope theory. By consequence, if the feasible region of an LP problem is bounded, one can solve the problem by comparing the vertices of the feasible region (a polytope) to the objective function, as we did in Example 1.5.2. This is the starting point of a algorithm for solving LP problems, called the *Simplex Method*.

(a) S (b) $\operatorname{conv}(S)$

Figure 1.4: A set S and its convex hull $\operatorname{conv}(S)$

1.6 Graph theory

In this section, we will introduce graph theory from the works of Bondy & Murty [2], Brualdi & Ryser [4], and Dahl [7]. as it is relevant to the work in this thesis. There are various uses of graph theory, including transportation and communication networks. A graph may, among other things, be used to describe databases in computer science, or to represent a network of people and their association with each other. In some cases, like transportation networks, it is necessary to include a direction between elements of a graph. While in other cases, for example when describing relationships in a network of people, the relation itself has no direction. In this sense, we distinguish between *directed* and *undirected* graphs. It is also common to make a correspondence between certain graphs and matrices, which we will include in this section.

A graph G consists of a set of vertices V and a set of edges E. We denote this $G = (V, E)$. The set of edges E consists of pairs of elements of V (not necessarily disjoint), see [7]. For example,

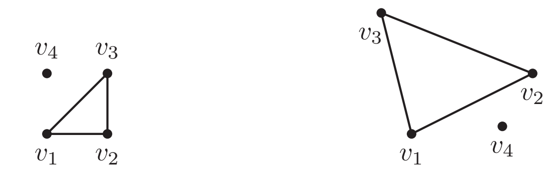
$$V = \{v_1, v_2, v_3, v_4\} \quad E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}.$$

In some cases, we will denote an edge $\{v_i, v_j\}$ as e_{ij}. A graph is *finite* if the sets V, E are finite, see [2]. In this thesis we will only be focusing on finite graphs, and for the remaining theory we will assume any graph G is finite.

A graph may be represented as a drawing in the plane, where vertices and edges are drawn as points and lines. This is exemplified in Example 1.6.1. If a graph is directed, the edges may be drawn as arrows. Note that there are several ways to draw a graph, and the edges may be drawn by either curved or straight lines.

Example 1.6.1. Let $G = (V, E)$ be a graph where $V = \{v_1, v_2, v_3, v_4\}$, $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}$ Then the graph may be represented as the drawings in Figure 1.5.

△



(i) Graph G (ii) Graph G

Figure 1.5: Two drawings of the graph $G = (V, E)$

For all following discussions of graph theory, graphs may be represented by similar drawings.

We will now continue with additional standard definitions related to graph theory, following [2]. The *order* of a graph is the number of vertices in the graph. We say that an edge $\{v_i, v_j\}$ *connects* the vertices v_i, v_j. If $v_i = v_j$, the edge is called a *loop*. Two vertices are *adjacent* if they are connected by an edge, and *nonadjacent* otherwise. For example in Figure 1.5, the vertex v_1 is adjacent to vertices v_2 and v_3. Moreover, the *neighbors* of a vertex v are the vertices adjacent to v. In figure Figure 1.5, v_2 and v_3 are neighbors of v_1.

As mentioned earlier, we distinguish between directed and undirected graphs. In directed graphs, the edges are assigned an orientation and we call them *arcs*, following [2]. This means we distinguish between the arcs (v_i, v_j) and (v_j, v_i) connecting vertices v_i, v_j, and the pair of vertices are ordered. For an arc (v_i, v_j) from v_i to v_j, we call v_i the *tail* and v_j the *head*. This is often illustrated by an arrow in a drawing of a graph. Directed graphs are often abbreviated to *digraphs*, and are denoted $D = (V, E)$.

Let $G = (V, E)$, $H = (\tilde{V},\ \tilde{E})$ be two graphs, and let $\tilde{V} \subseteq V$, $\tilde{E} \subseteq E$. Then H is a *subgraph* of G and we say that H is contained in G, or $H \subseteq G$, see [2]. This means that H can be obtained by removing vertices and/or edges of G. A *spanning subgraph* of G is a graph where edges of G are removed. This means that a spanning subgraph contains all the vertices of G, but possibly not all edges. Figure 1.6 exemplifies subgraph and spanning subgraph.

We will now continue with some important terms in graph theory, still following [2]. First, let $G = (V, E)$ be a graph. If for every nonempty partition I, J of the vertices V, there exists an edge that connects a vertex in I to a vertex in J, the graph is *connected*. If this does not hold, the graph is *disconnected*. The vertices in a disconnected graph can be partitioned into two nonempty sets I, J such that there does not exist any edges connecting a vertex in I to a vertex in J [2]. Figure 1.7 illustrates an example of a connected and a disconnected

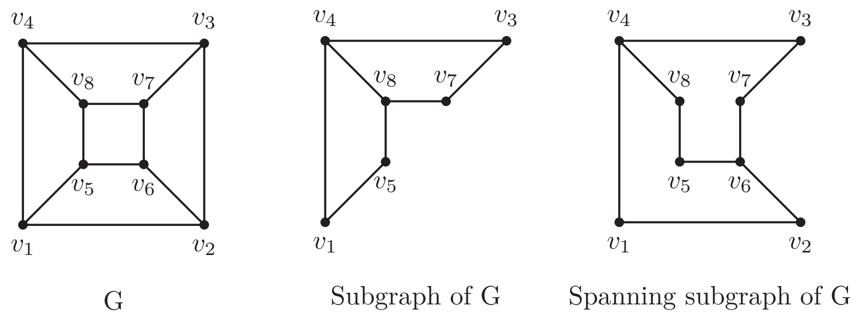


Figure 1.6: Example of graph, subgraph, spanning subgraph

graph. A *path* is a graph with distinct vertices which can be arranged in a linear sequence (v_1, $v_2, \ldots,$ v_m), such that each consecutive vertex is adjacent, and nonadjacent otherwise [2]. The *length* of a path is the number of edges in the path. For example, the disconnected graph in Figure 1.7 has two paths of length 3 and 2. Such a path is called a *cycle* if the first vertex is equal to the last vertex of the path. Figure 1.8 illustrates an example of a graph which contains a cycle, and one which does not.

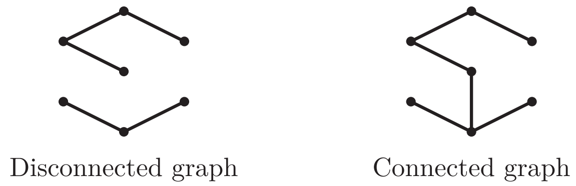


Figure 1.7: Example of a disconnected and a connected graph

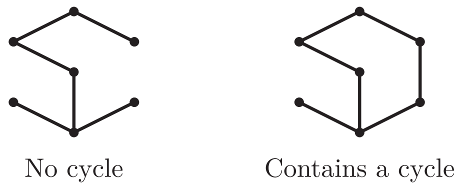


Figure 1.8: Example of a graph which contains a cycle and one which does not

In the next chapter, we will present fully indecomposable matrices. In that regard, we will discuss strongly connected directed graphs, and will now present relevant theory for that matter, following [4]. A *walk* from vertex v_1 to v_m is a sequence of arcs $((v_1, v_2), (v_2, v_3), \ldots, (v_{m-1}, v_m))$. Put simply, one can travel from vertex v_1 to v_m provided there are arcs connecting each vertex on the walk in the direction one is traveling. As apposed to paths, a walk may include loops. Two vertices v_i, v_j are strongly connected if there exists a walk from vertex v_i to v_j. Finally, a digraph is *strongly connected* if and only if each pair of vertices are strongly connected. Figure 1.9 illustrates an example of a graph which is strongly connected, and one which is not.

(a) Strongly connected digraph D (b) Not strongly connected

Figure 1.9: Example of strongly connected digraph

Let $D(V, E)$ be a digraph with vertices $(v_1, v_2, \ldots, v_n)$. The adjacency matrix of D is an $n \times n$ $(0,1)$-matrix $A(D) = [a_{ij}]$ such that a_{ij} equals the multiplicity $m(v_i, v_j)$ of arcs from vertices v_i to v_j, for $i, j \leq n$, following [4]. For example, the adjacency matrix $A(D)$ of the strongly connected digraph D in Figure 1.9 (a) is

$$A(D) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In the next section, we will dive deeper into graph theory which is relevant for this thesis. This entails bipartite graphs, matchings and weights in a graph.

1.7 Bipartite graphs, matchings and weights

In this section, we will introduce bipartite graphs, matchings and weights in graphs. The theory which is presented in this section is cited from the works of Bondy & Murty [2] and Shader [12].

If the vertices in a graph $G = (V, E)$ can be partitioned into two disjoint sets I, J such that each edge in G connects a vertex in I to a vertex in J, then G is called is called a *bipartite graph*, following [2]. We often denote bipartite graphs $G(I, J)$ or $G = (I, J, E)$ where $V = I \cup J$. Moreover, if there exists edges in G such that all vertices in I are connected to all vertices in J, then G is a *complete bipartite graph*. A complete bipartite graph may be defined as the following.

Definition 1.7.1. Let $G = (I, J, E)$ be a graph with vertices $V = I \cup J$, where $I = \{i_1, i_2, \ldots, i_m\}$, $J = \{j_1, j_2, \ldots, j_n\}$, and edges E. G is a complete bipartite graph if the following hold

(i) I and J are disjoint sets

(ii) $E = \{\{i_k, j_l\}\}$ for all $k \leq m$, $l \leq n$

We denote the complete bipartite graphs $K_{m,n}$.

Figure 1.10 exemplifies a bipartite graph and the complete bipartite graph $K_{2,3}$.

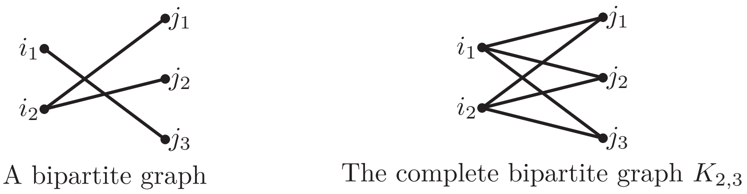


Figure 1.10: Example of a bipartite and a complete bipartite graph

A *matching* in a graph G is a set of edges such that no two edge share the same vertex. If all vertices in G are covered by a matching, it is called a *perfect matching*, following [2]. Notice that a perfect matching is only possible when the number of vertices in a graph is even. Figure 1.11 illustrates a graph G with a matching and a perfect matching.

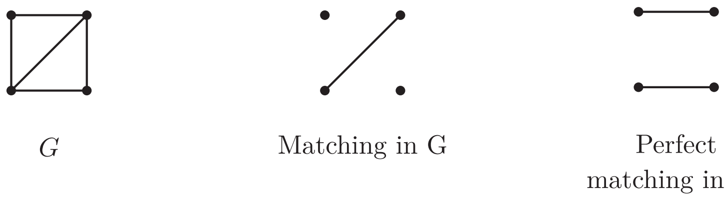


Figure 1.11: Example of matching and perfect matching

In this thesis, we will discuss matchings and perfect matchings in bipartite graphs, and for this purpose we will introduce a few theorems.

Theorem 1.7.2 (**Hall's theorem**). *[2, Theorem 16.4] Let $G(I,J)$ be a bipartite graph, let $S \subseteq I$ be a set of vertices and let $N(S)$ be the set of all neighbors of S. $G(I,J)$ has a matching which covers every vertex in I if and only if*

$$|N(S)| \geq |S| \quad \textit{for all} \quad S \subseteq I \tag{1.5}$$

Inequality 1.5 is often called Hall's criterion.

Corollary 1.7.3. *[2, Corollary 16.5] Let $G(I,J)$ be a bipartite graph, let $S \subseteq I$ be a set of vertices and let $N(S)$ be the set of all neighbors of S. $G(I,J)$ has a perfect matching if and only if $|I| = |J|$ and $|N(S)| \geq |S| \quad \textit{for all} \quad S \subseteq I$.*

We will present some examples that illustrate the theorem and corollary.

Example 1.7.4. Let $G_1(I_1, J_1)$, $G_2(I_2, J_2)$ be two bipartite graphs drawn in Figure 1.12. If we choose the subset $S_1 = \{i_1, i_2\} \subseteq I_1$, then $|S_1| = 2$, $N(S_1) = \{j_1\}$, $|N(S_1)| = 1$ and thus $|N(S_1)| \not\geq |S_1|$. Hall's criterion does not hold for G_1 and thus there is not a matching which covers each vertex in I_1. Note that G_1 has matchings that do not cover all vertices in I_1, for example $\{\{i_1, j_1\}, \{i_3, j_2\}, \{i_4, j_5\}\}$.

In comparison, Hall's criterion holds for graph G_2. No matter which subset $S \subseteq I_2$ one chooses, $|N(S)| \geq |S|$ for all S. An example of a matching which covers all vertices in I_2 is $\{\{i_1, j_1\}, \{i_2, j_2\}, \{i_3, j_4\}, \{i_4, j_5\}\}$.

$\triangle$

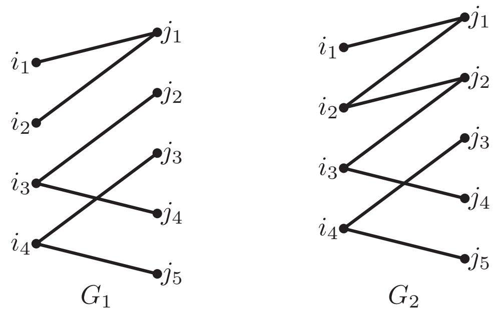


Figure 1.12: Two bipartite graphs

We will also present an example of a bipartite graph which has a perfect matching.

Example 1.7.5. Let $G(I, J)$ be a bipartite graph drawn in Figure 1.13. No matter which subset $S \subseteq I$ of vertices one chooses, $|N(S)| \geq |S|$ for all S, and thus Hall's criterion holds. Since $|I| = |J|$, G has a perfect matching. An example of a perfect matching in G is $\{\{i_1, j_3\}, \{i_2, j_1\}, \{i_3, j_4\}, \{i_4, j_2\}\}$.

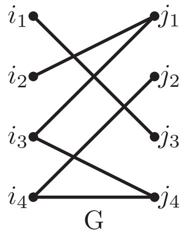


Figure 1.13: Bipartite graph that has perfect matching

△

Among the many applications of graph theory, we have the classical *optimal assignment problem*. Assume we have a certain amount of people and tasks to fill, but not all people are qualified for all the tasks. One can construct a bipartite graph which represents each person (the vertices in I), the tasks (the vertices in J), and which person is qualified for which task (the edges in the graph). Each perfect matching of this graph represents a possible assignment of people to the different tasks such that each person is assigned a task and each task is covered by a person. A formal definition of the optimal assignment problem is introduced later in the thesis, under Section 2.5.

Graph theory is often used in optimization problems like the optimal assignment problem, or other practical problems like communication and transportation problems. In these kinds of problems, one might need to take additional factors, like costs or profits, into account. These factors are often represented as a real number, which we call *weight* $w(e)$, associated to each edge e of a graph, following [2]. A graph G with weights w is called a *weighted graph* and is often denoted (G, w). We will often be interested in finding the sum of all weights in a subgraph F of G, in which case one computes $\sum_{e \in E(F)} w(e)$. In the optimal assignment problem, the weights may represent costs corresponding to a certain person and task, and the aim is to find the perfect matching which gives the minimum sum of weights.

We will now introduce a correspondence between bipartite graphs and matrices. For a bipartite graph $G(I, J)$, where $I = \{i_1, i_2, \ldots i_m\}$, $J = \{j_1, j_2, \ldots, j_n\}$, the *biadjacency matrix* of G is an $m \times n$ matrix $A(G) = [a_{kl}]$, where $a_{kl} = 1$ for each edge $\{v_k, v_l\}$, $k \leq m$, $l \leq n$, and $a_{kl} = 0$ if there does not exist an edge between the vertices v_k, v_l, see [12].

Similarly, for a *weighted* bipartite graph (G, w), with vertices $I = \{i_1, i_2, \ldots i_m\}$, $J = \{j_1, j_2, \ldots, j_n\}$, the biadjacency matrix of G is an $m \times n$ matrix $A(G) = [a_{kl}]$, where $a_{kl} = w(e_{kl})$ and $a_{kl} = 0$ if there does not exist an edge between the vertices v_k, v_l, see [12]. We will give an example of a weighted bipartite graph and its associated biadjacency matrix.

Example 1.7.6. Let G be the weighted bipartite graph in Figure 1.14.

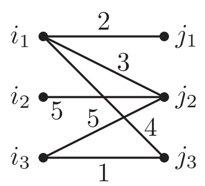


Figure 1.14: Weighted bipartite graph G

Then, the following matrix $A(G)$ is the biadjacency matrix associated to G.

$$A(G) = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

Notice that the graph G in Figure 1.14 contains exactly one perfect matching, $\{e_{11}, e_{22}, e_{33}\}$. This perfect matching corresponds to the main diagonal of matrix $A(G)$. The weight and diagonal sum equals to $2 + 5 + 1 = 8$. We will discuss matrix diagonals, diagonal sums and their correspondence to perfect matchings later in this thesis. △

In the previous example, we observed a correspondence between a perfect matching in a graph and the diagonal of the biadjacency matrix of the graph. All possible perfect matchings in a bipartite graph correspond to the diagonals of the biadjacency matrix after suitable permutations of its row and column vector. In the next section, we will present theory on permutations of matrices.

1.8 Permutations

Permutations of matrices, and their diagonals, are an important aspect of our work in later chapters. We will therefore give a brief overview over the theory of permutations in this section. The theory presented in this section is cited from Brualdi [5], Brualdi & Dahl [3] and Terras [14].

Let $p = \{p_1, p_2, \ldots, p_n\}$ be a set of n elements. A *permutation* of p is any ordered set of the elements of p. We can define a permutation as a bijective to itself, $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ for $n = 2, 3, \ldots$, following [14]. For example, let $p = \{1, 2, 3\}$. Then all possible permutations of p are $\sigma_1 = (1, 2, 3)$, $\sigma_2 = (1, 3, 2)$, $\sigma_3 = (2, 1, 3)$, $\sigma_4 = (2, 3, 1)$, $\sigma_5 = (3, 1, 2)$, $\sigma_6 = (3, 2, 1)$. Moreover, one can find the inverse σ^{-1} of a permutation σ by interchanging each position with its corresponding element, i.e. for the k'th position in σ, if $\sigma(k) = l$, then $\sigma^{-1}(l) = k$. This is exemplified in Example 1.8.1.

Permutations may be written with matrix notation, and we call these matrices *permutation matrices*. For a permutation $\sigma = (k_1, k_2, \ldots, k_n)$, we define an $n \times n$ permutation matrix $P_\sigma = [p_{ij}]$ such that $p_{ij} = 1$ if $j = k_i$ for each i, and $[p_{ij}] = 0$ otherwise, following [3]. Note that a permutation matrix has exactly one entry in each row and column, and thus there are $n!$ permutation matrices of size n.

Example 1.8.1. Let $\sigma = (1, 4, 2, 3)$ be a permutation. When finding the inverse of the permutation, we first look at the first position $\sigma(1) = 1$. Since the position equals the element, $\sigma^{-1}(1) = 1$. Furthermore, $\sigma(2) = 4$ gives $\sigma^{-1}(4) = 2$, $\sigma(3) = 2$ gives $\sigma^{-1}(2) = 3$, and lastly $\sigma(4) = 3$ gives $\sigma^{-1}(3) = 4$. Thus, the inverse σ^{-1} is equal to $(1, 3, 4, 2)$. Moreover, the permutation matrices P_σ, $P_{\sigma^{-1}}$ are

$$P_\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad P_{\sigma^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Notice that $P_{\sigma^{-1}} = P_\sigma^T$. This is the case for all permutation matrices, see [5], and will be used in the following discussion.

△

In this thesis, we will often discuss permuted matrices and their diagonals. We can permute row and column vectors of a matrix A by left and right matrix multiplication. Left-multiplying a matrix A by P, i.e. PA, will permute the row vectors of A. Right-multiplying A by P, i.e. AP, will however permute the column vectors according to the inverse permutation σ^{-1}. The permutation matrix $P_{\sigma^{-1}}$ is equal to P^T, and thus AP^T permutes the column vectors according to σ. Thus, one can simultaneously permute row and column vectors of the matrix A by PAP^T, see [5]. When permuting a matrix, the positions of the diagonals will clearly change, which will be discussed in later chapters of this thesis.

Example 1.8.2. Let P_σ be the permutation matrix from Example 1.8.1, let $P_\sigma^T = P_{\sigma^{-1}}$ be the permutation matrix of σ^{-1}, and let A be the following 4×4 matrix.

$$P_\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad P_\sigma^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

We have the following matrix products

$$P_\sigma A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 12 & 13 & 14 & 15 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \qquad AP_\sigma^T = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 5 & 8 & 6 & 7 \\ 9 & 12 & 10 & 11 \\ 13 & 16 & 14 & 15 \end{bmatrix}$$

We can see that the row vectors in P_σ are permuted according to the permutation $\sigma = (1,4,2,3)$ and the column vectors in AP_σ^T are permuted according to the permutation $\sigma^{-1} = (1,3,4,2)$. Lastly, the product $P_\sigma AP_\sigma^T$ simultaneously permutes the row and column vectors.

$$P_\sigma AP_\sigma^T = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 13 & 16 & 14 & 15 \\ 5 & 8 & 6 & 7 \\ 9 & 12 & 10 & 11 \end{bmatrix}$$

△

1.9 Doubly stochastic matrices

We will now introduce doubly stochastic matrices, which is widely used in this thesis. Doubly stochastic matrices are square nonnegative matrices where the sum of each row and column vector equals to 1. These are a special kind of matrices with many applications, one of them being the optimal assignment problem. Since the sum of all row and column vectors equal to one, these matrices are often found in work with statistics and probability. A useful property of these matrices is that all permutations of doubly stochastic matrices are also doubly stochastic. In the end of this section, we will provide a very important theorem which states that the convex hull of all $n \times n$ permutation matrices is indeed the $n \times n$ doubly stochastic matrices. This means that any convex combination of doubly stochastic matrices are also doubly stochastic, and the set of all $n \times n$ doubly stochastic matrices form a convex polytope. But first, we give the formal definition of a doubly stochastic matrix, following [3].

Definition 1.9.1. [3] Let A be an $n \times n$ real matrix. We call A *doubly stochastic* if all elements are nonnegative and each column and row sum is equal to 1, i.e. the following conditions hold

(i) $A = [a_{ij}] \geq 0$

(ii) $\sum_{i=1}^n a_{ij} = 1, \ j = 1, \dots, n$

(iii) $\sum_{j=1}^n a_{ij} = 1, \ i = 1, \dots, n$

The definition uses the notation $A \geq 0$ which means that all elements of A are nonnegative. We denote the set of all $n \times n$ doubly stochastic matrices as Ω_n, and Ω_n is a subset of the vector space M_n. In the end of this section, we will present a theorem which states that the permutation matrices are in fact the extreme points of Ω_n. We will now present a few examples of doubly stochastic matrices, where we also exemplify the relation between doubly stochastic matrices and bipartite graphs as introduced previously.

Example 1.9.2. Let P_σ be the permutation matrix from Example 1.8.1 corresponding to the permutation $\sigma = (1,4,2,3)$.

$$P_\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Then, the bipartite graph G in Figure 1.15 is associated to matrix P_σ, such that P_σ is the biadjacency matrix of G. Notice that the edges in the graph is a perfect matching, which is the case for all bipartite graphs associated to permutation matrices.

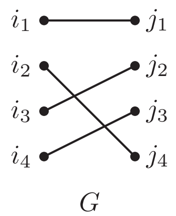


Figure 1.15: Bipartite graph G. Matrix P_σ is the biadjacency matrix of G

△

Example 1.9.3. Let B be the following doubly stochastic matrix,

$$B = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.5 & 0.1 \end{bmatrix}$$

Then, the weighted bipartite graph F in Figure 1.16 is associated to matrix B, such that B is the biadjacency matrix of F. Notice that F is the complete bipartite graph $K_{3,3}$

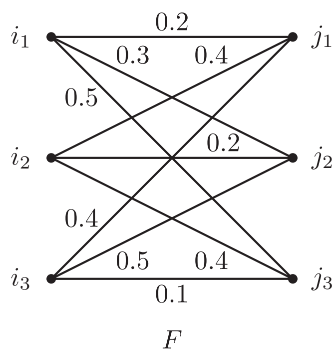


Figure 1.16: Weighted bipartite graph F. Matrix B is the biadjacency matrix of F

△

We will now present an important theorem within convexity theory, with relevance to doubly stochastic matrices. Before we present it, we must give some definitions. First, we will give a definition of an extreme point, cited from Lay et al. [8].

Definition 1.9.4. Let C be a convex set. A point $p \in C$ is called an extreme point of C if p is not in the interior of any line segment that lies in C.

We denote the set of all extreme points of a set C as $ext(C)$. The following definitions are cited from Dahl [6]. A set $S \in \mathbb{R}^n$ is *bounded* if there exists a number M such that $||x|| \leq M$ for all $x \in \mathbb{R}^n$. Moreover, an *open ball* $B(a, r)$ is a set with points x such that for $r > 0$, $a \in \mathbb{R}^n$, the distance between x and a is strictly less than r, i.e. $B(a, r) = \{x \in \mathbb{R}^n : ||x - a|| < r\}$. A set S is *open* if for each $x \in S$ there exists an $\epsilon > 0$ such that $B(a, \epsilon) \in S$. A set S if *closed* if its complement $\overline{S} = \{x \in \mathbb{R}^n : x \notin S\}$ is open. Lastly, a set is *compact* if it is both closed and bounded. We are now ready to present the theorems.

Theorem 1.9.5 (**Minkowski's theorem**). *[6, Corollary 4.3.4] Let $C \subseteq \mathbb{R}^n$ be a compact convex set. Then C is the convex hull of its extreme points, i.e. $C = \text{conv}(ext(C))$.*

From polyhedra theory, it is known that polytopes are both compact and convex sets, see [6]. The Birkhoff-von Neumann theorem provides a better understanding of the close connection between convexity, permutation matrices and doubly stochastic matrices.

Theorem 1.9.6 (**Birkhoff-von Neumann theorem**). *[7, Theorem 2.6] Let $\mathcal{P}_n$ be the set of all $n \times n$ permutation matrices, and conv($\mathcal{P}_n$) be the convex hull of $\mathcal{P}_n$. The set of all doubly stochastic matrices Ω_n is the convex hull of the set of all permutation matrices $\mathcal{P}_n$, i.e.*

$$\Omega_n = \text{conv}(\mathcal{P}_n)$$

Moreover, $\mathcal{P}_n$ is the set of vertices of Ω_n. So every doubly stochastic matrix may be written as a convex combination of permutation matrices.

Theorem 1.9.6 may be used to construct doubly stochastic matrices, which we have done for the work in this thesis. We will therefore provide an example of how this can be done.

Example 1.9.7. Let P_1, P_2, P_3 be the following 3×3 permutation matrices.

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} P_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then the convex combination $1/12\, P_1 + 5/12\, P_2 + 1/2\, P_3$ is a doubly stochastic matrix.

$$\begin{aligned} 1/12 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} &+ 5/12 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 1/2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} 5/12 & 1/12 & 1/2 \\ 1/12 & 1/2 & 5/12 \\ 1/2 & 5/12 & 1/12 \end{bmatrix} \end{aligned}$$

△

We have written a code in MATLAB which generates doubly stochastic matrices in this way, and the code may be found in Appendix A.1. We have used the code to generate matrices for the work in this thesis.

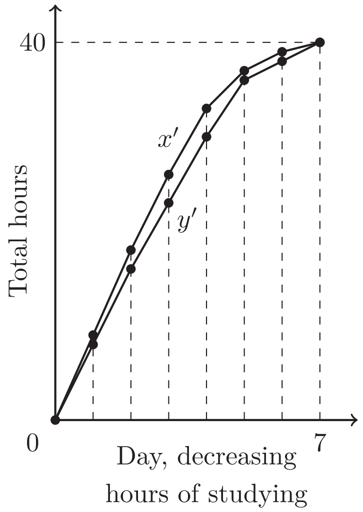


Figure 1.17: Distribution of hours for student x, y

1.10 Majorization

Within the theory of doubly stochastic matrices, there is an important connection to majorization. Majorizaton is central within linear algebra and matrix theory, of which we will provide the formal definition of. First, we will give a simple explanation of the concept. Consider two mathematics students who study the same amount of hours each weak. The first student distributes their working hours evenly throughout the week, and the second student studies minimally on Mondays and Tuesdays, and more extensively later in the week. Let x, y be the hours they work each day for seven days. Assume they study for a total of 40 hours each week, and the distribution of hours for the first student is $x = (7, 8, 8, 7, 6, 2, 2)$, and the second student $y = (1, 2, 8, 9, 9, 7, 4)$. Notice that the components in vector x is more evenly spread out than the components in vector y. Moreover, if we reorder the components such that they are in decreasing order, i.e. $x' = (8, 8, 7, 7, 6, 2, 2)$, $y' = (9, 9, 8, 7, 4, 2, 1)$, we see that the sum up to the i'th component in y' is larger than in x', for $i \leq 7$. The reordered distribution is illustrated in Figure 1.17. In this example, y *majorizes* x. This can also be seen geometrically in Figure 1.17 as both curves start and end in the same point, but otherwise one curve is below the other. We will now give the formal definition of majorization, which is cited from Dahl [7].

Definition 1.10.1. Let x, y be vectors in $\mathbb{R}^n$, and let $x_{[i]}$ and $y_{[i]}$ denote the i'th largest elements of the vectors. We say that x is majorized by y if the following hold

(i) $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 0, 1, \ldots n-1$

(ii) $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_j$

and we write $x \preceq y$.

We will present a short example of majorization.

Example 1.10.2.

$$(\frac{1}{2}, \frac{1}{6}, \frac{1}{4}, \frac{1}{12}) \preceq (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0) \preceq (\frac{1}{2}, \frac{1}{2}, 0, 0) \preceq (1, 0, 0, 0)$$

Notice that the sum of up to the i'th largest component of each vector is less than or equal to the corresponding sum of the majorizing vector. Also, the sum of components is equal for all vectors. △

Now, we proceed to the connection between majorization and doubly stochastic matrices. In fact, an $n \times n$ matrix P is doubly stochastic if and only if $xP \preceq x$ for all $x \in \mathbb{R}^n$, see [9]. We also have the following theorem which shows the important connection between majorization and doubly stochastic matrices, cited from Dahl [7].

Theorem 1.10.3. *Let $x, y \in \mathbb{R}^n$. Then the following are equivalent*

(i) $x \preceq y$

(ii) *There exists a doubly stochastic matrix A such that $x = Ay$*

(iii) *For all convex functions $g : \mathbb{R} \to \mathbb{R}$, the following inequality holds*

$$\sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i)$$

We will demonstrate this theorem with an example.

Example 1.10.4. Let $A \in M_4$, $y \in \mathbb{R}^4$ be the following 4×4 doubly stochastic matrix and vector respectively.

$$A = \begin{bmatrix} 1/4 & 0 & 1/2 & 1/4 \\ 1/4 & 3/4 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/2 \\ 1/4 & 0 & 1/2 & 1/4 \end{bmatrix} \quad y = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

Then we can find a vector x such that $x \preceq y$ by computing the matrix product.

$$x = Ay = (2, 2/3, 7/2, 3)$$

We take a convex function $g(x) = \{x^2 : x \in \mathbb{R}\}$ and see that

$$\sum_{i=1}^{n} g(x_i) = 2^2 + (2/3)^2 + (7/2)^2 + 3^9 = 25/2$$
$$\leq \sum_{i=1}^{n} g(y_i) = 0^2 + 2^2 + 1^1 + 6^6 = 41$$

△

This concludes the introduction chapter. In the next chapter, we will introduce RCDS matrices and explore the diagonals of these matrices.

CHAPTER 2

Diagonals of RCDS matrices

In this chapter, we will explore a class of matrices called *restricted constant diagonal sum* matrices. These are often abbreviated to RCDS matrices. These matrices have the special property that all diagonals of a given matrix, that do not contain any zeros, have the same sum. RCDS matrices are intricate, and there does not exist a way to find them all. However, Brualdi & Dahl [3] have provided us with ways to construct some types of RCDS matrices. We will introduce a few of these, and then focus of on one specific class of RCDS matrices for the remaining thesis. We will also use MATLAB to generate random RCDS matrices in order to study them. Moreover, we will see how RCDS matrices correspond to bipartite graphs and the optimal assignment problem.

We will start by discussing matrix diagonals and zero patterns of matrices. In our work, we have only studied fully indecomposable matrices, which we will provide a definition of before we move on to RCDS matrices. Later, we will introduce a specific class of RCDS matrices and explore the diagonals and constant diagonal sum of these matrices. This work leads us to the next chapter, where we will proceed to make some modifications to the matrices to see how this alters the diagonal sums.

2.1 Diagonals of matrices

Matrix diagonals are widely used in mathematics. We have previously discussed how matrix diagonals are associated to perfect matchings in bipartite graphs, the optimal assignment problem and permutation matrices. The diagonals of a matrix is a main topic in this thesis, and in this section we will present some important definitions and theory.

From linear algebra, the diagonal of an $n \times n$ matrix is usually defined as the set of n positions $\{(1,1),(2,2),\dots(n,n)\}$. Recall from Section 1.8 that permutations of matrices change the the diagonals. In this thesis, we are interested in certain diagonals of a matrix that can be obtained after all permutations of a matrix. For an $n \times n$ matrix, we will denote the diagonal D_σ of the matrix to be a set of n diagonal positions corresponding to a permutation σ of the matrix. We will use the term *diagonal* when discussing the matrix positions of a diagonal, and the term *diagonal entries* when discussing their elements. We will now present an example of matrix diagonals.

Example 2.1.1. Let A be the following 5×5 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

Then, some examples of diagonals may be

$$\begin{aligned} D_{\sigma_1} &= \{(1,1),(2,2),(3,3),(4,4),(5,5)\} \\ D_{\sigma_2} &= \{(4,1),(1,2),(2,3),(5,4),(3,5)\} \\ D_{\sigma_3} &= \{(3,1),(2,2),(1,3),(4,4),(5,5)\} \end{aligned}$$

where σ_1, σ_2, σ_3 correspond to permutations of A such that these diagonals are obtained. △

Recall that permutation matrices contain exactly one 1 in each row and column and 0's otherwise. Thus, the set of positions with 1-entries in permutation matrices constitutes to a diagonal of the permutation matrix. Two permutation matrices are *pairwise disjoint* if they do not have a 1 in the same position, following [3]. Similarly, we say that their associated diagonals are pairwise disjoint. In this thesis we will discuss *diagonal sums*, which are the sums of the diagonal entries. More specifically, we are interested in studying diagonals which do not contain any zero positions, and the sums of these diagonals. For a matrix X, we denote the set of all zero positions $\xi(X)$, and we call diagonals disjoint from $\xi(X)$ *nonzero diagonals*. If all diagonals of a matrix X that are disjoint from $\xi(X)$ have the same sum, we say the matrix has *restricted constant diagonal sums*. If we have two matrices X, Y such that $\xi(X) = \xi(Y)$, we say that X and Y have the same *pattern*. Moreover, we will sometimes refer to *zero diagonals*, which entails diagonals that only consist of zero positions. We will now introduce a few theorems which are relevant for further discussion.

Theorem 2.1.2. *[3, Thoerem 1.1] Let $X \in \Omega_n$, and let $D = \{D_{\sigma_1}, D_{\sigma_2}, \dots D_{\sigma_k}\}$ be a set of k pairwise disjoint zero diagonals of X. Assume that every diagonal of X that is disjoint from the diagonals in D have constant sum. Then all entries of X that is not in any of the diagonals in D equal $\frac{1}{n-k}$*

Theorem 2.1.2 is proved by Sinkhorn [13] and Balasubramanian [1]. From this theorem, one can construct a doubly stochastic matrix with constant diagonal sums from certain $(0,1)$-matrices. We will present an example of this.

Example 2.1.3. Let A be a 5×5 $(0,1)$-matrix with three 1's in each row and column.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

This matrix has $k = 2$ disjoint zero diagonals. We then define X such that $\xi(X) = \xi(A)$ and each nonzero entry of X is equal to $\frac{1}{5-2} = \frac{1}{3}$. Then X is a doubly stochastic matrix with constant diagonal sums.

$$X = \begin{bmatrix} 1/3 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \end{bmatrix}$$

△

The following theorem and corollary provides a useful connection between RCDS matrices and their zero positions.

Theorem 2.1.4. *[3, Theorem 1.2] Let* $X, Y \in \Omega_n$ *and let* $Z_X \subseteq \xi(X)$, $Z_Y \subseteq \xi(Y)$ *be subsets of zero positions. Assume all diagonals of* X *disjoint from* Z_X *have constant sum* α *and all diagonals of* Y *disjoint from* Z_Y *have constant sum* β. *Then the following two conditions hold*

(i) *If* $Z_X \subseteq Z_Y$, *then* $\alpha \leq \beta$

(ii) *If* $Z_X = Z_Y$, *then* $\alpha = \beta$ *and* $X = Y$

Corollary 2.1.5. *[3, Corollary 1.3] An* $n \times n$ *doubly stochastic matrix* X, *with a specified set* Z *of zeros all of whose diagonal sums avoiding* Z *are equal, is uniquely determined.*

We will refer to Theorem 2.1.4 and Corollary 2.1.5 in later parts of this thesis, in connection with some of our results.

Recall that for a weighted bipartite graph (G, w), with vertices $I = \{i_1, i_2, \ldots i_m\}$, $J = \{j_1, j_2, \ldots, j_n\}$, the biadjacency matrix associated to G is an $m \times n$ matrix $A(G) = [a_{kl}]$ where a_{kl} correspond to the weights in G, for $k \leq m$, $l \leq n$. Moreover, recall that a perfect matching in a bipartite graph G, when $n = m$, is a set of disjoint edges such that each vertex in G is contained in the set of edges. Thus, all possible perfect matchings of a bipartite graph G correspond to the nonzero diagonals of the matrix $A(G)$.

In the next section, we will provide a further specification to the matrices we are studying in this thesis, namely fully indecomposable matrices.

2.2 Total support and fully indecomposable matrices

Recall that the aim of this thesis is to explore nonzero diagonals of doubly stochastic matrices, and the sums of these diagonals. Without certain restrictions to the zero positions of the matrices we are studying, it may prove difficult to find certain patterns of the matrices. We will therefore only study fully indecomposable matrices, of which we will provide a formal definition in this section. For this matter, we will first introduce the term total support.

A matrix A has *total support* if each of its nonzero elements belong to a nonzero diagonal of the matrix, following [5]. For example, the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

does not have total support, since the 1 in position $(1, 2)$ only belongs to a diagonal including a 0. For our work, this means that for matrices with total support, all nonzero elements will be included in the nonzero diagonals we are studying. However, matrices with total support, along with their permutations, may be a direct sum of fully indecomposable matrices, see [5]. When studying the properties and diagonals of matrices, it is therefore mostly interesting to study fully indecomposable matrices (which also have total support). Now, we will explain what this entails.

A matrix A is *partly decomposable* if, after suitable permutations of rows and columns, it obtains the form

$$\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \tag{2.1}$$

where A_1 and A_3 are square nonempty matrices, following [5]. If a matrix is not partly decomposable, then it is *fully indecomposable.* Notice that if a matrix is fully indecomposable, all possible permutations are as well. Equivalently, a matrix is fully indecomposable if it does not have a zero submatrix of size $k \times (n-k)$ for any $0 < k < n$. In other words, a fully indecomposable matrix does not have a zero submatrix of size $k \times l$ such that $k + l \geq n$. For small matrices, one can easily check if a matrix is fully indecomposable by checking the dimension of zero submatrices (if there are any), but this is not efficient for larger matrices. We will provide an example of both a partly decomposable matrix and a fully indecomposable matrix.

Example 2.2.1. Let A, B be the following 4×4 matrices.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

We can determine that A is partly decomposable since it has a 1×3 zero submatrix. However, it is not possible to permute row and column vectors of B such that it provides the same form as 2.1. One can also see that B does not contain any zero submatrices of size $k \times (4-k)$ for any $0 < k < 4$. Thus, B is fully indecomposable.

△

In the Example 2.2.1, we could easily check if the matrices were fully indecomposable by checking the dimensions of its zero submatrices. In general, there is a more efficient method for this purpose which we have also used in this thesis. We will first present two theorems for this matter.

Theorem 2.2.2. *[4, Theorem 3.2.1] Let A be an $n \times n$ matrix. Then A is irreducible if and only it its digraph D is strongly connected.*

In Theorem 2.2.2, A is the adjacency matrix of digraph D.

Theorem 2.2.3. *[4, Theorem 4.2.3] Let A be an $n \times n$ $(0,1)$-matrix. Let A' be the matrix obtained from A by replacing each entry on the main diagonal with a 1. Then, A is irreducible if and only if A' is fully indecomposable.*

For the work in this thesis, we have generated random doubly stochastic matrices in MATLAB. In this case, we have ensured that all matrices are fully indecomposable by first making sure that the main diagonal is nonzero, and then checking that the associated digraph is strongly connected.

If a matrix A is fully indecomposable, then there exists a doubly stochastic matrix with the same pattern as A, see [3]. If a doubly stochastic matrix is not fully indecomposable, then after suitable permutations of its row and column vectors it can obtain form in 2.1. However, since it is doubly stochastic, the sum of all column vectors in A_1 equal to 1, and thus the sum of all row vectors also equal to one. Hence, A_2 can only consist of zeros, and the matrix is a direct sum of doubly stochastic matrices. In that case, it is more constructive to study the matrices A_1, A_3 separately. For this reason, we will only study fully indecomposable matrices.

2.3 RCDS matrices

So far, we have discussed some different kinds of matrices, namely doubly stochastic matrices, permutation matrices, and $(0, 1)$-matrices. We have also briefly discussed diagonals with constant sums, which we will discuss more extensively in the remaining thesis. First, we give the following formal definition.

Let $X \in M_n$ and let $\xi(X)$ be the set of all 0-positions of X. If all diagonals disjoint from $\xi(X)$ have constant sum, we denote X to be a *restricted constant diagonal sum* (RCDS) matrix, following [3]. Notice that RCDS are not necessarily doubly stochastic, but we will primarily focus on doubly stochastic RCDS matrices in this thesis.

For an RCDS matrix X, we have an associated weighted bipartite graph G such that X is the biadjacency matrix of G, as discussed previously. Since all diagonals disjoint from $\xi(X)$ are constant, all perfect matchings of the bipartite graph will equivalently have a constant weight sum. If such is the case for an optimal assignment problem, all possible perfect matchings will solve the problem.

Our goal is to investigate certain characteristics of RCDS matrices and try to find some patterns within them. These patterns may then apply to the associating weighted bipartite graphs, and the optimal assignment problem. So far, there exists certain algorithms to construct these matrices but a general method to finding them all is not yet known, see [3]. There are some patterns that clearly are RCDS, namely all permutation matrices and the all 1-matrices. Since the 1-matrices have RCDS pattern, we can establish that an RCDS matrix need not have any zeros. However, from Theorem 2.1.4, we know that certain zero patterns may give certain conditions to the diagonal sums. Moreover, we can scale the all 1-matrices such that they are additionally doubly stochastic. If A is an $n \times n$ all 1-matrix, and hence RCDS, then $\frac{1}{n}A$ is RCDS doubly stochastic. We will now provide an example of an RCDS matrix.

Example 2.3.1. Let A be the following 5×5 matrix,

$$A = \begin{bmatrix} 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 2 & 0 & 0 & 3 \\ 0 & 0 & 5 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

No matter which nonzero diagonal one chooses, the diagonal sum will equal 17. Thus, matrix A is an RCDS matrix. Each row and column vectors have sum 5, and we can scale this matrix such that it is also doubly stochastic,

$$\frac{1}{5}A = \begin{bmatrix} 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0.2 & 0 & 0.4 & 0.4 \\ 0 & 0.4 & 0 & 0 & 0.6 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

△

It is worth noting that permutations of RCDS matrices are also RCDS matrices. Recall that a permutation of a matrix may change the order of its row and column vectors. This will however not change the possible diagonals of the matrix, since the elements of the permuted rows and columns remain the same. We will now give a characterization of RCDS doubly stochastic matrices.

Theorem 2.3.2. *[3, Theorem 2.1] Let $A = a_{ij}$ be a fully indecomposable $n \times n$ $(0,1)$-matrix and let $R = (r_1, r_2, \dots, r_n)$ and $S = (s_1, s_2, \dots, s_n)$ be the row and column sum vectors of A.*

(i) *Let $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ be real vectors. Define $Y = Y(u,v) = [y_{ij}] \in M_n$ by $y_{ij} = u_i + v_j$ when $a_{ij} = 1$, and $y_{ij} = 0$ when $a_{ij} = 0$. Assume that $y_{ij} > 0$ whenever $a_{ij} = 1$ and that all row and column sums of Y are equal to some positive number α, i.e.*

$$u_i r_i + \sum_{j:a_{ij}=1} v_j = \alpha \quad (i \leq n) \tag{2.2}$$

$$v_j s_j + \sum_{i:a_{ij}=1} u_i = \alpha \quad (j \leq n) \tag{2.3}$$

Then $X = (1/\alpha)Y(u,v)$ is an RCDS doubly stochastic matrix of A.

(ii) *Conversely, assume X is an RCDS doubly stochastic matrix of A. Then, $X = (1/\alpha)Y(u,v)$, as in (i), for some vectors u, v and α is the common line sum of $Y(u,v)$.*

We will illustrate this with an example.

Example 2.3.3. Let A be the following $(0,1)$-matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

We choose vectors $u = (1,0,0,1,0)$, $v = (1,1,2,2,1)$ and obtain $Y = [y_{ij}]$ where $y_{ij} = u_i + v_j$ when $a_{ij} = 1$ and $y_{ij} = 0$ otherwise.

$$Y = \begin{bmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{bmatrix}$$

Notice that each row and column vector sum to 4. Thus, by Theorem 2.3.2, Y is RCDS and $1/4Y$ is RCDS doubly stochastic. △

This algorithm for constructing RCDS matrices is straightforward, except for the choices of vectors u, v. However, these vectors may be found by solving a system of linear equations. We will discuss this further in the next section.

2.4 Constructing RCDS doubly stochastic matrices from $(0,1)$-matrices

In the previous section, we presented a characterization of RCDS matrices and constructed an example. Now, we will provide an algorithm for constructing RCDS matrices using this characterization. The following construction is directly cited from Brualdi & Dahl [3].

Let A be a fully indecomposable $(0,1)$-matrix. Let $R(A) = (r_1, r_2, \ldots, r_n)$ and $C(A) = (c_1, c_2, \ldots, c_n)$ be row and columns sum vectors of A. Then let D_R, D_C be the diagonal matrices with main diagonals $R(A)$ and $C(A)$, respectively. Define

$$R_i(A) = \{j : a_{ij} = 1\}, \quad (i \leq n) \quad \text{and} \quad C_j(A) = \{i : a_{ij} = 1\}, \quad (j \leq n)$$

Thus, $r_i = |R_i(A)|$ and $c_j = |C_i(A)|$ for each i, j. From Theorem 2.3.2 we assume that $\alpha = 1$. This gives

$$r_i u_i + \sum_{j \in R_i(A)} v_j = 1 \quad (i \leq n) \tag{2.4}$$

$$c_j v_j + \sum_{i \in C_j(A)} u_i = 1 \quad (j \leq n) \tag{2.5}$$

This is a system of linear equations with $2n$ variables and $2n$ constraints, which we can solve to find the vectors u, v from Theorem 2.3.2 and thus construct an RCDS matrix. The system of linear equations correspond to

$$Hx = e, \quad \text{where} \quad H = \begin{bmatrix} D_R & A \\ A^T & D_C \end{bmatrix} \text{and } x = \begin{bmatrix} u \\ v \end{bmatrix} \tag{2.6}$$

Here, e is the all 1's vector. Following this construction, Brualdi & Dahl [3] presented and proved the following result.

Theorem 2.4.1. *[3, Theorem 2.6] Let A be an $n \times n$ fully indecomposable $(0,1)$-matrix. Then the folllowing holds,*

(i) *There exists* $u = (u_1, u_2, \ldots, u_n)$ *and* $v = (v_1, v_2, \ldots, v_n)$ *such that* (u, v) *is a solution to the system 2.6. The solution is unique up to adding a constant to each component in* u *and subtracting the same constant from each component in* v*.*

(ii) A *is the pattern of an RCDS doubly stochastic matrix if and only if* $u_i + v_j > 0$ *for all* (i, j) *with* $a_{ij} = 1$*, where* (u, v) *is an arbitrary solution of 2.6.*

We have written a code in MATLAB that uses this construction, which we have used to generate random RCDS doubly stochastic matrices for the work in this thesis. The code can be found in Appendix A.3. We will demonstrate this construction by an example. Actually, we used this construction to find the RCDS matrix in Example 2.3.3, so we will use the same matrix in the following example.

Example 2.4.2. Let A be the following $(0,1)$-matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

First, we obtain $R(A)$, $C(A)$ by the row and column vector sums of A.

$$R(A) = (2, 3, 3, 2, 2) \quad C(A) = (2, 3, 2, 2, 3)$$

The matrices D_R, D_C are diagonal matrices where $R(A)$, $R(C)$ are the elements along the main diagonal.

$$D_R = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \qquad D_C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Then, we find the matrix H,

$$H = \begin{bmatrix} D_R & A \\ A^T & D_C \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Lastly, we solve the system of linear equations

$$\begin{bmatrix} D_R & A \\ A^T & D_C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = e, \quad \text{where } e \text{ is the all 1's vector}$$

using the code in Appendix A.3, we obtain the vectors

$$u = (-0.2115, -0.4615, -0.4615, -0.2115, -0.4615),$$
$$v = (0.7115, 0.7115, 0.9615, 0.9615, 0.7115)$$

Since the solution is unique up to adding a constant to each component in u and subtracting the same constant from each component in v, we add and subtract 0.4615 to u and from v. We obtain $u' = (0.25, 0, 0, 0.25, 0) = \frac{1}{4}(1, 0, 0, 1, 0)$, $v' = (0.25, 0.25, 0.5, 0.5, 0.25) = \frac{1}{4}(1, 1, 2, 2, 1)$ which are equal to the vectors u, v from Example 2.3.3 before scaling the matrix to be doubly stochastic. △

This algorithm for constructing RCDS matrices is based on the pattern of the starting matrix A, which does not necessarily have an RCDS pattern. In those cases, the algorithm does not provide an RCDS matrix. However, if there exists an RCDS matrix X such that $\xi(A) \subseteq \xi(X)$, i.e. there exists an RCDS matrix which contains the zero pattern of A, then the algorithm will provide us with this RCDS matrix. We will present a short example which demonstrates this.

Example 2.4.3. Let A be the following $(0, 1)$-matrix.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Constructing an RCDS matrix based on A gives the following matrix X,

$$X = \begin{bmatrix} 0 & 0.6 & 0.4 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.4 & 0.4 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The matrices A, X do not have the same pattern, but $\xi(A) \subseteq \xi(X)$. Positions $(3, 1)$, $(3, 2)$, $(3, 4)$ are nonzero in A and zero in X. △

In the next section, we will move on to a method for checking if a matrix is RCDS doubly stochastic.

2.5 The optimal assignment problem and RCDS matrices

RCDS matrices are intricate and it may be a complicated matter to check if a matrix is RCDS. One can use the optimal assignment problem from linear programming to check this property, by computing the maximum and minimum nonzero diagonal sum. If the maximum and minimum nonzero diagonal sum are equal, the matrix is RCDS. For this purpose, will now give a formal definition of the optimal assignment problem, which is directly cited from Vanderbei [15].

Let S be a set of n people, and T be a set of n tasks. For each $i \in S$, $j \in T$ there is a cost c_{ij} associated with assigning person i to task j. The optimal assignment problem is to assign each person to exactly one task such that each task is covered and the total cost of assignments is minimized.

We let

$$x_{ij} = \begin{cases} 1 & \text{if person } i \text{ is assigned task } j \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

The objective function can be written as

$$\min \sum_{i \in S} \sum_{j \in T} c_{ij} x_{ij} \tag{2.8}$$

The constraints are defined such that each person is assigned exactly one task, and each task is covered by exactly one person,

$$\begin{aligned} \sum_{j \in T} x_{ij} = 1 \quad \text{for all } i \in S \\ \sum_{i \in S} x_{ij} = 1 \quad \text{for all } j \in T \end{aligned} \tag{2.9}$$

Notice that the constraints are equivalent to $X = [x_{ij}]$ being a permutation matrix. Since the assumed variables x_{ij} are integral, any feasibly solution to this problem will be integral. We will use this LP problem to find the minimum and maximum diagonal sums of an doubly stochastic matrix B. If the optimal minimum value equals the optimal maximum value, the matrix is RCDS.

Let $B = [b_{ij}]$ be a fully indecomposable doubly stochastic matrix, and $X = [x_{ij}]$ a permutation matrix. We want to find the minimum diagonal sum avoiding all zeros. Since B is fully indecomposable, such a nonzero diagonal must exist. In order to avoid all zeros in the linear programming solution, we replace all 0's in B with M, where M is an large enough number. We denote this matrix $B' = [b'_{ij}]$. This gives the following LP problem,

$$\begin{aligned} \min \sum_{i \leq n} \sum_{j \leq n} b'_{ij} x_{ij} \\ \text{s.t.} \sum_{j \leq n} x_{ij} = 1 \quad \text{for all } i \leq n \\ \sum_{i \leq n} x_{ij} = 1 \quad \text{for all } j \leq n \end{aligned} \tag{2.10}$$

Similarly, we want to find the maximum diagonal sum of B. In that case, we replace all 0's in B with $-M$ and denote this matrix B''. This gives the following LP problem,

$$\begin{aligned} \max \sum_{i \leq n} \sum_{j \leq n} b''_{ij} x_{ij} \\ \text{s.t.} \sum_{j \leq n} x_{ij} = 1 \quad \text{for all } i \leq n \\ \sum_{i \leq n} x_{ij} = 1 \quad \text{for all } j \leq n \end{aligned} \tag{2.11}$$

We are interested in the integer solutions which give us permutation matrices such that the 1's indicate the positions of the elements which give the respective maximum and minimum diagonal sums. We will demonstrate this LP problem with a 3×3 doubly stochastic matrix B. For this demonstration, we will only focus on the minimization problem.

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

The solution of the LP problem will be a 3×3 matrix X where the positions of the 1's provide the positions of B which give the minimum diagonal sum. The system of equations looks like the following

$$\begin{array}{llllllllllll}
\textit{min} & b_{11}x_{11} + & b_{21}x_{21} + & b_{31}x_{31} + & b_{12}x_{12} + & b_{22}x_{22} + & b_{32}x_{32} + & b_{13}x_{13} + & b_{23}x_{23} + & b_{33}x_{33} & & \\
\textit{s.t.} & x_{11} + & x_{21} + & x_{31} & & & & & & & = 1 \\
 & & & & x_{12} + & x_{22} + & x_{32} & & & & = 1 \\
 & & & & & & & x_{13} + & x_{23} + & x_{33} & = 1 \\
 & x_{11} & & + & x_{12} & & + & x_{13} & & & = 1 \\
 & & x_{21} & & + & x_{22} & & + & x_{23} & & = 1 \\
 & & & x_{31} + & & + & x_{32} & & + & x_{33} & = 1 \\
 & & & & 0 \leq & x_{11}, & x_{21}, x_{31}, & x_{12}, x_{22}, & x_{32}, x_{13}, & x_{23}, x_{33} & \leq 1
\end{array} \tag{2.12}$$

We have used this LP problem to write a code in MATLAB which computes the maximum and minimum diagonal sums of a matrix. This code can be found in Appendix A.5.

2.6 The matrix class $X^{(r,s,n)}$

We will now begin to explore the number of nonzero diagonals of RCDS doubly stochastic matrices. The number of nonzero diagonals is equal to to the number of perfect matchings of a weighted bipartite graph associated to the matrix. Since RCDS matrices have proven to be quite complicated, we will study a specific class of RCDS doubly stochastic matrices in order to explore this problem. We denote this class of matrices as $X^{(r,s,n)}$ and start by defining this class. The following definition and preposition is cited from Brualdi & Dahl [3].

Definition 2.6.1. Let r, s, n be positive integers such that $s < r < n$. We define the matrix $X^{(r,s,n)} = [x_{ij}] \in M_n$ to be a $n \times n$ matrix with the following elements

$$x_{ij} = \begin{cases} 1/r & (i \leq r,\, j \leq s) \\ (r-s)/(r(n-s)) & (i \leq r,\, s < j \leq n) \\ 0 & (r < i \leq n,\, j \leq s) \\ 1/(n-s) & (r < i \leq n,\, s < j \leq n) \end{cases} \tag{2.13}$$

Proposition 2.6.2. *[3, Proposition 5.2] $X^{(r,s,n)}$ is an RCDS doubly stochastic matrix for each $s < r < n$.*

Notice that that the definition of the matrix class $X^{(r,s,n)}$ ensures that the matrices are fully indecomposable. Recall that a fully indecomposable matrix does not have a zero submatrix of size $k \times l$ such that $k+l \geq n$. $X^{(r,s,n)}$-matrices have zero submatrices of size $(n-r) \times s$. Since $s < r < n$, it follows that $r \leq n-1$, $s \leq n-2$, and the largest possible zero submatrix is

$$(n-r) + s \leq n - (n-1) + (n+2) = n - 1$$

Thus $X^{(r,s,n)}$-matrices are fully indecomposable. We will provide an example of an RCDS doubly stochastic matrix $X^{(3,2,6)}$ and study the possible diagonals of this matrix. This matrix was also used in an example by Brualdi & Dahl [3].

Example 2.6.3. We will consider the matrix $X^{(3,2,6)}$ given by

$$X^{(3,2,6)} = \begin{bmatrix} 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

One can easily check that $X^{(3,2,6)}$ is RCDS doubly stochastic. First, we see that all column and row vectors sum to 1. Furthermore, when choosing a diagonal excluding zero elements, we see that we must choose 1/3 from both the first and second column. No matter which rows we choose 1/3 from, the remaining row will only have 1/12 as an element. We then proceed to the bottom three rows, and they all have the same element 1/4. Thus, the constant diagonal sum is

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{12} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{2}$$

Using enumerative combinatorics, we can compute the number of ways to choose diagonals of the matrix. There are three ways to choose a position from column 1, thereafter two ways to choose a position from column 2, and then four remaining ways to choose a position with the element 1/12. Equivalently, there are $3 \cdot 2 \cdot 1$ ways to choose positions with elements 1/4 in the bottom three rows. Thus, for the matrix $X^{(3,2,6)}$, we can choose a diagonal in $3 \cdot 2 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 144$ ways. We may for example choose a diagonal in the following way, where the entries are marked in bold.

$$X^{(3,2,6)} = \begin{bmatrix} 1/3 & \mathbf{1/3} & 1/12 & 1/12 & 1/12 & 1/12 \\ \mathbf{1/3} & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & \mathbf{1/12} \\ 0 & 0 & \mathbf{1/4} & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & \mathbf{1/4} & 1/4 \\ 0 & 0 & 1/4 & \mathbf{1/4} & 1/4 & 1/4 \end{bmatrix}$$

$\triangle$

In the next section, we will study the number of diagonals for general matrices from the class $X^{(r,s,n)}$.

2.7 The number of diagonals of $X^{(r,s,n)}$-matrices

With the results from Example 2.6.3 in the previous section, it is interesting to further explore the number of diagonals of RCDS doubly stochastic matrices.

Recall that there are $n!$ permutations of an $n \times n$ matrix, and thus $n!$ diagonals. However, we are mainly interested in finding nonzero diagonals, and computing the number of these can be complicated. As we have seen in the previous section, the number of nonzero diagonals can vary by the size and zero positions of a matrix. Nevertheless, certain classes of RCDS matrices have certain restrictions to their zero positions which may provide a pattern of the number of their diagonals. In this section, we will explore further the matrix class $X^{(r,s,n)}$ from Definition 2.6.1 in search for a pattern in their diagonals.

We have written a code in MATLAB to compute the sums of all nonzero diagonals of a matrix, which we used to find all nonzero diagonal sums, and the number of diagonals from which they were obtained. See Appendix A.7 for the code. In later parts of this thesis, we will explore how minor modifications to an RCDS matrix alters the diagonal sums, but first our goal is to find the total number of nonzero diagonals of the matrix class $X^{(r,s,n)}$. When this is found, we can use the result to find the number of diagonal sums of modified $X^{(r,s,n)}$-matrices. We begin with an example, where we used the code in Appendix A.7 to compute the number of nonzero diagonals.

Example 2.7.1. In this example, we will look at all possible $X^{(r,s,n)}$-matrices where $n = 5$. The number of nonzero diagonals for each matrix X is denoted $N(X)$.

$$X^{(2,1,5)} = \begin{bmatrix} 1/2 & 1/8 & 1/8 & 1/8 & 1/8 \\ 1/2 & 1/8 & 1/8 & 1/8 & 1/8 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}, \; N(X^{(2,1,5)}) = 48$$

$$X^{(3,1,5)} = \begin{bmatrix} 1/3 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/3 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/3 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}, \; N(X^{(3,1,5)}) = 72$$

$$X^{(4,1,5)} = \begin{bmatrix} 1/4 & 3/16 & 3/16 & 3/16 & 3/16 \\ 1/4 & 3/16 & 3/16 & 3/16 & 3/16 \\ 1/4 & 3/16 & 3/16 & 3/16 & 3/16 \\ 1/4 & 3/16 & 3/16 & 3/16 & 3/16 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}, \; N(X^{(4,1,5)}) = 96$$

$$X^{(3,2,5)} = \begin{bmatrix} 1/3 & 1/3 & 1/9 & 1/9 & 1/9 \\ 1/3 & 1/3 & 1/9 & 1/9 & 1/9 \\ 1/3 & 1/3 & 1/9 & 1/9 & 1/9 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}, \; N(X^{(3,2,5)}) = 36$$

$$X^{(4,2,5)} = \begin{bmatrix} 1/4 & 1/4 & 1/6 & 1/6 & 1/6 \\ 1/4 & 1/4 & 1/6 & 1/6 & 1/6 \\ 1/4 & 1/4 & 1/6 & 1/6 & 1/6 \\ 1/4 & 1/4 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}, \; N(X^{(4,2,5)}) = 72$$

$$X^{(4,3,5)} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/8 & 1/8 \\ 1/4 & 1/4 & 1/4 & 1/8 & 1/8 \\ 1/4 & 1/4 & 1/4 & 1/8 & 1/8 \\ 1/4 & 1/4 & 1/4 & 1/8 & 1/8 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}, \; N(X^{(4,3,5)}) = 48$$

△

From this example, we studied the nonzero diagonals of general matrices from the class $X^{(r,s,n)}$ and arrived at the following result.

Theorem 2.7.2. *Let $X^{(r,s,n)}$ be a matrix by Definition 2.6.1. Then, the number of nonzero diagonals is $N(X) = \frac{r!}{(r-s)!}\,(n-s)!$.*

Proof. When finding the number of all diagonals of $X^{(r,s,n)}$, one can divide the matrix into four sections.

$$X^{(r,s,n)} = \left[\begin{array}{c|c} \frac{1}{r} & \frac{r-s}{r(n-s)} \\ \hline 0 & \frac{1}{n-s} \end{array}\right] \begin{array}{l} \}\, r \\ \}\, n-r \end{array}$$
$$\qquad\qquad\underbrace{\qquad}_{s}\;\underbrace{\qquad\quad}_{n-s}$$

Each chosen diagonal will have n elements. The upper left section is an $r \times s$ submatrix of $X^{(r,s,n)}$. Since $s < r$, a total of s positions will be chosen from this section. When choosing diagonal positions from this section, there are $r \cdot (r-1) \cdots (r-s+1) = \frac{r!}{(r-s)!}$ choices. Since s positions are chosen in the lop left section, there remains $n-s$ positions to choose from $(n-s)$ columns in the whole right section. This leaves a total of $(n-s)!$ combinations, and thus the total number of nonzero diagonals is $\frac{r!}{(r-s)!}\,(n-s)!$. □

This result can be applied to the number of perfect matchings of a bipartite graph. First, we will give the formal definition of the class of weighted bipartite graph associated to the matrix class $X^{(r,s,n)}$.

Definition 2.7.3. Let r, s, n be positive integers such that $s < r < n$. We define the weighted bipartite graph $G^{(r,s,n)}$ with disjoint sets of vertices $I = \{i_1, i_2, \ldots, i_n\}$, $J = \{j_1, j_2, \ldots, j_n\}$ and edges with weights such that

$$w(\{i_k, j_l\}) = \begin{cases} 1/r & (k \le r,\ l \le s) \\ (r-s)/(r(n-s)) & (k \le r,\ s < l \le n) \\ \text{no edge} & (k < i \le n,\ l \le s) \\ 1/(n-s) & (k < i \le r,\ s < l \le n) \end{cases} \tag{2.14}$$

Applying Theorem 2.7.2 to the weighted bipartite graphs $G^{(r,s,n)}$ gives us the following corollary.

Corollary 2.7.4. *Let $G^{(r,s,n)}$ be a weighted bipartite graph by Definition 2.7.3. Then, the number er perfect matchings in the graph is $N(G) = \frac{r!}{(r-s)!}\,(n-s)!$.*

Proof. The corollary is a direct result from Theorem 2.7.2. □

2.8 The constant diagonal sum of $X^{(r,s,n)}$-matrices

Based on the definition of $X^{(r,s,n)}$-matrices from Definition 2.6.1, one can compute the constant diagonal sum of the matrices. Since $s < r < n$, there are no zeros along the main diagonal of the matrix and we can use the main diagonal to compute the sum. Again, we will divide the matrix onto four sections for this computation,

$$X^{(r,s,n)} = \left[\begin{array}{c|c} \frac{1}{r} & \frac{r-s}{r(n-s)} \\ \hline 0 & \frac{1}{n-s} \end{array}\right] \begin{array}{l} \}\, r \\ \}\, n-r \end{array}$$
$$\qquad\qquad \underbrace{\qquad}_{s} \;\; \underbrace{\qquad\qquad}_{n-s}$$

Since $s < r$, there will be s elements in the top left section in the main diagonal of $X^{(r,s,n)}$. Moreover, the main diagonal will contain $n-r$ elements from the bottom right section. This leaves $n-s-(n-r) = r-s$ elements in the main diagonal, which must belong in the top right section. Thus, the constant diagonal sum of $X^{(r,s,n)}$-matrices is

$$S(X) = s\,\frac{1}{r} + (r-s)\,\frac{r-s}{r(n-s)} + (n-r)\,\frac{1}{n-s} = \frac{nr+ns-2rs}{r(n-s)} \tag{2.15}$$

From Theorem 2.1.4 we can make in interesting observation of the constant diagonal sums of $X^{(r,s,n)}$-matrices. The theorem implies that for two RCDS doubly stochastic matrices X_1, X_2 with constant diagonal sums S_1, S_2, if $\xi(X_1) \subseteq \xi(X_2)$ then $S_1 \leq S_2$. This means that if n is fixed, the zero positions will influence the constant diagonal sum. Notice that the number of zero positions of $X^{(r,s,n)}$-matrices is $s\,(n-r)$. More specifically, for two matrices $X^{(r_1,s_1,n)}, X^{(r_2,s_2,n)}$ where n is fixed, if $r_1 \geq r_2$ and $s_1 \leq s_2$, then $S_1 \leq S_2$. We will demonstrate this with an example.

Example 2.8.1. Let $n = 7$, $r_1 = 2$, $r_2 = 3$, $s_1 = 6$, $s_2 = 4$. This gives the following matrices of the class $X^{(r,s,n)}$.

$$X^{(6,2,7)} = \begin{bmatrix} 1/6 & 1/6 & 2/15 & 2/15 & 2/15 & 2/15 & 2/15 \\ 1/6 & 1/6 & 2/15 & 2/15 & 2/15 & 2/15 & 2/15 \\ 1/6 & 1/6 & 2/15 & 2/15 & 2/15 & 2/15 & 2/15 \\ 1/6 & 1/6 & 2/15 & 2/15 & 2/15 & 2/15 & 2/15 \\ 1/6 & 1/6 & 2/15 & 2/15 & 2/15 & 2/15 & 2/15 \\ 1/6 & 1/6 & 2/15 & 2/15 & 2/15 & 2/15 & 2/15 \\ 0 & 0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$$

The constant diagonal sum of $X^{(6,2,7)}$ is $S_1 = 16/15$.

$$X^{(4,3,7)} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/16 & 1/16 & 1/16 & 1/16 \\ 1/4 & 1/4 & 1/4 & 1/16 & 1/16 & 1/16 & 1/16 \\ 1/4 & 1/4 & 1/4 & 1/16 & 1/16 & 1/16 & 1/16 \\ 1/4 & 1/4 & 1/4 & 1/16 & 1/16 & 1/16 & 1/16 \\ 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

The constant diagonal sum of $X^{(4,3,7)}$ is $S_2 = 25/15$. Moreover, we see that $\xi(X^{(6,2,7)}) \subseteq X^{(4,3,7)}$, $s_1 \geq s_2$, $r_1 \leq r_2$ and $S_1 \leq S_2$. △

Keeping in mind that $s < r < n$, one can also see from 2.15 that the diagonal sum will increase when s increases or when r decreases. Conversely, the sum will decrease when s decreases or when r increases.

The results from this section can also be applied to weighted bipartite graphs $G^{(r,s,n)}$. Since the nonzero diagonals of a matrix correspond to the perfect matchings of the associated bipartite graph, the weight sum of all perfect matchings in the weighted bipartite graphs $G^{(r,s,n)}$ is

$$S(G) = \frac{nr + ns - 2rs}{r(n-s)}$$

The sum of weights in the weighted bipartite graph is equal to the constant solution to the optimal assignment. The optimal solution to the problem will thus increase or decrease depending on the values of r, s.

The number of diagonals in $X^{(r,s,n)}$-matrices and their sums conclude this chapter. In the next chapter, we will make some modifications to RCDS matrices and explore how the modifications alter the nonzero diagonals and their respective sums.

CHAPTER 3

Modification of RCDS doubly stochastic matrices

In the previous chapter, we found the number of nonzero diagonals of the matrix class $X^{(r,s,n)}$. These diagonals have the same sum, since they are restricted constant diagonal sum matrices. Now, we will explore the following problem. Let X be an $n \times n$ RCDS doubly stochastic matrix. We then proceed to make minor changes to some of the nonzero elements in the matrix, and name the new matrix X'. How will these changes affect the sum of the diagonals? We know from Corollary 2.1.5 that the diagonal sums will no longer be constant, since RCDS matrices are uniquely determined by their zero positions.

There are of course many ways to explore this problem. For the sake of continuity in our work, we defined an operation, ε-modification, which changes the value of four nonzero elements in a matrix. This operation is defined in the first section of this chapter. We then proceed to explore the diagonals of ε-modified $X^{(r,s,n)}$-matrices and randomly generated RCDS doubly stochastic matrices from MATLAB. This lead us to some results which are presented in later sections of this chapter. Lastly, we will see how the results apply to weighted bipartite graphs and the optimal assignment problem.

3.1 ε-modification

In this section, we will define the operation ε-modification which we used in our work to explore the modified diagonals. First, we need to define a submatrix. A *submatrix* of a matrix A is any matrix that results in deleting rows and/or columns from matrix A, see [8]. ε-modification is an operation performed on $n \times n$ matrices where we assume $n \geq 2$, and the matrices are RCDS.

We start by defining the following 2×2 matrix

$$E_\varepsilon = \begin{bmatrix} \varepsilon & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix} \quad , \varepsilon > 0$$

Then, we choose a 2×2 nonzero submatrix Y of an RCDS matrix $X = [x_{ij}]$, and compute the sum $Y' = Y + E_\varepsilon$. We replace the values in X, in the positions of X corresponding to submatrix Y, with the values of Y', and denote the new matrix X'. Notice that Y' is a submatrix of X'. Also, notice that adding and subtracting ε to the same row and column in X' maintains the row and column sums of X. If ε is small enough, X' also maintains the property of being doubly stochastic.

With the new matrix X', our problem is to determine the diagonal sums of X', and how the diagonal sums differ from the constant diagonal sum of X. For a general result of this problem, the submatrix Y will be chosen at random. We illustrate this with the following matrices X, Y, Y', X',

$$X = \begin{bmatrix} x_{11} & \cdots & \cdots & \cdots & \cdots & \cdots & x_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ & \cdots & x_{ac} & \cdots & x_{ad} & \cdots & \\ \vdots & & \vdots & & \vdots & & \vdots \\ & \cdots & x_{bc} & \cdots & x_{bd} & \cdots & \\ \vdots & & \vdots & & \vdots & & \vdots \\ x_{n1} & \cdots & \cdots & \cdots & \cdots & \cdots & x_{nn} \end{bmatrix}$$

where a, b, c, d are some integers such that $1 \leq a, b, c, d \leq n$ and x_{ac}, x_{bc}, x_{ad}, $x_{bd} \neq 0$. Note that x_{ac}, x_{bc}, x_{ad}, x_{bd} may be in the corners or outskirts of the matrix. We obtain the submatrices Y, Y',

$$Y = \begin{bmatrix} x_{ac} & x_{ad} \\ x_{bc} & x_{bd} \end{bmatrix}$$

$$Y' = \begin{bmatrix} x_{ac} + \varepsilon & x_{ad} - \varepsilon \\ x_{bc} - \varepsilon & x_{bd} + \varepsilon \end{bmatrix}$$

Lastly, we obtain the matrix X' by replacing the elements of X with the elements of Y' in the positions corresponding to the submatrix Y.

$$X' = \begin{bmatrix} x_{11} & \cdots & \cdots & \cdots & \cdots & \cdots & x_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ & \cdots & x_{ac} + \varepsilon & \cdots & x_{ad} - \varepsilon & \cdots & \\ \vdots & & \vdots & & \vdots & & \vdots \\ & \cdots & x_{bc} - \varepsilon & \cdots & x_{bd} + \varepsilon & \cdots & \\ \vdots & & \vdots & & \vdots & & \vdots \\ x_{n1} & \cdots & \cdots & \cdots & \cdots & \cdots & x_{nn} \end{bmatrix}$$

Performing ε-modification on a matrix X to obtain X' will be used several times in this thesis. We will now give a formal definition of the operation. Notice that the definition applies to RCDS matrices that are not restricted to being doubly stochastic.

Definition 3.1.1. Let X be an $n \times n$ RCDS matrix where $n \geq 2$, which contains at least four nonzero entries. Let Y be any 2×2 nonzero submatrix of X, and let E_ε be the following 2×2 matrix

$$E_\varepsilon = \begin{bmatrix} \varepsilon & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix} \quad , \varepsilon \in \mathbb{R}$$

We define the operation *ε-modification* to be the procedure of replacing the elements in the positions of X which correspond to submatrix Y with the elements in $Y' = Y + E_\varepsilon$. We denote the new matrix X'.

Notice that we will use E_ε in all our examples and computations, but the results would be the same if we used $-E_\varepsilon$ instead. In the next section, we will provide an example of ε-modification.

3.2 An example of an ε-modified matrix

We will now present an example of ε-modification, and start exploring the diagonals of the modified matrix. In this example, we will use the matrix $X^{(3,2,6)}$ from Example 2.6.3.

Example 3.2.1.

$$X^{(3,2,6)} = \begin{bmatrix} 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

We choose the following submatrix Y, and obtain the matrix X' by ε-modification.

$$Y = \begin{bmatrix} x_{11} & x_{15} \\ x_{31} & x_{35} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/12 \\ 1/3 & 1/12 \end{bmatrix}$$

$$X' = \begin{bmatrix} 1/3+\varepsilon & 1/3 & 1/12 & 1/12 & 1/12-\varepsilon & 1/12 \\ 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 1/3-\varepsilon & 1/3 & 1/12 & 1/12 & 1/12+\varepsilon & 1/12 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

Now, the question is how this change affects the diagonal sums of matrix X'. We start by finding the maximum sum. In this case, we must choose both ε-modified elements corresponding to $+\varepsilon$. Elements from the first three rows must be chosen in the following way,

$$X' = \begin{bmatrix} \mathbf{1/3+\boldsymbol{\varepsilon}} & 1/3 & 1/12 & 1/12 & 1/12-\varepsilon & 1/12 \\ 1/3 & \mathbf{1/3} & 1/12 & 1/12 & 1/12 & 1/12 \\ 1/3-\varepsilon & 1/3 & 1/12 & 1/12 & \mathbf{1/12+\boldsymbol{\varepsilon}} & 1/12 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

After elements from the first three rows are chosen, there are $3 \cdot 2 \cdot 1 = 6$ ways the choose elements from the bottom three rows. Thus, there are six diagonals with sum $3/2 + 2\varepsilon$. Equivalently, there are six ways to choose a diagonal with minimum sum $3/2 - 2\varepsilon$.

We will now compute the number of ways to choose a diagonal with sum $3/2 + \varepsilon$. This means we must choose either position $(1,1)$ or $(3,5)$ but not both. If we choose position $(1,1)$ from the first column, we must choose either position $(2,2)$ or $(3,2)$ from the second column. This gives us two further options. In case we choose $(2, 2)$, we may choose between three elements on row 3 with element $1/12$ and then $3 \cdot 2 \cdot 1$ choices for the bottom three rows. In case we choose $(3, 2)$, we may choose between four elements on row 2 and then $3 \cdot 2 \cdot 1$ choices for the bottom three rows. This gives us $3 \cdot 3 \cdot 2 \cdot 1 + 4 \cdot 3 \cdot 2 \cdot 1 = 42$ combinations so far. However, if we start by choosing position $(3, 5)$, we must choose the elements $(1, 2)$ and $(2, 1)$ from the first two columns, and as usual there are $3 \cdot 2 \cdot 1$ ways to choose elements from the bottom three rows. In total, this gives us $42 + 6 = 48$ ways to choose a diagonal with sum $3/2 + \varepsilon$. The computation for finding the number of ways to choose a diagonal with sum $3/2 - \varepsilon$ is equivalent, thus the number of combinations is also 48.

Lastly, we will compute the number of ways to choose a diagonal with sum $3/2$. We first note that it is not possible to choose a diagonal entry that includes ε-modified elements corresponding to both $+\ \varepsilon$ and $-\ \varepsilon$, so we are restricted to position $(2, 1)$ in the first column. Then there are two choices for column 2, three choices for the element $1/12$ and $3 \cdot 2 \cdot 1$ choices for the bottom three rows. This gives us $2 \cdot 3 \cdot 3 \cdot 2 \cdot 1 = 36$ combinations.

Notice that $6 + 6 + 48 + 48 + 36 = 144$ which equals the total number of ways to choose a diagonal for matrix $X^{(3,2,6)}$, as shown in Example 2.6.3. We can summarize our findings in Table 3.1.

Diagonal sum	**Number of diagonals**
$3/2$	36
$3/2 + \varepsilon$	48
$3/2 - \varepsilon$	48
$3/2 + 2\varepsilon$	6
$3/2 - 2\varepsilon$	6

Table 3.1: Possible diagonal sums for X'

$\triangle$

In the example, our findings were restricted to the case where we perform ε-modification to elements in the indices $(1, 1), (3, 1), (1, 5), (3, 5)$. In case we choose other elements, the number of combinations for the different sums will differ but the sums will be the same. We will explore this further in the following sections.

3.3 An example of an ε-modified RCDS matrix using MATLAB

The previous section explored how ε-modification of an $X^{(r,s,n)}$-matrix altered the diagonal sums of the matrices. We were interested in exploring this further, especially for more general RCDS doubly stochastic matrices. We used MATLAB to generate random RCDS doubly stochastic matrices and thereafter select random 2×2 nonzero submatrices to proceed with ε-modification, as described in the previous section.

We discovered the exact same changes in the diagonal sums, although the number of diagonals varied. Moreover, not all five diagonal sums were attainable for each matrix. Let S be the constant diagonal sum of an RCDS doubly stochastic matrix. Then, after performing ε-modification to the matrix, the diagonal sums were restricted to S, $S+\varepsilon$, $S-\varepsilon$, $S+2\varepsilon$, $S-2\varepsilon$. We will demonstrate this with an example from the MATLAB program.

Example 3.3.1. In this example, we will generate a random RCDS matrix in MATLAB, perform ε-modification, and check the diagonal sums of the modifed matrix. See Chapter 4 for explanations of the MATLAB codes. First, we chose the matrix dimension $n = 5$, and generated a random doubly stochastic matrix using the code in Appendix A.1. This gave the following matrix A

$$A = \begin{bmatrix} 0.1941 & 0 & 0.0259 & 0 & 0.7800 \\ 0.0259 & 0 & 0 & 0.9741 & 0 \\ 0 & 0.8600 & 0.1141 & 0.0259 & 0 \\ 0 & 0.0259 & 0.8600 & 0 & 0.1141 \\ 0.7800 & 0.1141 & 0 & 0 & 0.1059 \end{bmatrix}$$

Then we used the code in Appendix A.3 to generate an RCDS matrix X based on the pattern of matrix X. This matrix has 9 nonzero diagonals and constant diagonal sum $S = 1.8636$.

$$X = \begin{bmatrix} 0.2955 & 0 & 0.3636 & 0 & 0.3409 \\ 0.4091 & 0 & 0 & 0.5909 & 0 \\ 0 & 0.2955 & 0.2955 & 0.4091 & 0 \\ 0 & 0.3409 & 0.3409 & 0 & 0.3182 \\ 0.2955 & 0.3636 & 0 & 0 & 0.3409 \end{bmatrix}$$

Now, we performed ε-modification where $\varepsilon = 0.001$, using the code in Appendix A.6. The changes were made to positions $(1,1), (5,1), (1,5), (5,5)$. The ε-modified matrix X' is

$$X' = \begin{bmatrix} 0.2945 & 0 & 0.3636 & 0 & 0.3419 \\ 0.4091 & 0 & 0 & 0.5909 & 0 \\ 0 & 0.2955 & 0.2955 & 0.4091 & 0 \\ 0 & 0.3409 & 0.3409 & 0 & 0.3182 \\ 0.2965 & 0.3636 & 0 & 0 & 0.3399 \end{bmatrix}$$

Using the code in Appendix A.8, we found the number of diagonals and their respective sums. The results are presented in Table 3.2

Diagonal sum	Number of diagonals
1.8636	1
$1.8636 + 0.001$	2
$1.8636 - 0.001$	2
$1.8636 + 0.002$	2
$1.8636 - 0.002$	2

Table 3.2: Possible diagonal sums for X'

△

The previous examples and computations lead us to a result which will be presented in the next section.

3.4 Diagonal sums of an ε-modified RCDS matrices

In the previous sections, we explored how the diagonals and diagonal sums change when performing ε-modification to RCDS doubly stochastic matrices. Using similar computations and various RCDS doubly stochastic matrices generated from Matlab, we arrived at the following result. Notice that the result applies to all RCDS matrices, not only RCDS doubly stochastic matrices.

Theorem 3.4.1. *Let $X = [x_{ij}]$ be an $n \times n$ RCDS matrix, and let S be the constant diagonal sum of X. Assume $n \geq 2$. Perform ε-modification on X and obtain X'. Then, X' is not RCDS. Furthermore, X' has up to five distinct diagonal sums which equal to S, $S+\varepsilon$, $S-\varepsilon$, $S+2\varepsilon$, $S-2\varepsilon$.*

Proof. It follows from Corollary 2.1.5 that X' is not RCDS, since it is assumed that X is RCDS and $\xi(X') = \xi(X)$.

When selecting a diagonal of X' with nonzero elements, there are up to three different cases depending on the size and zero positions of X. Let $F = \{x_1+\varepsilon,\ x_2-\varepsilon,\ x_3+\varepsilon,\ x_4-\varepsilon\}$ be the set of ε-modified elements in X'.

Case 1: If possible, choose a diagonal D_σ such that $D_\sigma \cap F = \emptyset$. Then the diagonal sum will equal the constant diagonal sum of X, namely S.

Case 2: If possible, choose a diagonal D_σ which contains exactly one element of F. This element will have a difference of either $+\varepsilon$ or $-\varepsilon$ from the initial element in the same position in X. The sum of these diagonals will thus equal either $S+\varepsilon$ or $S-\varepsilon$.

Case 3: If possible, choose a diagonal D_σ which contains exactly two elements of F. Since it is not possible to select two elements from the same row or column for a diagonal entry, D_σ must include either $\{x_1+\varepsilon,\ x_3+\varepsilon\}$ or $\{x_2-\varepsilon,\ x_4-\varepsilon\}$. The sum of these diagonals will thus equal either $S+2\varepsilon$ or $S-2\varepsilon$.

Apart from these three cases, there are no other ways to choose a nonzero diagonal of X. □

In Example 3.2.1 and Example 3.3.1, we saw cases of a matrices where all five diagonal sums were attainable. However, ε-modified RCDS matrices may have less than five distinct diagonal sums. We will present two examples of this.

Example 3.4.2. Let X be the following 2×2 RCDS doubly stochastic matrix.

$$X = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

After ε-modification, we obtain X',

$$X' = \begin{bmatrix} 0.5+\varepsilon & 0.5-\varepsilon \\ 0.5-\varepsilon & 0.5+\varepsilon \end{bmatrix}$$

There are only two possible diagonals for X', namely $D_{\sigma 1} = \{(1,1),(2,2)\}$ and $D_{\sigma 2} = \{(2,1),(1,2)\}$. These diagonals yield diagonal sums $S+2\varepsilon$ and $S-2\varepsilon$, where S is the constant diagonal sum of X. △

Example 3.4.3. Let $X^{(3,2,6)}$ be the RCDS doubly stochastic matrix from Example 3.2.1.

$$X^{(3,2,6)} = \begin{bmatrix} 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

We perform ε-modification in positions $(1,5),(1,6),(2,5),(2,6)$ and obtain X',

$$X' = \begin{bmatrix} 1/3 & 1/3 & 1/12 & 1/12 & 1/12+\varepsilon & 1/12-\varepsilon \\ 1/3 & 1/3 & 1/12 & 1/12 & 1/12-\varepsilon & 1/12+\varepsilon \\ 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

X' has three distinct diagonal sums, namely $S,\ S+\varepsilon,\ S-\varepsilon$. It is not possible to find a nonzero diagonal with diagonal sum $S \pm 2\varepsilon$. In order to yield the diagonal sum including both $+\varepsilon$ elements or both $-\varepsilon$ elements, one has to choose either positions $(1,5),(2,6)$ or $(1,6),(2,6)$ which forces a zero position in the diagonal. △

In the next section, we will continue to explore ε-modified $X^{(r,s,n)}$-matrices and find the number of diagonals corresponding to each diagonal sum.

3.5 The number of diagonals of ε-modified $X^{(r,s,n)}$-matrices

In Chapter 2, we found the number of diagonals and constant diagonal sums of $X^{(r,s,n)}$-matrices, and in the previous section we established how the diagonal sums change when the matrices of these class are ε-modified. Recall that the diagonal sums of an ε-modified $X^{(r,s,n)}$-matrix are a subset of $\{S,\ S+\varepsilon,\ S-\varepsilon,\ S+2\varepsilon,\ S-2\varepsilon\}$. We will now finally explore the number of nonzero diagonals that yield the different diagonal sums in ε-modified $X^{(r,s,n)}$-matrices. This was computed by hand in Example 3.2.1, but we will now find a way to compute this in general for all matrices of the class. For this discussion, we will again divide the matrix into four sections.

$$X^{(r,s,n)} = \left[\begin{array}{c|c} \frac{1}{r} & \frac{r-s}{r(n-s)} \\ \hline 0 & \frac{1}{n-s} \end{array}\right] \begin{array}{l} \}\, r \\ \}\, n-r \end{array}$$
$$\qquad\qquad\underbrace{\qquad}_{s}\ \underbrace{\qquad\quad}_{n-s}$$

For ε-modified matrices of the class $X^{(r,s,n)}$, the number of diagonals per diagonal sum will vary based on which of the three sections contain modified elements. Since there are three possible sections to perform ε-modification on, and the modification involves two rows and columns, this gives a total of five outcomes. We will explore one of these outcomes where the ε-modification is performed in the top left section, i.e. on elements x_{ij} such that $i \leq r$ and $j \leq s$.

$$X' = \left[\begin{array}{cc|c} \frac{1}{r}+\varepsilon & \frac{1}{r}-\varepsilon & \\ & & \frac{r-s}{r(n-s)} \\ \frac{1}{r}-\varepsilon & \frac{1}{r}+\varepsilon & \\ \hline \multicolumn{2}{c|}{0} & \frac{1}{n-s} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} r \\ \left.\vphantom{1}\right\} n-r \end{array}$$

$$\underbrace{\qquad\qquad}_{s}\;\underbrace{\qquad\qquad}_{n-s}$$

In order to perform ε-modification on elements x_{ij} where $i \leq r, j \leq s$, the section must include at least four elements. Since $s < r < n$, this requires that $s \geq 2$, $r \geq 3$, $n \geq 4$ for all matrices $X^{(r,s,n)}$. When counting the nonzero diagonals of X', there are up to five different cases of diagonal sums depending on which positions one chooses. Let $F = \{x_1 + \varepsilon,\, x_2 - \varepsilon,\, x_3 + \varepsilon,\, x_4 - \varepsilon\}$ be the set of ε-modified elements in X'.

Case 1: Diagonal sum = S
When choosing positions in the columns of ε-modified elements that ensures diagonal sum equal to S, one must avoid all elements in F and thus there are $(r-2)\,(r-2-1) = (r-2)\,(r-3)$ choices. When choosing among the remaining positions in the top left section, there are $(r-2)\cdot(r-3)\cdots(r-s+1) = \frac{(r-2)!}{(r-s)!}$ possible combinations. Furthermore, when choosing diagonal positions from the top right and bottom right section, there are $(n-s)!$ choices, by the same argument as in the proof in Theorem 2.7.2. This results in a total of $(r-2)\,(r-3)\,\frac{(r-2)!}{(r-s)!}\,(n-s)!$ combinations. In the special case where $s = 2$, $r = 3$, $n = 4$, it is not possible to avoid at least one element from F, and thus it is not possible to attain any diagonal sum equal to S.

Case 2: Diagonal sum = $S + \varepsilon$
When choosing diagonal positions from the top left section that ensures diagonal sum equal to $S + \varepsilon$, one must choose exactly one element from F which corresponds to an ε-modified element of $+\varepsilon$. Thus there are $2\,(r-2)$ choices when choosing positions in the columns of ε-modified elements. When choosing among the remaining positions in the top left section, there are $(r-2)\cdot(r-3)\cdots(r-s+1) = \frac{(r-2)!}{(r-s)!}$ possible combinations. Furthermore, when choosing diagonal positions from the top right and bottom right section, there are $(n-s)!$ choices, by the same argument as in the proof in Theorem 2.7.2. This results in a total of $2\,(r-2)\,\frac{(r-2)!}{(r-s)!}\,(n-s)!$ combinations.

Case 3: Diagonal sum $= S - \varepsilon$
This case follows the same argument as in case 2, except one must choose exactly one element from F which corresponds to an ε-modified element of $-\varepsilon$. Like case 2, this results in a total of $2\,(r-2)\,\frac{(r-2)!}{(r-s)!}\,(n-s)!$ combinations.

Case 4: Diagonal sum $= S + 2\varepsilon$
When choosing diagonal positions from the top left section that ensures diagonal sum equal to $S + 2\varepsilon$, one must choose both elements from F which corresponds to ε-modified elements of $+\varepsilon$. Thus there is only one choice when choosing positions in the columns of ε-modified elements. When choosing among the remaining positions in the top left section, there are $(r-2)\cdot(r-3)\cdots(r-s+1) = \frac{(r-2)!}{(r-s)!}$ possible combinations. Furthermore, when choosing diagonal positions from the top right and bottom right section, there are $(n-s)!$ choices, by the same argument as in the proof in Theorem 2.7.2. This results in a total of $\frac{(r-2)!}{(r-s)!}\,(n-s)!$ combinations.

Case 5: Diagonal sum $= S - 2\varepsilon$
This case follows the same argument as in case 4, one must choose both elements from F which corresponds to ε-modified elements of $-\varepsilon$. Like case 4, this results in a total of $\frac{(r-2)!}{(r-s)!}\,(n-s)!$ combinations.

The results are summarized in Table 3.3, and gives us the following theorem.

Diagonal sum	**Number of nonzero diagonals**
S	$(r-2)\,(r-3)\,\frac{(r-2)!}{(r-s)!}\,(n-s)!$
$S+\varepsilon$	$2\,(r-2)\,\frac{(r-2)!}{(r-s)!}\,(n-s)!$
$S-\varepsilon$	$2\,(r-2)\,\frac{(r-2)!}{(r-s)!}\,(n-s)!$
$S+2\varepsilon$	$\frac{(r-2)!}{(r-s)!}\,(n-s)!$
$S-2\varepsilon$	$\frac{(r-2)!}{(r-s)!}\,(n-s)!$

Table 3.3: Possible diagonal sums for X' when ε-modification is performed on elements x_{ij} such that $i \le r$, $j \le s$. These diagonal sums are attainable when $s \ge 2$, $r \ge 3$, $n \ge 4$.

Based on the discussion above, we arrived at the following result.

Theorem 3.5.1. *Let $X^{(r,s,n)}$ be a matrix by Definition 2.6.1. Assume $s \ge 2$, $r \ge 3$, $n \ge 4$. Perform ε-modification on $X = [x_{ij}]$ restricted to elements x_{ij} such that $i \le r, j \le s$. Then, the number of nonzero diagonals and their respective diagonal sums are presented in Table 3.3.*

Proof. See the discussion of the five cases above for a proof of this theorem. □

For the purpose of further discussion of the number of diagonals, we will denote them $N(S)$, $N(S+\varepsilon)$, $N(S-\varepsilon)$, $N(S+2\varepsilon)$, $N(S-2\varepsilon)$. First, notice that the sum of number of diagonals in Table 3.3 is equal to the total number of distinct diagonal positions in Theorem 2.7.2, N.

$$\begin{aligned}
&N(S) + N(S+\varepsilon) + N(S-\varepsilon) + N(S+2\varepsilon) + N(S-2\varepsilon) \\
&= \frac{(r-2)!}{(r-s)!}\,(n-s)!\,\big((r-s)\,(r-3) + 4(r-2) + 2\big) \\
&= \frac{(r-2)!}{(r-s)!}\,(n-s)!\,\big(r\,(r-1)\big) \\
&= \frac{r!}{(r-s)!}\,(n-s) \\
&= N(X)
\end{aligned}$$

From Table 3.3, we also notice that r is the only variable that impacts the difference between the number of diagonals. For example, when $r = 3$, then $N(S) = 0$ and $N(S+2\varepsilon)$, $N(S-2\varepsilon)$ are strictly less than $N(S+\varepsilon)$, $N(S-\varepsilon)$. When $r = 4$ or $r = 5$, we will also obtain special cases, but when $r > 5$, then $N(S+2\varepsilon)$, $N(S-2\varepsilon)$ are strictly less than $N(S+\varepsilon)$, $N(S-\varepsilon)$ which are strictly less than $N(S)$. All cases of how the diagonal sums depend on r is presented in Table 3.4.

r	Comparison of number of diagonals
$r = 3$	$N(S) = 0 < N(S+2\varepsilon) = N(S-2\varepsilon) < N(S+\varepsilon) = N(S-\varepsilon)$
$r = 4$	$N(S+2\varepsilon) = N(S-2\varepsilon) < N(S) < N(S+\varepsilon) = N(S-\varepsilon)$
$r = 5$	$N(S+2\varepsilon) = N(S-2\varepsilon) < N(S) = N(S+\varepsilon) = N(S-\varepsilon)$
$r > 5$	$N(S+2\varepsilon) = N(S-2\varepsilon) < N(S+\varepsilon) = N(S-\varepsilon) < N(S)$

Table 3.4: A comparison of number of diagonals depending on the value of r

We will now present some examples of $X^{(r,s,n)}$-matrices with the number of their nonzero diagonals and their respective diagonal sums. The examples will include cases where $r = 3$, $r = 5$ and $r > 5$.

Example 3.5.2. Let $X^{(3,2,4)}$ be a matrix from the class $X^{(r,s,n)}$ where $r = 3$, $s = 2$, $n = 4$.

$$X^{(3,2,4)} = \begin{bmatrix} 1/3 & 1/3 & 1/6 & 1/6 \\ 1/3 & 1/3 & 1/6 & 1/6 \\ 1/3 & 1/3 & 1/6 & 1/6 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

The matrix has 12 nonzero diagonals, and the constant diagonal sum is 4/3. For any ε-modified matrix of X, where the modified elements are restricted to x_{ij} such that $i \leq 3$, $j \leq 2$, the number of nonzero diagonals and their respective sums are summarized in Table 3.5.

△

Diagonal sum	Number of diagonals
$4/3$	0
$4/3+\varepsilon$	4
$4/3-\varepsilon$	4
$4/3+2\varepsilon$	2
$4/3-2\varepsilon$	2

Table 3.5: Diagonal sums for ε-modified $X^{(3,2,4)}$ restricted to elements x_{ij} such that $i \leq 3$, $j \leq 2$

Example 3.5.3. Let $X^{(5,4,7)}$ be a matrix from the class $X^{(r,s,n)}$ where $r = 5$, $s = 4$, $n = 7$.

$$X^{(5,4,7)} = \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/15 & 1/15 & 1/15 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/15 & 1/15 & 1/15 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/15 & 1/15 & 1/15 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/15 & 1/15 & 1/15 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/15 & 1/15 & 1/15 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \end{bmatrix}$$

The matrix has 720 nonzero diagonals, and the constant diagonal sum is $23/15$. For any ε-modified matrix of X, where the modified elements are restricted to x_{ij} such that $i \leq 5$, $j \leq 4$, the number of nonzero diagonals and their respective sums are summarized in Table 3.6.

Diagonal sum	Number of diagonals
$23/15$	216
$23/15+\varepsilon$	216
$23/15-\varepsilon$	216
$23/15+2\varepsilon$	36
$23/15-2\varepsilon$	36

Table 3.6: Diagonal sums for ε-modified $X^{(5,4,7)}$ restricted to elements x_{ij} such that $i \leq 5$, $j \leq 4$

$\triangle$

Example 3.5.4. Let $X^{(8,5,10)}$ be a matrix from the class $X^{(r,s,n)}$ where $r = 8$, $s = 5$, $n = 10$.

$$X^{(8,5,10)} = \begin{bmatrix} 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 3/40 & 3/40 & 3/40 & 3/40 & 3/40 \\ 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 3/40 & 3/40 & 3/40 & 3/40 & 3/40 \\ 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 3/40 & 3/40 & 3/40 & 3/40 & 3/40 \\ 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 3/40 & 3/40 & 3/40 & 3/40 & 3/40 \\ 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 3/40 & 3/40 & 3/40 & 3/40 & 3/40 \\ 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 3/40 & 3/40 & 3/40 & 3/40 & 3/40 \\ 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 3/40 & 3/40 & 3/40 & 3/40 & 3/40 \\ 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 3/40 & 3/40 & 3/40 & 3/40 & 3/40 \\ 0 & 0 & 0 & 0 & 0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 0 & 0 & 0 & 0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$$

The matrix has 806,400 nonzero diagonals and the constant diagonal sum is 5/4. For any ε-modified matrix of X, where the modified elements are restricted to x_{ij} such that $i \leq 8$, $j \leq 5$, the number of nonzero diagonals and their respective sums are summarized in Table 3.7.

Diagonal sum	**Number of diagonals**
$5/4$	$432,000$
$5/4 + \varepsilon$	172,800
$5/4 - \varepsilon$	172,800
$5/4 + 2\varepsilon$	14,400
$5/4 - 2\varepsilon$	14,400

Table 3.7: Diagonal sums for ε-modified $X^{(8,5,10)}$ restricted to elements x_{ij} such that $i \leq 8$, $j \leq 5$

△

The previous three examples demonstrated how the comparison of number of diagonals $N(S)$, $N(S+\varepsilon)$, $N(S-\varepsilon)$, $N(S+2\varepsilon)$, $N(S-2\varepsilon)$ vary based on the the value r.

So far in this chapter, we have seen how ε-modified RCDS matrices have changed the diagonal sums, and we have counted the number of diagonals which yield each sum for ε-modified $X^{(r,s,n)}$-matrices. In the final section of this chapter, we will see how that results apply to weighted bipartite graphs and the optimal assignment problem.

3.6 ε-modification on weighted bipartite graphs

We have previously discussed the correspondence between matrices and weighted bipartite graphs. The results from ε-modified RCDS matrices also apply to equivalent changes made to weighted bipartite graphs associated to the matrices. Before we present these results, we will give a formal definition of ε-modified weighted bipartite graphs.

Definition 3.6.1. Let (G, w) be a weighted bipartite graph associated to an RCDS matrix. Let $I = \{i_1, i_2, \ldots, i_n\}$, $J = \{j_1, j_2, \ldots, j_n\}$ be the sets of vertices in (G, w). Assume (G, w) has a complete bipartite subgraph of order $m \geq 4$. We define the operation ε-*graph modification* to be the procedure of choosing two indices $k, l \leq n$ such that there exists edges $\{i_k, j_k\}$, $\{i_k, j_l\}$, $\{i_l, j_k\}$, $\{i_l, j_l\}$ in (G, w), and make the following changes to the weights

$$\begin{aligned} w'(\{i_k, j_k\}) &= w(\{i_k, j_k\}) + \varepsilon \\ w'(\{i_k, j_l\}) &= w(\{i_k, j_l\}) - \varepsilon \\ w'(\{i_l, j_k\}) &= w(\{i_l, j_k\}) + \varepsilon \\ w'(\{i_l, j_l\}) &= w(\{i_l, j_l\}) - \varepsilon \end{aligned} \tag{3.1}$$

With these changes, we denote the graph (G', w').

Figure 3.1 shows an illustration of ε-graph modification of weighted bipartite graphs.

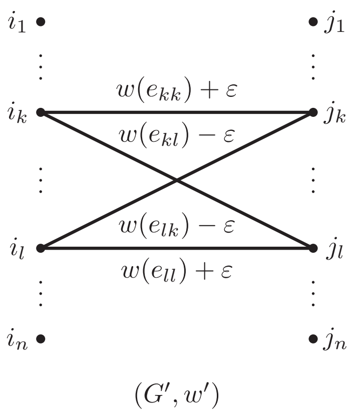


Figure 3.1: ε-graph modification of a weighted bipartite graph

We will now present how the results from this chapter applies to weighted bipartite graphs. The first is a corollary following from Theorem 3.4.1.

Corollary 3.6.2. *Let (G, w) be a weighted bipartite graph associated to an RCDS matrix. Let S be the constant weight sum of the perfect matchings in G, and assume (G, w) has a complete bipartite subgraph of order $m \geq 4$. Perform ε-graph modification on (G, w) and obtain (G', w'). Then, the perfect matchings in (G', w') have up to five distinct weight sums which equal to S, $S + \varepsilon$, $S - \varepsilon$, $S + 2\varepsilon$, $S - 2\varepsilon$.*

The proof follows from the proof of Theorem 3.4.1. Furthermore, if we study weighted bipartite graphs of the form $G^{(r,s,n)}$, we can find the numbers of perfect matchings corresponding to each sum. From Theorem 3.5.1, we have the following corollary.

Corollary 3.6.3. *Let $G^{(r,s,n)}$ be a weighted bipartite graph by Definition 2.7.3 with vertices $I = \{i_1, i_2, \ldots, i_n\}$, $J = \{j_1, j_2, \ldots, j_n\}$. Assume $s \geq 2$, $r \geq 3$, $n \geq 4$. Perform ε-graph modification on $G^{(r,s,n)}$ restricted to edges $\{\{i_k, j_l\} : k \leq r,\, l \leq s\}$ and obtain G'. Then, the number of perfect matchings and their respective weight sums are presented in table Table 3.8.*

The proof follows from the proof of Theorem 3.5.1. These corollaries have possible applications for the optimal assignment problem. In the case where all feasible solutions to the optimal assignment problem yield the constant optimal value S, ε-modification will decrease the optimal value by 2ε. However, it is more likely that the feasible solutions of the assignment problem do not yield a constant value. In that case, ε-modification of a bipartite graph will alter the weight sum of a varied number of perfect matchings by $\pm\,\varepsilon$ and $\pm\,2\,\varepsilon$.

3.7 Further work on the diagonals of RCDS matrices

In this thesis, we have have made some modifications to RCDS matrices and established how the modifications alter the diagonal sums. However, the converse problem could also be an interesting study. For some doubly stochastic matrices

Weight sum	**Number of perfect matchings**
S	$(r-2)(r-3)\frac{(r-2)!}{(r-s)!}(n-s)!$
$S+\varepsilon$	$2(r-2)\frac{(r-2)!}{(r-s)!}(n-s)!$
$S-\varepsilon$	$2(r-2)\frac{(r-2)!}{(r-s)!}(n-s)!$
$S+2\varepsilon$	$\frac{(r-2)!}{(r-s)!}(n-s)!$
$S-2\varepsilon$	$\frac{(r-2)!}{(r-s)!}(n-s)!$

Table 3.8: Possible weight sums for (G', w') when ε-modification is performed on edges edges $\{\{i_k, j_l\} : k \leq r,\, l \leq s\}$. These weight sums are attainable when $s \geq 2$, $r \geq 3$, $n \geq 4$.

which have *almost* constant diagonal sum, which modifications could be made in order for the matrices to obtain the RCDS property?

In MATLAB, we generated 10,000 random doubly stochastic matrices and computed the difference δ_S between their maximum and minimum nonzero diagonal sum S_{max}, S_{min}. We were interested in the number of doubly stochastic matrices which had approximately constant diagonal sums, and found the number of matrices where the relative difference between maximum and minimum diagonal sum, $\delta_S \;/\; S_{min}$, was less than 5 %. We found that this applied to 2.17 % of the 10,000 doubly stochastic matrices. In order to gather an overview over the general diagonal difference δ_S of these 10,000 matrices, we computed some of the percentiles. These are summarized in Table 3.9.

Percentile	**Relative diagonal sum difference,** δ_S/S_{min}
5	7.5%
25	19.1%
50	30.5%
75	47.9%
95	104%
99	234%

Table 3.9: Overview over the relative difference between maximum and minimum diagonal sums of doubly stochastic matrices, divided into percentiles

For further work, it would be interesting to see which modifications one could make to matrices with a small relative difference in diagonal sums in order for them to obtain the RCDS property.

This concludes the chapter, and the results from this thesis. In the next chapter, we will present an explanation of the work we have done in MATLAB.

CHAPTER 4

Explanations of the work in MATLAB

For the work done in this thesis, we have written several codes in MATLAB [10] to support us in our investigation in doubly stochastic matrices, RCDS matrices, and their diagonals. In this chapter, we will provide some brief explanations to the codes we used, and how they contributed to our work. All codes that have been used in the work of this thesis can be found in Appendix A

4.1 Generate a random doubly stochastic matrix

For many of our investigations, we needed random doubly stochastic matrices. To generate these, we used Birkhoff-von Neumann theorem (Theorem 1.9.6), by computing convex combinations of permutation matrices. First, one chooses the order n of the desired matrix. Then, we need to decide how many terms the convex combination shall have. In general, a larger number of terms will give less zero positions in the matrix. In our work, we decided it was most beneficial to let the number of zero positions vary. We therefore decided to let the number of terms be chosen at random between 2 and n. If there is only 1 term, the resulting matrix would be a permutation matrix, which would provide an insignificant result.

Moreover, the code generates n nonnegative numbers such their sum equals 1, which are to be used as the coefficients in the convex combination. Lastly, it computes the convex combination of randomly generated permutation matrices and the coefficients. Notice that this code does not necessarily generate fully indecomposable doubly stochastic matrices, although it turned out to be highly likely. In our work, we very rarely came across partly decomposable matrices using this code. However, we wrote a code that took this into account, which is presented in the next section. For the purpose of determining if the matrix is fully indecomposable, we ensured that the main diagonal is nonzero by letting the first permutation matrix in this code be the identity matrix.

In the code, we have used MATLAB functions `randi`, `rand zeros`, `sum`, `eye`, and `randperm` [10].

4.2 Generate a random fully indecomposable doubly stochastic matrix

This code takes in a randomly generated doubly stochastic matrix and determines if it is fully indecomposable. Recall that a matrix is fully indecomposable if its main diagonal consists only of nonzero elements, and the associated digraph is strongly connected. The construction of random doubly stochastic matrices in the previous code ensures that the main diagonal in nonzero. Thus, it remains to determine if the associated digraph is strongly connected. If the associated digraph is not strongly connected, the code generates a new doubly stochastic matrix until it arrives at a fully indecomposable doubly stochastic matrix.

In the code, we have used MATLAB functions `size`, `digraph`, and `conncomp` [10].

4.3 Generate an RCDS matrix based on a $(0, 1)$-matrix

The aim of this code is to generate random RCDS matrices based on randomly generated $(0, 1)$-matrices. In most of our work, we generated the RCDS matrices based on doubly stochastic matrices generated from the previous codes, since their zero positions proved more likely to generate a matrix with RCDS pattern. In that case, the code first replaces all nonzero elements with 1. The code uses the algorithm explained in Section 2.4 and solves the linear system of equations,

$$Hx = e, \quad \text{where} \quad H = \begin{bmatrix} D_R & A \\ A^T & D_C \end{bmatrix} \text{and } x = \begin{bmatrix} u \\ v \end{bmatrix} \tag{4.1}$$

and computes the RCDS matrix using vectors u, v and the pattern of initial matrix A.

In the code, we have used MATLAB functions `size`, `zeros` `transpose`, `linsolve`, and `sum` [10].

4.4 Generate $X^{(r,s,n)}$-matrix

This code uses Definition 2.6.1 to generate an $X^{(r,s,n)}$-matrix. Here, one only needs to enter values for r, s, n and the code will generate a matrix $X = [x_{ij}]$ of the form

$$x_{ij} = \begin{cases} 1/r & (i \leq r,\, j \leq s) \\ (r-s)/(r(n-s)) & (i \leq r,\, s < j \leq n) \\ 0 & (r < i \leq n,\, j \leq s) \\ 1/(n-s) & (r < i \leq n,\, s < j \leq n) \end{cases} \tag{4.2}$$

In the code, we have used MATLAB function `zeros` [10].

4.5 Using linear programming to check if a matrix is RCDS

The aim of this code is to check if a given matrix is an RCDS matrix. This is done by using linear programming as explained in Section 2.5. The following linear programs are to be solved

$$\begin{aligned}
\min \sum_{i \le n} \sum_{j \le n} b'_{ij} x_{ij} & \\
\text{s.t.} \sum_{j \le n} x_{ij} = 1 \quad & \text{for all } i \le n \\
\sum_{i \le n} x_{ij} = 1 \quad & \text{for all } j \le n
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\max \sum_{i \le n} \sum_{j \le n} b''_{ij} x_{ij} & \\
\text{s.t.} \sum_{j \le n} x_{ij} = 1 \quad & \text{for all } i \le n \\
\sum_{i \le n} x_{ij} = 1 \quad & \text{for all } j \le n
\end{aligned} \tag{4.4}$$

The code takes in a matrix C, and finds the maximum and minimum diagonal sums of the matrix. It also provides the positions in C which gives the respective maximum and minimum diagonal sums.

In the code, we have used MATLAB functions `reshape`, `ones`, `zeros`, `permute`, and `linprog` [10].

4.6 ε-modification of a matrix

This code takes in a matrix A, and finds a random nonzero submatrix of A. Then, it performs ε-modification on A with a chosen ε. For most of our computations, we used RCDS matrices, and chose $\varepsilon = 0.001$ to ensure that the ε-modified matrix remained doubly stochastic.

In the code, we have used MATLAB functions `size` and `randsample` [10].

4.7 Finding number of diagonals and their sums of a matrix

This code computes the number of nonzero diagonals of a matrix, and the diagonal sums. After it takes in a matrix, it checks all possible permutations of the matrix and finds each nonzero diagonal. This is clearly not as efficient as using linear programming, but it provided us with each distinct diagonal sum which was useful in our work. Since we only worked with relatively small matrices, it was adequate.

In the code, we have used MATLAB functions `size`, `reshape flipud`, `perms`, `factorial`, `diag`, `sum`, `unique`, `sort`, `accumarray` [10].

4.8 Finding number of diagonals and their sums of an ε-modified matrix

This code is almost equivalent to the previous, except it finds the nonzero diagonals of two matrices simultaneously. We used this code when we were interested in comparing the nonzero diagonals of a matrix and a corresponding ε-modified matrix.

In the code, we have used MATLAB functions `size`, `reshape flipud`, `perms`, `factorial`, `diag`, `sum`, `unique`, `sort`, `accumarray` [10].

4.9 Gathering some statistics on the diagonals of doubly stochastic matrices

The aim of this code is to gather some statistics of the diagonal sums of doubly stochastic matrices. The code generates a number of doubly stochastic matrices, and computes the maximum and minimum diagonal sums. Then, it finds the relative difference in diagonal sums of the matrices. We used 10×10 matrices, and thus scaled the results by 10. The code also finds the number of matrices which have a relative difference in diagonal sums of less than $5\,\%$, and some percentiles.

In the code, we have used MATLAB functions `zeros` and `prctile` [10].

CHAPTER 5

Conclusion

In this thesis, we have studied the nonzero diagonals of doubly stochastic matrices with constant diagonal sums. With the basis of the work by Brualdi & Dahl [3], we constructed some of these matrices and used them in our work. More specifically, we have studied the matrix class $X^{(r,s,n)}$, and obtained the constant diagonal sum and number of diagonals of these matrices. RCDS matrices are intricate, and it could be useful to further explore the diagonals of other RCDS matrices, and possibly find more patterns that may lie within them.

Furthermore, we defined an operation, ε-modification, where one adds and subtracts a nonnegative number ε to four of the matrix elements in a specific way. We established that the constant diagonal sums are altered by these modificatons, and the changes are restricted to $\pm\,\varepsilon$ and $\pm\,2\,\varepsilon$. We also found the number of diagonals of $X^{(r,s,n)}$-matrices which correspond to each diagonal sum, where the modifications were made to a restricted submatrix of $X^{(r,s,n)}$. By ε-modification, four values of the matrix are modified. However, it could prove beneficial to make modifications to more values, and establish how the diagonal sums alter with these modifications. Additionally, it could be useful to explore the converse problem, by studying doubly stochastic matrices which have approximately constant diagonal sums. One could possibly determine which modifications could be made to these matrices in order for them to obtain the RCDS property.

APPENDIX A

MATLAB codes

A.1 Generate a random doubly stochastic matrix

```
% This program generates a random doubly stochastic matrix using convex
% combinations of permutation matrices

function[A] = genererDSmatrise()

% decide the dimension of the permutation matrix
n = 10;

% N is the number of terms in the convex combination
% coeff is a vector with coeffictient for the convex combination
% coeff is scaled such that all elements sum to 1
N=randi([2,n]);
coeff = zeros(1,N);
for i = 1:N
        coeff(i) = rand();
end
coeff = (1/sum(coeff))*coeff;


% generate convex combination of permutation matrices
I = eye(n);
% the first term of the convex combination uses the identity matrix
A = coeff(1)*I;
% the remaining terms uses random permutation matrices
for i = 2:N
        A = A + coeff(i)*I(randperm(n),:);
end

end
```

A.2 Generate a random fully indecomposable doubly stochastic matrix

```
% This code determines if a matrix is fully indecomposable, by checking
% if the associated graph is strongly
% connected. If it is not strongly connected, it provides a new
% DS matrix which is.

function[A] = fullyindecomposable()
A = genererDSmatrise();
```

```
[n,n] = size(A);

% define the digraph G and find strongly connected subgraphs
G = digraph(A);
subg = conncomp(G);

% check that G is strongly connected. If not, generate a new DS matrix
% until we have a fully indecomposable matrix
for i = 1:n
        if subg(i) == 1
                i = 2;
        else
                A = genererDSmatrise4;
                G = digraph(A);
                subg = conncomp(G);
                i = 1;
        end
end
```

A.3 Generate an RCDS matrix based on a $(0, 1)$-matrix

```
% This program takes a DS matrix and uses the pattern to
% generate a RCDS matrix when possible

% generate a random (0,1)-matrix
        % n = 4;
        % A = zeros(n,n);
        % for i = 1:n
        %     for j = 1:n
        %         A(i,j) = randi([0 1],1);
        %     end
        % end

% use fully indecomposable DS matrix, and change nonzero elements to 1
function[Y] = fraDStilRCDS()
A = fullyindecomposable();
[n m] = size(A);
for i = 1:n
        for j = 1:n
                if A(i,j) > 0
                A(i,j) = 1;
                end
        end
end

% compute the vectors r, c
r = zeros(n, 1);
c = zeros(m, 1);
for i = 1:n
        for j = 1:m
                if A(i,j) == 1
                r(i) = r(i) + 1;
                c(j) = c(j) + 1;
                end
        end
end


% defining the diagonal matrices Dr and Dc
DR = zeros(n,n);
```

```
DC = zeros(n,n);

for i = 1:n
        DR(i,i) = r(i);
        DC(i,i) = c(i);
end


% defining H
H = [DR A; transpose(A) DC];

% Solving the system of linear equations
e = ones(2*n,1);
% w = linsolve(H,e);
w = H\e;

% adjustment of u av v such that they are nonnegative
u = zeros(n,1);
v = zeros(n,1);
for i = 1:n
        u(i) = w(i) ;
        v(i) = w(i+n) ;
end

% computing the RCDS matrix
Y = zeros(n,n);
for i = 1:n
        for j = 1:n
                if A(i,j) == 1
                Y(i,j) = u(i) + v(j);
                end
        end
end
SumCol = sum(Y)
SumRow = sum(Y,2)
end
```

A.4 Generate $X^{(r,s,n)},$-matrix

```
function[A] = generer_rsn_RCDSmatrise()

s = 3;
r = 3;
n = 6;
A = zeros(n);
        for i = 1:n
                for j = 1:n
                        if i <= r && j <= s
                        A(i,j) = 1/r;
                        elseif i <= r && s <= j <= n
                        A(i,j) = (r-s)/(r*(n-s));
                        elseif r <= i <= n && j <= s
                        A(i,j) = 0;
                        else
                        A(i,j) = 1/(n-s);
                        end
        end
end
```

A.5 Using linear programming to check if a matrix is RCDS

```
% The program generates a random DS-matrix and finds both min and max
% of the diagonals

% Max and min values differ by changing the signs of
%     the array c
%     the variable M
%     the variable optimum
%          where max has negative sign, and min has positive sign


% define a DS matrix for diagonal computation
C1 = epsilon_2x2();

C2 = C1;
[n m] = size(C1);
%C1 is the matrix for finding the maximum (negative M)
%C2 is the matrix for finding the minimum (positive M)

% Replace 0's with arbitrary large number
M1 = -10;
for i=1:n
        for j = 1:n
                if C1(i,j) == 0
                C1(i,j) = M1;
                end
        end
end

M2 = 10;
for i=1:n
        for j = 1:n
                if C2(i,j) == 0
                C2(i,j) = M2;
                end
        end
end


% Convert elements of matrix C to array c
% c1 is the vector for finding the maximum (negative elements)
% c2 is the vector for finding the minimum (positive elements)
c1 = -reshape(C1, 1, []);
c2 = reshape(C2, 1, []);

%vector b consists of 2n elements equal to 1
b = ones(2*n,1);

% creates the top half of matrix A
A1 = zeros(n,n,n) ;
for i = 1:n
        A1(i,:,i) = ones(n,1) ;
end
A1 = reshape(permute(A1,[1,2,3]),size(A1,2),[]);


% creates the bottom half of matrix A
A2 = zeros(n,n*n);
for i = 1:n
        for j = 0:n-1
                for k = 1:n
```

```
                A2(i,i+j*n) = 1;
                end
        end
end

% combine A1 and A2 to create matrix A
A = [A1; A2];

% Solving the LP problem
%   x1 finds the maximum
%   x2 finds the minimum
x1=linprog(c1, [], [], A, b, zeros(size(c1)), ones(size(c1)));
x2=linprog(c2, [], [], A, b, zeros(size(c2)), ones(size(c2)));


% Converts optimal solution from vector til matrix, in order to see
% the positions of the diagonal elements
%   Y1 shows the positions of the maximum
%   Y2 shows the positions of the minimum
Y1 = reshape(x1,n,[]);
Y2 = reshape(x2,n,[]);

% Compute the optimum
% optimum1 is the maximum value
% optimum2 is the minimum value
optimum1 = 0;
optimum2 = 0;

for i = 1:n^2
        optimum1 = optimum1 + c1(i)*x1(i);
end
% change sign of optimum1
optimum1 = -optimum1;

for i = 1:n^2
        optimum2 = optimum2 + c2(i)*x2(i);
end

if optimum1 == optimum2
        disp('The matrix is RCDS')
        else
        disp('The matrix is not RCDS')
end

optimum1
optimum2
diff = abs(optimum2 - optimum1)
end
```

A.6 ε-modification of a matrix

```
% this function takes an nxn matrix and adds/subtracts epsilon to four
% elements from two rows/colums

function[A] = epsilon_2x2()

% define an RCDS matrix or generate from defined function
A = fraDStilRCDS();
[n,n] = size(A);
epsilon = 0.001;
```

```
a = randsample(n,1);
b = randsample(n,1);
c = randsample(n,1);
d = randsample(n,1);

while (A(a,c) == 0 || A(b,c) == 0 || A(a,d) == 0 || A(b,d) == 0 )
        || (a == b || c == d)
                a = randsample(n,1);
                b = randsample(n,1);
                c = randsample(n,1);
                d = randsample(n,1);
end

A(a,c) = A(a,c) + epsilon;
A(b,c) = A(b,c) - epsilon;
A(a,d) = A(a,d) - epsilon;
A(b,d) = A(b,d) + epsilon;
```

A.7 Finding number of diagonals and their sums of a matrix

```
% define the matrix
X = generer_rsn_RCDSmatrise()
[n,n] = size(X);

%find all permutations of A
B = reshape(X(:,flipud(perms(1:n)).'),n,n,factorial(n));

% compute diagonal sums with only nonzero elements of X and store
% values in dgnl
dgnl = [];
for i = 1:factorial(n)
        if all(diag(B(:,:,i))~=0)
        dgnl = [dgnl,sum(diag(B(:,:,i)))];
        end
end

% count the number of unique diagonal sums
[~,~,ix] = unique(dgnl);
C = [sort(unique(dgnl)); accumarray(ix,1).']
```

A.8 Finding number of diagonals and their sums of an ε-modified matrix

```
% use function from other algorithms to produce RCDS matrix and an
% epsilon-modified version of the matrix

% X is RCDS, X_ is epsilon-modified of X
[X, X_] = epsilon_2x2()
[n,n] = size(X_);

%find all permutations of X and X_
B = reshape(X(:,flipud(perms(1:n)).'),n,n,factorial(n));
B_ = reshape(X_(:,flipud(perms(1:n)).'),n,n,factorial(n));

% compute diagonal sums with only nonzero elements of X and store
```

```
% values in dgnl
dgnl = [];
for i = 1:factorial(n)
        if all(diag(B(:,:,i))~=0)
                dgnl = [dgnl,sum(diag(B(:,:,i)))];
        end
end
dgnl;

% compute diagonal sums with only nonzero elements of X_ and store
% values in dgnl_
dgnl_ = [];
for i = 1:factorial(n)
        if all(diag(B_(:,:,i))~=0)
                dgnl_ = [dgnl_,sum(diag(B_(:,:,i)))];
        end
end
dgnl_;

% count the number of unique diagonal sums and store them in
% matrices C, C_
[~,~,ix] = unique(dgnl);
[~,~,ix_] = unique(dgnl_);
C = [sort(unique(dgnl)); accumarray(ix,1).']
C_ = [sort(unique(dgnl_)); accumarray(ix_,1).']
```

A.9 Gathering some statistics on the diagonals of doubly stochastic matrices

```
N = 10000;
relative = zeros(1,N);
delta = zeros(1,N);
max_ = zeros(1,N);
min_ = zeros(1,N);
for i=1:N
        [diff, optimum1, optimum2] = LPnxn_maxmin();
        delta(i) = diff;
        max_(i) = optimum1;
        min_(i) = optimum2;
        relative(i) = diff/optimum2;
end

x = 1:N;
delta = (1/10)*delta;
max_ = (1/10)*max_;
min_ = (1/10)*min_;
relative = (1/10)*relative;
tol = 0.05 ;
x5 = 0;
for i = 1:N
        if relative(i) < tol
                x5 = x5 + 1;
        end
end

p = [5 25 50 75 95 99]
P = prctile(relative, p)
```

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