UiO Department of Mathematics
University of Oslo

Linear and Non-Linear Illiquidity Models in Finance and Insurance

Thomas Balser

Master's Thesis, Spring 2022

Abstract

Not being able to price the illiquidity costs of a portfolio can often be an expensive gambit for investors. Yet, mathematical models of illiquidity are rare in the literature. The object of this thesis is to explore the discrete-time illiquidity framework of both Christodoulou [Chr20] and Schweizer et al. [LPS98] to understand the impact of illiquidity under various market regimes. Secondly, we will add the illiquidity framework to unit-linked insurance, and finally, we will explore if the conclusions of Schweizer et al. [LPS98] hold under a non-linear supply curve. We found that the illiquidity cost in the model of Christodoulou [Chr20] was influenced by the length to maturity of the derivative, the size of the illiquidity parameter ϵ, and the amount of stock purchased. Additionally, we found that the illiquidity cost was subject to a desaturation point and we succeeded in expanding the model of Schweizer et al. [LPS98] to a non-linear supply curve. The results of this thesis apply to a broad class of trading strategies and our findings can be used to find the behavior and bounds of an illiquid trading strategy.

Acknowledgements

I wish to thank my supervisor Prof. Dr. Frank Norbert Proske for letting me explore the field of illiquidity to my hearts content and always being available for a meeting in my hour of need.

I would also like to thank my friends and family for supporting me with encouraging words and moral support.

My most heartfelt thanks goes to my dad. Without him I would never have tried to become a mathematician. I dedicate this thesis to him.

Contents

CHAPTER 1

Introduction

Oscar Wilde defined a cynic as “A man who knows the price of everything and the value of nothing”. The aim of a financial mathematician should then be to become halfway cynical, since much of the discipline is devoted to finding the correct market price.

This thesis is taking aim at pricing illiquidity risk. Illiquidity is loosely speaking when a stock incurs a cost when sold at a particular point. This can be due to various reasons such as the market being saturated with the given stock, a liquidity squeeze making money less available etc. The risk of illiquidity impacts the price of an asset and must somehow be estimated such that investors are compensated fairly when buying on markets with illiquidity.

Specially, this thesis will investigate hedging methods in incomplete markets, then move on to summarize important results from the discrete-time model introduced by Schweizer et al.[LPS98] and expanded on by Christodoulou [Chr20]. Then we will investigate the behavior of illiquidity given by the model developed by Christodoulou under various assumptions and lastly we will expand the model to a non-linear supply curve and see which assumptions of Schweizer are still valid.

We found an explicit representation of the value process under illiquidity, created a numerical method to calculate the illiquidity cost, incorporated this method into a unit-linked policy and finally we managed to expand Schweizer et al.[LPS98] linear supply curve model into a the non-linear setting.

CHAPTER 2

Mathematical prerequisites

Before diving into the topic of the paper, we make a short pit stop and equip ourselves with some useful mathematical concepts.

2.1 Measure theory

We start in the realm of measure theory. Measure theory is a field of mathematics exploring the concept of a measure. A measure gives a formalization of the concepts of length, area and volume. Measures form the foundation of probability theory and integration theory which are corner stones of stochastic equations.

σ-algebra

Let A be a set. The σ-algebra on A, called $\mathcal{A}$, is a non-empty collection of subsets of A, which satisfy the following three conditions:

1. $\emptyset \in \mathcal{A}$.
2. If $B \in \mathcal{A}$, then $B^c \in \mathcal{A}$.
3. If $B_1, B_2, B_3, ... \in \mathcal{A}$, and $B^c = \mathcal{A} \setminus B$, then $\cup_n B_n \in \mathcal{A}$.

Then smallest σ-algebra is defined as $\{\Omega, \emptyset\}$ and the largest is of order 2^N also known as the power set.

See [Dam07].

Measure

Let $\mathcal{A}$ be a σ-algebra on a set A. A measure is a function $\mu : \mathcal{A} \to [0, +\infty]$ if and only if it satisfied the following conditions:

1. Null empty set: $\mu(\emptyset) = 0$.
2. σ-additivity: For $n \geq 1$, if B_n is a sequence of disjoint subsets in $\mathcal{A}$, then $\mu(\cup_n B_n) = \sum_n \mu(B_n)$.

A probability measure is a mapping $P : \mathcal{A} \to [0, 1]$ for,

1. $\mathcal{P}(\emptyset) = 0$.
2. $\mathcal{P}(\Omega) = 1$.

3. For disjoint events B_n of the subsets of $\mathcal{A}$ the $P(\cup_{n=1}^{\infty} B_n) = \sum_{n\geq 1} P(B_n)$.

See [Dam07].

Discrete probability space

A probability space $(\Omega, \mathcal{A}, \mathcal{P})$ is discrete if and only if:

1. Ω is finite or countably infinite.
2. The σ-algebra is a collection of subsets of Ω.
3. The probability measure is defined for every subset of Ω st. $P(A) = \sum_{\omega_A} P(\omega)$ and $\sum_{\omega\in\Omega} P(\omega = 1)$.

The discrete probability space is the sum of probability measure of singletons.
See [Dup].

Borel σ-algebra

The Borel σ-algebra on $\mathbb{R}$, noted by $\mathcal{B}(\mathbb{R})$, is the σ-algebra generated by open sets in $\mathbb{R}$, or equivalently by open intervals (a, b) of $\mathbb{R}$.

Similarly the Borel σ-algebra on $\mathbb{R}^n$, noted by σ-algebra on $\mathbb{R}^n$, is the σ-algebra generated by the sets $\prod_{i=1}^{n}(a_i, b_i)$ of $\mathbb{R}^n$. See [Dup].

Measurable function

Let $(X, \mathcal{A})$ and (Y, Σ) be two measurable spaces. Where X and Y are two measurable sets equipped with the σ-algebras $\mathcal{A}$ and Σ respectively. A subset $B \in \Sigma$, and a function $f : X \to Y$ is measurable if

$$f^{-1}(B) = \{x \in X | f(x) \in B\} \in \mathcal{A} \quad \text{for all} \quad B \in \Sigma. \tag{2.1}$$

See [Dup].

2.2 Stochastic analysis

Stochastic analysis allows one to understand stochastic dynamics and models which will be important in the later sections.

Stochastic process

A stochastic process describes how a variable changes over time due in part to some random variation.

Formally speaking: Let $T \neq \emptyset$ be an index set (e.g. $T = [0, \infty)$ time interval). Then a collection of random variables $\{X_t\}_{t\in T}$ is called a stochastic process with parameter space T.

Filtration

A filtration is a family of σ-algebras on a measurable space.

Given a $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration $\mathcal{F}_n$ is an increasing sequence of σ-algebras of $\mathcal{F}$, such that $\mathcal{F}_t \in \mathcal{F}$ and $t_1 \leq t_2 \to \mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$.

See [Dam07].

Predictable process

Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration. The process $(\mathcal{X}_n)_{n\in\mathbb{N}}$ is a predictable process for the filtration if $\mathcal{X}_0$ is $\mathcal{F}_0$ measurable and $\mathcal{X}_n$ is $\mathcal{F}_{n-1}$ measurable for all $n > 0$.

In other words: By knowing the value of the process at a previous time we also know its value at a future time such that given $X_{n+1} \in \mathcal{F}_n$ and X_n is a predictable process we have

$$E[X_{n+1}|\mathcal{F}_n] = X_{n+1}. \tag{2.2}$$

See [Dam07].

Adapted process

The process $\{\mathcal{X}_t\}_{t\in\mathbb{N}}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ if the random variable $X_t : \Omega \to S$ is a $(\mathcal{F}_t, \mathcal{A})$-measurable function for each $t \in \mathbb{N}$ with respect to the measurable space $(S, \mathcal{A})$.

Informally speaking the adapted process cannot see into the future and cannot reveal more information than the σ-algebra. See [Dam07].

Stopping time

Loosely speaking a stopping time is a rule that decides whether a process continues or stops on the basis of its present position and past events.

Specifically, let τ be a random variable, which is defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, P)$. τ is called a stopping time when $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in T$. See [Dam07].

Martingale

A martingale is the mathematical way of describing a fair money game in the sense that if we play this game and want to estimate how much money we will have in the future, our best guess is simply how much money we have now. So the expected value of each game is 0. Mathematically we define it as follows:

A Stochastic process, defined on a filtered probability space $(\Omega, \mathcal{F}, \{\sqcup\}_{t\in T}, P)$, $Y : T \times \Omega \to S$ with values in a separable Banach space S is a martingale if

1. Y is adapted to the filtration $\mathcal{F}$ such that for each $t \in T$ the random variable Y_t is a $\mathcal{F}_t$-measurable function.
2. For each t, Y_t lies in L^1 space i.e.
$$E_1(|Y_t|) < +\infty.$$
3. For all s and t where $s < t$ and for all $F \in \mathcal{F}_s$,
$$E_1([Y_t - Y_s]\mathbf{1}_F) = 0,$$
where $\mathbf{1}_F$ is the indicator function of the event F.

In discrete-time the conditional expected value of the next observation, given all past observations is equal to the most recent observation i.e.

$$E(X_{n+1}|X_1, ..., X_n) = X_n. \tag{2.3}$$

See [Dam07].

Super- and Submartingale

A supermartingale is a stochastic process $\{X_n; n \geq 1\}$ satisfies the relations

$$E[|X_n|] < \infty; \quad E[X_n|X_{n-1}, X_{n-2}, ..., X_1] \leq X_{n-1}; \quad n \geq 1. \tag{2.4}$$

A submartingale is a stochastic process $\{X_n; n \geq 1\}$ satisfies the relations

$$E[|X_n|] < \infty; \quad E[X_n|X_{n-1}, X_{n-2}, ..., X_1] \geq X_{n-1}; \quad n \geq 1. \tag{2.5}$$

See [Dam07].

Local martingale

Given a probability space $(\Omega, \mathcal{F}, P)$, let $\mathcal{F}_* = \{\mathcal{F}_t\}_{t \geq 0}$ be a filtration on $\mathcal{F}$. Let $X : [0, \infty) \times \Omega \rightarrow S$ be a stochastic process with values in S. Then X is called an $\mathcal{F}_*$-local martingale if there exist a sequence of $\mathcal{F}_*$-stopping time $\tau_k : \Omega \rightarrow [0, \infty)$ such that

1. The τ_k's are almost surely increasing: $P(\tau_k \leq \tau_{k+1}) = 1$.
2. The τ_k diverges almost surely: $P(lim_{k\rightarrow\infty}\tau_k = \infty) = 1$.
3. the stopped process
$$X_t^{\tau_k} := X_{min(t,\tau_k)}$$
is an $\mathcal{F}_*$-martingale for every k.

See [Dam07].

Semimartingale

A real valued process X on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is a semimartingale if it can be decomposed as

$$X_t = M_t + A_t, \tag{2.6}$$

where M is a local martingale and A is a càdlàg adapted process of locally bounded variation. See [Dam07].

Doobs decomposition

Let $(X_n, \mathcal{F}_n)_{n \in \mathcal{N}}$ be a submartingale. Then the doobs decomposition of X_n is given by

$$X_n = X_0 + M_n + A_n, \tag{2.7}$$

where $A_n = \sum_{k=1}^{n} E[X_k - X_{k-1}|\mathcal{F}_{k-1}]$ and is a increasing predictable process and $M_n = X_n - X_0 - A_n$ is a martingale. See [Dam07].

Martingale representation theorem

The Martingale representation theorem states that a random variable that is square integrable functional of a Brownian motion can be expressed as an *itô* integral with respect to that Brownian motion. Let $\{B_t\}_{t\geq 0}$ be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ and let G_t be a complete and right continuous filtration generated by B. Let X be a square integrable random variable that is measurable with respect to G_∞, then there exists a predictable process C, which is adapted to G_t such that

$$X = E[X] + \int_0^\infty C_s dB_s. \tag{2.8}$$

See [Dam07].

Discrete-time processes

A discrete-time financial model considers financial assets at a finite number of time points, which may be unrealistic in markets where price changes occur so frequently that the discrete model cannot follow the moves.

Consider a finite probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\{\mathcal{F}_n\}_{1\geq n\geq N}$.

The market consist of (d+1) assets, whose prices are positive random variables $S_n^0, S_n^1, ..., S_n^d$ at a time n. These random variables are measurable with respect to $\{\mathcal{F}_n\}_{1\geq n\geq N}$. See [Dam07].

Continuous-time processes

Let (A, δ) be a measurable space, then a continuous-time stochastic process with a state space (A, δ) is a set $(X_t)_{t\in\mathbb{R}^+}$ of random variables on a probability space $(\Omega, \mathbb{A}, \mathbb{P})$ with values in (A, δ).

An important precondition for the Black-Scholes model is a continuous-time process. See [Dam07].

Brownian motion

A Brownian motion is a stochastic process $(\mathcal{B}_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{A}, P)$ with the following properties:

1. $B_0 = 0$.
2. The path $(t \mapsto B_t)$ is continuous in t with probability 1.
3. The process $(\mathcal{B}_t)_{t\geq 0}$ has stationary and independent increments.
4. The increment $B_{t+1} - B_t$ is normally distributed with mean 0 and variance 1.

Independent increments means that increments of random variables are jointly independent. While stationary increments means that the distribution of $B_{t+1} - B_t = B_1$ is the same as $B_1 - B_0 = B_1$.

The Brownian motion is often used to model random stock movements and drunk people walking. See [Dam07].

Geometric Brownian motion

A process $S_t, t \geq 1$ is a geometric Brownian motion (or exponential Brownian motion) with drift μ and volatility σ if it can be written as

$$S_t = S_0 exp(\mu t + \sigma W_t), \quad t \in \mathbb{R}_+, \tag{2.9}$$

where W is a standard Brownian motion.

Note that the law of a geometric Brownian motion is not actually Gaussian. Instead the random variable S_t is lognormally distributed with mean μt and variance $\sigma^2 t$. Such that it is only its relative increments that are stationary and independent ie.

$$\frac{S_{t_n} - S_{t_{n-1}}}{S_{t_{n-1}}}, \frac{S_{t_{n-1}} - S_{t_{n-2}}}{S_{t_{n-2}}}, ..., \frac{S_{t_1} - S_{t_0}}{S_{t_0}}, \quad 0 \leq t_0 < t_1 < ... < t_n, \tag{2.10}$$

are stationary and independent. This is the same as saying

$$\frac{S_{t_n}}{S_{t_{n-1}}}, \frac{S_{t_{n-1}}}{S_{t_{n-2}}}, ..., \frac{S_{t_1}}{S_{t_0}} \quad 0 \leq t_0 < t_1 < ... < t_n, \tag{2.11}$$

furthermore, the log-returns

$$log(\frac{S_{t_n}}{S_{t_{n-1}}}), log(\frac{S_{t_{n-1}}}{S_{t_{n-2}}}), ..., log(\frac{S_{t_1}}{S_{t_0}}) \quad 0 \leq t_0 < t_1 < ... < t_n, \tag{2.12}$$

are stationary and independent. Additionally, the law of $\frac{S_t}{S_s}$ with $s < t$ is lognormal distributed with parameters $\mu(t-s)$ and $\sigma^2(t-s)$ and the law of $log(\frac{S_t}{S_s})$,$s < t$ is normally distributed with $\mathcal{N}(\mu(t-s), \sigma^2(t-s))$

This process is often used to model stock prices, a prominent application is the famous/infamous Black-Scholes theory of option pricing. See [Dam07].

Option pricing for geometric Brownian motion

If we consider a European call option with expiration date $t = T$, at a strike price K; the payoff will be $C_T = (S(T) - K)^+$. Assume that the stock price follows a geometric Brownian motion and we work in continuous-time.

The discounted price of an option with expiration date $t = n$ is given by the discounted expected value

$$C_0 = \frac{1}{(1+r)^n} E^*(S_n - K)^+, \tag{2.13}$$

where E^* is the expected value under a risk-neutral probability measure P. Under a risk neutral measure the expected value of the stock equals the risk-free interest rate r, i.e. $E(S_1) = (1+r)S_0$. That is to say that the discounted stock price is the "fair" price and form a martingale. Another way to write this is

$$C_0 = e^{-rT} E^*(S(T) - K)^+. \tag{2.14}$$

See [Uni].

Risk-neutral measure with respect to S(t)

One often wants to have the price only in terms of the price of the stock and not by the risk preferences of the people in the market. This is where the risk neutral measure can help, since it adjusts the current value of the stock such that it is worth the present value of the expected future returns on that stock. Formally speaking: Let $S(t) = S_0 e^{X(t)}$, where $X(t) = \mu t + \sigma B(t)$ is a Brownian motion. We find new values for μ and σ (calling them μ^*,σ^*), where the pricing is "fair" such that $e^{-rt}S(t)$ forms a martingale and $E(S(t)) = e^{rt}S_0$. We have that $E(S(t)) = e^{\overline{r}t}S_0$, where $\overline{r} = \mu + \sigma^2/2$ so we need to have

$$\mu + \sigma^2/2 = r. \tag{2.15}$$

This occurs when $\sigma^* = \sigma$, but changing the drift term μ to

$$\mu^* = r - \sigma^2/2. \tag{2.16}$$

Which can be called the risk-neutral drift. In effect we are replacing $S(t)$ by its risk neutral version such that $S^*(t) = S_0 e^{X^*(t)}$, where

$$X^*(t) = \mu^* t + \sigma B(t) = (r - \sigma^2/2)t + \sigma B(t), \tag{2.17}$$

and then

$$\begin{aligned} C_0 &= e^{-rT} E^*(S(T) - K)^+ \\ &= e^{-rT} E(S(T)^* - K)^+ \\ &= e^{-rT} E(S_0 e^{(r-\sigma^2/2)T + \sigma B(T)} - K)^+, \end{aligned} \tag{2.18}$$

such that the price only depends on the real variance term σ^2. See [Uni].

Equivalent measures

Consider a probability space $(\Omega, \mathcal{F})$. Two probability measures $\mathcal{P}$ and $\mathcal{Q}$ are equivalent on $\mathcal{F}$ if

$$\mathcal{P}(A) = 0 \Leftrightarrow \mathcal{Q}(A) = 0, \quad \forall A \in \mathcal{F}.$$

See [NIE].

Rodyn-Nikodym theorem

Rodyn-Nikodym theorem states that $P(A) = 0 \rightarrow \mathcal{Q}(A) = 0, \quad \forall A \in \mathcal{F}$ only occurs if and only if there exists an $\mathcal{F}$-measurable mapping $\epsilon : \Omega \rightarrow [0, \infty)$ such that

$$\int_A d\mathcal{Q}(w) = \int_A \epsilon(w) d\mathcal{P}(w), \forall A \in \mathcal{F}. \tag{2.19}$$

The above can be written as $\epsilon = d\mathcal{Q}/d\mathcal{P}$, ϵ is called the likelihood ratio between $\mathcal{P}$ and $\mathcal{Q}$ or the Radon-Nikodym derivative. Some useful consequences of this theorem are:

1. For any random variable X on $L^1(\Omega, \mathcal{F}, \mathcal{Q}) : E^{\mathcal{Q}}[X] = E^{\mathcal{P}}[\epsilon X]$ and likewise $E^{\mathcal{Q}}[\epsilon^{-1} X] = E^{\mathcal{P}}[X]$.

2. Assume $\mathcal{Q}$ is absolutely continuous w.r.t. $\mathcal{P}$ on $\mathcal{F}$ and that $\mathcal{G} \subseteq \mathcal{F}$, then the likelihood ratios $\epsilon^{\mathcal{F}}$ and $\epsilon^{\mathcal{G}}$ are connected through $\epsilon^{\mathcal{G}} = E[\epsilon^{\mathcal{F}}|\mathcal{G}]$.

3. Assume X is a random variable on $(\Omega, \mathcal{F}, \mathcal{P})$ and let $\mathcal{Q}$ be another measure on $(\Omega, \mathcal{F})$ with Radon-Nikodym derivative $\epsilon = d\mathcal{Q}/dP$ on $\mathcal{F}$. Assume now $X \in L^1(\Omega, \mathcal{F}, P)$ and let $\mathcal{G} \subseteq \mathcal{F}$ then

$$E^{\mathcal{Q}}[X|\mathcal{G}] = \frac{E^P[\epsilon X|\mathcal{G}]}{E^P[\epsilon|\mathcal{G}]}, \mathcal{Q} - a.s. \tag{2.20}$$

See [NIE].

Girsanov's Theorem

Suppose $B_t, 0 \geq t \geq T$ is defined on some probability space $(\Omega, \mathcal{F}, P)$ is a Brownian motion with respect to the filtration $(\mathcal{F}_t, t \geq 0)$, and let $(\mu(t), t \geq 0)$ be adapted. If there exists

$$\overline{B}(t) = B(t) + \int_0^t \mu(u)du, \tag{2.21}$$

and

$$Z(t) = exp(-\int_0^t \mu(u)dB(u) - \frac{1}{2}\int_0^t \mu(u)^2 du), \quad Z = Z(T). \tag{2.22}$$

Suppose that $Z_t, 0 \geq t \geq T$ is a martingale.

Then under the probability measure $\overline{P}$ with the Radon-Nikodym density $\frac{d\overline{P}}{dP} = Z$ the process $(\overline{\mathcal{B}}, t \in [0, T])$ is a standard Brownian motion.

Proof: Using the Levy theorem, which states that a continuous martingale starting at 0 and having quadratic variation equal to t for every $t > 0$ is a Brownian Motion. We see that $\overline{B}(0) = B(0) = 0$ and the quadratic variation is $\langle\overline{B}\rangle(t) = t$. It remains to show that $\overline{B}$ is a martingale under $\overline{P}$. We defined $Z(t) = e^{X(t)}$ where

$$X(t) = -\int_0^t \mu(u)dB(u) - \frac{1}{2}\int_0^t \mu(u)^2 du. \tag{2.23}$$

Using the Ito formula

$$dZ(t) = de^{X(t)} = e^{X(t)}(-\mu(t)dB(t) - \frac{1}{2}\mu(t)^2dt) + \frac{1}{2}e^{X(t)}\mu(t)^2dt = -\mu(t)Z(t)dB(t). \tag{2.24}$$

Which is solved by the stochastic integral

$$Z(t) = Z(0) - \int_0^t \mu(u)Z(u)dB(u). \tag{2.25}$$

So $Z(t)$ is a martingale under $\overline{P}$, since $E[Z(T)] = Z(0) = 1$. We assume further that the stochastic integral is square-integrable and thus well defined.

From the martingale property we have

$$Z(t) = E[Z(T)|\mathcal{F}_t] = E[Z|\mathcal{F}_t] \tag{2.26}$$

is the Radon-Nikodym derivative process.

Using the product rule $d(XY) = XdY + YdX + dXdY$ we have

$$\begin{aligned} d(\overline{B}(t)Z(t)) =& \overline{B}(t)dZ(t) + Z(t)d\overline{B}(t) + d\overline{B}(t)dZ(t) \\ =& -\overline{B}(t)\mu(t)Z(t)dB(t) + Z(t)dB(t) + Z(t)\mu(t)dt \\ &+ (dB(t) + \mu(t)dt)(-\mu(t)Z(t)dB(t)) \\ =& (-\overline{B}(t)\mu(t) + 1)Z(t)dB(t). \end{aligned} \tag{2.27}$$

Now we see that there is no drift dt-term, so $\overline{B}(t)Z(t)$ is a martingale under P.

Due to the martingale property of $\overline{B}(t)Z(t)$ under P we have for $0 \leq s \leq t \leq T$ we have

$$\overline{E}[\overline{B}(t)|F_s] = \frac{1}{Z(s)}E[\overline{B}(t)Z(t)|\mathcal{F}_s] = \frac{1}{Z(s)}\overline{B}(s)Z(s) = \overline{B}(s). \tag{2.28}$$

So $\overline{B}$ under P.

See [NIE].

Black-Scholes Model

The Black-Scholes model describes the behavior of prices in a continuous-time setting with one risky asset (a stock with price S_t at time t) and a riskless asset (with price S_t^0 at time t).

Suppose the riskless asset is described by the differential equation

$$dS_t^0 = rS_t^0 dt, \tag{2.29}$$

where r is a non-negative constant. Let $S_0^0 = 1$ and then $S_t^0 = e^{rt}$

It is assumed that the stock price is determined by the stochastic differential equation:

$$dS_t = S_t(\mu dt + \sigma dB_t), \tag{2.30}$$

where μ and σ are two constants and (B_t) is a Brownian motion.

We consider an interval $[0, T]$, where T is the terminal time of the option. The solution to (2.30) is

$$S_t = S_0 exp(\mu t - \frac{\sigma^2}{2}t + \sigma B_t) \tag{2.31}$$

where S_0 is the spot price at time 0.

A limitation of the model is that when used in discrete-time the standard Black-Scholes hedge is no longer perfect, in the sense that the expected return of the hedged portfolio only goes to zero in expectation and not almost surely, such that the hedge is not perfect. See [Dam07].

2.3 Life insurance mathematics

We take a brief detour into the land of life insurance mathematics so that we have the theoretical tools at hand to calculate the reserve of a unit linked insurance. Later on we want to use a unit linked insurance with the value process of christodoulou's discrete-time framework [Chr20].

Markov process

A stochastic process $X = \{X_t\}_{t\in\mathbb{R}_+}$ is a Markov process if

$$P(X_t \in B|\sigma(X_{s_1}, X_{s_2}, ..., X_{s_n})) = P(X_t \in B|\sigma(X_{s_n})), \tag{2.32}$$

for all $0 \leq s_1 < s_2 < ... < s_n \leq t$ and $B \in \mathcal{B}(\mathbb{R})$.

A Markov process is independent of past events given the present, such that the process $X_{t_{n+1}}$ at time t_{n+1} only remembers its last position $X_{t_n} = i_n$.

See [Bañ22].

Transition probability

A continuous-time Markov chain with transition probabilities is defined as

$$p_{ij}(s,t) = \mathbb{P}(X_t = j|X_s = i), \quad s \leq t, \quad i,j \in S, \tag{2.33}$$

where $p_{ij}(s,t)$ is the probability that X will be in state j at time t given that X was in state i at a previous time s.

Discrete-time Markov chain with transition probabilities. We now evaluate $X = \{X_n, n \geq 0\}$ in discrete-time such that we have pointwise transition rates i.e

$$p_{ij}(n,m) = P[X_m = j|X_n = i], \quad i,j \in S, \quad m \geq n. \tag{2.34}$$

This framework is often used to calculate survival probabilities in a life insurance setting.

See [Bañ22].

Chapman-Kolmogorov equation

Let $\{X_t\}_{t\in J}$, be a Markov process and let $P(s,t) = \{p_{ij}(s,t)\}_{i,j\in S}$ be its matrix of transition probabilities. Then

$$p_{ij}(s,t) = \sum_{k\in S} p_{ik}(s,u)p_{k,j}(u,t), \tag{2.35}$$

for all $s \leq u \leq t$ and $i,j \in S$ with $\mathbb{P}(X_s = i), \mathbb{P}(X_t = j) \neq 0$, where the following is true

1. $p_{ij}(s,t) \geq 0$
2. $\sum_{j\in S} p_{ij}(s,t) \quad \text{for all} \quad i \in S$
3. $p_{ij}(s,s) = \mathbf{1}_{i=j} \quad \text{provided that} \quad \mathbb{P}(X_s = i) \neq 0$

See [Bañ22].

Markov process characterization

A stochastic process $X = \{X_t\}_{t\in J}$ is a Markov process if and only if

$$\mathbb{P}(X_{t_1} = i_1, ..., X_{t_n} = i_n) = \mathbb{P}(X_{t_1} = i_1)p_{i_1,i_2}(t_1,t_2)p_{i_2,i_3}(t_2,t_3)...p_{i_{n-1},i_n}(t_{n-1},t_n) \tag{2.36}$$

for all $t_1 < t_2 < ... < t_n \in J, i_1, ..., i_n \in S, n \geq 1$. See [Bañ22].

Transition rates

Let $X = \{X_t, t \in J\}$ be a Markov process with finite state space S. The transition rates $\mu_i, \mu_{ij}, i, j \in S, j \neq i$ are the functions defined by

$$\mu_i(t) = lim_{h \to 0, h>0} \frac{1 - p_{ii}(t, t+h)}{h}, t \in J, i \in S, \tag{2.37}$$

and

$$\mu_{ij}(t) = lim_{h \to 0, h>0} \frac{p_{ij}(t, t+h)}{h}, t \in J, i \in S, i \neq j, \tag{2.38}$$

when they exist and are finite. See [Bañ22].

Kolmogorov equations

Assume that $X = \{X_t, t \in J\}$ is regular, i.e. the transition rates $\mu_i, \mu_{ij}, t \in J, i \in S, i \neq j$ exist and are continuous with respect to t, then:

1. Backward Kolmogorov equation

$$\frac{d}{ds} p_{ij}(s,t) = \mu_i(s) p_{ij}(s,t) - \sum_{k \in S, k \neq i} \mu_{ik}(s) p_{kj}(s,t) \tag{2.39}$$

2. Forward Kolmogorov equation

$$\frac{d}{ds} p_{ij}(s,t) = \mu_j(s) p_{ij}(s,t) + \sum_{k \in S, k \neq i} \mu_{kj}(s) p_{ik}(s,t) \tag{2.40}$$

See [Bañ22].

Discount factor

$v : [0, \infty) \to [0, \infty)$ is the discount factor defined as

$$v(t) = exp(-\int_0^t r_u du), \quad t \geq 0, \tag{2.41}$$

where $r : [0, \infty) \to \mathbb{R}$ is a deterministic integrable function modelling the interest rate.

See [Bañ22].

Policy functions in continuous-time

Let $a_i, a_{ij} : [0, \infty) \to \mathbb{R}, i, j \in S, i \neq j$ be functions of bounded variation. Here

$a_i(t)$= the accumulated payments from the insurer to the insured up to time t, given that we know that the insured has always been in state i.

$a_{ij}(t)$= the punctual payments which are due when the insured switches from state i to j at time t.

See [Bañ22].

Policy functions in discrete-time

Let a_i^{Pre}, $a_{ij}^{Post} : \mathbb{N} \to \mathbb{R}, i, j \in S$ be functions. We call them policy functions and they are defined as follows:

a_i^{Pre}= pension payment which is due at time n, given that the insured is at time n in i,

a_{ij}^{Post}= capital benefits which are due when switching from i at time n to j at time n+1.

See [Bañ22].

Mathematical Reserve

The mathematical reserve is the cost of the insurance for the insurance company at each time t. It is calculated as the present value of the insurers future obligations minus the present value of future obligations of the insured. You can calculate the reserve for various insurance policies. The reserve can be described in continuous-time and discrete-time where the major difference is that in the first case we use integrals while in the latter we use sums. See [Bañ22].

Reserve formula in continuous-time

The formula for reserves in continuous-time is given by:

$$V_i(t, A) = \frac{1}{v(t)} \int_t^{\infty} (v(s)p_{ij}(t, s)da_j(s) + v(s)p_{ij}(t, s)\mu_{jk}(s)a_jk(s)ds, \quad (2.42)$$

where the $\int_0^t da_j(s)$ corresponds to the accumulated liabilities while the insured is in state j, and $\int_0^t a_jk(s)ds$ is the accumulated liabilities when the insurer switches from state j to k.

See [Bañ22].

Reserve formula in discrete-time

The reserve of a stochastic cash flow is given as

$$V_i(t, A) = \frac{1}{v(t)}[\sum_{j\in S}\sum_{n\geq t} v(n)p_{ij}(t, n)a_j^{Pre}(n) + \sum_{j,k\in S,k\neq j}\sum_{n\geq t} v(n+1)p_{ij}(t, n)p_{jk}(n, n+1)a_{jk}^{Post}(n)] \quad (2.43)$$

See [Bañ22].

Unit linked policy based on the Black-Scholes model

A unit linked policy is an insurance contract which is linked to the performance of a stock market or fund. Such that the amount insurers liability corresponds to some underlying stock.

We consider a regular Markov process $X = \{X_t, t \in [0, T]\}, T \in \mathbb{R}, T > 0$ with a filtration $\mathcal{F}^X$ generated by X. The state space of X is denoted $\mathbb{I}$ (i.e. the states of the insured).

Additionally, we consider a stochastic process S with the SDE-dynamics

$$\frac{dS_t}{S_t} = \mu(t, S_t)dt + \sigma(t, S_t)dW_t, \quad S_0 > 0, \quad t \in [0, T], \quad (2.44)$$

which we assume to have a unique strong solution, such that S is adapted to the filtration F, i.e. S_t is a measurable function of $W_s, s \leq t$.

Let $\mathcal{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be the filtration generated by W. Let $\mathbb{Q}$ denote the risk neutral martingale measure.

Assume now that a_i is differentiable almost everywhere with at most a discontinuity at the end of the contract $t = T$ and let $\Delta a_i(T) = a_i(T) - a_i(T-)$ be the jump caused by the discontinuity. Assume also that the functions $f_i, g_i, h_{ij} : [0, T] \times \mathbb{R} \to \mathbb{R}$ are

$$\Delta a_i(T) = f_i(T, S_T), \quad \widehat{a}_i(t) = g_i(t, S_t), \quad a_{ij}(t) = h_{ij}(t, S_t), \quad t \in [0, T], \tag{2.45}$$

where $\widehat{a}_i(t)$ is the (weak) derivative of $a_i(t)$ with respect to time.

The mathematical reserve $V_i^+(t, S_t)$ of the contract linked to the fund S at time t, given that the insured is in state i at time t, is given by

$$\begin{aligned} V_i^+(t, S_t) =& \frac{1}{v(t)}(\sum_{j\in F} v(T)p_{ij}(t, T)E_Q[f_j(T, S_T))|\mathcal{F}_t] \\ &+ \sum_{j\in F}\int_t^T v(s)p_{ij}(t, T)E_Q[g_j(s, S_T)|\mathcal{F}_t]ds \\ &+ \sum_{j,k\in F,k\neq j}\int_t^T v(s)p_{ij}(t, T)\mu_{jk}^x(s)E_Q[h_{jk}(s, S_s)|\mathcal{F}_t])ds. \end{aligned} \tag{2.46}$$

Further assume that $S_0 > 0$, $\frac{dB_t}{B_t} = rdt$, $B_0 = 1$. The fair value of a European call option based on the Black-Scholes model is given by

$$E_Q[e^{-r(T-t)}(S_T - K)|\mathcal{F}_t] = N(A)S_t - N(B)Ke^{-rt}, \tag{2.47}$$

where

$$A = \frac{ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt(t)} \tag{2.48}$$

$$B = A - \sigma\sqrt{t} \tag{2.49}$$

and

$$N = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du \tag{2.50}$$

See [Bañ22].

Unit linked insurance in discrete-time

Let $a_j^{Pre}(n) = f_j(n, S_t)$ and $a_j^{Post}k(n) = g_{jk}(n, S_t)$ where $f_j, g_{jk}\mathbb{N} \to \mathbb{R}, j, k \in S$.

Then the mathematical reserve of a unit linked policy in discrete-time is given by

$$\begin{aligned} V_i^+(t, S_t) =& \frac{1}{v(t)}(\sum_{j\in S}\sum_{n\geq t} v(n)p_{ij}(t, n)E_Q[f_j(n, S_T)|\mathcal{F}_t] \\ &+ \sum_{j,k\in S,k\neq j}\sum_{n\geq t} v(n+1)p_{ij}(t, n)p_{jk}(n, n+1)E_Q[g_{jk}(n, S_T)|\mathcal{F}_t]) \end{aligned} \tag{2.51}$$

See [Bañ22].

The equivalence principle

The equivalence principle is a method to make the present value of a policy null as seen against the value of a premium or premiums. See [Bañ22].

2.4 Financial mathematics

We will now pass in review important theorems and definitions from financial mathematics which will be the backbone for many of the later results.

First fundamental theorem of asset pricing

A discrete market, on a probability space $(\Omega, \mathcal{A}, P)$ is arbitrage free if, and only if, there exists at least one risk neutral probability measure that is equivalent to the original probability measure P.

See [Dam07].

Second fundamental theorem of asset pricing

An arbitrage-free market (S,B) consisting of a collection of stocks S and a risk-free bond B is complete if and only if there exists a unique risk-neutral measure that is equivalent to P and has numeraire B.

See [Dam07].

Self Financing Strategy

A trading strategy is said to be self-financing if

$$dV_t = \phi_t dB_t + \psi_t dS_t. \tag{2.52}$$

In other words purchasing new assets can only be financed by the sale of assets in the portfolio and not through outside infusion or withdrawal of money, i.e the variation in the strategy value are only caused by gains/losses in the asset price and does not depend on the fluctuations of portfolio weight.

See [Dam07].

Attainable claim

Let X be a contingent claim. X is a non-negative valued, $\mathcal{F}_t$-measurable, random variable. The contingent claim X is attainable if and only if its expected value is the same under all equivalent martingale measures.

See [Dam07].

Arbitrage opportunity

An arbitrage opportunity arises if there exist a trading strategy φ such that

$$V_0(\varphi) = 0 \quad \text{and} \quad V_T(\varphi) > 0 \quad \text{and} \quad P[V_T(\varphi) > 0] > 0, \tag{2.53}$$

i.e., there is a strategy such that money is generated out of nothing. See [Dam07].

Complete market

A market is complete if every contingent claim is attainable. See [Dam07].

Incomplete market

In an incomplete market perfect risk transfer is impossible. There is no unique martingale measure in an incomplete market and we have for any given non-replicable claim an interval of prices that are all compatible with the no arbitrage condition. An example of an incomplete market is when we have two sources of randomness (stock and volatility) but only one instrument that we can use to hedge (stock). There are more states of the world than underlying securities to hedge them. So that you have a source of risk which cannot be gotten rid of. See [Dam07].

Pricing in incomplete markets

We have to price the risk somehow in the incomplete market setting. Intuitively, the more risk you take on the higher the expected return should be in excess of the risk-free rate. We thus have to find a market price of risk. We can introduce the concept of market price of volatility risk, which measures the excess expected return of this unhedgeable risk. We have to create a market pricing model which contains the market price of risk parameter, such that the prices of all options are consistent with each other through a 'universal' measure. See [Dam07].

CHAPTER 3

Literature review of hedging and pricing methods in finance

3.1 Hedging options

In this section we give an overview of some methods for pricing and hedging options by means of a quadratic criterion. The quadratic criterion will then later be extended to capture illiquidity effects in markets as well. The problem of pricing and hedging can be described as follows: What price should the seller of an option charge the buyer at time 0? and having sold the option, how can the seller insure himself against a random loss at a time T?

To answer these questions it is useful to consider a portfolio strategy of the form $(X, Y) = (X_t, Y_t)_{0 \leq t \leq T}$ where X is a d-dimensional predictable process and Y is adapted. X_t^i describes the number of units of asset i held at time t and Y_t is the amount invested in the riskless asset at time t. We can value the portfolio (X_t, Y_t) by

$$V_t = X_t S_t + Y_t. \tag{3.1}$$

The cumulative gains in the portfolio up to a time t is $G_t(X) = \int_0^t X_s dS_s$. It is assumed that $G_t(X)$ is well-defined and that S is a semimartingale. The cumulative cost up to a time t is given by

$$C_t = V_t - \int_0^t X_s dS_s = V_t - G_t(X). \tag{3.2}$$

The self-financing condition imply that the cumulative cost process C is constant over time and/or the value process V is given by

$$V_t = V_0 + \int_0^t X_s dS_s = V_0 + G_t(X) = C_0 + G_t(X) \tag{3.3}$$

Where C_0 is the initial outlay at the start of the strategy. A contingent claim H is attainable if there exist a self-financing strategy with $V_T = H$ P-a.s. H is then given by

$$H = H_0 + \int_0^T X_s dS_s \quad \text{P-a.s.}, \tag{3.4}$$

In a setting of incomplete markets we can give a range of possible prices for H which are consistent with an arbitrage free market. The risk process of the portfolio φ is defined as

$$R_t(\varphi) = E[(C_T(\varphi) - C_t(\varphi))^2|\mathcal{F}_t] \quad 0 \leq t \leq T. \tag{3.5}$$

A risk minimizing strategy (RMS) is a strategy with $V_T(\varphi) = H$ that also minimized the risk process. A formal definition of RMS for a strategy φ is if for any RMS strategy $\overline{\varphi}$ where $V_T(\overline{\varphi}) = V_T(\varphi)$ P-a.s., we have

$$R_t(\varphi) \leq R_t(\overline{\varphi}) \quad \text{P-a.s. for every t} \in [0, T]. \tag{3.6}$$

where $\overline{\varphi} = \overline{X}, \overline{Y}$ is a perturbation of the portfolio. The intuition behind this notion of risk minimization is that any small change in the underlying stock should not be able to create a new risk minimizing strategy. In addition a RMS is called mean-self-financing if its cost process $C(\varphi)$ is a P-martingale. It holds true that any RMS is also mean-self-financing. See [Sch99].

Quadratic hedging

In general a non-attainable contingent claim can by definition not be obtained by a strategy with final value $V_T = H$ that is also self-financing. We can insist that $V_T = H$ by a choice of Y_T, because Y is adapted. Since this strategy cannot be self-financing, we have to find a "good" approximation of a self-financing strategy by minimizing the cost process C. To this end we introduce a quadratic criterion which measures riskiness when S is a martingale.

See [Sch99].

Locally risk-minimizing strategy

One method of quadratic hedging is local risk-minimization (LRM) in both discrete and continuous-time. This method is somewhat narrow since it requires the price process S to be a local P-martingale.

To exclude arbitrage opportunities it is assumed that S has an equivalent local martingale measure (ELMM) Q, i.e., that there exists a probability measure $Q \sim P$ such that S is a local Q-martingale. $\mathcal{P}$ is the convex set of all ELMMs Q for S, thus $\mathcal{P}$ is non empty. When S is a local P-martingale we mean that the measure P is in $\mathcal{P}$.

See [Sch99].

Discrete-time framework

In this scenario trading can only occur at the dates $k = 0, 1, ..., T \in \mathbb{N}$. At a time k, we choose a number of shares X_{k+1} being held in the period $(k, k+1]$ and the amount of Y_k units of riskless asset held in the period $[k, k+1)$. We can determine the holdings of X_{k+1} at a time k due to the predictability of X. We further assume that $S = (S_k)_{0,1,...,T}$ is a square-integrable process adapted to the filtration $\mathcal{F} = (\mathcal{F}_k)_{0,1,..,T}$.

The portfolio $\varphi_k = (X_k, Y_k)$ at time k has as before the value $V_k(\varphi) = X_k S_k + Y_k$. We want to minimize risk locally so we have to consider incremental cost of switching the portfolio from φ_k to φ_{k+1}. X_{k+1} is chosen at time k with a given price S_k. The incremental cost is

$$\begin{aligned} C_{k+1}(\varphi) - C_k(\varphi) =&(X_{k+1} - X_k)S_k + Y_{k+1} - Y_k \\ =&V_{k+1}(\varphi) - V_k(\varphi) - X_{k+1}(S_{k+1} - S_k) \\ =&\Delta V_{k+1}(\varphi) - X_{k+1}\Delta S_{k+1}, \end{aligned} \tag{3.7}$$

where the difference operator $\Delta U_{k+1} := U_{k+1} - U_k$ for any discrete-time stochastic process U.

The aim of local risk-minimization is to minimize $E[(C_{k+1}(\varphi) - C_k(\varphi))^2|\mathcal{F}_k]$ with respect to time k. Since the $\mathcal{F}_k$-measurable term $V_k(\varphi)$ does not affect the conditional variance of the cost given $\mathcal{F}_k$, we can write

$$\begin{aligned} E[(\Delta C_{k+1}(\varphi))^2|\mathcal{F}_k] =&Var[V_{k+1}(\varphi) - X_{k+1}\Delta S_{k+1}|\mathcal{F}_k] \\ &+ (E[V_{k+1}(\varphi) - X_{k+1}\Delta S_{k+1}|\mathcal{F}_k] - V_k(\varphi))^2. \end{aligned} \tag{3.8}$$

Since the first term on the right hand side does not depend on Y_k we can optimize by choosing Y_k such that

$$V_k(\varphi) = E[V_{k+1}(\varphi) - X_{k+1}\Delta S_{k+1}|\mathcal{F}_k], \tag{3.9}$$

which is the same as

$$0 = E[\Delta V_{k+1}(\varphi) - X_{k+1}\Delta S_{k+1}|\mathcal{F}_k] = E[\Delta C_{k+1}(\varphi)|\mathcal{F}_k]. \tag{3.10}$$

Since $V_T(\varphi) = H$ is fixed we can through the use of backwards induction in the previous equation take the value $V_{k+1}(\varphi)$ as given. It remains to minimize $Var[V_{k+1}(\varphi) - X_{k+1}\Delta S_{k+1}|\mathcal{F}_k]$ with respect to the $\mathcal{F}_k$-measurable X_{k+1} which is only achieved if and only if

$$Cov(V_{k+1}(\varphi) - X_{k+1}\Delta S_{k+1}, \Delta S_{k+1}|\mathcal{F}_k) = 0. \tag{3.11}$$

We can use the Doobs decomposition of S to write it as a martingale $\overline{M}$ and a predictable process $\overline{A}$ such that $\overline{M}_0 = 0 = \overline{A}_0$, $\Delta\overline{A}_{k+1} = E[\Delta S_{k+1}|\mathcal{F}_k]$ and $\Delta\overline{M}_{k+1} = \Delta S_{k+1} - \Delta\overline{A}_{k+1}$.

We can now rewrite (3.11) as

$$0 = Cov(\Delta C_{k+1}(\varphi), \Delta\overline{M}_{k+1}|\mathcal{F}_k) = E[\Delta C_{k+1}(\varphi)\Delta\overline{M}_{k+1}|\mathcal{F}_k]. \tag{3.12}$$

This is saying that the product of two martingales $C(\varphi)$ and $\overline{M}$ must also be a martingale.

The general statement in discrete-time is thus: A suitable integrable strategy φ is locally risk-minimizing if and only if its cost process $C(\varphi)$ is a martingale and the product of it and the martingale part (here $\overline{M}$) is also a martingale. See [Sch99].

3.2 Christodoulou's discrete-time framework for hedging in illiquid markets

This section gives an outline the framework for estimating a locally risk minimizing strategy under illiquidity in discrete-time developed by Christodoulou [Chr20] and some important properties of this strategy.

The basic model

We assume that we have a probability space that is filtered $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and a financial market that has d+1 assets. P is the objective measure and the filtration is $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,...,T}$. The indices $k = 0, 1, ..., T$ are discrete-time points where $t_0 < t_1 < ... < t_T$. The discounted (marginal) price of d risky assets is an F-adapted, non-negative d-dimensional stochastic process $S = (S_k)_{k=0,1,...,T}$. S_k^j denotes the price of an asset j at time t_k. Assume that the risk-less asset (Y) exists and has a discounted price of 1.

Then we assume that we have a non-negative, d-dimensional supply curve $S_k(x) = (S_k(x)^1, ..., S_k(x)^d)$, for $x^j \in \mathbb{R}^d$, where $S_k(x)^j = S_k^j(x^j)$ is the j-th stock price per share at a time k for sale when $x^j < 0$ and to purchase when $x^j > 0$. The supply curve determines the price that each market participants has to pay or receive for a illiquidity of size x^j at time k. We further assume that the supply curve is independent of participants past actions and assume that the supply curve is measurable with respect to the filtration $\mathbb{F}$.

We also assume that the illiquidity costs are non-decreasing in the number of shares x i.e. for each k and j, then $S_k(x)^j \leq S_k(y)^j$, P-a.s. for $x^j \leq y^j$. For $x \in \mathbb{R}^d$ we let $|x|$ be the Euclidean norm. Let $\langle x, y \rangle$ is the inner product of $x, y \in \mathbb{R}^d$.

Let $L_T^p(\mathbb{R}^d)$ be the space of all $\mathcal{F}_T-$measurable random variable $Z : \Omega \to \mathbb{R}^d$ satisfying $||Z||^p = E(|Z|^p) < \infty$. We define $\Delta S_k = S_k - S_{k-1}$. Lastly, $\Theta_d(S)$ is the space of all $\mathbb{R}^d-$valued predictable strategies $X = (X_k)_{k=1,2,...T+1}$ such that $X_k \Delta S_k \in L_T^{2,1}$ and $\Delta X_{k+1}[S_k(\Delta X_{k+1}) - S_k(0)] \in L_T^{1,1}$ for $k = 1, 2, ..., T$.

See [Chr20].

Trading strategy

A trading strategy describes the buying and selling of stocks in a market in at a time T.

Definition 1 A pair $\varphi = (X, Y)$ is called a trading strategy if:

1. $Y = (Y_k)_{k=0,1,...,T}$ is a real-valued F-adapted process.
2. $X \in \Theta_d(S)$
3. $V_k(\varphi) = X_{k+1}S_k + Y_k \in \mathbb{L}_T^{1,2}$ for $k = 0, 1, ..., T$

Where X_{k+1}^j is the number of shares held in the risky asset S_k^j and Y_k is the units in the non-risky asset in the time interval $(k, k+1]$. $V_k(\varphi)$ is the marked-to-market value or book value of the portfolio (X_{k+1}, Y_k) at a time k.

Cost and Risk process

A contingent claim H in $\mathbb{L}_T^{1,2}$is defined as

$$H = \overline{X}_{T+1}S_T + \overline{Y}_T, \quad \text{with} \quad \overline{X}_{T+1}S_T, \overline{X}_{T+1} \in \mathbb{L}_T^{1,2}, \tag{3.13}$$

where both $\overline{X}_{T+1}$ and $\overline{Y}_T$ are $\mathcal{F}_t$-measurable random variables. $\overline{X}_{T+1}$ is the quantity of risky assets that the option seller will provide the buyer at the expiration date T of the financial contract H. Similarly, $\overline{Y}_T$ is the committed number of bonds at time T. The total outlay under liquidity costs is

$$\Delta Y_k + \Delta X_{k+1} S_k(\Delta X_{k+1}) = \Delta Y_k + \Delta X_{k+1}[S_k(\Delta X_{k+1}) - S_k(0)], \quad (3.14)$$

where $\Delta Y_k = Y_k - Y_{k-1}$ and $\Delta X_k = X_k - X_{k-1}$ at a time $k \in 1, 2, ..., T$. The last term in (3.14) is the illiquidity cost stemming from illiquidity, where $S_k(0) = S_k$ is the marginal price.

(3.14) can be rewritten as the relation

$$\Delta Y_k + \Delta X_{k+1} S_k(\Delta X_{k+1}) = \Delta V_k(X, Y) - X_k \Delta S_k + \Delta X_{k+1}[S_k(\Delta X_{k+1}) - S_k(0)]. \quad (3.15)$$

Note that a self-financing trading strategy occur when the total outlay is zero at time k.

To define the cumulative cost of the trading strategy we let $\widehat{C}_0(\varphi) = V_0(\varphi)$ be the initial cost and define the cost process under illiquidity $\widehat{C}(\varphi) = (\widehat{C}_k(\varphi))_{k=0,1,...,T}$ by

$$\widehat{C}_k(\varphi) = V_k(\varphi) - \sum_{m=1}^{k} X_m \Delta S_m + \sum_{m=1}^{k} \Delta X_{m+1}[S_m(\Delta X_{m+1}) - S_m(0)]. \quad (3.16)$$

When the cost process is square integrable the quadratic risk process under illiquidity $\widehat{R}(\varphi) = (\widehat{R}_k(\varphi))_{k=0,1,...,T}$ is defined as

$$\widehat{R}(\varphi) = \mathbb{E}[(\widehat{C}_T(\varphi) - \widehat{C}_k(\varphi))^2 | \mathcal{F}_k]. \quad (3.17)$$

A local risk-minimization approach aims to find a locally risk-minimizing strategy $\varphi = (X, Y)$ s.t. $V_T(\varphi) = H$ where $X_{T+1} = \overline{X}_{T+1}$ and $Y_T = \overline{Y}_T$

Let $C(\varphi) = (C_k(\varphi))_{k=0,1,...,T}$ be the cost process without liquidity cost (i.e. $S(x) = S(0)$), defined as

$$C_k(\varphi) = V_k(\varphi) - \sum_{m=1}^{k} X_m \Delta S_m. \quad (3.18)$$

We then get the following relation

$$\widehat{C}_T(\varphi) - \widehat{C}_k(\varphi) = C_T(\varphi) - C_k(\varphi) + \sum_{m=k+1}^{T} \Delta X_{m+1}[S_m(\Delta X_{m+1}) - S_m(0)]. \quad (3.19)$$

Another way to define the risk process is the linear risk process under illiquidity

$$\widehat{R}_k(\varphi) = \mathbb{E}[|\widehat{C}_T(\varphi) - \widehat{C}_k(\varphi)| | \mathcal{F}_k]. \quad (3.20)$$

Still another approach is the quadratic-linear risk process (QLRP) under illiquidity

$$T_k(\varphi) = \mathbb{E}[(C_T(\varphi) - C_k(\varphi))^2 | \mathcal{F}_k] + \mathbb{E}[\sum_{m=k+1}^{T} \bigtriangleup X_{m+1}[S_m(\bigtriangleup X_{m+1}) - S_m(0)] | \mathcal{F}_k], \quad (3.21)$$

Which measures the quadratic difference of the cost process and the liquidity costs linearly. See [Chr20].

Locally risk minimization under illiquidity

The aim for this section is to locally minimize the risk associated with random fluctuations of the stock price while reducing the liquidity costs brought about by the strategy. We want to find the current optimal choice of strategy by fixing the portfolio at past or future times, such that Y_k and X_{k+1} is only minimized locally at a time k.

Definition 2 A local perturbation $\varphi' = (X', Y')$ of a strategy $\varphi = (X, Y)$ at a time $k \in [0, 1, ..., T-1]$ is a trading strategy such that $X_{m+1} = X^{'}_{m+1}$ and $Y_{m+1} = Y^{'}_{m+1}$ for all $m \neq k$.

We can define the QLRP as

$$T_k^{\alpha}(\varphi) = \mathbb{E}[(C_T(\varphi) - C_k(\varphi))^2|\mathcal{F}_k] + \alpha\mathbb{E}[\Delta X_{k+2}[S_{k+1}(\Delta X_{k+2}) - S_{k+1}(0)|\mathcal{F}_k]. \tag{3.22}$$

Which we will use to find a local risk minimizing strategy under illiquidity for some $\alpha \in \mathbb{R}^+$.

Definition 3 A trading strategy $\varphi = (X, Y)$ is called locally risk-minimizing (LRM) under illiquidity if for any time $k \in 0, 1, ..., T-1$

$$T_k^{\alpha}(\varphi) \leq T_k^{\alpha}(\varphi') \quad \text{P-a.s. for any local perturbation at time k.} \tag{3.23}$$

Definition 3 takes into account the liquidity costs only at the current time, since the risk is only minimized locally. This is well-defined since the cost process is square-integrable and the liquidity costs are integrable.

The α represent the traders sentiment towards liquidity risk. $\alpha > 1$ represent a severe risk aversion towards liquidity risk and $\alpha < 1$ means a severe risk aversion to the risk of miss-hedging.

An important property of the local risk-minimizing strategy is that the cost process is a martingale.

Lemma 1 For a LRM-strategy under illiquidity, the cost process $C(\varphi)$ is a martingale. The martingale property of the cost process give us the representation,

$$R_k(\varphi) = E[R_{k+1}(\varphi)|\mathcal{F}_k] + Var(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) \quad \text{P-a.s. for} \quad k = 0, 1, ..., T-1. \tag{3.24}$$

Proof:

Fix a date $k \in [0, 1, ..., T-1]$ and define a pair $\varphi' = (X', Y')$ by letting $X' = X$, $Y'_j = Y_j$ for $j \neq k$ and

$$Y'_k = E[C_T(\varphi) - C_k(\varphi)|\mathcal{F}_k] + Y_k. \tag{3.25}$$

Then $Y^{'}_k$ is adapted since both of its terms are adapted to the filtration $\mathbb{F}$. Additionally we have

$$V_k(\varphi') = V_k(\varphi) + E[C_T(\varphi) - C_k(\varphi)|\mathcal{F}_k]. \tag{3.26}$$

which adapted to $\mathbb{F}$ and so φ' is a strategy. Indeed from $V_k(\varphi')$ we have that under a local perturbation of φ at date k we have

$$C_T(\varphi') - C_k(\varphi') = C_T(\varphi) - C_k(\varphi) - E[C_T(\varphi) - C_k(\varphi)|\mathcal{F}_k], \tag{3.27}$$

and

$$\begin{aligned}R_k(\varphi') =&E[(C_T(\varphi) - C_k(\varphi) - E[C_T(\varphi) - C_k(\varphi)|\mathcal{F}_k])^2|\mathcal{F}_k] \\ =&Var[C_T(\varphi) - C_k(\varphi)|\mathcal{F}_k] \leq E[(C_T(\varphi) - C_k(\varphi))^2|\mathcal{F}_k] \\ =&R_k(\varphi).\end{aligned} \tag{3.28}$$

However, we assumed that φ is a locally risk-minimizing, so we must have $R_k(\varphi') = R_k(\varphi)$ P-a.s. so

$$E[C_T(\varphi) - C_k(\varphi)|\mathcal{F}_k] = E[(C_T(\varphi) - C_k(\varphi))\mathbb{I}_{\mathcal{F}_k}] = 0 \quad \text{P-a.s.} \tag{3.29}$$

Then the cost process must be a martingale.

Lemma 2 If $C(\varphi)$ is a martingale and φ' is a local perturbation of φ at time k then

$$\begin{aligned}T_k(\varphi') =&E[R_{k+1}(\varphi)|\mathcal{F}_k] + E[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k] \\ &+ \alpha E[(X_{k+2} - X'_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k].\end{aligned} \tag{3.30}$$

Proof:

From Lemma 1 we have that since $C(\varphi)$ is a martingale and

$$R_k(\varphi') = E[R_{k+1}(\varphi)|\mathcal{F}_k] + E[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k], \tag{3.31}$$

where $\Delta C_{k+1}(\varphi') = E[C_T(\varphi') - C_k(\varphi')]$.

In addition since φ' is a local perturbation and $R_k(\varphi) = R_k(\varphi')$ then

$$\begin{aligned}&E[(X'_{k+2} - X'_{k+1})[S_{k+1}(X'_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k] \\ &= E[(X_{k+2} - X'_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k],\end{aligned} \tag{3.32}$$

which completes the proof.

Since $R_{k+1}(\varphi) = R_{k+1}(\varphi')$ holds for any local perturbation at a time k, we then get from Lemma 2 that we can minimize over the expression

$$Var(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) + \alpha E[\Delta X_{k+2} - S_{k+1}(0)|\mathcal{F}_k] \quad \text{at time k.} \tag{3.33}$$

Proposition 1 A trading strategy $\varphi = (X, Y)$ is LRM under illiquidity if and only if the following two properties are satisfied:

1. $C(\varphi)$ is a martingale.
2. For each $k \in [0, 1, ..., T-1]$, X_{k+1} minimizes:

   $$Var(V_{k+1}(\varphi) - (X'_{k+1})\Delta S_{k+1}|\mathcal{F}_k) + \alpha E[(X_{k+2} - X'_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k], \tag{3.34}$$

   over all $\mathcal{F}_k$-measurable random variables X'_{k+1} so that $X'_{k+1}\Delta S_{k+1} \in \mathbb{L}^{1,2}_T$ and $(X_{k+2} - X'_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)] \in \mathbb{L}^1_T$.

Proof:

First we prove that $\varphi = (X, Y)$ is a LRM-strategy under illiquidity. To do this we need to show that $T_k(\varphi) \leq T_k(\varphi')$ P-a.s. for any time $k \in (0, 1, .., T-1)$ and for any local perturbation φ' of φ at time k.

We know that $C(\varphi)$ is a martingale and φ' is a local perturbation of φ at time k then we know from Lemma 2 that

$$\begin{aligned} T_k(\varphi') =& E[R_{k+1}(\varphi)|\mathcal{F}_k] + E[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k] \\ &+ \alpha E[(X_{k+2} - X'_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k]. \end{aligned} \tag{3.35}$$

From the definition of conditional variance we know

$$E[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k] \geq Var(\Delta C_{k+1}(\varphi')|\mathcal{F}_k), \tag{3.36}$$

and so we can evaluate

$$\begin{aligned} T_k(\varphi') \geq& E[R_{k+1}(\varphi)|\mathcal{F}_k] + Var(\Delta C_{k+1}(\varphi')|\mathcal{F}_k) \\ &+ \alpha E[(X_{k+2} - X'_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k]. \end{aligned} \tag{3.37}$$

For φ' then $X'_{k+2} = X_{k+2}$ and $Y'_{k+1} = Y_{k+1}$ and so

$$\begin{aligned} Var(\Delta C_{k+1}(\varphi')|\mathcal{F}_k) =& Var(C_{k+1}(\varphi')|\mathcal{F}_k) \\ =& Var(V_{k+1}(\varphi') - (X'_{k+1})\Delta S_{k+1}|\mathcal{F}_k) \\ =& Var(V_{k+1}(\varphi) - (X'_{k+1})\Delta S_{k+1}|\mathcal{F}_k). \end{aligned} \tag{3.38}$$

which yields

$$\begin{aligned} T_k(\varphi') \geq& E[R_{k+1}(\varphi)|\mathcal{F}_k] + Var(V_{k+1}(\varphi) - (X'_{k+1})\Delta S_{k+1}|\mathcal{F}_k) \\ &+ \alpha E[(X_{k+2} - X'_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k], \end{aligned} \tag{3.39}$$

and in fact we have

$$\begin{aligned} T_k(\varphi') \geq& E[R_{k+1}(\varphi)|\mathcal{F}_k] + Var(V_{k+1}(\varphi) - (X_{k+1})\Delta S_{k+1}|\mathcal{F}_k) \\ &+ \alpha E[(X_{k+2} - X_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k]. \end{aligned} \tag{3.40}$$

However, we defined $T_k(\varphi)$ as

$$T_k(\varphi) = R_k(\varphi) + \alpha E[\Delta X_{k+2}[S_{k+1}(\Delta X_{k+2}) - S_{k+1}(0)]|\mathcal{F}_k]. \tag{3.41}$$

Since $C(\varphi)$ is a martingale we can use Lemma 2 to characterize $T_k^\alpha(\varphi)$ as

$$\begin{aligned} T_k(\varphi) =& E[R_{k+1}(\varphi)|\mathcal{F}_k] + Var(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) \\ &+ \alpha E[\Delta X_{k+2}[S_{k+1}(\Delta X_{k+2}) - S_{k+1}(0)]|\mathcal{F}_k]. \end{aligned} \tag{3.42}$$

Then $T_k(\varphi') \geq T_k(\varphi)$ and thus φ is a LRM-strategy under illiquidity.

Proving the opposite direction we start by assuming that φ is a LRM-strategy under illiquidity ($T_k(\varphi') \geq T_k(\varphi)$) for any perturbation φ' at time k. Property (1) is satisfied due to Lemma 1. To show Property (2) is satisfied we observe that since $C(\varphi)$ is martingale and φ' is a local perturbation we have that equation (3.30) in Lemma 2 holds and since $C(\varphi)$ is a martingale, equation (3.41) also holds and $T_k^\alpha(\varphi') \geq T_k^\alpha(\varphi)$ we have

$$\begin{aligned} T_k^\alpha(\varphi') =& E[R_{k+1}(\varphi)|\mathcal{F}_k] + E[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k] \\ &+ \alpha E[(X_{k+2} - X'_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k] \\ &\geq T_k^\alpha(\varphi) = E[R_{k+1}(\varphi)|\mathcal{F}_k] + Var(\Delta C_{k+1}(\varphi))|\mathcal{F}_k) \\ &+ \alpha E[(X_{k+2} - X_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k], \end{aligned} \tag{3.43}$$

using the definition of conditional variance we have

$$\begin{aligned}&Var(\Delta C_{k+1}(\vartheta')|\mathcal{F}_k) + (E[\Delta C_{k+1}(\vartheta')|\mathcal{F}_k])^2\\ &+ \alpha E[(X_{k+2} - X_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k]\\ &\geq Var(\Delta C_{k+1}(\vartheta)|\mathcal{F}_k)\\ &+ \alpha E[(X_{k+2} - X_{k+1})[S_{k+1}(X_{k+2} - X_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k].\end{aligned} \tag{3.44}$$

for any $\mathcal{F}_k$-measurable choice of Y'_k and X'_{k+1}. We can fix X'_{k+1} and pick Y'_k such that $E[\Delta C_{k+1}(\varphi')|\mathcal{F}_k] = 0$ and φ' is a local perturbation of φ at time k, we have the inequality

$$\begin{aligned}&Var(V_{k+1}(\varphi) - (X'_{k+1})\Delta S_{k+1}|\mathcal{F}_k)\\ &+ \alpha E[(X_{k+2} - X'_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k]\\ &\geq Var(V_{k+1}(\varphi) - (X_{k+1})\Delta S_{k+1}|\mathcal{F}_k)\\ &+ \alpha E[(X_{k+2} - X_{k+1})[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k],\end{aligned} \tag{3.45}$$

so condition (2) is satisfied and the proof is completed. See [Chr20].

Linear supply curve

The liquidity cost is related to the depth of the limit order book (LOB). In this scenario assume that the LOB's ability to recover itself is infinite, so that no feedback from the hedging strategy is taken into account. The supply curve $S_k(x) = (S_k^1(x^1), ..., S_k^d(x^d))$ is given by

$$S_k^j(x^j) = S_k^j + x^j \epsilon_k^j S_k^j \tag{3.46}$$

where x^j is again the illiquiditys size x (of the j-th asset) at time k. S is assumed to be a non-negative semimartingale price process and $\epsilon_k = (\epsilon_k)_{k=0,1,..,T}$ is a positive deterministic $\mathbb{R}^d$-valued process, s.t. the price process does not take on negative values. A 1-dimensional, time independent LOB can be described with a density function q, where $q(x)dx$ is the bid or ask offers at a given price level xS_k. In addition, $F(p) = \int_1^p q(x)dx$ is the quantity available at a price pS_k. So that an order of $x = F(p)$ shares at a time k shifts up the quoted price in the LOB to $S_k(x)^+ = g(x)S_k$ where $g(x)$ solves the equation $x = \int_1^{g(x)} q(y)dy$, hence $g(x) = F^{-1}(x)$. Since we in this scenario does not take into account price impact, the price returns to S_k. The cost of x shares is $S_k \int_1^{g(x)} pdF(p)$ which should be equal to $xS_k(x) = xS_k + \epsilon_k x^2 S_k$ for a suitable choice of q i.e depth of the order book.

Such a choice could be

$$q(x) = \frac{1}{2\epsilon_k} \tag{3.47}$$

which is independent from price. Notice that as ϵ_k tends to zero the liquidity cost vanishes. ϵ_k is thus a measure of illiquidity.

The aim is to construct an optimal strategy satisfying the LRM-criterion under illiquidity ie. at time k, minimize

$$Var(V_{k+1}(X,Y) - (X'_{k+1})\Delta S_{k+1}|\mathcal{F}_k) + \alpha E[\sum_{j=1}^{d} \epsilon_k S_{k+1}^j (X_{k+2}^j - (X'_{k+1})^j)^2|\mathcal{F}_k]. \tag{3.48}$$

for all X'_{k+1} with a chosen Y_k so that the cost process C becomes a martingale.

We can expand the above expression as:

$$
\begin{aligned}
&Var[V_{k+1}(X,Y)|\mathcal{F}_k] + Var[(X'_{k+1})\Delta S_{k+1}|\mathcal{F}_k] - 2Cov[V_{k+1}(X,Y),(X'_{k+1})\Delta S_{k+1}|\mathcal{F}_k] \\
&+ \alpha E[\sum_{j=1}^{d} \epsilon^j_{k+1} S^j_{k+1} (X^j_{k+2})^2 - 2(X^j_{k+1})((X'_{k+1})^j) + ((X'_{k+1})^j)^2|\mathcal{F}_k] \\
&= Var[V_{k+1}(X,Y)|\mathcal{F}_k] + |(X'_{k+1})|^2 Var[\Delta S_{k+1}|\mathcal{F}_k] \\
&- 2(X'_{k+1})Cov[V_{k+1}(X,Y),\Delta S_{k+1}|\mathcal{F}_k] + \alpha \sum_{j=1}^{d} |(X'_{k+1})^j|^2 E[\epsilon^j_{k+1} S^j_{k+1}|\mathcal{F}_k] \\
&- 2\alpha \sum_{j=1}^{d} E[\epsilon^j_{k+1} S^j_{k+1}(X'_{k+1})^j X^j_{k+2}|\mathcal{F}_k] + \alpha \sum_{j=1}^{d} E[\epsilon^j_{k+1} S^j_{k+1}|X^j_{k+2}|^2|\mathcal{F}_k],
\end{aligned}
\tag{3.49}
$$

where we used the following relation $Var[X-Y] = Var[X] + Var[Y] - 2Cov[X,Y]$.

We introduce the following notation:

$$A^0_{k;j} = Var(\Delta S^j_{k+1}|\mathcal{F}_k), \quad A^{\epsilon}_{k;j} = E[\epsilon^j_{k+1} S^j_{k+1}|\mathcal{F}_k], \quad A_{k;j} = A^0_{k;j} + A^{\epsilon_k}_{k;j},$$

$$b^0_{k;j} = Cov(V_{k+1}\Delta S^j_{k+1}|\mathcal{F}_k), \quad b^{\epsilon}_{k;j} = E[\epsilon^j_{k+1} S^j_{k+1} X^j_{k+2}|\mathcal{F}_k], \quad b_{k;j} = b^0_{k;j} + b^0_{k;j}$$

$$D_{k;j,i} = Cov(\Delta S^j_{k+1}\Delta S^i_{k+1}|\mathcal{F}_k)$$

for $i \neq j$, for all $i,j = 1,...,d$ and $k = 0,...,T-1$

For simplicity assume $\alpha = 1$ then (3.49) can be rewritten by defining the function $F_k : \mathbb{R}^d \times \Omega \to \mathbb{R}^+$ as

$$
\begin{aligned}
F_k(c,w) = &\sum_{j=1}^{d} |c_j|^2 A_{k;j}(w) - 2\sum_{j=1}^{d} c_j b_{k;j}(w) + \sum_{j\neq i} c_j c_i D_{k;j,i}(w) \\
&+ Var(V_{k+1}|\mathcal{F}_k)(w) + \sum_{j=1}^{d} E[\epsilon^j_k S^j_{k+1}|X^j_{k+2}|^2|\mathcal{F}_k](w).
\end{aligned}
\tag{3.50}
$$

For a fixed w we can find the gradient of the function F_k. The extreme points, found by solving $grad(F_k) = 0$, will be the solution to the linear system of equations $F_k c = b_k$

Where $F_k \in \mathcal{R}^{d\times d}$ with $F_{k;i,j} = D_{k;i,j}$ for $i \neq j$, $F_{k;i,j} = A_{k;j}$ for $i = j$ and $b_k = (b_{k;1},...,b_{k;d}) \in \mathcal{R}^d$. See [Chr20].

3.3 Schweizer's bounded mean-variance and substantial risk conditions

Schweizer et. al [LPS98] showed that in a discrete-time non-complete market we can impose appropriate bounds such that a locally risk-minimizing strategy with illiquidity cost exists for every square-integrable contingent claim. Such that one can hedge a contingent claim using square-integrability rather than other more stringent conditions such as convexity or concavity etc.

Another important result is that one can create a fictitious asset price process which lies between the bid and ask price processes and if we hedge our contingent claim by a LRM strategy without illiquidity costs in this fictitious model, we obtain the exact same LRM strategy with illiquidity costs in the original model with bid-ask spread. This adds a robustness to the LRM criterion described earlier.

Consider now the presence of a bid-ask spread in a market due to illiquidity costs. Such that for some fixed illiquidity parameter $\epsilon_k \in [0,1)$ the bid and ask prices will be $(1-\epsilon_k)S_k$ and $(1+\epsilon_k)S_k$ respectively for one share of a stock at date k.

Definition 4 Let Γ be the class of all adapted processes $x = (x_k)_{k=0,1,...,T}$ with values in $[-1,+1]$. For $x \in \Gamma$, the linear supply curve S^x is defined by $S^x_k = S_k + \epsilon_k x_k S_k$ for $k = 0,1,...,T$.

if $\varphi = (X,Y)$ is a strategy, the process $V^x(\varphi)$ is defined by

$$V^x_k(\varphi) = X_{k+1}S^x_k + Y_k \quad \text{k=0,1,...,T.} \tag{3.51}$$

The process X^x is non-negative, since $\epsilon_k \in [0,1]$, adapted and assumed to be square-integrable.

Jouini/Kallal [EH95] showed that the bid ask processes $S^- = (1-\epsilon_k)S$ and $S^+ = (1+\epsilon_k)S$ is an arbitrage-free system of bid and ask prices if and only if there exists a process $x \in \Gamma$ and a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that S^x is a $\mathbb{Q}-$martingale. $\mathbb{Q}$ is then a price system which is compatible with both the bid and the ask price process and S^x becomes a re-valuation of the stock, while $V^x(\varphi)$ is the value process of the strategy φ in terms of S^x.

The objective with the rest of this section is to impose suitable bounds on S and ϵ_k such that $\Theta(S^x) = \Theta(S)$ for all $x \in \Gamma$. Meaning that one strategy can be used for all reasonable choices of units.

Proposition 2 A strategy $\varphi = (X,Y)$ is LRM if and only if it has the following two properties:

1. $C(\varphi)$ is a martingale.
2. For each $k \in [0,1,...,T-1]$, X_{k+1} minimizes

$$Var[V_{k+1}(\varphi) - X'_{k+1}\Delta S_{k+1} + \epsilon S_{k+1}|X_{k+2} - X'_{k+1}||\mathcal{F}_k]$$

   over all $\mathcal{F}_k-$measurable random variables X'_{k+1} such that $X'_{k+1}\Delta S_{k+1} \in L^2(P)$ and $X'_{k+1}S_{k+1} \in L^2(P)$.

which is essentially the same definition used in (3.34) by Christodoulou, except that the investors risk appetite α is exchanged with the illiquidity risk ϵ which now depends on a variance term instead. The proof follows by similar reasoning as in Proposition 1 in the previous section, and is therefor omitted.

Definition 5 S has substantial risk if there exist a constant $c \leq \infty$ such that

$$\frac{S^2_{k-1}}{E[\Delta S^2_k|\mathcal{F}_{k-1}]} \leq c \quad \text{P-a.s. for} \quad k = 1,...,T. \tag{3.52}$$

The smallest constant satisfying (3.52) we call c_{SR}

This condition provides a lower bound on the conditional variance of increments of S.

Before moving further we establish which spaces we are working in when S has substantial risk.

Lemma 5 Assume S has substantial risk. Then:

1. $\Theta(S^x) \supseteq \Theta(S)$ for every $x \in \Gamma$, where $\Theta(S)$ is the space of all predictable processes $X = (X_k)_{k=1,..,T+1}$ such that $X_k \Delta S_k \in L^2(P)$ for $k = 1, ..., T$ and Γ is the class of all adapted processes $x = (x_k)_{k=0,1,...,T}$ in $[-1, +1]$.
2. $V_k^x(\varphi) \in L^2(P)$ for $k = 0, 1, ..., T$, for every $x \in \Gamma$ and for every strategy φ.
3. $X_{k+1} S_k \in L^2(P)$ for $k = 0, 1, ..., T$ for every $X \in \Theta(X)$.
4. $C_k(\varphi) \in L^2(P)$ for $k = 0, 1, ..., T$ and for every strategy φ.

Proof:

We have that

$$X_k \Delta S_k^x = X_k \Delta S_k + \epsilon_k x_k X_k \Delta S_k + \epsilon_k X_k S_{k-1} \Delta x_k, \tag{3.53}$$

and each x is bounded by 1, 1) then follows from 3) since each $X_k \Delta S_k \in L^2(P)$ in $\Theta(S^x)$ is also in $\Theta(S)$.

We also have that

$$V_k^x(\varphi) = V_k(\varphi) + \epsilon_k x_k X_{k+1} S_k, \tag{3.54}$$

and $V_k(\varphi) = X_{k+1} S_k + Y_k \in L^2(P)$ for $k = 0, 1, ..., T$. so 2) follows from 3).

We further have a useful relation

$$\begin{aligned} \Delta C_k(\varphi) &= \Delta V_k(\varphi) - X_k \Delta S_k + \epsilon_k S_k |\Delta X_{k+1}| \\ &= X_{k+1} S_k^x + Y_k - X_k S_k^x - Y_{k-1} \\ &= \Delta V_k^x(\varphi) - X_k \Delta S_k^x. \end{aligned} \tag{3.55}$$

$X \in \Gamma$ and x is predictable so (3.55) and $V_k(\vartheta) = X_{k+1} S_k + Y_k \in L^2(P)$ for $k = 0, 1, ..., T$ makes 4) follow from 2) and 1).

To prove 3) we observe that

$$E[(X_{k+1} S_k)^2] = E[(X_{k+1} \Delta S_{k+1})^2 \frac{S_k^2}{E[\Delta S_{k+1}^2 | \mathcal{F}_k]}] \le c_{SR} E[(X_{k+1} \Delta S_{k+1})^2] < \infty \tag{3.56}$$

since $X \in \Theta(S)$ and S has substantial risk so 3) follows. Which completes the proof.

To prove that $\Theta(S^x) \subseteq \Theta(S)$ we look at the mean-variance tradeoff process of S^x

Definition 6 S has bounded mean-variance tradeoff process if for some constant $C > 0$

$$\frac{(E[\Delta S_{k+1}^x | \mathcal{F}_k])^2}{Var(\Delta S_{k+1}^x | \mathcal{F}_k)} \le c \quad \text{P-a.s. for} \quad x = 1, ..., T. \tag{3.57}$$

Proposition 4 Assume S has bounded mean-variance tradeoff and substantial risk. For a fixed $x \in \Gamma$ and assume that there is a constant $c > 0$ such that

$$Var[\Delta S_k^x|\mathcal{F}_{k-1}] \geq cVar[\Delta S_k|\mathcal{F}_{k-1}] \quad \text{P-a.s. for} \quad k = 1, ..., T, \tag{3.58}$$

then S^j has bounded mean-variance tradeoff and $\Theta(S^x) = \Theta(S)$

Proof: We first show that (3.58) implies that S^j has bounded mean-variance trade off. From (3.57) we know that this will be the case if

$$(E[\Delta S_k^x|\mathcal{F}_{k-1}])^2 \leq const.Var[\Delta S_k|\mathcal{F}_{k-1}] \quad \text{P-a.s. for} \quad k = 1, ..., T. \tag{3.59}$$

Let $c_{MVT}(0)$ be the lowest possible value satisfying the inequality.

We also have that

$$\Delta S_k^x = \Delta S_k + x^j \epsilon_k S_k - \epsilon_k x_{k-1}^j S_{k-1}^x = \Delta S_k + x^j \epsilon_k^j \Delta S_k^x + \epsilon_k \Delta x^j S_{k-1}^x \tag{3.60}$$

and

$$\begin{aligned} E[(\Delta S_k^x)|\mathcal{F}_{k-1}]^2 &\leq 2(1+\epsilon_k)^2 E[\Delta S_k^2|\mathcal{F}_{k-1}] + 8\epsilon_k^2 S_{k-1}^2 \\ &\leq const.E[\Delta S_k|\mathcal{F}_{k-1}]^2 \leq const.(1 + c_{MVT}(0))Var[\Delta S_k|\mathcal{F}_{k-1}], \end{aligned} \tag{3.61}$$

where we used that ϵ_k is bounded by 1 and $c_{MVT}(0) = \frac{E[(\Delta S_k)^2|\mathcal{F}_{k-1}]^2}{Var[\Delta S_k|\mathcal{F}_{k-1}]}$, and then (3.52) along with (3.57).

We have from lemma 5 that $\Theta(S^x) \supseteq \Theta(S)$ so we need only show that $\Theta(S^x) \subseteq \Theta(S)$.

First let $S^x = S_0^x + M^x + A^x$ be the Doob decomposition of S^x then

$$X_k \Delta S_k^x = X_k \Delta M_k^x + X_k \Delta A_k^x = X_k \Delta M_k^x + X_k E[\Delta S_k^x|\mathcal{F}_{k-1}], \tag{3.62}$$

and

$$Var[\Delta S_k^x|\mathcal{F}_{k-1}] = E[(\Delta M_k^x)^2|\mathcal{F}_{k-1}]. \tag{3.63}$$

S^x has bounded mean-variance tradeoff, (3.57) gives that $X \in \Theta(S^x)$ if and only if $X_k \Delta M_k^x \in L^2(P)$ for $k = 1, ..., T$ which will be written as $X \in L^2(M^x)$ The same is true for $S = S^0$. When X is predictable and (3.58) holds, then

$$\begin{aligned} E[(X_k \Delta M_k)^2|\mathcal{F}_{k-1})] &= X_k^2 Var[\Delta S_k|\mathcal{F}_{k-1}] \\ &\leq \frac{1}{c} X_k^2 Var[\Delta S_k^x|\mathcal{F}_{k-1}] = \frac{1}{c} E[(X_k \Delta M_k^x)^2|\mathcal{F}_{k-1}], \end{aligned} \tag{3.64}$$

which means that $L^2(M^x) \subseteq L^2(M)$, then $\Theta(S^x) \subseteq \Theta(S)$ since both have mean-variance tradeoffs which are bounded. This completes the proof.

If one knows x Proposition 4 gives an estimate of (3.58), but we need to impose additional conditions on S and x such that (3.58) holds uniformly over all $x \in \Gamma$.

Proposition 3 If there is a constant $\delta < 1$ such that

$$2\epsilon_k \sqrt{\frac{E[S_k^2|\mathcal{F}_{k-1}]}{Var[\Delta S_k|\mathcal{F}_{k-1}]}} \leq \delta \quad \text{P-a.s. for} \quad k = 1, ..., T. \tag{3.65}$$

Then (3.58) holds simultaneously for all $x \in \Gamma$, with $c = 1 - \delta$. Specifically, (3.65) holds if S has bounded mean-variance tradeoff and substantial risk and if ϵ_k satisfies

$$4\epsilon_k^2(1 + 2c_{MVT}(0) + 2c_{SR}(1 + c_{MVT}(0)) < 1. \tag{3.66}$$

Proof: We know that $S_k^x = S_k(1 + x_k\epsilon_k)$ and leaving aside the $\mathcal{F}_{k-1}$-measurable terms from the conditional variance yields

$$\begin{aligned} Var[\Delta S_k^x|\mathcal{F}_{k-1}] &= Var[\Delta S_k + x_k\epsilon_k S_k|\mathcal{F}_{k-1}] \\ &= Var[\Delta S_k|\mathcal{F}_{k-1}] + Var[x_k\epsilon_k S_k|\mathcal{F}_{k-1}] + 2Cov[\Delta S_k, x_k\epsilon_k S_k|\mathcal{F}_{k-1}] \\ &\geq Var[\Delta S_k|\mathcal{F}_{k-1}] - 2\epsilon_k\sqrt{Var[\Delta S_k|\mathcal{F}_{k-1}]Var[x_k S_k|\mathcal{F}_{k-1}]}, \end{aligned} \tag{3.67}$$

where we used the Cauchy-Schwarz inequality ie. $|Cov(X,Y)|^2 \leq Var(X)Var(Y) \rightarrow |Cov(X,Y)| \geq -\sqrt{Var(X)Var(Y)}$ since the variance is non-negative.

When we assume that x_k is bounded by 1, (3.65) gives us

$$\begin{aligned} Var[x_k S_k|\mathcal{F}_{k-1}] &= E[(x_k S_k)^2|\mathcal{F}_{k-1}] - E[x_k S_k|\mathcal{F}_{k-1}]^2 \\ &\leq E[x_k^2 S_k^2|\mathcal{F}_{k-1}] \leq \frac{\delta^2}{4\epsilon_k^2} Var[\Delta S_k|\mathcal{F}_{k-1}]. \end{aligned} \tag{3.68}$$

Then (3.58) with $c = 1 - \delta$. We get (3.65) from (3.66) when we take

$$E[X_k^2|\mathcal{F}_{k-1}] = Var[\Delta X_k|\mathcal{F}_{k-1}] + (X_{k-1} + E[\Delta X_k|\mathcal{F}_{k-1}])^2 \tag{3.69}$$

and use the following estimate from (3.52)

$$X_{k-1}^2 \leq c_{SR} E[\Delta X_k^2|\mathcal{F}_{k-1}] \tag{3.70}$$

and (3.57)

$$E[\Delta X_k^2|\mathcal{F}_{k-1}] \leq Var[\Delta X_k|\mathcal{F}_{k-1}](1 + c_{MVT}(0)). \tag{3.71}$$

Then the statement follows from these inequalities.

Condition (3.65) informally states that illiquidity costs have to be small enough for the next theorems to hold.

The explicit calculation is as follows: we must show that

$$\frac{E[S_k^2|\mathcal{F}_{k-1}]}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \leq (1 + 2c_{MVT}(0) + 2c_{SR}(1 + c_{MVT}(0))), \tag{3.72}$$

for the inequality to hold.

We have

$$\begin{aligned} \frac{E[S_k^2|\mathcal{F}_{k-1}]}{Var[\Delta S_k|\mathcal{F}_{k-1}]} &\leq \frac{Var[\Delta S_k|\mathcal{F}_{k-1}] + (S_{k-1} + E[\Delta S_k|\mathcal{F}_{k-1}])^2}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \\ &\leq \frac{Var[\Delta S_k|\mathcal{F}_{k-1}] + 2S_{k-1}^2 + 2E[\Delta S_k|\mathcal{F}_{k-1}]^2}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \\ &\leq \frac{Var[\Delta S_k|\mathcal{F}_{k-1}] + 2c_{SR}Var[\Delta S_k|\mathcal{F}_{k-1}](1 + c_{MVT}(0)) + 2E[\Delta S_k|\mathcal{F}_{k-1}]^2}{Var[\Delta S_k|\mathcal{F}_{k-1}]}, \end{aligned} \tag{3.73}$$

and

$$\begin{aligned} Var[\Delta S_k^x|\mathcal{F}_{k-1}] &\geq Var[\Delta S_k|\mathcal{F}_{k-1}] - 2\epsilon_k \sqrt{Var[\Delta S_k|\mathcal{F}_{k-1}]Var[x_k S_k|\mathcal{F}_{k-1}]} \\ &= E[\Delta S_k^2|\mathcal{F}_{k-1}] - E[\Delta S_k|\mathcal{F}_{k-1}]^2 - 2\epsilon_k \sqrt{Var[\Delta S_k|\mathcal{F}_{k-1}]Var[x_k S_k|\mathcal{F}_{k-1}]}, \end{aligned} \tag{3.74}$$

then use

$$Var[\Delta S_k^x|\mathcal{F}_{k-1}] \leq (1-\delta)Var[\Delta S_k|\mathcal{F}_{k-1}], \tag{3.75}$$

such that

$$\begin{aligned} (1-\delta)Var[\Delta S_k|\mathcal{F}_{k-1}] &\geq E[\Delta S_k^2] - E[\Delta S_k|\mathcal{F}_{k-1}]^2 - \delta Var[\Delta S_k|\mathcal{F}_{k-1}] \\ &\geq Var[\Delta S_k|\mathcal{F}_{k-1}](1 + c_{MVT}(0)) - E[\Delta S_k|\mathcal{F}_{k-1}]^2 - \delta Var[\Delta S_k|\mathcal{F}_{k-1}]. \end{aligned} \tag{3.76}$$

Then

$$E[\Delta S_k|\mathcal{F}_{k-1}]^2 \geq c_{MVT}(0)Var[\Delta S_k|\mathcal{F}_{k-1}], \tag{3.77}$$

and finally

$$\begin{aligned} \frac{E[S_k^2|\mathcal{F}_{k-1}]}{Var[\Delta S_k|\mathcal{F}_{k-1}]} &\leq \frac{Var[\Delta S_k|\mathcal{F}_{k-1}] + 2c_{SR}Var[\Delta S_k|\mathcal{F}_{k-1}](1 + c_{MVT}(0)) + 2E[\Delta S_k|\mathcal{F}_{k-1}]^2}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \\ &\leq \frac{Var[\Delta S_k|\mathcal{F}_{k-1}] + 2c_{SR}Var[\Delta S_k|\mathcal{F}_{k-1}](1 + c_{MVT}(0)) + 2Var[\Delta S_k|\mathcal{F}_{k-1}]c_{MVT}(0)}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \\ &= (1 + 2c_{MVT}(0) + 2c_{SR}(1 + c_{MVT}(0))), \end{aligned} \tag{3.78}$$

which yields the desired inequality. See [LPS98].

3.4 Solving for the optimal strategy

In this section we prove the existence of a LRM strategy under illiquidity cost. The method for finding the optimal strategy is to solve for every k in the conditional variance minimization problem in (2) of Proposition 2 for the optimal X value. For this section we assume that S is a square-integrable process.

We define

$$V_k^x(\varphi) = X_{k+1}S_k^x + Y_k = V_k(\varphi) + \epsilon_k x X_{k+1} S_k. \tag{3.79}$$

Theorem 1 Let S have bounded mean-variance tradeoff, substantial risk, satisfy (118) and

$$Var[\Delta X_k|\mathcal{F}_{k-1}] > 0 \quad \text{P-a.s. for} \quad k = 1, ..., T. \tag{3.80}$$

Then for any contingent claim $(\overline{X}_{T+1}, \overline{Y}_T)$ there exist a LRM strategy $\varphi^* = (X^*, Y^*)$ where $X^*_{T+1} = \overline{X}_{T+1}$ and $Y^*_T = \overline{Y}_T$.

Where X^* can be characterized as follows: There exists a process $\delta^* \in \Gamma$ such that we can define $v \in \Gamma$ by

$$v_k = sign(X^*_{k+1} - X^*_k) + \delta^*_k I_{X^*_{k+1} = X^*_k} \quad \text{for} \quad k = 1, ..., T, 3.1 \tag{3.81}$$

and

$$V^v_k(\varphi^*) = V_k(\varphi) + \epsilon_k(sign(X^*_{k+1} - X^*_k) + \delta^*_k I_{X^*_{k+1} = X^*_k})X_{k+1}S_k \tag{3.82}$$

then

$$X^*_k = \frac{Cov(\Delta V^v_k(\varphi^*), \Delta S^v | \mathcal{F}_{k-1})}{Var[\Delta S^v | \mathcal{F}_{k-1}]} \quad \text{P-a.s. for} \quad k = 1, ..., T. \tag{3.83}$$

We can view S^v as a suitable strategy within the bid-ask range.

Proof:

The proof relies on backward induction to prove the existence of a predictable process X^* with $X^*_{T+1} = \overline{X}_{T+1}$ and satisfying assertions 1), 2) below for $k = 0, 1, ..., T$ and 3), 4), 5) for $k = 1, ..., T$:

1. $X^*_{k+1} S_k \in L^2(P)$.
2. $W^*_k = H - \sum_{j=k+1}^T X^*_j \Delta S_j + \sum_{j=k+1}^T \epsilon_{kj} S_j |\Delta X^*_{j+1}| \in L^2(P)$.
3. There exists an $\mathcal{F}_{k-1}-$measurable random variable X^*_k with values in $[-1, +1]$ such that if we define v_k by (3.82), then we have

$$X^*_k = \frac{Cov(E[W^*_k | \mathcal{F}_k] + \epsilon_k v_k S_k X^*_{k+1}, S_k(1 + \epsilon_k v_k) | \mathcal{F}_{k-1})}{Var[S_k(1 + \epsilon_k v_k | \mathcal{F}_{k-1}]} \quad \text{P-a.s.}$$

4. $X^*_k \Delta S_k \in L^2(P)$.
5. X^*_k minimizes $Var[E[W^*_k | \mathcal{F}_k] - X_k \Delta S_k + \epsilon_k S_k |X^*_{k+1} - X_k| | \mathcal{F}_{k-1}]$ for all $\mathcal{F}_{k-1}$

When these are established we can define Y^* by

$$Y^*_k = E[W^*_k | \mathcal{F}_k] - X^*_{k+1} S_k \quad \text{for} \quad k = 0, 1, ..., T. \tag{3.84}$$

Then Y^* is adapted and $X^*_{k+1} S_k + Y^*_k \in L^2(P)$ by 2). Using 4), $\varphi^* = (X^*, Y^*)$ does satisfy $X^*_{T+1} = \overline{X}_{T+1}$ and $Y^*_T = \overline{Y}_T$. From the definitions of Y^* and W^*_k, $V_k(\varphi^*) = E[W^*_k | \mathcal{F}_k]$ for all k and $C(\varphi^*)$ is a martingale. Then using 5) and Proposition 1 we have that φ^* is LRM. Indeed, since Y^* suggests that

$$E[W^*_k | \mathcal{F}_k] + \epsilon_k v_k S_k X^*_{k+1} = V^v_k(\varphi^*),$$

Then (3.84) is just a restatement of X^*_k in 3). In order to complete the proof we only need to establish 1)- 5). For that purpose define $X^*_{T+1} = \overline{X}_{T+1}$, then if we assume that $\overline{X}_{T+1} S_T \in L^2(P)$ and $H = \overline{X}_{T+1} S_T + \overline{Y}_T \in L^2(P)$ then 1) and 2) holds for $k = T$. We now want to show that when 1) and 2) holds for any k then there exist an $\mathcal{F}_{k-1}-$measurable random variable X^*_k satisfying 3)-5) for k, which in turn imply the validity of 1) and 2) for $k - 1$.

Assume that 1) and 2) hold for k. Let

$$\begin{aligned}
\overline{sign}(x) &= sign(x) + \mathbf{1}_{x=0} = \begin{cases} +1 & \text{for} \quad x \geq 0 \\ -1 & \text{for} \quad x < 0 \end{cases}, \\
\underline{sign}(x) &= sign(x) - \mathbf{1}_{x=0} = \begin{cases} +1 & \text{for} \quad x > 0 \\ -1 & \text{for} \quad x \leq 0 \end{cases},
\end{aligned} \tag{3.85}$$

and define the function

$$f_k(c, \omega) = Var[E[W_k^*|\mathcal{F}_k] - cS_k + \epsilon_k S_k |X_{k+1}^* - c||\mathcal{F}_{k-1}](\omega) \tag{3.86}$$

and

$$\begin{aligned}
g_k(c, \alpha, \omega) = &Cov(E[W_k^*|\mathcal{F}_k] + \epsilon_k S_k X_{k+1}^* G_k^{(\alpha,c)}, S_k(1 + \epsilon_k G_k^{(\alpha,c)})|\mathcal{F}_{k-1})(\omega) \\
&- cVar[S_k(1 + \epsilon_k G_k^{(\alpha,c)})|\mathcal{F}_{k-1})(\omega)
\end{aligned} \tag{3.87}$$

where

$$G_k^{(\alpha,c)}) = \alpha\overline{sign}(X_{k+1}^* - c) + (1 - \alpha)\underline{sign}(X_{k+1}^* - c), \tag{3.88}$$

where the conditional variance and covariance are calculated with respect to the distribution of $(E[W_k^*|\mathcal{F}_k], S_k, X_{k+1}^*)$ given $\mathcal{F}_{k-1}$ Schweizer et al. [LPS98] showed and we will take for granted that we can obtain the existence of an $\mathcal{F}_{k-1}-$measurable random variable X_k^* and an $\mathcal{F}_{k-1}-$measurable random variable α_k^* with values in $[0, 1]$ such that

$$f_k(X_k^*(\omega), \omega) \leq f_k(c, \omega) \quad \text{P-a.s. for all c} \tag{3.89}$$

and

$$g_k(X_k^*(\omega), \alpha_k^*(\omega), \omega) = 0 \quad \text{P-a.s.} \tag{3.90}$$

If we then define $\delta_k^* = 2\alpha_k^* - 1$, then we have

$$G_k^{(\alpha^*, X_k^*)} = sign(X_{k+1}^* - X_k^*) + \delta_k^* \mathbf{1}_{X_{k+1}^* = X_k^*} = v_k \tag{3.91}$$

with all that in place we can get X_k^* from 3) by rewriting (3.90) such that 3) holds for k.

Next we prove that 4) hold for k. Let x be any process in Γ with $x_k = v_k$ and define

$$W_k^x = E[W_k^*|\mathcal{F}_k] + \epsilon_k v_k S_k X_{k+1}^*. \tag{3.92}$$

By 1) and 2) for k, $W_k^x \in L^2(P)$, we can write X_k^* in 3) as

$$X_k^* = \frac{Cov(W_k^x, \Delta S_k^x|\mathcal{F}_{k-1})}{Var[\Delta S^x|\mathcal{F}_{k-1}]} \tag{3.93}$$

because $\mathcal{F}_{k-1}-$measurable terms do matter for the conditional variance and covariance.

The Cauchy-Schwarz inequality and Proposition 5 imply that

$$\begin{aligned} E[(X_k^*\Delta S_k)^2] &\leq E[\frac{Var[W_k^x|\mathcal{F}_{k-1}]}{Var[\Delta S_k^x|\mathcal{F}_{k-1}]}E[\Delta S_k^2|\mathcal{F}_{k-1}]] \\ &\geq \frac{1}{c}E[E[(W_k^x)^2|F_{k-1}]\frac{Var[\Delta S_k^2|\mathcal{F}_{k-1}]}{Var[\Delta S_k|\mathcal{F}_{k-1}]}] \\ &\geq \frac{1}{c}(1+c_{MVT}(0))E[(W_k^x)^2] < \infty \end{aligned} \tag{3.94}$$

then 4) holds for k. Because S has substantial risk, we can conclude as in Lemma 3 that $X_k^* S_{k-1} \in L^2(P)$ which establishes 1) for $k-1$. While at the same time we have $X_k^* S_k = X_k^* \Delta S_k + X_k^* S_{k-1} \in L^2(P)$ as required in 5).

That 5) holds for k is then evident. In fact, if X_k is $\mathcal{F}_{k-1}-$measurable and satisfies $X_k \Delta S_k \in L^2(P)$ and $X_k S_k \in L^2(P)$, then we have

$$Var[E[W_k^*|\mathcal{F}_k] - X_k\Delta X_k + \epsilon_k S_k|X_{k-1}^* - X_k||\mathcal{F}_{k-1}](\omega) = f_k(X_k(\omega),\omega) \quad \text{P-a.s.} \tag{3.95}$$

and so 5) for k follows from (3.89). Finally,

$$W_{k-1}^* = W_k^* - X_k^*\Delta S_k + \epsilon_k S_k|X_{k+1}^* - X_k^*| \in L^2(P) \tag{3.96}$$

due to 2) for k, 4) for k, 1) for k and the square-integrability of $X_k^* S_k$. Then b) holds for $k-1$, and the induction along with the proof is complete.

Theorem 2 Assume Theorem 1 holds. The strategy φ^* which is LRM for a price process S with illiquidity costs, is also a strategy which is LRM for a price process S^v without illiquidity costs, where v is given by (3.81).

Proof:

We first want to show that under no illiquidity costs we have that the strategy described in Theorem 1, namely $\varphi^* = (X^*, Y^*)$ has the same value process, cost process and risky stock amount X as a new strategy $\overline{\varphi}$ which will be defined below. With this result in hand we can prove the statement of Theorem 2.

Consider when $\epsilon_k = 0$ so there are no illiquidity costs. Recall that the value and cost process of a strategy $\varphi = (X, Y)$ are defined as

$$\overline{V}_k(\varphi) = X_k S_k + Y_k \quad \text{for} \quad k = 0, 1, ..., T, \tag{3.97}$$

with $X_0 = 0$ and

$$\overline{C}_k(\varphi) = \overline{V}_k(\varphi) - \sum_{j=1}^{k} X_j \Delta S_j \quad \text{for} \quad k = 0, 1, ..., T, \tag{3.98}$$

and a contingent claim H which is a $\mathcal{F}_t$-measurable random variable in $L^2(P)$ space.

With these conditions S has bounded mean-variance tradeoff and is a LRM strategy $\overline{\varphi}$ for H. We can then characterize $\overline{V}_T(\overline{\varphi}) = H$ P-a.s., $\overline{C}(\overline{\varphi})$ is a martingale and so

$$\overline{X_k} = \frac{Cov(\Delta\overline{V}_k(\overline{\varphi}), \Delta S_k|\mathcal{F}_{k-1})}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \quad \text{P-a.s. for} \quad k = 0, 1, ..., T. \tag{3.99}$$

When $\epsilon_k = 0$, the φ^* strategy from theorem 1 yields $V(\varphi^*) = \overline{V}(\overline{\varphi})$ when $X^* = \overline{X}$. Then when $H = \overline{X}_{T+1}S_T + \overline{Y}_T$. It is then given that

$$V_T(\varphi^*) = H = \overline{V}_T(\overline{\varphi}) \quad \text{P-a.s.,} \tag{3.100}$$

and so

$$\overline{X_T} = \frac{Cov(H, \Delta S_T|\mathcal{F}_{t-1})}{Var[\Delta S_T|\mathcal{F}_{t-1}]} = X_T^* \quad \text{P-a.s.,} \tag{3.101}$$

and when $\epsilon_k = 0$ we have $S^v = S$.

Furthermore, When $V_{k+1}(\varphi^*) = \overline{V}_{k+1}(\overline{\varphi})$ and $X_{k+1}^* = \overline{X}_{k+1}$, the martingale property of both $C(\varphi^*)$ and $\overline{C}(\overline{\varphi})$ yields

$$E[\Delta V_{k+1}(\varphi^*)|\mathcal{F}_k] = X_{k+1}^* E[\Delta X_{k+1}|\mathcal{F}_k] = E[\Delta \overline{V}_{k+1}(\overline{\varphi})|\mathcal{F}_k], \tag{3.102}$$

so therefore $V_k(\varphi^*) = \overline{V}_k(\overline{\varphi})$ P-a.s. which implies that $X_k^* = \overline{X}_k$. It follows from backwards induction that this is true for all $k = 1, ..., T$.

With this in hand we turn to prove the general statement of Theorem 2.

Let $\varphi = (X, Y)$ be a LRM for the price process S^v without illiquidity costs and let

$$\overline{V}_k(\varphi) = X_{k+1}S_k^v + Y_k, \tag{3.103}$$

and

$$\overline{C}_k(\varphi) = \overline{V}_k(\varphi) - \sum_{j=1}^{k} X_j \Delta S_j^v, \tag{3.104}$$

which is the value and cost processes when the price process is S^v. We have that S^v has bounded mean-variance tradeoff by Proposition 3 and its cost process is a martingale from Proposition 1 and

$$X_k = \frac{Cov(\Delta \overline{V}_k(\varphi), \Delta S_k^v|\mathcal{F}_{k-1})}{Var[\Delta S_k^v|\mathcal{F}_{k-1}]} \quad \text{P-a.s. for} \quad k = 1, ..., T. \tag{3.105}$$

We have already showed that $V^v(\varphi) = \overline{V}(\varphi)$ in the previous paragraphs and so by (136) we must have that $X = X^*$ and since $\Delta C_k(\varphi) = \Delta V_k^v - X_k \Delta S_k^v$ we also have that

$$\Delta \overline{C}_k(\varphi) = \Delta \overline{V}_k^v - X_k \Delta S_k^v = \Delta V_k^v - X_k \Delta S_k^v = \Delta C_k(\varphi), \tag{3.106}$$

then $\overline{C}(\varphi) = C(\varphi)$, and since φ is mean-self financing both $\overline{C}(\varphi) = C(\varphi)$ is a martingale.

We also have that $C(\varphi^*)$ is a martingale with the same terminal value as $C(\varphi)$ namely,

$$H - \sum_{j=1}^{T} X_j \Delta S_j^v = H - \sum_{j=1}^{T} X_j^* \Delta S_j^v, \tag{3.107}$$

hence $Y = Y^*$.

This theorem proves that we can construct a LRM strategy φ^* for the price process S including illiquidity cost by re-valuing the stock at a suitable price S^v within the bid-ask range and then minimize the risk locally for the illiquidity cost free prices with S^v.

This result is important since it shows that the LRM strategy have a robustness property under illiquidity costs. See [Sch99].

3.5 The F-property

This section will define the F-property described by Christodoulou [Chr20] which will be used to give an exact formula for $\overline{X}_k$.

Definition 7 The process $S_t, t \geq 1$ has the F-property if there exists some $\delta \in (0,1)$ such that

$$det(F_k) - (1-\delta)det(F_k^A) \geq 0. \tag{3.108}$$

Note that this is not to be confused with the filtration $\mathcal{F}_k$.

Where $F_k^A = diag(A_{k;1}, ..., A_{k;d})$ and we recall that $A_{k;j} = var(\Delta S_{k+1}|\mathcal{F}_k) + E[\epsilon_{k\,k+1} S_{k+1}|\mathcal{F}_k]$.

In the 1 dimensional setting F_k becomes

$$F_k = Var(\Delta S_{k+1}|\mathcal{F}_k) + E[\epsilon_{k\,k+1} S_{k+1}|\mathcal{F}_k] \geq 0, \tag{3.109}$$

which is what we will build on.

Now we will describe properties of the F-property.

Definition 8 S has below bounded mean-variance tradeoff if there exist a constant $\overline{C} > 0$ such that in the 1 dimensional setting which translates into

$$\frac{(E[S_{k+1}^x|\mathcal{F}_k])^2}{Var(S_{k+1}^x)} \geq \overline{C} \quad \text{P-a.s. for all} \quad x = 1, ..., d. \tag{3.110}$$

Definition 9 S satisfies the F condition if for some constant $C > 0$ we have

$$\sqrt{Var(\Delta S_{k+1}^x|\mathcal{F}_k)} + \frac{E[S_{k+1}^x|\mathcal{F}_k]}{\sqrt{Var(S_{k+1}^x|\mathcal{F}_k)}} \geq C \quad \text{P-a.s. for all} \quad x = 1, ..., d, \tag{3.111}$$

uniformly in k and w and if for some $\overline{C} > 0$.

$$\frac{\sqrt{Var(\Delta S_{k+1}^x|\mathcal{F}_k)}}{E[S_{k+1}^x|\mathcal{F}_k]} + \frac{1}{\sqrt{Var(\Delta S_{k+1}^x|\mathcal{F}_k)}} \geq C \quad \text{P-a.s. for all} \quad x = 1, ..., d, \tag{3.112}$$

uniformly in k and w.

Proposition 4 For S satisfying $\overline{C} \leq Var(\Delta S_{k+1}^x|\mathcal{F}_k) \leq C$ for some positive constants $C, \overline{C}$ for all $x = 1, ..., d$ then the F property holds. In particular if S has independent increments then S has bounded mean variance and satisfy the F property.

Proof:

The claim follows directly from $\overline{C} \leq Var(\Delta S_{k+1}^x) \leq C$.

Proposition 5 For S having bounded mean variance tradeoff from above and below then the F condition holds. In particular, if S has independent returns then S has bounded mean-variance tradeoff and satisfies the F property.

Proof:

The claim is a direct consequence of S having bounded mean variance tradeoff from above and below.

We can use the F property to prove the existence of a local risk-minimizing strategy under illiquidity. In addition we will use backwards induction to give an explicit representation of the optimal strategy

Theorem 3 Assume S has the F-property and bounded mean variance tradeoff from below and above. Assume also that the covariance matrix F_k^0 is positive definite at all time $k = 0, 1, ..., T-1$. Then for any contingent claim $H = \overline{X}_{T+1}S_T + \overline{Y}_T \in L_T^{2,1}$ with $\overline{X}_{T+1}S_T \in L_T^{1,2}$ and $\overline{X}_{T+1} \in L_T^{2,d}$, there exists a local risk-minimizing strategy $\widehat{\varphi} = (\widehat{X}, \widehat{Y})$ under illiquidity with $\hat{X}_{T+1} = \overline{X}_{T+1}$ and $\hat{Y}_T = \overline{Y}_T$. Assume further that the strategy is represented by

$$\widehat{X}_{k+1} = F_k^{-1}b_k \quad \text{P-a.s. for} \quad k = 0, ..., T-1, \tag{3.113}$$

$$\widehat{Y}_{k+1} = E[\widehat{W}_k|\mathcal{F}_k] - \widehat{X}_{k+1}S_k \quad \text{P-a.s. for} \quad k = 0, ..., T-1, \tag{3.114}$$

where

$$\widehat{W}_k = H - \sum_{m=k+1}^{T} \widehat{X}_m \Delta S_m. \tag{3.115}$$

Proof:

The proof relies on a backward induction argument on $k = 0, 1, ..., T$. Start by setting $\widehat{X}_{T+1} = \overline{X}_{T+1}$ and $\widehat{Y}_T = \overline{Y}_T$. Fix some $k \in \{0, 1, ..., T-2\}$ and assume that at times $l = k, ..., T-2$

1. $\widehat{X}_{l+2}\Delta S_{l+2}^x \in L_T^{2,1}$ and $\widehat{X}_{l+2} \in L_T^{2,1}$
2. $|\widehat{X}_{l+2}|^2 S_{l+1}^x \in L_T^{1,1}$,
3. $\widehat{X}_{l+2}S_{l+2} + \widehat{Y}_{l+1} \in L_T^{2,1}$, $\quad \widehat{Y}_{l+1} \in F_{l+2}$,

At time k we want to minimize

$$Var(V_{k+1}(\varphi) - (X'_{k+1})\Delta S_{k+1}|\mathcal{F}_k) + \alpha E[\sum_{j=1}^{d} \epsilon_{k\,k+1} S_{k+1}^x (X_{k+2} - X'_{k+1})^2 |\mathcal{F}_k], \tag{3.116}$$

over all X'_{k+1} and show that the following properties are fulfilled.

1. $X'_{k+1}\Delta S_{k+1}^x \in L_T^{2,1}$ and $X'_{k+1} \in L_T^{2,1}$,
2. $|X'_{k+1}|^2 S_k^x \in L_T^{1,1}$,
3. $(X'_{k+1})S_k + Y'_k \in L_T^{2,1}$, $\quad Y'_k \in \mathcal{F}_k$

The properties 1)-3) ensures that $(\widehat{X}, \widehat{Y}) \in \Theta(S)$. First define the function F_k as in (3.50), where all the terms in F_k are integrable. Since F_k is positive definite then there exist a unique solution to the minimization problem and an $\mathcal{F}_k-$measurable minimizer $\widehat{X}_{k+1}$ can be created which equal $F_k^{-1}b_k$. Assume also that $\widehat{Y}_k$ as in (3.114). $\widehat{Y}_k$ is $\mathcal{F}_k-$measurable since all of its elements are $\mathcal{F}_k-$measurable.

$\widehat{X}_{k+1}S_k + \widehat{Y}_k = E[\widehat{W}_k|\mathcal{F}_k] \in L_T^{2,1}$ follows from $H \in L_T^{2,1}$, the induction hypothesis $\sum_{m=k+2}^{T} \widehat{X}_m \Delta S_m \in L_T^{2,1}$ and $\widehat{X}^*_{k+1}\Delta S_{k+1}$, which will be shown below.

First we show that $\widehat{X}_{k+1}\Delta S^x_{k+1} \in L_T^{2,1}$. By the inequality

$$E[|\Delta S^x_{k+1}|^2|\mathcal{F}_k] = Var(\Delta S^x_{k+1}|\mathcal{F}_k) + E[|\Delta S^x_{k+1}||\mathcal{F}_k]^2 \leq CVar(\Delta S^x_{k+1}|\mathcal{F}_k) \tag{3.117}$$

We then know that for a constant $C > 0$,

$$\begin{aligned} E[(X_{k+1}\Delta S^x_{k+1}|^2] \leq CE[Var(\widehat{X}_{k+2}S_{k+1} + \widehat{Y}_{k+1}|\mathcal{F}_k) \sum_{i=1}^{d} \frac{|Var(\Delta S^x_{k+1})|^2}{|Var(\Delta S^x_{k+1})} \\ + E[\epsilon_{k\,k+1}S^x_{k+1}]|^2 + \sum_{i=1}^{d}(c(\epsilon_{k\,k+1}) \frac{|Var(\Delta S^x_{k+1})|^2}{|Var(\Delta S^x_{k+1}) + E[\epsilon_{k\,k+1}S^x_{k+1}]|^2} \\ + E[\epsilon_{k\,k+1}S^x_{k+1}] \frac{Var(\Delta S^x_{k+1})|^2}{|Var(\Delta S^x_{k+1}) + E[\epsilon_{k\,k+1}S^x_{k+1}]|^2})E[|X_{k+2}|^2|\mathcal{F}_k], \end{aligned} \tag{3.118}$$

holds. We have that both $\widehat{X}_{k+2}S_{k+1} + \widehat{Y}_{k+1}$ and $\widehat{X}_{k+2}$ are both in $L_T^{2,1}$. Christodoulou [Chr20] also showed that

$$\frac{|Var(\Delta S^x_{k+1})|^2}{|Var(\Delta S^x_{k+1}) + E[\epsilon_{k\,k+1}S^x_{k+1}]|^2} + E[\epsilon_{k\,k+1}S^x_{k+1}] \frac{Var(\Delta S^x_{k+1})|^2}{|Var(\Delta S^x_{k+1}) + E[\epsilon_{k\,k+1}S^x_{k+1}]|^2}, \tag{3.119}$$

is uniformly bounded in k and w when S has the F-property and bounded mean variance tradeoff from below and above, which we will take as a given. Then $\widehat{X}_{k+1} \in L_T^{2,1}$ due to (3.114).

The next step is to show that the liquidity costs

$$E[\sum_{j=1}^{d} \epsilon_{k\,k+1}S^x_{k+1}|\widehat{X}_{k+2} - \widehat{X}_{k+1}|^2|\mathcal{F}_k], \tag{3.120}$$

are integrable.

To construct the optimal strategy according to the LRM-criterion under illiquidity at time k we need to minimize

$$\begin{aligned} &Var(V_{k+1}(\varphi) - (\widehat{X}_{k+1})\Delta S_{k+1}|\mathcal{F}_k) \\ &+ \alpha E[\sum_{j=1}^{d} \epsilon_{k\,k+1}S^x_{k+1}(\widehat{X}_{k+2} - (\widehat{X}_{k+1})^2|\mathcal{F}_k], \end{aligned} \tag{3.121}$$

for an appropriate minimizer $\widehat{X}_{k+1}$.

When $\alpha = 1$ we get

$$\begin{aligned} &Var(\widehat{X}_{k+2}S_{k+1} + \widehat{Y}_{k+1} - \widehat{X}'_{k+1})^*\Delta S_{k+1}|\mathcal{F}_k) \\ &+ E[\sum_{x=1}^{d} \epsilon_{k+1}S^x_{k+1}|\widehat{X}_{k+2} - \widehat{X}_{k+1}|^2|\mathcal{F}_k] \\ &\leq Var(\widehat{X}_{k+2}S_{k+1} + \widehat{Y}_{k+1}|\mathcal{F}_k) + E[\sum_{x=1}^{d} \epsilon_{k+1}S^x_{k+1}|\widehat{X}_{k+2}|^2|\mathcal{F}_k], \end{aligned} \tag{3.122}$$

holds when we choose $X_{k+1} = 0$, since

$$E[\sum_{x=1}^{d} \epsilon_{k+1} S_{k+1}^x |\widehat{X}_{k+2} - \widehat{X}_{k+1}|^2 | \mathcal{F}_k] \leq E[|\widehat{X}_{k+2} S_{k+1} + \widehat{Y}_{k+1}|^2] + E[\sum_{x=1}^{d} \epsilon_{k+1} S_{k+1}^x |\widehat{X}_{k+2}|^2], \tag{3.123}$$

where we used that $Var(X) \leq E[X]^2$. We have from earlier that $\widehat{X}_{k+2} S_{k+1} + \widehat{Y}_{k+1} \in L_T^{2,1}$ and $S_{k+1}^x |\widehat{X}_{k+2}|^2 \in L_T^{1,1}$, then the illiquidity cost $\sum_{x=1}^{d} \epsilon_{k+1} S_{k+1}^x |\widehat{X}_{k+2} - \widehat{X}_{k+1}|^2$ is also in $L_T^{1,1}$.

So $\epsilon_{k+1} S_{k+1}^x |\widehat{X}_{k+2} - \widehat{X}_{k+1}|^2 \in L_T^{1,1}$. This holds for a deterministic process ϵ_k and the marginal price process S are both non-negative by assumption.

The only thing which remains to show is that $|\widehat{X}_{k+1}|^2 S_k^x \in L_T^{1,1}$. When this is done we can by an induction argument show that the liquidity costs in the next step are again integrable. From the equality

$$|\widehat{X}_{k+1}|^2 S_k^x = -|\widehat{X}_{k+1}|^2 \Delta S_{k+1}^x + |\widehat{X}_{k+1}|^2 S_{k+1}^x, \tag{3.124}$$

We need to show that $|\widehat{X}_{k+1}|^2 \Delta S_{k+1}^x$ and $|\widehat{X}_{k+1}|^2 S_{k+1}^x$ are both in $L_T^{1,1}$. Since we already showed that the liquidity costs are integrable and $|\widehat{X}_{k+2}|^2 S_{k+1}^x \in L_T^{1,1}$ then the inequality

$$0 \leq |\widehat{X}_{k+1}|^2 S_{k+1}^x \leq 2|\widehat{X}_{k+2} - \widehat{X}_{k+1}|^2 S_{k+1}^x + 2|\widehat{X}_{k+2}|^2 S_{k+1}^x, \tag{3.125}$$

follows. Because $\epsilon_k \geq 0$ imply that $|\widehat{X}_{k+1}|^2 S_{k+1}^x$ is integrable. The term $|\widehat{X}_{k+1}|^2 \Delta S_{k+1}^x$ is integrable as well due to the fact that $\widehat{X}_{k+1} \Delta S_{k+1}^x$ and $\widehat{X}_{k+1}$ are both in $L_T^{2,1}$.

Then we must have

$$\begin{aligned} E[|\widehat{X}_{k+1}|^2 \Delta S_{k+1}^x] &\leq E[|\widehat{X}_{k+1}|^2 \mathbf{1}_{\{|\Delta S_{k+1}^x| \leq 1\}}] + E[|\widehat{X}_{k+1} \Delta S_{k+1}^x|^2 \mathbf{1}_{\{|\Delta S_{k+1}^x| \geq 1\}} \\ &\leq E[|\widehat{X}_{k+1}|^2] + E[|\widehat{X}_{k+1} \Delta S_{k+1}^x|^2], \end{aligned} \tag{3.126}$$

and this finally proves and completes the induction step at time k. Finally, define

$$\widehat{Y}_{T-1} = E[H - \widehat{X}_T \Delta S_T | \mathcal{F}_k] - \widehat{X}_T S_{T-1}, \tag{3.127}$$

then $\widehat{Y}_{T-1}$ is $\mathcal{F}_{t-1}$−measurable and $\widehat{X}_T S_{T-1} + \widehat{Y}_{T-1} = E[H - \widehat{X}_T \Delta S_T | \mathcal{F}_k]$ belongs to $L_T^{2,1}$.

For the strategy $\widehat{\varphi}$ to be LRM we need it also to have a cost process $C(\widehat{\varphi})$ has the martingale property. The martingale property of $C(\widehat{\varphi})$ follows from the construction of $\widehat{Y}$ since at each time k we have

$$E[C_T(\widehat{\varphi}) - C_k(\widehat{\varphi}) | \mathcal{F}_k] = 0. \tag{3.128}$$

Then $\widehat{\varphi} = (\widehat{X}, \widehat{Y})$ is a LRM under illiquidity. See [Chr20].

3.6 1-dimensional Black-Scholes model

Consider the 1-dimensional Black-Scholes model of a geometric Brownian motion W, that is

$$S_t = S_0 exp(\mu t + \sigma W_t), \tag{3.129}$$

which is lognormally distributed. This is a process of i.i.d. random variables and with bounded mean-variance tradeoff satisfying the F-property. see [Chr20].

3.7 1-dimensional value process under illiquidity

Consider the 1-dimensional case.

We remember that

$$b_{k;1} = Cov(V_{k+1}, \Delta S_{k+1}|\mathcal{F}_k) + E[\epsilon_{k\,k+1} S_{k+1} X_{k+2}|\mathcal{F}_k],$$

$$F_k = (\Delta S_{k+1}|\mathcal{F}_k) + E[\epsilon_{k\,k+1} S_{k+1}|\mathcal{F}_k],$$

and

$$\overline{X}_{k+1} = F_k^{-1} b_{k;1}.$$

The representation of a LRM-strategy $\overline{\varphi} = (\overline{X}, \overline{Y})$ under illiquidity is thus the following:

$$\overline{X}_{k+1} = \frac{Cov(V_{k+1}(\overline{\varphi}), \Delta S_{k+1}|\mathcal{F}_k) + E[\epsilon_{k+1} S_{k+1} \overline{X}_{k+2}|\mathcal{F}_k]}{Var(\Delta S_{k+1}|\mathcal{F}_k) + E[\epsilon_{k+1} S_{k+1}|\mathcal{F}_k]}, \tag{3.130}$$

and

$$V_k(\overline{\varphi}) = E[H - \sum_{m=k+1}^{T} \overline{X}_m \Delta S_m|\mathcal{F}_k]. \tag{3.131}$$

See [Chr20].

CHAPTER 4

New Contributions

We are now ready to present some new results in connection with hedging in illiquid markets obtained in this master thesis.

We have explicitly derived the value process with illiquidity for a geometric Brownian motion and developed a numerical program which can calculate the illiquidity cost for $k \in [0, T]$ under various conditions. We have coupled the numerical program to a unit linked insurance and investigated the impact of illiquidity on the reserve. Finally, we have expanded Schweizer et al.'s [LPS98] framework to include a non-linear supply curve.

4.1 Value process of a geometric Brownian motion

We start by using the two expression (3.130) and (3.131) to deduce an explicit general formula for the value process under illiquidity with the geometric Brownian motion.

Expanding (3.131) yields

$$V_k(\overline{\varphi}) =$$
$$E[H - \sum_{m=k+1}^{T} \frac{E[V_m(\overline{\varphi})\Delta S_m|\mathcal{F}_{m-1}] - E[V_m(\overline{\varphi})|\mathcal{F}_{m-1}]E[\Delta S_m|\mathcal{F}_{m-1}] + E[\epsilon_m S_m X_{m+1}|\mathcal{F}_{m-1}]}{E[(\Delta S_m)^2|\mathcal{F}_{m-1}] - E[\Delta S_m|\mathcal{F}_{m-1}]^2 + E[\epsilon_m S_m|\mathcal{F}_{m-1}]} \Delta S_m|\mathcal{F}_k]. \tag{4.1}$$

We highlight several useful calculations below:

We have

$$\begin{aligned} E[S_m|\mathcal{F}_{m-1}] &= E[S_0 e^{(\mu m + \sigma W_m)}|\mathcal{F}_{m-1}] \\ &= E[S_0 e^{(\mu m + \sigma(W_m - W_{m-1} + W_{m-1})}|\mathcal{F}_{m-1}] \\ &= S_0 e^{(\mu m + \sigma W_{m-1})} e^{(\sigma^2/2)}. \end{aligned} \tag{4.2}$$

Where we used that W has independent increments and is normally distributed with N(0,1) such that $E[\sigma(W_m - W_{m-1})|\mathcal{F}_{m-1}] = E[e^{(\sigma(m-m+1)}] = e^{(\sigma^2/2)}$, and $S_0 e^{(\mu(m-1)+\sigma W_{m-1})}$ is $\mathcal{F}_{m-1}$ measurable so it can be moved out of the conditional expectation, Secondly, we also have

$$\begin{aligned}
E[\Delta S_m|\mathcal{F}_{m-1}] &= E[S_m - S_{m-1}|\mathcal{F}_{m-1}] \\
&= E[S_{m-1}(\frac{S_m}{S_{m-1}} - 1)|\mathcal{F}_{m-1}] \\
&= E[S_0 e^{(\mu(m-1)+\sigma W_{m-1})}(S_0 e^{(\mu(m-m+1)+\sigma(W_m - W_{m-1}))} - 1)|\mathcal{F}_{m-1}] \\
&= S_0^2 e^{(\mu m+\sigma W_{m-1}+\sigma^2/2)} - S_0 e^{(\mu(m-1)+\sigma W_{m-1})}.
\end{aligned} \tag{4.3}$$

Thirdly, we have that

$$\begin{aligned}
E[(S_m)^2|\mathcal{F}_{m-1}] &= E[S_0^2 e^{(2\mu m+2\sigma W_m)}|\mathcal{F}_{m-1}] \\
&= S_0^2 e^{(2\mu m)} E[e^{(2\sigma W_m)}|\mathcal{F}_{m-1}] \\
&= S_0^2 e^{(2\mu m+2\sigma W_{m-1})} E[e^{(2\sigma(W_m - W_{m-1}))}|\mathcal{F}_{m-1}] \\
&= S_0^2 e^{(2\mu m+2\sigma W_{m-1})} e^{(2\sigma^2)},
\end{aligned} \tag{4.4}$$

and

$$E[(S_m)|\mathcal{F}_{m-1}]^2 = (S_0 e^{(\mu m+\sigma W_{m-1})} e^{(\sigma^2/2)})^2 = S_0^2 e^{(2\mu m+2\sigma W_{m-1})} e^{(\sigma^2)}. \tag{4.5}$$

Finally, we have that

$$E[S_m S_{m-1}|\mathcal{F}_{m-1}] = E[S_0 e^{(\mu m+\sigma W_m)} S_0 e^{(\mu(m-1)+\sigma W_{m-1})}|\mathcal{F}_{m-1}] = S_0^2 e^{(\mu(2m-1)+2\sigma W_{m-1})} e^{(\sigma^2/2)}. \tag{4.6}$$

We use a backwards induction schema to find an explicit formula:

Take $K = T$ then

$$V_T(\varphi) = E[H - 0|F_T] = E[H|F_T]. \tag{4.7}$$

Take $K = T - 1$ then

$$V_{T-1}(\varphi) = E[H - \frac{E[V_T(\varphi)\Delta S_T|\mathcal{F}_{T-1}] - E[V_T(\varphi)|\mathcal{F}_{T-1}]E[\Delta S_T|\mathcal{F}_{T-1}] + E[\epsilon_T S_T X_{T+1}|\mathcal{F}_{T-1}]}{E[(\Delta S_T)^2|\mathcal{F}_{T-1}] - E[\Delta S_T|\mathcal{F}_{T-1}]^2 + E[\epsilon_T S_T|\mathcal{F}_{T-1}]}\Delta S_T|\mathcal{F}_{T-1}]. \tag{4.8}$$

Take $K = T - 2$ then

$$\begin{aligned}
V_{T-2}(\varphi) = E[H &- \frac{E[V_T(\varphi)\Delta S_T|\mathcal{F}_{T-1}] - E[V_T(\varphi)|\mathcal{F}_{T-1}]E[\Delta S_T|\mathcal{F}_{T-1}] + E[\epsilon_T S_T X_{T+1}|\mathcal{F}_{T-1}]}{E[(\Delta S_T)^2|\mathcal{F}_{T-1}] - E[\Delta S_T|\mathcal{F}_{T-1}]^2 + E[\epsilon_T S_T|\mathcal{F}_{T-1}]}\Delta S_T \\
&- \frac{E[V_{T-1}(\varphi)\Delta S_{T-1}|\mathcal{F}_{T-2}] - E[V_{T-1}(\varphi)|\mathcal{F}_{T-2}]E[\Delta S_{T-1}|\mathcal{F}_{T-2}] + E[\epsilon_{T-1} S_{T-1} X_T|\mathcal{F}_{T-2}]}{E[(\Delta S_{T-1})^2|\mathcal{F}_{T-2}] - E[\Delta S_{T-1}|\mathcal{F}_{T-2}]^2 + E[\epsilon_{T-1} S_{T-1}|\mathcal{F}_{T-2}]}\Delta S_{T-1}|\mathcal{F}_{T-2}].
\end{aligned} \tag{4.9}$$

Take $K = T - 3$ then

$$\begin{aligned}
V_{T-3}(\varphi) = E[H &- \frac{E[V_T(\varphi)\Delta S_T|\mathcal{F}_{T-1}] - E[V_T(\varphi)|\mathcal{F}_{T-1}]E[\Delta S_T|\mathcal{F}_{T-1}] + E[\epsilon_T S_T X_{T+1}|\mathcal{F}_{T-1}]}{E[(\Delta S_T)^2|\mathcal{F}_{T-1}] - E[\Delta S_T|\mathcal{F}_{T-1}]^2 + E[\epsilon_T S_T|\mathcal{F}_{T-1}]}\Delta S_T \\
&- \frac{E[V_{T-1}(\varphi)\Delta S_{T-1}|\mathcal{F}_{T-2}] - E[V_{T-1}(\varphi)|\mathcal{F}_{T-2}]E[\Delta S_{T-1}|\mathcal{F}_{T-2}] + E[\epsilon_{T-1} S_{T-1} X_T|\mathcal{F}_{T-2}]}{E[(\Delta S_{T-1})^2|\mathcal{F}_{T-2}] - E[\Delta S_{T-1}|\mathcal{F}_{T-2}]^2 + E[\epsilon_{T-1} S_{T-1}|\mathcal{F}_{T-2}]}\Delta S_{T-1} \\
&- \frac{E[V_{T-2}(\varphi)\Delta S_{T-2}|\mathcal{F}_{T-3}] - E[V_{T-2}(\varphi)|\mathcal{F}_{T-3}]E[\Delta S_{T-2}|\mathcal{F}_{T-3}] + E[\epsilon_{T-2} S_{T-2} X_{T-1}|\mathcal{F}_{T-3}]}{E[(\Delta S_{T-2})^2|\mathcal{F}_{T-3}] - E[\Delta S_{T-2}|\mathcal{F}_{T-3}]^2 + E[\epsilon_{T-2} S_{T-2}|\mathcal{F}_{T-3}]}\Delta S_{T-2}|\mathcal{F}_{T-3}].
\end{aligned} \tag{4.10}$$

Note that X_{T+1} is not defined for time t up to T.

Take $K = 0$ then

$$V_0(\varphi) = E[H - \sum_{m=1}^{T} \frac{E[V_m(\varphi)\Delta S_m|\mathcal{F}_{k-1}] - E[V_m(\varphi)|\mathcal{F}_{m-1}]E[\Delta S_m|\mathcal{F}_{m-1}] + E[\epsilon_m S_m X_{m+1}|\mathcal{F}_{m-1}]}{E[(\Delta S_m)^2|\mathcal{F}_{m-1}] - E[\Delta S_m|\mathcal{F}_{m-1}]^2 + E[\epsilon_m S_m|\mathcal{F}_{m-1}]} \Delta S_m|\mathcal{F}_0]. \tag{4.11}$$

Writing out $V_{T-1}(\varphi)$ we get

$$V_{T-1}(\varphi) = E[H - \frac{E[V_T(\varphi)\Delta S_T|\mathcal{F}_{T-1}] - E[V_T(\varphi)|\mathcal{F}_{T-1}]E[\Delta S_T|\mathcal{F}_{T-1}] + E[\epsilon_T S_T X_{T+1}|\mathcal{F}_{T-1}]}{E[(\Delta S_T)^2|\mathcal{F}_{T-1}] - E[\Delta S_T|\mathcal{F}_{T-1}]^2 + E[\epsilon_T S_T|\mathcal{F}_{T-1}]} \Delta S_T|\mathcal{F}_{T-1}]. \tag{4.12}$$

We define some shorthand notation:

$$E[\Delta S_T|\mathcal{F}_{T-1}] = S_0^2 e^{(\mu T + \sigma W_{T-1} + \sigma^2/2)} - S_0 e^{(\mu(T-1) + \sigma W_{T-1})}, \tag{4.13}$$

as A.

Then using that the geometric Brownian motion is independent of the value process we have

$$E[V_T(\varphi)\Delta S_T|\mathcal{F}_{T-1}] = E[V_T(\varphi)|\mathcal{F}_{T-1}]E[\Delta S_T|\mathcal{F}_{T-1}], \tag{4.14}$$

such that

$$V_{T-1}(\varphi) =$$
$$E[H - \frac{AE[V_T(\varphi)|\mathcal{F}_{T-1}] - AE[V_T(\varphi)|\mathcal{F}_{T-1}] + S_0 e^{(\mu T + \sigma W_{T-1})} e^{(\sigma^2/2)} E[\epsilon_T X_{T+1}|\mathcal{F}_{T-1}]}{E[S_T^2 - 2S_T S_{T-1} + S_{T-1}^2|\mathcal{F}_{T-1}] - E[S_T - S_{T-1}|\mathcal{F}_{T-1}]^2 + E[\epsilon_T S_T|\mathcal{F}_{T-1}]}$$
$$* (S_T - S_{T-1})|\mathcal{F}_{T-1}]. \tag{4.15}$$

Use linearity of expectation on

$$E[S_T^2 - 2S_T S_{T-1} + S_{T-1}^2|\mathcal{F}_{T-1}] = E[S_T^2|\mathcal{F}_{T-1}] - 2E[S_T S_{T-1}|\mathcal{F}_{T-1}] + E[S_{T-1}^2|\mathcal{F}_{T-1}], \tag{4.16}$$

and ϵ_k is assumed to be predicable with respect to $\mathcal{F}_k$ so

$$E[\epsilon_T S_T|\mathcal{F}_{T-1}] = \epsilon_T S_0 e^{(\mu T + \sigma W_{T-1})} e^{(\sigma^2/2)}, \tag{4.17}$$

which we will call B.

Then

$$V_{T-1}(\varphi) = E[H - \frac{S_0 e^{(\mu T + \sigma W_{T-1})} e^{(\sigma^2/2)} E[\epsilon_T X_{T+1}|\mathcal{F}_{T-1}]}{E[S_T^2|\mathcal{F}_{T-1}] - 2E[S_T S_{T-1}|\mathcal{F}_{T-1}] + E[S_{T-1}^2|\mathcal{F}_{T-1}] - A^2 + B}$$
$$* (S_T - S_{T-1})|\mathcal{F}_{T-1}]$$
$$= E[H - \frac{S_0 e^{(\mu T + \sigma W_{T-1})} e^{(\sigma^2/2)} E[\epsilon_T X_{T+1}|\mathcal{F}_{T-1}]}{S_0^2 e^{(2\mu T + 2\sigma W_{T-1})} e^{(2\sigma^2)} - 2S_0^2 e^{(\mu(2T-1) + 2\sigma W_{T-1})} e^{(2\sigma^2)} + S_0^2 e^{(2\mu(T-1) + 2\sigma W_{T-2})} e^{(\sigma^2/2)} - A^2 + B}$$
$$* (S_T - S_{T-1})|\mathcal{F}_{T-1}]$$
$$= E[H|\mathcal{F}_{T-1}]$$
$$- \frac{E[\epsilon_T X_{T+1}|\mathcal{F}_{T-1}]}{S_0 e^{(\mu T + \sigma W_{T-1})} e^{(\frac{3}{2}\sigma^2)} - 2S_0 e^{(\mu(T-1) + \sigma W_{T-1})} + S_0 e^{(\mu(T-2) + \sigma W_{T-2})} e^{2\sigma^2} - C_T + E[\epsilon_T|\mathcal{F}_{T-1}]} A, \tag{4.18}$$

Where we used that

$$\begin{aligned} A^2 &= E[\Delta S_T|F_{T-1}]^2 \\ &= (S_0^2 e^{(\mu T+\sigma W_{T-1}+\sigma^2/2)} - S_0 e^{(\mu(T-1)+\sigma W_{T-1})})^2 \\ &= S_0^4 e^{(2\mu T+2\sigma W_{T-1}+\sigma^2)} - 2S_0^3 e^{(\mu(2T-1)+2\sigma W_{T-1}+\sigma^2/2)} + S_0^2 e^{(2\mu(T-1)+2\sigma W_{T-1}}, \end{aligned} \tag{4.19}$$

And

$$S_0^3 e^{(\mu T+\sigma W_{T-1})} e^{(\sigma^2)} + 2S^2 e^{(\mu(T-1)+\sigma W_{T-1})} - S_0 e^{(\mu(T-2)+\sigma W_{T-1})}, \tag{4.20}$$

which we called C_T for $m = T$.

The general closed formula for $V_k(\varphi)$ is thus,

$$\begin{aligned} &V_k(\varphi) = E[H|\mathcal{F}_k] \\ &- \sum_{m=k}^{T} \frac{\epsilon_{m+1} E[X_{m+2}|\mathcal{F}_m]}{S_0 e^{\mu(m+1)+\sigma W_m} e^{\frac{3}{2}\sigma^2} - 2S_0 e^{\mu m+\sigma W_m} + S_0 e^{\mu(m-1)+2\sigma W_{m-1}} e^{2\sigma^2} - C_m + \epsilon_{m+1}} \\ &* (S_0^2 e^{(\mu m+\sigma W_{m-1}+\sigma^2/2)} - S_0 e^{(\mu(m-1)+\sigma W_{m-1})}), \end{aligned} \tag{4.21}$$

In the case where the illiquidity ϵ_{k+1} tends to 0. We can define a trading strategy $\overline{\varphi} = (\overline{X}, \overline{Y})$ and observe that $V_k(\overline{\varphi}) = E[H|\mathcal{F}_k]$.

In the case where the illiquidity $\epsilon_{k+1} \to \infty$ we have

$$X_{k+1} \to E[\frac{S_{k+1}...S_T X_{T+1}}{E[S_{k+1}|\mathcal{F}_k]...E[S_T|\mathcal{F}_{T-1}]}|\mathcal{F}_k]. \tag{4.22}$$

4.2 Pricing the value process

We can derive a risk-neutral measure to price derivative securities. The derivation follows the standard method found in [Unk] and the references therein. The important implication of this derivation is that for any risk neutral measure the illiquidity cost becomes zero.

Stock price under risk-neutral measure

The stock price is given by

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dB(t), \quad t \in [0, T], \tag{4.23}$$

were both $\alpha(t), \sigma(t)$ are adapted. Its integral form is,

$$S(t) = S(0)exp[\int_0^t \alpha(u)dB(u) + \int_0^t (\alpha(u) - \sigma^2(u)/2)du]. \tag{4.24}$$

Let $R(t)$ be adapted interest rate, and the discounted process

$$D(t) = e^{-\int_0^t R(u)du}, \tag{4.25}$$

satisfies

$$dD(t) = -R(t)D(t)dt = -R(t)e^{-\int_0^t R(u)du}, \tag{4.26}$$

The discounted stock price is

$$D(t)S(t) = S(0)exp(\int_0^t \sigma(u)dB(u) + \int_0^t (\alpha(u) - R(u) - \frac{1}{2}\sigma(u)^2)du), \quad (4.27)$$

and

$$\begin{aligned} d(D(t)S(t)) &= (\alpha(t) - R(t))D(t)S(t)dt + \sigma(t)D(t)S(t)dB(t) \\ &= \alpha(t)D(t)S(t)(\mu(t)dt + dB(t)), \end{aligned} \quad (4.28)$$

where

$$\mu(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}, \quad (4.29)$$

where we have used that $(dD(t))(dS(t)) = 0$. This can be viewed as the price of risk in the market.

Using this μ, the Z from earlier and $\overline{B}(t) = B(t) + \int_0^t \mu(t)dt$ we have that

$$d(D(t)S(t)) = \sigma(t)D(t)S(t)d\overline{B}(t), \quad (4.30)$$

which is defined on the measure $\mathbb{P}$ with the Radon-Nikodym density Z with respect to $\mathbb{P}$. Using the Girsanov's theorem we have that $\overline{B}$ is a Brownian motion. From

$$D(t)S(t) = S(0) + \int_0^t \sigma(u)D(u)S(u)d\overline{B}(u), \quad (4.31)$$

we have that $(D(t)S(t))$ is a martingale under $\mathbb{P}$.

The undiscounted stock price under $\mathbb{P}$ is

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\overline{B}(t). \quad (4.32)$$

See [Unk].

Pricing through the risk neutral measure

Continuing with the stock model $dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dB(t)$ we can construct a hedging strategy. Let $X(t)$ be the value of self-financing portfolio with $\Delta(t)$ shares of stock and a differential equation

$$dX(t) = \Delta(t)dS(t)+R(t)(X(t)-\Delta(t)S(t))dt = R(t)X(t)dt+\Delta(t)\sigma(t)S(t)(\mu(t)dt+dB(t)) \quad (4.33)$$

Using equation (4.26) and (4.27) we get

$$d(D(t)X(t)) = \Delta(t)\sigma(t)D(t)S(t)[\mu(t)dt + dB(t)] = \Delta(t)d(D(t)S(t)), \quad (4.34)$$

Using equation (4.30) we get

$$d(D(t)X(t)) = \Delta(t)\sigma(t)D(t)S(t)d\overline{B}(t), \quad (4.35)$$

such that under the discounted stock portfolio the value $D(t)X(t)$ is a martingale.

Assume that we have a self-financing portfolio with value $(X(t), t \in [0, T])$, hedging the option $V(t)$ such that $X(T) = V(T)$. In addition assume that

$V(T)$ is $\mathcal{F}_T$-measurable and assuming the existence of this option. Then as $(D(t)X(t), t \in [0, Y])$ is a martingale under $\mathbb{P}$ we get

$$D(t)X(t) = \overline{E}[D(T)X(T)|\mathbb{F}_t] = \overline{E}[D(T)V(T)|\mathcal{F}_t]. \tag{4.36}$$

If we assume an arbitrage free strategy then the value of the option at time t should be taken as $V(t) = X(t)$. $V(t)$ under the risk neutral measure is thus equal to

$$D(t)V(t) = \overline{E}[D(T)V(T)|\mathbb{F}_t] \quad t \in [0, T], \tag{4.37}$$

and so

$$V(t) = \overline{E}[e^{-\int_t^T R(u)du} V(T)|\mathbb{F}_t] \quad t \in [0, T]. \tag{4.38}$$

See [Unk].

Pricing a European call option

Assume that we have a European call option, which can be expressed as $C = (S(T) - K)^+$. We let the contingent claim be the call option $H = (S(T) - K)^+$ such that

$$V_k^*(\overline{\varphi}) = \overline{E}[H - \sum_{m=k+1}^{T} \overline{X}_m \Delta S_m |\mathcal{F}_k] = \overline{E}[(S(T) - K)^+ - \sum_{m=k+1}^{T} \overline{X}_m \Delta S_m |\mathcal{F}_k], \tag{4.39}$$

and so

$$\begin{aligned} V_t^*(\overline{\varphi}) &= \overline{E}[e^{-\int_t^T R(u)du} V_T(\overline{\varphi})|\mathbb{F}_t] \\ &= \overline{E}[e^{-\int_t^T R(u)du}((S(T) - K)^+ - \sum_{m=T+1}^{T} \overline{X}_m \Delta S_m)|\mathcal{F}_k] \\ &= \overline{E}[e^{-\int_t^T R(u)du}(S(T) - K)^+|\mathcal{F}_k] \\ &= S(t)N(d_+) - e^{-r(T-t)}KN(d_-). \end{aligned} \tag{4.40}$$

Which is the regular Black-Scholes formula. This risk neutral measure makes the illiquidity cost component of the value process zero. In fact any risk neutral measure will create a similar situation.

To describe the value process subject to illiquidity costs in a meaningful way we simply take the expectation under the physical probability measure P of the European call option such that

$$\begin{aligned} V_t(\overline{\varphi}) &= E[e^{-\int_t^T R(u)du} V_T(\overline{\varphi})|\mathbb{F}_t] \\ &= E[e^{-\int_t^T R(u)du}((S(T) - K)^+ - \sum_{m=T+1}^{T} \overline{X}_m \Delta S_m)|\mathcal{F}_k]. \end{aligned} \tag{4.41}$$

4.3 Numerical analysis of illiquidity cost

We now want to compare the value process $V_t(\varphi)$ under the risk neutral measure with the Black-Scholes formula as its solution against the value process subject to the regular expectation seen below.

$$\begin{aligned}V_t(\overline{\varphi}) =& S(t)N(d_+) - e^{-r(T-t)}KN(d_-)\\ &- \sum_{m=k}^{T} \frac{\epsilon_{m+1}E[X_{m+2}|\mathcal{F}_m]}{S_0e^{\mu(m+1)+\sigma W_m}e^{\frac{3}{2}\sigma^2} - 2S_0e^{\mu m+\sigma W_m} + S_0e^{\mu(m-1)+2\sigma W_{m-1}}e^{2\sigma^2} - C_m + \epsilon_{m+1}}\\ &* (S_0^2e^{(\mu m+\sigma W_{m-1}+\sigma^2/2)} - S_0e^{(\mu(m-1)+\sigma W_{m-1})}).\end{aligned} \tag{4.42}$$

Where

$$\begin{aligned}d_\pm((T-t),x) &= \frac{1}{\sigma\sqrt{(T-t}}(ln(\frac{x}{K}) + (r \pm \frac{\sigma^2}{2}(T-t)),\\ N(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy = \frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{-y^2/2}dy,\\ \mu &= r - \frac{\sigma^2}{2}.\end{aligned} \tag{4.43}$$

The difference between the two value processes is the illiquidity cost

$$\begin{aligned}V_t(\overline{\varphi}) - V_t(\overline{\varphi})^* = &- \sum_{m=k}^{T} \frac{\epsilon_{m+1}E[X_{m+2}|\mathcal{F}_m]}{S_0e^{\mu(m+1)+\sigma W_m}e^{\frac{3}{2}\sigma^2} - 2S_0e^{\mu m+\sigma W_m} + S_0e^{\mu(m-1)+2\sigma W_{m-1}}e^{2\sigma^2} - C_m + \epsilon_{m+1}}\\ &* (S_0^2e^{(\mu m+\sigma W_{m-1}+\sigma^2/2)} - S_0e^{(\mu(m-1)+\sigma W_{m-1})}).\end{aligned} \tag{4.44}$$

Since ϵ_k is assumed to be measurable with respect to $\mathcal{F}_{t-1}$ we can take it out of the expectation. We further set $E[X_T|\mathcal{F}_{t-1}] = 1$ for simplicity. One could compute the conditional expectation as

$$\mu_{X_T|Y=\mathcal{F}_{t-1}} = E[X_T|Y = \mathcal{F}_{t-1}] = \sum_x x f_{X_T|Y}(x|\mathcal{F}_{t-1}),$$

such that $E[X_T|Y = \mathcal{F}_{t-1}]$ is the mean value of X, when Y is fixed at $\mathcal{F}_{t-1}$. However since we do not use any empirical data to estimate $E[X_T|\mathcal{F}_{t-1}]$ we just put it to 1 arbitrarily. Indicating that the expected obligated risky stock to sell at time T is 1 unit. The R-code used to create the following tables and graphs can be found in Appendices. We will display the illiquidity cost as a positive value in the following tables.

We used the following parameters to obtain these data from each table $\sigma = 0.05, K = 1, r = 0.05, S(0) = 1, S(t) = 10$.

Time	Illiquidity cost
1	1.041116
2	1.041116
3	1.041116
4	1.041116
5	1.041116
6	1.041116
7	1.041116
8	1.041116
9	1.041116
10	1.041116
11	1.041116
12	1.041116
13	1.041116
14	1.041116
15	1.041116
16	1.041116
17	1.041116
18	1.041116
19	1.041116
20	1.041116
21	1.041116
22	1.041116
23	1.041116
24	1.041116
25	1.041116
26	1.041116
27	1.041116
28	1.041116
29	1.041116
30	1.041116
31	1.041116
32	1.041116
33	1.041116
34	1.041116
35	1.041116
36	1.041116
37	1.041116
38	1.041116
39	1.041116
40	1.041116
41	1.041116
42	1.041115
43	1.041110
44	1.040968
45	1.040728
46	1.038900
47	1.034155
48	1.020368
49	1.000000
50	0.000000

Table 4.1: Illiquidity parameter $\epsilon = 0.1$.

Time	Illiquidity cost
1	1.160306
2	1.160306
3	1.160306
4	1.160306
5	1.160306
6	1.160306
7	1.160306
8	1.160306
9	1.160306
10	1.160306
11	1.160306
12	1.160306
13	1.160306
14	1.160306
15	1.160306
16	1.160306
17	1.160306
18	1.160306
19	1.160306
20	1.160306
21	1.160306
22	1.160306
23	1.160306
24	1.160306
25	1.160306
26	1.160306
27	1.160306
28	1.160306
29	1.160306
30	1.160306
31	1.160306
32	1.160306
33	1.160306
34	1.160306
35	1.160306
36	1.160306
37	1.160306
38	1.160306
39	1.160304
40	1.160302
41	1.160297
42	1.160220
43	1.160122
44	1.158324
45	1.157111
46	1.146953
47	1.138065
48	1.109465
49	1.000000
50	0.000000

Table 4.2: Illiquidity parameter $\epsilon = 0.6$.

Time	Illiquidity cost
1	3.547406
2	3.547406
3	3.547406
4	3.547406
5	3.547406
6	3.547406
7	3.547406
8	3.547406
9	3.547406
10	3.547406
11	3.547406
12	3.547406
13	3.547406
14	3.547406
15	3.547406
16	3.547406
17	3.547406
18	3.547406
19	3.547406
20	3.547406
21	3.547406
22	3.547406
23	3.547406
24	3.547406
25	3.547406
26	3.547406
27	3.547406
28	3.547406
29	3.547406
30	3.547406
31	3.547406
32	3.547406
33	3.547406
34	3.547406
35	3.547406
36	3.547403
37	3.547401
38	3.547401
39	3.547394
40	3.547392
41	3.547277
42	3.545802
43	3.526572
44	3.510709
45	3.477497
46	3.384790
47	3.353414
48	2.544754
49	1.000000
50	0.000000

Table 4.3: Illiquidity parameter $\epsilon = 1$.

Discussion of Table 4.1, 4.2 and 4.3 From the formula and data we observe that at the final time $T = 50$ there is no illiquidity cost since the X_{T+1} in the illiquidity cost is not defined and taken to be 0. Additionally we assume that $X_T = 1$ corresponding to having one stock of the risky asset in the portfolio at time T.

We see that as the illiquidity parameter increases so does the illiquidity cost. At $\epsilon = 0.1$ and $t = 1$ the illiquidity cost is 10.41% while at $\epsilon = 1$ and $t = 1$ the cost increases to 35.47% of the underlying stock price.

It is also evident that time to terminal plays an important role. The illiquidity cost remain almost constant between time $1 \leq t \leq 39$ using all illiquidity parameters, and the largest change occur in the final steps $t \geq 46$. There seem to be an illiquidity saturation range in which only negligible changes to the illiquidity cost occur. The illiquidity saturation range always starts at the $t = 1$ but ends at different times depending on the illiquidity parameter ϵ and the length of T.

In our simulation an illiquidity desaturation point with $\epsilon = 0.1$ occurs at time $t = 39$ before which the illiquidity saturation range is in effect. For $\epsilon = 1$ the illiquidity desaturation point occurs at $t = 35$. Pinning down the exact desaturation point can be done by solving the following equation for t and finding the maximum value for t:

$$Max_t(\Delta t = \frac{E[\sum_{m=t}^{T} X_m \Delta S_m | \mathcal{F}_t]}{E[\sum_{m=t+1}^{T} X_m \Delta S_m | \mathcal{F}_t]}), \tag{4.45}$$

where Δt is the rate of change in illiquidity cost, which we want to be approximately zero.

In reserve calculations one often uses the equivalence principle which states that the value of premiums must be equal to the value of the pay-outs at time $t = 0$. If we applied a similar principle to the value of the portfolio, such that the value of the illiquidity cost should be equal to an illiquidity premium we would have a type of risk pricing in an illiquid market.

An economic interpretation A stochastic process that moves randomly in a state space will given enough time visit all possible states. Some of these states are scenarios where the market is illiquid to some extend. When you are at the initial time $t = 1$ there are the most risk associated with illiquidity in the future, and at the final time $t = T$ there is no more possibility of entering any further states. This tells us that the illiquidity cost at the final time should be zero and largest at the initial time, which we see in our model. We see a rapid decrease in illiquidity cost towards the end, which corresponds to the number of state spaces decreases towards the final one.

The observed desaturation point can be viewed within the framework of the Lindy effect. The Lindy effect proposes that the longer something survives the more likely it is to have a longer remaining life expectancy. In regards to illiquidity, an asset is Lindy if it has survived a long time without illiquidity, then the best future prediction is that it is more unlikely that the stock will experience illiquidity. The desaturation point is then when the stock becomes "Lindy" and as more time passes it becomes more "Lindy".

One criticism to this proposed explanation is that the illiquidity regime is constant in each period, so we are not talking about a risk but rather a certainty.

Following this logic we would expect that there would be a constant decrease in illiquidity cost till the final time. If we instead see the illiquidity cost as the cost of uncertainty for future market illiquidity, then we would expect that the Lindy effect is present and our explanation fits the observed data.

Limitations of the model The model suffers from a number of limitations that might shift our conclusions. We will list some limitations and the effect on the model.

The illiquidity parameter is constant in all periods in this model which is unrealistic since variation certainly exist, especially in the energy market which is dependent on many factors such as geopolitical conflict, seasonal variation, global supply chains etc. However, this model can be used to bound the illiquidity cost associated with a given portfolio, since we can calculate a maximum and minimum illiquidity cost for the whole period.

This model is in discrete-time and has quite large jumps towards the final time. The size of the jumps are primarily due to the assumption of the size of the final X_T value, which we arbitrarily choose to be 1 for this example. However, a jump will still occur for $X_T > 0$ which will be large compared with the earlier illiquidity costs. This is due to the discrete framework that the formula is derived in. The most intuitive way to remedy this problem would be to change from discrete-time to continuous-time. However this would in turn change the underlying mathematical framework that the explicit formulas are based on.

The formula is a stochastic formula and the illiquidity cost will change based on the random walk of the Brownian motion such that for practical implementation the program should be run thousands of times to establish a mean and variance on the illiquidity cost in each period. Nevertheless, the example above does provide a pattern of behavior which each illiquidity cost run will exhibit.

The formula is time dependent in that illiquidity cost changes quite based on the length of the time interval. $T = 50$ was chosen arbitrarily, but higher values of T would see the initial illiquidity cost increase substantially, even to the point where it becomes more than the cost of the entire underlying stock. It is unrealistic that illiquidity costs could be higher than the underlying stock and so a illiquidity cost limit has to be imposed to prevent this outcome. A possible solution could be that any illiquidity cost which is higher than the underlying stock is simply equal to the value of the stock. Which would result in the stock not being purchased before a certain point in time.

No method for estimating the illiquidity parameter ϵ is given. The size of ϵ is of crucial importance when estimating the value of the portfolio and the lack of a method for finding ϵ limits the real world applications of this framework. A possible solution to estimating ϵ would be to compare a given portfolio in the market of interest with itself over time to gauge at which times the illiquidity in the market is high and at which times it is low based on the underlying stocks prices fluctuation. This method would only provide a rough estimate of the illiquidity parameter since changes in the price of the stock is not only due to illiquidity costs, but also due to randomness and other factors.

Historical data is needed to estimate the conditional expectation $E[X_t|\mathcal{F}_{t-1}]$, which could be done by looking at various stock portfolio on a given market and examine how many units of stock on average is being held at various times given some estimate of illiquidity level relative to the size of the portfolio. Obviously our choice of $E[X_T|\mathcal{F}_{t-1}]$ drives the illiquidity cost in that lower values would cause the illiquidity cost to reduce substantially, and a higher value would likewise increase the cost. However, we are more interested in the pattern of behavior that the illiquidity cost takes under this model which is independent of the size of $E[X_T|\mathcal{F}_{t-1}]$. Further research could be done to elucidate a realistic estimate for $E[X_t|\mathcal{F}_{t-1}]$ using the approach outlined above.

Unit linked policy

We decided to use the unit linked policy formula for continuous-time since the surface of the reserve using the discrete formula became very "blocky" to graph due to the limited time interval.

The result is thus not strictly mathematically correct, but the graphs do illustrate the same point in a more aesthetically pleasing way.

Lets assume that a_i is almost everywhere differentiable with at most a discontinuity at the end of the contract $t = T$ and that $\Delta a_i(T) = a_i(T) - a_i(T-)$ is this jump. Assume also that the functions $f_i, g_i, h_{ij} : [0,T] \times \mathbb{R} \to \mathbb{R}$ are defined as follows

$$\Delta a_i(T) = f_i(T, S_T), \quad \widehat{a}(t) = g_i(t, S_t), \quad a_{ij}(t) = h_{ij}(t, S_t), \quad t \in [0,T]. \tag{4.46}$$

The mathematical reserve $V_i^+(t, S_t)$ of the contract with the above policy functions (f_i, g_i and h_{ij}) linked to the fund S at time t, assuming that the insured is in state i at time t, is:

$$\begin{aligned} V_i^+(t, S_t) =& \frac{1}{v(t)}[\sum_{j\in\mathbb{J}} v(T)p_{ij}(t,T)E_{\mathbb{Q}}[f_j(T,S_T)|\mathcal{F}_t] \\ &+ \sum_{j\in\mathbb{J}} \int_t^T v(s)p_{ij}(t,s)E_{\mathbb{Q}}[g_j(s,S_s)|\mathcal{F}_t]ds \\ &+ \sum_{j,k\in\mathbb{J},k\neq j} \int_t^T v(s)p_{ij}(t,s)\mu_{jk}(s)E_{\mathbb{Q}}[h_{jk}(s,S_s)|\mathcal{F}_t]ds], \end{aligned} \tag{4.47}$$

where

$$\frac{v(T)}{v(t)}E_{\mathbb{Q}}[f_j(T,S_T)|\mathcal{F}_t], \quad \frac{v(s)}{v(t)}E_{\mathbb{Q}}[g_j(s,S_s)|\mathcal{F}_t], \quad \frac{v(s)}{v(t)}E_{\mathbb{Q}}[h_{jk}(s,S_s)|\mathcal{F}_t], \tag{4.48}$$

are the risk neutral prices of European options $f_j(T, S_T)$ with terminal time T, $g_j(s, S_s)$ with terminal time $s \geq t$ and $h_{jk}(s, S_s)$ with $s \geq t$ respectively.

Let φ be the pay-off function and let $H = \varphi(s, S_s)$ be the pay-off of an option at terminal time s. Let price of H with terminal time T be

$$U_s^{\varphi}(t, S_t) = \frac{v(s)}{v(t)}E_{\mathbb{Q}}[\varphi(s,S_s)|\sigma(S_t)], t \in [0,s]. \tag{4.49}$$

where we used that S has the Markov property. See [Bañ22].

The stochastic reserve is then

$$\begin{aligned} V_i^+(t,S_t) =& [\sum_{j\in\mathbb{J}} p_{ij}(t,T)U_T^{f_j}(t,S_t) \\ &+ \sum_{j\in\mathbb{J}} \int_t^T p_{ij}(t,s)U_s^{g_j}(t,S_t)ds \\ &+ \sum_{j,k\in\mathbb{J},k\neq j} \int_t^T p_{ij}(t,s)\mu_{jk}(s)U_s^{h_{jk}}(t,S_t)ds]. \end{aligned} \tag{4.50}$$

We have that the expected value of the contingent claim with illiquidity is $E[H|\mathcal{F}_{t-1}] = V_T(\varphi)$ and we know the value process at time $t = 0$ from the recursive algorithm developed in section (4.1). We let the price of a European call option at time t be the price of the value process of the European call option at that time. Such that

$$\begin{aligned} U_s^{\varphi}(t,S_t) &= \frac{v(s)}{v(t)} E_{\mathbb{Q}}[\varphi(s,S_s)|\sigma(S_t)] \\ &= \frac{v(s)}{v(t)} E_{\mathbb{Q}}[H|\sigma(S_t)] \\ &= \frac{v(s)}{v(t)} E[H - \sum_{m=t+1}^{T} \overline{X}_m \Delta S_m|\sigma(S_t)] \\ &= \frac{v(s)}{v(t)} V_t(\varphi). \end{aligned} \tag{4.51}$$

With this formulation we have included the illiquidity cost into the reserve calculation.

We switched from the risk neutral measure to the measure used in section (4.2) which is not a risk neutral measure, since that would eliminate the illiquidity cost as explained earlier.

Pure endowment with and without illiquidity cost

We can describe a unit-linked endowment without illiquidity cost as follows:

Lets consider a contract of T years for a person who is x years old which pays the maximum between G and the value of the fund S_T at terminal time T upon survival. Such that the pay-off is

$$H = max(G, S_T) \tag{4.52}$$

Where G is the guaranteed amount.

Assume that the fund S is modelled by the Black-Scholes model and has the following policy function

$$a_*(t) = \begin{cases} 0, & t \in [0,T) \\ max(G,S_T), & t \geq T. \end{cases} \tag{4.53}$$

and

$$\Delta a_*(T) = a_*(T) - a_*(T-) = f_*(T,S_T) = max(G,S_T) = (S_T - G)_+ + G. \tag{4.54}$$

The price at time t of a European call option is given by

$$U_T(t, S_t) = \frac{v(T)}{v(t)} E_{\mathbb{Q}}[max(G, S_T)|\sigma(S_t)] = BS(t, T, S_t, G) + Ge^{-r(T-t)}, \quad (4.55)$$

where BS is shorthand for the Black-scholes price of the option at time t with terminal T and strike price G. Then the price of the endowment at time t is

$$V_t(t, S_t) = p^x_{**}(t, T)[BS(t, T, S_t, G) + Ge^{-r(T-t)}], \quad t \in [0, T]. \quad (4.56)$$

See [Bañ22]. The unit-linked endowment with illiquidity cost is described similarly but with the $U^{\varphi}_s(t, S_t)$ as defined in (4.51). Such that we have

$$\begin{aligned} U_T(t, S_t) =& \frac{v(T)}{v(t)} E_{\mathbb{Q}}[max(G, S_T)|\sigma(S_t)] - \frac{v(T)}{v(t)} E[\sum_{m=t+1}^{T} \overline{X}_m \Delta S_m|\sigma(S_t)] \\ &= BS(t, T, S_t, G) + Ge^{-r(T-t)} - E[\sum_{m=t+1}^{T} \overline{X}_m \Delta S_m|\sigma(S_t)]e^{-r(T-t)}, \end{aligned} \quad (4.57)$$

Let T = 120 years, G = 1 and let the fund have the following dynamics
$\sigma = 5\%, r = 5\%$.

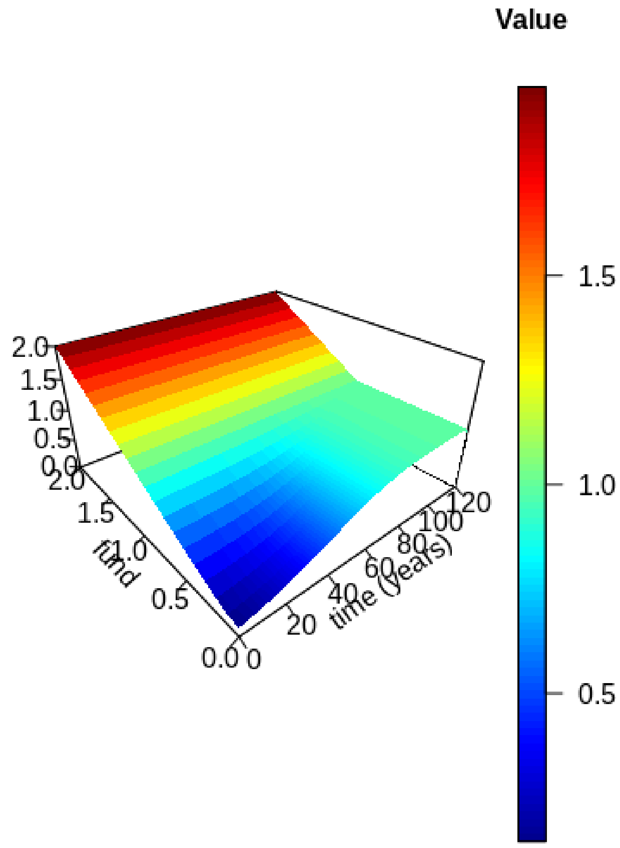

Figure 1: Reserve surface of a unit-linked endowment.

Discussion of Figure 1 The figure describes the reserve surface for the unit-linked insurance. We see that the larger the value of the fund at the start of the contract ($t = 0$), the higher the price of the policy. This is reasonable since the fund outperforms the guarantee $G = 1$. When the stock price is low at the beginning we see that it behaves as a regular endowment which increases deterministic.

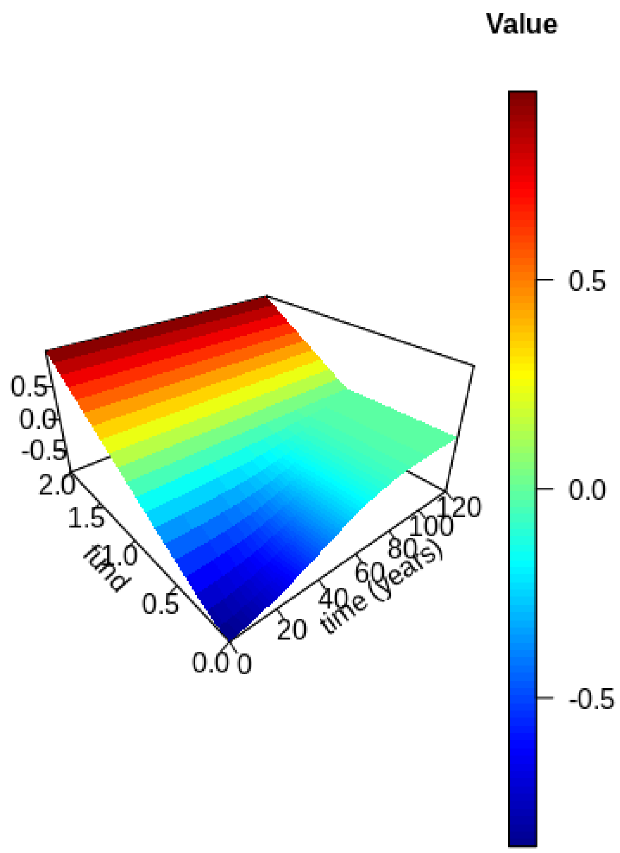

Figure 2: Reserve surface of a unit-linked endowment with illiquidity cost included where $\epsilon = 1$ and the final expected stock amount was set at $E[X_T|\mathcal{F}_{T-1}] = 0.1$.

Discussion of Figure 2 The overall behavior of the reserve surface moves much like the reserve with no illiquidity cost. The significant difference is that the price level is much lower due to the illiquidity cost. This is most keenly seen at the beginning of the contract where the reserve becomes negative. A lower reserve means that the projected pay-out is lower. A negative price would indicate the projected pay-out is negative as well, so no costumer would want to buy such a policy. To compensate for this illiquidity cost an insurer would have to pay the costumer with an equivalent money pay-out.

Multiple illiquidity parameters

As mentioned earlier the illiquidity parameter ϵ may undergo predictable periodic changes during the time interval. A real world example of this could be seasonal change in energy access. One could imagine that solar power becomes too abundant during summer and is unable to be sold at a positive price (in fact negative energy prices has been seen in Germany on certain summer days). The stock of solar companies may experience a certain degree of illiquidity due to this.
In this section we want to see the effect of 2 illiquidity parameters on the illiquidity cost.
From time $t \in [1, x]$ we have ϵ_1, while from $t \in [x+1, T]$ we use ϵ_2.
In general we have:

$$\overline{X}_{k+1} = \frac{Cov(V_{k+1}(\overline{\varphi}), \Delta S_{k+1}|\mathcal{F}_k) + E[\epsilon_{k+1} S_{k+1} \overline{X}_{k+2}|\mathcal{F}_k]}{Var(\Delta S_{k+1}|\mathcal{F}_k) + E[\epsilon_{k+1} S_{k+1}|\mathcal{F}_k]} \tag{4.58}$$

and

$$V_k(\overline{\varphi}) = E[H - \sum_{m=k+1}^{T} \overline{X}_m \Delta S_m|\mathcal{F}_k] \tag{4.59}$$

specifically the illiquidity cost is

$$E[\sum_{m=k+1}^{T} \overline{X}_m \Delta S_m|\mathcal{F}_k] = E[\sum_{m=k+1}^{T} \frac{Cov(V_{k+1}(\overline{\varphi}), \Delta S_{k+1}|\mathcal{F}_k) + E[\epsilon_{k+1} S_{k+1} \overline{X}_{k+2}|\mathcal{F}_k]}{Var(\Delta S_{k+1}|\mathcal{F}_k) + E[\epsilon_{k+1} S_{k+1}|\mathcal{F}_k]} \Delta S_m|\mathcal{F}_k] \tag{4.60}$$

The recursive nature of $\overline{X}_t$ makes it clear that recursion starts at $\overline{X}_{18}$ which depends on $\overline{X}_{19}$ and has ϵ_2 associated with it. Then $\overline{X}_{17}$ depends on $\overline{X}_{18}$ and $\overline{X}_{19}$ with ϵ_2 and so on, until we reach $\overline{X}_{10}$ which relies on ϵ_1, but the rest of the recursion $\overline{X}_{11}, ..., \overline{X}_{19}$ still depends on ϵ_2.

Time	Illiquidity cost
1	1.014692
2	1.014692
3	1.014692
4	1.014692
5	1.014692
6	1.014692
7	1.014692
8	1.014692
9	1.014692
10	1.014692
11	1.014692
12	1.014692
13	1.014692
14	1.014691
15	1.014687
16	1.014665
17	1.014311
18	1.013714
19	1.009810
20	1.000000
21	0.000000

Table 4.4: the following parameters were used $\epsilon_1 = 0.5$ for $t \in [1, 10]$, $\epsilon_2 = 0.1$ for $t \in [11, 20]$, $\sigma = 0.05$, $S(0) = 1, X_T = 1, r = 0.05$.

Time	Illiquidity cost
1	1.000072
2	1.000072
3	1.000072
4	1.000072
5	1.000072
6	1.000072
7	1.000072
8	1.000072
9	1.000072
10	1.000072
11	1.000072
12	1.000072
13	1.000072
14	1.000072
15	1.000072
16	1.000072
17	1.000072
18	1.000072
19	1.000071
20	1.000000
21	0.000000

Table 4.5: the following parameters were used $\epsilon_1 = 0.5$ for $t \in [1, 10]$, $\epsilon_2 = 0.1$ for $t \in [11, 20]$, $\sigma = 0.05$, $S(0) = 1, X_T = 1, r = 0.05$.

Discussion of table 4.4 and 4.5 From both table 4.4 and 4.5 we see that each added recursive link to $\overline{X}_t$ produce less and less of an effect on the illiquidity cost. Such that ϵ_1 impacts the cost much less than ϵ_2. Having the second half of the time interval be governed by a smaller illiquidity parameter makes the effect of later increases in the illiquidity parameter negligible. Another effect that becomes evident is that reducing the length of the time interval decreases the illiquidity cost substantially as compared with the previous tables (4.1,4.2 and 4.3).
We observe the familiar desaturation point in both tables like we did in the table 4.1, 4.2 and 4.3.

Linking a Brownian motion to the illiquidity parameter

In this section we want the illiquidity parameter to depend on a Brownian motion $\epsilon = f(B_t)$ where

$$f(x) = \begin{cases} 0.5 \text{ when} & \mathbf{1}(\mathrm{B}_t)_{(-\infty,0)} \\ 1 \text{ when} & \mathbf{1}(\mathrm{B}_t)_{[0,\infty)}. \end{cases} \tag{4.61}$$

Where $\mathbf{1}$ is the indicator function.
The program used generated the following illiquidity parameter

$$\epsilon = [0.5\ 1\ 1\ 0.5\ 0.5\ 0.5\ 1\ 0.5\ 0.5\ 0.5\ 0.5\ 0.5\ 0.5\ 0.5\ 0.5\ 0.5\ 0.5\ 1\ 1]$$

which gave the following data:

Time	Illiquidity cost
1	1.047070
2	1.047070
3	1.047070
4	1.047070
5	1.047070
6	1.047070
7	1.047070
8	1.047070
9	1.047070
10	1.047070
11	1.047070
12	1.047070
13	1.047070
14	1.047069
15	1.047066
16	1.046982
17	1.045400
18	1.027774
19	1.000000
20	0.000000

Table 4.6: the following parameters were used $\epsilon = 1$ for $\mathbf{1}(B_t)_{(-\infty,0]}$, $\epsilon = 0.5$ for $\mathbf{1}(B_t)_{[0,\infty]}$, $\sigma = 0.05$, $S(0) = 1, X_T = 1, r = 0.05$.

Discussion of Table 6 We see a familiar pattern of a desaturation point at time $t = 14$, and a rapid decrease until time $t = 20$. $\overline{X}_{20}$ carries the largest impact on the illiquidity cost and subsequent terms are contribute much less. According to this model illiquidity cost is almost independent of the illiquidity risk at any other time than the last couple of time periods.

Economic interpretation Tying the illiquidity parameter to a Brownian motion illustrates a market going through random economic shocks in which the illiquidity shoots up. It also captures the idea that there is a certain "stickiness" to economic shock, such that when the Brownian motion is highly positive it becomes less likely that it will become negative and change the illiquidity parameter in the next period. A real world could the housing crisis in 2008, which made banks less willing to buy housing derivatives causing a regime of higher illiquidity in the housing market. [GH20]

4.4 Non-linear supply curves

Let the supply curve $S_k(x) = (S_k^1(x), ..., S_k^d(x^d))$ be non-linear. Then

$$S_k^j(x^j) = S_k^j + x_k^{2,j} \epsilon_k S_k^j. \tag{4.62}$$

is an example of a non linear supply curve.

We once again assume that S is a non-negative semimartingale price process and $\epsilon_k = (\epsilon_k)_{k=0,1,..,T}$ is a positive deterministic $\mathbb{R}^d$-valued process, s.t. the price process does not take on negative values. For the sake of simplicity and easy of notation we shall consider one asset x at a time k, such that the non-linear supply curve becomes:

$$S_k^x(x) = S_k + x_k^2 \epsilon_k S_k. \tag{4.63}$$

The following lemmas, theorems and proofs can also be applied to the case of d number of assets, but the notation become quite unwieldy.
Returning to Schwiezer et al. [LPS98] we will rework the bounds and check whether their conclusions can be proven for the non linear supply curve.
Assume S has substantial risk, i.e. there is a constant $c < \infty$ such that

$$\frac{S_{k-1}^2}{E[\Delta S_k^2 | \mathcal{F}_{k-1}]} \leq c \quad \text{P-a.s. for} \quad k = 1, ..., T. \tag{4.64}$$

Lemma 3 Assume S has substantial risk. Then:

1. $\Theta(S^x) \supseteq \Theta(S)$ for every $x \in \Gamma$.
2. $V_k^x(\varphi) \in L^2(P)$ for $k = 0, 1, ..., T$, for every $x \in \Gamma$ and for every strategy φ.
3. $X_{k+1} S_k \in L^2(P)$ for $k = 0, 1, ..., T$, for every $X \in \Theta(S)$.
4. $C_k(\varphi) \in L^2(P)$ for $k = 0, 1, ..., T$ and for every strategy φ.

Proof:
By definition of S^x, we have

$$\begin{aligned} X_k \Delta S_k^x &= X_k S_k + X_k x_k^2 \epsilon_k S_k - X_k S_{k-1} - X_k x_{k-1}^2 \epsilon_k S_{k-1} \\ &= X_k \Delta S_k + X_k \epsilon_k \Delta x_k^2 S_{k-1} + X_k \epsilon_k x_k^2 \Delta S_k, \end{aligned} \tag{4.65}$$

x_k, x_{k-1} and ϵ_k are bounded by 1, and we remind ourselves that $\Theta(Y)$ is a space of all predictable processes $X = (X_k)_{k=1,...,T+1}$ such that $X_k \Delta Y_k \in L^2(P)$ for $k = 1, ..., T$. Then 1) follows from 3) since $X_k \epsilon_k \Delta x_k^2 S_{k-1} \in L^2(P)$ for $k = 0, 1, ..., T$ and so every $X_k \Delta S_k \in L^2(P)$ is also in $X_k \Delta S_k^j$ so $\Theta(S^x) \supseteq \Theta(S)$
The definition of $V^x(\varphi)$ is

$$V_k^x(\varphi) = X_{k+1} S_k^x + Y_k \leq V_k(\varphi) + X_{k+1} x_k^2 \epsilon_k S_k \leq V_k(\varphi) + X_{k+1} S_k, \tag{4.66}$$

since $V_k(\varphi) = X_{k+1} S_k + Y_k \in L^2(P)$ for $k = 0, 1, .., T$ and $x_k, \epsilon_k \leq 1$ we have that 2) follows from 3).
The cost process is defined as

$$C(\varphi) = V_k(\varphi) - \sum_{j=1}^{k} x_j \Delta S_j + \epsilon_k \sum_{j=1}^{k} S_j |\Delta x_{j+1}|, \tag{4.67}$$

which implies that

$$\begin{aligned}
\Delta C_k(\varphi) &= \Delta V_k(\varphi) - X_k\Delta S_k + \epsilon_k S_k|\Delta X_{k+1}| \\
&= X_{k+1}(S_k + \epsilon_k x_k S_k) + Y_k - X_k(S_k + \epsilon_k x_k S_k) - Y_{k-1} \\
&= \frac{x_k(X_{k+1}(S_k + \epsilon_k x_k S_k) + Y_k - X_k(S_k + \epsilon_k x_k S_k) - Y_{k-1})}{x_k} \\
&\leq \frac{X_{k+1}S_k + X_{k+1}\epsilon_k x_k^2 S_k + Y_k - X_k S_k + \epsilon_k x_k^2 S_k - Y_{k-1}}{x_k} \\
&= \frac{\Delta V_k^x(\varphi) - X_k\Delta S_k^x}{x_k},
\end{aligned} \tag{4.68}$$

as $x_k \to 0$ then $\Delta C_k(\varphi) \to \infty$. We have to impose the condition that x_k cannot go to zero then the above (4.68) shows that 4) follows from 2) and 1). 3) is proven by the assumption of substantial risk i.e.

$$E[(X_{k+1}S_k)^2] = E[(X_{k+1}\Delta S_{k+1})^2\frac{S_k^2}{E[\Delta S_{k+1}^2|\mathcal{F}_k]}] \leq cE[(X_{k+1}\Delta S_{k+1})^2] < \infty. \tag{4.69}$$

Proposition 4 Assume that S has bounded mean-variance tradeoff and substantial risk. Fix $x \in \Gamma$ and assume that there is a constant $C > 0$ such that

$$Var[\Delta S_k^x|\mathcal{F}_{k-1}] \geq cVar[\Delta S_k|\mathcal{F}_{k-1}] \quad \text{P-a.s. for} \quad k = 1, ..., T. \tag{4.70}$$

Then S^x has bounded mean-variance tradeoff, and $\Theta(S^x) = \Theta(S)$.
Proof:
We first show that (4.70) implies that S^x has bounded mean-variance trade off. This will be the case if

$$(E[\Delta S_k^x|\mathcal{F}_{k-1}])^2 \leq cVar[\Delta S_k|\mathcal{F}_{k-1}] \quad \text{P-a.s. for} \quad k = 1, ..., T. \tag{4.71}$$

We have that

$$\begin{aligned}
\Delta S_k^x &= S_k + x_k^2\epsilon_k S_k - S_{k-1} - x_{k-1}^2\epsilon_k S_{k-1} \\
&= \Delta S_k + \epsilon_k\Delta x_k^2 S_{k-1} + \epsilon_k x_k^2\Delta S_k.
\end{aligned} \tag{4.72}$$

Then

$$\begin{aligned}
(\Delta S_k^x)^2 &= (\Delta S_k + \epsilon_k\Delta x_k^2 S_{k-1} + \epsilon_k x_k^2\Delta S_k)^2 \\
&= (\Delta S_k)^2 x_k^4\epsilon_k^2 + 2(\Delta S_k)^2 x_k^2\epsilon_k + (\Delta S_k)^2 + 2\Delta S_k\Delta x_k^2 x_k^2\epsilon_k + (\Delta x_k^2)^2 S_{k-1}^2\epsilon_k^2 \\
&= S_k^2 x_k^4\epsilon_k^2 + 2S_k^2 x_k^2\epsilon_k + S_k^2 - 2S_{k-1}S_k x_k^2 x_{k-1}^2\epsilon_k^2 - 2S_k S_{k-1}x_k^2\epsilon_k \\
&- 2S_k S_{k-1}x_{k-1}^2\epsilon_k - 2S_k S_{k-1} + S_{k-1}^2 x_{k-1}^4\epsilon_k^2 + 2S_{k-1}^2 x_k^2\epsilon_k + S_{k-1}^2.
\end{aligned} \tag{4.73}$$

and assuming linearity of expectation, $x_k, x_{k-1} \leq 1$ and S_k is measurable w.r.t. the filtration $\mathcal{F}$ we get that

$$\begin{aligned}
E[(\Delta S_k^x)|\mathcal{F}_{k-1}]^2 \leq\ & \epsilon_k^2 E[S_k^2|\mathcal{F}_{k-1}] + 2\epsilon_k E[S_k^2|\mathcal{F}_{k-1}] + E[S_k^2|\mathcal{F}_{k-1}] \\
& - 2\epsilon_k^2 E[S_k|\mathcal{F}_{k-1}]S_{k-1} - 2\epsilon_k E[S_k^2|\mathcal{F}_{k-1}]S_{k-1} - 2\epsilon_k E[S_k^2|\mathcal{F}_{k-1}]S_{k-1} \\
& - 2E[S_k^2|\mathcal{F}_{k-1}]S_{k-1} + \epsilon_k^2 S_{k-1}^2 + 2\epsilon_k S_{k-1}^2 + S_{k-1}^2 \\
=\ & (1 + 2\epsilon_k + \epsilon_k^2)E[\Delta S_k^2|\mathcal{F}_{k-1}] - 2(1 + 2\epsilon_k + \epsilon_k^2)S_{k-1} \\
\leq\ & constant.E[\Delta S_k^2|\mathcal{F}_{k-1}]] \\
\leq\ & constant.(1 + c_{MVT}(0))Var[\Delta X_k|\mathcal{F}_{k-1}],
\end{aligned} \tag{4.74}$$

Where we used that ϵ_k is bounded by 1, then (4.71) and (4.64).
We had from Lemma 3 that $\Theta(S^x) \supseteq \Theta(S)$, so we have to show that $\Theta(S) \supseteq \Theta(S^x)$. We again let $S^x = S_0^x + M^x + A^x$ be the Doobs decomposition of S^x such that

$$X_k \Delta S_k^x = X_k \Delta M_k^x + X_k \Delta A_k^x = X_k \Delta M_k^x + X_k E[\Delta S_k^x|\mathcal{F}_{k-1}], \tag{4.75}$$

and

$$Var[\Delta S_k^x|\mathcal{F}_{k-1}] = E[(\Delta M_k^x)^2|\mathcal{F}_{k-1}]. \tag{4.76}$$

In this case S^x has bounded mean-variance tradeoff, (4.71) gives that $X \in \Theta(S^x)$ if and only if $X_k \Delta M_k^x \in L^2(P)$ for $k = 1, ..., T$ which will be written as $X \in L^2(M^x)$ The same is true for $X = X^0$. When X is predictable and (4.70) holds, then

$$\begin{aligned}
E[(X_k \Delta M_k)^2|\mathcal{F}_{k-1})] &= X_k^2 Var[\Delta X_k|\mathcal{F}_{k-1}] \\
&\leq \frac{1}{c} X_k^2 Var[\Delta S_k^x|\mathcal{F}_{k-1}] \\
&= \frac{1}{c} E[(X_k \Delta M_k^x)^2|\mathcal{F}_{k-1}].
\end{aligned} \tag{4.77}$$

This implies that $L^2(M^x) \subseteq L^2(M)$, then $\Theta(S^x) \subseteq \Theta(S)$ since both have mean-variance tradeoffs which are bounded.
This completes the proof.

Proposition 5 If there is a constant $\delta < 1$ such that

$$2\epsilon_k \sqrt{\frac{E[S_k^2|\mathcal{F}_{k-1}]}{Var[\Delta S_k|\mathcal{F}_{k-1}]}} \leq \delta \quad \text{P-a.s. for} \quad k = 1, ..., T. \tag{4.78}$$

Then (4.70) holds simultaneously for all ϵ_k, with

$$\begin{aligned}
c = (1 &+ \frac{\epsilon_k^2 k \Delta x_k^4 Var[S_{k-1}|\mathcal{F}_{k-1}]}{Var[S_{k-1}|\mathcal{F}_{k-1}]} + \frac{\epsilon_k^2 x_k^4 Var[\Delta S_k|\mathcal{F}_{k-1}]}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \\
&- 2\frac{\sqrt{Var[\Delta S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}]}}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \\
&- 2\frac{\sqrt{Var[\Delta S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}]}}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \\
&- 2\frac{\sqrt{Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}]}}{Var[\Delta S_k|\mathcal{F}_{k-1}]}).
\end{aligned} \tag{4.79}$$

Specifically, (4.78) has bounded mean-variance tradeoff and substantial risk and if x satisfies

$$4x^2(1 + 2c_{MVT}(0) + 2c_{SR}(1 + c_{MVT}(0)) < 1. \tag{4.80}$$

Proof:
We know that $S_k^x = S_k(1 + x_k^2\epsilon_k)$ and leaving aside the $\mathcal{F}_{k-1}$-measurable terms from the conditional variance yields

$$\begin{aligned}
Var[\Delta S_k^x|\mathcal{F}_{k-1}] &= Var[\Delta S_k + \epsilon_k \Delta x_k^2 S_{k-1} + \epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}] \\
&= Var[\Delta S_k|\mathcal{F}_{k-1}] + Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}] + Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}] \\
&+ 2Cov[\Delta S_k, \epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}] \\
&+ 2Cov[\Delta S_k, \epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}] \\
&+ 2Cov[\epsilon_k \Delta x_k^2 S_{k-1}, \epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}] \\
&\geq Var[\Delta S_k|\mathcal{F}_{k-1}] + Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}] + Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}] \\
&- 2\sqrt{Var[\Delta S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}]} \\
&- 2\sqrt{Var[\Delta S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}]} \\
&- 2\sqrt{Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}]},
\end{aligned} \tag{4.81}$$

where we used the formula $Var(\sum_i A_i) = \sum_i Var(A_i) + 2\sum_{i<j} Cov(A_i, A_j)$ and the Cauchy-Schwartz inequality.
Use now

$$Var[\Delta S_k^x|\mathcal{F}_{k-1}] \leq cVar[\Delta S_k|\mathcal{F}_{k-1}] \tag{4.82}$$

and (4.81) such that

$$\begin{aligned}
cVar[\Delta S_k|\mathcal{F}_{k-1}] &\geq Var[\Delta S_k|\mathcal{F}_{k-1}] + Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}] + Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}] \\
&- 2\sqrt{Var[\Delta S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}]} \\
&- 2\sqrt{Var[\Delta S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}]} \\
&- 2\sqrt{Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}]} \\
&= E[\Delta S_k^2|\mathcal{F}_{k-1}] - E[\Delta S_k|\mathcal{F}_{k-1}]^2 \\
&+ \epsilon_k^2 k \Delta x_k^4 Var[S_{k-1}|\mathcal{F}_{k-1}] + \epsilon_k^2 x_k^4 Var[\Delta S_k|\mathcal{F}_{k-1}] \\
&- 2\sqrt{Var[\Delta S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}]} \\
&- 2\sqrt{Var[\Delta S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}]} \\
&- 2\sqrt{Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}]},
\end{aligned} \tag{4.83}$$

where we used the Cauchy-Schwarz inequality. Then let

$$\begin{aligned}
c = (1 + &\frac{\epsilon_k^2 k \Delta x_k^4 Var[S_{k-1}|\mathcal{F}_{k-1}]}{Var[S_{k-1}|\mathcal{F}_{k-1}]} + \frac{\epsilon_k^2 x_k^4 Var[\Delta S_k|\mathcal{F}_{k-1}]}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \\
&- 2\frac{\sqrt{Var[\Delta S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}]}}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \\
&- 2\frac{\sqrt{Var[\Delta S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}]}}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \\
&- 2\frac{\sqrt{Var[\epsilon_k \Delta x_k^2 S_{k-1}|\mathcal{F}_{k-1}]Var[\epsilon_k x_k^2 \Delta S_k|\mathcal{F}_{k-1}]}}{Var[\Delta S_k|\mathcal{F}_{k-1}]}).
\end{aligned} \tag{4.84}$$

Then plugging (4.84) into (4.83) we get

$$E[\Delta S_k|\mathcal{F}_{k-1}]^2 \geq c_{MVT}(0) Var[\Delta S_k|\mathcal{F}_{k-1}]. \tag{4.85}$$

Finally, we have

$$\begin{aligned}
\frac{E[S_k^2|\mathcal{F}_{k-1}]}{Var[\Delta S_k|\mathcal{F}_{k-1}]} &\leq \frac{Var[\Delta S_k|\mathcal{F}_{k-1}] + (S_{k-1} + E[\Delta S_k|\mathcal{F}_{k-1}])^2}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \\
&\leq \frac{Var[\Delta S_k|\mathcal{F}_{k-1}] + 2S_{k-1}^2 + 2E[\Delta S_k|\mathcal{F}_{k-1}]^2}{Var[\Delta S_k|\mathcal{F}_{k-1}]} \\
&\leq \frac{Var[\Delta S_k|\mathcal{F}_{k-1}] + 2c_{SR} Var[\Delta S_k|\mathcal{F}_{k-1}](1 + c_{MVT}(0)) + 2E[\Delta S_k|\mathcal{F}_{k-1}]^2}{Var[\Delta S_k|\mathcal{F}_{k-1}]},
\end{aligned} \tag{4.86}$$

which completes the proof.
It should be noted that the conditions for c and Lemma 3 are very restrictive, such that the class of supply curves which satisfy them in a real market may be very small indeed.
With these properties in place the rest of Schweizer et al. [LPS98] conclusions follows. Namely Theorem 1 and Theorem 2 highlighted earlier.

Theorem 4 Let S have bounded mean-variance tradeoff, substantial risk, satisfy (4.78) and

$$Var[\Delta X_k|\mathcal{F}_{k-1}] > 0 \quad \text{P-a.s. for} \quad k = 1, ..., T. \tag{4.87}$$

Then for any contingent claim $(\overline{X}_{T+1}, \overline{Y}_T)$ there exist a LRM strategy $\varphi^* = (X^*, Y^*)$ where $X_{T+1}^* = \overline{X}_{T+1}$ and $Y_T^* = \overline{Y}_T$.
Where X^* can be characterized as follows: There exists a process $\delta^* \in \Gamma$ such that we can define $v \in \Gamma$ by

$$v_k = sign(X_{k+1}^* - X_k^*) + \delta_k^* I_{X_{k+1}^* = X_k^*} \quad \text{for} \quad k = 1, ..., T, 3.1 \tag{4.88}$$

and

$$V_k^v(\varphi^*) = V_k(\varphi) + \epsilon_k(sign(X_{k+1}^* - X_k^*) + \delta_k^* I_{X_{k+1}^* = X_k^*}) X_{k+1} S_k \tag{4.89}$$

then

$$X_k^* = \frac{Cov(\Delta V_k^v(\varphi^*), \Delta S^v|\mathcal{F}_{k-1})}{Var[\Delta S^v|\mathcal{F}_{k-1}]} \quad \text{P-a.s. for} \quad k = 1, ..., T. \tag{4.90}$$

Proof:
The proof of Theorem 4 is the same as the proof Theorem 1 since the construction of X_k^* does not depend on the non-supply curve. The exact proof can be found in section 3.4 on page 31.

Theorem 5 Assume the conditions of Theorem 4. The strategy φ^* is a LRM for the price process S inclusive of illiquidity costs is then also the strategy which is locally risk-minimizing for the price process S^v without illiquidity costs, where v is given by (4.88)
Proof:
The proof of Theorem 5 with a non-linear supply curve follows exact same argument as in Theorem 2.
The proof starts by assuming that the illiquidity parameter ϵ_k is equal to zero which makes the linear and non-linear supply curve coincide, and the rest follows by virtue of making the non-linear supply curve obey the same bounds as the linear supply curve does, namely (4.64), Lemma 3, (4.70), (4.71) and (4.78). Then the exact same recursive argument can be used on the non-linear supply curve to show that the LRM strategy φ^* with illiquidity costs is equal to the same strategy for the price process S^v without illiquidity costs.
The complete proof can be found in section 3.4 on page 33.
We could use similar arguments on a non linear supply curve of a higher order polynomial and the theorems would still hold true, however the bounds would become even more restrictive and you would in all likelihood not be describing a set of price processes which exist in reality.
The neat thing about this approach is that you only assume that S is square-integrable along with the bounds, so even though it seems rather restrictive it might ultimately be less so than other approaches, which must impose additional conditions on the supply curve such as convexity.

4.5 Conclusion

We found that the illiquidity cost associated with the value process followed a similar pattern under the various illiquid scenarios. The illiquidity cost is almost constant until a desaturation point occurs, then the illiquidity cost decreases rapidly before it reaches zero at the terminal time. In regards to the size of the illiquidity cost we found that increasing the time interval length and the illiquidity parameter ϵ both increases the illiquidity cost.
The largest cost components are always the first few expressions following $\overline{X}_{T-1}$ in

$$\overline{X}_{k+1} = \frac{Cov(V_{k+1}(\overline{\varphi}), \Delta S_{k+1}|\mathcal{F}_k) + E[\epsilon_{k+1} S_{k+1} \overline{X}_{k+2}|\mathcal{F}_k]}{Var(\Delta S_{k+1}|\mathcal{F}_k) + E[\epsilon_{k+1} S_{k+1}|\mathcal{F}_k]}. \tag{4.91}$$

Such that $\overline{X}_{T-2}$ contributes much more to the cost than $\overline{X}_{T-3}$, which contributes much more to the cost than $\overline{X}_{T-4}$ and so on. This is the reason for the desaturation point, since illiquidity cost terms before a certain time adds next to nothing to the cost.
The illiquidity cost causes the unit linked policy reserve surface to become negative at the initial point, for low fond values. In comparison to the reserve surface of the unit linked policy without illiquidity costs we saw that the whole

reserve surface was suppressed by the illiquidity cost, in particular at the initial time.
The model developed by Christodoulou captures many intuitive aspects of illiquidity such as the time dependence, the illiquidity parameter dependence and the dependence on the length of the time interval. It would be interesting to compare the model to data from the energy market and see whether the model can retrodict the illiquidity parameter of the market in various periods. Additionally one could expand Christodoulou's model into a non-linear supply curve setting using non-linear stochastic calculus.
We found that the framework of Schweizer et al. could be extended to a non-linear supply curve setting by imposing additional conditions on x in Lemma 3 and c in Proposition 5.

Appendices

R codes

Code for Table 4.1,4.2,4.3

```
#parameters
T = 50
S0 <- 1
sigma <- 0.05
mu <- r- sigma**2/2
xi <- 0.6
x = 10
K = 1
r = 0.05

X <- rep(0,T)
X[T] <- 1


#Illiquidity formula

for (i in seq((T-1),0,-1)){
  Bt = sqrt(t)*cumsum(rnorm(T-1,0,1))
  St = S0*exp(mu*i+sigma*Bt+sigma^2/2)
  St1 = (S0**2)*exp(2*mu*i+2*sigma*Bt+2*sigma^2)
  St2 = (S0**2)*exp(mu*(2*i-1)+2*sigma*Bt+sigma^2/2)
  St3 = (S0**2)*exp(2*mu*(i-1)+2*sigma*Bt)
  St4 = (S0**2)*exp(mu*(i)+sigma*Bt+sigma^2/2)- S0*exp(mu*(i-1)+sigma*Bt)
  St5 = -(S0**4)*exp(2*mu*i+2*sigma*Bt+sigma^2)
  -2*(S0**3)*exp(mu*(2*i-1)+2*sigma*Bt+sigma^2/2)
  +(S0**2)*exp(2*mu*(i-1)+2*sigma*Bt)
  X[i]<- (St[i]*xi[i]*X[i+1]/(St1[i]-2*St2[i]+St3[i]+St5[i]+xi*St[i]))*St4[i]
}

result <- rep(0,T)
result[]=X[T]
result[T] = 0
for (i in seq((T-2),1,-1)){
  result[i]=result[i+1]+X[i]
}


#Black-Scholes function
BS <- function(t,T,x,K){
  d1 <- (log(x/K)+(r+0.5*sigma*sigma)*(T-t))/(sigma*sqrt(T-t))
  d2 <- (log(x/K)+(r-0.5*sigma*sigma)*(T-t))/(sigma*sqrt(T-t))

  return(x*pnorm(d1)-K*exp(-r*(T-t))*pnorm(d2))
}

BSE <- function(t,T,x,K){ return(BS(t,T,x,K)+K*exp(-r*(T-t)))}

for (i in seq(0,T,1)){
 print(BSE(i,T,x,K)-exp(-r*(T-i))*result[i])
}
```

```
for (i in seq(0,T,1)){
  print(BSE(i,T,x,K))
}
```

Code for Figure 1 and Figure 2

```
#transition rates
a <- -9.13275
b <- 0.0809438
c <- 0.000011018

#Gompertz-Makeham law
mort <- function(u){
  return(exp(a+b*u-c*u^2))
}

#Survival probability: s to t
surv_prob <- function(s,t){
  mu <- b/(2*c)
  sigma <- sqrt(1/(2*c))
  #if(s>t){ return("s>t") }
  val <- exp(-sigma*exp(a+(b*b)/(4*c))*sqrt(2*3.14)*
  (pnorm((t-mu)/sigma,0,1)-pnorm((s-mu)/sigma,0,1)))
  return(val)
}

#model parameters
T = 21
S0 <- 1
sigma <- 0.05
mu <- r- sigma**2/2
xi <- 1
x = 10
sig = 0.05
K = 1
r = 0.05

X <- rep(0,T)
X[T] <- 0.1

#surface for pure endowment

h <- 1/12
time <- seq(0,T,by=h)
n.t <- length(time)-1
fund <- seq(0,2,by=0.1)
surf <- matrix(rep(0,(length(time))*(length(fund))), nrow=length(time))

for(i in 1:length(time)){
  for(j in 1:length(fund)){
    int1 <- function(s){
      result <- rep(0,length(fund))
      }
```

```
      val <- surv_prob(x0+time[i],x0+s)*mort(x0+s)*
      BSE(time[i],s,fund[j],G)
      return(val)
    }
    surf[i,j] <- as.numeric(integrate(int1,time[i],T)[1])
  }
}

#surface for pure endowment with illiquidity cost

h <- 1/12
time <- seq(0,T,by=h)
n.t <- length(time)-1
fund <- seq(0,2,by=0.1)
surf <- matrix(rep(0,(length(time))*(length(fund))), nrow=length(time))

for(i in 1:length(time)){
  for(j in 1:length(fund)){
    int1 <- function(s){
      result <- rep(0,length(fund))
      result[]=X[T]
      result[length(fund)] = 0
      for (h in seq((length(fund)-2),1,-1)){
        result[h]=result[h+1]+X[h]
      }

      val <- surv_prob(x0+time[i],x0+s)*mort(x0+s)*
      (BSE(time[i],s,fund[j],G)-result[h])
      return(val)
    }
    surf[i,j] <- as.numeric(integrate(int1,time[i],T)[1])
  }
}
```

Code for Table 4.4 and Table 4.5

```
T = 21
S0 <- 1
sigma <- 0.05
mu <- r- sigma**2/2


sig = 0.05
r = 0.05

X <- rep(0,T)
X[T] <- 1

xi = rep(0,T-1)
v1 = c(1,2,3,4,5,6,7,8,9,10)
v11 = c(11,12,13,14,15,16,17,18,19,20)
v2 = c(0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5)
```

```
v3 = c(1,1,1,1,1,1,1,1,1,1)
v4 = c(0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1)
xi[v1] = v4
xi[v11] = v2




for (i in seq((T-1),0,-1)){
  Bt = sqrt(t)*cumsum(rnorm(T-1,0,1))
  St = S0*exp(mu*i+sigma*Bt+sigma^2/2)
  St1 = (S0**2)*exp(2*mu*i+2*sigma*Bt+2*sigma^2)
  St2 = (S0**2)*exp(mu*(2*i-1)+2*sigma*Bt+sigma^2/2)
  St3 = (S0**2)*exp(2*mu*(i-1)+2*sigma*Bt)
  St4 = (S0**2)*exp(mu*(i)+sigma*Bt+sigma^2/2)- S0*exp(mu*(i-1)+sigma*Bt)
  St5 = -(S0**4)*exp(2*mu*i+2*sigma*Bt+sigma^2)
  -2*(S0**3)*exp(mu*(2*i-1)+2*sigma*Bt+sigma^2/2)
  +(S0**2)*exp(2*mu*(i-1)+2*sigma*Bt)
  X[i]<- (St[i]*xi[i]*X[i+1]/(St1[i]-2*St2[i]+St3[i]+St5[i]+xi*St[i]))*St4[i]
}

result <- rep(0,T)
result[]=X[T]
result[T] = 0
for (i in seq((T-2),1,-1)){
  result[i]=result[i+1]+X[i]
}
```

Code for Table 4.6

```
#parameters
T = 20
S0 <- 1
sigma <- 0.05
mu <- r- sigma**2/2
r = 0.05

X <- rep(0,T)
X[T] <- 1

xi1 <- 1
xi2 <- 0.5

xi <- sqrt(T)*cumsum(rnorm((T-1),0, 1))
xi <- ifelse(xi < 0, xi1,xi2)

#Illiquidity formula

for (i in seq((T-1),0,-1)){
  Bt = sqrt(T)*cumsum(rnorm(T-1,0,1))
  St = S0*exp(mu*i+sigma*Bt+sigma^2/2)
  St1 = (S0**2)*exp(2*mu*i+2*sigma*Bt+2*sigma^2)
  St2 = (S0**2)*exp(mu*(2*i-1)+2*sigma*Bt+sigma^2/2)
```

```
  St3 = (S0**2)*exp(2*mu*(i-1)+2*sigma*Bt)
  St4 = (S0**2)*exp(mu*(i)+sigma*Bt+sigma^2/2)- S0*exp(mu*(i-1)+sigma*Bt)
  St5 = -(S0**4)*exp(2*mu*i+2*sigma*Bt+sigma^2)
  -2*(S0**3)*exp(mu*(2*i-1)+2*sigma*Bt+sigma^2/2)
  +(S0**2)*exp(2*mu*(i-1)+2*sigma*Bt)
  X[i]<- (St[i]*xi[i]*X[i+1]/(St1[i]-2*St2[i]+St3[i]+St5[i]+xi*St[i]))*St4[i]
}

result <- rep(0,T)
result[]=X[T]
result[T] = 0
for (i in seq((T-2),1,-1)){
  result[i]=result[i+1]+X[i]
}
```

Bibliography

ref12 [Bañ22] Baños, D. R. *STK4500: Life Insurance and Finance*. https://www.uio.no/studier/emner/matnat/math/STK4500/v22/lecture-notes/lecturenotes2022.pdf. Accessed: 2022-05-09. 2022.

ref1 [Chr20] Christodoulou, P. *Local risk-minimization under illiquidity and consistent specification of credit migration models*. Ludwig-Maximilians-Universität München, July 2020.

ref5 [Dam07] Damien Lamberton, B. L. *Introduction to Stochastic Calculus Applied to Finance, 2nd edition*. Chapman and Hall/CRC, November 30, 2007.

ref9 [Dup] Dupraz, S. *Measure Theory and Probability Theory*. http://www.stephaneduprazecon.com/measuretheory.pdf. Accessed: 2022-05-09.

ref10 [EH95] Elyes, J. and Hedi, K. 'Martingales and Arbitrage in Securities Markets with Transaction Costs'. In: *Journal of Economic Theory* vol. 66, no. 1 (June 1995), pp. 178–197.

ref3 [GH20] Garriga, C. and Hedlund, A. 'Mortgage Debt, Consumption, and Illiquid Housing Markets in the Great Recession'. In: *American Economic Review* vol. 110, no. 6 (June 2020), pp. 1603–34.

ref2 [LPS98] Lamberton, D., Pham, H. and Schweizer, M. *Local Risk-Minimization under Transaction Costs*. Vol. 23. 3. INFORMS, 1998, pp. 585–612.

ref8 [NIE] NIELSEN, S. E. *THE MARTINGALE METHOD DEMYSTIFIED*. https://sellersgaard.files.wordpress.com/2013/09/martingale.pdf. Accessed: 2022-05-09.

ref6 [Sch99] Schweizer, M. *A guided tour through quadratic hedging approaches*. SFB 373 Discussion Papers 1999,96. Humboldt University of Berlin, Interdisciplinary Research Project 373: Quantification and Simulation of Economic Processes, May 1999.

ref4 [Uni] University, C. *Geometric Brownian motion*. http://www.columbia.edu/~ks20/FE-Notes/4700-07-Notes-GBM.pdf. Accessed: 2022-05-09.

ref11 [Unk] Unkown, U. *Risk-neutral pricing*. https://webspace.maths.qmul.ac.uk/a.gnedin/StochCalcDocs/StochCalcSection4.pdf. Accessed: 2022-05-09.