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Multi-view geometry of the plane

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

The theory of multi-view geometry concerns the reconstruction of a landscape from multiple images.

We will analyze the reconstruction of $\mathbb{P}^2$ from n images, for $n = 2, 3, 4$. A reconstruction of $\mathbb{P}^2$ consists of finding a surface $S \subseteq (\mathbb{P}^1)^n$ and a birational map α from S back to $\mathbb{P}^2$, when S is embedded into $\mathbb{P}^{2^n-1}$ using the Segre embedding. When $n = 2$, then $S = (\mathbb{P}^1)^2$ and α is a projection from a specific point q on S in $\mathbb{P}^3$. When $n = 3$ and $n = 4$, then $S \subset (\mathbb{P}^1)^n$, and α is a projection from the span $\langle C \rangle$, restricted to S, where C is a curve in S in $\mathbb{P}^{2^n-1}$.

As we will see, when the number of images used to reconstruct $\mathbb{P}^2$ increases, there is less ambiguity in the reconstruction. We will analyze the ambiguity for $n = 2, 3, 4$ images.

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CHAPTER 1

Introduction

Multi-view geometry is the geometry of multiple images. The theory concerns how to reconstruct a 3-dimensional (3D) landscape from 2D images of the landscape. When taking an image, we obtain a projection from a 3D space to a 2D plane. Thus, from multiple images, we obtain multiple projections.

In this thesis, we will study 1D images of the 2D plane. The images are obtained by projections from points in the plane. The question is whether we can reconstruct the plane from these images. We will consider the reconstruction of the projective plane and get:

Problem. Can we reconstruct the projective plane $\mathbb{P}^2$ based on 1D images from camera centers in $\mathbb{P}^2$?

We are interested in finding a unique reconstruction of $\mathbb{P}^2$. The reconstruction is based on 1D images from different camera centers $q_i \in \mathbb{P}^2$, i.e. projections $\pi_{q_i} : \mathbb{P}^2 \to \mathbb{P}^1$. We will consider n camera centers, for $n = 2, 3, 4$. By composing an image from each camera center, we can identify points in $(\mathbb{P}^1)^n$ that corresponds to points in $\mathbb{P}^2$.

Definition 1.0.1. A reconstruction of $\mathbb{P}^2$ is obtained, if we can find:

i) a surface $S \subseteq (\mathbb{P}^1)^n$ birationally equivalent to $\mathbb{P}^2$, where S consists of identified points in $(\mathbb{P}^1)^n$ from each 1D image, and

ii) an inverse birational map $\alpha : S \dashrightarrow \mathbb{P}^2$.

It may be tedious to determine S based on identified points in $(\mathbb{P}^1)^n$, thus we can embed identified points into $\mathbb{P}^{2^n-1}$ using the Segre embedding, and search for S and α in $\mathbb{P}^{2^n-1}$. When embedded into $\mathbb{P}^{2^n-1}$ $S = \langle S \rangle \cap (\mathbb{P}^1)^n$, where $\langle S \rangle$ is the span of S, i.e. the intersection of all projective subspaces containing S.

When $n = 2$, then $S = (\mathbb{P}^1)^2$, as $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birationally equivalent. To find α we embed S into $\mathbb{P}^3$, where the image of S is the quadric Q in $\mathbb{P}^3$. Then, α is the projection from the point $q \in Q$, where q is the image of the line between the two camera centers $q_0, q_1 \in \mathbb{P}^2$, i.e. $q = p(\overline{q_0 q_1})$. The projection from any point on the quadric is birational to $\mathbb{P}^2$, thus to find α we need to find the specific $q = p(\overline{q_0 q_1})$.

When $n = 3$, then $S \subset (\mathbb{P}^1)^3$. To find S we want to identify enough points in $(\mathbb{P}^1)^3$, such that they span a $\mathbb{P}^6$ when embedded into $\mathbb{P}^7$, as the image of S spans a $\mathbb{P}^6$ in $\mathbb{P}^7$. To find α in $\mathbb{P}^7$, we search for a curve C of degree 3 that spans $\mathbb{P}^3$. However, there are two different classes of curves of degree 3 that spans $\mathbb{P}^3$. Thus to find α we need to find the right class. The map α will be the projection from $\langle C \rangle = \mathbb{P}^3$, restricted to S.

When $n = 4$, then $S \subset (\mathbb{P}^1)^4$. To find S we want to identify enough points in $(\mathbb{P}^1)^4$, such that they span a $\mathbb{P}^{10}$ when embedded into $\mathbb{P}^{15}$, as the image of S spans a $\mathbb{P}^{10}$. To find α in $\mathbb{P}^{15}$, we search for a curve C of degree 8 that spans $\mathbb{P}^7$. Then, α will be the projection from $\mathbb{P}^7 \supset C$, restricted to S.

For $n = 3$ and $n = 4$, as $S \subset (\mathbb{P}^1)^n$ is strictly contained, we need to identify enough linearly independent points to determine S in $\mathbb{P}^{2^n-1}$. If we cannot identify sufficiently many linearly independent points, we cannot uniquely reconstruct $\mathbb{P}^2$. However, we might identify some points or a curve in $\mathbb{P}^{2^n-1}$, that corresponds to points or curves in $\mathbb{P}^2$. We call these critical configurations of points. We will classify different critical configurations and study the information we obtain regarding $\mathbb{P}^2$ in these cases.

We find that the reconstruction of $\mathbb{P}^2$ might be ambiguous for both $n = 2$ and $n = 3$, even after S is identified in $\mathbb{P}^{2^n-1}$. For $n = 2$, there exists a birational map to $\mathbb{P}^2$ from every point on the quadric in $\mathbb{P}^3$, α may be any of these maps. Hence, it is crucial to find the exact point $q = p(\overline{q_0 q_1})$ to reconstruct the $\mathbb{P}^2$ we started out with. For $n = 3$, α is the projection from the span of a curve of degree 3 in $\mathbb{P}^6$. But such a curve may belong to either of two classes. Thus, to uniquely reconstruct the $\mathbb{P}^2$ we started out with, it will be crucial to find a curve of the right class. For $n = 4$, if we can find S, there is no ambiguity. Thus, $n = 4$ is the least amount of images required to uniquely reconstruct $\mathbb{P}^2$.

1.1 Outline

In Chapter 2 we consider some useful results regarding the Segre embedding, divisors, the Picard group and the Chow ring.

In Chapter 3, we consider the reconstruction of $\mathbb{P}^2$ from two projections, where each projection is an image from a camera center $q_i \in \mathbb{P}^2$. As $S = \mathbb{P}^1 \times \mathbb{P}^1$, the problem of reconstruction is reduced to finding α. We describe the birational map between the projective plane $\mathbb{P}^2$ and the image of S when embedded into $\mathbb{P}^3$. Thereafter, we study the correspondence between curves in $\mathbb{P}^2$ and curves in the image of S in $\mathbb{P}^3$.

In Chapter 4 we consider the reconstruction of $\mathbb{P}^2$ from three projections, where each projection is an image from a camera center $q_i \in \mathbb{P}^2$. In this chapter, we are looking for the surface $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ birationally equivalent to $\mathbb{P}^2$, and the inverse birational map α from S to $\mathbb{P}^2$. To determine S in $\mathbb{P}^7$ we need to identify enough points so that they span a hyperplane of $\mathbb{P}^7$. If they do not, the reconstruction will be ambiguous. Lastly, we end the chapter by classifying different critical configurations of points and study the information we obtain about $\mathbb{P}^2$ in these cases.

In Chapter 5 we consider the reconstruction of $\mathbb{P}^2$ from four projections, where each projection is an image from a camera center $q_i \in \mathbb{P}^2$. In this chapter, we are looking for the surface $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ birationally equivalent to $\mathbb{P}^2$, and the inverse birational map α from S to $\mathbb{P}^2$. To determine S in $\mathbb{P}^{15}$, we want to identify enough points so that they span $\mathbb{P}^{10}$. The chapter ends with a classification of different critical configurations, cases where the points do not span $\mathbb{P}^{10}$, and study the information we obtain about $\mathbb{P}^2$ in these cases.

1.2 Prerequisites

We will assume familiarity with algebraic geometry at the level of [Har77]. The thesis is self-contained, presenting the definitions and results we need.

1.3 Further questions

We have yet to describe how to find the curve C that is necessary for finding α in $\mathbb{P}^{2^n-1}$. Although relevant, this will not be discussed in this thesis.

Further, we have studied a reconstruction of the projective plane $\mathbb{P}^2$ based on 1D images. An interesting generalization would be to study a reconstruction of the projective space $\mathbb{P}^3$ based on 2D images. However, such a generalization is not included.

CHAPTER 2

Preliminaries

In this chapter we first recall some useful results regarding the Segre embedding, then we move on to describing divisors, the Picard group and the Chow ring.

2.1 The Segre embedding

The Segre map $\sigma_{n,m}$ is a map from the product of two projective spaces $\mathbb{P}^n \times \mathbb{P}^m$ into the projective space $\mathbb{P}^{(nm+n+m)}$, i.e.

$$\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n \cdot m)+n+m} \tag{2.1}$$

Given the homogeneous coordinates $[x] = (x_0 : \cdots : x_n)$ and $[y] = (y_0 : \cdots : y_m)$ in each projective space $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively, the Segre map becomes

$$\sigma_{n,m}\big([x], [y]\big) = \big(x_0 y_0 : \cdots : x_i y_j : \cdots : x_n y_m\big) \tag{2.2}$$

where $x_i y_j$ represents any coordinate such that $0 \leq i \leq n$ and $0 \leq j \leq m$.

The double indexing in (2.2) makes it possible to represent the coordinates of all points in the image of $\sigma_{n,m}$ as entries in a $(n+1) \times (m+1)$-matrix. This is shown in (2.3), where we denote the matrix as M.

$$M = (x_0 : \cdots : x_n)^t (y_0 : \cdots : y_m) = \begin{pmatrix} x_0 y_0 & x_0 y_1 & \dots & x_0 y_m \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_0 & x_n y_1 & \dots & x_n y_m \end{pmatrix}$$

$$= \begin{pmatrix} t_{00} & t_{01} & \dots & t_{0m} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n0} & t_{n1} & \dots & t_{nm} \end{pmatrix} \tag{2.3}$$

All entries $t_{i,j}$ in (2.3) are homogeneous coordinates in the projective space $\mathbb{P}^{(nm+n+m)}$, indexed by pairs (i, j) with $0 \leq i \leq n$ and $0 \leq j \leq m$.

Definition 2.1.1 (Closed embedding)**.** A morphism $\iota : X \to Y$ is a closed embedding if X is isomorphic to the image $\iota(X)$ which is closed in Y.

Proposition 2.1.2 ([EOa], Proposition 5.20)**.** *The Segre map $\sigma_{n,m}$ is a closed embedding of the product $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^{nm+n+m}$. The image, usually denoted $S_{n,m}$, of $\sigma_{n,m}$ is the locus where the 2×2-minors of the matrix M vanish. $S_{n,m}$ will therefore be a projective variety called a Segre variety.*

In more generality, the Segre map extends to an embedding of multiple projective spaces, by repeated application of Proposition 2.1.2 above.

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^{((n_1+1)\dots(n_k+1))-1}$$

2.2 Divisors

Let C be a non-singular curve in $\mathbb{P}^2$. For each line $L_i \in \mathbb{P}^2$, the intersection $C \cap L_i$ is equal to a finite number of points on C. We can obtain the number of intersection points by the following theorem.

Theorem 2.2.1 ([Har77], Theorem 7.7-Corollary 7.8 (Bézout's theorem))**.** *Let Y, Z be two distinct curves in $\mathbb{P}^2$ of degree u, v respectively. If $Y \cap Z$ contains finitely many points, then $Y \cap Z = \{P_0, \dots P_r\}$ will be the intersection points of Y and Z and consists of $u \cdot v$ points, counting with multiplicities.*

Let d be the degree of C. Then from Theorem 2.2.1, we know that $C \cap L_i$ will contain exactly d points, when counting with multiplicities. Thus,

$$C \cap L_i = \sum a_j P_j \tag{2.4}$$

where $P_j \in C$ represents the intersection points and $a_j \in \mathbb{Z}$ is the number of multiplicities of each point. In fact, Equation (2.4) is called a divisor on C.

Definition 2.2.2 (Divisor)**.** A divisor D of a variety X is a finite formal sum of irreducible subvarieties $Y_i \subseteq X$ of codimension 1, i.e.

$$D = \sum a_i Y_i \tag{2.5}$$

where $a_i \in \mathbb{Z}$. The divisor D is an effective divisor if all $a_i \in D$ are non-negative in Equation (2.5).

We define $\mathrm{Div}\,(X)$ as the group of divisors D_i of X.

$$\mathrm{Div}(X) = \bigoplus_{i \in I} D_i = \{\sum a_i Y_i \mid a_i \in \mathbb{Z}, Y_i \subseteq X\}$$

Then in Equation (2.4), for each L_i in $\mathbb{P}^2$, we obtain a new divisor on C. Such that $\mathrm{Div}(C)$ contains the group of all divisors of C.

The group $\mathrm{Div}(X)$ is extensive and quite unmanageable. Therefore, we often consider the quotient group $\mathrm{Div}\,(X)/\sim$, i.e. the group of divisors of X modulo linear equivalences. See [Har77, pp. 129–149] for more details.

In fact,

Proposition 2.2.3 ([EOa], Proposition 15.14)**.** *When X is a non-singular variety, the map $\rho : \big(\mathrm{Div}\,(X)/\sim\big) \to \mathrm{Pic}\,X$ is an isomorphism.*

2.3 Picard group

In this section we describe the Picard group of $(\mathbb{P}^1)^n$, by considering the Picard group for $n = 4$. This will be of particular relevance to the content in Chapter 5. The theory and approach however is entirely equivalent for $n = 2$ and $n = 3$, that will be of relevance in Chapter 3 and Chapter 4 repectively.

We use a similar definition as [EOb, p. 222]

Definition 2.3.1 (Picard group)**.** The Picard group of non-singular varieties X, denoted as $\mathrm{Pic}\,(X)$, is the group of isomorphism classes of invertible sheaves (or line bundles) on X. It has tensor product as the group operation and the structure sheaf as the identity.

Consider the projections π_i from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to each $\mathbb{P}^1$

$$\begin{tikzcd} & & & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dlll, "\pi_0"'] \arrow[dll, "\pi_1"] \arrow[drr, "\pi_2"'] \arrow[drrr, "\pi_3"] & & & \\ \mathbb{P}^1 & \mathbb{P}^1 & & & & \mathbb{P}^1 & \mathbb{P}^1 \end{tikzcd}$$

Figure 2.1: Projections π_i from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to each of the $\mathbb{P}^1$'s

We let

$$h_0 = \pi_0^*([\mathrm{pt}]), \qquad h_1 = \pi_1^*([\mathrm{pt}]), \qquad h_2 = \pi_2^*([\mathrm{pt}]), \qquad h_3 = \pi_3^*([\mathrm{pt}]) \tag{2.6}$$

where π_i^* is the inverse image of π_i, such that π_i^* sends the class of a point in $\mathbb{P}^1$ to the class of a threefold in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We have

$$\begin{aligned} h_0 &= \pi_0^*([\mathrm{pt}]) = \left[\{\mathrm{pt}\} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\right] \\ h_1 &= \pi_1^*([\mathrm{pt}]) = \left[\mathbb{P}^1 \times \{\mathrm{pt}\} \times \mathbb{P}^1 \times \mathbb{P}^1\right] \\ h_2 &= \pi_2^*([\mathrm{pt}]) = \left[\mathbb{P}^1 \times \mathbb{P}^1 \times \{\mathrm{pt}\} \times \mathbb{P}^1\right] \\ h_3 &= \pi_2^*([\mathrm{pt}]) = \left[\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \{\mathrm{pt}\}\right] \end{aligned} \tag{2.7}$$

where each h_i is of codimension 1, as it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Thus, h_0, h_1, h_2, h_3 are divisors of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The class of the hyperplane section in the Segre embedding is given by

$$h_0 + h_1 + h_2 + h_3$$

We are interested in the Picard group of the domain and codomain of π^*, i.e. $\mathrm{Pic}\,(\mathbb{P}^1)$ and $\mathrm{Pic}\,(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ respectively. That is

$$\begin{aligned} \pi_i^* : \mathrm{Pic}\,(\mathbb{P}^1) &\rightarrow \mathrm{Pic}\left(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\right) \\ [D] &\mapsto \pi_i^*\,[D] \end{aligned}$$

The Picard group of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$\mathrm{Pic}\left(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\right) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \langle h_0, h_1, h_2, h_3 \rangle$$

such that $\langle h_0, h_1, h_2, h_3 \rangle$ forms a basis, where each h_i represents a family of lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The Picard group includes any multiple of the free module generated by $\langle h_0, h_1, h_2, h_3 \rangle$. That is, any linear combination of this free module with integer coefficients, i.e.

$$a_0h_0 + a_1h_1 + a_2h_2 + a_3h_3 \tag{2.8}$$

for $a_i \in \mathbb{Z}$.

As the elements of the Picard group are divisors and spanned by $\langle h_0, h_1, h_2, h_3 \rangle$, each element represents a class of a threefold.

A threefold in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ has a multi degree of the form (a_0, a_1, a_2, a_3), where each a_i represents the corresponding a_i in Equation (2.8). To determine the multi degree of a threefold in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we intersect the class of the threefold with $h_ih_jh_k$ for $i, j, k = 0, 1, 2, 3$ and $i \neq j \neq k$. Here $h_ih_jh_k$ represents the class of a curve isomorphic to $\mathbb{P}^1$. By intersecting the class of a threefold with $h_1h_2h_3$, $h_0h_2h_3$, $h_0h_1h_3$ and $h_0h_1h_2$ separately, we obtain each component of the multi degree, i.e. a_0, a_1, a_2 and a_3 respectively.

The Picard group of the blow-up of $\mathbb{P}^2$ in four points

The Picard group of the blow-up of $\mathbb{P}^2$ in four points, i.e. $\overline{\Gamma_p}$, is generated by

$$\operatorname{Pic}\left(\overline{\Gamma_p}\right) = \langle L, e_0, e_1, e_2, e_3 \rangle$$

Such that any element of the Picard group will be a linear combination of these generators with integer coefficients, i.e.

$$\beta L + \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$$

for $\alpha_i, \beta \in \mathbb{Z}$.

We have a map from $\overline{\Gamma_p} \xrightarrow{\phi} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that

$$\begin{aligned} \phi^* : \operatorname{Pic}\left(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\right) &\to \operatorname{Pic}\left(\overline{\Gamma_p}\right) \\ h_i &\mapsto (L - e_i) \end{aligned}$$

where $(L - e_i)$ represents a line in the blow-up of $\mathbb{P}^2$ that intersects the exceptional divisor e_i. The blow-down of $(L - e_i)$ will be a line in $\mathbb{P}^2$ going through the point q_i.

The class of a hyperplane section in $\operatorname{Pic}\left(\overline{\Gamma_p}\right)$ is given by

$$\begin{aligned} \sum_{i=0}^{3}(L - e_i) &= (L - e_0) + (L - e_1) + (L - e_2) + (L - e_3) \\ &= 4L - e_0 - e_1 - e_2 - e_3 \end{aligned}$$

2.4 The Chow ring

In chapter Chapter 4 we consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^7$, where the image $V = \psi(S)$ in $\mathbb{P}^7$ is a divisor of the image of $U = \psi(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ in $\mathbb{P}^7$, as V is of codimension 1 in U. Therefore, to find the class of V we study the Picard group of U.

However, in Chapter 5 we consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^{15}$, where the image $V = \Psi(S) \subset \mathbb{P}^{15}$ no longer is a divisor of the image

$U = \Psi(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^{15}$, as V is of codimension 2 in U. Thus, we turn to the Chow ring of U, that includes the Picard group of U.

The Chow ring of U considers all codimensions of U, such that $\mathrm{CH}^1(U)$ represents the elements in the Chow ring of codimension 1, modulo linear equivalences, and thereby corresponds to the Picard group of U. Further, $\mathrm{CH}^2(U)$ represents the elements in the Chow ring of codimension 2, modulo linear equivalences. $\mathrm{CH}^3(U)$ represents the elements of the Chow ring of codimension 3, modulo linear equivalences. And lastly $\mathrm{CH}^4(U)$ represents the elements in the chow ring of codimension 4, modulo linear equivalences.

As stated above $\mathrm{CH}^1(U) = \mathrm{Pic}(U)$, thus the elements are classes generated by

$$\langle h_0, h_1, h_2, h_3 \rangle$$

where we regard each element h_i as in Equation (2.7).

It can be shown that the elements of $\mathrm{CH}^2(U)$ are classes generated by

$$\langle h_0h_1, h_0h_2, h_0h_3, h_1h_2, h_1h_3, h_2h_3 \rangle$$

i.e. the classes are spanned by products of two classes in $\mathrm{CH}^1(U)$. Each element h_ih_j is given by

$$\begin{aligned}
h_0h_1 &= [\{\mathrm{pt}\} \times \{\mathrm{pt}\} \times \mathbb{P}^1 \times \mathbb{P}^1] \\
h_0h_2 &= [\{\mathrm{pt}\} \times \mathbb{P}^1 \times \{\mathrm{pt}\} \times \mathbb{P}^1] \\
h_0h_3 &= [\{\mathrm{pt}\} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \{\mathrm{pt}\}] \\
h_1h_2 &= [\mathbb{P}^1 \times \{\mathrm{pt}\} \times \{\mathrm{pt}\} \times \mathbb{P}^1] \\
h_1h_3 &= [\mathbb{P}^1 \times \{\mathrm{pt}\} \times \mathbb{P}^1 \times \{\mathrm{pt}\}] \\
h_2h_3 &= [\mathbb{P}^1 \times \mathbb{P}^1 \times \{\mathrm{pt}\} \times \{\mathrm{pt}\}]
\end{aligned}$$

In $\mathrm{CH}^3(U)$, the elements are classes generated by

$$\langle h_0h_1h_2, h_0h_1h_3, h_0h_2h_3, h_1h_2h_3 \rangle$$

i.e. the classes are spanned by products of *three* classes in $\mathrm{CH}^1(U)$. Each element $h_ih_jh_k$ is given by

$$\begin{aligned}
h_0h_1h_2 &= [\{\mathrm{pt}\} \times \{\mathrm{pt}\} \times \{\mathrm{pt}\} \times \mathbb{P}^1] \\
h_0h_1h_3 &= [\{\mathrm{pt}\} \times \{\mathrm{pt}\} \times \mathbb{P}^1 \times \{\mathrm{pt}\}] \\
h_0h_2h_3 &= [\{\mathrm{pt}\} \times \mathbb{P}^1 \times \{\mathrm{pt}\} \times \{\mathrm{pt}\}] \\
h_1h_2h_3 &= [\mathbb{P}^1 \times \{\mathrm{pt}\} \times \{\mathrm{pt}\} \times \{\mathrm{pt}\}]
\end{aligned}$$

Lastly, there is only one element in $\mathrm{CH}^4(U)$, that is

$$\langle h_0h_1h_2h_3 \rangle$$

which is the class of a single point in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, thus $h_0h_1h_2h_3 = 1$.

Thus, the Chow ring of U is the polynomial ring in the four variables h_0, h_1, h_2, h_3 modulo the ideal representing the relations $h_0^2 = h_1^2 = h_2^2 = h_3^2 = 0$, i.e.

$$\mathrm{CH}\left(U\right) = \frac{\mathbb{Z}[h_0, h_1, h_2, h_3]}{(h_0^2, h_1^2, h_2^2, h_3^2)}$$

All subsets $\mathrm{CH}^i(U) \subset \mathrm{CH}(U)$ are closed under addition, that is $h_i + h_j \in \mathrm{CH}^1(U)$ for $h_i, h_j \in \mathrm{CH}^1(U)$. Further, the intersection $h_i h_j$ of two elements h_i, h_j in the Chow ring of codimension a and codimension b respectively, will be an element of Chow ring of codimension $a + b$.

The Chow ring of the blow-up of $\mathbb{P}^2$ in four points

In the Chow ring of $\overline{\Gamma_p}$ the relations are $LL = 1$, $Le_i = Le_j = 0$, $e_i e_j = 0$, $e_i e_i = -1$, such that

$$\mathrm{CH}\left(\overline{\Gamma_p}\right) = \frac{\mathbb{Z}[L, e_0, e_1, e_2, e_3]}{\left(LL - 1,\ Le_i,\ Le_j,\ e_i e_j,\ e_i e_i + 1\right)}$$

2.5 Intersection products

A curve in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a variety of dimension 1. We are not able to determine the degree of the curve as easily. $\mathbb{P}^1$ for instance, is isomorphic to various curves of different degrees, in different projective spaces, e.g. a conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. Thus, both the embedding of the curve and the projective space the curve is embedded into, are crucial to determine the degree of the curve.

Consider the Segre embedding

$$\Psi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^{15}$$

Let $\mathbb{P}^1 \times \{\mathrm{pt}\} \times \{\mathrm{pt}\} \times \{\mathrm{pt}\}$ be a curve embedded into $\mathbb{P}^{15}$ with the Segre embedding. We can calculate the degree of such a curve by using the theory above.

The class of a curve in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ will be a linear combination of the generators in $\mathrm{CH}^3(U)$, i.e.

$$\sum_{i,j,k=0}^{3} \alpha_{ijk} h_i h_j h_k = a_{012} h_0 h_1 h_2 + a_{013} h_0 h_1 h_3 + a_{023} h_0 h_2 h_3 + a_{123} h_1 h_2 h_3 \tag{2.9}$$

Thus, a curve in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ has a multi degree $(a_{012}, a_{013}, a_{023}, a_{123})$, where each a_{ijk} represents the corresponding a_{ijk} in Equation (2.9). By intersecting the class of the curve with h_3, h_3, h_1 and h_0 separately, we obtain a_{012}, a_{013}, a_{023} and a_{123} respectively.

To obtain the degree of a curve C in the Segre embedding, we intersect the class of the curve $[C]$ with the class of the hyperplane section $h_0 + h_1 + h_2 + h_3$.

$$\begin{aligned}
[C] \cdot (h_0 + h_1 + h_2 + h_3) =& (a_{012} h_0 h_1 h_2 + a_{013} h_0 h_1 h_3 + a_{023} h_0 h_2 h_3 + a_{123} h_1 h_2 h_3) \\
& \cdot (h_0 + h_1 + h_2 + h_3) \\
=& (a_{012} + a_{013} + a_{023} + a_{123}) \cdot h_0 h_1 h_2 h_3 \\
=& (a_{012} + a_{013} + a_{023} + a_{123})
\end{aligned}$$

since $h_i^2 = 0$ for $i = 0, 1, 2, 3$ and $h_0 h_1 h_2 h_3 = 1$. Thus the degree of the curve C in the Segre embedding is $(a_{012} + a_{013} + a_{023} + a_{123})$.

Further, we can study the class of a surface $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ when embedded into $\mathbb{P}^{15}$. A surface will be of codimension 2 in U, so we consider $\mathrm{CH}^2(U)$. The generators of $\mathrm{CH}^2(U)$

$$\langle h_0h_1, h_0h_2, h_0h_3, h_1h_2, h_1h_3, h_2h_3\rangle$$

Then, the class of S will be a linear combination of these generators with integer coefficients, i.e.

$$\sum_{i,j=0}^{3} \alpha_{ij}h_ih_j = \alpha_{01}h_0h_1 + \alpha_{02}h_0h_2 + \alpha_{03}h_0h_3 + \alpha_{12}h_1h_2 + \alpha_{13}h_1h_3 + \alpha_{23}h_2h_3 \tag{2.10}$$

Thus, a surface in U will have a multi degree of six components $(\alpha_{01}, \alpha_{02}, \alpha_{03}, \alpha_{12}, \alpha_{13}, \alpha_{23})$, where each a_{ij} represents the corresponding a_{ij} in Equation (2.10). By intersecting the class of a surface with h_2h_3, h_1h_3, h_1h_2, h_0h_3, h_0h_2 and h_0h_1 separately, we obtain each component of the multi degree, that is α_{01}, α_{02}, α_{03}, α_{12}, α_{13} and α_{23} respectively.

To determine the degree of a surface S in U, we intersect the class of S with the intersection of two hyperplane sections.

$$\begin{aligned}[V]\cdot(h_0+h_1+h_2+h_3)^2 &= 2\cdot(\alpha_{01}+\alpha_{02}+\alpha_{03}+\alpha_{12}+\alpha_{13}+\alpha_{23})\cdot h_0h_1h_2h_3\\ &= 2\cdot(\alpha_{01}+\alpha_{02}+\alpha_{03}+\alpha_{12}+\alpha_{13}+\alpha_{23})\end{aligned}$$

since $h_i^2 = 0$ for $i = 0, 1, 2, 3$ and $h_0h_1h_2h_3 = 1$. Thus, the degree of a surface S in the Segre embedding is $2\cdot(\alpha_{01}+\alpha_{02}+\alpha_{03}+\alpha_{12}+\alpha_{13}+\alpha_{23})$.

CHAPTER 3

Reconstruction from two camera centers

We will reconstruct $\mathbb{P}^2$ from two projections $\pi_{q_i} : \mathbb{P}^2 \to \mathbb{P}^1$, where each projection is an image from a camera center $q_i \in \mathbb{P}^2$ for $i = 0, 1$. We will refer to the two camera centers as the points q_0, q_1 in $\mathbb{P}^2$ from now on.

In this case, $S = \mathbb{P}^1 \times \mathbb{P}^1$ as $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birationally equivalent. Thus, the problem related to finding a reconstruction of $\mathbb{P}^2$ is reduced to finding α. We will, in addition to finding α, describe the birational map between the projective plane $\mathbb{P}^2$ and the image of S, when S is embedded into $\mathbb{P}^3$. As we will see, the map α is the projection from a point $q = p(\overline{q_0 q_1})$ on S in $\mathbb{P}^3$. However, finding q will be difficult, as any point on S in $\mathbb{P}^3$ will have a birational correspondence to a $\mathbb{P}^2$.

Then, we will study the correspondence between curves in $\mathbb{P}^2$ and curves in the image of S in $\mathbb{P}^3$. In particular, the correspondence between degrees of curves in $\mathbb{P}^2$ and bidegrees of curves in the image of S in $\mathbb{P}^3$.

3.1 A rational map from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we want to find a rational map from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$. First, we construct the map p from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$. Then, we show that p is a rational map, and find that it is in fact birational. Lastly, we consider the closure of the graph of p, and recognize the correspondence to the blow-up of $\mathbb{P}^2$ in two points.

Construction

In $\mathbb{P}^2$, the equation $a_0x_0 + a_1x_1 + a_2x_2 = 0$ represents a line, which we denote by the coordinates $(a_0 : a_1 : a_2)$.

Given two points q_0, $q_1 \in \mathbb{P}^2$, where $q_0 = (1 : 0 : 0)$ and $q_1 = (0 : 1 : 0)$. The variety of lines in $\mathbb{P}^2$ through q_i is isomorphic to $\mathbb{P}^1$, for a fixed $i = 0$, 1. To see this, consider the lines $m_0 = (1 : 0 : 0)$ and $m_1 = (0 : 1 : 0)$. Since m_i is a line in $\mathbb{P}^2$ it will be isomorphic to $\mathbb{P}^1$, and in particular the maps

$$p_0 : (0 : x_1 : x_2) \mapsto (-x_2 : x_1)$$

$$p_1 : (x_0 : 0 : x_2) \mapsto (-x_2 : x_0)$$

are isomorphisms between m_i and $\mathbb{P}^1$.

Now, for any line $l \neq m_i$ in $\mathbb{P}^2$ passing through q_i, consider the mapping

$$l \to l \cap m_i$$

which is well-defined as any two lines in $\mathbb{P}^2$ intersect in a single point.
It also has an inverse map, given by

$$q \to \overline{q_i q}$$

for any point $q \in m_i$ and $q_i \in \{q_0, q_1\}$. Hence, the variety of lines through a fixed point q_i in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

For any point $r \in \mathbb{P}^2$, where $r \neq q_0$, q_1, consider the lines $\overline{q_0 r}$ and $\overline{q_1 r}$. By the previous discussion, this defines a map

$$p : \mathbb{P}^2 \setminus \{q_0, q_1\} \to \mathbb{P}^1 \times \mathbb{P}^1$$

as depicted in Figure 3.1 below.

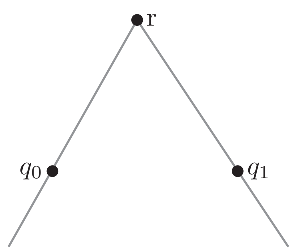

Figure 3.1: Lines to r through q_0 and q_1

Given a point $r = (x_0 : x_1 : x_2)$ we want to compute where r is mapped.

$$p\big((x_0 : x_1 : x_2)\big) = (u_0 : v_0) \times (u_1 : v_1)$$

Firstly, we need to find the equation for the line $l_i = \overline{q_i r}$, for $i = 0$, 1.

Consider the line l_0 with coordinates $(a_0 : a_1 : a_2)$. By definition, both q_0 and r are points on l_0. Thus, we obtain the two equations

$$\begin{aligned} a_0 \cdot 1 + a_1 \cdot 0 + a_2 \cdot 0 = 0 \\ a_0 x_0 + a_1 x_1 + a_2 x_2 = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow a_1 x_1 &= -a_2 x_2 \\ \frac{a_1}{a_2} &= -\frac{x_2}{x_1} \end{aligned}$$

which yields the solution $(a_0 : a_1 : a_2) = (0 : -x_2 : x_1)$.

Similarly, we can find the coordinates of the line $l_1 = (b_0 : b_1 : b_2)$, by solving the two equations

$$\begin{aligned} b_0 \cdot 0 + b_1 \cdot 1 + b_2 \cdot 0 = 0 \\ b_0 x_0 + b_1 x_1 + b_2 x_2 = 0 \end{aligned}$$

$$\Rightarrow b_0 x_0 = -b_2 x_2$$

Again, which yields the solution $(b_0 : b_1 : b_2) = (-x_2 : 0 : x_1)$.

Hence, by our identifications

$$l_i \mapsto l_i \cap m_i \overset{p_i}{\longmapsto} \mathbb{P}^1$$

we have found the map

$$p : (x_0 : x_1 : x_2) \mapsto (-x_2 : x_1) \times (-x_2 : x_0)$$

Birational equivalence

First, we define a rational map, and use the same definition as in [EOa, p. 130].

Definition 3.1.1 (Rational map)**.** A rational map between two varieties X and Y, consists of an open subset $U \subseteq X$ and a morphism $f : U \to Y$. A rational map is usually indicated by a broken arrow, i.e. $f : X \dashrightarrow Y$.

As p is defined on the open subset $U = \mathbb{P}^2 \setminus \{q_0, q_1\}$, the map is only a morphism from U. However, by Definition 3.1.1 p is a rational map between $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, we write p as

$$p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

The line defined by $x_2 = 0$ in $\mathbb{P}^2$ is in fact the fiber of the point $(0 : 1) \times (0 : 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, p is not injective at any point on the line $x_2 = 0$ in $\mathbb{P}^2$.

On the other hand, when $x_2 \neq 0$, then p is injective. Furthermore, p is an isomorphism between the two subsets $U' = \mathbb{P}^2 \setminus \{\overline{q_0 q_1}\} \subseteq \mathbb{P}^2$ and $V' = p(U') \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, where U' ensures $x_2 \neq 0$.

To see that $p : U' \to V'$ is an isomorphism, we need to exhibit a map $q : V' \to U'$ such that $q \circ p = \mathrm{id}_{U'}$ and $p \circ q = \mathrm{id}_{V'}$. That is, we want to show that the inverse map p^{-1} is well-defined on V'.

Let us consider the point $(u_0 : v_0) \times (u_1 : v_1) \in V'$, i.e. where $u_0, u_1 \neq 0$. We have that

$$\begin{aligned} p^{-1}\big((u_0 : v_0) \times (u_1 : v_1)\big) &= p^{-1}\big((u_0u_1 : v_0u_1) \times (u_0u_1 : u_0v_1)\big) \\ &= (u_0v_1 : v_0u_1 : -u_0u_1) \end{aligned} \tag{3.1}$$

since $u_0, u_1 \neq 0$. As $u_0u_1 \neq 0$, the fibre $(u_0v_1 : v_0u_1 : -u_0u_1)$ is a well-defined point in $U' \subseteq \mathbb{P}^2$.

Similarly, given a point $(x_0 : x_1 : x_2) \in U'$ i.e. any point such that $x_2 \neq 0$. We obtain

$$p\big((x_0 : x_1 : x_2)\big) = (-x_2 : x_1) \times (-x_2 : x_0)$$

As $x_2 \neq 0$, the image $(-x_2 : x_1) \times (-x_2 : x_0)$ is contained in V'. Hence, $p(U') = V'$, p^{-1} is well-defined on V' and $p : U \to V'$ is an isomorphism.

■

Definition 3.1.2 (Birational Map)**.** A rational map that has a rational inverse is called a birational map. Two varieties X and Y are said to be birationally equivalent if there exists open sets $U \subseteq X$, $V \subseteq Y$ and an isomorphism $\phi : U \to V$ between them.

We want to prove that $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birationally equivalent using the map p, previously shown to be rational.

We choose the open subsets $U' = \mathbb{P}^2 \setminus \{\overline{q_0 q_1}\} \subseteq \mathbb{P}^2$ and $V' = p(U') \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.

To see that p is an isomorphism between U' and V', we refer to our previous discussion.

■

Now, the inverse birational map q, where $q : V' \xrightarrow{\simeq} U'$ is an isomorphism, is in fact the map α. In Section 3.2, we will embed $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$, where the image of $\mathbb{P}^1 \times \mathbb{P}^1$ becomes the quadric Q in $\mathbb{P}^3$. Then, α becomes the projection from the point $q = (0:1) \times (0:1) \in Q$ in $\mathbb{P}^3$ back to $\mathbb{P}^2$, where q is the image of the line $x_2 = 0$ in $\mathbb{P}^2$. This is illustrated in Figure 3.2.

$$\begin{tikzcd}
 & \mathbb{P}^3 \\
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[ur, hook] \arrow[r, "\sigma_{1,1}"] \arrow[dr, dashed, shift left, "\alpha"] & Q \subseteq \mathbb{P}^3 \arrow[u, hook] \arrow[d, dashed, "\pi_q = \alpha"] \\
 & \mathbb{P}^2 \arrow[ul, dashed, shift left, "p"]
\end{tikzcd}$$

Figure 3.2: The birational correspondence between $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

However, given only the quadric Q, we cannot uniquely determine α, since for every point on Q there is a corresponding birational map to a $\mathbb{P}^2$.

The graph of the rational map

We now want to study W, the complement of $p(U)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. the points in $\mathbb{P}^1 \times \mathbb{P}^1$ which are not in the image of p. As previously shown, p is an isomorphism between U' and V'. i.e.

$$p : U' \to V' = \{(u_0 : v_0) \times (u_1 : v_1) \mid u_0 \neq 0 \wedge u_1 \neq 0\}$$

Thus, we only need to consider points on the form $(0 : v_0) \times (u_1 : v_1)$ and $(u_0 : v_0) \times (0 : v_1)$. Howevever, if $(0 : v_0) \times (u_1 : v_1) \in p(U)$, then $u_1 = 0$ as well. As the point where $u_0 = u_1 = 0$ is in the image of p, we exclude the point where both u_0 and u_1 are equal to zero.

Thus, the points in the compliment of $p(U)$ are $(u_0 : v_0) \times (u_1 : v_1)$ satisfying exactly $u_0 = 0$ or $u_1 = 0$. Then, if $u_0 = 0$, then u_1 can be anything except 0. Similarly, if $u_1 = 0$, then u_0 can be anything except 0. So we have

$$W = \Big((0:1) \times \mathbb{P}^1 \cup \mathbb{P}^1 \times (0:1) \Big) \setminus \{(0:1) \times (0:1)\}$$

which are two lines, i.e. two copies of $\mathbb{P}^1$ with the exception of the point $(0:1) \times (0:1)$, representing the intersection of the two lines.

Definition 3.1.3 (Graph)**.** If $\phi : X \to Y$ is a morphism between varieties, the graph is the subset $\Gamma_\phi = \{(x, \phi(x)) \mid x \in X\}$ of the product $X \times Y$.

We have the rational map,

$$p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

such that p is a morphism between the open subsets U of $\mathbb{P}^2$ and V of $\mathbb{P}^1 \times \mathbb{P}^1$.

Consider the graph Γ_p of the morphism $p : U \to V$. As the subsets U and V are open, the graph will be an open subset of $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ and we therefore look at the closure $\overline{\Gamma_p}$.

By considering the closure of the graph $\overline{\Gamma_p}$, we have an object that maps surjectively onto both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ through the canonical projections $\pi_{\mathbb{P}^2}$ and $\pi_{\mathbb{P}^1\times\mathbb{P}^1}$ of the product $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$.

We denote a general point in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ by $(x_0 : x_1 : x_2) \times (u_0 : v_0) \times (u_1 : v_1)$.

Moreover, the polynomials in the product topology must be polyhomogeneous i.e. that the polynomials are homogeneous with respect to each factor.

$$\begin{tikzcd}
 & \overline{\Gamma_p} \arrow[d, hook] \arrow[ddl, two heads] \arrow[ddr, two heads] & \\
 & \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl, "\pi_{\mathbb{P}^2}"'] \arrow[dr, "\pi_{\mathbb{P}^1\times\mathbb{P}^1}"] & \\
\mathbb{P}^2 \arrow[rr, dashed, "p"'] & & \mathbb{P}^1 \times \mathbb{P}^1
\end{tikzcd}$$

Thus, we want to know which points in $\overline{\Gamma_p}$ are sent to $\mathbb{P}^2 \backslash U$, i.e $q_0 = (1 : 0 : 0)$ and $q_1 = (0 : 1 : 0)$, and which points are sent to $\mathbb{P}^1 \times \mathbb{P}^1 \setminus V$, i.e. the lines $(0 : 1) \times \mathbb{P}^1$ and $\mathbb{P}^1 \times (0 : 1)$. We denote by $\alpha_{\mathbb{P}^2}$ and $\alpha_{\mathbb{P}^1\times\mathbb{P}^1}$ the canonical map from $\overline{\Gamma_p}$ to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ respectively.

Consider the two equations $u_1x_0 + v_1x_2 = 0$ and $u_0x_1 + v_0x_2 = 0$, which are well-defined in the product as they are homogeneous in the three factors, with tridegrees $(1, 0, 1)$ and $(1, 1, 0)$ respectively. Now, for any $(x_0 : x_1 : x_2) \in U$, we have

$$p\big((x_0 : x_1 : x_2)\big) = (-x_2 : x_1) \times (-x_2 : x_0)$$

and hence

$$\Gamma_p \subseteq Z_+(u_1x_0 + v_1x_2) \Rightarrow \overline{\Gamma_p} \subseteq Z_+(u_1x_0 + v_1x_2)$$

$$\Gamma_p \subseteq Z_+(u_0x_1 + v_0x_2) \Rightarrow \overline{\Gamma_p} \subseteq Z_+(u_0x_1 + v_0x_2)$$

Hence, $\overline{\Gamma_p} \subseteq Z_+(u_1x_0 + v_1x_2,\, u_0x_1 + v_0x_2)$.

By definition of closure, the variety $\overline{\Gamma_p}$ is irreducible. Further, we see from the defining equations that $Z_+(u_1x_0 + v_1x_2,\, u_0x_1 + v_0x_2)$ is irreducible.

Then, we can utilize Krulls Hauptidealsatz, to compute the dimension of $Z_+(u_1x_0 + v_1x_2,\, u_0x_1 + v_0x_2)$. Firstly, we see that the variety $Z_+(u_1x_0 + v_1x_2)$ is of dimension $\dim \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 - 1 = 4 - 1 = 3$. Secondly, since $Z_+(u_1x_0 + v_1x_2,\, u_0x_1 + v_0x_2) \supseteq \overline{\Gamma_p}$ is non-empty, it must be of dimension:

$$\dim Z_+(u_1x_0 + v_1x_2,\, u_0x_1 + v_0x_2) = \dim Z_+(u_1x_0 + v_1x_2) - 1 = 3 - 1 = 2$$

Next, we compute the dimension of $\overline{\Gamma_p}$. As Γ_p is a dense open subset of its closure $\overline{\Gamma_p}$, we have $\dim \overline{\Gamma_p} = \dim \Gamma_p$. Moreover, $\dim \Gamma_p = \dim \mathbb{P}^2 \setminus \{q_0, q_1\} = 2$.

Then as,

i) both $\overline{\Gamma_p}$ and $Z_+(u_1x_0+v_1x_2,\, u_0x_1+v_0x_2)$ are irreducible and closed

ii) $\overline{\Gamma_p} \subseteq Z_+(u_1x_0+v_1x_2,\, u_0x_1+v_0x_2)$

iii) $\dim \overline{\Gamma_p} = 2 = \dim Z_+(u_1x_0+v_1x_2,\, u_0x_1+v_0x_2)$

they must be equal, by definition of dimension.

Fibres of the projection from the closure of the graph

If we look at points $(1:0:0)\times(u_0:v_0)\times(u_1:v_1) \in \overline{\Gamma_p} = Z_+(u_1x_0 + v_1x_2,\, u_0x_1+v_0x_2)$.

As $x_1 = x_2 = 0$, we obtain

$$\begin{aligned} 0 &= u_1x_0+v_1x_2 = u_1 \\ 0 &= u_0x_1+v_0x_2 = 0 \end{aligned}$$

Thus,

$$\alpha_{\mathbb{P}^2}^{-1}\big((1:0:0)\big) = (1:0:0)\times\mathbb{P}^1\times(0:1) \in \mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1$$

as u_0 and v_0 can be anything.

Similarly,

$$\alpha_{\mathbb{P}^2}^{-1}\big((0:1:0)\big) = (0:1:0)\times(0:1)\times\mathbb{P}^1 \in \mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1$$

Then, we want to look at the fibres of the morphism $\alpha_{\mathbb{P}^1\times\mathbb{P}^1}$.

First, consider

$$\alpha_{\mathbb{P}^1\times\mathbb{P}^1}^{-1}\big((0:1)\times(0:1)\big)$$

which corresponds to $(x_0:x_1:x_2)\times(0:1)\times(0:1) \in \overline{\Gamma_p} = Z_+(u_1x_0 + v_1x_2,\, u_0x_1+v_0x_2)$. These equations imply $x_2 = 0$, which yields that the fibre equals

$$\alpha_{\mathbb{P}^1\times\mathbb{P}^1}^{-1}\big((0:1)\times(0:1)\big) = (x_0:x_1:0)\times(0:1)\times(0:1) \in \mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1$$

which is a line in $\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1$. When projecting this line down to $\mathbb{P}^2$ using $\alpha_{\mathbb{P}^2}$ we obtain the line $(0:0:1)$ i.e. the line through the points q_0 and q_1.

Further, if we look at the inverse mapping of $(u_0:v_0)\times(0:1)$, for $u_0 \neq 0$, we get

$$(x_0:x_1:x_2)\times(u_0:v_0)\times(0:1) \in Z_+(u_1x_0+v_1x_2,\, u_0x_1+v_0x_2)$$

which, corresponds to

$$\alpha_{\mathbb{P}^1\times\mathbb{P}^1}^{-1}\big((u_0:v_0)\times(0:1)\big) = (1:0:0)\times(u_0:v_0)\times(0:1) \in \mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1$$

Hence, the fibre of the missing line $(u_0:v_0)\times(0:1)$ is a line in $\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1$, which is in fact the same line as the fibre of $(1:0:0)$ under the projection $\alpha_{\mathbb{P}^2}$.

A similar statement is true for the fibre of the line $(0:1)\times(u_1:v_1)$ for $u_1 \neq 0$.

$$\alpha_{\mathbb{P}^1\times\mathbb{P}^1}^{-1}\big((0:1)\times(u_1:v_1)\big) = (0:1:0)\times(0:1)\times(u_1:v_1) \in \mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1$$

The first line $C_0 = (1:0:0) \times (u_0 : v_0) \times (0:1)$ is mapped from the closure of the graph $\overline{\Gamma_p}$ to q_0 and the second line $C_1 = (0:1:0) \times (0:1) \times (u_1 : v_1)$ is mapped from the $\overline{\Gamma_p}$ to q_1.

Since the two lines $(1:0:0) \times (u_0 : v_0) \times (0:1)$ and $(0:1:0) \times (0:1) \times (u_1 : v_1)$ differ in the first coordinates $(1:0:0) \neq (0:1:0)$, C_0 and C_1 have no common points in the closure of the graph $\overline{\Gamma_p}$. Thus, they will not intersect.

In fact, what we just studied, corresponds to the mathematical concept called *blowing up*, where C_0 and C_1 in this case corresponds to the exceptional divisors of q_0 and q_1 respectively.

Definition 3.1.4 (Blowing up)**.** Blowing up or blow-up is a type of geometric transformation which replaces a subspace of a given space with all the directions pointing out of that subspace.

By blowing up of $\mathbb{P}^2$ in the two points q_0 and q_1, we can extend the rational map $p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ into a morphism between the blow-up of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

The theory related to *blowing up* does in fact originate from examples similar to this, where the points outside the domain and codomain of a rational map as well as the associated fibers, are the topic of interest.

3.2 An embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$

We have a map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$ given by

$$\begin{aligned} \sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^3 \\ (u_0 : v_0) \times (u_1 : v_1) &\mapsto (u_0u_1 : u_0v_1 : v_0u_1 : v_0v_1) \\ &\mapsto (y_0 : y_1 : y_2 : y_3) \end{aligned} \tag{3.2}$$

The map $\sigma_{1,1}$ is a closed embedding called the Segre embedding, see Section 2.1 for more details. According to Proposition 2.1.2, the image $S_{1,1} = \sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1)$ is equal to the hypersurface $Z_+(y_0y_3 - y_1y_2)$, obtained by

$$y_0y_3 = u_0u_1v_0v_1 = y_1y_2$$

This hypersurface is known as the quadric in $\mathbb{P}^3$. As $\sigma_{1,1}$ is a closed embedding, the quadric $Z_+(y_0y_3 - y_1y_2) \subseteq \mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

3.3 The image of $\mathbb{P}^2$ in $\mathbb{P}^3$

First, we have the rational map p from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$, such that $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birationally equivalent. Then, we have the Segre embedding $\sigma_{1,1}$ from $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$, where the image $Z_+(y_0y_3 - y_1y_2) = \sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

We compose these two maps, to consider the image of $\mathbb{P}^2$ in $\mathbb{P}^3$. The coordinates in $\mathbb{P}^3$, when mapped from $\mathbb{P}^2$ are given by

$$\begin{aligned} y_0 &= u_0u_1 = (-x_2)(-x_2) = x_2^2 \\ y_1 &= u_0v_1 = (-x_2)(x_0) = -x_0x_2 \\ y_2 &= v_0u_1 = (x_1)(-x_2) = -x_1x_2 \\ y_3 &= v_0v_1 = (x_1)(x_0) = x_0x_1 \end{aligned}$$

where every monomial y_i is of degree 2.

Hence, the composition of the two maps is given by

$$\begin{array}{ccccc} \mathbb{P}^2 & \overset{p}{\dashrightarrow} & \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\sigma_{1,1}} & \mathbb{P}^3 \\ (x_0 : x_1 : x_2) & \mapsto & (-x_2 : x_1) \times (-x_2 : x_0) & \longmapsto & (x_2^2 : -x_0x_2 : -x_1x_2 : x_0x_1) \end{array}$$

Then, to reconstruct $\mathbb{P}^2$, the surface $S = \mathbb{P}^1 \times \mathbb{P}^1$ we want to find is isomorphic to the quadric, i.e. $S \cong Z_+(y_0y_3 - y_1y_2)$.

3.4 Lines in the quadric

We are now interested in the lines on the quadric in $\mathbb{P}^3$. We denote coordinates in $\mathbb{P}^3$ by $(y_0 : y_1 : y_2 : y_3)$.

Recall that the image of the Segre embedding $\sigma_{1,1}$ will be the quadric $Z_+(y_0y_3 - y_1y_2) \subseteq \mathbb{P}^3$.

Consider the matrix representation of the point $(y_0 : y_1 : y_2 : y_3)$

$$M_1 = \begin{pmatrix} y_0 & y_1 \\ y_2 & y_3 \end{pmatrix}$$

Whenever the determinant of M_1 $\det(M_1)$ is equal to zero the point $(y_0 : y_1 : y_2 : y_3)$ will be on the quadric $Z_+(y_0y_3 - y_1y_2)$. This is for example the case when either a column or a row in M_1 is equal to zero.

For instance, by a linear transformation of the first row of M_1 by adding a multiple of the second row, we obtain the matrix M_1'

$$M_1' = \begin{pmatrix} y_0 + 2y_2 & y_1 + 2y_3 \\ y_2 & y_3 \end{pmatrix}$$

We can easily see that the determinant of M_1 and M_1' are equal.

In general, adjusting a matrix by using linear combination of the rows, will make the determinant of the modified matrix unchanged.

Using this matrix representation, we will show that fixing one of the coordinates of a point in $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding, yields lines in the quadric.

Consider the closed embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ in Equation (3.2). The point $(u_0 : v_0) \times (u_1 : v_1)$ is sent to $(u_0u_1 : u_0v_1 : v_0u_1 : v_0v_1)$. Which can be represented as the matrix

$$\begin{pmatrix} u_0u_1 & u_0v_1 \\ v_0u_1 & v_0v_1 \end{pmatrix} \tag{3.3}$$

Notice that u_0 and v_0 are mutual factors in the first and second *row* respectively. Thus, if we fix u_0 and v_0, then these can be extracted from the matrix, such that

$$\begin{pmatrix} u_0u_1 & u_0v_1 \\ v_0u_1 & v_0v_1 \end{pmatrix} = u_0 \begin{pmatrix} u_1 & v_1 \\ 0 & 0 \end{pmatrix} + v_0 \begin{pmatrix} 0 & 0 \\ u_1 & v_1 \end{pmatrix}$$

proving that the image of the line $(u_0 : v_0) \times (u_1 : v_1)$, for a fixed u_0, v_0 in $\mathbb{P}^1 \times \mathbb{P}^1$ is in fact a line in the quadric. In fact, we get $\mathbb{P}^1$ versions of this line, by varying the fixed point u_0, v_0 in $\mathbb{P}^1 \times \mathbb{P}^1$.

Consider Equation (3.3) once again. Notice that u_1 and v_1 are mutual factors in the first and second *column* respectively. If we now fix u_1 and v_1, then this time u_1 and v_1 can be extracted from the matrix, such that

$$\begin{pmatrix} u_0u_1 & u_0v_1 \\ v_0u_1 & v_0v_1 \end{pmatrix} = u_1 \begin{pmatrix} u_0 & 0 \\ v_0 & 0 \end{pmatrix} + v_1 \begin{pmatrix} 0 & u_0 \\ 0 & v_0 \end{pmatrix}$$

proving that similarly, the image of the line $(u_0 : v_0) \times (u_1 : v_1)$, for a fixed u_1, v_1 in $\mathbb{P}^1 \times \mathbb{P}^1$ is a line in the quadric. Again, we get $\mathbb{P}^1$ versions of this line, by varying the fixed point u_1, v_1 in $\mathbb{P}^1 \times \mathbb{P}^1$.

Hence, we see that lines contained in the quadric in $\mathbb{P}^3$, are exactly those going through pair of points on the form $A = (u_1 : v_1 : 0 : 0)$, $B = (0 : 0 : u_1 : v_1)$ and $A = (u_0 : 0 : v_0 : 0)$, $B = (0 : u_0 : 0 : v_0)$.

Thus, there are two families of lines in the quadric, generated by either of the factors in $\mathbb{P}^1 \times \mathbb{P}^1$, by fixing one of the coordinates and varying the other.

3.5 The pullback from $\mathbb{P}^1 \times \mathbb{P}^1$

Consider our rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

Previously, we looked at points in $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, and studied the fibers of these points in the closure of the graph $\overline{\Gamma_p}$. Now we want to consider curves, and in particular lines, in $\mathbb{P}^1 \times \mathbb{P}^1$, and are interested in the inverse mapping of such a curve in $\mathbb{P}^2$, factoring through $\overline{\Gamma_p}$.

Lines in $\mathbb{P}^1 \times \mathbb{P}^1$

Since $\mathbb{P}^1 \times \mathbb{P}^1$ has dimension 2, a line in $\mathbb{P}^1 \times \mathbb{P}^1$ is a subvariety of dimension 1, which in this case corresponds to codimension $1 = 2 - 1$ i.e. a hyperplane.

Hence, a line can be described by

$$Z_+\big(f(u_0, v_0; u_1, v_1)\big)$$

where $f(u_0, v_0; u_1, v_1)$ is a bihomogeneous polynomial of degree 1.
Since $1 = \deg f(u_0, v_0; u_1, v_1) = \operatorname{bideg} f(u_0, v_0) + \operatorname{bideg} f(u_1, v_1)$
we must have that

$$\{\operatorname{bideg} f(u_0, v_0), \operatorname{bideg} f(u_1, v_1)\} = \{0, 1\}$$

Hence, a line looks like something on the form

$$uu_0 + vv_0 + wu_1 + zv_1 = 0$$

for constants a, b, c, d where either $a = b = 0$ or $c = d = 0$. We will denote these cases by $(0 : 0) \times (c : d)$ and $(a : b) \times (0 : 0)$ respectively.

The pullback of lines

Consider a line in $\mathbb{P}^1 \times \mathbb{P}^1$, for instance $(a : b) \times (0 : 0)$, i.e. the locus of points on the form $(-b : a) \times (u_1 : v_1)$, where u_1, v_1 are arbitrary.

To find the inverse mapping of this line we factor through the closure of the graph.

Then, independent of the choice of c and d, the points in the preimage of this line in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ becomes $(x_0 : x_1 : x_2) \times (-b : a) \times (u_1 : v_1)$.

This is a hyperplane represented by the equation $(0 : 0 : 0) \times (a : b) \times (0 : 0)$. For each choice of u_1 and v_1 we get a point in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$.

To intersect the hyperplane with the closure of the graph, we consider the system of equations.

$$\begin{aligned} u_1x_0 + v_1x_2 &= 0 \\ -bx_1 + ax_2 &= 0 \end{aligned}$$

Further, we want to find values for x_0, x_1 and x_2 such that the equations are equal to zero. We let $x_2 = bu_1$, then

$$\begin{aligned} u_1x_0 + v_1(bu_1) = 0 &\Rightarrow x_0 = -bv_1 \\ -bx_1 + a(bu_1) = 0 &\Rightarrow x_1 = au_1 \end{aligned}$$

so we get $(x_0 : x_1 : x_2) = (-bv_1 : au_1 : bu_1)$.

Thus, the inverse mapping of the line $(a : b) \times (0 : 0)$ becomes the locus of the points $(-bv_1 : au_1 : bu_1)$ in $\mathbb{P}^2$.

Consider the equation of a line $a_0x_0 + a_1x_1 + a_2x_2$. We are interested in the choice of a_0, a_1, and u_0 that will satisfy the equation

$$a_0(-bv_1) + a_1(au_1) + a_2(bu_1) = 0$$

So the inverse of the line $(a : b) \times (0 : 0)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is the line in $\mathbb{P}^2$ with coordinates $(0 : b : -a)$.

Similarly, we can find the inverse mapping in $\mathbb{P}^2$ of the line $(0 : 0) \times (c : d)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. This turns out to be the line with coordinates $(d : 0 : -c)$.

The pullback of general polynomials

Now, consider a bihomogenous polynomial equation $f(u_0, v_0; u_1, v_1) = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$. E.g.

$$u_0^4u_1^2 + v_0^4v_1^2 = 0 \tag{3.4}$$

To find the preimage of this equation under the rational map $p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ we will again factor through the closure of graph of p.

To do this we consider the preimage of $f(u_0, v_0; u_1, v_1) = 0$ under the projection $\alpha_{\mathbb{P}^1 \times \mathbb{P}^1}$ i.e. the hypersurface $Z_+(f)$ in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, intersected with the closure of the graph $\overline{\Gamma_p}$.

Hence, consider the equation system

$$\begin{aligned} f(u_0, v_0; u_1, v_1) = u_0^4u_1^2 + v_0^4v_1^2 &= 0 \\ u_1x_0 + v_1x_2 &= 0 \\ u_0x_1 + v_0x_2 &= 0 \end{aligned}$$

We will solve the equations in the the open affine chart $D_+(v_0u_1)$.

Thus, we may assume $v_0 = 1$ and $u_1 = 1$, and thereby solve the equations above for v_1 and u_0 and obtain

$$\begin{aligned} v_1 &= -\frac{x_0}{x_2} \\ u_0 &= -\frac{x_2}{x_1} \end{aligned}$$

Then, we can substitute the above results into Equation (3.4)

$$\left(-\frac{x_2}{x_1}\right)^4 + \left(-\frac{x_0}{x_2}\right)^2 = 0$$

$$\overline{f}(x_0, x_1, x_2) = x_2^6 + x_0^2x_1^4 = 0$$

to obtain one homogeneous polynomial in the variables of $\mathbb{P}^2$. Hence by projecting down to $\mathbb{P}^2$ using $\alpha_{\mathbb{P}^2}$ we obtain a homogeneous polynomial in $\mathbb{P}^2$ which equals the closure of the preimage $p^{-1}\big(f(u_0, v_0; u_1, v_1) = 0\big)$.

In addition we can prove that when the original equation f is irreducible, then the same holds for $\overline{f}$.

Now, we will prove that equation $\overline{f}$ has singularities in the points q_0 and q_1. To do this we introduce the Jacobian criterion in the projective case.

Theorem 3.5.1 ([EOa], Proposition 8.15)**.** *Let $X = Z_+(F_1, \ldots, F_r) \subseteq \mathbb{P}^n$ be a closed algebraic set, and let*

$$J = \left(\frac{\partial F_i}{\partial f z_j}(p) \right)$$

Then the the rank of J does not depend on the choice of representative for p. Moreover X is non-singular at p if and only if $\operatorname{rank} J = n - \dim X$.

To see this, we consult our example. Hence, we consider the Jacobian Matrix

$$J = \left[\tfrac{\partial f}{\partial x_0} = 2x_0 x_1^4 \qquad \tfrac{\partial f}{\partial x_1} = 4x_0^2 x_1^3 \qquad \tfrac{\partial f}{\partial x_2} = 6x_2^5 \right] = \mathbf{0}$$

Solving these equations yields

$$x_2 = 0 \wedge (x_0 = 0 \vee x_1 = 0)$$

i.e. $q_0 = (1:0:0)$ or $q_1 = (0:1:0)$. Since both of the points are on the curve $\overline{f}(x_0, x_1, x_2) = 0$, they are singular.

Lemma 3.5.2. *The equation $\overline{f}$ will always be singular in the points q_0 and q_1.*

Next, we compare the degree of the curves $f(u_0, v_0; u_1, v_1)$ and $\overline{f}(x_0, x_1, x_2)$.

Denote by m, n the bidegrees of the equation $f(u_0, v_0; u_1, v_1)$. E.g. in the example

$$f(u_0, v_0; u_1, v_1) = u_0^4 u_1^2 + v_0^4 v_1^2$$

the bidegrees equal

$$m = 4\,, \quad n = 2\,, \quad m + n = 6$$

The multiplicities

Now, we are interested in the multiplicities and the associated connection to the bidegree.

Recall our homogeneous polynomial in the variables of $\mathbb{P}^2$

$$\overline{f}(x_0, x_1, x_2) = x_2^6 + x_0^2 x_1^4 = 0$$

Generally we have

$$\begin{aligned} \overline{f}_{x_0=1} &= \overline{f}(1, x_1, x_2) \\ &= \overline{f}_0(x_1, x_2) + \overline{f}_1(x_1, x_2) + \overline{f}_2(x_1, x_2) + \overline{f}_3(x_1, x_2) \dots \end{aligned}$$

where the $\overline{f}_i$'s represents homogeneous polynomials of degree i in $\overline{f} \in \mathbb{P}^2$. Here $f_0(x_1, x_2)$ represents the constant polynomial, $f_1(x_1, x_2)$ represents the polynomial of degree 1, $f_2(x_1, x_2)$ represents the polynomial of degree 2, and so on. In such a representation, the degree of the first non-zero polynomial represents the multiplicity of the root in a given point.

As x_0, x_1 and x_2 are the homogeneous coordinates of $\mathbb{P}^2$, *one* of the coordinates must be different from zero. In the above case, x_0 represents this coordinate. We let $x_0 = 1$. Thus

$$x_2^6 + x_1^4 = 0 \tag{3.5}$$

The only integer solution is such that $x_1 = x_2 = 0$. In this case we are on the point $q_0 = (1 : 0 : 0)$.

In Equation (3.5) the f_i's $\neq 0$ when $i = 4, 6$. Which means that the multiplicity of the point $q_0 = (1 : 0 : 0) \in \mathbb{P}^2$ is 4.

Similarly, if we instead let $x_1 = 1$, then

$$x_2^6 + x_0^2 = 0$$

The only integer solution of this equation is $x_0 = x_2 = 0$. Which means we are on q_1. Now, the f_i's $\neq 0$ is when $i = 2, 6 \neq 0$, all other f_i's are equal to zero. Thus, the multiplicity of the point $q_1 = (0 : 1 : 0) \in \mathbb{P}^2$ is 2.

Thus, the curve $\overline{f}(x_0, x_1, x_2) \in \mathbb{P}^2$ have multiplicity equal to 4 and 2 when on the two points q_0 and q_1 respectively. These multiplicities are in fact identical to the bidregree of the curve we started out with $f(u_0, v_0; u_1, v_1) \in \mathbb{P}^1 \times \mathbb{P}^1$, i.e. bidegree (4,2).

As it turns out, this will (almost) always be the case. First we will consider the case where this is true. Then, we will come back to *when* we need to be more careful with the correspondence between the bidregree of a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ and the multiplicity of points on the curve in the preimage. Figure 3.3 illustrates a curve $f(u_0, v_0; u_1, v_1) \in \mathbb{P}^1 \times \mathbb{P}^1$ with bidegree (4,2).

If we fix $(u_0 : v_0) = (a : b)$, then $f(a, b; u_1, v_1)$ has two zeros. This can be illustrated in Figure 3.3, by studying the line $(a : b) \times (u_1 : v_1)$ and count the number of intersections between this line and the curve. Similarly, if we instead fix $(u_1 : v_1) = (c : d)$, the degree of the curve will be 4. And in a similar manner, we can observe this by studying the line $(u_0 : v_0) \times (c : d)$ and count the number of intersections between this line and the curve.

Generally, as we know, the preimage of lines in $\mathbb{P}^1 \times \mathbb{P}^1$ are lines in $\mathbb{P}^2$. But as we saw, the preimage of some of the lines are in fact contracted

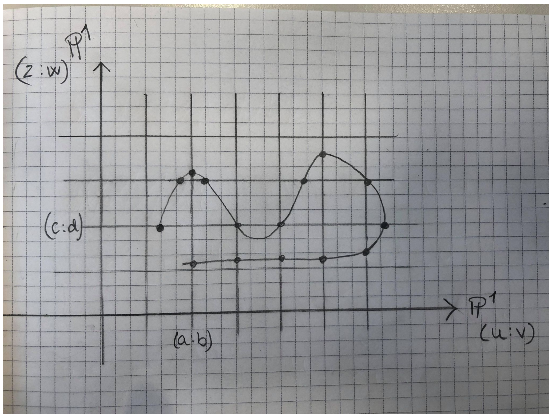

Figure 3.3: Curve with bidegree (4,2) in $\mathbb{P}^1 \times \mathbb{P}^1$

to points in $\mathbb{P}^2$. For instance, the preimage of the line $(u_0 : v_0) \times (0 : 1)$ is $(u_0 : 0 : 0) \cong (1 : 0 : 0) = q_0$, and similarly the preimage of the line $(0 : 1) \times (u_1 : v_1)$ is $(0 : u_1 : 0) \cong (0 : 1 : 0) = q_1$.

Thus, the two lines $(u_0 : v_0) \times (0 : 1)$ and $(0 : 1) \times (u_1 : v_1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, with degree 4 and 2 respectively, corresponds to the points q_0 and q_1 in $\mathbb{P}^2$, with multiplicity 4 and 2 respectively. This is illustrated in Figure 3.4, where the curve $\overline{f}(x_0 : x_1 : x_2)$ passes though the points q_0 and q_1, 4 and 2 times respectively.

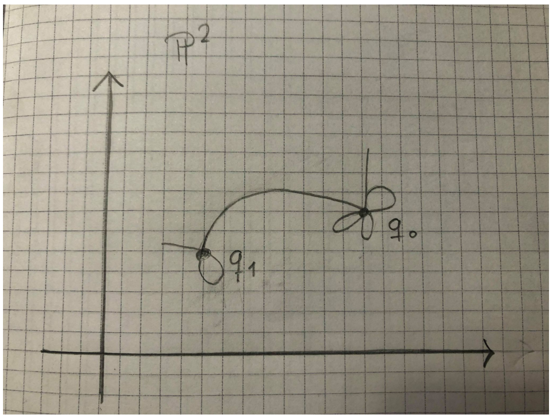

Figure 3.4: Curve with passing through q_0 and q_1 in $\mathbb{P}^2$.

Once again, we can consult the curve $f(u_0, v_0; u_1, v_1) \in \mathbb{P}^1 \times \mathbb{P}^1$, but this time rather study where the two lines $(u_0 : v_0) \times (0 : 1)$ and $(0 : 1) \times (u_1 : v_1)$ intersect, namely $(0 : 1) \times (0 : 1)$. As we have seen earlier, the fiber of this point corresponds to a line in the closure of the graph. When projected down to $\mathbb{P}^2$ we obtained the line $(0 : 0 : 1)$ i.e. the line through the points q_0 and q_1, denoted $\overline{q_0 q_1}$. Hence, the number of intersections between $\overline{f}(x_0, x_1, x_2)$ and the line $\overline{q_0 q_1}$ in $\mathbb{P}^2$, with the exception of q_0 and q_1, is determined by the number of times $f(u_0, v_0; u_1, v_1)$ intersects the point $(0 : 1) \times (0 : 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

We started out by considering a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ and its associated bidegree, and used the bidegree to find the multiplicities of certain points in the preimage in $\mathbb{P}^2$. Instead, it is possible to do this reversed, by considering the multiplicities of certain points in $\mathbb{P}^2$, and using these to determine the associated bidegree of curve in the image in $\mathbb{P}^1 \times \mathbb{P}^1$.

Now, let us consider the case when the correspondence between the bidregree of a curve and the multiplicity of points in the preimage, may not be straight forward.

Consider a curve $\overline{f'}(x_0, x_1, x_2)$ in $\mathbb{P}^2$ of degree 4, that passes through the point q_0 twice, the point q_1 once, and in addition intersects the line $\overline{q_0 q_1}$ in one more point. Since $\overline{q_0 q_1}$ is intersected once outside q_0 and q_1, $f(u_0, v_0; u_1, v_1)$ passes through $(0 : 1) \times (0 : 1)$ one time. Further, when fixing $(u_0 : v_0)$ for any a and b, $f(u_0, v_0; u_1, v_1)$ will intersect $(a : b) \times (u_1 : v_1)$ once. Similarly, when fixing $(u_1 : v_1)$ for any c and d, $f(u_0, v_0; u_1, v_1)$ will intersect $(u_0 : v_0) \times (c : d)$ two times. Thus, the bidegree of $f(u_0, v_0; u_1, v_1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is $(3, 2)$. As we know, $\overline{f'}(x_0, x_1, x_2)$ only intersects q_0 twice and q_1 once. Thus, we need to be aware when a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ intersects with $(0 : 1) \times (0 : 1)$ and when a curve in $\mathbb{P}^2$ intersects with $\overline{q_0 q_1}$ at any other point than q_0 and q_1.

Bidregrees in $\mathbb{P}^1 \times \mathbb{P}^1$

Let $\overline{f}(x_0, x_1, x_2)$ be a curve of degree d in $\mathbb{P}^2$, with multiplicity a in point q_0 and b in point q_1. Then, the number of intersection points between $\overline{f}(x_0, x_1, x_2)$ and the line $\overline{q_0q_1}$ will be $\overline{f}(x_0, x_1, x_2) \cap \overline{q_0q_1} = a + b + c$, where c is the number of intersection points not in q_0 and q_1.

Theorem 3.5.3 ([EOa], Theorem 11.5 (Bezout's theorem))**.** *Let $Z_1, \dots, Z_n$ be hypersurfaces in $\mathbb{P}^n$ with only finitely many points in common. Then*

$$\deg Z_1 \cdots \deg Z_n = \sum_p \mu_p(Z_1, \dots, Z_n).$$

According to Bézout, the number of points in $\overline{f}(x_0, x_1, x_2) \cap \overline{q_0q_1}$, counted with multiplicities, is equal to the product of the degrees of $\overline{f}(x_0, x_1, x_2)$ and $\overline{q_0q_1}$. As $\overline{q_0q_1}$ is a line, it is of degree 1. Thus, the number of intersection points is equal to d. If $d > a + b$, then $\overline{f}(x_0, x_1, x_2)$ intersects $\overline{q_0q_1}$ at more points than q_0 and q_1. Thus, if $d = a + b$, then the curve will only intersect $\overline{q_0q_1}$ in q_0 and q_1. As mentioned, the multiplicities of $\overline{f}(x_0, x_1, x_2)$ in q_0 and q_1 in $\mathbb{P}^2$ is denoted a and b. We denote the corresponding bidregree of the curve $f(u_0, v_0; u_1, v_1) \in \mathbb{P}^1 \times \mathbb{P}^1$ as (α, β).

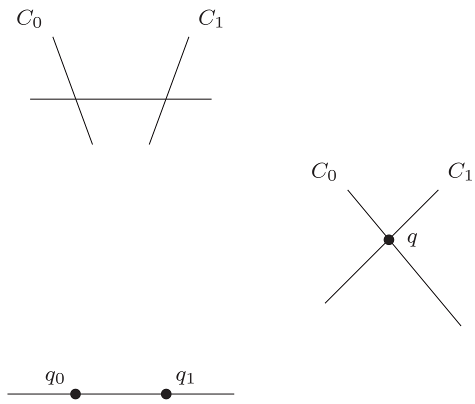

Figure 3.5: Blow up of $q_0, q_1 \in \mathbb{P}^2$ and two lines in $\mathbb{P}^1 \times \mathbb{P}^1$

Consider Figure 3.5, in $\mathbb{P}^2$ the multiplicity of q_0 and q_1 is a and b respectively. Since $\overline{f}(x_0, x_1, x_2)$ is of degree d, the curve intersects each line in $\mathbb{P}^2$ d times. By subtracting the multiplicities of q_0 and q_1 we get the number of times $\overline{f}(x_0, x_1, x_2)$ intersects the line outside q_0 and q_1.

When $d \geq a + b$ we have two possible cases of different bidegrees of $f(u_0, v_0; u_1, v_1) \in \mathbb{P}^1 \times \mathbb{P}^1$.

Let α denote the first bidegree, β the second and δ the number of times the curve intersects $C = (0 : 1) \times (0 : 1)$.

To see the relationship between a, b, d and α, β, δ we have that

$$\alpha = a + (d - (a + b)) = d - b$$
$$\beta = b + (d - (a + b)) = d - a$$

$$\delta = d - (a + b) = d - a - b$$

To prove this, consider two lines A_0 and A_1, each going through q_0 and q_1 respectively, as depicted below in Figure 3.6.

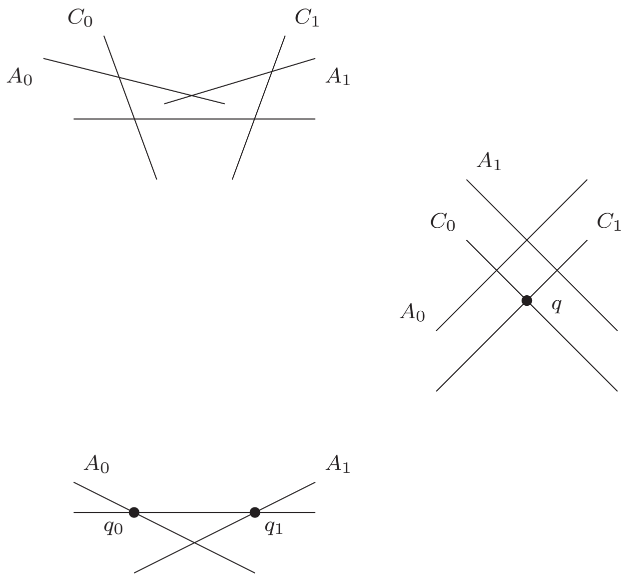

Figure 3.6: Blow up of q_0, q_1 and lines in $\mathbb{P}^2$ and lines in $\mathbb{P}^1 \times \mathbb{P}^1$

As before, $\overline{f}(x_0, x_1, x_2)$ is a curve of degree d. In the figure, we see that A_0 intersects $\overline{q_0q_1}$ in q_0. As a result, $\overline{f}(x_0, x_1, x_2)$ intersects the line A_0 everywhere but q_0, $(d-a)$ times. Similarly, A_1 intersects $\overline{q_0q_1}$ in q_1. Therefore, $\overline{f}(x_0, x_1, x_2)$ will intersect A_1 everywhere but q_1, $(d-b)$ times. Further, $\overline{f}(x_0, x_1, x_2)$ will intersect $\overline{q_0q_1}$ everywhere but q_0 and q_1, $\big(d - (a + b)\big)$ times.

Now, let us consider the bidegrees in $\mathbb{P}^1 \times \mathbb{P}^1$. A_0 will be mapped to a line in $\mathbb{P}^1 \times \mathbb{P}^1$ parallel with C_1, where C_1 is the image of q_1. Since, A_0 intersects the curve $(d-a)$ times in $\mathbb{P}^2$, outside q_0, this is also the case in $\mathbb{P}^1 \times \mathbb{P}^1$. Hence, the second bidegree of the curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is $(d-a)$. A similar argument yields that the first bidegree must equal $(d-b)$.

Additionally, we can calculate that the curve must pass through the point $(0:1) \times (0:1)$ exactly $(d-a-b)$ times, as the line C_0, with bidegree $(d-b)$, has exactly a intersections outside of the point $(0:1) \times (0:1)$, which stems from the multiplicity of the curve in the point q_0.

■

Hence, knowing either a, b, d or α, β, δ we can compute the other three unknowns, using that the equations

$$\begin{aligned} \alpha &= a + (d - (a + b)) = d - b \\ \beta &= b + (d - (a + b)) = d - a \\ \delta &= d - (a + b) = d - a - b \end{aligned}$$

imply

$$\begin{aligned}
a &= (d-b)-(d-a-b) = \alpha - \delta \\
b &= (d-a)-(d-a-b) = \beta - \delta \\
d &= (d-b)+(d-a)-(d-a-b) = \alpha + \beta - \delta
\end{aligned}$$

CHAPTER 4

Reconstruction from three camera centers

We will reconstruct $\mathbb{P}^2$ from three projections $\pi_{q_i} : \mathbb{P}^2 \to \mathbb{P}^1$, where each projection is an image from a camera center $q_i \in \mathbb{P}^2$ for $i = 0, 1, 2$. We will refer to the camera centers as the points q_0, q_1, q_2 in $\mathbb{P}^2$ from now on.

To reconstruct $\mathbb{P}^2$, we want to find the surface $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ birationally equivalent to $\mathbb{P}^2$, and the inverse birational map α from S to $\mathbb{P}^2$. We consider $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding in $\mathbb{P}^7$. In this chapter, we will first show that the image of S in $\mathbb{P}^7$ is $S = (\mathbb{P}^1)^3 \cap \mathbb{P}^6$. Then, we will show that the map α is a projection from $\mathbb{P}^3$ restricted to S, such that $\mathbb{P}^3 \cap S$ is a curve C of degree 3 where $\langle C \rangle = \mathbb{P}^3$.

To determine S in $\mathbb{P}^7$ we need to identify enough points so that they span a hyperplane of $\mathbb{P}^7$, i.e. $\mathbb{P}^6$. As we will see, even though we find enough points to span $\mathbb{P}^6$, the reconstruction of $\mathbb{P}^2$ is ambiguous.

Further, there are cases where we do not find enough points to determine S in $\mathbb{P}^7$. We call these critical configurations. We will classify different critical configurations and study the information we obtain of $\mathbb{P}^2$ in such cases.

4.1 A rational map from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Consider the points q_0, q_1, $q_2 \in \mathbb{P}^2$, where $q_0 = (1 : 0 : 0)$, $q_1 = (0 : 1 : 0)$ and $q_2 = (0 : 0 : 1)$.

The variety of lines in $\mathbb{P}^2$ through q_i is isomorphic to $\mathbb{P}^1$, for a fixed $i = 0,\ 1,\ 2$. Hence, by choosing a point $r \in \mathbb{P}^2$, and considering the lines $\overline{rq_i}$ we obtain a map from $p : \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined similarly as the case for two points.

The map p can be described explicitly by

$$\begin{aligned} p : \mathbb{P}^2 &\to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ (x_0 : x_1 : x_2) &\mapsto (-x_2 : x_1) \times (-x_2 : x_0) \times (-x_1 : x_0) \end{aligned}$$

so that p fits into the diagram in Figure 4.1.

From discussion in Section 3.1, we see that p is a morphism between the open subset $U = \mathbb{P}^2 \setminus \{q_0, q_1, q_2\}$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then by Definition 3.1.1, we recognize p as a rational map from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, i.e.

$$p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

$$\begin{tikzcd} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "\pi_0 \times \pi_1"] \\ \mathbb{P}^2 \arrow[ur, dashed, "p"] \arrow[r, dashed, "p"'] & \mathbb{P}^1 \times \mathbb{P}^1 \end{tikzcd}$$

Figure 4.1: Correspondence between rational maps from $\mathbb{P}^2$ into $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

4.2 An embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^7$

We have a map from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^7$ given by

$$\begin{aligned} \psi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^7 \\ (u_0 : v_0) \times (u_1 : v_1) \times (u_2 : v_2) &\mapsto (u_0 u_1 u_2 : u_0 u_1 v_2 : \cdots : v_0 v_1 v_2) \\ &\mapsto (y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7) \end{aligned} \tag{4.1}$$

The map ψ is a closed embedding called the Segre embedding, see Section 2.1, where the image $U = \psi(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ is closed and isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Moreover, the image U as a variety in $\mathbb{P}^7$ satisfies the following relations

$$\begin{aligned} y_0 y_7 &= (u_0 u_1 u_2)(v_0 v_1 v_2) = u_0 u_1 u_2 v_0 v_1 v_2 \\ y_1 y_6 &= (u_0 u_1 v_2)(v_0 v_1 u_2) = u_0 u_1 u_2 v_0 v_1 v_2 \\ y_2 y_5 &= (u_0 v_1 u_2)(v_0 u_1 v_2) = u_0 u_1 u_2 v_0 v_1 v_2 \\ y_3 y_4 &= (u_0 v_1 v_2)(v_0 u_1 u_2) = u_0 u_1 u_2 v_0 v_1 v_2 \end{aligned}$$

in addition to

$$\begin{aligned} y_0 y_3 &= u_0^2 u_1 u_2 v_1 v_2 = y_1 y_2 \\ y_4 y_7 &= u_1 u_2 v_0^2 v_1 v_2 = y_5 y_6 \\ y_0 y_5 &= u_0 u_1^2 u_2 v_0 v_2 = y_1 y_4 \\ y_2 y_7 &= u_0 u_2 v_0 v_1^2 v_2 = y_3 y_6 \\ y_0 y_6 &= u_0 u_1 u_2^2 v_0 v_1 = y_2 y_4 \\ y_1 y_7 &= u_0 u_1 v_0 v_1 v_2^2 = y_3 y_5 \end{aligned}$$

Thus, the image U of ψ is contained in the twelve hypersurfaces

$$\begin{gathered} Z_+(y_0 y_7 - y_1 y_6) \\ Z_+(y_0 y_7 - y_2 y_5) \\ Z_+(y_0 y_7 - y_3 y_4) \\ Z_+(y_1 y_6 - y_2 y_5) \\ Z_+(y_1 y_6 - y_3 y_4) \\ Z_+(y_2 y_5 - y_3 y_4) \\ Z_+(y_0 y_3 - y_1 y_2) \\ Z_+(y_4 y_7 - y_5 y_6) \\ Z_+(y_0 y_5 - y_1 y_4) \\ Z_+(y_2 y_7 - y_3 y_6) \end{gathered}$$

$$Z_+(y_0y_6 - y_2y_4)$$
$$Z_+(y_1y_7 - y_3y_5)$$

Hence, also in their intersection.

Lemma 4.2.1. *The image U of ψ equals the intersection of the twelve hypersurfaces $U^* = \bigcap Z_+(\mathbf{a_i})$.*

Proof. We will begin with the observation $U \subseteq U^*$, since U is contained in each of the twelve separate hypersurfaces. Thus, together with $U^* \subseteq U$, it is sufficient to conclude $U^* = U$.

Hence, we move on to prove $U^* \subseteq U$, which is equivalent to $U^* \cap D_+(y_i) \subseteq U \cap D_+(y_i)$ $\forall i$, since $\{D_+(y_i)\}$ make up an open cover of $\mathbb{P}^7$.

The twelve hypersurfaces are symmetric in terms of the y_i, and we may therefore without loss of generality prove $U^* \cap D_+(y_0) \subseteq U \cap D_+(y_0)$.

We will prove that for $\mathbf{y} = (y_0 : y_1 : \cdots : y_7) \in U^* \cap D_+(y_0)$, we have $\psi \circ \psi_0^{-1}(\mathbf{y}) = \mathbf{y}$ and hence have

$$U^* \cap D_+(y_0) \subseteq \psi\big(\psi_0^{-1}(D_+(y_0))\big) = \psi\big(D_+(u_0v_0w_0)\big) = U \cap D_+(y_0) \tag{4.2}$$

To see $\psi \circ \psi_0^{-1}(\mathbf{y}) = \mathbf{y}$ note that

$$\begin{aligned}\psi \circ \psi_0^{-1}\big((y_0 : y_1 : \cdots : y_7)\big) &= \psi\big((y_0 : y_4) \times (y_0 : y_2) \times (y_0 : y_1)\big) \\ &= (y_0^3 : y_0^2y_1 : y_0^2y_2 : y_0y_1y_2 : y_0^2y_4 : y_0y_1y_4 : y_0y_2y_4 : y_1y_2y_4)\end{aligned}$$

Now, the crucial observation is that since $\mathbf{y} \in U^*$, we have

$$\begin{aligned}y_1y_2 = y_0y_3 &\Rightarrow y_0y_1y_2 = y_0^2y_3 \\ y_1y_4 = y_0y_5 &\Rightarrow y_0y_1y_4 = y_0^2y_5 \\ y_2y_4 = y_0y_6 &\Rightarrow y_0y_2y_4 = y_0^2y_6 \\ y_1y_6 = y_0y_7 &\Rightarrow y_1y_2y_4 = y_0y_1y_6 = y_0^2y_7\end{aligned}$$

Hence

$$\begin{aligned}\psi \circ \psi_0^{-1}(\mathbf{y}) &= (y_0^3 : y_0^2y_1 : y_0^2y_2 : y_0y_1y_2 : y_0^2y_4 : y_0y_1y_4 : y_0y_2y_4 : y_1y_2y_4) \\ &= (y_0^3 : y_0^2y_1 : y_0^2y_2 : y_0^2y_3 : y_0^2y_4 : y_0^2y_5 : y_0^2y_6 : y_0^2y_7) \\ &= (y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7) = \mathbf{y}\end{aligned}$$

since $y_0 \neq 0$ for $\mathbf{y} \in U^* \cap D_+(y_0) \subset D_+(y_0)$.

Hence, $U^* \cap D_+(y_0) \subseteq U \cap D_+(y_0)$, from Equation (4.2), and by symmetry we obtain $U^* \cap D_+(y_i) \subseteq U \cap D_+(y_i)$ which is sufficient to prove $U = U^*$. ∎

Notice that

$$\begin{aligned}&(y_0y_7 - y_1y_6) + (y_1y_6 - y_2y_5) = (y_0y_7 - y_2y_5) \\ &(y_1y_6 - y_2y_5) + (y_2y_5 - y_3y_4) = (y_1y_6 - y_3y_4) \\ &(y_0y_7 - y_1y_6) + (y_1y_6 - y_2y_5) + (y_2y_5 - y_3y_4) = (y_0y_7 - y_3y_4)\end{aligned}$$

Therefore, the following three hypersurfaces

$$Z_+(y_0y_7 - y_2y_5),\ Z_+(y_1y_6 - y_3y_4),\ Z_+(y_0y_7 - y_3y_4) \tag{4.3}$$

are redundant. Thus, when disregarding the three hypersurfaces in Equation (4.3), the nine remaining hypersurfaces are linearly independent.

In fact, by using Hilbert's Nullstellensatz, we show that there is a redundancy in the nine hypersurfaces as well, as U can be expressed as the intersection of only 8 hypersurfaces.

Theorem 4.2.2 ([EOa], Theorem 1.12 (Hilbert's Nullstellensatz))**.** *For ideals* $\mathbf{a}, \mathbf{b}$ *we have the equivalence*

$$Z_+(\mathbf{b}) \subseteq Z_+(\mathbf{a}) \Leftrightarrow \sqrt{\mathbf{a}} \subseteq \sqrt{\mathbf{b}}$$

We will prove that the hypersurface $Z_+(y_1y_6 - y_2y_5)$ contains the intersection of the eight remaining hypersurfaces. We will prove this explicitly by showing $(y_1y_6 - y_2y_5)^2 \in \mathbf{u}$, where $\mathbf{u}$ is the ideal generated by the eight remaining polynomials.

Proof. This can be done in the quotient ring modulo $\mathbf{u}$, by considering

$$\begin{aligned} y_1^2y_6^2 + y_2^2y_5^2 &= y_0y_1y_6y_7 + y_2y_3y_4y_5, \text{ by } y_1y_6 = y_0y_7 \text{ and } y_2y_5 = y_3y_4 \\ &= y_1y_2y_4y_7 + y_0y_3y_5y_6, \text{ by } y_2y_4 = y_0y_6 \text{ used in both directions} \\ &= y_0y_2y_5y_7 + y_1y_3y_4y_6, \text{ by } y_0y_5 = y_1y_4 \text{ used in both directions} \\ &= 2y_1y_2y_5y_6, \text{ by } y_1y_6 = y_0y_7 \text{ and } y_2y_5 = y_3y_4 \end{aligned}$$

Hence we have

$$y_1^2y_6^2 + y_2^2y_5^2 = 2y_1y_2y_5y_6 \implies (y_1y_6 - y_2y_5)^2 = 0$$

in the quotient ring, and we may conclude

$$(y_1y_6 - y_2y_5)^2 \in \mathbf{u} \Rightarrow (y_1y_6 - y_2y_5) \in \sqrt{\mathbf{u}} \Rightarrow (\mathbf{y_1y_6 - y_2y_5}) \subseteq \sqrt{\mathbf{u}}$$

Thus, the hypersurface $Z_+(y_1y_6 - y_2y_5)$ is redundant. ∎

Now, it is rather simple to see that the remaining hypersurfaces are all necessary i.e. that none contain the entire intersection of the remaining seven, due to the observation that each monomial appear only once in the eight different ideals.

That is, we may consider the points $x_{ij} \in \mathbb{P}^7$ on the form $(0 : \cdots : 1 : \cdots : 1 : \cdots : 0)$, where the i-th and j-th index are 1. Now, for the ideal $(y_iy_j - y_ky_l)$, we may consider the points x_{ij} and x_{kl}, and finish by noting that

$$x_{ij} \notin Z_+(y_iy_j - y_ky_l)\ ,\ x_{ij} \in Z_+(\mathbf{a_x})$$

since the monomial y_iy_j does not appear in any of the other ideals, and for any other monomial $y_ay_b(x_{ij}) = 0$.

We summarize this discussion in the result below

Lemma 4.2.3. *U can be written as the intersection of exactly eight of the twelve hypersurfaces, but no fewer than eight.*

4.3 The image of $\mathbb{P}^2$ in $\mathbb{P}^7$

Then, we have the image rational map p from $\mathbb{P}^2$ into $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, i.e.

$$\begin{aligned} p : \mathbb{P}^2 &\to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ (x_0 : x_1 : x_2) &\mapsto (-x_2 : x_1) \times (-x_2 : x_0) \times (-x_1 : x_0) \end{aligned}$$

and the Segre embedding ψ from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^7$, i.e.

$$\begin{aligned} \psi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^7 \\ (u_0 : v_0) \times (u_1 : v_1) \times (u_2 : v_2) &\mapsto (u_0u_1u_2 : u_0u_1v_2 : \cdots : v_0v_1v_2) \\ &\mapsto (y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7) \end{aligned}$$

The coordinates in $\mathbb{P}^7$, when mapped from $\mathbb{P}^2$ is given by

$$\begin{aligned} y_0 &= u_0u_1u_2 = (-x_2)(-x_2)(-x_1) = -x_1x_2^2 \\ y_1 &= u_0u_1v_2 = (-x_2)(-x_2)(x_0) = x_0x_2^2 \\ y_2 &= u_0v_1u_2 = (-x_2)(x_0)(-x_1) = x_0x_1x_2 \\ y_3 &= u_0v_1v_2 = (-x_2)(x_0)(x_0) = -x_0^2x_2 \\ y_4 &= v_0u_1u_2 = (x_1)(-x_2)(-x_1) = x_1^2x_2 \\ y_5 &= v_0u_1v_2 = (x_1)(-x_2)(x_0) = -x_0x_1x_2 \\ y_6 &= v_0v_1u_2 = (x_1)(x_0)(-x_1) = -x_0x_1^2 \\ y_7 &= v_0v_1v_2 = (x_1)(x_0)(x_0) = x_0^2x_1 \end{aligned}$$

where every monomial y_i is of degree 3.

Hence, the composition of the two maps is a map from $\mathbb{P}^2 \to \mathbb{P}^7$ given by

$$\begin{aligned} &\psi \circ p : \mathbb{P}^2 \to \mathbb{P}^7 \\ &(x_0 : x_1 : x_2) \mapsto (-x_1x_2^2 : x_0x_2^2 : x_0x_1x_2 : -x_0^2x_2 : x_1^2x_2 : -x_0x_1x_2 : -x_0x_1^2 : x_0^2x_1) \end{aligned}$$

Now, it becomes natural to wonder what the image $V = \psi \circ p(\mathbb{P}^2)$ looks like. This V will in fact be the surface S we are looking for, when $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is embedded into $\mathbb{P}^7$.

Clearly it is contained in U, as the mapping factors through $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, but we expect it to have dimension $\dim \mathbb{P}^2 = 2$, unlike U, which have dimension $\dim U = 3$.

By observation, we also have $V \subseteq Z_+(y_2 + y_5)$, and we will prove that this is sufficient to determine V.

Lemma 4.3.1. *V, that is the image of $\mathbb{P}^2$ in $\mathbb{P}^7$, fulfils $V = U \cap Z_+(y_2 + y_5)$.*

Proof. We have already shown $V \subseteq U \cap Z_+(y_2 + y_5)$, and what remains is therefore demonstrating $U \cap Z_+(y_2 + y_5) \subseteq V$.

To prove this, we only consider the case $y_0 \neq 0$, i.e. within $D_+(y_0)$ as the others are analogous.

Hence, consider the point $\mathbf{y} = (y_0 : \cdots : y_7) \in U \cap Z_+(y_2 + y_5) \cap D_+(y_0)$, we will prove that $\mathbf{y} \in (\psi \circ p)(\mathbb{P}^2) = V$.

We already know that the preimage of $\mathbf{y}$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ equals

$$\psi^{-1}\big((y_0 : \cdots : y_7)\big) = (y_0 : y_4) \times (y_0 : y_2) \times (y_0 : y_1) = \mathbf{z}$$

However, to take into account that $\psi(\mathbf{z}) \in \mathbb{Z}_+(y_2+y_5)$, we have the equation

$$y_0^2 y_2 + y_0 y_4 y_1 = 0 \Rightarrow y_0 y_2 + y_1 y_4 = 0$$

which is equivalent to saying $(y_0 : y_1) = (-y_4 : y_2)$.
Hence,

$$\mathbf{z} = (y_0 : y_4) \times (y_0 : y_2) \times (-y_4 : y_2) = p^{-1}\big((y_2 : y_4 : -y_0)\big)$$

which proves that

$$\mathbf{y} = (\psi \circ p)(y_2 : y_4 : -y_0) \in (\psi \circ p)(\mathbb{P}^2) = V$$

and finally $V = U \cap Z_+(y_2 + y_5)$. ∎

Thus, the surface S we are looking for, to reconstruct $\mathbb{P}^2$, is $S \cong V = U \cap Z_+(y_2 + y_5)$.

4.4 The class of the image of $\mathbb{P}^2$

In this section we want to determine the class of the image of $\mathbb{P}^2$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The notation and theory in this section is based on results in Chapter 2. See this chapter for more details.

Let the image of $\mathbb{P}^2$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be

$$\begin{aligned} S &= \overline{p(\mathbb{P}^2)} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ &= \psi^{-1}(V) \end{aligned}$$

As S is of codimension 1 in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, S is a divisor. The class of S, can be represented as $[S] = a_0h_0 + a_1h_1 + a_2h_2$, where $\langle h_0, h_1, h_2 \rangle$ is the basis for the Picard group $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $a_0, a_1, a_2 \in \mathbb{Z}$.

To determine the class $[S]$, we consider the intersection of $[S]$ and classes h_ih_j for $i \neq j$, where each h_ih_j represents the class of a curve going through q_i and q_j. Recall that $h_0h_1 = [\{pt\} \times \{pt\} \times \mathbb{P}^1]$, $h_0h_2 = [\{pt\} \times \mathbb{P}^1 \times \{pt\}]$ and $h_1h_2 = [\mathbb{P}^1 \times \{pt\} \times \{pt\}]$. First, we intersect $[S]$ with the class of curves represented by h_0h_1, to obtain

$$\begin{aligned} [S] \cdot h_0h_1 &= (a_0h_0 + a_1h_1 + a_2h_2) \cdot h_0h_1 \\ &= a_2 \cdot h_0h_1h_2 \\ &= a_2 \end{aligned}$$

As $h_i^2 = 0$ and $h_0h_1h_2 = 1$. Here a_2 corresponds to the number of intersection points between S and a curve of class h_0h_1.

To determine a_2, we consider this intersection in S. Each class h_0, h_1, h_2 have restrictions to S, where h_i restricted to S is $(L - e_i)$. Thus, we get

$$\begin{aligned} [S] \cdot h_0h_1 = h_0|_S \cdot h_1|_S &= (L - e_0) \cdot (L - e_1) \\ &= L^2 - Le_1 - Le_0 + e_0e_1 \\ &= 1 \end{aligned}$$

Hence, the number of intersection points between S and curves of class h_0h_1 in $\mathbb{P}^2$ is $a_2 = 1$. In a similar manner, we find that $a_0 = 1$ and $a_1 = 1$. Thus, the class of S is given by $[S] = h_0 + h_1 + h_2$.

To obtain the degree of a surface of class $[S] = h_0 + h_1 + h_2$ in the Segre embedding, we intersect $[S]$ with the class of the intersection of two hyperplane sections, i.e.

$$\begin{aligned} [S] \cdot (h_0 + h_1 + h_2)^2 =& (h_0 + h_1 + h_2) \cdot \big(h_0h_1 + h_0h_2 + h_1h_0 + h_1h_2 + h_2h_0 + h_2h_1\big) \\ =& 6 \cdot h_0h_1h_2 \\ =& 6 \end{aligned}$$

Thus, the degree of the surface $V = \psi(S)$ in the Segre embedding is 6.

Intersection of two surfaces

Now, if we are to look at two separate blow-ups of $\mathbb{P}^2$ of three different points, we obtain two different images, S_0 and S_1, when mapped into $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

However, the class of both surfaces will still be $(h_0 + h_1 + h_2)$. Thus, their intersection, a subvariety of dimension 1, will have class

$$\begin{aligned}[S]_0 \cdot [S]_1 &= (h_0 + h_1 + h_2) \cdot (h_0 + h_1 + h_2) \\ &= h_0h_1 + h_0h_2 + h_1h_0 + h_1h_2 + h_2h_0 + h_2h_1 \\ &= 2 \cdot (h_0h_1 + h_1h_2 + h_2h_0)\end{aligned}$$

Thus, we have calculated the class of the intersection of two different blow-ups of $\mathbb{P}^2$, S_1 and S_2, in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Now, we want to study the intersection of two such blow-ups when embedded into $\mathbb{P}^7$. Recall that

$$S_1 = \psi^{-1}(V_1), \qquad S_2 = \psi^{-1}(V_2)$$

In $\mathbb{P}^7$ we assume V_1 to be equal to the image V we found in Lemma 4.3.1, i.e $V_1 = U \cap Z_+(y_2 + y_5)$. Then, we let V_2 be a blow-up of three other points in $\mathbb{P}^2$ given by

$$V_2 = U \cap \mathbb{Z}_+(a_0y_0 + a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 + a_5y_5 + a_6y_6 + a_7y_7)$$

for some $a_i \in k$. To explain that the expression is linear, the class of $\psi^{-1}(V_2)$ equals the class of $\psi^{-1}(V_1)$, namely $(h_0 + h_1 + h_2)$, such that the degree of V_2 in $\mathbb{P}^7$ equals 6 as well. Implying that the hypersurface U is intersected with to obtain V_2, is of the same degree as $Z_+(y_2 + y_5)$, i.e. degree 1. Thus, it is a hyperplane as well.

As above, we want to consider the intersection of the two separate blow-ups of $\mathbb{P}^2$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, when embedded into $\mathbb{P}^7$. Thus,

$$V_1 \cap V_2 = U \cap \mathbb{Z}_+(y_2 + y_5) \cap \mathbb{Z}_+(a_0y_0 + a_1y_1 + \cdots + a_7y_7) \tag{4.4}$$

An intersection of two blow-ups when embedded into $\mathbb{P}^7$ will in general be of codimension $4 + 1 + 1 = 6$ and thus be a curve in $\mathbb{P}^7$.

We can calculate the degree of the curve in $\mathbb{P}^7$ by intersecting the class of the intersection of the two blow-ups in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with the class of a hyperplane section $(h_0 + h_1 + h_2)$. Thus,

$$\begin{aligned}[S_0] \cdot [S_1] \cdot (h_0 + h_1 + h_2) &= 2 \cdot (h_0h_1 + h_1h_2 + h_2h_0) \cdot (h_0 + h_1 + h_2) \\ &= 2 \cdot (h_0h_1h_2 + h_0h_1h_2 + h_0h_1h_2) \\ &= 6\end{aligned}$$

Thus, $V_1 \cap V_2$ will in general be a curve in $\mathbb{P}^7$ of degree 6.

The pullback of the class of V

As V is a hyperplane section of U and the class of a hyperplane section is given by $(h_0 + h_1 + h_2)$, then the corresponding class in the blow-up of $\mathbb{P}^2$ in three points is given by

$$(L - e_0) + (L - e_1) + (L - e_2) = 3L - e_0 - e_1 - e_2$$

As h_0, h_1, h_2 corresponds to $(L - e_0)$, $(L - e_1)$, $(L - e_2)$. A curve of class $3L - e_0 - e_1 - e_2$ will in $\mathbb{P}^2$ represent a cubic curve through the points q_0, q_1, q_2.

L is both the class of a line in $\mathbb{P}^2$ and the class of the image of a line in V. Thus, we can find α, first by identifying curves of class $A = H - L$ in $\mathbb{P}^7$, and then project from the span of a curve of this class. Given the class of a hyperplane section $3L - e_0 - e_1 - e_2$, we obtain the class A by subtracting L, i.e.

$$A = (3L - e_0 - e_1 - e_2) - L = 2L - e_0 - e_1 - e_2$$

Further, the degree of the curve of class A in $\mathbb{P}^7$ will be

$$(3L - e_0 - e_1 - e_2) \cdot (2L - e_0 - e_1 - e_2) = 6L^2 + e_0^2 + e_1^2 + e_2^2 = 3$$

As it turns out, the curve of class $3L - e_0 - e_1 - e_2$ will span $\mathbb{P}^3$. Thus, α will be the projection from $\langle A \rangle = \mathbb{P}^3$ restricted to V.

For similar discussion, see Chapter 5.

4.5 Critical configurations as hyperplane sections

In this section, we want to study curves in the blow-up of $\mathbb{P}^2$ in q_0, q_1, q_2. We are interested in identifying curves that can be mapped to cubic curves in $\mathbb{P}^2$ going through q_0, q_1, q_2, i.e. curves in the blow-up satisfying the class $\sum_{i=1}^{3}(L - e_i) = 3L - e_0 - e_1 - e_2$.

Every curve is associated to a class. If a curve is reducible, i.e. a union of different components, then the class of the curve is equal to the sum of the classes of all components. As we want to study curves of class $3L - e_0 - e_1 - e_2$, we will consider both irreducible and reducible curves. To be a reducible curve of this class, the sum of all component classes must add up to $3L - e_0 - e_1 - e_2$. We call each unique composition of $3L - e_0 - e_1 - e_2$ a *partition*.

Definition 4.5.1. A partition of a class C is a unique decomposition of C, such that each component itself is a class of a curve.

Before we begin to study partitions of $3L - e_0 - e_1 - e_2$, we will comment on the possibilities of singularities of curves of different degrees. This is important, as it restricts the number of possible partitions.

An application of Bézout's theorem lets us determine the maximal number of singular points that an irreducible plane curve can have. We can determine the number by using the following proposition.

Proposition 4.5.2 ([EOa], Proposition 12.7)**.** *An irreducible curve C of degree d cannot have more than $\binom{d-1}{2}$ singular points.*

As a result, neither lines nor conics can contain singular points. However a cubic curve on the other hand may have *one* singularity, and the multiplicity of this point cannot exceed 2. Thus, an irreducible curve of class $3L - e_0 - e_1 - e_2$ can at most have *one* singularity. Reducible curves of this class will have no singularities.

Partitions of class $3L - e_0 - e_1 - e_2$

In this section, we start by describing some of the partitions of class $3L - e_0 - e_1 - e_2$. In the end of the section we list all partitions of this class in a summarizing table, i.e. Table 4.1.

In total, when including symmetric classes, there are 55 unique partitions of class $3L - e_0 - e_1 - e_2$ in the blow-up of $\mathbb{P}^2$ in three points. By symmetric classes, we refer to obtaining different types of a specific partition, by rearranging e_0, e_1 and e_2 in the expression for the given partition.

We describe the partitions in the order determined by the number of components in each partition. We start with the partition consisting of one single component, the irreducible curve.

Partition consisting of one component

The only partition consisting of one component is the irreducible curve of class $3L - e_0 - e_1 - e_2$. The class represents cubic curves going through the points q_0, q_1, q_2 exactly one time, see Figure 4.2.

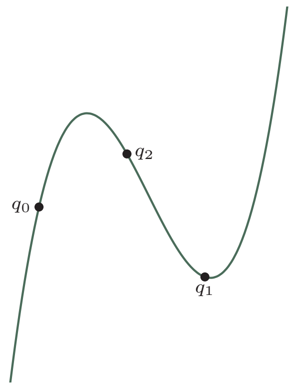

Figure 4.2: Cubic curve going through q_0, q_1, q_2 in $\mathbb{P}^2$ exactly once.

Partition consisting of two components

By rearranging the elements of class $3L - e_0 - e_1 - e_2$, we can find one of the partitions consisting of two components, each being its own class, i.e.

$$3L - e_0 - e_1 - e_2 = (3L - 2e_0 - e_1 - e_2) + e_0$$

The curve is still of degree three, but is now a combination of two components, i.e. a cubic curve and the exceptional divisor e_0, that is a blow-up of the point q_0. From the expression, we see that the curve goes through q_0 twice and further q_1 and q_2 once, see Figure 4.3.

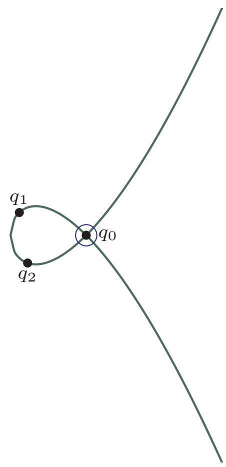

Figure 4.3: Nodal curve in $\mathbb{P}^2$

By symmetry, we can find two other types of this class, that is by rearranging e_0, e_1 and e_2 in the expression above.

Partition consisting of three components

We continue to combine a quadratic curve with other elements to obtain a curve of class $3L - e_0 - e_1 - e_2$.

$$3L - e_0 - e_1 - e_2 = (2L - e_0 - e_1) + (L - e_0 - e_2) + e_0$$

which is a quadratic curve going through q_0, q_1, a line going through q_0 and q_2 and the exceptional divisor e_0, see Figure 4.4.

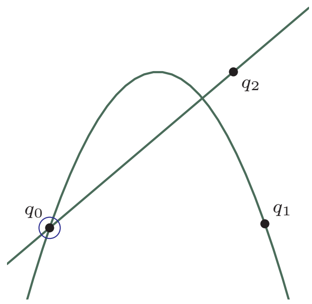

Figure 4.4: Curve in $\mathbb{P}^2$ of partition containing three components

By symmetry, there will be 6 partitions of this combination, given that we change the position of the points q_0, q_1, q_2.

Partition consisting of four components

$$3L - e_0 - e_1 - e_2 = (L - e_0 - e_1) + e_0 + L + (L - e_0 - e_2)$$

This curve is a combination of a line going through q_0, q_1, a line going through q_0, q_2, an arbitrary line $L \in \mathbb{P}^2$ and the exceptional divisor e_0, that is a blow-up of q_0, see Figure 4.5.

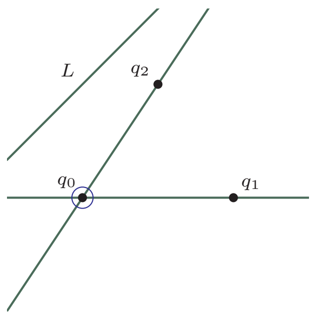

Figure 4.5: Curve in $\mathbb{P}^2$ of partition containing four components

By symmetry, there will exist in total 3 different partitions of this combination.

Partition consisting of five components

$$3L - e_0 - e_1 - e_2 = (L - e_1) + (L - e_0 - e_2) + e_1 + (L - e_0 - e_1) + e_0$$

This curve is a combination of a line going through q_1, a line going through q_0, q_1, a line going through q_0, q_2 and the two exceptional divisors e_0, e_1. The two exceptional divisors are blow-ups of q_0, q_1 respectively. See Figure 4.6.

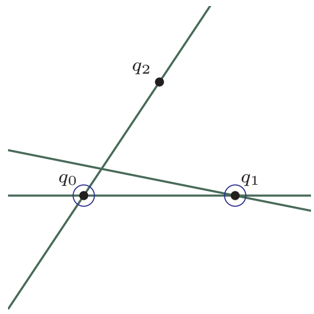

Figure 4.6: Curve in $\mathbb{P}^2$ of partition containing five components

By symmetry, we can obtain in total 6 different partitions of this combination.

Partition consisting of six components

If we consider a curve that contain all three lines, and that each line goes through two of the three points q_0, q_1, q_2. That is

$$3L - e_0 - e_1 - e_2 = (L - e_0 - e_1) + e_1 + (L - e_0 - e_2) + e_2 + (L - e_1 - e_2) + e_0$$

Here, the curve in the blow-up is a combination of six components, i.e. the lines $\overline{q_0q_1}$, $\overline{q_0q_2}$ and $\overline{q_1q_2}$, and the exceptional divisors e_0, e_1 and e_2, that are blow-ups of q_0, q_1, q_2 respectively. There is no symmetric variation of this partition.

Given the combination of the lines $\overline{q_0q_1}$, $\overline{q_0q_2}$ and $\overline{q_1q_2}$, the curve will now have a multiplicity of 2 in each point q_0, q_1, q_2, see Figure 4.7.

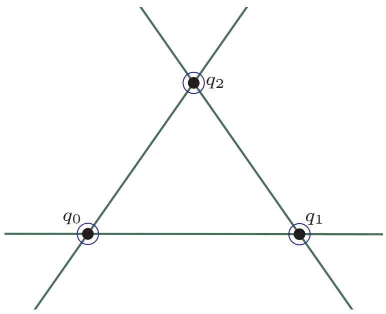

Figure 4.7: Curve in $\mathbb{P}^2$ of partition containing six components

Summary

In Table 4.1 we summarize the partitions of class $(3L - e_0 - e_1 - e_2)$. We only consider the 18 different partitions, where the symmetric variations of each partition is disregarded. The order of the partitions in the table are based on the number of components in each partition.

Table 4.1: All partitions

Number of components	Partition
1	$(3L - e_0 - e_1 - e_2)$
2	$(3L - 2e_0 - e_1 - e_2) + e_0$
2	$(2L - e_0 - e_1 - e_2) + L$
2	$(2L - e_0 - e_1) + (L - e_2)$
2	$(2L - e_0) + (L - e_1 - e_2)$
3	$(2L - e_0 - e_1 - e_2) + (L - e_0) + e_0$
3	$(2L - e_0 - e_1) + (L - e_0 - e_2) + e_0$
3	$(L - e_0 - e_1) + (L - e_2) + L$
3	$(L - e_0) + (L - e_1) + (L - e_2)$
4	$(L - e_0 - e_1) + e_0 + L + (L - e_0 - e_2)$
4	$(L - e_0 - e_1) + (L - e_2) + e_0 + (L - e_0)$
4	$(L - e_0 - e_1) + (L - e_2) + (L - e_2) + e_2$
4	$(2L - e_0 - e_1 - e_2) + e_1 + e_0 + (L - e_0 - e_1)$
5	$(L - e_1) + (L - e_0 - e_2) + e_1 + (L - e_0 - e_1) + e_0$
5	$(L - e_0 - e_1) + (L - e_0 - e_1) + (L - e_2) + e_0 + e_1$
5	$(L - e_0 - e_1) + (L - e_0 - e_2) + (L - e_0) + e_0 + e_0$
6	$(L - e_0 - e_1) + (L - e_0 - e_2) + (L - e_1 - e_2) + e_0 + e_1 + e_2$
6	$(L - e_0 - e_1) + (L - e_0 - e_2) + (L - e_0 - e_2) + e_0 + e_0 + e_2$

Degree of partitions of class $3L - e_0 - e_1 - e_2$ when mapped into $\mathbb{P}^5$

In Section 4.4 we found that two different blow-ups when embedded into $\mathbb{P}^7$ intersect in a curve of degree 6. When two hyperplane sections are intersected, i.e. $(h_0 + h_1 + h_2)^2$, we end up in $\mathbb{P}^5$. Thus, the curve of degree 6 is an intersection of two blow-ups is in $\mathbb{P}^5$.

In Section 4.5, we found 18 unique partitions of the class $3L - e_0 - e_1 - e_2$ in the blow-up of $\mathbb{P}^2$, when the symmetric variations are disregarded. Here, we are interested in the degree of curves of these partitions in $\mathbb{P}^5$. We will focus on the same 6 partitions as we did in Section 4.5, but summarize the results for all partitions in upcoming tables.

To determine the degree of each component of a partition in $\mathbb{P}^5$, we intersect each component of a partition with the class $3L - e_0 - e_1 - e_2$. The degree of a curve in a partition is equal to the sum of the degree of each component. Here, curves of each partition must necessarily be of degree 6, as 6 is the degree of the curve we started out with in $\mathbb{P}^5$.

We start by calculating the degree of the partition consisting of only one component.

The degree of the partition containing one component

We intersect the partition with the class of the curve, that is

$$(3L - e_0 - e_1 - e_2) \cdot \Big((3L - e_0 - e_1 - e_2)\Big) = (9L^2 + e_0^2 + e_1^2 + e_2^2)$$

$$= (9 - 1 - 1 - 1)$$
$$= 6$$

as $L^2 = 1$, $e_i^2 = -1$ and the product of two non-identical element are equal to zero. Thus, this partition, consisting of only one element, corresponds to a curve of degree 6 in $\mathbb{P}^5$.

In Table 4.2 below, we list the only partition that consists of *one* component and the degree of this component.

Table 4.2: Partition with 1 component and degree of the component (c)

Partition	Deg(c)
$(3L - e_0 - e_1 - e_2)$	6

The degree of a partition with two components

We continue to intersect each partition with the class $3L - e_0 - e_1 - e_2$, that is

$$\begin{aligned}(3L - e_0 - e_1 - e_2) \cdot \Big((3L - 2e_0 - e_1 - e_2) + e_0\Big) &= 5 + 1 \\ &= 6\end{aligned}$$

This partition corresponds to a curve in $\mathbb{P}^5$ of degree 6, where each component has degree 5, 1 respectively.

In Table 4.3 below, we summarize the degrees of each component in the partitions that consists of two components.

Table 4.3: Partitions with 2 components and the degree of each component (c_i)

Partition	Deg(c_1)	Deg(c_2)
$(3L - 2e_0 - e_1 - e_2) + e_0$	5	1
$(2L - e_0 - e_1 - e_2) + L$	3	3
$(2L - e_0 - e_1) + (L - e_2)$	4	2
$(2L - e_0) + (L - e_1 - e_2)$	5	1

The degree of a partition with three components

$$\begin{aligned}(3L - e_0 - e_1 - e_2) \cdot \Big((2L - e_0 - e_1) + (L - e_0 - e_2) + e_0\Big) &= 4 + 1 + 1 \\ &= 6\end{aligned}$$

The partition corresponds to a curve in $\mathbb{P}^5$ of degree 6, where each component has degree $4, 1, 1$ respectively.

In Table 4.4 below, we summarize the degrees of each component in the partitions that consists of three components.

Table 4.4: Partitions with 3 components and degree of each component (c_i)

Partition	Deg(c_1)	Deg(c_2)	Deg(c_3)
$(2L - e_0 - e_1 - e_2) + (L - e_0) + e_0$	3	2	1
$(2L - e_0 - e_1) + (L - e_0 - e_2) + e_0$	4	1	1
$(L - e_0 - e_1) + (L - e_2) + L$	1	2	3
$(L - e_0) + (L - e_1) + (L - e_2)$	2	2	2

The degree of a partition with four components

$$\begin{aligned}(3L - e_0 - e_1 - e_2) \cdot \Big((L - e_0 - e_1) + e_0 + L + (L - e_0 - e_2)\Big) &= 1 + 1 + 3 + 1 \\ &= 6\end{aligned}$$

The partition corresponds to a curve in $\mathbb{P}^5$ of degree 6, where each component has degree $1, 1, 3, 1$ respectively.

In Table 4.5 below, we summarize the degrees of each component in the partitions that consists of four components.

Table 4.5: Partitions with 4 components and degree of each component (c_i)

Partition	Deg(c_1)	Deg(c_2)	Deg(c_3)	Deg(c_4)
$(L - e_0 - e_1) + e_0 + L + (L - e_0 - e_2)$	1	1	3	1
$(L - e_0 - e_1) + (L - e_2) + e_0 + (L - e_0)$	1	2	1	2
$(L - e_0 - e_1) + (L - e_2) + (L - e_2) + e_2$	1	2	2	1
$(2L - e_0 - e_1 - e_2) + e_1 + e_0 + (L - e_0 - e_1)$	3	1	1	1

The degree of a partition with five components

$$\begin{aligned}(3L - e_0 - e_1 - e_2) \cdot \\ \Big((L - e_1) + (L - e_0 - e_2) + e_1 + (L - e_0 - e_1) + e_0\Big) &= 2 + 1 + 1 + 1 + 1 \\ &= 6\end{aligned}$$

The partition corresponds to a curve in $\mathbb{P}^5$ of degree 6 with five components, where each component has degree $2, 1, 1, 1, 1$ respectively.

In Table 4.6 below, we summarize the degrees of each component in the partitions that consists of five components.

Table 4.6: Partitions with 5 components and the degree of each component (c_i)

Partition	Degree of				
	c_1	c_2	c_3	c_4	c_5
$(L - e_1) + (L - e_0 - e_2) + e_1 + (L - e_0 - e_1) + e_0$	2	1	1	1	1
$(L - e_0 - e_1) + (L - e_0 - e_1) + (L - e_2) + e_0 + e_1$	1	1	2	1	1
$(L - e_0 - e_1) + (L - e_0 - e_2) + (L - e_0) + e_0 + e_0$	1	1	2	1	1

The degree of a partition with six components

$$(3L-e_0-e_1-e_2)\cdot\Big((L-e_0-e_1)+e_1+(L-e_0-e_2)+e_2+(L-e_1-e_2)+e_0\Big)=6$$

The partition corresponds to a curve in $\mathbb{P}^5$ of degree 6 with six components, where each component is of degree 1.

In Table 4.7 below, we list the partitions that consist of six components, and the belonging degree of each of the components.

Table 4.7: Partitions with 6 components and the degree of each component (c_i)

Partition	Degree of					
	c_1	c_2	c_3	c_4	c_5	c_6
$(L-e_0-e_1)+e_1+(L-e_0-e_2)+e_2+(L-e_1-e_2)+e_0$	1	1	1	1	1	1
$(L-e_0-e_1)+(L-e_0-e_2)+(L-e_0-e_2)+e_0+e_0+e_2$	1	1	1	1	1	1

Can two different blow-ups be mapped to the same curve in $\mathbb{P}^5$?

In this section we want to study the preimage of the intersection of two surfaces in $\mathbb{P}^7$. In Section 4.4 we found that the intersection of two such surfaces is a curve of degree 6 in $\mathbb{P}^5$.

We will examine whether two different partitions of the class $3L-e_0-e_1-e_2$, i.e. two blow-ups of $\mathbb{P}^2$, can be mapped to the same curve of degree 6 in $\mathbb{P}^5$.

$$\begin{tikzcd}
 & & \mathbb{P}^7 & & \\
 & \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1 \arrow[ur, hook, "\psi"] & & \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1 \arrow[ul, hook', "\psi"'] & \\
\overline{\Gamma}_p \arrow[ur, hook, "\omega"] & & & & \overline{\Gamma}_p \arrow[ul, hook', "\omega"']
\end{tikzcd}$$

Figure 4.8: Two blow-ups of $\mathbb{P}^2$ in three points

When we consider the preimage of such a curve in $\mathbb{P}^5$, the map will pull back to two different blow-ups of $\mathbb{P}^2$, each blow-up corresponding to a curve of degree 3 in $\mathbb{P}^2$ going through three points, q_0, q_1, q_2 and q_0', q_1', q_2' respectively. Thus, the preimage will correspond to curves in the blow-up of $\mathbb{P}^2$ of class $(3L-e_0-e_1-e_2)$ and $(3L'-e_0'-e_1'-e_2')$ respectively. These classes behave similarly, as both correspond to curves going through three points in $\mathbb{P}^2$. Thus, there are 18 partitions of each such class, if we omit the symmetric partitions.

We are interested in studying different partitions of $(3L-e_0-e_1-e_2)$ that becomes the same intersection curve in $\mathbb{P}^5$ when embedded into $\mathbb{P}^7$. Thus, we study the 18 partitions corresponding to the classes of the two blow-ups, to examine whether we can obtain the intersection curve in $\mathbb{P}^5$, by intersecting different partitions.

We want to find partitions that can be matched, and the necessary conditions for two partitions to be matched are defined below.

Definition 4.5.3 (Matching partitions)**.**

i) The partitions have the same number of components

ii) The partitions consists of components of the same degrees

iii) The intersection between the components in each partition must be equal in both partitions that are matched.

When we search for possible matches, as both classes of the two blow-ups are identical up to variable change, we only consider the partitions corresponding to the class $3L - e_0 - e_1 - e_2$ that we found in Section 4.5.

By studying each table in Section 4.5, we can find possible candidates of matching that satisfies both condition 1 and 2. Then subsequently, we can check whether the last condition is met. We start by studying the tables in succeeding order based on the number of components in a partition. As there is only one partition in Table 4.2, we skip this table.

Possible matches in partitions with two components

In Table 4.3, we can observe that partition number 2 and partition number 4 are possible candidates to be matched, i.e.

$$\text{partition \#1: } (3L - 2e_0 - e_1 - e_2) + e_0$$
$$\text{partition \#4: } (2L - e_0) + (L - e_1 - e_2)$$

They are possible candidates as both partitions consist of two components, and in each partition there is one component of degree 1 and the other is of degree 5.

We can study the intersection points between the components in each of the partitions. We start with partition number 1.

$$(3L - 2e_0 - e_1 - e_2) \cdot e_0 \quad = \quad -2e_0 \quad = \quad 2$$

As we can see, there are two intersection points. Thus, the cubic curve with a singularity in q_0 intersects the exceptional divisor e_0 twice in partition number 1. This is visualized in the dual graph in Figure 4.9 below. In a dual graph, each node represents a curve of some class, corresponding to a component c_i of a partition. Inside each node, in addition to c_i, there is a number representing the degree of the related curve. The lines between the nodes represents the intersections between the curves, and the number associated to each line represents the number of times the curves intersect. Two separate lines between the same two nodes, represents two different intersection points between the curves.

Further, we can consider the intersection points of the components of partition number four, that is $(2L - e_0) + (L - e_1 - e_2)$.

$$(2L - e_0) \cdot (L - e_1 - e_2) \quad = \quad 2L^2 \quad = \quad 2$$

Also here, the two components intersect exactly twice. The quadratic curve intersect with the line going through q_0, q_1 in two different points. This is visualized in the dual graph in Figure 4.10 below.

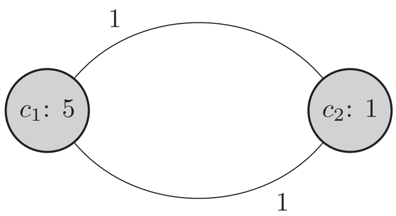

Figure 4.9: Dual graph of $(3L - 2e_0 - e_1 - e_2) + e_0$

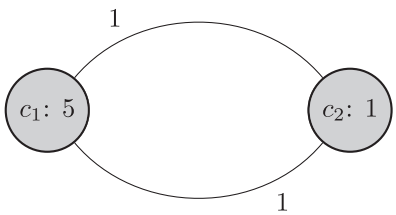

Figure 4.10: Dual graph of $(2L - e_0) + (L - e_1 - e_2)$

Thus, both partitions have the same number of intersection points, and the intersection between the components in each partition seems to be identical, as is visualized in the dual graphs Figure 4.9 and Figure 4.10. Thus, they meet all three necessary conditions for matching.

Possible matches in partitions with three components

In Table 4.4 partition number 1 and 3 are two partitions that may be matched, i.e.

$$\text{partition \#1: } (2L - e_0 - e_1 - e_2) + (L - e_0) + e_0$$
$$\text{partition \#3: } (L - e_0 - e_1) + (L - e_2) + L$$

Both partitions have three components, and each partition has one component of degree 1, a second component of degree 2 and a third component of degree 3.

We can study the intersection points between the components in each of the partitions. We start with partition 1.

$$\begin{array}{ccccc} (2L - e_0 - e_1 - e_2) \cdot (L - e_0) & = & 2L^2 + e_0^2 & = & 1 \\ (2L - e_0 - e_1 - e_2) \cdot e_0 & = & -e_1^2 & = & 1 \\ (L - e_0) \cdot e_0 & = & -e_1^2 & = & 1 \end{array}$$

As we can see, each of the components intersects each of the other components exactly once. This is visualized in the dual graph in Figure 4.11 below.

Further, we can consider the intersection points of the components of partition 3

$$\begin{array}{ccccc} (L - e_0 - e_1) \cdot (L - e_2) & = & L^2 & = & 1 \\ (L - e_0 - e_1) \cdot L & = & L^2 & = & 1 \\ (L - e_2) \cdot L & = & L^2 & = & 1 \end{array}$$

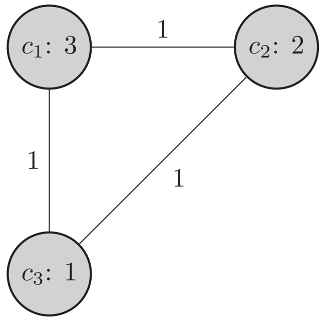

Figure 4.11: Dual graph of $(2L - e_0 - e_1 - e_2) + (L - e_0) + e_0$

Also here, each of the components intersects each of the other components exactly once. Thus, the two partitions satisfies the three necessary conditions.

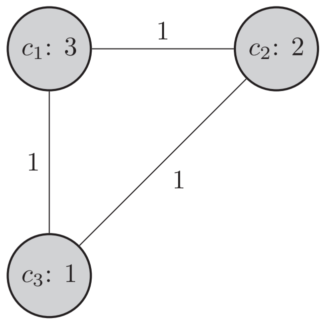

Figure 4.12: Dual graph of $(L - e_0 - e_1) + (L - e_2) + L$

From the study of intersection points above, and also visualized in the dual graphs Figure 4.11 and Figure 4.12, the two partitions have identical intersection points. Thus, they meet all three necessary conditions for matching.

Possible matches in partitions with four components

In Table 4.5 partition number 1 and 4 are two possible matches. Both partitions have four components, and three of the components is of degree 1 and one component is of degree 3.

$$\begin{aligned}
&\text{partition \#1: } (L - e_0 - e_1) + e_0 + L + (L - e_0 - e_2)\\
&\text{partition \#4: } (2L - e_0 - e_1 - e_2) + e_1 + e_0 + (L - e_0 - e_1)
\end{aligned}$$

We can study the intersection points between the components in each of the partitions. We start with partition 1, that is $(L-e_0-e_1)+e_0+L+(L-e_0-e_2)$

$$\begin{array}{lcccccc}
c_1 \cdot c_2 & = & (L - e_0 - e_1) \cdot e_0 & = & -e_0^2 & = & 1\\
c_1 \cdot c_3 & = & (L - e_0 - e_1) \cdot L & = & L^2 & = & 1\\
c_1 \cdot c_4 & = & (L - e_0 - e_1) \cdot (L - e_0 - e_2) & = & L^2 + e_0^2 & = & 0\\
c_2 \cdot c_3 & = & e_0 \cdot L & = & - & = & 0\\
c_2 \cdot c_4 & = & e_0 \cdot (L - e_0 - e_2) & = & -e_0^2 & = & 1\\
c_3 \cdot c_4 & = & L \cdot (L - e_0 - e_2) & = & L^2 & = & 1
\end{array}$$

As we can see, the four components intersect with two of the other components exactly once. The line $\overline{q_0q_1}$ and the line $\overline{q_0q_2}$ do not intersect in $\mathbb{P}^5$. Further, the arbitrary line L does not intersect the exceptional divisor e_0. This is visualized in the dual graph in Figure 4.13 below.

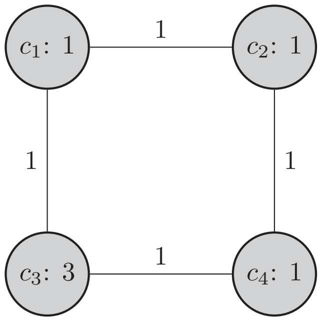

Figure 4.13: Dual graph of $(L - e_0 - e_1) + e_0 + L + (L - e_0 - e_2)$

Then, we study the intersection points between the components in partition number 4, i.e. $(2L - e_0 - e_1 - e_2) + e_1 + e_0 + (L - e_0 - e_1)$.

$$\begin{array}{rcccccl}
c_1 \cdot c_2 & = & (2L - e_0 - e_1 - e_2) \cdot e_1 & = & -e_1^2 & = & 1 \\
c_1 \cdot c_3 & = & (2L - e_0 - e_1 - e_2) \cdot e_0 & = & -e_0^2 & = & 1 \\
c_1 \cdot c_4 & = & (2L - e_0 - e_1 - e_2) \cdot (L - e_0 - e_1) & = & 2L^2 + e_0^2 + e_1^2 & = & 0 \\
c_2 \cdot c_3 & = & e_1 \cdot e_0 & = & - & = & 0 \\
c_2 \cdot c_4 & = & e_1 \cdot (L - e_0 - e_1) & = & -e_1^2 & = & 1 \\
c_3 \cdot c_4 & = & e_0 \cdot (L - e_0 - e_1) & = & -e_0^2 & = & 1
\end{array}$$

Also here, all components intersect with two of the other components, exactly once. The quadratic curve going through all three points q_0, q_1, q_2 and the line going through q_0 and q_1 do not intersect in $\mathbb{P}^5$. In addition, the two exceptional divisors e_0, e_1 do not intersect either. This is visualized in the dual graph in Figure 4.14 below.

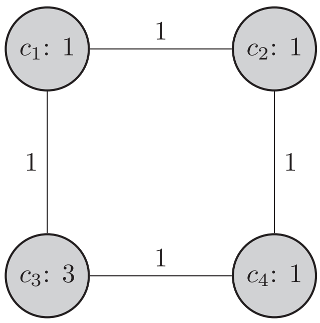

Figure 4.14: Dual graph of $(2L - e_0 - e_1 - e_2) + e_1 + e_0 + (L - e_0 - e_1)$

From the study of intersection points above, and also visualized in the dual graphs Figure 4.13 and Figure 4.14, the two partitions have identical intersection points. Thus, they meet all three necessary conditions for matching.

Further, we have another possible match in Table 4.5, that is partition number 2 and 3. Both of the partitions have four components, and both have two components of degree 1 and two components of degree 2.

$$\begin{aligned}\text{partition \#2: } & (L-e_0-e_1)+(L-e_2)+e_0+(L-e_0)\\ \text{partition \#3: } & (L-e_0-e_1)+(L-e_2)+(L-e_2)+e_2\end{aligned}$$

We look at the intersection points between the components in each of the partitions. We start with partition 2, i.e. $(L-e_0-e_1)+(L-e_2)+e_0+(L-e_0)$.

$$\begin{array}{ccccccc} c_1\cdot c_2 & = & (L-e_0-e_1)\cdot(L-e_2) & = & L^2 & = & 1\\ c_1\cdot c_3 & = & (L-e_0-e_1)\cdot e_0 & = & -e_0^2 & = & 1\\ c_1\cdot c_4 & = & (L-e_0-e_1)\cdot(L-e_0) & = & L^2+e_0^2 & = & 0\\ c_2\cdot c_3 & = & (L-e_2)\cdot e_0 & = & - & = & 0\\ c_2\cdot c_4 & = & (L-e_2)\cdot(L-e_0) & = & L^2 & = & 1\\ c_3\cdot c_4 & = & e_0\cdot(L-e_0) & = & -e_0^2 & = & 1 \end{array}$$

Again, all components intersect two of the other components exactly once. Here, as we can see, the line $\overline{q_0q_1}$ and the line going through q_0 do not intersect in $\mathbb{P}^5$. Further, the line going through q_2 do not intersect the exceptional divisor e_0. This is visualized in the dual graph in Figure 4.15 below.

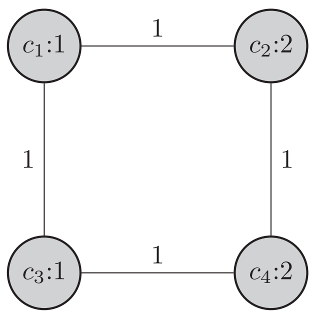

Figure 4.15: Dual graph of $(L-e_0-e_1)+(L-e_2)+e_0+(L-e_0)$

Then, we study the intersection points between the components in partition 3, i.e. $(L-e_0-e_1)+(L-e_2)+(L-e_2)+e_2$.

$$\begin{array}{ccccccc} c_1\cdot c_2 & = & (L-e_0-e_1)\cdot(L-e_2) & = & L^2 & = & 1\\ c_1\cdot c_3 & = & (L-e_0-e_1)\cdot(L-e_2) & = & L^2 & = & 1\\ c_1\cdot c_4 & = & (L-e_0-e_1)\cdot e_2 & = & - & = & 0\\ c_2\cdot c_3 & = & (L-e_2)\cdot(L-e_2) & = & L^2+e_2^2 & = & 0\\ c_2\cdot c_4 & = & (L-e_2)\cdot e_2 & = & -e_2^2 & = & 1\\ c_3\cdot c_4 & = & (L-e_2)\cdot e_2 & = & -e_2^2 & = & 1 \end{array}$$

Also here, all components intersect with two of the other components, again exactly once. This time, the line $\overline{q_0q_1}$ and the exceptional divisor e_0 do not intersect in $\mathbb{P}^5$. Further, the two lines both going through q_2 do not intersect. This is visualized in the dual graph in Figure 4.16 below.

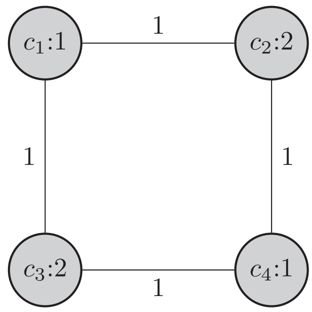

Figure 4.16: Dual graph of $(L-e_0-e_1)+(L-e_2)+(L-e_2)+e_2$

From the calculation of intersection points, and also visualized in the dual graphs Figure 4.15 and Figure 4.16, partition number two and three have the same intersection points. Thus, they meet all three necessary conditions for matching.

Possible matches in partitions with five components

In Table 4.6, all three partitions that consist of five components are in fact all possible matches. That is, all three partitions have five components and each partition have four components of degree 1 and one component of degree 2.

$$\begin{aligned}
&\text{partition \#1: } (L-e_1)+(L-e_0-e_2)+e_1+(L-e_0-e_1)+e_0\\
&\text{partition \#2: } (L-e_0-e_1)+(L-e_0-e_1)+(L-e_2)+e_0+e_1\\
&\text{partition \#3: } (L-e_0-e_1)+(L-e_0-e_2)+(L-e_0)+e_0+e_0
\end{aligned}$$

We consider the intersection points between the components in each of the partitions. We start with the first partition, i.e. $(L-e_1)+(L-e_0-e_2)+e_1+(L-e_0-e_1)+e_0$.

$$\begin{array}{cccccccc}
c_1\cdot c_2 & = & (L-e_1)\cdot(L-e_0-e_2) & = & L^2 & = & 1\\
c_1\cdot c_3 & = & (L-e_1)\cdot e_1 & = & -e_1^2 & = & 1\\
c_1\cdot c_4 & = & (L-e_1)\cdot(L-e_0-e_1) & = & L^2+e_1^2 & = & 0\\
c_1\cdot c_5 & = & (L-e_1)\cdot e_0 & = & - & = & 0\\
c_2\cdot c_3 & = & (L-e_0-e_2)\cdot e_1 & = & - & = & 0\\
c_2\cdot c_4 & = & (L-e_0-e_2)\cdot(L-e_0-e_1) & = & L^2+e_0^2 & = & 0\\
c_2\cdot c_5 & = & (L-e_0-e_2)\cdot e_0 & = & -e_0^2 & = & 1\\
c_3\cdot c_4 & = & e_1\cdot(L-e_0-e_1) & = & -e_1^2 & = & 1\\
c_3\cdot c_5 & = & e_1\cdot e_0 & = & - & = & 0\\
c_4\cdot c_5 & = & (L-e_0-e_1)\cdot e_0 & = & -e_0^2 & = & 1
\end{array}$$

As we can see, in $\mathbb{P}^5$, each component intersects two other components exactly once. The line through q_1 intersects both the line $\overline{q_0q_2}$ and the exceptional divisor e_1. In addition, the line $\overline{q_0q_2}$ intersects the exceptional divisor e_0. Further, the exceptional divisor e_1 also intersects the line $\overline{q_0q_1}$. Lastly, the line $\overline{q_0q_1}$ intersects the exceptional divisor e_0. This is visualized in the dual graph in Figure 4.17 below.

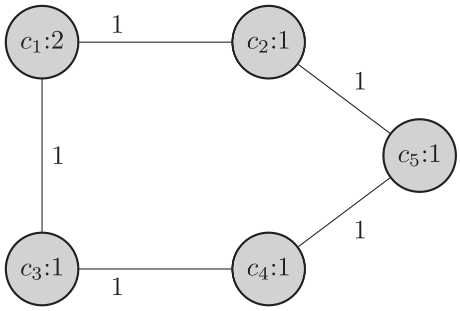

Figure 4.17: Dual graph of $(L - e_1) + (L - e_0 - e_2) + e_1 + (L - e_0 - e_1) + e_0$

Then we study the intersection points between the components in partition number 2, i.e. $(L - e_0 - e_1) + (L - e_0 - e_1) + (L - e_2) + e_0 + e_1$.

$$
\begin{array}{lcccccc}
c_1 \cdot c_2 & = & (L - e_0 - e_1) \cdot (L - e_0 - e_1) & = & L^2 + e_0^2 + e_1^2 & = & -1 \\
c_1 \cdot c_3 & = & (L - e_0 - e_1) \cdot (L - e_2) & = & L^2 & = & 1 \\
c_1 \cdot c_4 & = & (L - e_0 - e_1) \cdot e_0 & = & -e_0^2 & = & 1 \\
c_1 \cdot c_5 & = & (L - e_0 - e_1) \cdot e_1 & = & -e_1^2 & = & 1 \\
c_2 \cdot c_3 & = & (L - e_0 - e_1) \cdot (L - e_2) & = & L^2 & = & 1 \\
c_2 \cdot c_4 & = & (L - e_0 - e_1) \cdot e_0 & = & -e_0^2 & = & 1 \\
c_2 \cdot c_5 & = & (L - e_0 - e_1) \cdot e_1 & = & -e_1^2 & = & 1 \\
c_3 \cdot c_4 & = & (L - e_2) \cdot e_0 & = & - & = & 0 \\
c_3 \cdot c_5 & = & (L - e_2) \cdot e_1 & = & - & = & 0 \\
c_4 \cdot c_5 & = & e_0 \cdot e_1 & = & - & = & 0
\end{array}
$$

In $\mathbb{P}^5$, as component number one and two are equal, the intersection between them is equal to -1. Further, both of these components intersect each of the three other components exactly once. Thus, the line $\overline{q_0 q_1}$ intersects the line going through q_2, the exceptional divisor e_0 and the exceptional divisor e_1. This is visualized in the dual graph in Figure 4.18 below.

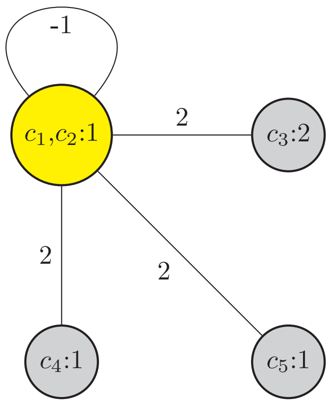

Figure 4.18: Dual graph of $(L - e_0 - e_1) + (L - e_0 - e_1) + (L - e_2) + e_0 + e_1$

Then we study the intersection points between the components in partition number 2, i.e. $(L - e_0 - e_1) + (L - e_0 - e_2) + (L - e_0) + e_0 + e_0$.

$$
\begin{array}{rclcrcr}
c_1 \cdot c_2 & = & (L - e_0 - e_1) \cdot (L - e_0 - e_2) & = & L^2 + e_0^2 & = & 0 \\
c_1 \cdot c_3 & = & (L - e_0 - e_1) \cdot (L - e_0) & = & L^2 + e_0^2 & = & 0 \\
c_1 \cdot c_4 & = & (L - e_0 - e_1) \cdot e_0 & = & -e_0^2 & = & 1 \\
c_1 \cdot c_5 & = & (L - e_0 - e_1) \cdot e_0 & = & -e_0^2 & = & 1 \\
c_2 \cdot c_3 & = & (L - e_0 - e_2) \cdot (L - e_0) & = & L^2 + e_0^2 & = & 0 \\
c_2 \cdot c_4 & = & (L - e_0 - e_2) \cdot e_0 & = & -e_0^2 & = & 1 \\
c_2 \cdot c_5 & = & (L - e_0 - e_2) \cdot e_0 & = & -e_0^2 & = & 1 \\
c_3 \cdot c_4 & = & (L - e_0) \cdot e_0 & = & -e_0^2 & = & 1 \\
c_3 \cdot c_5 & = & (L - e_0) \cdot e_0 & = & -e_0^2 & = & 1 \\
c_4 \cdot c_5 & = & e_0 \cdot e_0 & = & e_0^2 & = & -1
\end{array}
$$

Now, component number four and five are equal, thus the intersection between them is equal to -1. Also here, both component four and five intersect each of the other three components exactly once. Hence, the exceptional divisor e_0 intersects the line $\overline{q_0 q_1}$, the line $\overline{q_0 q_2}$ and the line going through q_0. This is visualized in the dual graph in Figure 4.19 below.

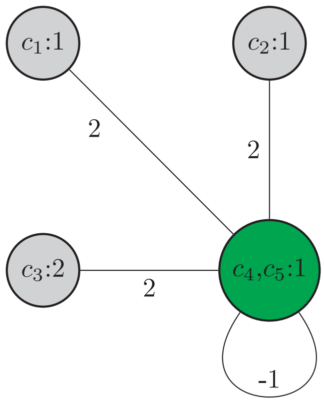

Figure 4.19: Dual graph of $(L - e_0 - e_1) + (L - e_0 - e_2) + (L - e_0) + e_0 + e_0$

By the calculation of intersection points between the components in each partition, and thus as visualized in Figure 4.17, Figure 4.18 and Figure 4.19, we can conclude that the first partition does not match the two remaining partitions, as they differ in some of the intersection points. However, partition number two and three have similar intersection points. Thus, they meet all three necessary conditions for matching.

Possible matches in partitions with six components

Lastly, in Table 4.7 both partitions are possible matches. That is, both partitions have six components and all components in each partition is of degree 1.

partition #1: $(L - e_0 - e_1) + e_1 + (L - e_0 - e_2) + e_2 + (L - e_1 - e_2) + e_0$

partition #2: $(L - e_0 - e_1) + (L - e_0 - e_2) + (L - e_0 - e_2) + e_0 + e_0 + e_2$

We consider the intersection points between the components in each of the partitions. We start with the first partition, i.e. $(L - e_0 - e_1) + e_1 + (L - e_0 - e_2) + e_2 + (L - e_1 - e_2) + e_0$.

$$
\begin{array}{rclcccc}
c_1 \cdot c_2 & = & (L - e_0 - e_1) \cdot e_1 & = & -e_1^2 & = & 1 \\
c_1 \cdot c_3 & = & (L - e_0 - e_1) \cdot (L - e_0 - e_2) & = & L^2 + e_0^2 & = & 0 \\
c_1 \cdot c_4 & = & (L - e_0 - e_1) \cdot e_2 & = & - & = & 0 \\
c_1 \cdot c_5 & = & (L - e_0 - e_1) \cdot (L - e_1 - e_2) & = & L^2 + e_1^2 & = & 0 \\
c_1 \cdot c_6 & = & (L - e_0 - e_1) \cdot e_0 & = & -e_0^2 & = & 1 \\
c_2 \cdot c_3 & = & e_1 \cdot (L - e_0 - e_2) & = & - & = & 0 \\
c_2 \cdot c_4 & = & e_1 \cdot e_2 & = & - & = & 0 \\
c_2 \cdot c_5 & = & e_1 \cdot (L - e_1 - e_2) & = & -e_1^2 & = & 1 \\
c_2 \cdot c_6 & = & e_1 \cdot e_0 & = & - & = & 0 \\
c_3 \cdot c_4 & = & (L - e_0 - e_2) \cdot e_2 & = & -e_2^2 & = & 1 \\
c_3 \cdot c_5 & = & (L - e_0 - e_2) \cdot (L - e_1 - e_2) & = & L^2 + e_2^2 & = & 0 \\
c_3 \cdot c_6 & = & (L - e_0 - e_2) \cdot e_0 & = & -e_0^2 & = & 1 \\
c_4 \cdot c_5 & = & e_2 \cdot (L - e_1 - e_2) & = & -e_2^2 & = & 1 \\
c_4 \cdot c_6 & = & e_2 \cdot e_0 & = & - & = & 0 \\
c_5 \cdot c_6 & = & (L - e_1 - e_2) \cdot e_0 & = & - & = & 0
\end{array}
$$

In this partition, all components are lines and all lines intersect exactly once with two of the other lines. In regard to three of the lines, each line goes through two of the points q_0, q_1, q_2. These three lines do not intersect with each other, they only intersect with the exceptional divisors associated with the points they go through.

Thus, the line $\overline{q_0 q_1}$ intersects the exceptional divisors e_0 and e_1. Further, $\overline{q_0 q_2}$ intersects the exceptional divisors e_0 and e_2. Lastly, $\overline{q_1 q_2}$ intersects the exceptional divisors e_1 and e_2. This is visualized in the dual graph in Figure 4.20 below.

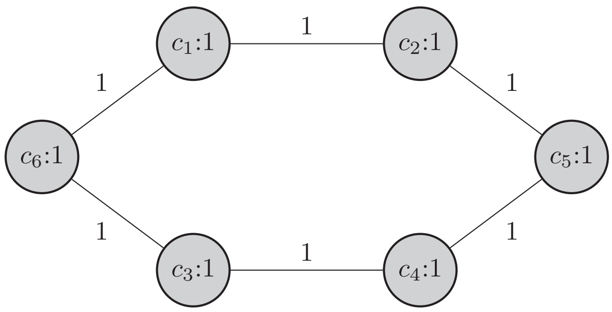

Figure 4.20: Dual graph of $(L-e_0-e_1)+e_1+(L-e_0-e_2)+e_2+(L-e_1-e_2)+e_0$

We continue to study the intersection points between the components in partition number 2, i.e. $(L-e_0-e_1)+(L-e_0-e_2)+(L-e_0-e_2)+e_0+e_0+e_2$.

$$
\begin{array}{lclclcl}
c_1 \cdot c_2 & = & (L-e_0-e_1)\cdot e_1 & = & -e_1^2 & = & 1 \\
c_1 \cdot c_3 & = & (L-e_0-e_1)\cdot e_2 & = & - & = & 0 \\
c_1 \cdot c_4 & = & (L-e_0-e_1)\cdot(L-e_1-e_2) & = & L^2+e_1^2 & = & 0 \\
c_1 \cdot c_5 & = & (L-e_0-e_1)\cdot(L-e_0-e_2) & = & L^2+e_0^2 & = & 0 \\
c_1 \cdot c_6 & = & (L-e_0-e_1)\cdot e_0 & = & -e_0^2 & = & 1 \\
c_2 \cdot c_3 & = & e_1 \cdot e_2 & = & - & = & 0 \\
c_2 \cdot c_4 & = & e_1 \cdot (L-e_1-e_2) & = & -e_1^2 & = & 1 \\
c_2 \cdot c_5 & = & e_1 \cdot (L-e_0-e_2) & = & - & = & 0 \\
c_2 \cdot c_6 & = & e_1 \cdot e_0 & = & - & = & 0 \\
c_3 \cdot c_4 & = & e_2 \cdot (L-e_1-e_2) & = & -e_2^2 & = & 1 \\
c_3 \cdot c_5 & = & e_2 \cdot (L-e_0-e_2) & = & -e_2^2 & = & 1 \\
c_3 \cdot c_6 & = & e_2 \cdot e_0 & = & - & = & 0 \\
c_4 \cdot c_5 & = & (L-e_1-e_2)\cdot(L-e_0-e_2) & = & L^2+e_2^2 & = & 0 \\
c_4 \cdot c_6 & = & (L-e_1-e_2)\cdot e_0 & = & - & = & 0 \\
c_5 \cdot c_6 & = & (L-e_0-e_2)\cdot e_0 & = & -e_0^2 & = & 1
\end{array}
$$

In this partition there are two double components, i.e. component number two and three are equal, and component number four and five are equal. The intersection between the equal components are -1. In total, there are four components that are different from each other. Each of these components intersects each of the other three components exactly once. This is visualized in the dual graph in Figure 4.21 below.

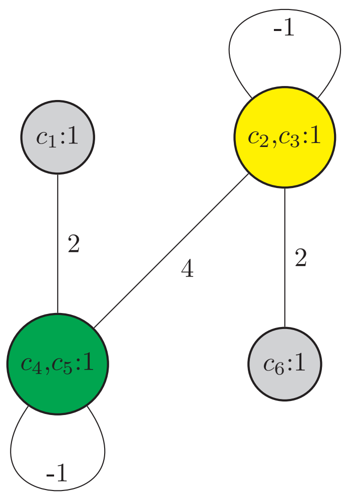

Figure 4.21: Dual graph of$(L-e_0-e_1)+(L-e_0-e_2)+(L-e_0-e_2)+e_0+e_0+e_2$

By the calculation of intersection points between the components in both partition, and as visualized in Figure 4.20 and Figure 4.21, we can conclude that the partitions do not match, as they differ in most of the intersection points.

4.6 Reconstruction of $\mathbb{P}^2$

In this section we want to investigate if we can reconstruct $\mathbb{P}^2$. First, we consider the image of the blow up of $\mathbb{P}^2$ in the three points q_0, q_1, q_2 when embedded into $\mathbb{P}^7$, and consider the preimage of this surface. As before, the blow up of $\mathbb{P}^2$ is denoted by $\overline{\Gamma}_p$. Then we consider the preimage of $\overline{\Gamma}_p$ in $\mathbb{P}^2$. As we will see, there are two possible reconstructions of $\overline{\Gamma}_p$ in $\mathbb{P}^2$. Consequently, we might end up with a different image than we started with. This means that there exists a Cremona transformation.

Definition 4.6.1 ([Har77], p. 30)**.** A Cremona transformation is a birational map of $\mathbb{P}^2$ into itself, i.e. $\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

We have a map

$$\begin{aligned}\tau : \mathbb{P}^2 \setminus \{q_0, q_1, q_2\} &\to \mathbb{P}^2 \setminus \{q_0, q_1, q_2\}\\ (x_0 : x_1 : x_2) &\mapsto (x_1x_2 : x_0x_2 : x_0x_1)\end{aligned}$$

where the transformation becomes an isomorphism of coordinate axis that is undefined on the three fundamental points $q_0 = (1,0,0)$, $q_1 = (0,1,0)$ and $q_2 = (0,0,1)$, and maps each coordinate axis onto the unique point not contained in that axis.

Recall the map

$$\begin{aligned}&\psi \circ p : \mathbb{P}^2 \to \mathbb{P}^7\\ &(x_0 : x_1 : x_2) \mapsto (-x_1x_2^2 : x_0x_2^2 : x_0x_1x_2 : -x_0^2x_2 : x_1^2x_2 : -x_0x_1x_2 : -x_0x_1^2 : x_0^2x_1)\end{aligned}$$

where p is the rational map from $\mathbb{P}^2$ to $(\mathbb{P}^1)^3$ and ψ is the Segre embedding from $(\mathbb{P}^1)^3$ to $\mathbb{P}^7$, such that $V = \psi \circ p(\mathbb{P}^2)$ in $\mathbb{P}^7$.

In Section 4.3 by Lemma 4.3.1, we found that $V = U \cap Z_+(y_2 + y_5)$, i.e. the image of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ intersected with a hyperplane of $\mathbb{P}^7$. Thus, V is a surface of dimension two in $\mathbb{P}^7$.

The rational map p from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, factors through $\omega \circ \alpha_{\mathbb{P}^2}^{-1}$. Here $\alpha_{\mathbb{P}^2}$ represents the *blow down* from $\overline{\Gamma}_p$ to $\mathbb{P}^2$ and ω represents the morphism from $\overline{\Gamma}_p$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, thus $S = \omega(\overline{\Gamma}_p) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. This is visualised in Figure 4.22 below.

$$\begin{tikzcd} & \overline{\Gamma}_p \arrow[dr, "\omega"] & & \\ \mathbb{P}^2 \arrow[ur, "\alpha_{\mathbb{P}^2}^{-1}"] \arrow[rr, dashed, "p"] & & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "\psi"] & \mathbb{P}^7 \end{tikzcd}$$

Figure 4.22: Composition of the maps from $\mathbb{P}^2$ to $\mathbb{P}^7$

As ω and ψ are morphisms, the image of $\overline{\Gamma}_p$ in $\mathbb{P}^7$ is isomorphic to V. Thus, we can reconstruct $\overline{\Gamma}_p$ from V.

The pull back from $\mathbb{P}^7$ to $\overline{\Gamma}_p$ is given by

$$\begin{aligned}(\psi \circ \omega)^{-1} &: \mathbb{P}^7 \to \overline{\Gamma}_p\\ (\psi \circ \omega)^{-1}(V) &= \overline{\Gamma}_p\end{aligned}$$

Then, it remains to map $\overline{\Gamma}_p$ back to $\mathbb{P}^2$, we have the morphism

$$\alpha_{\mathbb{P}^2} : \overline{\Gamma}_p \to \mathbb{P}^2$$
$$\alpha_{\mathbb{P}^2}(\overline{\Gamma}_p) = \mathbb{P}^2$$

where each exceptional divisor e_0, e_1, e_2 in $\overline{\Gamma}_p$ are contracted to the corresponding point in $\mathbb{P}^2$, q_0, q_1, q_2 respectively.

Identification of 7 points in $\mathbb{P}^7$

Recall that the surface in $\mathbb{P}^7$ birationally equivalent to $\mathbb{P}^2$ is $V = U \cap Z_+(y_2+y_5)$. Here $Z_+(y_2 + y_5)$ is one specific hyperplane of $\mathbb{P}^7$. However, most hyperplane sections will be isomorphic to $Z_+(y_2 + y_5)$.

In general, 7 points is enough to span a hyperplane in $\mathbb{P}^7$, i.e. $\mathbb{P}^6$, as long as they are linearly independent. Assume we are able to identify 7 linearly independent points in U in $\mathbb{P}^7$, and thereby obtain the hyperplane $Z_+(y_2 + y_5)$ of $\mathbb{P}^7$. By intersecting the hyperplane with U, we find the surface V in $\mathbb{P}^7$.

Given the divisor $(3L - e_0 - e_1 - e_2)$, there exists 6 different classes that alone represents a line in $\mathbb{P}^6$, i.e. $(L - e_0 - e_1)$, $(L - e_0 - e_2)$, $(L - e_1 - e_2)$, e_0, e_1, e_2. Each of these classes will only intersect with two of the other classes in $\mathbb{P}^6$, that is $(L - e_i - e_j) \cdot e_i = 1$ and $(L - e_i - e_j) \cdot e_j = 1$ for every $i \neq j$ and $i, j = 0, 1, 2$.

Thus,

$$(L - e_i - e_j) \cdot (L - e_j - e_k) = 0 \qquad \text{for } i, j, k = 0, 1, 2$$
$$e_i \cdot e_j = 0 \qquad \text{for } i, j = 0, 1, 2$$

This is important as neither of the exceptional divisors e_i intersect in $\overline{\Gamma}_p$.

As every other line $L - e_i - e_j$ and every other exceptional divisor intersect, we obtain a hexagon in $\mathbb{P}^7$. Such a hexagon is one of the possible partitions of $3L - e_0 - e_1 - e_2$ with six components, that we considered in Section 4.5.

The hexagon consists of two sets of triple lines, see figure Figure 4.23, where the two set of triple lines are separated by red and blue colors respectively. As the curve is visualized by a dual graph the lines are depicted as nodes, and the intersection points as lines between the nodes.

However, without the distinction of name and color on each line, it will be impossible to distinguish between the two sets of triple lines.

Given the surface $V \subseteq \mathbb{P}^7$, there are two ways to map the surface back to $\mathbb{P}^2$, i.e. we can blow down either set of three lines. Thus, the reconstruction of $\mathbb{P}^2$ is ambiguous.

To see how this is possible, consider the six lines in $V \subseteq \mathbb{P}^7$ as mentioned above, and consider the two maps from $V \subseteq \mathbb{P}^7$ to $\mathbb{P}^2$ as in Figure 4.24, where we either multiply with the class L or the class $(2L - e_0 - e_1 - e_2)$.

Only three of the lines in $\mathbb{P}^7$ are mapped to lines in $\mathbb{P}^2$. To see how the two maps affects each line, we calculate the degree of the image in $\mathbb{P}^2$ by intersecting

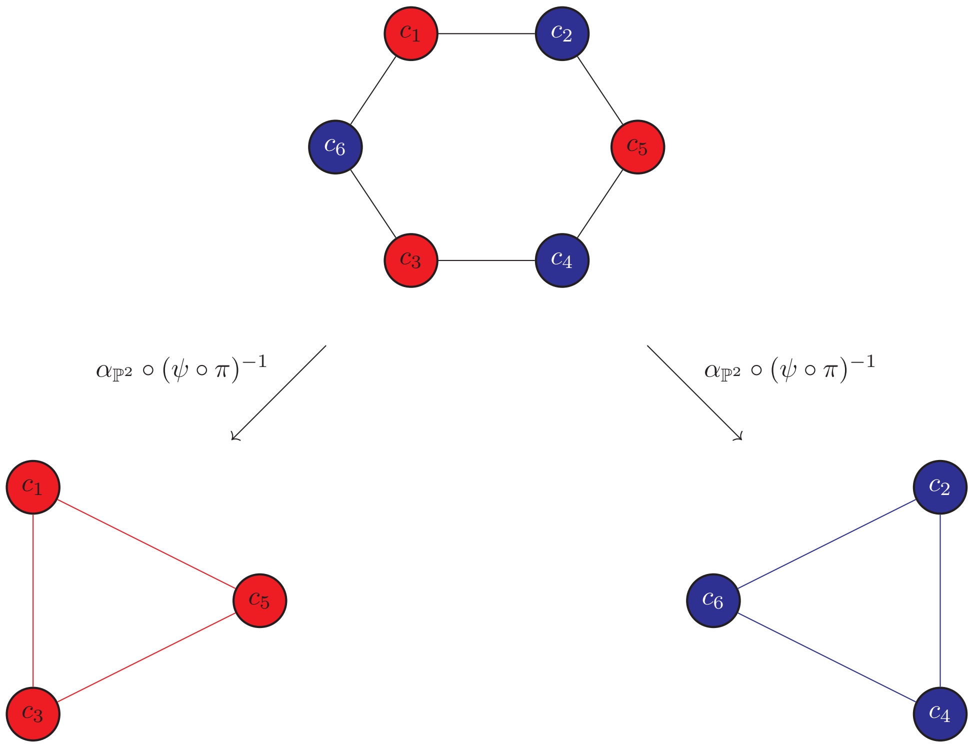

Figure 4.23: Dual graph of $(L-e_0-e_1)+e_1+(L-e_0-e_2)+e_2+(L-e_1-e_2)+e_0$ in $\mathbb{P}^7$ mapped back to $\mathbb{P}^2$, where $c_1 = (L-e_0-e_1)$, $c_2 = e_1$, $c_3 = (L-e_0-e_2)$, $c_4 = e_2$, $c_5 = (L-e_1-e_2)$, $c_6 = e_0$.

$$\begin{tikzcd}
 & V \simeq \overline{\Gamma}_p \simeq \overline{\Gamma}_{p'} \arrow[dl, shift right, "\cdot(L)"'] \arrow[dr, shift left, "\cdot(2L-e_0-e_1-e_2)"] & \\
\mathbb{P}^2 \arrow[ur, dashed, shift right, "p"'] \arrow[rr, dashed, "\tau"] & & \mathbb{P}^2 \arrow[ul, dashed, shift left, "p'"]
\end{tikzcd}$$

Figure 4.24: Two blow-downs from $V \subseteq \mathbb{P}^7$ that are isomorphic to $\overline{\Gamma}_p$.

it with both L and $C = 2L - e_0 - e_1 - e_2$ respectively, i.e.

$$\begin{aligned}
C \cdot e_i &= 1 && \text{for } i = 0,1,2 \\
C \cdot (L - e_i - e_j) &= 0 && \text{for } i,j = 0,1,2 \\
L \cdot e_i &= 0 && \text{for } i = 0,1,2 \\
L \cdot (L - e_i - e_j) &= 1 && \text{for } i,j = 0,1,2
\end{aligned}$$

Thus, by mapping all six lines with the left map in Figure 4.24, only $(L-e_0-e_1)$, $(L-e_0-e_2)$ and $(L-e_1-e_2)$ become lines in $\mathbb{P}^2$, as they obtain degree 1 in $\mathbb{P}^2$. The exceptional divisors e_i on the other hand, becomes points as they obtain degree 0 in $\mathbb{P}^2$. This is visualized in Figure 4.25 below.

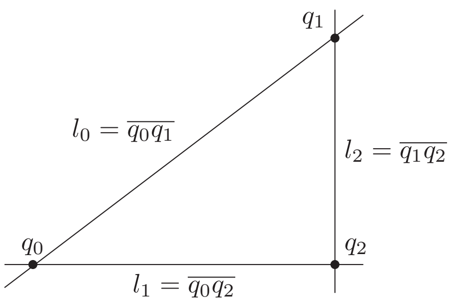

Figure 4.25: The curve consisting of the three lines $l_0 = \overline{q_0 q_1}$, $l_1 = \overline{q_0 q_2}$, $l_2 = \overline{q_1 q_2}$ combined with nodes in q_0, q_1, q_2 in $\mathbb{P}^2$

On the other hand, by mapping the lines to $\mathbb{P}^2$ with the right map, now the exceptional divisors e_i becomes the lines in $\mathbb{P}^2$, and the lines $L - e_i - e_j$ becomes points.

Thus, there is a possibility that we start out with the blow-up of $\mathbb{P}^2$ in three points, then embed it into $\mathbb{P}^7$ where the image is a surface V, and when we blow it back down we obtain a different $\mathbb{P}^2$ than the one we started out with.

This is indeed a Cremona transformation as discussed above, i.e.

$$\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

where $\tau = (\psi \circ \omega) \circ \alpha_{\mathbb{P}^2}^{-1}$. Thus, even though we have identified 7 linearly independent points, we might not be able to uniquely reconstruct $\mathbb{P}^2$.

Identification of 6 points in $\mathbb{P}^7$

There is however no guarantee that we can find 7 linearly independent points. Assume that of the points we identify, only 6 of them are linearly independent points. These points will then span a $\mathbb{P}^5$.

We have that a $\mathbb{P}^5$ is the intersection of two hyperplanes in $\mathbb{P}^7$. When $\mathbb{P}^5$ is intersected with U we obtain, as previously found, a curve of degree 6 in $\mathbb{P}^7$.

As we have seen, this curve may consist of anywhere from 1 to 6 components. In Section 4.5, we saw that if the curve consists of either 2, 3, 4 or 5 components there may be two possible partitions of the class $3L - e_0 - e_1 - e_2$ in $\overline{\Gamma}_p$. In addition, the curve with 6 components that we discussed above, can be its own match, by varying the lines and the exceptional divisors. Thus, the preimage of the curves may not be unique, as we may obtain a different curve than the curve we started out with.

In addition, for each of these possibilities, as we saw in the previous section there will be two ways to map it back down to $\mathbb{P}^2$ from $\overline{\Gamma}_p$. This is depicted in Figure 4.26 below.

Consider the possible matches of partitions containing two components, i.e.

$$(3L - 2e_0 - e_1 - e_2) + e_0$$
$$(2L - e_0) + (L - e_1 - e_2)$$

$$\begin{tikzcd}
 & & V_0 \cap V_1 \subseteq \mathbb{P}^5 \arrow[dl, hook, "\iota_0"'] \arrow[dr, hook, "\iota_1"] & & \\
 & V_0 \arrow[dl, "\cdot(L)"'] \arrow[d, "\cdot(2L-e_0-e_1-e_2)"] & & V_1 \arrow[d, "\cdot(L)"'] \arrow[dr, "\cdot(2L-e_0-e_1-e_2)"] & \\
\mathbb{P}^2 & \mathbb{P}^2 & & \mathbb{P}^2 & \mathbb{P}^2
\end{tikzcd}$$

Figure 4.26: Two blow-downs from each V_0, V_1 into $\mathbb{P}^2$.

Both partitions are classes of curves, where one component is of degree 1 and the other is of degree 5 in $\mathbb{P}^5$. As mentioned, they are mapped back to different curves in $\mathbb{P}^2$. The first is mapped to the nodal cubic that goes through all three points, with a multiplicity of two in q_0, combined with a node in q_0. The second partition is mapped to the conic going through point q_0, combined with a line going through the two remaining points.

Then, in addition, each of these curves in $\mathbb{P}^7$ will be mapped back to $\mathbb{P}^2$ in two different ways, by the two maps described in the previous section. To see how these maps affect the curves in $\mathbb{P}^2$ we calculate the degree of each component in $\mathbb{P}^2$ by intersection each component with both L and $C = 2L - e_0 - e_1 - e_2$ respectively, i.e.

$$\begin{aligned}
(2L - e_0 - e_1 - e_2) \cdot \Big((2L - e_0) + (L - e_1 - e_2)\Big) &= 3 + 0 = 3 \\
L \cdot \Big((2L - e_0) + (L - e_1 - e_2)\Big) &= 2 + 1 = 3
\end{aligned}$$

$$\begin{aligned}
(2L - e_0 - e_1 - e_2) \cdot \Big((3L - 2e_0 - e_1 - e_2) + e_0\Big) &= 2 + 1 = 3 \\
L \cdot \Big((3L - 2e_0 - e_1 - e_2) + e_0\Big) &= 3 + 0 = 3
\end{aligned}$$

CHAPTER 5

Reconstruction from four camera centers

We will reconstruct $\mathbb{P}^2$ from four projections $\pi_{q_i} : \mathbb{P}^2 \to \mathbb{P}^1$, where each projection is an image from a camera center $q_i \in \mathbb{P}^2$ for $i = 0, 1, 2, 3$.

To reconstruct $\mathbb{P}^2$, we want to find the surface $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, birationally equivalent to $\mathbb{P}^2$, and the inverse birational map α from S to $\mathbb{P}^2$. We consider $(\mathbb{P}^1)^4$ in its Segre embedding in $\mathbb{P}^{15}$. In this chapter, we will show that the image of S in $\mathbb{P}^{15}$ is $S = (\mathbb{P}^1)^4 \cap \mathbb{P}^{10}$. Thus, to determine S we need to identify enough points in $(\mathbb{P}^1)^4$ so that they span $\mathbb{P}^{10}$ in $\mathbb{P}^{15}$. As we will show, the map α is the projection from $\mathbb{P}^7$ restricted to S, such that $\mathbb{P}^7 \cap S$ is a curve C of degree 8, where $\langle C \rangle = \mathbb{P}^7$ in $\mathbb{P}^{10}$.

However, there are cases where we do not find enough points to determine S in $\mathbb{P}^{15}$. Consequently, we can not unambiguously reconstruct $\mathbb{P}^2$. We call these critical configurations. So in addition to finding S and α, we will classify different critical configurations and study the information we obtain of $\mathbb{P}^2$ in such cases.

5.1 A rational map from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Now we want to construct a map given the four points $q_0, q_1, q_2, q_3 \in \mathbb{P}^2$, where $q_0 = (1 : 0 : 0)$, $q_1 = (0 : 1 : 0)$, $q_2 = (0 : 0 : 1)$ and $q_3 = (1 : 1 : 1)$.

Recall the map from $\mathbb{P}^2$ into $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, given by

$$
\begin{aligned}
p : \mathbb{P}^2 &\to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\
(x_0 : x_1 : x_2) &\mapsto (-x_2 : x_1) \times (-x_2 : x_0) \times (-x_1 : x_0)
\end{aligned}
$$

It remains to find the coordinates of the last $\mathbb{P}^1$ satisfying $x_0 = x_1 = x_2 = 1$. As a line in $\mathbb{P}^2$ is given by $u_0 x_0 + u_1 x_1 + u_2 x_2 = 0$, we get $u_0 = -u_1 - u_2$. By rewriting the equation of the line we get

$$
\begin{aligned}
(-u_1 - u_2)x_0 + u_1 x_1 + u_2 x_2 &= 0 \\
u_1(x_1 - x_0) + u_2(x_2 - x_0) &= 0 \\
u_1(x_1 - x_0) &= u_2(x_0 - x_2)
\end{aligned}
$$

Such that $u_1 = (x_0 - x_2)$ and $u_2 = (x_1 - x_0)$.

Thus, a map from $\mathbb{P}^2$ into $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ can be described explicitly by

$$\begin{aligned} p : \mathbb{P}^2 &\to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ (x_0 : x_1 : x_2) &\mapsto (-x_2 : x_1) \times (-x_2 : x_0) \times (-x_1 : x_0) \times (x_0 - x_2 : x_1 - x_0) \end{aligned}$$

where $(x_0 - x_2 : x_1 - x_0)$ is undefined when $x_0 = x_1 = x_2$, as is the case at point $q_3 = (1 : 1 : 1)$.

By discussion in Section 3.1, we see that p is a morphism between the open subset $\mathbb{P}^2 \setminus \{q_0, q_1, q_2, q_3\}$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then by Definition 3.1.1, we recognize p as a rational map from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, i.e.

$$p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

5.2 An embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^{15}$

We have a map from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^{15}$ given by

$$\begin{aligned} \Psi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^{15} \\ (u_0 : v_0) \times (u_1 : v_1) \times (u_2 : v_2) \times (u_3 : v_3) &\mapsto \begin{pmatrix} u_0u_1u_2u_3 : & u_0u_1u_2v_3 : & u_0u_1v_2u_3 : & u_0u_1v_2v_3 : \\ u_0v_1u_2u_3 : & u_0v_1u_2v_3 : & u_0v_1v_2u_3 : & u_0v_1v_2v_3 : \\ v_0u_1u_2u_3 : & v_0u_1u_2v_3 : & v_0u_1v_2u_3 : & v_0u_1v_2v_3 : \\ v_0v_1u_2u_3 : & v_0v_1u_2v_3 : & v_0v_1v_2u_3 : & v_0v_1v_2v_3 \end{pmatrix} \\ &\mapsto (y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7 : \cdots : y_{15}) \end{aligned}$$

The map Ψ is a closed embedding called the Segre embedding, see Section 2.1, where the image $U = \Psi(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ is closed and isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

5.3 The image of $\mathbb{P}^2$ in $\mathbb{P}^{15}$

Then, we have the rational map p from $\mathbb{P}^2$ into $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, i.e.

$$\begin{aligned} p : \mathbb{P}^2 &\to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ (x_0 : x_1 : x_2) &\mapsto (-x_2 : x_1) \times (-x_2 : x_0) \times (-x_1 : x_0) \times (x_0 - x_2 : x_1 - x_0) \end{aligned}$$

and the Segre embedding ψ from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^{15}$, i.e.

$$\begin{aligned} \Psi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^{15} \\ (u_0 : v_0) \times (u_1 : v_1) \times (u_2 : v_2) \times (u_3 : v_3) &\mapsto (u_0u_1u_2u_3 : u_0u_1u_2v_3 : u_0u_1v_2u_3 : u_0u_1v_2v_3 : \\ &\quad u_0v_1u_2u_3 : u_0v_1u_2v_3 : u_0v_1v_2u_3 : u_0v_1v_2v_3 : \\ &\quad v_0u_1u_2u_3 : v_0u_1u_2v_3 : v_0u_1v_2u_3 : v_0u_1v_2v_3 : \\ &\quad v_0v_1u_2u_3 : v_0v_1u_2v_3 : v_0v_1v_2u_3 : v_0v_1v_2v_3) \end{aligned}$$

Thus, when mapped from $\mathbb{P}^2$ the coordinates in $\mathbb{P}^{15}$ become

$$\begin{aligned} y_0 = u_0u_1u_2u_3 &= (-x_2)(-x_2)(-x_1)(x_0 - x_2) = -x_0x_1x_2^2 + x_1x_2^3 \\ y_1 = u_0u_1u_2v_3 &= (-x_2)(-x_2)(-x_1)(x_1 - x_0) = -x_1^2x_2^2 + x_0x_1x_2^2 \\ y_2 = u_0u_1v_2u_3 &= (-x_2)(-x_2)(x_0)(x_0 - x_2) = x_0^2x_2^2 - x_0x_2^3 \end{aligned}$$

$$\begin{aligned}
y_3 &= u_0u_1v_2v_3 &&= (-x_2)(-x_2)(x_0)(x_1-x_0) = x_0x_1x_2^2 - x_0^2x_2^2\\
y_4 &= u_0v_1u_2u_3 &&= (-x_2)(x_0)(-x_1)(x_0-x_2) = x_0^2x_1x_2 - x_0x_1x_2^2\\
y_5 &= u_0v_1u_2v_3 &&= (-x_2)(x_0)(-x_1)(x_1-x_0) = x_0x_1^2x_2 - x_0^2x_1x_2\\
y_6 &= u_0v_1v_2u_3 &&= (-x_2)(x_0)(x_0)(x_0-x_2) = -x_0^3x_2 + x_0^2x_2^2\\
y_7 &= u_0v_1v_2v_3 &&= (-x_2)(x_0)(x_0)(x_1-x_0) = -x_0^2x_1x_2 + x_0^3x_2\\
y_8 &= v_0u_1u_2u_3 &&= (x_1)(-x_2)(-x_1)(x_0-x_2) = -x_0x_1^2x_2 + x_1^2x_2^2\\
y_9 &= v_0u_1u_2v_3 &&= (x_1)(-x_2)(-x_1)(x_1-x_0) = -x_1^3x_2 + x_0x_1^2x_2\\
y_{10} &= v_0u_1v_2u_3 &&= (x_1)(-x_2)(x_0)(x_0-x_2) = -x_0^2x_1x_2 + x_0x_1x_2^2\\
y_{11} &= v_0u_1v_2v_3 &&= (x_1)(-x_2)(x_0)(x_1-x_0) = -x_0x_1^2x_2 + x_0^2x_1x_2\\
y_{12} &= v_0v_1u_2u_3 &&= (x_1)(x_0)(-x_1)(x_0-x_2) = -x_0^2x_1^2 + x_0x_1^2x_2\\
y_{13} &= v_0v_1u_2v_3 &&= (x_1)(x_0)(-x_1)(x_1-x_0) = -x_0x_1^3 + x_0^2x_1^2\\
y_{14} &= v_0v_1v_2u_3 &&= (x_1)(x_0)(x_0)(x_0-x_2) = x_0^3x_1 - x_0^2x_1x_2\\
y_{15} &= v_0v_1v_2v_3 &&= (x_1)(x_0)(x_0)(x_1-x_0) = x_0^2x_1^2 - x_0^3x_1
\end{aligned}$$

where every monomial y_i is of degree 4.

Hence, the composition of the two maps is a map from $\mathbb{P}^2 \to \mathbb{P}^{15}$ given by

$$\begin{aligned}
&\Psi \circ p : \mathbb{P}^2 \to \mathbb{P}^{15}\\
(x_0 : x_1 : x_2) &\mapsto \big((-x_0x_1x_2^2 + x_1x_2^3) : (-x_1^2x_2^2 + x_0x_1x_2^2) : \cdots : (x_0^2x_1^2 - x_0^3x_1)\big)\\
&= (y_0 : y_1 : \cdots : y_{15})
\end{aligned}$$

Now, we want to know what $V = \Psi \circ p(\mathbb{P}^2)$ of $\mathbb{P}^{15}$ looks like. Clearly V is contained in U, as the map factors through $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, but we expect it to have dimension $\dim V = \dim \mathbb{P}^2 = 2$, unlike U, which is of dimension $\dim U = 4$.

From the expression of each coordinate y_i above, we obtain the following relations

$$\begin{aligned}
y_4 + y_{10} &= 0\\
y_5 + y_{11} &= 0\\
y_3 + y_4 + y_6 + y_7 &= 0\\
y_1 + y_4 + y_5 - y_8 &= 0\\
-y_5 + y_{12} + y_{14} + y_{15} &= 0
\end{aligned}$$

that give the hyperplanes of $\mathbb{P}^{15}$,

$$\begin{aligned}
&Z_+(y_4 + y_{10})\\
&Z_+(y_5 + y_{11})\\
&Z_+(y_3 + y_4 + y_6 + y_7)\\
&Z_+(y_1 + y_4 + y_5 - y_8)\\
&Z_+(-y_5 + y_{12} + y_{14} + y_{15})
\end{aligned} \tag{5.1}$$

By observation, V is a subset of each hyperplane listed above. We claim that these five hyperplanes are sufficient to determine V.

Lemma 5.3.1. *V, i.e. the image of $\mathbb{P}^2$ in $\mathbb{P}^{15}$, fulfils*

$$\begin{aligned}V =& U \cap Z_+(y_4 + y_{10}) \cap Z_+(y_5 + y_{11}) \cap Z_+(y_3 + y_4 + y_6 + y_7) \\ & \cap Z_+(y_1 + y_4 + y_5 - y_8) \cap Z_+(-y_5 + y_{12} + y_{14} + y_{15})\end{aligned}$$

For easier notation we write $V = U \cap \mathbb{P}^{10}_1$.

If this is true, V spans a space of dimension

$$\dim(\mathbb{P}^{15}) - 1 - 1 - 1 - 1 - 1 = 10$$

where we subtract five dimensions, one for each hyperplane. Thus, V would span a $\mathbb{P}^{10}$.

Proof. Clearly, V is a subset of U and of each of the five hyperplanes in Equation (5.1). Thus, we only need to show that

$$U \cap \mathbb{P}^{10}_1 \subseteq V$$

The coordinates of the inverse image Ψ^{-1} from $\mathbb{P}^{15}$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are given by

$$(y_0 : y_8) \times (y_0 : y_4) \times (y_0 : y_2) \times (y_0 : y_1) \tag{5.2}$$

And when mapped from $\mathbb{P}^2$ the coordinates of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$(-x_2 : x_1) \times (-x_2 : x_0) \times (-x_1 : x_0) \times (x_0 - x_2 : x_1 - x_0) \tag{5.3}$$

where these all are written in terms of the three coordinates of $\mathbb{P}^2$.

To show that $U \cap \mathbb{P}^{10}_1 \subseteq V$, we want the coordinates of the inverse image $\Psi^{-1}(V)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to be written in terms of three coordinates only, such that the surface can be mapped back to $\mathbb{P}^2$. From Equation (5.2) and Equation (5.3), we observe that

$$y_0 = -x_2, \quad y_8 = x_1, \quad y_4 = x_0$$

To be able to map $V \subset \mathbb{P}^{15}$ back to $\mathbb{P}^2$ we want to obtain the following coordinates of the inverse image $\Psi^{-1}(V)$

$$(y_0 : y_8) \times (y_0 : y_4) \times (-y_8 : y_4) \times (y_0 + y_4 : y_8 - y_4)$$

For this to be true, the following relations must be satisfied

$$(y_0 : y_2) = (-y_8 : y_4) \;\; \Rightarrow \;\; y_0 y_4 + y_2 y_8 = 0 \tag{5.4}$$

and

$$(y_0 : y_1) = \big((y_0 + y_4) : (y_8 - y_4)\big) \;\Rightarrow\; y_1 y_0 + y_1 y_4 - y_0 y_8 + y_0 y_4 = 0 \tag{5.5}$$

To prove Equation (5.4) and Equation (5.5), we consider the case $y_0 \neq 0$, i.e. the open $D_+(y_0)$. We consider a point of the variety U intersected with these five hyperplanes, i.e. $\mathbf{x} = (y_0 : \cdots : y_{15}) \in U \cap \mathbb{P}^{10}_1$.

We already know that the preimage of $\mathbf{x}$ equals

$$\Psi^{-1}\big((y_0 : \cdots : y_{15})\big) = (y_0 : y_8) \times (y_0 : y_4) \times (y_0 : y_2) \times (y_0 : y_1) = \mathbf{z}$$

However, we must take into account that $\Psi(\mathbf{z}) \in Z_+(y_4 + y_{10})$, $Z_+(y_5 + y_{11})$, $Z_+(y_3 + y_4 + y_6 + y_7)$, $Z_+(y_1 + y_4 + y_5 - y_8)$, $Z_+(-y_5 + y_{12} + y_{14} + y_{15})$.

Thus, we need to consider these relations when looking at the preimage. First, we consider $y_4 + y_{10} = 0$. With a change of coordinates into $\Psi^{-1}\big((y_0 : \cdots : y_{15})\big)$ we obtain

$$\begin{aligned} y_0^3 y_4 + y_0^2 y_2 y_8 &= 0 \\ y_0 y_4 + y_2 y_8 &= 0 \end{aligned}$$

which is equivalent to $(y_0 : y_2) = (-y_8 : y_4)$. Thus Equation (5.4) is satisfied.

Further, we consider the relation $y_1 + y_4 + y_5 - y_8 = 0$. By change of coordinates, we obtain

$$\begin{aligned} y_0^3 y_1 + y_0^3 y_4 + y_0^2 y_1 y_4 - y_0^3 y_8 &= 0 \\ y_0 y_1 + y_0 y_4 + y_1 y_4 - y_0 y_8 &= 0 \end{aligned}$$

which is equivalent to $(y_0 : y_1) = \big((y_0 + y_4) : (y_8 - y_4)\big)$. Thus Equation (5.5) is satisfied.

Thus, we obtained exactly what we wanted. Now, $\mathbf{z}$ can be written as

$$\mathbf{z} = (y_0 : y_8) \times (y_0 : y_4) \times (-y_8 : y_4) \times (y_0 + y_4 : y_8 - y_4)$$

which proves that

$$\mathbf{x} = \Psi(\mathbf{z}) = (\Psi \circ p)\big((y_4 : y_8 : -y_0)\big) \subseteq (\Psi \circ p)(\mathbb{P}^2) = V$$

By symmetry, we need only to consider the case $D_+(y_i)$ for $i = 0$. For which we exhibit the inverse morphism.

When considering the remaining $D_+(y_i)$ for $i \neq 0$, we might need to replace the hyperplanes used above with some of the other hyperplanes in $\mathbb{P}_1^{10}$ to determine the coordinates of the last two $\mathbb{P}^1$'s, but in the case of $D_+(y_0)$, we are done. Thus, we can argue that $V = U \cap \mathbb{P}_1^{10}$. ■

Hence, the surface S that we need to reconstruct $\mathbb{P}^2$ is $S \cong V = U \cap \mathbb{P}_1^{10}$.

5.4 The class of the image of $\mathbb{P}^2$

In this section, we want to determine class of the surface V. In addition we will also determine the degree of U and V in $\mathbb{P}^{15}$. The notation and theory in this section is based on results in Chapter 2. See this chapter for more details.

First, we want to determine the degree of U in $\mathbb{P}^{15}$, where U is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^{15}$. To calculate the degree of a space, we intersect the class of the space with the class of a hyperplane section, until the dimension is reduced to zero. In general, when the dimension is reduced to zero, the number of intersection points will represent the degree of the space. Thus, to find the degree of U, we intersect U with four hyperplane sections.

The class of a hyperplane section is given by $(h_0 + h_1 + h_2 + h_3)$. Thus, to calculate the degree of U, we calculate the class of the intersection of four hyperplane sections, given by $(h_0 + h_1 + h_2 + h_3)^4$. That is

$$\begin{aligned} (h_0 + h_1 + h_2 + h_3)^4 =& \alpha \cdot h_0 h_1 h_2 h_3 \\ =& \alpha \end{aligned}$$

where α, in addition to being the coefficient of $h_0h_1h_2h_3$ in $(h_0+h_1+h_2+h_3)^4$, represents the number of intersection points, and thereby the degree of $U \subset \mathbb{P}^{15}$. Thus, the degree of U is $\alpha = 4! = 24$.

Then, we want to determine the class of V and the degree of V in $\mathbb{P}^{15}$. In Section 2.5, we showed that the class of a surface S is given by

$$[S] = \sum_{i,j=0}^{3} \alpha_{ij} h_i h_j \tag{5.6}$$

for $i \neq j$. Further, we showed that to find the degree of a surface in the Segre embedding, we intersect the class of the surface with the intersection of two hyperplane sections, i.e.

$$\begin{aligned}[S] \cdot (h_0 + h_1 + h_2 + h_3)^2 &= 2 \cdot (\alpha_{01} + \alpha_{02} + \alpha_{03} + \alpha_{12} + \alpha_{13} + \alpha_{23}) \cdot h_0h_1h_2h_3 \\ &= 2 \cdot (\alpha_{01} + \alpha_{02} + \alpha_{03} + \alpha_{12} + \alpha_{13} + \alpha_{23})\end{aligned}$$

Thus, the degree of the surface is given by $2 \cdot (\alpha_{01} + \alpha_{02} + \alpha_{03} + \alpha_{12}, \alpha_{13} + \alpha_{23})$.

In Section 4.4 we described that each class h_0, h_1, h_2, h_2 have restrictions to V, such that $h_i|_S = (L - e_i)$. Thus,

$$a_{23} = [V] \cdot h_0h_1 = h_0|_S \cdot h_1|_S = (L - e_0) \cdot (L - e_1) = 1$$

In fact, we find that each $\alpha_{ij} = 1$ for all $i \neq j$. Then, as every $\alpha_{ij} = 1$ in Equation (5.6), we can determine the class of the surface V, i.e.

$$[V] = \sum_{i,j=0}^{3} h_i h_j = h_0h_1 + h_0h_2 + h_0h_3 + h_1h_2 + h_1h_3 + h_2h_3$$

And the degree of the surface V is $2 \cdot (1 + 1 + 1 + 1 + 1 + 1) = 12$.

We can verify that $\deg V = 12$ in a different way. Recall that the class of a hyperplane section in $\operatorname{Pic}\left(\overline{\Gamma_p}\right)$ is given by $(4L - e_0 - e_1 - e_2 - e_3)$. Thus, when we restrict the intersection of two hyperplane sections to V, we find that the degree of V is 12, i.e.

$$\begin{aligned}[V] \cdot (h_0 + h_1 + h_2 + h_3)^2 &= (4L - e_0 - e_1 - e_2 - e_3)^2 \\ &= 12\end{aligned}$$

5.5 Intersection of $\langle V \rangle$ and $\langle V' \rangle$

In this section we want to study the intersection of two surfaces V and V' in U and the span of this intersection in the span of U, i.e. $\mathbb{P}^{15}$, where $\langle V \cap V' \rangle = \langle V \rangle \cap \langle V' \rangle$. The notation and theory in this section is based on results in Chapter 2.

Recall the result in Lemma 5.3.1, where we found that

$$\begin{aligned} V = U \cap Z_+(y_4 + y_{10}) \cap Z_+(y_5 + y_{11}) \cap Z_+(y_3 + y_4 + y_6 + y_7) \\ \cap Z_+(y_1 + y_4 + y_5 - y_8) \cap Z_+(-y_5 + y_{12} + y_{14} + y_{15}) \end{aligned} \tag{5.7}$$

As in the lemma, we use $V = U \cap \mathbb{P}^{10}_1$ for easier notation. The span of V is the span of U intersected with five linearly independent hyperplanes, and

$$\dim \langle V \rangle = \dim \mathbb{P}^{15} - 5 = 10$$

As $V = U \cap \mathbb{P}^{10}_1$, then $\langle V \rangle = \mathbb{P}^{10}$.

Then, consider another surface V', that is isomorphic to the blow up of $\mathbb{P}^2$ in four different points. Thus, V' will be equal to U intersected with 5 other hyperplanes, i.e.

$$\begin{aligned} V' = U \cap Z_+(a_0y_0 + \cdots + a_{15}y_{15}) \cap Z_+(a_0y_0 + \cdots + a_{15}y_{15}) \\ \cap Z_+(a_0y_0 + \cdots + a_{15}y_{15}) \cap Z_+(a_0y_0 + \cdots + a_{15}y_{15}) \cap Z_+(a_0y_0 + \cdots + a_{15}y_{15}) \end{aligned}$$

where each $Z_+(a_0y_0 + \cdots + a_{15}y_{15})$ represents a hyperplane of $\mathbb{P}^{15}$ that is linearly independent of the other four hyperplanes. For easier notation we write $V' = U \cap \mathbb{P}^{10}_2$, where $\mathbb{P}^{10}_2$ represents the intersection of these five hyperplanes.

Now, the span of V' is the span of U intersected with five linearly independent hyperplanes. Thus, $\langle V' \rangle = \mathbb{P}^{10}_2$. If the hyperplanes that determines V' are equal to the hyperplanes that determines V, then $V = V'$.

Classification by dimension of the intersection

V and V' are both of codimension 2 with respect to U. Thus, V intersected with V' will at most be of codimension $2 + 2 = 4$. However, if the intersection is of codimension 2, then $V = V'$.

As both $\langle V \rangle$ and $\langle V' \rangle$ is a $\mathbb{P}^{10}$, then both is of codimension 5 in $\mathbb{P}^{15}$. Thus, the span of V intersected with the span of V' is at most of codimension $5+5 = 10$, i.e. $\langle V \rangle \cap \langle V' \rangle = \mathbb{P}^5$. Thus, the intersection may of $\langle V \rangle$ and $\langle V' \rangle$ may range from a $\mathbb{P}^5$ to a $\mathbb{P}^{10}$.

Now, consider the class of each surface V and V'. Both surfaces is of class

$$[V] = \sum_{i,j=0}^{3} h_ih_j = h_0h_1 + h_0h_2 + h_0h_3 + h_1h_2 + h_1h_3 + h_2h_3$$

By considering the sum of the class of V and V', we get

$$\begin{aligned} [V] + [V] &= \sum_{i,j=0}^{3} h_ih_j + \sum_{i,j=0}^{3} h_ih_j \\ &= 2 \cdot (h_0h_1 + h_0h_2 + h_0h_3 + h_1h_2 + h_1h_3 + h_2h_3) \end{aligned} \tag{5.8}$$

Then, consider the class of a hyperplane section in $\mathbb{P}^{15}$, i.e. $(h_0+h_1+h_2+h_3)$. If we intersect the class of two hyperplane sections, we obtain

$$\begin{aligned}(h_0+h_1+h_2+h_3)^2 =& 2h_0h_1+2h_0h_2+2h_0h_3+2h_1h_2+2h_1h_3+2h_2h_3\\ =& 2\cdot(h_0h_1+h_0h_2+h_0h_3+h_1h_2+h_1h_3+h_2h_3)\end{aligned}$$

which is the exact same result as in Equation (5.8). Thus, $2\cdot[V]$, the sum of the class of V and V', is precisely the class of the intersection of two hyperplane sections.

This means that the span of V and the span of V' must be contained in the intersection of two hyperplanes in $\mathbb{P}^{15}$. The intersection of two linearly independent hyperplanes is a $\mathbb{P}^{13}$.

Let us consider $\mathbb{P}^{15}$ where V, $V' \subseteq \mathbb{P}^{15}$. As the span of V and the span of V' are both equal to a $\mathbb{P}^{10}$, there are five hyperplanes that contains each surface V, V'. We pick two hyperplanes such that each hyperplane contains both V and V'. Then, the intersection of these hyperplanes will contain the span of both V and V'.

We can now reconsider the intersection of the span of V and V'. As the span of both V and V' being contained in the intersection of two hyperplanes, the codimension of the intersection can at most be of codimension $3+3=6$. Thus, span of V and V' will intersect in no less than a space of dimension $\dim\mathbb{P}^{13}-6=7$, i.e. $\mathbb{P}^7$. Thus, we can disregard the possibility of intersecting in $\mathbb{P}^5$ or $\mathbb{P}^6$. We will examine each of the possibilities from $\mathbb{P}^{10}$ to $\mathbb{P}^7$ below.

Intersection of $\langle V\rangle$ and $\langle V'\rangle$ is a $\mathbb{P}^{10}$

If the span of V intersected with the span of V' is a $\mathbb{P}^{10}$, then $V=V'$.

As both $\langle V\rangle$ and $\langle V'\rangle$ are equal to a $\mathbb{P}^{10}$ and

$$\langle V\rangle\cap\langle V'\rangle=\mathbb{P}^{10}_1\cap\mathbb{P}^{10}_2=\mathbb{P}^{10}$$

Thus, $\mathbb{P}^{10}_1=\mathbb{P}^{10}_2$ and thereby $V=V'$. Then, the intersection of V and V' is given by $V\cap V'=\mathbb{P}^{10}_1\cap U$.

Intersection of $\langle V\rangle$ and $\langle V'\rangle$ is a $\mathbb{P}^{9}$

If the intersection of $\langle V\rangle$ and $\langle V'\rangle$ is a $\mathbb{P}^9$, then the span of V and the span of V' will both contain this $\mathbb{P}^9$. The intersection of V and V' is given by $V\cap V'=\mathbb{P}^9\cap U$.

In this case, $\langle V\cap V'\rangle$ is a hyperplane of $\mathbb{P}^{10}$. The class of a hyperplane section, denoted by H, is given by

$$4L-e_0-e_1-e_2-e_3$$

i.e. a class of quartic curves going through q_0,q_1,q_2,q_3 in $\mathbb{P}^2$.

Intersection of $\langle V\rangle$ and $\langle V'\rangle$ is a $\mathbb{P}^{8}$

If the intersection of $\langle V\rangle$ and $\langle V'\rangle$ is a $\mathbb{P}^8$, then the span of V and the span of V' will both contain this $\mathbb{P}^8$. The intersection of V and V' is given by $V\cap V'=\mathbb{P}^8\cap U$.

In this case, $\langle V \cap V' \rangle$ equal the intersection of two hyperplanes. The class of the intersection of two hyperplane sections is given by

$$(4L - e_0 - e_1 - e_2 - e_3)^2$$

By Bezout, the intersection will either contain a component or consists of some finite number of points equal to the degree of V.

If the intersection contain a component, we can write the class of each hyperplane section as

$$H = A + B$$

where A represents the class of the common component that the two classes share and B represents the class of the residual part after A is subtracted. Then, each hyperplane section can be written as

$$H_1 = (A + B_1), \quad H_2 = (A + B_2)$$

where H_1, H_2 are curves of class H, A is a curve of class A representing the common component in each hyperplane section, and B_1, B_2 are curves of class B. As B_i represents different curves of the class B, we call each B_i the variable component. Further, for A' to be a common component of *two* hyperplane sections in $\mathbb{P}^8$, there needs to be at least two different curves of class B, such that $\dim |B| = \dim |H - A'| \geq 2$.

Then, $V \cap V'$ either contains a component of class A or it is equal to 12 points, as $\deg V = 12$.

Intersection of $\langle V \rangle$ and $\langle V' \rangle$ is a $\mathbb{P}^7$

If the intersection of $\langle V \rangle$ and $\langle V' \rangle$ is a $\mathbb{P}^7$, then the span of V and the span of V' will both contain this $\mathbb{P}^7$. The intersection of V and V' is given by $V \cap V' = \mathbb{P}^7 \cap U$.

In this case, $\langle V \cap V' \rangle$ is equal to the intersection of three hyperplanes. The class of the intersection of three hyperplane sections is given by

$$(4L - e_0 - e_1 - e_2 - e_3)^3$$

The intersection will either contain a component or consists of some finite number of points, strictly less than the degree of V. If the intersection contains a component, each hyperplane section can be written as

$$H_1 = (A + B_1), \quad H_2 = (A + B_2), \quad H_3 = (A + B_3)$$

where A is a curve of class A, representing the common component in each hyperplane section, and B_i representing three different curves of class B, being the residual part of the hyperplane section H_i after A is subtracted. Further, for each A', there needs to be at least three different curves of class B, for A' to be a common component of *three* hyperplane sections. Thus, $\dim|B'| = \dim|H - A'| \geq 3$.

Then, $V \cap V'$ either contains a component A or consist of strictly less than 12 points.

The span of A

For each possible common component A, we are interested in finding the projective span of this component.

We use the dimension of the vector space of B to determine the span of A. The dimension of the vector space is equal to the number of linearly independent hyperplane sections H_i that contains the common component A. That is, the number of linearly independent curves B_i of class B such that $A + B_i = H_i$ for each hyperplane section H_i containing A.

As we need $n + 1$ vectors to span a projective space of dimension n, the dimension of the projective space of B, denoted $\dim |B|$ is one less than the dimension of the vector space of B.

We obtain the dimension of the vector space by the following formula

$$\binom{n+d}{n} - \binom{n+a_0-1}{n} - \binom{n+a_1-1}{n} - \cdots - \binom{n+a_j-1}{n} \tag{5.9}$$

where n represent the dimension of space $\mathbb{P}^n$, d represent the degree of the curve and $a_0, a_1, \dots, a_j$ represents the multiplicity of the curve in points $q_0, q_1, \dots, q_j$, respectively.

We subtract $\binom{n+a_i-1}{n}$ for each point q_i we impose conditions on. For $a_i = 1$ we subtract 1 linear condition for restricting the curve to the associated point. For $a_i \geq 2$, the curve is singular in the associated point, such that $\binom{n+a_i-1}{n}$ represents the number of partial derivatives of order $a_i - 1$ that vanish in the point.

For $a_i = 2$, there are three first order partial derivatives in three variables, i.e. f_{x_0}, f_{x_1} and f_{x_2}. By Clairaut's theorem we know that $f_{x_0x_1} = f_{x_1x_0}$. Thus, for $a_i = 3$, there are six second order partial derivatives in three variables, i.e. $f_{x_0x_0}$, $f_{x_0x_1}$, $f_{x_0x_2}$, $f_{x_1x_1}$, $f_{x_1x_2}$ and $f_{x_2x_2}$. For $a_i = 4$ there are ten third order partial derivatives, and so on.

Example 5.5.1. Let H represents the class of quartic curves in $\mathbb{P}^2$ with no linear conditions, i.e. $H = 4L$. We get $A = 0$, as there is no restrictions on these curves, such that $B_i = H_i$. Then, the number of linearly independent quartic curves is given by

$$\binom{4+2}{2} = 15$$

i.e. the dimension of the vector space is 15, while the dimension of the projective space is 14.

In the example above, there are no linear conditions imposed. However in this chapter, we are studying quartic curves in $\mathbb{P}^2$ that are required to go through the four points q_0, q_1, q_2, q_3. If a curve is reducible, as in the case of $H = A + B$, then every component of the curve may not go through all four points, or some of the components may go through some of points more than once. We take a closer look at the latter, in the next example.

Example 5.5.2. Now $H = 4L - e_0 - e_1 - e_2 - e_3$ and represents the class of quartic curves in $\mathbb{P}^2$ going through the points q_0, q_1, q_2, q_3. Let $A = e_0$, then $B = 4L - 2e_0 - e_1 - e_2 - e_3$, where a curve of class B goes through the point q_0 twice. Thus, the number of linearly independent quartic curves in this class is given by

$$\binom{4+2}{2} - \binom{2+1}{2} - \binom{2+0}{2} - \binom{2+0}{2} - \binom{2+0}{2} = 15 - 3 - 1 - 1 - 1 = 9$$

Such that the dimension of the vector space is 9, and the projective dimension is 8.

Now we can use the number of hyperplane sections that contains A to determine the projective span of $A = H_i - B_i$. As the span of V is equal to a $\mathbb{P}^{10}$, we intersect the span of V with all hyperplanes that contains A. This will result in *the projective span of* A.

Example 5.5.3. Returning to the example of $A = e_0$, where we found that there were nine hyperplane sections containing A. Then the projective span of A equals $\mathbb{P}^1$ as we reduce the dimension of $\mathbb{P}^{10}$ by nine hyperplanes.

By using similar computations, the projective span of all possible candidates for the common component A can be found in column 3 in Table 5.1 below.

Table 5.1: Hyperplane section of class $H = A + B$, span of A, degree of curves in class A and intersection points of curves B_i and B_j of class B

Common component A	Variable component B	Span of A	Degree of A	$B_i \cap B_j = B^2$
e_0	$4L - 2e_0 - e_1 - e_2 - e_3$	$\mathbb{P}^1$	1	9
$e_0 + e_1$	$4L - 2e_0 - 2e_1 - e_2 - e_3$	$\mathbb{P}^3$	2	6
$e_0 + e_0$	$4L - 3e_0 - e_1 - e_2 - e_3$	$\mathbb{P}^4$	2	4
$e_0 + e_1 + e_2$	$4L - 2e_0 - 2e_1 - 2e_2 - e_3$	$\mathbb{P}^5$	3	3
$e_0 + e_1 + e_2 + e_3$	$4L - 2e_0 - 2e_1 - 2e_2 - 2e_3$	$\mathbb{P}^7$	4	0
$L - e_0 - e_1$	$3L - e_2 - e_3$	$\mathbb{P}^2$	2	7
$L - e_0$	$3L - e_1 - e_2 - e_3$	$\mathbb{P}^3$	3	6
L	$3L - e_0 - e_1 - e_2 - e_3$	$\mathbb{P}^4$	4	5
$L + e_0$	$3L - 2e_0 - e_1 - e_2 - e_3$	$\mathbb{P}^6$	5	2
$2L - e_0 - e_1 - e_2 - e_3$	$2L$	$\mathbb{P}^4$	4	4
$2L - e_0 - e_1 - e_2$	$2L - e_3$	$\mathbb{P}^5$	5	3
$2L - e_0 - e_1$	$2L - e_2 - e_3$	$\mathbb{P}^6$	6	2
$2L - e_0$	$2L - e_1 - e_2 - e_3$	$\mathbb{P}^7$	7	1
$2L$	$2L - e_0 - e_1 - e_2 - e_3$	$\mathbb{P}^8$	8	0
$3L - e_0 - e_1 - e_2 - e_3$	L	$\mathbb{P}^7$	8	1
$3L - e_0 - e_1 - e_2$	$L - e_3$	$\mathbb{P}^8$	9	0

Returning to the classification of intersection

Recall, when the intersection of the span of V and V' is either a $\mathbb{P}^8$ or a $\mathbb{P}^7$, the intersection is equal to either two or three hyperplanes respectively. In both cases, each hyperplane section can be written as

$$H_i = A + B_i$$

where A represents the common component in the hyperplane section and B_i represents the variable component in each hyperplane section.

Common components in $\mathbb{P}^8$

When the intersection of the span of V and V' is a $\mathbb{P}^8$, the class B needs to contain at least two independent curves. Thus, there are at least two linearly independent hyperplane sections of class H that contain a common component. If it is spanned by at least two hyperplanes, the dimension of $|B|$ must be greater than or equal to 1.

As the span of A in Table 5.1 is at most $\mathbb{P}^8$ for each A, then $|B|$ must be at least 1 for all A. Thus, the intersection of two quartic curves in $\mathbb{P}^2$ going through q_0, q_1, q_2, q_3 may share any of the common components of class A in Table 5.1.

Common components in $\mathbb{P}^7$

When the intersection of the span of V and V' is a $\mathbb{P}^7$, there are at least three independent curves of class B. Thus, there are at least three linearly independent hyperplane sections of class H that contain a common component. If it is spanned by at least three linearly independent hyperplanes, the dimension of $|B|$ must be greater than or equal to 2.

As the span of A in Table 5.1 is at most $\mathbb{P}^8$ for each A, then $|B|$ must be at least 2 for all A. Thus, the intersection of two quartic curves in $\mathbb{P}^2$ going through q_0, q_1, q_2, q_3 may share any of the common components of class A in Table 5.1.

In Table 5.1, we see that there are two decompositions of hyperplane sections, where the span of A is a $\mathbb{P}^8$. In these cases, the residual part of class B cannot be a variable components in $\mathbb{P}^7$, as we need at least three different hyperplanes in $\mathbb{P}^7$. If these two decompositions are disregarded, the remaining decompositions of hyperplane sections contain at least three linearly independent curves of class B.

Thus, the intersection of three quartic curves in $\mathbb{P}^2$ going through q_0, q_1, q_2, q_3 may share one of the common components A in Table 5.1, provided that the span of A is at most $\mathbb{P}^7$.

Intersections of hyperplanes with common components

In this section we want to study the cases where the intersection of the span of V and the span of V' consist of a common component. That is, they intersect in either $\mathbb{P}^8$ or $\mathbb{P}^7$.

If the span of V and V' intersect in $\mathbb{P}^8$ ($\mathbb{P}^7$), the intersection consists of two (three) hyperplane sections that share a common component A, and where the residual components B_i of each hyperplane section varies. However, B_1 and B_2 (B_1, B_2 and B_3) may share some common points. Thus, the intersection of the span of V and V' may consist of a common component A and some intersection points in $B_1 \cap B_2$ ($B_1 \cap B_2 \cap B_3$).

First we study the possible common components A. For each A in Table 5.1 we choose a curve of this class and determine the degree of the curve in the intersection, i.e. either $\mathbb{P}^8$ or $\mathbb{P}^7$.

Consider the example of the common component of class $A = e_0$ and the variable component of class $B = 4L - 2e_0 - e_1 - e_2 - e_3$. Below, we describe how we obtain the degree of A and the number of intersection points in B in both $\mathbb{P}^8$ and $\mathbb{P}^7$.

Similar computations are used on the remaining decompositions in Table 5.1, and can be found in column 4 and 5 in Table 5.1.

Example 5.5.4. We let $A = e_0$. If the intersection equals either $\mathbb{P}^8$ or $\mathbb{P}^7$, then a curve of such a class is a line. In general, to obtain the degree of the curve, we intersect the class A with the class of a hyperplane section. Thus,

$$e_0 \cdot (4L - e_0 - e_1 - e_2 - e_3) = -e_0^2 = 1$$

i.e. the degree of a curve in class $A = e_0$ is 1. When we map the curve back to $\mathbb{P}^2$ we obtain one single point q_0.

Above we studied the common component A of the hyperplane section, now we want to study the residual part of the hyperplane section, i.e curves of class B, such that $B = H - A$.

In $\mathbb{P}^8$ ($\mathbb{P}^7$), we need at least two (three) curves of class B, i.e. the dimension of the vector space of B must be at least 2 (3).

First consider $\mathbb{P}^8$. The number of intersection points of two arbitrary curves of class B is B^2. We return to our previous example to calculate the intersection in $\mathbb{P}^8$.

Example 5.5.5. In $\mathbb{P}^8$, the number of intersection points of two arbitrary curves of class B is given by B^2 such that

$$\begin{aligned}(4L - 2e_0 - e_1 - e_2 - e_3)^2 &= 16L^2 + 4e_0^2 + e_1^2 + e_2^2 + e_3^2 \\ &= 16 - 4 - 1 - 1 - 1 \\ &= 9\end{aligned}$$

Thus, the number of intersection points of two curves of class $B = 4L - 2e_0 - e_1 - e_2 - e_3$ is 9.

Then consider $\mathbb{P}^7$, the number of intersection points of three arbitrary curves of class B is strictly less than B^2 and they do not need to intersect at all. Thus, in $\mathbb{P}^7$ we do not obtain the actual number of intersection points, only that the range of intersection points is between 0 and $B^2 - 1$.

Again, we return to our previous example, now to calculate the intersection in $\mathbb{P}^7$.

Example 5.5.6. In $\mathbb{P}^7$, the number of intersection points of three arbitrary curves of class B is strictly less than B^2. Thus, the number of intersection points of three curves of class $B = 4L - 2e_0 - e_1 - e_2 - e_3$ is less than 9.

5.6 Reconstruction

In this section we want to study the reconstruction of $\mathbb{P}^2$. The reconstruction is based on image points captured by 1D images from four camera centers in $\mathbb{P}^2$. By composing an image from each camera center, we can identify points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ that corresponds to points in $\mathbb{P}^2$. If these points span a $\mathbb{P}^{10}$ when embedded into $\mathbb{P}^{15}$, i.e. the span of V, it will be possible to uniquely reconstruct $\mathbb{P}^2$. If we cannot identify enough linearly independent points, such that they span a $\mathbb{P}^{10}$, the reconstruction will be ambiguous. However, we might identify some points or a curve in $\mathbb{P}^{10}$, that corresponds to points or curves in $\mathbb{P}^2$, we call these critical configurations of points. We will classify different critical configurations in Section 5.7.

A birational map from $\mathbb{P}^2$

The four points $q_0, q_1, q_2, q_3 \in \mathbb{P}^2$ represents the positions of the camera centers in $\mathbb{P}^2$. Each camera captures 1D images of $\mathbb{P}^2$ from different angles, such that an image is represented by a $\mathbb{P}^1$.

We distinguish between *scene points*, that represents points in $\mathbb{P}^2$, and *image points*, that represents points on each image $\mathbb{P}^1$ of scene points in $\mathbb{P}^2$. When composing these images we obtain

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

All points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are denoted *quadruple points*, as each quadruple point can be projected down to four image points, one image point in each $\mathbb{P}^1$.

If one image point in each $\mathbb{P}^1$ represents the same scene point in $\mathbb{P}^2$, the point in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is denoted a *quadruple image point.* We denote the set of all quadruple image points as S. Thus, S represents the blow-up of $\mathbb{P}^2$ in q_0, q_1, q_2, q_3 embedded into $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Let $\mathbf{x} = (x_0 : x_1 : x_2)$ be any scene point in $\mathbb{P}^2$ apart from the four camera centers. Then each image will contain the image point $(a_i : b_i)$ for all $i = 0, 1, 2, 3$.

By composing the images, the scene point $\mathbf{x}$ can be identify as a quadruple image point in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and will be of the form

$$(a_0 : b_0) \times (a_1 : b_1) \times (a_2 : b_2) \times (a_3 : b_3)$$

In fact, there exists a biratonal map from $\mathbb{P}^2$ to S sending each scene point, except the points representing the camera centers, unambiguously to one quadruple image point, i.e.

$$p : \mathbb{P}^2 \dashrightarrow S$$

where $U = \mathbb{P}^2 \setminus \{q_0, q_1, q_2, q_3\}$ and $V = S \setminus \{e_0, e_1, e_2, e_3\}$ such that $\phi : U \to V$ is an isomorphism.

As p is a biratonal map, there exists an inverse map. Then, the inverse map from S to $\mathbb{P}^2$ will be a birational map as well. We denote the inverse map α.

Definition 5.6.1. α is the birational map from S to $\mathbb{P}^2$, taking quadruple image points in $V \subset S$ to unique scene points in $U \subset \mathbb{P}^2$.

Thus, once we have S and α we are able to reconstruct the $\mathbb{P}^2$, as stated in Chapter 1. We rephrase the definition slightly, and get

Definition 5.6.2. The reconstruction of $\mathbb{P}^2$ from these four images consist of finding the surface S, that contain all quadruple image points, *and* finding the rational map α, that takes quadruple image points in S to unique scene points in $\mathbb{P}^2$.

The hard part of the reconstruction is finding S. If we cannot find S based on the quadruple image points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we embed these points into $\mathbb{P}^{15}$. When embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^{15}$, then $S = V$. Recall the previous definition of $V = U \cap \mathbb{P}^{10}_1$, where $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \cong U \subset \mathbb{P}^{15}$ and $\mathbb{P}^{10}_1$ is the span of V.

Assume we identify n quadruple image points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. If these quadruple image points span a $\mathbb{P}^{10}$, when embedded into $\mathbb{P}^{15}$, we can easily find V by intersecting the $\mathbb{P}^{10}$ with U. Once we have found the surface $V \cong S$, then there exists a unique reconstruction of $\mathbb{P}^2$. However, to reconstruct $\mathbb{P}^2$, it remains to find the inverse rational map α.

As it turns out, α can be realized as the restriction of a projection. The rational map from $\mathbb{P}^{10}$ to $\mathbb{P}^2$ is a projection from $\mathbb{P}^7 \subset \mathbb{P}^{10}$.

$$\pi_{\mathbb{P}^7} : \mathbb{P}^{10} \dashrightarrow \mathbb{P}^2$$

We project from $\mathbb{P}^7$, as the family of hyperplanes of $\mathbb{P}^{10}$ containing $\mathbb{P}^7$ is given by

$$\{H \mid H \subseteq \mathbb{P}^7\} \simeq \mathbb{P}^2$$

By restricting the rational map $\pi_{\mathbb{P}^7}$ to V, we obtain a birational map isomorphic to α, i.e.

$$\begin{array}{c} \pi_{\mathbb{P}^7}|_V : \mathbb{P}^{10} \dashrightarrow \mathbb{P}^2 \\ \cong \\ \alpha : V \dashrightarrow \mathbb{P}^2 \end{array}$$

Consider the line $L \in \mathbb{P}^2$. Then, assuming that $\mathbb{P}^2 \cap \mathbb{P}^7 = \emptyset$, the span $\langle \mathbb{P}^7, L\rangle = \mathbb{P}^9$, that is a hyperplane. Thus, each hyperplane that contains $\mathbb{P}^7$ will intersect $\mathbb{P}^2$ in a line.

Consider the point $p \in \mathbb{P}^2$. Again assuming $\mathbb{P}^2 \cap \mathbb{P}^7 = \emptyset$, such that the span is $\langle \mathbb{P}^7, p\rangle = \mathbb{P}^8$. Thus, each intersection of two hyperplanes, i.e. a $\mathbb{P}^8$, that contains $\mathbb{P}^7$ will intersect $\mathbb{P}^2$ in a point.

By intersecting $\mathbb{P}^8$ with V, the intersection will either be 12 points or contain a component. Assuming the latter case, we choose $\mathbb{P}^7 \subset \mathbb{P}^8$ such that $\mathbb{P}^7$ is the span of the curve $C = (V \cap \mathbb{P}^8) \setminus \{p\}$ where p is a single point. Then, we disregard C, such that each point $p \in (V \cap \mathbb{P}^8) \setminus C$ is sent to a unique scene points in $\mathbb{P}^2$. Thus, by projecting from $\mathbb{P}^7$ and restricting the projection to V, we obtain a one-to-one correspondence between points in $(V \cap \mathbb{P}^8) \setminus C$ and scene points in $\mathbb{P}^2$. In a similar manner, we obtain the same result if $\mathbb{P}^8 \cap V$ equals 12 points, where we subtract 11 points instead of C.

The reconstruction of $\mathbb{P}^2$ is illustrated in the diagram in Figure 5.1, where the inverse birational map α back to $\mathbb{P}^2$ is given by the projection from $\mathbb{P}^7 \subset \mathbb{P}^{10}$ restricted to V, denoted $\pi_{\mathbb{P}^7}|_V$.

$$
\begin{tikzcd}
 & & \mathbb{P}^{15} \\
 & & \mathbb{P}^{10} \arrow[u, hook] \arrow[ddd, dashed, bend left=60, "\pi_{\mathbb{P}^7}"] \\
\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[uurr, hook] & & \\
 & S \arrow[ul] \arrow[uur, hook] \arrow[r, hook, two heads, "\simeq"] \arrow[dr, dashed, shift left, "\alpha"] & V \arrow[uu, hook] \arrow[d, dashed, "\pi_{\mathbb{P}^7}|_V"] \\
 & & \mathbb{P}^2 \arrow[uull, hook, bend left] \arrow[ul, dashed, shift left, "p"]
\end{tikzcd}
$$

Figure 5.1: Reconstruction of $\mathbb{P}^2$

We want to find a $\mathbb{P}^7 \subset \mathbb{P}^8$, such that $\mathbb{P}^8 \cap V$ is equal to C plus one single point. Then, C will act as the common component of the two hyperplane sections that represent $\mathbb{P}^8$ when intersected. Further, we know that each hyperplane containing $\mathbb{P}^7$ will intersect $\mathbb{P}^2$ in a line.

Thus, we are looking for hyperplane sections of class $H = A + L$, where L is the class of a line. Then, the curve C will be of the following class

$$(4L - e_0 - e_1 - e_2 - e_3) - L = (3L - e_0 - e_1 - e_2 - e_3)$$

In Table 5.1, there is exactly one composition that satisfies

$$H = (3L - e_0 - e_1 - e_2 - e_3) + L$$

In Table 5.1 we see that $B^2 = 1$ for this given class, representing that two lines of class L intersect in exactly one point in $\mathbb{P}^2$. This coincides with the result that each $\mathbb{P}^8$ containing $\mathbb{P}^7$ will intersect V in one single point outside C.

We can determine the degree of the curve C in $\mathbb{P}^{15}$ by intersection the class of the curve with the class of a hyperplane section, i.e.

$$(3L - e_0 - e_1 - e_2 - e_3) \cdot (4L - e_0 - e_1 - e_2 - e_3) = 8$$

Thus, the curve is of degree 8 in $\mathbb{P}^{15}$. This information can also be attained in Table 5.1.

Hence, to find the projection from $\mathbb{P}^{15}$ to $\mathbb{P}^2$ we need to identify a curve C of degree 8 that spans $\mathbb{P}^7$, such that the intersection of V and $\mathbb{P}^8$ in addition to C is one single point. It is not necessarily easy to find such a curve. A closer study on how to find a curve that will satisfy the conditions above, is left for another time.

The curve C when projected down to $\mathbb{P}^2$ will be a curve of class $3L - e_0 - e_1 - e_2 - e_3$, i.e. a cubic curve going through all four camera centers.

Identification of camera centers in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

If we identify a quadruple image point that corresponds to a camera center in $\mathbb{P}^2$, it is possible to detect the position of the scene point solely based on the information from the points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Assume we identify n quadruple image points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ that all represent one camera center. Then for each quadruple image point, the coordinates from three image points will be fixed, while the coordinates from the fourth image point will likely vary.

The quadruple image points representing the camera center located in q_0 will be given by

$$\begin{gathered}(a_0^1 : b_0^1) \times (a_1 : b_1) \times (a_2 : b_2) \times (a_3 : b_3)\\ (a_0^2 : b_0^2) \times (a_1 : b_1) \times (a_2 : b_2) \times (a_3 : b_3)\\ \vdots\\ (a_0^n : b_0^n) \times (a_1 : b_1) \times (a_2 : b_2) \times (a_3 : b_3)\end{gathered}$$

When mapped into $\mathbb{P}^{15}$, these quadruple image points will be points on a line of class e_0. The preimage of e_0 in $\mathbb{P}^2$ is the point q_0.

In a similar matter, by evaluating fixed coordinates of quadruple points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we are able to detect the other camera centers as well.

5.7 Critical configurations

We need to identify at least 11 points in $\mathbb{P}^{15}$ to span $\mathbb{P}^{10}$. However, if the identified quadruple image points span something less than $\mathbb{P}^{10}$, the surface V containing these points is not unique. We can still study the intersection of the span of these points and V. In such cases, we say there is a critical configuration of points. In this section, we will classify different critical configuration.

Assume now that we have identified n quadruple image points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, where $n > 12$. We embed the quadruple image points into $\mathbb{P}^{15}$ and determine the span of the points.

Let us first consider the case where the span of these n points is isomorphic to $\mathbb{P}^{10}$. In this case, these n points must all be on a surface V, as the span of a surface is isomorphic to $\mathbb{P}^{10}$. If all points lie on two surfaces V and V', then $V = V'$.

Then by intersecting the span of these n points with U, we obtain the surface V. Thus, we have uniquely reconstructed the surface $\mathbb{P}^2$ that contain these n points, as V is isomorphic to $\mathbb{P}^2$.

Then, consider the case where the span of n points is isomorphic to $\mathbb{P}^9$, i.e. the points are all in the same hyperplane. In this case there might be two reconstructions, as it is possible that the span $V \cap V'$ intersect in $\mathbb{P}^9$. To be able to reconstruct $\mathbb{P}^2$ unambiguously, we need to identify more quadruple image points such that the span of all identified points when embedded into $\mathbb{P}^{15}$ is $\mathbb{P}^{10}$, and thereby obtain a unique reconstruction.

Then, consider the case where the span of n points is isomorphic to either $\mathbb{P}^8$ or $\mathbb{P}^7$, i.e. an intersection of either two or three hyperplanes. As we know, there is a possibility that the span of $V \cap V'$ is isomorphic to $\mathbb{P}^8$ or $\mathbb{P}^7$. Thus, if we identify n points that is either isomorphic to $\mathbb{P}^8$ or $\mathbb{P}^7$, there might be two possible reconstruction of $\mathbb{P}^2$.

Further, as we have identified more than 12 points, some of these points must lie on a curve, where such a curve will be a common component in the two or three hyperplane sections.

In Table 5.1, there are in total 16 options of classes of hyperplane sections that contains a common component. Out of these 16 alternatives, we can disregard the common component of the following classes (e_0), (e_0+e_1), (e_0+e_0), $(e_0 + e_1 + e_2)$ and $(e_0 + e_1 + e_2 + e_3)$, as we are able to recognize these as the camera centers in $\mathbb{P}^2$ from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and thereby reconstruct these points without embedding them into $\mathbb{P}^{15}$.

We are left with 11 possible common components. For each common component in $\mathbb{P}^{15}$ we are able to determine the degree of the curve. If the degree in $\mathbb{P}^{15}$ is unique, then based on this degree, we can identify the class of the curve in the preimage $\mathbb{P}^2$, as we can see in Table 5.1.

However, as there are three cases where curves of two different classes in $\mathbb{P}^{15}$ is of the same degree, as seen in Table 5.1.

This will lead to even greater problems, as there are two options for reconstruction for each of the two surfaces V and V'. Thus, again the reconstruction of $\mathbb{P}^2$ is ambiguous. We will go through these three cases below.

Classes in $\mathbb{P}^2$ of quartic curves in $\mathbb{P}^{15}$

Curves of classes L and $2L - e_0 - e_1 - e_2 - e_3$ are all of degree 4 in $\mathbb{P}^{15}$. Thus, the preimage of a quartic curve in $\mathbb{P}^{15}$ may either be an arbitrary line in $\mathbb{P}^2$ or a conic that goes through all four camera centers in $\mathbb{P}^2$.

In column 3 in Table 5.1 we find the span of each common component. As the span of both $A = L$ and $A = 2L - e_0 - e_1 - e_2 - e_3$ are $\mathbb{P}^4$, there exist 6 linearly independent hyperplane sections that contain each class A. Here we are considering the cases where the span of n points either equals $\mathbb{P}^8$ and $\mathbb{P}^7$, thus we need at most 3 linearly independent hyperplanes. As we cannot uniquely reconstruct the preimage of a quartic curve in $\mathbb{P}^{15}$, there is a critical configuration in both $\mathbb{P}^8$ and $\mathbb{P}^7$.

If the span of V and V' intersect in $\mathbb{P}^8$, then for each surface V and V' there are two possibilities, the reconstruction is either

- $L + 5$ points
- $2L - e_0 - e_1 - e_2 - e_3 + 4$ points

The number of additional points is attained in column 5 in Table 5.1, i.e. B^2.

If the span of V and V' intersect in $\mathbb{P}^7$, then for each surface V, V' there are two possibilities, the reconstruction is either

- $L + k$ points, where $0 \geq k < 5$
- $2L - e_0 - e_1 - e_2 - e_3 + k$ points, where $0 \geq k < 4$

As there are three hyperplanes that intersect in $\mathbb{P}^7$, the number of additional intersection points must be strictly less than when there are two hyperplanes that intersect. Thus, in $\mathbb{P}^7$ the number of additional intersection points is strictly less than B^2, found in column five in Table 5.1.

Classes in $\mathbb{P}^2$ of curves of degree 5 in $\mathbb{P}^{15}$

The degree of class $L + e_0$ and $2L - e_0 - e_1 - e_2$ are both 5 in $\mathbb{P}^{15}$. The preimage of a curve of degree 5 in $\mathbb{P}^{15}$ may either be an arbitrary line and the point q_0 that represents one of the camera centers in $\mathbb{P}^2$, or a conic that goes through three of the camera centers in $\mathbb{P}^2$.

In column 3 in Table 5.1 we find the span of each common component A. The span of $A = L + e_0$ is $\mathbb{P}^6$ and the span of $A = 2L - e_0 - e_1 - e_2$ is $\mathbb{P}^5$, i.e. there exist 4 and 5 respectively, linearly independent hyperplane sections that contains each associated class. As need at most 3 linearly independent hyperplanes, both common components are possible in each $\mathbb{P}^8$ and $\mathbb{P}^7$. As we cannot uniquely reconstruct the preimage of a curve of degree 5 in in $\mathbb{P}^{15}$, there is a critical configuration in both $\mathbb{P}^8$ and $\mathbb{P}^7$.

If the span of V and V' intersect in $\mathbb{P}^8$, then for each surface V, V' there are two possibilities, the reconstruction is either

- $L + e_0 + 2$ points
- $2L - e_0 - e_1 - e_2 + 3$ points

The number of additional points is attained in column 5 in Table 5.1, i.e. B^2.

If the span of V and V' intersect in $\mathbb{P}^7$, then for each surface V, V' there are again two possibilities, the reconstruction is either

- $L + e_0 + k$ points, where $0 \geq k < 2$
- $2L - e_0 - e_1 - e_2 + k$ points, where $0 \geq k < 3$

As before, in $\mathbb{P}^7$ the number of additional intersection points that is derived from the variable components, are strictly less than B^2 attained in column five in Table 5.1.

Classes in $\mathbb{P}^2$ of curves of degree 8 in $\mathbb{P}^{15}$

The degree of class $2L$ and $3L - e_0 - e_1 - e_2 - e_3$ are both 8 in $\mathbb{P}^{15}$. Thus, the preimage of a curve of degree 8 in $\mathbb{P}^{15}$ may either be an arbitrary conic in $\mathbb{P}^2$, or a cubic curve that goes through all camera centers in $\mathbb{P}^2$.

In column 3 in Table 5.1 we find the span of each common component A. The span of $A = 2L$ is $\mathbb{P}^8$ and the span of $A = 2L - e_0 - e_1 - e_2$ is $\mathbb{P}^7$, i.e. the number of linearly independent hyperplane section that contains each associated class is 2 and 3 respectively. As need 3 linearly independent hyperplanes in $\mathbb{P}^7$, then $A = 2L$ is not a possible common component in $\mathbb{P}^7$. However, in $\mathbb{P}^8$ both common components are a possibility. Thus, there is a critical configuration in $\mathbb{P}^8$.

If the span of V and V' intersect in $\mathbb{P}^8$, then for each surface V, V' there are two possibilities, the reconstruction is either

- $2L + 0$ points
- $3L - e_0 - e_1 - e_2 - e_3 + 1$ points

The number of additional points is attained in column 5 in Table 5.1, i.e. B^2.

If the span of V and V' intersect in $\mathbb{P}^7$, then for each surface V, V' we can uniquely reconstruct the preimage of a curve of degree 5 in $\mathbb{P}^{15}$. The reconstruction is

- $3L - e_0 - e_1 - e_2 - e_3 + 0$ points

As before, in $\mathbb{P}^7$ the number of additional intersection points that is derived from the variable components are strictly less than B^2. We find this number in column five in Table 5.1, here $B^2 = 1$.

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