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The Theory of Minimal Surfaces in $\mathbb{R}^3$ with a look at the Area Minimizing Property

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Master's Thesis, Spring 2022

This master's thesis is submitted under the master's programme *Computational Science*, with programme option *Applied Mathematics and Risk Analysis*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 60 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

In this thesis we look at minimal surfaces in $\mathbb{R}^3$. We begin by looking at the theory of minimal surfaces. We look at the definition as well as examples of minimal surfaces. A minimal surface is defined as a surface that has zero mean curvature. We also look at the connection of minimal surfaces to harmonic functions, as well as the Weierstrass Enneper formulas. Then we look at the area minimizing property of minimal surfaces. A minimal surface does not always have minimal surface area. Thus we would like to find out when a minimal surface has minimal surface area. In order to look at this, we study the connection between minimal surfaces and the area functional. We also look at this problem by studying the connection between minimal surfaces and soap bubbles, which is given by Plateau's problem. This allows us to find some conditions that tell us when a minimal surface has minimal surface area. We demonstrate these concepts by looking at specific examples of minimal surfaces and finding out when they have minimal surface area. The examples we consider are the catenoid, Enneper's surface and higher order Enneper surfaces. We find out when these minimal surfaces have minimal surface area. We mention that this is a local property. By that we mean that minimal surfaces locally minimize their surface area.

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Introduction

Minimal surfaces arise from differential geometry. They are surfaces that have mean curvature equal to zero. Thus zero mean curvature is the defining property of minimal surfaces. The theory of minimal surfaces can be connected to different branches of mathematics. We will see how this allows us to define a minimal surface in different ways.

The aim of this thesis is to look at the theory of minimal surfaces in $\mathbb{R}^3$, as well as the area minimizing property. We begin by looking at the definition and theory of minimal surfaces. Then we look at the area minimizing property. Minimal surfaces tend to locally minimize their surface area. However, a minimal surface may not always have minimal surface area. In other words, a minimal surface is not always minimal in the sense of having minimal surface area. This leads us to our problem of interest, which can be phrased as 'When does a minimal surface have minimal surface area?'

We will study some situations that tell us when a minimal surface has minimal surface area, and when it does not. We will look at two different ways to study this problem. One way we will study this problem is by looking at the connection between minimal surfaces and the area functional. Another way we will look at this problem is by looking at the connection between minimal surfaces and soap films. We will see how these two methods will give us some conditions that tell us when a minimal is area minimizing and thus having minimal surface area. We will look at specific examples of minimal surfaces and see in which case they have minimal surface area. However before we can look at this area minimizing property of minimal surfaces, we need to understand what a minimal surface is. This is what we will begin with. To get an overview we present an outline of the thesis.

Chapter 1 In this chapter we recall the concepts from differential geometry. This will be useful for understanding the definition of a minimal surface.

Chapter 2 In this chapter we look at the definition of a minimal surface in $\mathbb{R}^3$. We make the connection of minimal surfaces to harmonic functions, which then leads to the Weierstrass Enneper formulas. These formulas allow us to parametrize a minimal surface with an isothermal parametrization. We also look at the connection with partial differential equations. This chapter gives a theoretical overview of minimal surfaces.

Chapter 3 In this chapter we look at some examples of minimal surfaces in $\mathbb{R}^3$. The examples we will look at are the catenoid, the helicoid, Scherk's first surface, Enneper's surface, higher order Enneper surfaces and Richmond's surface. Furthermore we look at some interesting properties of these surfaces.

Chapter 4 In this chapter we look at the area minimizing property of minimal surfaces. We consider the area functional and see how it is connected to minimal surfaces. We also consider the Gauss map of minimal surfaces. We study the first and second derivative of the area functional. Furthermore the second derivative of the area functional gives us a condition that allows to find when a minimal surface has minimal surface area. This condition is described by the stability of a minimal surface. We look at some examples to see when they are area minimizing.

Chapter 5 In this chapter we look at the area minimizing property of minimal surfaces. We consider soap films and see how it is related to minimal surfaces. This leads us to Plateau's problem. The soap films give us a condition that tells us when a minimal surface has minimal surface area. We also look at some examples and see when they are area minimizing.

Chapter 6 Here we present the conclusion and the main results related to the area minimizing property of minimal surfaces.

We make a short comment about the notation used throughout the thesis. To denote the partial derivatives of a function f with respect to the variables u and v, the following notation is used

$$\frac{\partial f}{\partial u} = f_u, \qquad \frac{\partial f}{\partial u\, \partial v} = f_{uv}, \qquad \frac{\partial^2 f}{\partial u^2} = f_{uu}$$

We mention that we use both notations, that is, the notation $\frac{\partial f}{\partial u}$, as well as the shorthand notation, that is, f_u. Note that f, u, v are arbitrary symbols, so this notation depends on the context we are using it in. In addition we mention an important point in regards to the figures in this thesis. The programming language that was used to produce the figures in this thesis is Python. All the figures have been produced using python, except for figure 5.1 (which was hand drawn).

CHAPTER 1

Preliminaries

We begin by recalling some concepts in $\mathbb{R}^3$. These are the inner product, cross product, and the norm of a vector. To make it clear, we present them here as they will be used throughout the thesis. This will also make the notation clear. The material presented here is sourced from [Do 16, chapter 1.2, 1.4]. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be two vectors in $\mathbb{R}^3$. Then their *inner product* is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \in \mathbb{R} \tag{1.1}$$

Furthermore their *cross product* is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1) \in \mathbb{R}^3 \tag{1.2}$$

Finally we mention the norm of a vector in $\mathbb{R}^3$. The Euclidean *norm* of $\mathbf{u}$ is

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{(u_1)^2 + (u_2)^2 + (u_3)^2} \tag{1.3}$$

Now that we have these concepts in place, we are ready to look at the theory of surfaces and regularity.

1.1 Surfaces and Regularity

We will recall some concepts from the differential geometry of surfaces in $\mathbb{R}^3$. The theory presented in this section is based on [Do 16, chapter 1.2, 1.3, 2]. The theory presented on curves is based on [Do 16, chapter 1.2, 1.3], while the theory presented on surfaces is based on [Do 16, chapter 2].

Before we can consider the theory for surfaces, we will recall some concepts from the differential geometry of curves. Our main focus will be on the local properties of curves. As we are working with differential geometry, it is only natural to begin by looking at the notion of a differentiable function. A function is said to be *differentiable* if it has derivatives of all orders at all of the points where the function is defined. Let $I = (a, b)$ be some interval of the real line $\mathbb{R}$. The interval I is an open interval. We allow a and b in the interval to take the values $+\infty$ or $-\infty$. We introduce the variable $t \in \mathbb{R}$. Let $t \in I$. Then for each point $t \in I$, we denote the curve $\alpha : I = (a, b) \to \mathbb{R}^3$ by

$$\alpha(t) = (x(t), y(t), z(t))$$

The individual functions $x(t), y(t)$ and $z(t)$ are called the component functions. We can now look at the notion of a parametrized differentiable curve, it is given by the following definition.

Definition 1.1.1. A *parametrized differentiable curve* is a differentiable map $\alpha : I = (a, b) \to \mathbb{R}^3$, given by $\alpha(t) = (x(t), y(t), z(t))$.

The next important vector we will consider is the tangent vector. This is given by the following definition.

Definition 1.1.2. The *tangent vector* of the curve α at the point t is given by

$$\alpha'(t) = (x'(t), y'(t), z'(t)),$$

where $x'(t)$ denotes the first derivative of the x component function with respect to t. $y'(t)$ and $z'(t)$ are defined in a similar manner.

We mention that the tangent vector is also called the *velocity vector*. We will now restrict ourselves to the case where $\alpha'(t) \neq 0$. This is because the condition $\alpha'(t) \neq 0$ will allow us to define the tangent line of the curve α. Let $t \in I$ such that $\alpha'(t) \neq 0$. Then there is a straight line that passes through $\alpha(t)$ and $\alpha'(t)$, in the direction of $\alpha'(t)$. This straight line is said to be the *tangent line* to α at t. This brings us to the concept of regularity in connection with curves.

Definition 1.1.3. A parametrized differentiable curve $\alpha : I \to \mathbb{R}^3$ is said to be *regular* if for all $t \in I$, we have that $\alpha'(t) \neq 0$.

This regularity condition tells us that $\alpha'(t) \neq 0$, which further tells us that a tangent line to α is well defined. The existence of the tangent line will allow to compute further properties of curves. Going further we will only consider regular parametrized differentiable curves. Thus regular curves allow us to define other geometric concepts on the curve.

We are now in a position to look at the differential geometry of surfaces in $\mathbb{R}^3$. We encounter many examples of surfaces in $\mathbb{R}^3$ in our daily life. To give a simple example, the ice cubes in our drinks are bounded by surfaces that form cubes. Ice cream cones have the shape of the surface which is a cone. Let us consider a formal mathematical description of surfaces. To be more specific, we are interested in regular surfaces in $\mathbb{R}^3$. Let us recall the definition of a homeomorphism. A continuous function f is said to be a *homeomorphism* if it is one-to-one, and has an inverse function which is also continuous. We are now ready to look at a mathematical definition of a regular surface in $\mathbb{R}^3$.

Definition 1.1.4. Let S be a subset of $\mathbb{R}^3$, $S \subset \mathbb{R}^3$. Then S is a *regular surface* if, for each point $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map

$$\mathbf{x} : U \to V \cap S$$

of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that the following conditions are satisfied:

1. $\mathbf{x}$ is differentiable. This means that if we write the map $\mathbf{x}$ in the form

   $$\mathbf{x}(u, v) = (x(u, v),\ y(u, v),\ z(u, v)), \quad (u, v) \in U$$

   then the individual component functions $x(u, v), y(u, v), z(u, v)$ have continuous partial derivatives of all orders in U.

2. $\mathbf{x}$ is a homeomorphism. From the first condition we know that $\mathbf{x}$ is continuous. Thus $\mathbf{x}$ is one-to-one and the inverse, $\mathbf{x}^{-1} : V \cap S \to U$ exists, and it is also continuous.

3. For each $q = (u, v) \in U$, the cross product

$$\left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right)_{(q)} \neq \mathbf{0}$$

   This is the regularity condition.

The map $\mathbf{x}$ is called a *parametrization* of the surface S at the point p. The parametrization is local. It provides local coordinates for S, in a neighborhood around the point p. Thus a local parametrization of the surface $S \subset \mathbb{R}^3$ is given by $\mathbf{x} : U \to V \cap S$ using the local coordinate variables u, v

$$\mathbf{x}(u, v) = (x(u, v),\ y(u, v),\ z(u, v))$$

The regularity condition mentioned in the third condition in the above definition can be interpreted in another way. We will look at the matrix of the linear map of the differential of $\mathbf{x}$ at p, $d\mathbf{x}_p$. The differential is a linear map $d\mathbf{x}_p : \mathbb{R}^2 \to \mathbb{R}^3$. The matrix of $d\mathbf{x}_p$ is given by the following 3×2 matrix:

$$d\mathbf{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial u} & \frac{\partial \mathbf{x}}{\partial v} \end{pmatrix}$$

The regularity condition can be satisfied if the two column vectors of this matrix are linearly independent. Thus $\mathbf{x}$ is regular if the the two vectors

$$\frac{\partial \mathbf{x}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \text{ and } \frac{\partial \mathbf{x}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

are linearly independent. This can stated as requiring the matrix $d\mathbf{x}_q$ to have full rank, where $q = (u_0, v_0) \in U \subset \mathbb{R}^2$. One example of a regular surface that can be useful to know is mentioned in the proposition below.

Proposition 1.1.5. *[Do 16, p. 60] If the function $f : U \to \mathbb{R}$ is differentiable in an open set $U \subset \mathbb{R}^2$, then the graph of f, which is the subset of $\mathbb{R}^3$*

$$\{(x, y, f(x, y)) : (x, y) \in U\}$$

is a regular surface. Furthermore it can be parametrized by the map $\mathbf{x} : U \to \mathbb{R}^3$, with $U \subset \mathbb{R}^2$ an open set where the map $\mathbf{x}(u, v)$ is given by

$$\mathbf{x}(u, v) = (u, v, f(u, v)), \quad (u, v) \in U$$

The proposition tells us that if $f(x, y)$ is a differentiable function, then the graph of $f = z(x, y)$ gives us a regular surface. The proposition also gives us a simple way to parametrize f in the local u, v coordinates by the parametrization $\mathbf{x}(u, v)$. We mention that this is a local parametrization. We have talked about a regular parametrized surface, we can also talk about a parametrized surface.

Definition 1.1.6. A parametrized surface

$$\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$$

is a differentiable map from an open set $U \subset \mathbb{R}^2$ into $\mathbb{R}^3$. The mapping $\mathbf{x}$ is regular if the vectors $\partial\mathbf{x}/\partial u$ and $\partial\mathbf{x}/\partial v$ are linearly independent at all $q \in U$, in other words $\partial\mathbf{x}/\partial u \times \partial\mathbf{x}/\partial v \neq \mathbf{0}$.

We will be working with surfaces locally, and a parametrized surface can always be restricted to a neighborhood around a point $q \in U \subset R^2$ such that the parametrized surface is regular in that neighborhood. Locally speaking, we can always find a small regular patch of our parametrized surface, allowing us to apply all the properties of regular parametrized surfaces to the regular patch.

We saw that the regularity condition for curves allows us to define the tangent line to α at t. There is a similar interpretation in the case of surfaces. The reason we want the regularity condition to be fulfilled for surfaces is so that we can talk about the tangent plane. The tangent plane of a surface S allows us to discover many geometric properties of S, hence it is of great importance. Before we can look into the tangent plane, we present the definition $d\mathbf{x}_p(v)$. $d\mathbf{x}_p(v)$ is called the differential of $\mathbf{x}$ at the point p [Tap16, p. 118]. Let S be a regular surface, this means that the map of the parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is differentiable. Let $v \in \mathbb{R}^2$ and a differentiable curve $\alpha : (-\epsilon, \epsilon) \to U$ be such that, $\alpha(0) = p$ and $\alpha'(0) = v$. Then the differential of $\mathbf{x}$ at p is given by,

$$d\mathbf{x}_p(v) = (\mathbf{x} \circ \alpha)'(0)$$

We mention the notion of tangent vectors. Tangent vectors are defined in the following manner.

- Let $S \subset \mathbb{R}^3$ be a regular surface. Let $p \in S$. Then a *tangent vector* $\mathbf{w} \in \mathbb{R}^3$ of the surface S at the point p, is $\mathbf{w} = \alpha'(0)$, where $\alpha : (-\epsilon, \epsilon) \to S$ is a parametrized differentiable curve with $\alpha(0) = p$.

Tangent vectors of a surface define the tangent plane of a surface. The *tangent plane* to S at p consists of all the tangent vectors of differentiable curves in S, that also pass through the point $p = \alpha(0)$. The tangent plane can be thought of as a set, denoted by $T_p(S)$. A vector that is an element of this set $\mathbf{w} \in T_p(S)$ is said to be a tangent vector for S at p. This can be written as (see [Tap16, p. 141]),

$$\begin{aligned} T_p(S) &= \{\mathbf{w} \mid \mathbf{w} \text{ is a tangent vector for } S \text{ at } p \in S\} \\ &= \{\mathbf{w} = \alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to S \text{ with } \alpha(0) = p\} \end{aligned}$$

There is a basis for the tangent plane $T_p(S)$ that consists of the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$. The basis of $T_p(S)$ is given by $\{\mathbf{x}_u(q), \mathbf{x}_v(q)\}$ where $q \in U$ and so $\mathbf{x}(q) = p \in S$. We mention that the basis of $T_p(S)$ is determined by the parametrization $\mathbf{x}(u, v)$. Thus at a point $q \in U$ the vectors give a basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ for the tangent plane $T_p(S)$, and $q = \mathbf{x}^{-1}(p)$.

1.2 The Gauss Map

If we take the cross product of the two basis vectors $\mathbf{x}_u$ and $\mathbf{x}_v$, we get another vector, namely the vector given by $\mathbf{x}_u \times \mathbf{x}_v$. This vector is orthogonal to both $\mathbf{x}_u$ and $\mathbf{x}_v$. In order for this cross product $\mathbf{x}_u \times \mathbf{x}_v$ to be nontrivial, we need to have $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$. This is why the regularity condition is so important as it allows us to define the nonzero vector $\mathbf{x}_u \times \mathbf{x}_v$. This vector is significant when it comes to studying the local properties of surfaces in $\mathbb{R}^3$, or perhaps we should say regular surfaces. The theory presented in this section is based on [Do 16, chapter 3].

Let S be a regular surface, which is given by the parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$. Then for a point $p \in S$, we are able to define a *unit normal vector* $\mathbf{N}$ at each point $q \in \mathbf{x}(U)$ given by

$$\mathbf{N}(q) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}(q) \tag{1.4}$$

The unit normal vector $\mathbf{N}$ is then a differentiable mapping

$$\mathbf{N} : \mathbf{x}(U) \subset S \to \mathbb{R}^3$$

For each point $q \in \mathbf{x}(U)$ we can find a unit normal vector $\mathbf{N}(q)$, which is a unit vector in $\mathbb{R}^3$. Note that we have two possible ways to choose the unit normal $\mathbf{N}$, it can be either $+\mathbf{N}$ or $-\mathbf{N}$. The choice of a unit normal vector $\mathbf{N}$ induces an *orientation* on the surface S. Thus S is said to be *orientable* if it is possible to define a continuous field of unit normal vectors $\mathbf{N}$ on the surface. If this is not possible, the surface is said to be *non-orientable*. However locally speaking, we can always define a continuous field of unit normal vectors $\mathbf{N}$ on S.

From this point onwards we will let S be a regular surface where we have chosen an orientation $\mathbf{N}$. Recall the unit sphere $S^2 \subset \mathbb{R}^3$ given by the set

$$S^2 = \{\ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

Now the concept of the unit normal vector leads to the definition of the Gauss map.

Definition 1.2.1. Let S be a regular surface where we have chosen an orientation $\mathbf{N}$. Let S be given by the parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$. Let $p \in S$. Then the map $\mathbf{N} : S \to S^2$ that is given by

$$\mathbf{N}(p) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}(x^{-1}(p))$$

is called the *Gauss map* of S. With regards to the parametrization of S, the Gauss map can be written as

$$\mathbf{N}(\mathbf{x}(u, v)) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}(u, v)$$

for u, v in the parameter domain.

We can interpret the Gauss map as follows. Let p be some point on the surface S. Then there is a unit normal vector at the point p, $\mathbf{N}(p) \in \mathbb{R}^3$. The Gauss map takes as input this unit normal vector, and sends it to the unit sphere S^2. The output vector is also a vector in $\mathbb{R}^3$ and it has unit length. It is clear that $\|\mathbf{N}(p)\| = 1$. An important point to consider is that the Gauss map is a differentiable map.

1.3 The First Fundamental Form

The theory presented in this section is based on [Do 16, chapter 2]. Consider the tangent plane $T_p(S)$. If $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ are two vectors in $T_p(S)$, then $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ is the inner product of $\mathbf{v}_1$ and $\mathbf{v}_2$. We can associate a quadratic form Q in the tangent plane $T_p(S)$ to this inner product. This leads us to the concept of the first fundamental form.

Definition 1.3.1. Let $\mathbf{v} \in T_p(S)$. Then the quadratic from $I_p : T_p(S) \to \mathbb{R}$, defined on $T_p(S)$ by the following inner product

$$I_p(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle$$

is called the *first fundamental form* of S at the point p.

Let us find the expression of the first fundamental form in terms of the local coordinates u, v for a parametrization $\mathbf{x}(u, v)$ and the associated basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of the tangent plane at a point p. A vector $\mathbf{w}$ in $T_p(S)$ is $\mathbf{w} = \alpha'(0)$, where α is a parametrized curve $\alpha(t) = \mathbf{x}(u(t), v(t))$ for $t \in (-\epsilon, \epsilon)$ and $p = \alpha(0) = \mathbf{x}(u_0, v_0)$. Then

$$\alpha'(t) = \mathbf{x}_u u'(t) + \mathbf{x}_v v'(t)$$

Let $t = 0$. We express the first fundamental form I_p as

$$\begin{aligned} I_p(\mathbf{w}) = \langle \mathbf{w}, \mathbf{w} \rangle = \langle \alpha'(0), \alpha'(0) \rangle &= \langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \rangle \\ &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle (u')^2 + 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle u'v' + \langle \mathbf{x}_v, \mathbf{x}_v \rangle (v')^2 \\ &= E(u')^2 + 2Fu'v' + G(v')^2 \end{aligned}$$

at a point p. Note that $u' = u'(0)$ and $v' = v'(0)$. The functions $E(u, v), F(u, v)$ and $G(u, v)$ are functions of $(u, v) \in U$. The values of these functions lead us to the next definition.

Definition 1.3.2. Let $\mathbf{x} : U \to V \cap S$ be the parametrization of a regular surface S. Then the quantities E, F and G are called the *coefficients of the first fundamental form*. In local coordinates u, v they are given as

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \|\mathbf{x}_u\|^2 \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = \|\mathbf{x}_v\|^2 \end{aligned}$$

The definition above gives E, F, G in the local coordinates u, v so we can make computations with them in the parameter domain. The significance of the first fundamental form is that many geometric quantities can be computed by only using E, F and G. One such property is the surface area. The *area element*, which is denoted by dA, of the local parametrization $\mathbf{x}(u, v)$ of the surface S is

$$dA = \|\mathbf{x}_u \times \mathbf{x}_v\| \; du \; dv \tag{1.5}$$

Hence we can compute the surface area corresponding to a region U of the parameter domain by the integral

$$Area(U) = \int dA = \int\int_U \|\mathbf{x}_u \times \mathbf{x}_v\| \; du \; dv \tag{1.6}$$

It is possible to express the norm of the cross product in terms of E, F and G by using Lagrange's identity. For any two vectors $\mathbf{v}, \mathbf{w}$ *Lagrange's Identity* is given by

$$\|\mathbf{v} \times \mathbf{w}\|^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - \langle \mathbf{v}, \mathbf{w} \rangle^2 \tag{1.7}$$

Due to Lagrange's identity in (1.7) with $\mathbf{v} = \mathbf{x}_u$ and $\mathbf{w} = \mathbf{x}_v$ and by definition 1.3.2, we get the following expression

$$\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{EG - F^2} \tag{1.8}$$

This means that we can write the area element dA in terms of E, F and G as

$$dA = \|\mathbf{x}_u \times \mathbf{x}_v\| \; du \; dv = \sqrt{EG - F^2} \; du \; dv \tag{1.9}$$

Therefore we can now compute the surface area of a region U of the surface S by computing the integral

$$Area(U) = \int dA = \int \int_U \sqrt{EG - F^2} \; du \; dv \tag{1.10}$$

We only need to know the values of the coefficients of the first fundamental form E, F, G of the surface in order to compute the integral (1.10).

Recall the concept of a diffeomorphism. A diffeomorphism is simply a bijective function that is differentiable and has an inverse which is also differentiable. Another notion related to the first fundamental form is the notion of an isometry. Let S_1 and S_2 be two regular surfaces. A diffeomorphism $f : S_1 \to S_2$ is called an *isometry* if its differential *df* preserves the inner product for all points $p \in S_1$. The important point to consider is that an isometry between two surfaces will preserve their first fundamental forms.

Proposition 1.3.3. *[Do 16, p. 223] Let S_1 and S_2 be regular surfaces given by the parametrizations*

$$\mathbf{x}_1 : U \to S_1, \quad \mathbf{x}_2 : U \to S_2$$

If the two surfaces have the same coefficients of the first fundamental form, that is they satisfy

$$E_1 = E_2, \quad F_1 = F_2, \quad G_1 = G_2$$

Then the function $f = \mathbf{x}_2 \circ \mathbf{x}_1^{-1} : \mathbf{x}_1(U) \to S_2$ gives us a local isometry.

Thus if two surface have the same first fundamental form, they are said to be locally isometric to each other. Note that this holds in the local sense.

1.4 The Second Fundamental Form and Curvatures

The theory presented in this section is based on [Do 16, chapter 3, 4.3].

Since the Gauss map $\mathbf{N} : S \to S^2$ is a differentiable map, we can compute its differential. Note that the differential $d\mathbf{N}_p$ of $\mathbf{N}$ is the linear map $d\mathbf{N}_p : T_p(S) \to T_{\mathbf{N}(p)}(S^2)$. However $T_{\mathbf{N}(p)}(S^2) \simeq T_p(S)$. This is because the tangent plane is the same in S and in S^2. The Gauss map maps the unit normal vector from the surface S to the unit sphere. The unit normal vectors before and after the mapping have unit length. The only thing that changes is the point of origin. The unit normal vector $\mathbf{N}(p)$ has origin on the surface S, and it gets mapped to S^2, where the origin is the centre of S^2. Due to this $T_{\mathbf{N}(p)}(S^2) \simeq T_p(S)$. Hence the differential $d\mathbf{N}_p$ of the Gauss map $\mathbf{N}$ at a point $p \in S$ is the linear map given by

$$d\mathbf{N}_p : T_p(S) \to T_p(S)$$

An important property of the linear map $d\mathbf{N}_p$ is that it is a self-adjoint linear map. This just means that given some basis $\{\mathbf{a}_1, \mathbf{a}_2\}$ for $T_p(S)$,

$$\langle d\mathbf{N}_p(\mathbf{a}_1), \mathbf{a}_2 \rangle = \langle \mathbf{a}_1, d\mathbf{N}_p(\mathbf{a}_2) \rangle$$

Due to this self adjoint property we can associate a quadratic form Q in the tangent plane $T_p(S)$ to the linear map $d\mathbf{N}_p$.

Definition 1.4.1. Let $\mathbf{v} \in T_p(S)$. Then the quadratic from $II_p : T_p(S) \to \mathbb{R}$, defined on $T_p(S)$ by the following inner product

$$II_p(\mathbf{v}) = -\langle d\mathbf{N}_p(\mathbf{v}), \mathbf{v} \rangle$$

is called the *second fundamental form* of S at the point p.

The self adjoint property of $d\mathbf{N}_p$ can further be utilized. Let $p \in S$. Then we can find an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $T_p(S)$ at the point p such that

$$d\mathbf{N}_p(\mathbf{e}_1) = -k_1\mathbf{e}_1$$
$$d\mathbf{N}_p(\mathbf{e}_2) = -k_2\mathbf{e}_2$$

where $-k_1$ and $-k_2$ are the eigenvalues of $d\mathbf{N}_p$, and $\mathbf{e}_1, \mathbf{e}_2$ are the corresponding eigenvectors of $d\mathbf{N}_p$. Considering the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ the eigenvalues are ordered such that $k_1 \geq k_2$. Moreover

- k_1 is the maximum of II_p when we restrict $T_p(S)$ to the unit circle
- k_2 is the minimum of II_p when we restrict $T_p(S)$ to the unit circle

This holds for every $p \in S$. To see the significance of this, we have to look at the idea of normal curvature. Let α be regular curve in S going through $p \in S$ and k denote the curvature of α at p. In addition let $\cos(\theta) = \langle \mathbf{n}, \mathbf{N} \rangle$, where $\mathbf{n}$ is a unit normal vector to α, and $\mathbf{N}$ is a normal vector to $p \in S$. Then

$$k_n = k\cos(\theta)$$

is called the *normal curvature* of $\alpha \subset S$ at $p \in S$. Thus the significance of k_1 and k_2 is that they are the maximum and minimum values of the normal curvature at $p \in S$. Let $p \in S$. Then

- the *principal curvatures at* p are given by the maximum normal curvature k_1 and the minimum normal curvature k_2, and
- the *principal directions at* p are given by vectors in their respective directions, that is, they are given by the eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

The linear map $d\mathbf{N}_p : T_p(S) \to T_p(S)$ has a matrix of dimension 2×2. In addition this matrix gives rise to two other types of curvatures. This leads to the following definition.

Definition 1.4.2. Let $A = (a_{ij})_{ij=1,2}$ denote the 2×2 matrix of the linear map $d\mathbf{N}_p : T_p(S) \to T_p(S)$ which is the differential of the Gauss map. Then for any point $p \in S$

- the determinant of A is called the *Gaussian curvature* K of S at $p \in S$:

$$det(A) = a_{11}a_{22} - a_{12}a_{21}$$

- the negative of half of the trace of A is called the *mean curvature* H of S at $p \in S$:

$$-\frac{1}{2}tr(A) = -\frac{1}{2}(a_{11} + a_{22})$$

In the orthonormal basis $\mathbf{e}_1, \mathbf{e}_2$ of $T_p(S)$, the matrix of $d\mathbf{N}_p$, which we denote by $A_{principal}$, is given as

$$A_{principal} = \begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix}$$

The Gaussian curvature K in terms of the principal curvatures is given as

$$K = det(A_{principal}) = (-k_1)(-k_2) - 0 = k_1\ k_2$$

The mean curvature H in terms of the principal curvatures is given as

$$H = -\frac{1}{2}tr(A_{principal}) = -\frac{1}{2}(-k_1 - k_2) = -\frac{1}{2}(-(k_1 + k_2)) = \frac{1}{2}(k_1 + k_2)$$

Thus we can express the Gaussian and mean curvature in terms of the principal curvatures as

$$\text{Gaussian curvature } K = k_1\ k_2, \tag{1.11}$$

$$\text{Mean curvature } H = \frac{1}{2}(k_1 + k_2) \tag{1.12}$$

We can see that the Gaussian curvature is the product of the two principal curvatures, and the mean curvature is half of the sum of the two principal curvatures. We have found the curvatures in terms of the principal curvatures, we would also like to find an expression for these curvatures in terms of the local coordinates u, v of our parametrization. We mention the equations of the vectors $\mathbf{N}_u$ and $\mathbf{N}_v$ in terms of $\mathbf{x}_u$ and $\mathbf{x}_v$. They are given by

$$\begin{aligned} \mathbf{N}_u &= a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v \\ \mathbf{N}_v &= a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v \end{aligned} \tag{1.13}$$

Let us compute the matrix of $d\mathbf{N}_p$ in terms of u and v. Consider the parametrized curve α on S given by $\alpha(t) = \mathbf{x}(u(t), v(t))$ with $\alpha(0) = p$. At $p \in S$, the tangent vector is $\mathbf{w} = \alpha'(0) = \mathbf{x}_u u' + \mathbf{x}_v v'$. Then using these equations we compute

$$\begin{aligned} d\mathbf{N}_p(\mathbf{w}) = d\mathbf{N}_p(\alpha'(0)) = \mathbf{N}'(u(t), v(t)) &= \mathbf{N}_u u' + \mathbf{N}_v v' \\ &= (a_{11}u' + a_{12}v')\mathbf{x}_u + (a_{21}u' + a_{22}v')\mathbf{x}_v \end{aligned}$$

Let us compute the second fundamental form II_p

$$\begin{aligned}
II_p(\mathbf{w}) &= -\langle d\mathbf{N}_p(\mathbf{w}), \mathbf{w}\rangle \\
&= -\langle d\mathbf{N}_p(\alpha'(0))\ , \alpha'(0)\rangle \\
&= -\langle \mathbf{N}_u u' + \mathbf{N}_v v',\ \mathbf{x}_u u' + \mathbf{x}_v v'\rangle \\
&= -\left(\langle \mathbf{N}_u, \mathbf{x}_u\rangle (u')^2 + \langle \mathbf{N}_u, \mathbf{x}_v\rangle u'v' + \langle \mathbf{N}_v, \mathbf{x}_v\rangle u'v' + \langle \mathbf{N}_v, \mathbf{x}_v\rangle (v')^2\right) \\
&= e\ (u')^2 + 2f\ u'v' + g\ (v')^2
\end{aligned}$$

at a point p. Note that we have used that $\langle \mathbf{N}_u, \mathbf{x}_v\rangle = \langle \mathbf{N}_v, \mathbf{x}_u\rangle$ (see definition 1.4.3). The functions $e(u,v)$, $f(u,v)$ and $g(u,v)$ are functions of $(u,v) \in U$. The values of these functions are the focus of the next definition.

Definition 1.4.3. Let $\mathbf{x} : U \to V \cap S$ be the parametrization of a regular surface S. Then the quantities e, f and g are called the *coefficients of the second fundamental form*. Referring to the fact that $\langle \mathbf{N}, \mathbf{x}_u\rangle = \langle \mathbf{N}, \mathbf{x}_v\rangle = 0$, in the local coordinates u, v they are given as

$$\begin{aligned}
e &= -\langle \mathbf{N}_u, \mathbf{x}_u\rangle = \langle \mathbf{N}, \mathbf{x}_{uu}\rangle \\
f &= -\langle \mathbf{N}_v, \mathbf{x}_u\rangle = \langle \mathbf{N}, \mathbf{x}_{uv}\rangle = \langle \mathbf{N}, \mathbf{x}_{vu}\rangle = -\langle \mathbf{N}_u, \mathbf{x}_v\rangle \\
g &= -\langle \mathbf{N}_v, \mathbf{x}_v\rangle = \langle \mathbf{N}, \mathbf{x}_{vv}\rangle
\end{aligned}$$

The definition above gives e, f, g in local coordinates u, v. Let us denote the matrix of $d\mathbf{N}_p$ in local coordinates by A_{local}. To obtain A_{local} we use the coefficients of the second fundamental form e, f, g that are given by definition 1.4.3, and the coefficients of the first fundamental form E, F, G that are given by definition 1.3.2, along with the equations (1.13). This leads to

$$\begin{aligned}
-f &= \langle \mathbf{N}_u, \mathbf{x}_v\rangle = a_{11}F + a_{21}G \\
-f &= \langle \mathbf{N}_v, \mathbf{x}_u\rangle = a_{12}E + a_{22}F \\
-e &= \langle \mathbf{N}_u, \mathbf{x}_u\rangle = a_{11}E + a_{21}F \\
-g &= \langle \mathbf{N}_v, \mathbf{x}_v\rangle = a_{12}F + a_{22}G
\end{aligned}$$

or in matrix form

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

Thus the matrix $A_{local} = a_{ij}$ can be found by

$$\begin{aligned}
\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} &= -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \\
&= -\frac{1}{EG - F^2} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\
&= -\frac{1}{EG - F^2} \begin{pmatrix} eG - fF & -eF + fE \\ fG - gF & -fF + gE \end{pmatrix} \\
&= \frac{1}{EG - F^2} \begin{pmatrix} fF - eG & fE - eF \\ gF - fG & gE - fF \end{pmatrix}
\end{aligned}$$

The matrix A_{local} is given by

$$A_{local} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

and the elements of this matrix are given by

$$a_{11} = \frac{fF - eG}{EG - F^2}, \quad a_{21} = \frac{eF - fE}{EG - F^2}$$
$$a_{12} = \frac{gF - fG}{EG - F^2}, \quad a_{22} = \frac{fF - gE}{EG - F^2}$$

Let us compute the curvatures in terms of the local coordinates. Let $W = EG - F^2$. Then the Gaussian curvature K is given as

$$\begin{aligned}
K = det(A_{local}) &= a_{11}a_{22} - a_{12}a_{21} \\
&= \left(\frac{fF - eG}{W}\right)\left(\frac{fF - gE}{W}\right) - \left(\left(\frac{eF - fE}{W}\right)\left(\frac{gF - fG}{W}\right)\right) \\
&= \frac{1}{W^2}\left((fF - eG)(fF - gE) - ((eF - fE)(gF - fG))\right) \\
&= \frac{1}{W^2}(f^2F^2 - fgFE - efGF + egEG - (egF^2 - gfFE \\
&- efGF + f^2EG)) \\
&= \frac{1}{W^2}(f^2F^2 - fgFE + fgFE - efGF + efGF \\
&+ egEG - egF^2 - f^2EG) \\
&= \frac{1}{W^2}(egEG - f^2EG - egF^2 + f^2F^2) \\
&= \frac{1}{W^2}(EG(eg - f^2) - F^2(eg - f^2)) \\
&= \frac{1}{W^2}((EG - F^2)(eg - f^2))
\end{aligned}$$

Substituting W back into the expression above leads to

$$K = det(A_{local}) = \frac{(EG - F^2)(eg - f^2)}{(EG - F^2)^2} = \frac{eg - f^2}{EG - F^2}$$

The mean curvature H is given as

$$\begin{aligned}
H = -\frac{1}{2}tr(A_{local}) &= -\frac{1}{2}(a_{11} + a_{22}) \\
&= -\frac{1}{2}\left(\frac{fF - eG}{W} + \frac{fF - gE}{W}\right) \\
&= -\frac{1}{2W}(-eG + 2fF - gE) \\
&= \frac{1}{2W}(eG - 2fF + gE)
\end{aligned}$$

Substituting W back into the expression above leads to

$$H = -\frac{1}{2}tr(A_{local}) = \frac{1}{2(EG - F^2)}(eG - 2fF + gE) = \frac{1}{2}\frac{Eg - 2fF + Ge}{EG - F^2}$$

The curvatures are functions of the local variables u and v, that is $K(u, v)$ and $H(u, v)$ for $u, v \in U$. These two curvatures tell us about the shape of the surface locally at a point $p = (u, v) \in U$. We summarize the main expressions for the Gaussian and mean curvature in the following definition.

Definition 1.4.4. Let $\mathbf{x} : U \to V \cap S$ be the parametrization of a regular surface S. Then in the local coordinates u, v the *Gaussian curvature* K is given as

$$K = \frac{eg - f^2}{EG - F^2}$$

and the *mean curvature* H is given as

$$H = \frac{1}{2}\frac{Eg - 2fF + Ge}{EG - F^2}$$

where E, F, G are the coefficients of the first fundamental form, and e, f, g are the coefficients of the second fundamental form for the surface S for $(u, v) \in U$.

There is something special about the Gaussian curvature. From the formula for K given in definition 1.4.4, it seems that K depends on the coefficients e, f, g and E, F, G. However Gauss discovered that the Gaussian curvature only depends on the coefficients of the first fundamental form, E, F and G, and their derivatives. In the case that $F = 0$, it is possible to compute the Gaussian curvature using the formula [Do 16, p. 240],

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right] \tag{1.14}$$

Another important concept that is related to the Gaussian curvature K, is the integral of the Gaussian curvature. This leads us to the next definition.

Definition 1.4.5. [ONe06, p. 304] Let $\mathbf{x} : U \to V \cap S$ be the parametrization of a regular surface S. Let $R \subset U$ be a bounded set. Then the *total Gaussian curvature* is given by

$$\int_R K dA$$

where dA is the area element, and we integrate over the region R.

The total Gaussian curvature is defined as the integral of the Gaussian curvature K over some bounded region in our surface S. We would like to compute the total Gaussian curvature in terms of local coordinates u, v. We can compute this integral in terms of the coefficients of the first fundamental form, E, F and G. Due to (1.9) and (1.10), we get that

$$\begin{aligned}\int_R K dA &= \int\int_R K \ \|\mathbf{x}_u \times \mathbf{x}_v\| \ dudv \\ &= \int\int_R K \ \sqrt{EG - F^2} \ dudv\end{aligned}$$

Therefore the total Gaussian curvature can be computed in terms of E, F and G by

$$\int_R K dA = \int\int_R K \ \sqrt{EG - F^2} \ dudv \tag{1.15}$$

We are now ready to look at the geometric object of interest, namely minimal surfaces.

CHAPTER 2

Theory of Minimal Surfaces

In this chapter we will look at the definition of a minimal surface in $\mathbb{R}^3$. The interesting part about the theory of minimal surfaces is that we can look at it from different areas of mathematics.

2.1 Mean Curvature

In this section we will look at the definition of minimal surfaces in $\mathbb{R}^3$ that is connected to the branch of differential geometry (see [Do 16, p. 200]). The main definition of minimal surfaces is from differential geometry. We have reviewed the important concepts needed for surfaces in $\mathbb{R}^3$ from differential geometry. We are now in a position to define minimal surfaces.

Definition 2.1.1. Let S be a regular surface in $\mathbb{R}^3$ that is parametrized by $\mathbf{x}(u,v)$. Then $\mathbf{x}$ is a minimal surface if $H = 0$, where H is the mean curvature.

The concept of curvatures is significant when it comes to minimal surfaces. In particular the mean curvature H is crucial. From the definition above we can see that zero mean curvature is the defining property for minimal surfaces. Consider H in terms of principal curvatures given by (1.12). Then zero mean curvature implies that

$$
\begin{aligned}
H = \frac{k_1 + k_2}{2} = 0 &\iff k_1 + k_2 = 0 \\
&\iff k_1 = -k_2
\end{aligned}
$$

This shows us that the principal curvatures for a minimal surface S are such that $k_1 = -k_2$, so they are equal and opposite. Consider K in terms of principal curvatures given by (1.12). Since the Gaussian curvature is the product of the principal curvatures,

$$K = k_1 \cdot k_2 = (-k_2) \cdot k_2 = -(k_2)^2 \leq 0$$

This shows us that the Gaussian curvature K of a minimal surface is always less than or equal to zero.

Consider the case where $H = 0$ and $K = 0$. In this case the surface we get is the plane. This makes sense as a plane does not have any curvature at all. This brings us to our first encounter with an example of a minimal surface. This example is the plane. The plane is the trivial example of a minimal surface [Pér17], as it has zero mean curvature, $H = 0$.

2.2 Minimal Surfaces and Harmonic Functions

In this section we will look at the connection between minimal surfaces and harmonic functions. This further makes the connection of minimal surfaces to the branch of complex analysis. The theory presented in this section is based on [Opr00, chapter 3.4, 3.5]. The theory presented on the review of the basics of complex analysis is based on [Opr00, chapter 2.4]. We start by recalling some basics from complex analysis. The set of complex numbers $\mathbb{C}$ is defined by

$$\mathbb{C} = \{w = x + iy \mid x, y \in \mathbb{R}\}$$

Let $w \in \mathbb{C}$. Then $w = u + iv$, where u is the real part of w denoted by $\text{Re}(w)$, and v is the imaginary part of w denoted by $\text{Im}(w)$. In addition i is the imaginary unit and $i^2 = -1$. We denote the complex conjugate by $\overline{w}$, it is defined as $\overline{w} = u - iv$. We also recall the following

$$e^{iv} = \cos(v) + i\sin(v) \tag{2.1}$$

as well as the expression for e^w,

$$e^w = e^{u+iv} = e^u e^{iv} = e^u(\cos(v) + i\sin(v)) \tag{2.2}$$

We recall some properties of complex valued functions. We had previously considered differentiable functions when the functions where real valued. Now we consider the differentiation of a complex function.

Definition 2.2.1. Let $f : \mathbb{C} \to \mathbb{C}$ and $D \subset \mathbb{C}$ be an open set. Then f is *complex differentiable* at a point $a \in D$ if the limit

$$\lim_{w \to a} \frac{f(w) - f(a)}{w - a} = f'(a)$$

exists for all $a \in D$, for all paths taken in approaching a. If this limit exists it is denoted by $f'(a)$. We then say that f is *holomorphic* in D.

We can write the function $f(w)$ as

$$f(w) = f(x + iy) = b(x, y) + ic(x, y)$$

where $b(x, y)$ and $c(x, y)$ real valued functions with x and y real. The function $b(x, y)$ is the real part of f, and $c(x, y)$ is the imaginary part of f. Next we recall the Cauchy-Riemann equations.

Definition 2.2.2. Let $f(x + iy) = b(x, y) + ic(x, y)$. The equations given by

$$\frac{\partial b}{\partial x} = \frac{\partial c}{\partial y}, \quad \frac{\partial b}{\partial y} = -\frac{\partial c}{\partial x}$$

are called the *Cauchy-Riemann equations*.

If f is complex differentiable at $a \in D$, it will satisfy the Cauchy-Riemann equations. This gives a more convenient way to tell if a function is holomorphic or not, as computing the limit every time is not so practical. Thus a function f is said to be holomorphic in D if the partial derivatives

$$\frac{\partial b}{\partial x}, \frac{\partial c}{\partial y}, \frac{\partial b}{\partial y}, \frac{\partial c}{\partial x}$$

exist and are continuous on D, and the Cauchy-Riemann equations are satisfied. If f is holomorphic, then the higher order derivatives of f are also holomorphic. By that we mean f', f'', f''' and so on are holomorphic as well. Next we will need to recall the Laplace equation.

Definition 2.2.3. The *Laplace equation* of a function $f(x, y)$, $f : \mathbb{R}^2 \to \mathbb{R}$ is the equation given by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

and the operator that is denoted by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is called the *Laplace operator*. Therefore the Laplace equation can be written as $\Delta f = 0$. The symbol Δf is sometimes read as 'the Laplacian of f'.

Now we are ready to look at harmonic functions.

Definition 2.2.4. A function $\Psi(x, y), \Psi : D \subset \mathbb{R}^2 \to \mathbb{R}$ is called *harmonic* on D if it has continuous partial derivatives of second order and it satisfies the Laplace equation, that is

$$\Delta\Psi = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$$

In addition we have the following theorem.

Theorem 2.2.5. *[Opr00, p. 46] Let $D \subset \mathbb{C}$ be an open set. If $f : D \to \mathbb{C}$ given by*

$$f(w) = b(x, y) + ic(x, y)$$

is holomorphic in D, then the functions $b(x, y)$ and $c(x, y)$ are harmonic.

Proof. Lets consider $b(x, y)$ first. To show that b is harmonic, we have to show that it satisfies the Laplace equation. Due to the Cauchy-Riemann equations we get that

$$\frac{\partial^2 b}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial b}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial c}{\partial y}\right) = \frac{\partial^2 c}{\partial x \partial y}$$
$$\frac{\partial^2 b}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial b}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial c}{\partial x}\right) = -\frac{\partial^2 c}{\partial x \partial y}$$

The Laplacian of $b(x, y)$ is

$$\Delta b = \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} = \frac{\partial^2 c}{\partial x \partial y} - \frac{\partial^2 c}{\partial x \partial y} = 0$$

Similarly for $c(x, y)$ we have to show that c satisfies the Laplace equation. Once again by using the Cauchy-Riemann equation we get that

$$\frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial c}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial b}{\partial y}\right) = -\frac{\partial^2 b}{\partial x \partial y}$$
$$\frac{\partial^2 c}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial c}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial b}{\partial x}\right) = \frac{\partial^2 b}{\partial x \partial y}$$

Thus the Laplacian of $c(x,y)$ is

$$\Delta c = \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = -\frac{\partial^2 b}{\partial x \partial y} + \frac{\partial^2 b}{\partial x \partial y} = 0$$

This is what we wanted to show, hence the proof is complete. ■

The harmonic functions $b(x,y)$ and $c(x,y)$ that form the holomorphic function $f(w) = b(x,y) + ic(x,y)$, are called *harmonic conjugates.* Let $f(u,v)$ be a function with complex coordinates, where $w = u + iv$ and $\overline{w} = u - iv$. Then we can compute the partial derivatives with respect to w and $\overline{w}$ by

$$\begin{aligned} \frac{\partial}{\partial w} &= \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \\ \frac{\partial}{\partial \overline{w}} &= \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right), \end{aligned} \tag{2.3}$$

The equations (2.3) provide us with yet another way to check if a function is holomorphic. A function f is holomorphic precisely when $\partial f/\partial \overline{w} = 0$. We have looked at a holomorphic function. We will also look at the notion of a meromorphic function.

Definition 2.2.6. Let $g : \mathbb{C} \to \mathbb{C}$ and $D \subset \mathbb{C}$ be an open set. Let

$$g(w) = \frac{p_1(w)}{p_2(w)}$$

where $p_1(w)$ and $p_2(w)$ are polynomials. If all the singularities of g are poles, then $g(w)$ is *meromorphic* in D. By singularities, we mean the points where $g(w)$ is not defined. This means that the singularities of g are the roots of $p_2(w)$, as this is where the function is not defined. In addition if the roots of $p_2(w)$ are not cancelled out by the roots of $p_1(w)$, then this point is a pole for g.

Next we mention an important point about the parametrization of a minimal surface. This is the parametrization by isothermal coordinates.

Definition 2.2.7. Let S be a regular surface in $\mathbb{R}^3$ with parametrization $\mathbf{x}(u,v)$ given by $\mathbf{x} : U \to \mathbb{R}^3$ with $U \subset \mathbb{R}^2$ an open set. Then $\mathbf{x}(u,v)$ is said to have an *isothermal parametrization* if,

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = G \text{ and } F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0$$

If we parametrize a minimal surface using an isothermal parametrization, many calculations become simpler. Luckily we can always parametrize a minimal surface in $\mathbb{R}^3$ using isothermal coordinates.

Theorem 2.2.8. *[Opr00, p. 73] Let S be a minimal surface in $\mathbb{R}^3$. It is always possible to parametrize a minimal surface $\mathbf{x}(u,v)$ using isothermal coordinates.*

This type of parametrization leads us to the connection between minimal surfaces and harmonic functions. In order to see the connection we begin by stating an essential theorem.

Theorem 2.2.9. *[Opr00, p. 75] Let S be a surface in $\mathbb{R}^3$. If the surface is parametrized $\mathbf{x}(u,v)$ using isothermal coordinates, then there is a relationship between the parametrized surface $\mathbf{x}(u,v)$ and the Laplacian of the parametrization $\Delta\mathbf{x}(u,v)$. This relationship is*

$$\Delta\mathbf{x}(u,v) = (2EH)\,\mathbf{N}$$

The theorem above tells us that if we have a surface which is parametrized by isothermal parameters, then there is a connection between the parametrization $\mathbf{x}(u,v)$ and the Laplacian of that parametrization $\Delta\mathbf{x}(u,v)$. This relationship between the two introduces the constants E and H, the coefficient of the first fundamental form and the mean curvature, into the picture. This brings us to the interesting relationship between an isothermal parametrization of a minimal surface and harmonic functions.

Corollary 2.2.10. *[Opr00, p. 76] Let S be a regular surface in $\mathbb{R}^3$. Furthermore let S be parametrized by isothermal coordinates $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$. Then S is a minimal surface if and only if the three component functions of the parametrization $x(u,v), y(u,v)$ and $z(u,v)$ are harmonic.*

Proof. Let S be a surface in $\mathbb{R}^3$ with isothermal parametrization $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)$. Then we have to prove implications of the corollary to prove the if and only if statement.

- $\Rightarrow$ Let us assume that S is a regular minimal surface. This means that it has $H = 0$, which by theorem 2.2.9 implies that $\Delta\mathbf{x}(u,v) = (2EH)\mathbf{N} = 0 \cdot \mathbf{N} = 0$. Thus $\Delta\mathbf{x} = 0$ so $x(u,v), y(u,v)$ and $z(u,v)$ are harmonic functions.

- $\Leftarrow$ Let us assume that $x(u,v), y(u,v)$ and $z(u,v)$ are harmonic functions. This means that they satisfy the Laplace equation so $\Delta\mathbf{x} = 0$. Then by using theorem 2.2.9 once more we have that $\Delta\mathbf{x} = (2EH)\mathbf{N}$. Since $\Delta\mathbf{x} = 0$ we must have $(2EH)\mathbf{N} = 0$. Since $\mathbf{x}$ is regular, the quantity E is $\|\mathbf{x}_u\|^2 > 0$ and the unit normal vector $\mathbf{N}$ is nonzero. The only way we can have $(2EH)\mathbf{N} = 0$ is if $H = 0$. This is precisely the definition of a minimal surface, and so S is a minimal surface.

■

Thus we have looked at the ideas that connect the theory of minimal surfaces to harmonic functions. All this is only possible thanks to the isothermal parametrization of $\mathbf{x}(u,v)$.

2.3 The Weierstrass Enneper Formulas

In this section we look at another consequence of an isothermal parametrization for a minimal surface in $\mathbb{R}^3$, and the fact that the component functions are harmonic. This leads us to another type of parametrization for minimal surfaces, known as the Weierstrass Enneper representation. The theory presented in this section is based on [Opr00, chapter 3.6]. Let S be a minimal surface in $\mathbb{R}^3$. Let

S have an isothermal parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, with U an open set, given by

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$$

We let the coordinates $(u, v) \in \mathbb{R}^2$ correspond to points in the complex plane $\mathbb{C}$. That is, a point $(u, v) \in \mathbb{R}^2$ is identified with $w = u + iv \in \mathbb{C}$. With this, $\mathbf{x}$ becomes the parametrization $\mathbf{x} : U \subset \mathbb{C} \to \mathbb{R}^3$, with U an open set, given by

$$\mathbf{x}(w) = (x(w),\ y(w),\ z(w))$$

where the component functions $x(w), y(w), z(w)$ are complex functions. Furthermore we define the parametrization ϕ as $\mathbf{x}_w$. Due to the formulas given in (2.3) we define ϕ as,

$$\phi = \frac{\partial \mathbf{x}}{\partial w} = \left(\frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w}\right) = (x_w, y_w, z_w)$$

where the component functions x_w, y_w, z_w are given as

$$\begin{aligned} x_w &= \frac{\partial x}{\partial w} = \frac{1}{2}\left(\frac{\partial x}{\partial u} - i\frac{\partial x}{\partial v}\right) = \frac{1}{2}\left(x_u - ix_v\right), \\ y_w &= \frac{\partial y}{\partial w} = \frac{1}{2}\left(\frac{\partial y}{\partial u} - i\frac{\partial y}{\partial v}\right) = \frac{1}{2}\left(y_u - iy_v\right), \\ z_w &= \frac{\partial z}{\partial w} = \frac{1}{2}\left(\frac{\partial z}{\partial u} - i\frac{\partial z}{\partial v}\right) = \frac{1}{2}\left(z_u - iz_v\right), \end{aligned}$$

Therefore the parametrization ϕ is

$$\phi = \frac{\partial \mathbf{x}}{\partial w} = (x_w, y_w, z_w) = \left(\frac{1}{2}\left(x_u - ix_v\right),\ \frac{1}{2}\left(y_u - iy_v\right), \frac{1}{2}\left(z_u - iz_v\right)\right)$$

Let us verify that ϕ is an isothermal parametrization. This is the topic of the following lemma.

Lemma 2.3.1. *Let S be a regular surface in $\mathbb{R}^3$ parametrized by $\mathbf{x}(u, v)$. Define $\phi = \frac{\partial \mathbf{x}}{\partial w}$. Then $\mathbf{x}(u, v)$ is isothermal if and only if $\phi^2 = (x_w)^2 + (y_w)^2 + (z_w)^2 = 0$.*

Proof. We compute ϕ^2 and see that

$$\begin{aligned} \phi^2 &= (x_w)^2 + (y_w)^2 + (z_w)^2 \\ &= \frac{1}{4}\left(x_u^2 - x_v^2 - 2ix_u x_v + y_u^2 - y_v^2 - 2iy_u y_v + z_u^2 - z_v^2 - 2iz_u z_v\right) \\ &= \frac{1}{4}\left((x_u^2 + y_u^2 + z_u^2) - (x_v^2 + y_v^2 + z_v^2) - 2i(x_u x_v + y_u y_v + z_u z_v)\right) \\ &= \frac{1}{4}\left(\|\mathbf{x}_u\|^2 - \|\mathbf{x}_v\|^2 - 2i\langle \mathbf{x}_u, \mathbf{x}_v\rangle\right) \\ &= \frac{1}{4}(E - G - 2iF) \end{aligned}$$

Let us assume that $\mathbf{x}(u, v)$ is an isothermal parametrization. Then $E = G$ and $F = 0$, so $\phi^2 = \frac{1}{4}(E - E - 0) = 0$. To prove the other direction assume that $\phi^2 = 0$. Then we must have $\frac{1}{4}(E - G - 2iF) = 0$. The right hand side has both, real and imaginary part equal to zero. As $E, G > 0$, the only way the expression $\frac{1}{4}(E - G - 2iF)$ can have its real and imaginary parts be equal to zero is, if $E = G$ and $F = 0$. Thus the parametrization is isothermal. ∎

As $\mathbf{x}(u, v)$ is an isothermal parametrization of a minimal surface, the component functions x_w, y_w, z_w must be harmonic. In order to show this we first need a preliminary result.

Lemma 2.3.2. *Let $f(w)$ be holomorphic function. Then the Laplacian of f, $\Delta f = f_{uu} + f_{vv}$ can be written in terms of $\frac{\partial}{\partial \overline{w}}$ and $\frac{\partial}{\partial w}$ as*

$$\Delta f = 4 \cdot \frac{\partial}{\partial \overline{w}} \left(\frac{\partial f}{\partial w} \right)$$

Proof. We compute the expression by using the formulas (2.3). This leads to

$$\begin{aligned} \frac{\partial}{\partial \overline{w}} \left(\frac{\partial f}{\partial w} \right) &= \frac{\partial}{\partial \overline{w}} \left(\frac{1}{2} \left(\frac{\partial f}{\partial u} - i \frac{\partial f}{\partial v} \right) \right) \\ &= \frac{1}{4} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \left(\frac{\partial f}{\partial u} - i \frac{\partial f}{\partial v} \right) \\ &= \frac{1}{4} \left(\frac{\partial^2 f}{\partial u^2} - i \frac{\partial^2 f}{\partial u \partial v} + i \frac{\partial^2 f}{\partial u \partial v} - i^2 \frac{\partial^2 f}{\partial v^2} \right) \\ &= \frac{1}{4} \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \right) \\ &= \frac{1}{4} \Delta f \end{aligned}$$

Therefore we get that

$$4 \cdot \frac{\partial}{\partial \overline{w}} \left(\frac{\partial f}{\partial w} \right) = 4 \cdot \frac{1}{4} \Delta f = \Delta f$$

■

We have an isothermal parametrization of a minimal surface $\mathbf{x}(u, v)$. Hence by corollary 2.2.10 we have that $\Delta x = 0$. We want to show that each component of $\mathbf{x}$ is holomorphic. In order for ϕ to be holomorphic, we need to have $\frac{\partial \phi}{\partial \overline{w}} = 0$. We compute

$$\frac{\partial \phi}{\partial \overline{w}} = \frac{\partial}{\partial \overline{w}} \underbrace{\left(\frac{\partial \mathbf{x}}{\partial w} \right)}_{\text{def of } \phi} = \frac{1}{4} \Delta \mathbf{x} = 0$$

To make the notation easier we denote each component of ϕ by ϕ_i for $i = 1, 2, 3$. Using this notation we have $\phi_1 = x_w,\ \phi_2 = y_w,\ \phi_3 = z_w$. Then the above calculation shows that ϕ_1, ϕ_2 and ϕ_3 are holomorphic, since $\frac{\partial \phi}{\partial \overline{w}} = 0$. We have shown that each component function $\phi_i, i = 1, 2, 3$ is holomorphic. Since each component function ϕ_i is holomorphic, their corresponding functions, that is the functions $x_u - i x_v, y_u - i y_v$ and $z_u - i z_v$, are harmonic. This is due to the result of theorem 2.2.5.

In order to create an isothermal parametrization for a minimal surface, we need to find three holomorphic functions ϕ_1, ϕ_2 and ϕ_3 such that $\phi = (\phi_1, \phi_2, \phi_3)$. The functions also need to satisfy the condition $\phi^2 = (\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2 = 0$.

Turns out this is possible if we choose a holomorphic function f and a meromorphic function g, such that fg^2 is holomorphic. Once we have such functions, we set up the parametrization by

$$\phi_1 = \frac{1}{2} f(1-g^2)$$
$$\phi_2 = \frac{i}{2} f(1+g^2)$$
$$\phi_3 = fg$$

With these functions ϕ is given by

$$\phi = (\phi_1, \phi_2, \phi_3) = \Big(\frac{1}{2} f(1-g^2),\ \frac{i}{2} f(1+g^2),\ fg\Big) \tag{2.4}$$

The parametrization given by ϕ is an isothermal parametrization as

$$\begin{aligned}\phi^2 = (\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2 &= (\frac{1}{2} f(1-g^2))^2 + (\frac{i}{2} f(1+g^2))^2 + (fg)^2 \\ &= \frac{1}{4} f^2(1-g^2)^2 - \frac{1}{4} f^2(1+g^2)^2 + f^2g^2 \\ &= \frac{1}{4} f^2(-4g^2) + f^2g^2 \\ &= -f^2g^2 + f^2g^2 = 0\end{aligned}$$

We get that $\phi^2 = 0$. By the result of lemma 2.3.1, we have that the parametrization is isothermal. This leads us to the main theorem which gives us formulas to form the parametrization.

Theorem 2.3.3. *(see [Opr00, p. 80] or [DHS10, p. 112]) Let S be a minimal surface in $\mathbb{R}^3$ that is different from a plane. Let $U \subset \mathbb{C}$ be an open set. In addition let $f(w)$ and $g(w)$ be complex functions $f, g : U \subset \mathbb{C} \to \mathbb{C}$. If f is holomorphic in U, g is meromorphic in U, and fg^2 is holomorphic in U, then S can be defined by the parametrization $\mathbf{x} : U \subset \mathbb{C} \to \mathbb{R}^3$ given as*

$$\mathbf{x}(w) = (x(w),\ y(w),\ z(w))$$

where the components are given by the integrals

$$x(w) = \text{Re} \int f(1-g^2)\ dw$$
$$y(w) = \text{Re} \int if(1+g^2)\ dw$$
$$z(w) = \text{Re} \int 2fg\ dw$$

Furthermore the parametrization $\mathbf{x}(w) = (x(w),\ y(w),\ z(w))$ is isothermal.

The remarkable theorem above allows us to build a minimal surface by using two functions $f(w)$ and $g(w)$ that satisfy the conditions in theorem 2.3.3. We will see how this theorem is used in the next chapter.

2.3.1 Geometric quantities in terms of the Weierstrass Enneper formulas

If a minimal surface is parametrized by the formulas in theorem 2.3.3, we can express the important geometric quantities in terms of the functions $f(w)$ and

$g(w)$. We mention that the theory presented here is also based on [Opr00, chapter 3.6]. One important quantity we would like to compute is the area element, namely $dA = \sqrt{EG - F^2}$ from (1.9). Note that due to the isothermal condition we have that $E = G$ and $F = 0$. This means that

$$\sqrt{EG - F^2} = \sqrt{E(E) - 0^2} = \sqrt{(E)^2 - 0} = \sqrt{E^2} = E$$

In order to compute the area element, we can simply compute E. To compute E in terms of the parametrization $\phi = (\phi_1, \phi_2, \phi_3)$ the following lemma is useful.

Lemma 2.3.4. *The coefficient of the first fundamental form E in terms of the isothermal parametrization $\phi = (\phi_1, \phi_2, \phi_3)$, where $\phi = \frac{\partial \mathbf{x}}{\partial w}$, is given by*

$$E = 2|\phi|^2$$

Proof. We begin by computing $|\phi|^2 = |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 = |x_w|^2 + |y_w|^2 + |z_w|^2$ and keep in mind that ϕ is isothermal, so $E = G$ and $F = 0$. Then

$$\begin{aligned}
|\phi|^2 &= |x_w|^2 + |y_w|^2 + |z_w|^2 \\
&= x_w \cdot \bar{x}_w + y_w \cdot \bar{y}_w + z_w \cdot \bar{z}_w \\
&= \left(\frac{1}{2}(x_u - ix_v)\frac{1}{2}(x_u + ix_v)\right) + \left(\frac{1}{2}(y_u - iy_v)\frac{1}{2}(y_u + iy_v)\right) \\
&+ \left(\frac{1}{2}(z_u - iz_v)\frac{1}{2}(z_u + iz_v)\right) \\
&= \frac{1}{4}\left((x_u^2 + y_u^2 + z_u^2) + (x_v^2 + y_v^2 + z_v^2)\right) \\
&= \frac{1}{4}(\|\mathbf{x}_u\|^2 + \|\mathbf{x}_v\|^2) \\
&= \frac{1}{4}(E + G) = \frac{1}{4}(2E) = \frac{1}{2}E
\end{aligned}$$

Therefore $E = 2|\phi|^2$ and this is what we wanted to show. ∎

We can now compute E in terms of $f(w)$ and $g(w)$ in the parametrization given by theorem 2.3.3, where ϕ is given by (2.4). We begin by finding $|\phi|^2 = |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2$. This leads to

$$\begin{aligned}
|\phi|^2 &= \left|\frac{1}{2}f(1 - g^2)\right|^2 + \left|\frac{i}{2}f(1 + g^2)\right|^2 + |fg|^2 \\
&= \frac{1}{4}|f|^2|1 - g^2|^2 + \frac{1}{4}|f|^2|1 + g^2|^2 + f\overline{f}g\overline{g} \\
&= \frac{1}{4}|f|^2\left(|1 - g^2|^2 + |1 + g^2|^2 + 4g\overline{g}\right) \\
&= \frac{1}{4}|f|^2\left(|1 - g^2|^2 + |1 + g^2|^2 + 4|g|^2\right) \\
&= \frac{1}{4}|f|^2\left(2 + 2|g|^4 + 4|g|^2\right)
\end{aligned}$$

The above expression is a result of the following calculations:

$$\begin{aligned}
&|1 - g^2|^2 = (1 - g^2)(1 - \overline{g}^2) = 1 - \overline{g}^2 - g^2 + g^2\overline{g}^2 = 1 - \overline{g}^2 - g^2 + |g|^4 \\
&|1 + g^2|^2 = (1 + g^2)(1 + \overline{g}^2) = 1 + \overline{g}^2 + g^2 + g^2\overline{g}^2 = 1 + \overline{g}^2 + g^2 + |g|^4 \\
&|1 - g^2|^2 + |1 + g^2|^2 = 2 + 2|g|^4
\end{aligned}$$

To further simplify $|\phi|^2$ we note that $(1+|g|^2)^2 = 1+|g|^4+2|g|^2$. This leads to the following

$$\begin{aligned}|\phi|^2 &= \frac{1}{4}|f|^2\left(2+2|g|^4+4|g|^2\right) = \frac{1}{4}|f|^2\left(2(1+|g|^4+2|g|^2)\right)\\ &= \frac{1}{2}|f|^2(1+|g|^2)^2\end{aligned}$$

Then due to lemma 2.3.4 we find that

$$E = 2|\phi|^2 = 2\cdot\frac{1}{2}|f|^2(1+|g|^2)^2 = |f|^2(1+|g|^2)^2$$

Thus the expression for E in terms of $f(w)$ and $g(w)$ is given by

$$E = |f|^2(1+|g|^2)^2 \tag{2.5}$$

We can also compute the Gaussian curvature K in terms of the functions $f(w)$ and $g(w)$. Due to the isothermal condition we have $F=0$. This means that we can use the formula given in (1.14), which is

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)+\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right]$$

to find an expression for K. In this case, the formula simplifies as we have $E=G$. This simplified formula for K is given by

$$K = -\frac{1}{2E}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{E}\right)+\frac{\partial}{\partial u}\left(\frac{G_u}{E}\right)\right]$$

Note that we are concerned with local coordinates u, v. So E is a function of u, v, that is, $E(u,v)$. Bear in mind that

$$\frac{\partial}{\partial u}\left(\ln(E(u,v))\right) = \frac{E_u}{E}, \text{ and } \frac{\partial}{\partial v}\left(\ln(E(u,v))\right) = \frac{E_v}{E},$$

Thus if the surface is parametrized with isothermal coordinates, the formula for K further simplifies to

$$\begin{aligned}K &= -\frac{1}{2E}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{E}\right)+\frac{\partial}{\partial u}\left(\frac{E_u}{E}\right)\right]\\ &= -\frac{1}{2E}\left[\frac{\partial}{\partial v}\left(\frac{\partial}{\partial v}\ln(E)\right)+\frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}\ln(E)\right)\right]\\ &= -\frac{1}{2E}\left[\frac{\partial^2}{\partial v^2}(\ln(E))+\frac{\partial^2}{\partial u^2}(\ln(E))\right]\\ &= -\frac{1}{2E}\Delta(\ln(E))\end{aligned} \tag{2.6}$$

where Δ is the Laplace operator. By using the expression for E given in (2.5) and inserting it into the formula for K given by (2.6), we can find an expression for K in terms of $f(w)$ and $g(w)$. Firstly we compute

$$\begin{aligned}\ln(E) &= \ln\left(|f|^2(1+|g|^2)^2\right)\\ &= \ln(|f|^2)+\ln((1+|g|^2)^2)\end{aligned}$$

Inserting this into the formula for K leads to

$$\begin{aligned} K = \frac{-1}{2E}\Delta(\ln(E)) &= \frac{-\Delta\Big(\ln(|f|^2) + \ln((1+|g|^2)^2)\Big)}{2|f|^2(1+|g|^2)^2} \\ &= \frac{-\Delta\ln(|f|^2) - \Delta\ln((1+|g|^2)^2)}{2|f|^2(1+|g|^2)^2} \end{aligned}$$

We find $\Delta\ln(|f|^2)$ first. To compute the Laplacian we use the formula in lemma 2.3.2. Note that f only depends on w, and $\overline{f}$ only depends on $\overline{w}$. Then we get that

$$\begin{aligned} \Delta\ln(|f|^2) &= \Delta\ln(f\overline{f}) = \Delta\ln(f) + \Delta\ln(\overline{f}) \\ &= 4\frac{\partial}{\partial\overline{w}}\left(\frac{\partial\ln(f)}{\partial w}\right) + 4\frac{\partial}{\partial\overline{w}}\left(\frac{\partial\ln(\overline{f})}{\partial w}\right) \\ &= 4\frac{\partial}{\partial\overline{w}}\left(\frac{f_w}{f}\right) + 4\frac{\partial}{\partial\overline{w}}(0) = 4\frac{\partial}{\partial\overline{w}}\left(\frac{f_w}{f}\right) \\ &= 4\cdot 0 = 0 \end{aligned}$$

Since the function f_w/f only depends on w, differentiating with respect to $\overline{w}$ gives 0. Thus $\Delta\ln(|f|^2) = 0$. Now we find $\Delta\ln((1+|g|^2)^2)$. Note that g only depends on w, and $\overline{g}$ only depends on $\overline{w}$. Then by using the formula for the Laplacian in lemma 2.3.2 we get that

$$\begin{aligned} \Delta\ln((1+|g|^2)^2) &= 2\Delta\ln(1+|g|^2) = 2\Delta\ln(1+g\overline{g}) \\ &= 2\cdot 4\frac{\partial}{\partial\overline{w}}\left(\frac{\partial\ln(1+g\overline{g})}{\partial w}\right) = 8\frac{\partial}{\partial\overline{w}}\left(\frac{\partial\ln(1+g\overline{g})}{\partial w}\right) \\ &= 8\frac{\partial}{\partial\overline{w}}\left(\frac{g'\overline{g}}{1+g\overline{g}}\right) = 8\left(\frac{g'\overline{g}' + g\overline{g}g'\overline{g}' - g\overline{g}g'\overline{g}'}{(1+g\overline{g})^2}\right) \\ &= 8\frac{g'\overline{g}'}{(1+g\overline{g})^2} = 8\frac{|g'|^2}{(1+|g|^2)^2} \end{aligned}$$

Now we can find K. We find that

$$\begin{aligned} K &= \frac{-\Delta\ln(|f|^2) - \Delta\ln((1+|g|^2)^2)}{2|f|^2(1+|g|^2)^2} \\ &= \frac{-1}{2|f|^2(1+|g|^2)^2}\cdot\frac{8|g'|^2}{(1+|g|^2)^2} = \frac{-8|g'|^2}{2|f|^2(1+|g|^2)^2(1+|g|^2)^2} \\ &= \frac{-4|g'|^2}{|f|^2(1+|g|^2)^4} \end{aligned}$$

Thus the expression for K in terms of $f(w)$ and $g(w)$ is given by

$$K = \frac{-4|g'|^2}{|f|^2(1+|g|^2)^4} \tag{2.7}$$

We saw that we can find the coefficient of the first fundamental form E, and the Gaussian curvature K, in terms of the functions $f(w)$ and $g(w)$ when a minimal surface is parametrized by the Weierstrass Enneper formulas given in theorem 2.3.3. Another geometric property that we would like to express in terms of $f(w)$ and $g(w)$ is the unit normal vector $\mathbf{N}$. From [DHS10, p. 113] we get that the unit normal vector $\mathbf{N}$ can be expressed in terms of $f(w)$ and $g(w)$ by the following expression,

$$\mathbf{N} = \left(\frac{2\mathrm{Re}(g)}{1+|g|^2}, \frac{2\mathrm{Im}(g)}{1+|g|^2}, \frac{|g|^2-1}{1+|g|^2} \right) \tag{2.8}$$

The reason the unit normal vector is significant is due to the fact that the unit normal vector is also the Gauss map. If we can find an expression for the unit normal vector $\mathbf{N}$, we will also be able to look at the Gauss map $\mathbf{N} : S \to S^2$ of a minimal surface paramterized by the formulas in theorem 2.3.3.

2.4 Other Ways to Look at Minimal Surfaces

In this section we will mention some other ways we can look at minimal surfaces. We look at the connection of minimal surfaces to other areas of mathematics.

- **Partial differential equations and minimal surfaces**
  Minimal surfaces have a connection with partial differential equations [Opr00, p. 62]. Let S be a regular surface given in non parametric form by $f = z(x, y)$. The graph of $f = z(x, y)$ gives us a regular surface due to the result of proposition 1.1.5. Note that we consider the graph of f locally. Then f is said to be a minimal surface if it satisfies the following partial differential equation:

  $$f_{xx}(1+f_y^2) - 2f_x f_y f_{xy} + f_{yy}(1+f_x^2) = 0 \tag{2.9}$$

  The partial differential equation given by (2.9) is called the *minimal surface equation*. Thus a function $f = z(x, y)$ is a minimal surface if it satisfies the minimal surface equation. This tells us that the solutions to the equation (2.9) allow us to find minimal surfaces.

- **Area functional and minimal surfaces**
  Minimal surfaces have a connection to the area functional [Do 16, p. 202]. To be more specific, minimal surfaces can be looked at by considering the first variation, that is, the first derivative of the area functional. This is further explained in chapter four.

- **Soap bubbles and minimal surfaces**
  Minimal surfaces have a connection to soap bubbles [Opr00, chapter 3.3]. In the context of minimal surfaces, we say soap films rather than soap bubbles. This is because minimal surfaces can be modelled by the shape or form of soap films. This is further explained in chapter five.

We have looked at the theory as well as the definition of a minimal surface in $\mathbb{R}^3$. Next we look at some examples.

CHAPTER 3

Examples of Minimal Surfaces with a look at Some Properties

In this chapter we will look at some examples of minimal surfaces in $\mathbb{R}^3$. This will allow us to further develop our understanding of minimal surfaces.

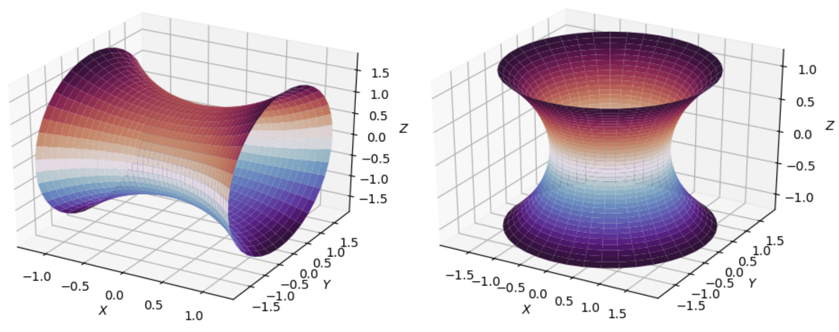
3.1 The Catenoid

The catenoid is a minimal surface that is a surface of revolution, see for example [Do 16, p. 204]. It is generated by rotating the catenary curve $y(x) = c\cosh(x/c)$. Let $c = 1$. If we rotate the catenary curve about the x-axis, that is, the curve $y(x) = \cosh(x)$, the catenoid is parametrized by

$$\begin{aligned} \mathbf{x}(u, v) = (u,\ \cosh(u)\cos(v),\ \cosh(u)\sin(v)), \\ -\infty < u < \infty,\ 0 \leq v < 2\pi \end{aligned} \tag{3.1}$$

If we rotate the catenary curve $f(y) = \cosh(y)$ about the y-axis, the catenoid is parametrized by

$$\begin{aligned} \mathbf{x}(u, v) = (\cosh(u)\cos(v),\ u,\ \cosh(u)\sin(v)), \\ -\infty < u < \infty,\ 0 \leq v < 2\pi \end{aligned} \tag{3.2}$$



(a) Rotation around the x-axis

(b) Rotation around the z-axis

Figure 3.1: The catenoid. This figure shows the catenoid in the region $u \in (-1.2, 1.2)$ and $v \in [0, 2\pi)$.

If we rotate the catenary curve $y(z) = \cosh(z)$ about the z-axis, the catenoid is parametrized by

$$\begin{aligned} \mathbf{x}(u, v) = (\cosh(u)\cos(v),\ \cosh(u)\sin(v),\ u), \\ -\infty < u < \infty,\ 0 \leq v < 2\pi \end{aligned} \tag{3.3}$$

The catenoid is shown in figure 3.1. Figure 3.1a shows the catenoid when it is given by the parametrization in (3.1). Figure 3.1b shows the catenoid when it is given by the parametrization in (3.3). Let us compute the mean curvature for the catenoid, as well as the other important geometric quantities. We will compute the geometric quantities using the parametrization where the rotation is around the z-axis, that is the parametrization given by (3.3). We begin by computing

$$\begin{aligned} \mathbf{x}_u &= (\sinh(u)\cos(v),\ \sinh(u)\sin(v),\ 1) \\ \mathbf{x}_v &= (-\cosh(u)\sin(v),\ \cosh(u)\cos(v),\ 0) \end{aligned}$$

Then we compute the second partial derivatives of the tangent vectors

$$\begin{aligned} \mathbf{x}_{uu} &= (\cosh(u)\cos(v),\ \cosh(u)\sin(v),\ 0) \\ \mathbf{x}_{uv} &= (-\sinh(u)\sin(v),\ \sinh(u)\cos(v),\ 0) \\ \mathbf{x}_{vu} &= (-\sinh(u)\sin(v),\ \sinh(u)\cos(v),\ 0) \\ \mathbf{x}_{vv} &= (-\cosh(u)\cos(v),\ -\cosh(u)\sin(v),\ 0) \end{aligned}$$

For the catenoid in this case, we get that $\mathbf{x}_{uv} = \mathbf{x}_{vu}$. We find the coefficients of the first fundamental form namely E, F and G.

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \cosh^2(u) \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0 \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = \cosh^2(u) \end{aligned}$$

We can see that $E = G$ and $F = 0$, hence this is an isothermal parametrization for the catenoid. It is useful to compute $EG - F^2$, since it is used for example in the denominator when computing curvatures.

$$EG - F^2 = \cosh^2(u) \cdot \cosh^2(u) - 0^2 = \cosh^4(u) \tag{3.4}$$

$$\sqrt{EG - F^2} = \sqrt{\cosh^4(u)} = \cosh^2(u) \tag{3.5}$$

To find the unit normal vector $\mathbf{N}$ we will have to find

$$\begin{aligned} \mathbf{x}_u \times \mathbf{x}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh(u)\cos(v) & \sinh(u)\sin(v) & 1 \\ -\cosh(u)\sin(v) & \cosh(u)\cos(v) & 0 \end{vmatrix} \\ &= \mathbf{i}(-\cosh(u)\cos(v)) - \mathbf{j}(\cosh(u)\sin(v)) \\ &\quad + \mathbf{k}(\sinh(u)\cosh(u)\cos(v)^2 + \sinh(u)\cosh(u)\sin(v)^2)) \\ &= (-\cosh(u)\cos(v),\ -\cosh(u)\sin(v),\ \sinh(u)\cosh(u)) \\ \|\mathbf{x}_u \times \mathbf{x}_v\| &= \sqrt{EG - F^2} = \cosh(u)^2 \end{aligned}$$

The unit normal vector $\mathbf{N}$ is given by

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \left(\frac{-\cos(v)}{\cosh(u)}, \frac{-\sin(v)}{\cosh(u)}, \frac{\sinh(u)}{\cosh(u)} \right) \tag{3.6}$$

and the derivatives of $\mathbf{N}$ with respect to u and v are:

$$\begin{aligned} \mathbf{N}_u &= \left(\frac{\cos(v)\sinh(u)}{\cosh^2(u)}, \frac{\sin(v)\sinh(u)}{\cosh^2(u)}, \frac{1}{\cosh^2(u)} \right) \\ \mathbf{N}_v &= \left(\frac{\sin(v)}{\cosh(u)}, \frac{-\cos(v)}{\cosh(u)}, 0 \right) \end{aligned}$$

Let us also compute the following inner products. This will be useful later.

$$\langle \mathbf{N}_u, \mathbf{N}_u \rangle = \frac{1}{\cosh^2(u)}, \quad \langle \mathbf{N}_u, \mathbf{N}_v \rangle = 0, \quad \langle \mathbf{N}_v, \mathbf{N}_v \rangle = \frac{1}{\cosh^2(u)} \tag{3.7}$$

Now that we have the unit normal vector we can find the coefficients of the second fundamental form, namely e, f and g.

$$\begin{aligned} e &= \langle \mathbf{N}, \mathbf{x}_{uu} \rangle = -1 \\ f &= \langle \mathbf{N}, \mathbf{x}_{uv} \rangle = 0 \\ g &= \langle \mathbf{N}, \mathbf{x}_{vv} \rangle = 1 \end{aligned}$$

We can now calculate the Gaussian and mean curvatures. The Gaussian curvature K is

$$K = \frac{eg - f^2}{EG - F^2} = \frac{(-1)(1) - 0^2}{\cosh^4(u)} = \frac{-1}{\cosh^4(u)} \tag{3.8}$$

The mean curvature H is

$$\begin{aligned} H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2} &= \frac{1}{2} \frac{(-1)(\cosh^2(u)) - 2(0)(0) + (1)(\cosh^2(u))}{\cosh^4(u)} \\ &= \frac{1}{2} \frac{-\cosh^2(u) + \cosh^2(u)}{\cosh^4(u)} = 0 \end{aligned} \tag{3.9}$$

The mean curvature is zero, thus the catenoid is a minimal surface. The Gaussian curvature is always negative.

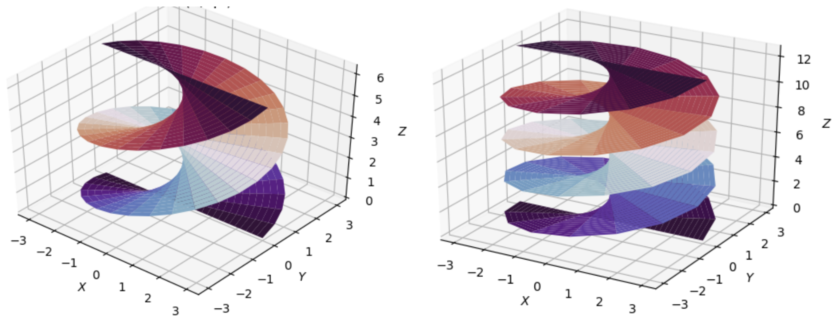
3.2 The Helicoid

The helicoid is a minimal surface, see for example [Opr00, p. 61]. This minimal surface is parametrized by

$$\begin{aligned} \mathbf{x}(u, v) = (u\cos(v),\ u\sin(v),\ v), \\ -\infty < u < \infty,\ 0 \leq v < 2\pi \end{aligned} \tag{3.10}$$

The helicoid is shown in figure 3.2. From the figure we can see that as the parameter v increases from 2π to 4π, the spirals of the helicoid increase. Let us compute the mean curvature for the helicoid. We begin with

$$\begin{aligned} \mathbf{x}_u &= (\cos(v),\ \sin(v),\ 0) \\ \mathbf{x}_v &= (-u\sin(v),\ u\cos(v),\ 1) \end{aligned}$$



(a) $v \in [0, 2\pi)$ (b) $v \in [0, 4\pi)$

Figure 3.2: The helicoid. This figure shows the helicoid in the region $u \in (-3, 3)$ and v as mentioned.

Then we compute the second partial derivatives of the tangent vectors

$$\begin{aligned}\mathbf{x}_{uu} &= (0,\ 0,\ 0)\\ \mathbf{x}_{uv} &= (-\sin(v),\ \cos(v),\ 0)\\ \mathbf{x}_{vv} &= (-u\cos(v),\ -u\sin(v),\ 0)\end{aligned}$$

The coefficients of the first fundamental form E, F and G are

$$\begin{aligned}E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1\\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0\\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 1 + u^2\end{aligned}$$

We can see that in this case, we did not get $E = G$ and $F = 0$. Therefore this is *not* an isothermal parametrization for the helicoid. In what follows we will see how we can get an isothermal parametrization for the helicoid. We compute the following

$$EG - F^2 = (1)(1 + u^2) - 0^2 = 1 + u^2 \tag{3.11}$$

$$\sqrt{EG - F^2} = \sqrt{1 + u^2} \tag{3.12}$$

To find the unit normal vector $\mathbf{N}$ we will have to find

$$\begin{aligned}\mathbf{x}_u \times \mathbf{x}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos(v) & \sin(v) & 0\\ -u\sin(v) & u\cos(v) & 1\end{vmatrix}\\ &= \mathbf{i}(\sin(v)) - \mathbf{j}(\cos(v)) + \mathbf{k}(u\cos^2(v) + u\sin^2(v))\\ &= (\sin(v),\ -\cos(v),\ u)\\ \|\mathbf{x}_u \times \mathbf{x}_v\| &= \sqrt{EG - F^2} = \sqrt{1 + u^2}\end{aligned}$$

The unit normal vector $\mathbf{N}$ is given by

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \left(\frac{\sin(v)}{\sqrt{1 + u^2}},\ \frac{-\cos(v)}{\sqrt{1 + u^2}},\ \frac{u}{\sqrt{1 + u^2}}\right) \tag{3.13}$$

Now that we have the unit normal vector we can find the coefficients of the second fundamental form e, f and g.

$$\begin{aligned} e &= \langle \mathbf{N}, \mathbf{x}_{uu} \rangle = 0 \\ f &= \langle \mathbf{N}, \mathbf{x}_{uv} \rangle = \frac{-1}{\sqrt{1+u^2}} \\ g &= \langle \mathbf{N}, \mathbf{x}_{vv} \rangle = 0 \end{aligned}$$

Finally we can compute the curvatures. The Gaussian curvature K is

$$\begin{aligned} K = \frac{eg - f^2}{EG - F^2} &= \frac{1}{1+u^2}\Big[0 \cdot 0 - (\frac{-1}{\sqrt{1+u^2}})^2\Big] \\ &= \frac{-1}{(1+u^2)^2} \end{aligned} \tag{3.14}$$

The mean curvature H is

$$\begin{aligned} H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2} &= \frac{1}{2}\frac{(0)(1+u^2) - 2(\frac{-1}{\sqrt{1+u^2}})(0) + (0)(1)}{1+u^2} \\ &= \frac{1}{2}\frac{0 - 0 + 0}{1+u^2} = 0 \end{aligned} \tag{3.15}$$

The mean curvature is zero, thus the helicoid is a minimal surface. The Gaussian curvature for the helicoid is always negative.

3.2.1 Relationship between the helicoid and the catenoid

The helicoid has a connection to the catenoid. In order to see this connection we consider the Weierstrass representation formulas from theorem 2.3.3. The catenoid can be represented by the functions [Opr00, p. 82]:

$$f(w) = \frac{-1}{2e^w}, \quad g(w) = -e^w$$

It will be useful to recall the following from complex analysis [Opr00, pp. 43–44]. Let w be a complex number, $w = u + iv$. Then

$$\sinh(w) = \frac{1}{2}(e^w - e^{-w}) \tag{3.16}$$

$$\cosh(w) = \frac{1}{2}(e^w + e^{-w}) \tag{3.17}$$

$$\sinh(w) = \sinh(u + iv) = \sinh(u)\cos(v) + i\cosh(u)\sin(v) \tag{3.18}$$

$$\cosh(w) = \cosh(u + iv) = \cosh(u)\cos(v) + i\sinh(u)\sin(v) \tag{3.19}$$

We begin by computing the integrals that are mentioned in theorem 2.3.3. First we just compute the integrals without taking the real parts. To compute the integrals we used the expressions in (3.16) and (3.17).

$$
\begin{aligned}
x(w) = \int f(1-g^2)\ dw &= \int \frac{-1}{2e^w}(1-(-e^w)^2)\ dw = \int \frac{-1}{2e^w}(1-e^{2w})\ dw \\
&= \frac{1}{2}\int (-e^{-w}+e^w)\ dw = \frac{e^{-w}+e^w}{2} \\
&= \cosh(w) \\
y(w) = \int if(1+g^2)\ dw &= \int \frac{-i}{2e^w}(1+(-e^w)^2)\ dw = \int \frac{-i}{2e^w}(1+e^{2w})\ dw \\
&= \frac{-i}{2}\int (e^{-w}+e^w)\ dw = -i\left[\frac{e^w-e^{-w}}{2}\right] \\
&= -i\sinh(w) \\
z(w) = \int 2fg\ dw &= \int 2\ \frac{-1}{2e^w}(-e^w)\ dw = \int \frac{2e^w}{2e^w}\ dw = \int 1\ dw = w
\end{aligned}
$$

To get to the next step we note that $w = u + iv$. Using this we find the parametrization in u, v coordinates. Here we have used the expressions given in (3.18) and (3.19).

$$
\begin{aligned}
x(w) &= \cosh(w) = \cosh(u+iv) = \cosh(u)\cos(v) + i\sinh(u)\sin(v) \\
y(w) &= -i\sinh(w) = -i\sinh(u+iv) = -i(\sinh(u)\cos(v) + i\cosh(u)\sin(v)) \\
&= \cosh(u)\sin(v) - i\sinh(u)\cos(v) \\
z(w) &= w = u + iv
\end{aligned}
$$

Consider the surface that is formed by taking the **real parts** of the parametrization. We will denote this surface by $\mathbf{x}_{real}$. This surface has the components

$$
\begin{aligned}
\mathbf{x}_{real}(u,v) &= (x(u,v),\ y(u,v),\ z(u,v)) \\
&= (\cosh(u)\cos(v),\ \cosh(u)\sin(v),\ u)
\end{aligned}
$$

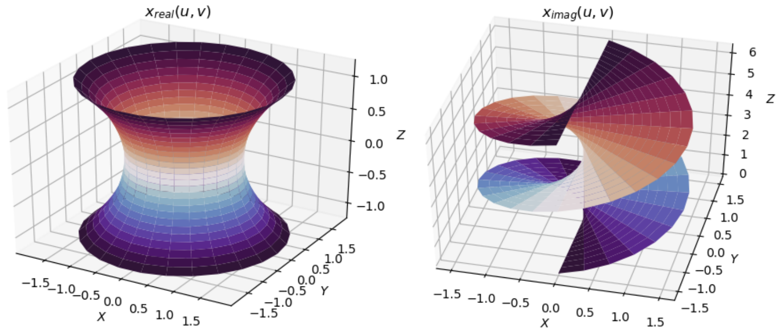
Similarly the surface that is formed by taking the **imaginary parts** of the parametrization will be denoted by $\mathbf{x}_{imag}$. This surface has the components

$$
\begin{aligned}
\mathbf{x}_{imag}(u,v) &= (x(u,v),\ y(u,v),\ z(u,v)) \\
&= (\sinh(u)\sin(v),\ -\sinh(u)\cos(v),\ v)
\end{aligned}
$$

The parameters of both of the surfaces are $-\infty < u < \infty,\ 0 \le v < 2\pi$. Hence $\mathbf{x}_{real}$ gives the parametrization for the catenoid. The functions $f(w) = \frac{-1}{2e^w},\ g(w) = -e^w$ represent the catenoid in the Weierstrass representation in theorem 2.3.3. The interesting point to note is that $\mathbf{x}_{imag}$ is a parametrization for the helicoid. These surfaces are shown in figure 3.3. To expand on this phenomenon let $\omega(t)$ be a function such that

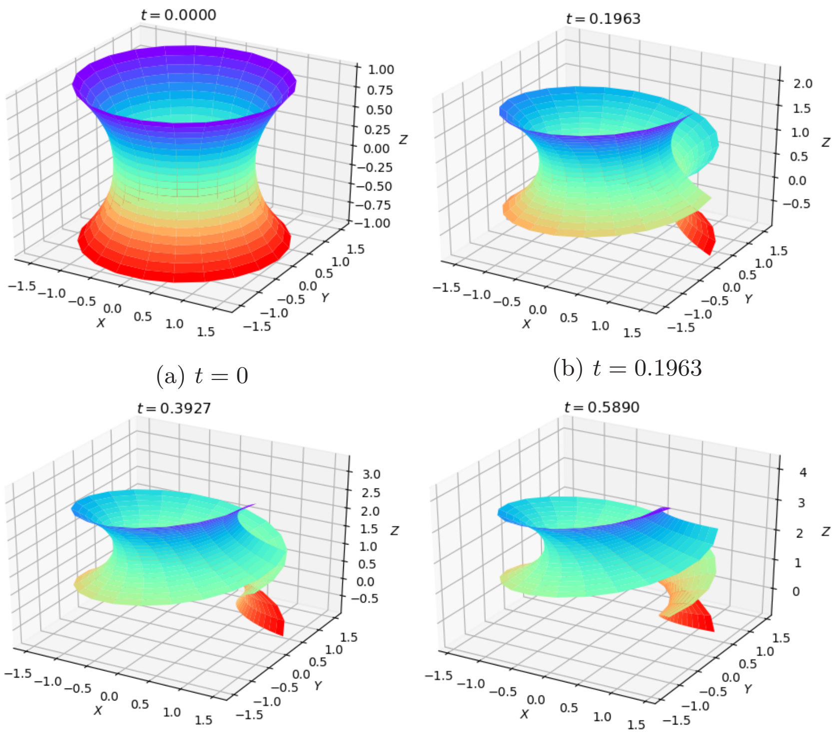
$$
\omega(u,v,t) = \mathbf{x}_{real}(u,v)\cos(t) + \mathbf{x}_{imag}(u,v)\sin(t), \quad t \in \left[0, \frac{\pi}{2}\right] \tag{3.20}
$$

Then the minimal surfaces $\mathbf{x}_{real}$ and $\mathbf{x}_{imag}$ are a part of a *one parameter family of minimal surfaces* [DHS10, p. 100] where t is the parameter and varies in the interval $t \in \left[0, \frac{\pi}{2}\right]$. Thus the catenoid and helicoid are minimal surfaces that



(a) $\mathbf{x}_{real}(u, v)$ (b) $\mathbf{x}_{imag}(u, v)$

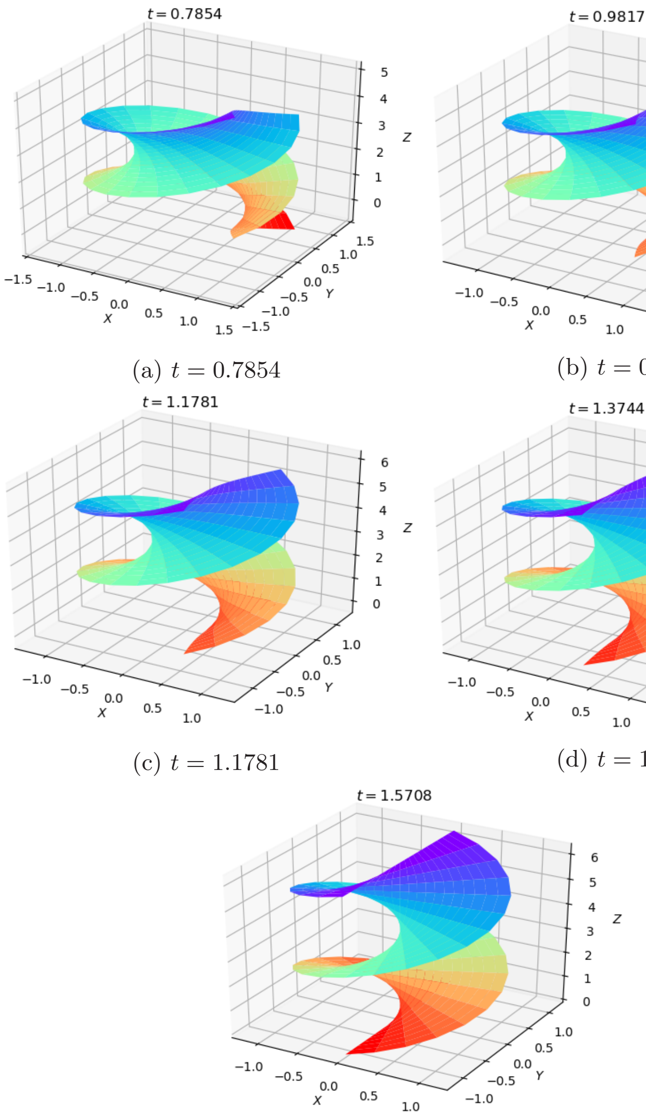
Figure 3.3: Surfaces parameterized by $\mathbf{x}_{real}(u, v)$ and $\mathbf{x}_{imag}(u, v)$. It is clear that figure a is a catenoid and figure b is a helicoid.



(a) $t = 0$ (b) $t = 0.1963$

(c) $t = 0.3927$ (d) $t = 0.5890$

Figure 3.4: Isometric transformation from the catenoid to the helicoid part 1. The parameter domain is $v \in [0, 2\pi)$ and $u \in [-1, 1]$ and $t \in [0, \pi/2]$. Figure a shows that at $t = 0$ we get a catenoid.



(a) $t = 0.7854$

(b) $t = 0.9817$

(c) $t = 1.1781$

(d) $t = 1.3744$

(e) $t = 1.5708$

Figure 3.5: Isometric transformation from the catenoid to the helicoid part 2. The parameter domain is $v \in [0, 2\pi)$ and $u \in [-1, 1]$ and $t \in [0, \pi/2]$. Figure e shows that at $t = \pi/2 \approx 1.570$ we get a helicoid.

span a one parameter family of minimal surfaces. The transformation from the catenoid to the helicoid is shown in figure 3.4 and 3.5. Note that

$$\omega(u,v,0) = \mathbf{x}_{real}(u,v), \qquad \omega(u,v,\frac{\pi}{2}) = \mathbf{x}_{imag}(u,v)$$

Minimal surfaces that are in the same family can be continuously deformed to one another as the parameter t varies from 0 to $\pi/2$. This means that the catenoid can be continuously transformed to the helicoid, and the helicoid can also be continuously transformed to the catenoid. When $t = 0$, the minimal surface $\omega(u,v,0)$ will parametrize the catenoid. When $t = \pi/2$, the minimal surface $\omega(u,v,\frac{\pi}{2})$ will parametrize the helicoid.

More can be said about this one parameter family. We have that $\mathbf{x}_{real}$ parametrises the catenoid, and $\mathbf{x}_{imag}$ parametrises the helicoid. Consider the geometric properties of $\mathbf{x}_{imag}$ and $\mathbf{x}_{real}$.

$$\begin{aligned} \mathbf{x}_{imag}(u,v) &= (\sinh(u)\sin(v),\ -\sinh(u)\cos(v),\ v) \\ \frac{\partial \mathbf{x}_{imag}}{\partial u} &= (\cosh(u)\sin(v),\ -\cosh(u)\cos(v),\ 0) \\ \frac{\partial \mathbf{x}_{imag}}{\partial v} &= (\sinh(u)\cos(v),\ \sinh(u)\sin(v),\ 1) \end{aligned}$$

Hence the coefficients of the first fundamental form are

$$E_{imag} = \cosh^2(u), \qquad F_{imag} = 0, \qquad G_{imag} = \cosh^2(u)$$

We know that $\mathbf{x}_{real}$ is the catenoid as mentioned in (3.3). We have already found the coefficients of the first fundamental form for the catenoid. They are given by

$$E_{real} = \cosh^2(u), \qquad F_{real} = 0, \qquad G_{real} = \cosh^2(u)$$

Therefore the minimal surfaces $\mathbf{x}_{imag}$ and $\mathbf{x}_{real}$ have the same first fundamental forms. Note that the two parametrizations have the same u, v domain. By proposition 1.3.3 we have that if two surfaces have the same first fundamental forms, then their is a local isometry between them. Since the catenoid and the helicoid given by $\mathbf{x}_{imag}$, have the same first fundamental forms, they are locally isometric to each other. Note that the helicoid that is given by the parametrization

$$\text{Helicoid } = \mathbf{x}_{imag}(u,v) = (\sinh(u)\sin(v),\ -\sinh(u)\cos(v),\ v) \tag{3.21}$$

has $E = G$ and $F = 0$. Therefore (3.21) is an isothermal parametrization for the helicoid. This is because the parametrization (3.21) is a result of of the Weierstrass Enneper representation in theorem 2.3.3. The Weierstrass Enneper formulas provide us with an isothermal parametrization for a minimal surface.

3.3 Scherk's First Surface

Scherk's first surface is a minimal surface that is defined in non-parametric form by a function $f = z(x,y)$, see for example [Opr00, p. 63]. The function f is given by

$$f = z(x,y) = c\ \ln\left(\frac{\cos(x/c)}{\cos(y/c)}\right) \tag{3.22}$$

where c is a constant. Let $c = 1$. Then Scherk's first surface is given by

$$f = z(x, y) = \ln\left(\frac{\cos(x)}{\cos(y)}\right) \tag{3.23}$$

Since Scherk's first surface is a minimal surface that is defined by the graph of a function $f = z(x, y)$, the function f satisfies the minimal surface equation given in (2.9), which is

$$f_{xx}(1 + f_y^2) - 2f_x f_y f_{xy} + f_{yy}(1 + f_x^2) = 0$$

To show this we compute the partial derivatives of f given in (3.23) and the quantities needed in the minimal surface equation. We begin by computing

$$f_x = -\frac{\sin(x)}{\cos(x)}, \quad f_x^2 = \frac{\sin^2(x)}{\cos^2(x)}, \quad f_{xy} = 0$$
$$f_y = \frac{\sin(y)}{\cos(y)}, \quad f_y^2 = \frac{\sin^2(y)}{\cos^2(y)}, \quad f_{yx} = 0$$

Next we compute

$$f_{xx} = -\frac{\sin^2(x)}{\cos^2(x)} - 1 = \frac{-\sin^2(x) - \cos^2(x)}{\cos^2(x)} = \frac{-(\sin^2(x) + \cos^2(x))}{\cos^2(x)} = \frac{-1}{\cos^2(x)}$$
$$f_{yy} = \frac{\sin^2(y)}{\cos^2(y)} + 1 = \frac{\sin^2(y) + \cos^2(y)}{\cos^2(y)} = \frac{1}{\cos^2(y)}$$

Due to the above calculations we get that,

$$1 + f_x^2 = 1 + \frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}, \quad 1 + f_y^2 = 1 + \frac{\sin^2(y)}{\cos^2(y)} = \frac{1}{\cos^2(y)}$$

Then computing the following quantities gives us,

$$f_{xx}(1 + f_y^2) = \frac{-1}{\cos^2(x)} \cdot \frac{1}{\cos^2(y)} = \frac{-1}{\cos^2(x)\cos^2(y)}$$
$$f_{yy}(1 + f_x^2) = \frac{1}{\cos^2(y)} \cdot \frac{1}{\cos^2(x)} = \frac{1}{\cos^2(x)\cos^2(y)}$$
$$-2f_x f_y f_{xy} = -2 \cdot 0 = 0$$

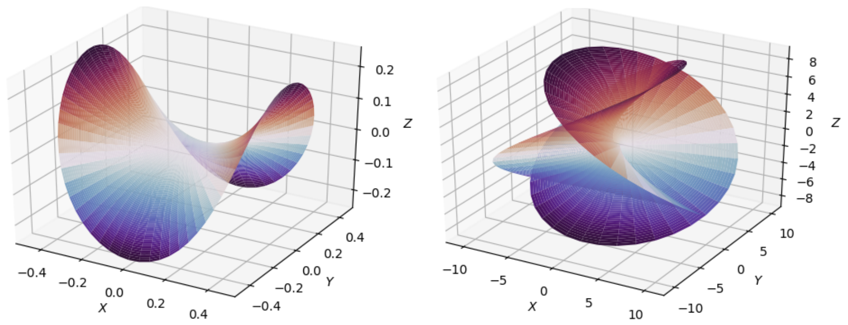
Inserting these calculations into the minimal surface equation leads to

$$f_{xx}(1 + f_y^2) - 2f_x f_y f_{xy} + f_{yy}(1 + f_x^2) = \frac{-1}{\cos^2(x)\cos^2(y)} + \frac{1}{\cos^2(x)\cos^2(y)}$$
$$= 0$$

Thus f in (3.23) satisfies the minimal surface equation. Therefore Scherk's first minimal surface is a minimal surface as it satisfies the minimal surface equation. It is possible to parametrize this surface by a map $\mathbf{x}(u, v)$, where $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Due to proposition 1.1.5, we can parametrize this surface by the parametrization $\mathbf{x}(u, v)$ given as

$$\mathbf{x}(u, v) = \left(u,\ v,\ \ln\left(\frac{\cos(u)}{\cos(v)}\right)\right)$$

The parameters u, v are defined such that $\cos(u)/\cos(v) > 0$.



(a) $R = 0.5$ (b) $R = 3$

Figure 3.6: Enneper's surface. This figure shows Enneper's surface in the region $\theta \in [0, 2\pi)$ and $r \in [0, R)$ as mentioned.

3.4 Enneper's Surface

Enneper's surface is a minimal surface, see for example [Do 16, p. 208]. It is parametrized by

$$\mathbf{x}(u,v) = (u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2), \qquad -\infty < u < \infty,\ -\infty < v < \infty \tag{3.24}$$

From the parametrization (3.24) we can see that Enneper's surface has component functions that are polynomials in u and v. Enneper's surface is shown in figure 3.6. Let us compute the mean curvature for Enneper's surface. We begin by computing the following vectors

$$\begin{aligned} \mathbf{x}_u &= (1 - u^2 + v^2,\ 2uv,\ 2u) \\ \mathbf{x}_v &= (2uv,\ 1 + u^2 - v^2,\ -2v) \end{aligned}$$

Then the second partial derivatives of the vectors are

$$\begin{aligned} \mathbf{x}_{uu} &= (-2u,\ 2v,\ 2) \\ \mathbf{x}_{uv} &= (2v,\ 2u,\ 0) \\ \mathbf{x}_{vv} &= (2u,\ -2v,\ -2) \end{aligned}$$

Thus we can find the coefficients of the first fundamental form E, F and G. They are given by

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = (1 + u^2 + v^2)^2 \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0 \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = (1 + u^2 + v^2)^2 \end{aligned}$$

We get that $E = G$ and $F = 0$, so this is an isothermal parametrization. Next we compute the quantity $EG - F^2$ as this will help in finding the curvatures.

$$EG - F^2 = (1 + u^2 + v^2)^2 \cdot (1 + u^2 + v^2)^2 - 0^2 = (1 + u^2 + v^2)^4 \tag{3.25}$$

$$\sqrt{EG - F^2} = \sqrt{(1 + u^2 + v^2)^4} = (1 + u^2 + v^2)^2 \tag{3.26}$$

To find the unit normal vector $\mathbf{N}$ we will have to find

$$\begin{aligned}\mathbf{x}_u \times \mathbf{x}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1-u^2+v^2 & 2uv & 2u \\ 2uv & 1-v^2+u^2 & -2v \end{vmatrix} \\ &= (-2u^3-2uv^2-2u, 2v^3+2u^2v+2v, 1-2u^2v^2-u^4-v^4) \\ &= \big(-2u(1+u^2+v^2),\ 2v(1+u^2+v^2),\ 1-(u^2+v^2)^2\big) \\ \|\mathbf{x}_u \times \mathbf{x}_v\| &= \sqrt{EG-F^2} = (1+u^2+v^2)^2\end{aligned}$$

The unit normal vector $\mathbf{N}$ is given by

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \left(\frac{-2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-(u^2+v^2)^2}{(1+u^2+v^2)^2}\right) \tag{3.27}$$

Now that we have the unit normal vector, we can find the coefficients of the second fundamental from, namely e, f and g.

$$\begin{aligned} e &= \langle \mathbf{N}, \mathbf{x}_{uu} \rangle = 2 \\ f &= \langle \mathbf{N}, \mathbf{x}_{uv} \rangle = 0 \\ g &= \langle \mathbf{N}, \mathbf{x}_{vv} \rangle = -2 \end{aligned}$$

Finally we can compute the curvatures. The Gaussian curvature K is

$$K = \frac{eg-f^2}{EG-F^2} = \frac{(2)(-2)-0^2}{(1+u^2+v^2)^4} = \frac{-4}{(1+u^2+v^2)^4} \tag{3.28}$$

The mean curvature H is

$$\begin{aligned} H = \frac{1}{2}\frac{eG-2fF+gE}{EG-F^2} &= \frac{1}{2}\frac{2(1+u^2+v^2)^2-2(0)(0)+(-2)(1+u^2+v^2)^2}{(1+u^2+v^2)^4} \\ &= \frac{1}{2}\frac{2(1+u^2+v^2)^2-2(1+u^2+v^2)^2}{(1+u^2+v^2)^4} = 0 \end{aligned} \tag{3.29}$$

The mean curvature is zero, thus the Enneper's surface is a minimal surface. The Gaussian curvature is also negative for all values of u and v. We saw that Enneper's surface has a parametrization which is in terms of polynomials. We can obtain this parametrization from the Weierstrass Enneper formulas in theorem 2.3.3. The functions $f(w)$ and $g(w)$ in the formulas in theorem 2.3.3 are given by [Opr00, p. 83]:

$$f(w) = 1, \quad g(w) = w$$

It will be convenient to find the parametrization in polar coordinates. We have that $w = u+iv$. Let $u = r\cos(\theta)$ and $v = r\sin(\theta)$. Then $w = r(\cos(\theta)+i\sin(\theta))$. Referring to (2.1) we can write w in polar form as

$$w = re^{i\theta} = r(\cos(\theta)+i\sin(\theta))$$

We compute the integrals and set $w = re^{i\theta}$ and then take the real part to find the parametrization in polar coordinates.

$$\int f(1-g^2)\ dw = \int (1-w^2)\ dw = w - \frac{w^3}{3} = re^{i\theta} - \frac{(re^{i\theta})^3}{3}$$

$$
\begin{aligned}
&= re^{i\theta} - \frac{r^3 e^{i3\theta}}{3} \\
x(r,\theta) &= \mathrm{Re} \int f(1-g^2)\ dw = r\cos(\theta) - \frac{r^3}{3}\cos(3\theta) \\
\int if(1+g^2)\ dw &= \int i(1+w^2)\ dw = i\Big[w + \frac{w^3}{3}\Big] = i\Big[re^{i\theta} + \frac{(re^{i\theta})^3}{3}\Big] \\
&= ire^{i\theta} + \frac{ir^3 e^{i3\theta}}{3} \\
y(r,\theta) &= \mathrm{Re} \int if(1+g^2)\ dw = -r\sin(\theta) - \frac{r^3}{3}\sin(3\theta) \\
\int 2fg\ dw &= \int 2w\ dw = 2\cdot\frac{w^2}{2} = w^2 = (re^{i\theta})^2 = r^2 e^{i2\theta} \\
z(r,\theta) &= \mathrm{Re} \int 2fg\ dw = r^2\cos(2\theta)
\end{aligned}
$$

Enneper's surface is parametrized in polar coordinates by the parametrization $\mathbf{x}(r,\theta)$ given as

$$
\mathbf{x}(r,\theta) = \Big(r\cos(\theta) - \frac{r^3}{3}\cos(3\theta),\ -r\sin(\theta) - \frac{r^3}{3}\sin(3\theta),\ r^2\cos(2\theta)\Big) \tag{3.30}
$$
$$
0 \le r < \infty,\ 0 \le \theta < 2\pi
$$

The figure 3.6 shows Enneper's surface given by the parametrization (3.30), which is in polar coordinates. We find the geometric properties for Enneper's surface in polar coordinates, as this will be useful later. Since this parametrization is obtained from the Weierstrass Enneper representation, we can compute the first fundamental form E and the Gaussian curvature K. Let $w = u + iv$. Then we note that

$$
w\ \overline{w} = (u+iv)(u-iv) = u^2 - i^2v^2 = u^2 - (-1)v^2 = u^2 + v^2
$$

We can find E and K in terms of $f(w)$ and $g(w)$. We wish to find E and K in polar coordinates, this means that $u^2 + v^2 = r^2$. Before we can compute E and K we will have to find the expressions for

$$
\begin{aligned}
|g|^2 &= g\cdot\overline{g} = (w)(\overline{w}) = w\ \overline{w} = u^2 + v^2 = r^2 \\
|g'|^2 &= g'\cdot\overline{g}' = 1 \\
|f|^2 &= f\cdot\overline{f} = 1
\end{aligned}
$$

Now we can find E from (2.5) as,

$$
E = |f|^2(1+|g|^2)^2 = 1\cdot(1+r^2)^2 = (1+r^2)^2 \tag{3.31}
$$

We can find K from (2.7) as,

$$
K = \frac{-4|g'|^2}{|f|^2(1+|g|^2)^4} = \frac{-4(1)}{1\cdot(1+r^2)^4} = \frac{-4}{(1+r^2)^4} \tag{3.32}
$$

We can also find the unit normal vector $\mathbf{N}$ in terms of the functions $f(w)$ and $g(w)$ by the expression given in (2.8). We compute the normal vector in polar coordinates. The formula for $\mathbf{N}$ in terms of $f(w)$ and $g(w)$ is

$$\mathbf{N} = \left(\frac{2\,\mathrm{Re}(g)}{1+|g|^2}, \frac{2\,\mathrm{Im}(g)}{1+|g|^2}, \frac{|g|^2-1}{1+|g|^2} \right)$$

We have that $g(w) = w = re^{i\theta} = r(\cos(\theta)+i\sin(\theta)) = r\cos(\theta)+ir\sin(\theta)$. Then

$$\mathrm{Re}(g) = r\cos(\theta), \quad \mathrm{Im}(g) = r\sin(\theta)$$

The unit normal vector $\mathbf{N}$ for Enneper's surface in polar coordinates is given by

$$\mathbf{N} = \left(\frac{2r\cos(\theta)}{1+r^2}, \frac{2r\sin(\theta)}{1+r^2}, \frac{r^2-1}{1+r^2} \right) \tag{3.33}$$

3.4.1 Self intersections

Enneper's surface has an interesting property. Let us take a look at Enneper's surface that is shown in figure 3.6. In this figure Enneper's surface is parametrized using polar coordinates, as in (3.30). Let us denote the radius by R.

The figure shows the surface for two different radiuses. Figure 3.6a shows the surface when the radius is $R = 0.5$. In figure 3.6b, the surface has a radius of $R = 3$. Note that the parameter $\theta \in [0, 2\pi)$ in both figures. From figures a and b, we can see that as the radius increases from $R = 0.5$ to $R = 3$, the shape of Enneper's surface changes dramatically. When $R = 0.5$, there are no self intersections, however as the radius begins to increase, Enneper's surface intersects itself. The Enneper surface is a self intersecting minimal surface.

This tells us that for some value of R in between $R = 0.5$ and $R = 3$, there is a value of R at which Enneper's surface intersects itself for the first time. Let us find the radius at which Enneper's surface intersects itself for the first time. The method used here to find this radius is based on the method presented in [Web18].

Self intersections occur in the z-plane. Consider the x-z plane. In this plane $y = 0$. To find the radius of intersection we will have to find when the y component of the parametrization (3.30) is equal to zero. Let us look closer at this. The y component from the parametrization in (3.30) is

$$y(r,\theta) = -r\sin(\theta) - \frac{r^3}{3}\sin(3\theta)$$

Hence we would like to find the minimum radius r at which $y(r,\theta) = 0$. We begin by rewriting the expression for $y(r,\theta)$ by finding the common denominator. This leads to

$$\begin{aligned} y(r,\theta) &= \frac{-3r\sin(\theta) - r^3\sin(3\theta)}{3} \\ &= \frac{-r}{3}(3\sin(\theta) + r^2\sin(3\theta)) \end{aligned}$$

We would like to find the minimum value of r such that

$$3\sin(\theta) + r^2\sin(3\theta) = 0$$

Let $P = r^2$. Then

$$\begin{aligned} 3\sin(\theta) + P\sin(3\theta) &= 0 \\ P\sin(3\theta) &= -3\sin(\theta) \\ P &= \frac{-3\sin(\theta)}{\sin(3\theta)} \end{aligned} \tag{3.34}$$

Let us denote by

$$P_1(\theta) = \frac{-3\sin(\theta)}{\sin(3\theta)} \tag{3.35}$$

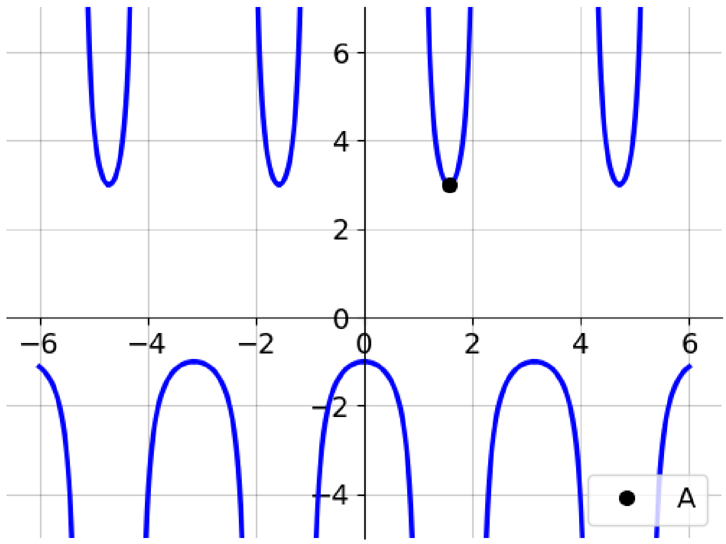


Figure 3.7: Graph of $P_1(\theta) = -3\sin(\theta)/\sin(3\theta)$. The point A is the first positive minimum point of $P_1(\theta)$. It is given by $A = (1.570796, 3)$

To find the radius r of intersection for the Enneper's surface, we wish to find the minimum value of θ for the function $P_1(\theta)$. We want to find the minimum point of the function $P_1(\theta)$. Since r is a radius, we will only consider $P_1(\theta) > 0$. This means that we want to find the first positive minimum point of $P_1(\theta)$. Let us denote the first positive minimum value of θ by θ_{min}. The graph of $P_1(\theta)$ is shown in figure 3.7. From the figure we can see that the first positive minimum point of $P_1(\theta)$ is the point A, which is $A = (1.570796, 3)$. Thus

$$\theta_{\text{min}} = 1.570796, \text{ and } P_1(\theta_{\text{min}}) = 3$$

We have found that $P_1(\theta_{\text{min}}) = 3$. Going back to the expression for P which was given in (3.34), we have found that the minimum value of P is

$$P = 3$$

Note that we had defined $P = r^2$. To find the radius r we have to find $(P)^{1/2}$ which gives us

$$\begin{aligned} r^2 &= 3 \\ r &= \sqrt{3} \approx 1.7321 \end{aligned}$$

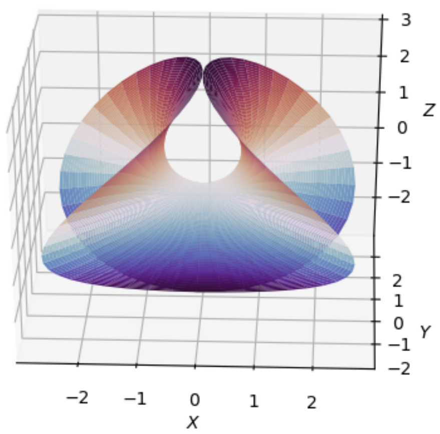


Figure 3.8: Enneper's surface with $R = \sqrt{3}$. This figure shows Enneper's surface in the region $r \in [0, \sqrt{3}]$ and $\theta \in [0, 2\pi)$.

Therefore the radius R at which the Enneper's surface intersects itself for the first time is

$$R = \sqrt{3} \approx 1.7321 \tag{3.36}$$

The Enneper surface with $R = \sqrt{3}$ is shown in figure 3.8. From the figure we can see that the Enneper's surface just intersects itself when $R = \sqrt{3}$. This means that for $R < \sqrt{3}$, Enneper's surface has no self intersections. Our result is consistent with the results of the following theorem.

Theorem 3.4.1. *[Nit89, p. 83] Enneper's surface does not have any self intersections for $R^2 < 3$, that is $R < \sqrt{3}$. This corresponds to $0 \leq r < \sqrt{3}$ in the parametrization* (3.30)*.*

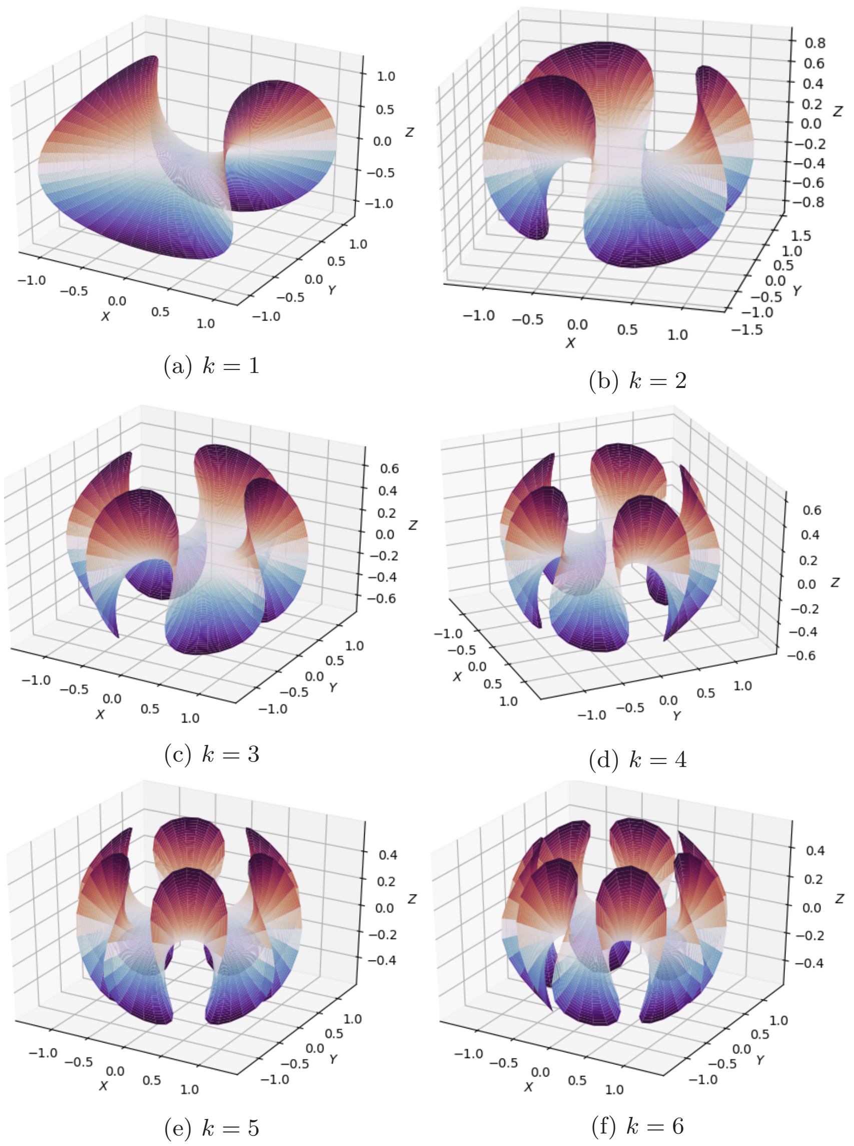
Therefore Enneper's surface has no self intersections for $R < \sqrt{3}$. It intersects itself for the first time when the radius is $R = \sqrt{3}$.

3.5 Higher Order Enneper Surfaces

The higher order Enneper surfaces of k^{th} order, are minimal surfaces that are defined through the Weierstrass Enneper representation formulas given in theorem 2.3.3, see [Opr00, p. 83]. The functions $f(w)$ and $g(w)$ in the formulas in theorem 2.3.3 are given by:

$$f(w) = 1, \quad g(w) = w^k$$

where k is the *degree or order* of the Enneper surface. When $k = 1$, we get the usual Enneper's surface parametrized by (3.30). We compute the integrals to find the parametrization. It will be convenient to find the parametrization in polar coordinates. We will compute the integrals and set $w = re^{i\theta}$. Then we will take the real parts to find the parametrization which will be in polar



(a) $k = 1$ (b) $k = 2$

(c) $k = 3$ (d) $k = 4$

(e) $k = 5$ (f) $k = 6$

Figure 3.9: Enneper surfaces of order k. This figure shows the higher order Enneper surfaces for $k = 1, 2, 3, 4, 5, 6$ in the region $r \in [0, 1.1)$ and $\theta \in [0, 2\pi)$.

coordinates. We begin by computing the x component

$$\begin{aligned}\int f(1-g^2)\ dw &= \int (1-w^{2k})\ dw = w - \frac{w^{2k+1}}{2k+1} = (re^{i\theta}) - \frac{(re^{i\theta})^{2k+1}}{2k+1} \\ &= re^{i\theta} - \frac{r^{2k+1}e^{i\theta(2k+1)}}{2k+1} \\ &= r\cos(\theta) + ir\sin(\theta) - \frac{r^{2k+1}}{2k+1}\Big(\cos((2k+1)\theta) \\ &+ i\sin((2k+1)\theta)\Big)\end{aligned}$$

Then taking the real parts

$$x(r,\theta) = \mathrm{Re}\int f(1-g^2)\ dw = r\cos(\theta) - \frac{r^{2k+1}}{2k+1}\cos((2k+1)\theta)$$

Similarly we compute the y component

$$\begin{aligned}\int if(1+g^2)\ dw &= \int i(1+w^{2k})\ dw = i\left[w + \frac{w^{2k+1}}{2k+1}\right] \\ &= i\left[re^{i\theta} + \frac{(re^{i\theta})^{2k+1}}{2k+1}\right] = i\left[re^{i\theta} + \frac{r^{2k+1}e^{i\theta(2k+1)}}{2k+1}\right] \\ &= i\Big[r\cos(\theta) + ir\sin(\theta) + \frac{r^{2k+1}}{2k+1}\Big(\cos((2k+1)\theta) \\ &+ i\sin((2k+1)\theta)\Big)\Big] \\ &= ir\cos(\theta) - r\sin(\theta) + \frac{r^{2k+1}}{2k+1}\Big(i\cos((2k+1)\theta) \\ &- \sin((2k+1)\theta)\Big)\end{aligned}$$

Then taking the real parts

$$y(r,\theta) = \mathrm{Re}\int if(1+g^2)\ dw = -r\sin(\theta) - \frac{r^{2k+1}}{2k+1}\sin((2k+1)\theta)$$

Finally we compute the z component

$$\begin{aligned}\int 2fg\ dw &= 2\int w^k\ dw = 2\left[\frac{w^{k+1}}{k+1}\right] = 2\left[\frac{(re^{i\theta})^{k+1}}{k+1}\right] = 2\frac{r^{k+1}e^{i\theta(k+1)}}{k+1} \\ &= \frac{2r^{k+1}}{k+1}\Big(\cos((k+1)\theta) + i\sin((k+1)\theta)\Big)\end{aligned}$$

Then taking the real parts

$$z(r,\theta) = \mathrm{Re}\int 2fg\ dw = \frac{2r^{k+1}}{k+1}\cos((k+1)\theta)$$

Therefore the higher order Enneper surfaces of order k are parametrized in polar coordinates by the parametrization $\mathbf{x}(r,\theta)$ given as

$$\mathbf{x}(r,\theta) = (x(r,\theta), y(r,\theta), z(r,\theta)), \tag{3.37}$$

$$0 \le r < \infty,\ 0 \le \theta < 2\pi$$

where the component functions $x(r,\theta), y(r,\theta)$ and $z(r,\theta)$ are given by

$$x(r,\theta) = r\cos(\theta) - \frac{r^{2k+1}}{2k+1}\cos((2k+1)\theta) \tag{3.38}$$

$$y(r,\theta) = -r\sin(\theta) - \frac{r^{2k+1}}{2k+1}\sin((2k+1)\theta) \tag{3.39}$$

$$z(r,\theta) = \frac{2r^{k+1}}{k+1}\cos((k+1)\theta) \tag{3.40}$$

The higher order Enneper surfaces are shown in figure 3.9 for $k = 1,2,3,4,5,6$. We find the geometric properties for the higher order Enneper surfaces. Since this parametrization is obtained from the Weierstrass Enneper representation, it is an isothermal parametrization. We can find the coefficient of the first fundamental form E, and the Gaussian curvature K in terms of $f(w)$ and $g(w)$. Note that $w = u + iv$, then

$$w\,\overline{w} = (u+iv)(u-iv) = u^2 - i^2v^2 = u^2 - (-1)v^2 = u^2 + v^2$$

We wish to find E and K in polar coordinates, this means that $u^2 + v^2 = r^2$. We first find the expressions for

$$\begin{aligned}
|g|^2 = g\cdot\overline{g} &= (w^k)(\overline{w}^k) = (w\,\overline{w})^k \\
&= (u^2+v^2)^k = (r^2)^k \\
&= r^{2k} \\
|g'|^2 = g'\cdot\overline{g}' &= (kw^{k-1})(k\overline{w}^{k-1}) = k^2\ w^{k-1}\overline{w}^{k-1} \\
&= k^2(w\,\overline{w})^{k-1} = k^2(u^2+v^2)^{k-1} = k^2(r^2)^{k-1} \\
&= k^2r^{2(k-1)} \\
&= k^2r^{2k-2} \\
|f|^2 = f\cdot\overline{f} &= 1
\end{aligned}$$

Now we can find E from (2.5) as,

$$E = |f|^2(1+|g|^2)^2 = 1\cdot(1+r^{2k})^2 = (1+r^{2k})^2 \tag{3.41}$$

We can find K from (2.7) as,

$$K = \frac{-4|g'|^2}{|f|^2(1+|g|^2)^4} = \frac{-4(k^2r^{2k-2})}{1\cdot(1+r^{2k})^4} = \frac{-4k^2\ r^{2k-2}}{(1+r^{2k})^4} \tag{3.42}$$

We can also find the unit normal vector $\mathbf{N}$ in terms of the functions $f(w)$ and $g(w)$ by the expression given in (2.8). We compute the normal vector in polar coordinates. The formula for $\mathbf{N}$ in terms of $f(w)$ and $g(w)$ is

$$\mathbf{N} = \left(\frac{2\,\mathrm{Re}(g)}{1+|g|^2},\ \frac{2\,\mathrm{Im}(g)}{1+|g|^2},\ \frac{|g|^2-1}{1+|g|^2}\right)$$

We have that $g(w) = w^k = (re^{i\theta})^k = r^k e^{ik\theta} = r^k(\cos(k\theta) + i\sin(k\theta)) = r^k\cos(k\theta) + ir^k\sin(k\theta)$. Then

$$\mathrm{Re}(g) = r^k\cos(k\theta), \quad \mathrm{Im}(g) = r^k\sin(k\theta)$$

The unit normal vector $\mathbf{N}$ for higher order Enneper surfaces of order k is given by

$$\mathbf{N} = \left(\frac{2r^k\cos(k\theta)}{1+r^{2k}},\ \frac{2r^k\sin(k\theta)}{1+r^{2k}},\ \frac{r^{2k}-1}{1+r^{2k}}\right) \tag{3.43}$$

3.6 Richmond's Surface

Richmond's minimal surface is a minimal surface that is defined through the Weierstrass Enneper representation formulas given in theorem 2.3.3, see [Opr00, p. 83]. The functions $f(w)$ and $g(w)$ in the formulas in theorem 2.3.3 are given by:

$$f(w) = w^2, \quad g(w) = \frac{1}{w^2}$$

We will find the parametrization in polar coordinates. We compute the integrals and let $w = re^{i\theta}$ and then we take the real parts.

$$\begin{aligned}
\int f(1-g^2)\ dw &= \int (w^2 - \frac{1}{w^2})\ dw = \frac{w^3}{3} + \frac{1}{w} = \frac{(re^{i\theta})^3}{3} + \frac{1}{(re^{i\theta})} \\
&= \frac{r^3e^{i3\theta}}{3} + \frac{1}{r}e^{-i\theta} \\
x(r,\theta) &= \mathrm{Re}\int f(1-g^2)\ dw = \frac{r^3}{3}\cos(3\theta) + \frac{1}{r}\cos(\theta) \\
\int if(1+g^2)\ dw &= \int i(w^2 + \frac{1}{w^2})\ dw = i(\frac{w^3}{3} - \frac{1}{w}) = i\frac{(re^{i\theta})^3}{3} - \frac{i}{re^{i\theta}} \\
&= \frac{ir^3e^{i3\theta}}{3} - \frac{i}{r}e^{-i\theta} \\
y(r,\theta) &= \mathrm{Re}\int if(1+g^2)\ dw = \frac{-r^3}{3}\sin(3\theta) - \frac{1}{r}\sin(\theta) \\
\int 2fg\ dw &= \int 2w^2\frac{1}{w^2}\ dw = \int 2\ dw = 2w = 2re^{i\theta} \\
z(r,\theta) &= \mathrm{Re}\int 2fg\ dw = 2r\cos(\theta)
\end{aligned}$$

Richmond's surface is parametrized in polar coordinates by the parametrization $\mathbf{x}(r,\theta)$ given as

$$\mathbf{x}(r,\theta) = \left(\frac{r^3}{3}\cos(3\theta) + \frac{1}{r}\cos(\theta),\ \frac{-r^3}{3}\sin(3\theta) - \frac{1}{r}\sin(\theta),\ 2r\cos(\theta)\right) \tag{3.44}$$
$$0 < r < \infty,\ 0 \le \theta < 2\pi$$

Thus we get to see another example of theorem 2.3.3. This shows us how the theorem is applied to find minimal surfaces. We started with two functions $f(w)$ and $g(w)$ that satisfy the requirements of theorem 2.3.3. We insert the functions into the formulas and obtain a parametrization for a minimal surface. In this case, a parametrization for Richmond's surface.

CHAPTER 4

Area Minimizing Property of Minimal Surfaces and The Area Functional

When we talk about minimal surfaces, we use the term *minimal*. However a minimal surface may not always have minimal surface area. We are interested in finding out when a minimal surface has minimal surface area. Let us take a closer look at the area minimizing property of minimal surfaces. In order to look at the problem of having minimal surface area we will consider the area functional. Let us begin by finding the area functional.

4.1 Variations of the Surface

The material presented on the variations of the surface is based on [Do 16, chapter 3.5B]. We will look at the notion of variations of a surface, which arise from the calculus of variations. Let $S \subset \mathbb{R}^3$ be a regular parametrized surface $\mathbf{x}(u,v)$, where $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Let $D \subset U$ be a bounded domain. We denote the boundary of D by ∂D. Recall that the closure of D, which is denoted by $\overline{D}$, is the union of D and the boundary of ∂D. Using set notation we can write this as $\overline{D} = D \cup \partial D$. Thus we let $\overline{D}$ be the closure of D. Let $h : \overline{D} \to \mathbb{R}$ be a differentiable function. We introduce a parameter $t \in \mathbb{R}$. Then for small values of t we define the map $\mathbf{x}^t(u,v)$ by

$$\mathbf{x}^t(u,v) = \mathbf{x}(u,v) + th(u,v)\mathbf{N}(u,v) \tag{4.1}$$

The map $\mathbf{x}^t(u,v)$ perturbs the surface $\mathbf{x}(u,v)$ in the direction of the normal vector $\mathbf{N}$, it gives us *normal variations* of $\mathbf{x}(u,v)$. The variational surfaces are created by $t\ h(u,v)$ as t varies in a small interval in $\mathbb{R}$. When $t = 0$, we get back our original surface $\mathbf{x}(u,v)$.

We would like to see whether the surface area of $\mathbf{x}^t$ is always greater than the surface area of $\mathbf{x}$ in the case that $\mathbf{x}$ is a minimal surface, in which case $\mathbf{x}$ has mean curvature $H = 0$ everywhere. From (1.10) the surface areas of $\mathbf{x}$ and $\mathbf{x}^t$ over the domain D are given respectively by

$$A = \int_D \|\mathbf{x}_u \times \mathbf{x}_v\|\, du\, dv = \int_D \sqrt{EG - F^2}\ du\, dv$$

$$A(t) = \int_D \|\mathbf{x}^t_u \times \mathbf{x}^t_v\|\, du\, dv = \int_D \sqrt{E^tG^t - (F^t)^2}\ du\, dv$$

Note that when $t = 0$, we get $A(0) = A$. If $t = 0$ is a critical point for the area functional $A(t)$, then we have that $A'(0) = 0$. For convenience we express the integrals above as

$$A = \int_D B^{1/2}\ du\,dv, \quad A(t) = \int_D (B^t)^{1/2}\ du\,dv \tag{4.2}$$

where B and B^t are given by

$$B = EG - F^2, \qquad B^t = E^tG^t - (F^t)^2 \tag{4.3}$$

The terms E^t, F^t, G^t are the coefficients of the first fundamental form of the map $\mathbf{x}^t$ in (4.1). They are defined in analogy to E, F, G, replacing $\mathbf{x}$ by $\mathbf{x}^t$. We will study the first and second derivatives of the area functional $A(t)$ at $t = 0$.

4.2 First Derivative of The Area Functional

We want to compute the first derivative of the area functional $A'(t)$ at $t = 0$. This is shown in [Do 16, p. 201], we will reproduce the proof here and consider the second derivative as well. The area functional is given in (4.2).

Lemma 4.2.1.

$$A'(0) = -2\int_D B^{1/2} hH\ du\,dv$$

If $\mathbf{x}$ is a minimal surface, that is, it has $H = 0$, then $A'(0) = 0$. Thus $t = 0$ is a critical point of the area functional $A(t)$.

Proof. We begin by differentiating $\mathbf{x}^t(u, v)$ given in (4.1) with respect to t. By using the chain rule we get that

$$\begin{aligned}
\mathbf{x}_u^t &= \frac{\partial \mathbf{x}^t}{\partial u} = \mathbf{x}_u + th_u\mathbf{N} + th\mathbf{N}_u \\
\mathbf{x}_v^t &= \frac{\partial \mathbf{x}^t}{\partial v} = \mathbf{x}_v + th_v\mathbf{N} + th\mathbf{N}_v
\end{aligned}$$

The coefficients of the first fundamental form are calculated as

$$\begin{aligned}
E^t &= \langle \mathbf{x}_u^t, \mathbf{x}_u^t \rangle \\
&= \underbrace{\langle \mathbf{x}_u, \mathbf{x}_u \rangle}_{E} + 2th \underbrace{\langle \mathbf{x}_u, \mathbf{N}_u \rangle}_{-e} + 2th_u \underbrace{\langle \mathbf{x}_u, \mathbf{N} \rangle}_{0} + t^2h_u^2 \underbrace{\langle \mathbf{N}, \mathbf{N} \rangle}_{1} + 2t^2hh_u \underbrace{\langle \mathbf{N}, \mathbf{N}_u \rangle}_{0} \\
&\quad + t^2h^2\langle \mathbf{N}_u, \mathbf{N}_u \rangle \\
&= E - 2eth + t^2h^2\langle \mathbf{N}_u, \mathbf{N}_u \rangle + t^2h_u^2 \\
F^t &= \langle \mathbf{x}_u^t, \mathbf{x}_v^t \rangle \\
&= \underbrace{\langle \mathbf{x}_u, \mathbf{x}_v \rangle}_{F} + th_v \underbrace{\langle \mathbf{x}_u, \mathbf{N} \rangle}_{0} + th \underbrace{\langle \mathbf{x}_u, \mathbf{N}_v \rangle}_{-f} + th_u \underbrace{\langle \mathbf{x}_v, \mathbf{N} \rangle}_{0} + th \underbrace{\langle \mathbf{x}_v, \mathbf{N}_u \rangle}_{-f} \\
&\quad + t^2h_uh_v \underbrace{\langle \mathbf{N}, \mathbf{N} \rangle}_{1} + t^2hh_u \underbrace{\langle \mathbf{N}, \mathbf{N}_v \rangle}_{0} + t^2hh_v \underbrace{\langle \mathbf{N}, \mathbf{N}_u \rangle}_{0} + t^2h^2\langle \mathbf{N}_u, \mathbf{N}_v \rangle \\
&= F - 2fth + t^2h^2\langle \mathbf{N}_u, \mathbf{N}_v \rangle + t^2h_uh_v
\end{aligned}$$

$$
\begin{aligned}
G^t &= \langle \mathbf{x}_v^t, \mathbf{x}_v^t \rangle \\
&= \underbrace{\langle \mathbf{x}_v, \mathbf{x}_v \rangle}_{G} + 2th_v \underbrace{\langle \mathbf{x}_v, \mathbf{N} \rangle}_{0} + 2th \underbrace{\langle \mathbf{x}_v, \mathbf{N}_v \rangle}_{-g} + 2t^2 hh_v \underbrace{\langle \mathbf{N}, \mathbf{N}_v \rangle}_{0} + t^2h^2 \langle \mathbf{N}_v, \mathbf{N}_v \rangle \\
&\quad + t^2 h_v^2 \underbrace{\langle \mathbf{N}, \mathbf{N} \rangle}_{1} \\
&= G - 2gth + t^2h^2\langle \mathbf{N}_v, \mathbf{N}_v \rangle + t^2h_v^2
\end{aligned}
$$

Thus the coefficients of the first fundamental from E^t, F^t, G^t are

$$
\begin{aligned}
E^t &= E - 2eth + t^2h^2\langle \mathbf{N}_u, \mathbf{N}_u \rangle + t^2h_u^2 \\
F^t &= F - 2fth + t^2h^2\langle \mathbf{N}_u, \mathbf{N}_v \rangle + t^2h_uh_v \\
G^t &= G - 2gth + t^2h^2\langle \mathbf{N}_v, \mathbf{N}_v \rangle + t^2h_v^2
\end{aligned} \tag{4.4}
$$

Then we obtain

$$
E^tG^t = EG - 2th(Eg + Ge) + t^2R_0 + O(t^3)
$$

where

$$
R_0 = Eh^2\langle \mathbf{N}_v, \mathbf{N}_v \rangle + Eh_v^2 + Gh^2\langle \mathbf{N}_u, \mathbf{N}_u \rangle + Gh_u^2 + 4h^2eg \tag{4.5}
$$

and

$$
(F^t)^2 = F^2 - 4thFf + t^2R_1 + O(t^3)
$$

where

$$
R_1 = 2Fh^2\langle \mathbf{N}_u, \mathbf{N}_v \rangle + 2Fh_uh_v + 4h^2f^2 \tag{4.6}
$$

Therefore we compute the expression for B^t as

$$
\begin{aligned}
B^t = E^tG^t - (F^t)^2 &= EG - F^2 - 2th(Eg - 2Ff + Ge) + t^2(R_0 - R_1) + O(t^3) \\
&= EG - F^2 - 2th(Eg - 2Ff + Ge) + t^2R + O(t^3)
\end{aligned}
$$

where we have $R = R_0 - R_1$. Furthermore by using the definition of the mean curvature H as mentioned in definition 1.4.4, we obtain that

$$
\begin{aligned}
B^t &= (EG - F^2)(1 - 4thH) + t^2R + O(t^3) \\
&= B(1 - 4thH) + t^2R + O(t^3) \\
&= B - 4tBhH + t^2R + O(t^3)
\end{aligned} \tag{4.7}
$$

Now differentiating through the integral sign in the definition of $A(t)$ as mentioned in (4.2) gives

$$
A'(t) = \frac{1}{2}\int_D (B^t)^{-1/2} \frac{d}{dt}(B^t)\; du\,dv, \tag{4.8}
$$

hence from (4.7) we compute

$$
\frac{d}{dt}(B^t) = -4BhH + 2tR + O(t^2) \tag{4.9}
$$

Then letting $t = 0$ gives

$$\frac{d}{dt}(B^t)\Big|_{t=0} = -4BhH, \qquad (B^t)^{-1/2}\Big|_{t=0} = B^{-1/2}$$

Therefore $A'(0)$ is

$$\begin{aligned} A'(0) &= \frac{1}{2}\int_D B^{-1/2}(-4BhH)\ du\,dv \\ &= -2\int_D B^{1/2}hH\ du\,dv \end{aligned}$$

This is what we wanted to proof. ■

This lemma shows us that $t = 0$ is a critical point for $A(t)$ when $\mathbf{x}$ has zero mean curvature. However it does not tell us anything about the type of critical point. The critical point could be a maximum, minimum or even a saddle point of $A(t)$. As we concerned with having minimum surface area, we would like to see if $t = 0$ is a minimum point of $A(t)$. To see this we consider the second derivative of the area functional.

4.3 Second Derivative of The Area Functional

We want to compute the second derivative of the area functional $A''(t)$. We will express $A''(t)$ in terms of Gaussian curvature K of $\mathbf{x}$. From definition 1.4.4 we know that in local u, v coordinates K can be expressed as,

$$K = \frac{eg - f^2}{EG - F^2}$$

and that K is the product of the two principal curvatures $K = k_1k_2$. The area functional is given in (4.2).

Lemma 4.3.1. *If $\mathbf{x}$ is a minimal surface, that is, it has $H = 0$, then*

$$A''(0) = \int_D \Big(2B^{1/2}Kh^2 + B^{-1/2}(\nabla h)^T M(\nabla h)\Big)\ du\,dv$$

where $\nabla h = [h_u, h_v]^T$ is the gradient of h, and M is given by the matrix

$$M = \begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$$

We will prove this lemma in two steps. The first step is to show the following lemma.

Lemma 4.3.2.

$$A''(0) = \int_D B^{-1/2}\Big(Sh^2 + (\nabla h)^T M(\nabla h)\Big)\ du\,dv \tag{4.10}$$

where the quantity S is given by,

$$S = -4BH^2 + E\langle \mathbf{N}_v, \mathbf{N}_v\rangle - 2F\langle \mathbf{N}_u, \mathbf{N}_v\rangle + G\langle \mathbf{N}_u, \mathbf{N}_u\rangle + 4(eg - f^2) \tag{4.11}$$

Proof. We begin by differentiating $A'(t)$, given by (4.8) with respect to t. This gives us the following

$$\begin{aligned} A''(t) &= \frac{1}{2}\int_D \left(\frac{d}{dt}\Big((B^t)^{-1/2}\Big)\frac{d}{dt}(B^t) + (B^t)^{-1/2}\frac{d^2}{dt^2}(B^t)\right)\, du\, dv \\ &= \frac{1}{2}\int_D \left(\frac{-1}{2}(B^t)^{-3/2}\Big(\frac{d}{dt}(B^t)\Big)^2 + (B^t)^{-1/2}\frac{d^2}{dt^2}(B^t)\right)\, du\, dv \end{aligned}$$

Now from (4.9) we compute

$$\frac{d^2}{dt^2}(B^t) = 2R + O(t).$$

Then letting $t = 0$ gives

$$\begin{aligned} (B^t)^{-3/2}\Big|_{t=0} &= B^{-3/2}, & \Big(\frac{d}{dt}(B^t)\Big)^2\Big|_{t=0} &= 16B^2h^2H^2 \\ (B^t)^{-1/2}\Big|_{t=0} &= B^{-1/2}, & \frac{d^2}{dt^2}(B^t)\Big|_{t=0} &= 2R \end{aligned}$$

Therefore $A''(0)$ is

$$\begin{aligned} A''(0) &= \frac{1}{2}\int_D \Big(-8B^{1/2}h^2H^2 + 2B^{-1/2}R\Big)\, du\, dv \\ &= \int_D \Big(-4B^{1/2}h^2H^2 + B^{-1/2}R\Big)\, du\, dv \\ &= \int_D B^{-1/2}(-4BH^2h^2 + R)\, du\, dv \end{aligned}$$

From (4.5) and (4.6), we get that

$$\begin{aligned} R &= R_0 - R_1 \\ &= h^2(E\langle \mathbf{N}_v, \mathbf{N}_v\rangle - 2F\langle \mathbf{N}_u, \mathbf{N}_v\rangle + G\langle \mathbf{N}_u, \mathbf{N}_u\rangle + 4eg - 4f^2) \\ &+ Eh_v^2 - 2Fh_uh_v + Gh_u^2 \end{aligned}$$

Let $Q = Eh_v^2 - 2Fh_uh_v + Gh_u^2$. Then we notice that Q is a quadratic form in the variables h_u and h_v. This means that we can write

$$\begin{aligned} Q &= \begin{bmatrix} h_u & h_v \end{bmatrix} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} h_u \\ h_v \end{bmatrix} \\ &= (\nabla h)^T M (\nabla h) \end{aligned}$$

where $\nabla h = [h_u, h_v]^T$ and the matrix of the quadratic form M is given by

$$M = \begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$$

Using this we see that R becomes

$$\begin{aligned} R &= h^2\Big(E\langle \mathbf{N}_v, \mathbf{N}_v\rangle - 2F\langle \mathbf{N}_u, \mathbf{N}_v\rangle + G\langle \mathbf{N}_u, \mathbf{N}_u\rangle + 4(eg - f^2)\Big) \\ &+ (\nabla h)^T M (\nabla h) \end{aligned}$$

Thus the expression for $A''(0)$ is shown, where R is given as above. ■

The second step in the proof is to show the following identity, which is stated in the lemma below.

Lemma 4.3.3.

$$E\langle \mathbf{N}_v, \mathbf{N}_v\rangle - 2F\langle \mathbf{N}_u, \mathbf{N}_v\rangle + G\langle \mathbf{N}_u, \mathbf{N}_u\rangle + 2(eg - f^2) = 4BH^2$$

Then substituting this identity into the expression for S in (4.11) shows that

$$S = -4BH^2 + 4BH^2 + 2(eg - f^2) = 2(eg - f^2) = 2BK$$

This gives the result and concludes the proof for lemma 4.3.1. It remains to show the proof for lemma 4.3.3. Thus we now show the proof of lemma 4.3.3.

Proof. Recall that from the formulas given in (1.13) we can write $\mathbf{N}_u$ and $\mathbf{N}_v$ in terms of $\mathbf{x}_u$ and $\mathbf{x}_v$ as

$$\begin{aligned} \mathbf{N}_u &= a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v \\ \mathbf{N}_v &= a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v \end{aligned} \tag{4.12}$$

In section 1.4, we expressed e, f, g in terms of the coefficients of the first fundamental form E, F, G by the following equations

$$\begin{aligned} -e &= \langle \mathbf{N}_u, \mathbf{x}_u\rangle = a_{11}E + a_{21}F, \\ -f &= \langle \mathbf{N}_u, \mathbf{x}_v\rangle = a_{11}F + a_{21}G \\ -f &= \langle \mathbf{N}_v, \mathbf{x}_u\rangle = a_{12}E + a_{22}F, \\ -g &= \langle \mathbf{N}_v, \mathbf{x}_v\rangle = a_{12}F + a_{22}G, \end{aligned}$$

We can express these equations in matrix form as

$$-\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

Let us denote the unknown matrix by $\mathbf{A} = (a_{ij})_{ij=1,2}$. In section 1.4 we found the matrix $\mathbf{A}$. We found that

$$\begin{aligned} \mathbf{A} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} &= -\begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \\ &= -\frac{1}{EG - F^2} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \\ &= -\frac{1}{EG - F^2} \begin{bmatrix} eG - fF & -eF + fE \\ fG - gF & -fF + gE \end{bmatrix} \end{aligned}$$

The trace of the matrix $\mathbf{A}$ is

$$tr(\mathbf{A}) = a_{11} + a_{22} = -\frac{eG - 2fF + gE}{EG - F^2} = -2H.$$

Due to definition 1.4.2 we can express K as

$$det(\mathbf{A}) = a_{11}a_{22} - a_{21}a_{12} = K$$

We also note that

$$\mathbf{N}_u \times \mathbf{N}_v = a_{11}a_{22} \cdot \mathbf{x}_u \times \mathbf{x}_v + a_{21}a_{12} \cdot \mathbf{x}_v \times \mathbf{x}_u$$

$$
\begin{aligned}
&= a_{11}a_{22} \cdot \mathbf{x}_u \times \mathbf{x}_v - a_{21}a_{12} \cdot \mathbf{x}_u \times \mathbf{x}_v \\
&= (a_{11}a_{22} - a_{21}a_{12}) \cdot \mathbf{x}_u \times \mathbf{x}_v = det(\mathbf{A}) \cdot \mathbf{x}_u \times \mathbf{x}_v \\
&= K \cdot \mathbf{x}_u \times \mathbf{x}_v
\end{aligned}
$$

Now we let

$$
T = E\langle \mathbf{N}_v, \mathbf{N}_v \rangle - 2F\langle \mathbf{N}_u, \mathbf{N}_v \rangle + G\langle \mathbf{N}_u, \mathbf{N}_u \rangle + 2(eg - f^2)
$$

Then by using the definitions of E, F, G given by definition 1.3.2, and the definitions of e, f, g given by definition 1.4.3, we rewrite T as

$$
\begin{aligned}
T = &\langle \mathbf{x}_u, \mathbf{x}_u \rangle\langle \mathbf{N}_v, \mathbf{N}_v \rangle - 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle\langle \mathbf{N}_u, \mathbf{N}_v \rangle + \langle \mathbf{x}_v, \mathbf{x}_v \rangle\langle \mathbf{N}_u, \mathbf{N}_u \rangle \\
&+ 2\langle \mathbf{x}_u, \mathbf{N}_u \rangle\langle \mathbf{x}_v, \mathbf{N}_v \rangle - \langle \mathbf{x}_u, \mathbf{N}_v \rangle^2 - \langle \mathbf{x}_v, \mathbf{N}_u \rangle^2
\end{aligned}
$$

Due to Lagrange's identity (1.7), we get $\langle \mathbf{x}_u, \mathbf{x}_u \rangle\langle \mathbf{N}_v, \mathbf{N}_v \rangle - \langle \mathbf{x}_u, \mathbf{N}_v \rangle^2 = \|\mathbf{x}_u \times \mathbf{N}_v\|^2$ and $\langle \mathbf{x}_v, \mathbf{x}_v \rangle\langle \mathbf{N}_u, \mathbf{N}_u \rangle - \langle \mathbf{x}_v, \mathbf{N}_u \rangle^2 = \|\mathbf{x}_v \times \mathbf{N}_u\|^2$. Then inserting this into the expression for T gives

$$
\begin{aligned}
T = &\|\mathbf{x}_u \times \mathbf{N}_v\|^2 + \|\mathbf{x}_v \times \mathbf{N}_u\|^2 \\
&- 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle\langle \mathbf{N}_u, \mathbf{N}_v \rangle + 2\langle \mathbf{x}_u, \mathbf{N}_u \rangle\langle \mathbf{x}_v, \mathbf{N}_v \rangle
\end{aligned}
$$

Next we will use the Binet-Cauchy identity. The *Binet-Cauchy identity* is the identity that is given by

$$
\langle \mathbf{a}, \mathbf{c} \rangle\langle \mathbf{b}, \mathbf{d} \rangle - \langle \mathbf{b}, \mathbf{c} \rangle\langle \mathbf{a}, \mathbf{d} \rangle = \langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d} \rangle \tag{4.13}
$$

Using the Binet-Cauchy identity with $\mathbf{a} = \mathbf{x}_u, \mathbf{b} = \mathbf{N}_v, \mathbf{c} = \mathbf{x}_v$ and $\mathbf{d} = \mathbf{N}_u$ gives us the following

$$
\langle \mathbf{x}_u, \mathbf{x}_v \rangle\langle \mathbf{N}_v, \mathbf{N}_u \rangle - \langle \mathbf{N}_v, \mathbf{x}_v \rangle\langle \mathbf{x}_u, \mathbf{N}_u \rangle = \langle \mathbf{x}_u \times \mathbf{N}_v, \mathbf{x}_v \times \mathbf{N}_u \rangle
$$

Then inserting this into the expression for T gives us

$$
\begin{aligned}
T &= \|\mathbf{x}_u \times \mathbf{N}_v\|^2 + \|\mathbf{x}_v \times \mathbf{N}_u\|^2 - 2\langle \mathbf{x}_u \times \mathbf{N}_v, \mathbf{x}_v \times \mathbf{N}_u \rangle \\
&= \|\mathbf{x}_u \times \mathbf{N}_v - \mathbf{x}_v \times \mathbf{N}_u\|^2
\end{aligned}
$$

We substitute the formulas for $\mathbf{N}_u$ and $\mathbf{N}_v$ that were mentioned in (4.12) into the expression for T above. We also use (1.8). This then leads to

$$
\begin{aligned}
T = (a_{22} + a_{11})^2\|\mathbf{x}_u \times \mathbf{x}_v\|^2 &= (-2H)^2(EG - F^2) \\
&= 4H^2(EG - F^2) = 4H^2B
\end{aligned}
$$

This is the identity that we wanted to prove. ■

It is possible to further simplify the formula for $A''(0)$ given in lemma 4.3.1. The simplification occurs in the case of $\mathbf{x}(u, v)$ having an isothermal parametrization, that is $E = G$ and $F = 0$. In this case we get that

$$
\begin{aligned}
&2B^{1/2}Kh^2 = 2EKh^2 \\
&B^{-1/2}(\nabla h)^T M(\nabla h) = h_u^2 + h_v^2 = \|\nabla h\|^2
\end{aligned}
$$

This shows us the advantage of having an isothermal parametrization for a minimal surface. We get a much simpler calculation due to this type of parametrization. Therefore if $\mathbf{x}$ is parametrized by isothermal coordinates, then

$$A''(0) = \int_D \Big(2EKh^2 + \|\nabla h\|^2\Big)\, du\, dv \tag{4.14}$$

Let us compute the expression for $A''(0)$ using some specific examples of minimal surfaces $\mathbf{x}(u, v)$.

4.3.1 The catenoid

We compute $A''(0)$ for the catenoid. Instead of using the formula in lemma 4.3.1, let us see how we can find $A''(0)$ from the area functional $A(t)$. We have the same premise as before. We define the variational surface by the map $\mathbf{x}^t(u, v)$ which is given in (4.1) as

$$\mathbf{x}^t(u, v) = \mathbf{x}(u, v) + th(u, v)\mathbf{N}(u, v)$$

The minimal surface $\mathbf{x}(u, v)$ is the catenoid which is parametrized by $\mathbf{x}(u, v)$ given in (3.3). In addition $t, h(u, v)$ and $\mathbf{N}(u, v)$ in the map $\mathbf{x}^t$ are defined as before. We begin by finding B^t for the catenoid. We first compute E^t, F^t and G^t, which were given in (4.4). We have computed all the necessary quantities to compute E^t, F^t and G^t for the catenoid in section 3.1 for the parametrization (3.3). From section 3.1 we get for the catenoid that

$$\begin{aligned} E^t &= \cosh^2(u) + 2th + t^2h^2\left(\frac{1}{\cosh^2(u)}\right) + t^2h_u^2 \\ G^t &= \cosh^2(u) - 2th + t^2h^2\left(\frac{1}{\cosh^2(u)}\right) + t^2h_v^2 \\ F^t &= t^2h_uh_v \end{aligned}$$

Then we have

$$\begin{aligned} E^tG^t &= \cosh^4(u) + t^2[-2h^2 + h_v^2\cosh^2(u) + h_u^2\cosh^2(u)] + O(t^3) \\ (F^t)^2 &= (t^2h_uh_v)^2 = t^4h_u^4h_v^4 = O(t^4) \end{aligned}$$

Hence $B^t = E^tG^t - (F^t)^2$ is,

$$B^t = \cosh^4(u) + t^2[-2h^2 + h_v^2\cosh^2(u) + h_u^2\cosh^2(u)] + O(t^3) \tag{4.15}$$

Thus the area functional and its derivatives are

$$\begin{aligned} A(t) &= \int_D (B^t)^{1/2}\, du\, dv \\ A'(t) &= \frac{1}{2}\int_D (B^t)^{-1/2}\frac{d}{dt}(B^t)\, dudv \\ A''(t) &= \frac{1}{2}\int_D \Big(\frac{-1}{2}(B^t)^{-3/2}\Big(\frac{d}{dt}(B^t)\Big)^2 + (B^t)^{-1/2}\frac{d^2}{dt^2}(B^t)\Big)\, dudv \end{aligned}$$

Now we compute the necessary quantities where B^t is given by (4.15),

$$\frac{d}{dt}(B^t) = 2t[-2h^2 + h_v^2\cosh^2(u) + h_u^2\cosh^2(u)] + O(t^2)$$

$$\frac{d^2}{dt^2}(B^t) = 2[-2h^2 + h_v^2 \cosh^2(u) + h_u^2 \cosh^2(u)] + O(t)$$

Then letting $t = 0$ we get that,

$$\begin{aligned}
\frac{d}{dt}(B^t)|_{t=0} &= 0 \\
\frac{d^2}{dt^2}(B^t)|_{t=0} &= 2[-2h^2 + h_v^2 \cosh^2(u) + h_u^2 \cosh^2(u)] \\
B^t|_{t=0} &= \cosh^4(u)
\end{aligned}$$

Finally we are in a position to compute the first and second derivative of $A(t)$ at $t = 0$. The first derivative is

$$A'(0) = \frac{1}{2}\int_D (\cosh^4(u))^{-1/2} \cdot 0 \; dudv = 0$$

The second derivative is

$$\begin{aligned}
A''(0) = \frac{1}{2}\int_D \Big(& \frac{-1}{2}(\cosh^4(u))^{-3/2} \cdot 0 \\
& + (\cosh^4(u))^{-1/2} \cdot (2[-2h^2 + h_v^2 \cosh^2(u) + h_u^2 \cosh^2(u)])\Big) \; dudv \\
= & \frac{1}{2}\int_D \Big(\cosh(u)^{-2} \cdot (2[-2h^2 + h_v^2 \cosh^2(u) + h_u^2 \cosh^2(u)])\Big) \; dudv \\
= & \int_D \Big(\frac{-2h^2 + h_v^2 \cosh^2(u) + h_u^2 \cosh^2(u)}{\cosh^2(u)}\Big) \; dudv \\
= & \int_D \Big(\frac{-2h^2}{\cosh^2(u)} + h_v^2 + h_u^2 \Big) \; dudv
\end{aligned}$$

For the catenoid we have found that $A''(0)$ is

$$A''(0) = \int_D \Big(\frac{-2h^2}{\cosh^2(u)} + h_v^2 + h_u^2 \Big) \; dudv \tag{4.16}$$

4.3.2 Enneper's surface

We compute $A''(0)$ for Enneper's surface. We will find an expression for $A''(0)$ directly by using the formula in lemma 4.3.1. From the lemma we have that

$$A''(0) = \int_D \Big(2B^{1/2} K h^2 + B^{-1/2} (\nabla h)^T M (\nabla h) \Big) \, du \, dv$$

Enneper's surface is parametrized by $\mathbf{x}(u, v)$ given in (3.24). We refer to section 3.4 for the quantities needed to find $A''(0)$ for Enneper's surface. Note that $B^{1/2} = (EG - F^2)^{1/2}$. Then from section 3.4 we find that

$$B^{1/2} = (1 + u^2 + v^2)^2, \quad B^{-1/2} = \frac{1}{(1 + u^2 + v^2)^2}, \quad K = \frac{-4}{(1 + u^2 + v^2)^4},$$

$$\begin{aligned}
(\nabla h)^T M (\nabla h) &= \begin{bmatrix} h_u & h_v \end{bmatrix} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} h_u \\ h_v \end{bmatrix} \\
&= \begin{bmatrix} h_u & h_v \end{bmatrix} \begin{bmatrix} (1 + u^2 + v^2)^2 & 0 \\ 0 & (1 + u^2 + v^2)^2 \end{bmatrix} \begin{bmatrix} h_u \\ h_v \end{bmatrix} \\
&= (1 + u^2 + v^2)^2 (h_u^2 + h_v^2)
\end{aligned}$$

Thus we find that

$$2B^{1/2}K = \frac{-4 \cdot 2(1+u^2+v^2)^2}{(1+u^2+v^2)^4} = \frac{-8}{(1+u^2+v^2)^2}$$

$$B^{-1/2}(\nabla h)^T M(\nabla h) = \frac{(1+u^2+v^2)^2(h_u^2+h_v^2)}{(1+u^2+v^2)^2} = h_u^2 + h_v^2$$

We insert these calculations into the formula for $A''(0)$. For Enneper's surface we find that

$$A''(0) = \int_D \left(\frac{-8h^2}{(1+u^2+v^2)^2} + h_u^2 + h_v^2 \right) du\, dv \tag{4.17}$$

From section 3.4 we saw that Enneper's surface parametrized by (3.24) is an isothermal parametrization. This means we can use the formula (4.14) to find an expression for $A''(0)$. Let us demonstrate a use of the formula (4.14) as well.

$$A''(0) = \int_D \left(2EKh^2 + \|\nabla h\|^2 \right) du\, dv$$

From section 3.4 we find that

$$E = (1+u^2+v^2)^2$$

Thus we find that

$$2EK = \frac{-4 \cdot 2(1+u^2+v^2)^2}{(1+u^2+v^2)^4} = \frac{-8}{(1+u^2+v^2)^2}$$

Inserting this into the formula for $A''(0)$ gives us

$$A''(0) = \int_D \left(\frac{-8h^2}{(1+u^2+v^2)^2} + \|\nabla h\|^2 \right) du\, dv \tag{4.18}$$

Thus we can see that we get the same expression for $A''(0)$ using the formula in lemma 4.3.1 and the formula (4.14). Thus the simplification in formula (4.14) is correct when we have an isothermal parametrization.

4.4 The Gauss Map for Minimal Surfaces

We will look at the Gauss map of minimal surfaces. This will be useful when looking at the concept of stability of minimal surfaces. We will be looking at minimal surfaces locally, thus we will look at the Gauss map locally. Let $\mathbf{x}(u, v)$ be a minimal surface, where $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Let $D \subset U$ be a bounded domain. Let p be a point in D, that is $p = (u, v) \in D$. From the definition 1.2.1, the Gauss map $\mathbf{N}$ is defined by the unit normal vector which is given by

$$\mathbf{N}(u, v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}(u, v)$$

at a point p. The Gauss map is defined from $\mathbf{N} : D \to S^2$, where D is a local region of the minimal surface $\mathbf{x}$ and S^2 is the unit sphere.

If the Gauss map is one-to-one, we have a formula to calculate the surface area of the image of the Gauss map. From [Fan96, p. 96] this formula is given by

$$Area(\mathbf{N}(D)) = -\int_D K\ dA \tag{4.19}$$

where the integral $\int_D K\ dA$ is the total Gaussian curvature given by the definition 1.15. If the Gauss map is one-to-one in the region D, then the negative of the total Gaussian curvature $-\int_D K\ dA$ allows us to calculate the surface area of the image of the Gauss map. It is not so often a surface has a one-to-one Gauss map. Things become much simpler for the case of two minimal surfaces. These two minimal surfaces are described in the next theorems.

Theorem 4.4.1. *[Opr00, p. 95] The Gauss map $\mathbf{N} : D \to S^2$ for Enneper's surface is one-to-one.*

Proof. It is convenient to look at the Gauss map for Enneper's surface in polar coordinates. From (3.33) we have that the Gauss map of Enneper's surface in polar coordinates is given by

$$\mathbf{N}(r,\theta) = \left(\frac{2r\cos(\theta)}{1+r^2}, \frac{2r\sin(\theta)}{1+r^2}, \frac{r^2-1}{1+r^2}\right)$$

In order to show that the Gauss map is one-to-one, we want to show that if $\mathbf{N}(r,\theta) = \mathbf{N}(\rho,\phi)$, then $r = \rho$ and $\theta = \phi$, where r and ρ are radiuses and θ and ϕ are angles. Thus we want to show that if $\mathbf{N}(r,\theta) = \mathbf{N}(\rho,\phi)$, which is given by

$$\left(\frac{2r\cos(\theta)}{1+r^2}, \frac{2r\sin(\theta)}{1+r^2}, \frac{r^2-1}{1+r^2}\right) = \left(\frac{2\rho\cos(\phi)}{1+\rho^2}, \frac{2\rho\sin(\phi)}{1+\rho^2}, \frac{\rho^2-1}{1+\rho^2}\right)$$

then $r = \rho$ and $\theta = \phi$. Since r and ρ are radiuses, $r, \rho \geq 0$. Keeping this in mind, from the third component we can see that

$$\begin{aligned}
\frac{r^2-1}{1+r^2} &= \frac{\rho^2-1}{1+\rho^2} \\
(r^2-1)(1+\rho^2) &= (\rho^2-1)(1+r^2) \\
r^2 + r^2\rho^2 - 1 - \rho^2 &= \rho^2 + \rho^2 r^2 - 1 - r^2 \\
2r^2 - 2\rho^2 &= 0 \\
r^2 &= \rho^2 \\
(r^2)^{1/2} &= (\rho^2)^{1/2} \Rightarrow r = \rho
\end{aligned}$$

Hence from the third component of $\mathbf{N}(r,\theta)$ and $\mathbf{N}(\rho,\phi)$ we get that $r = \rho$, and it is left to show that $\theta = \phi$. Looking at the first and second components of $\mathbf{N}(r,\theta)$ and $\mathbf{N}(\rho,\phi)$ we have to show that if,

$$\cos(\theta) = \cos(\phi) \text{ and } \sin(\theta) = \sin(\phi)$$

then it implies that $\theta = \phi$. The equations above can also be seen as requiring the following to hold,

$$(\cos(\theta),\ \sin(\theta)) = (\cos(\phi),\ \sin(\phi))$$

If we think about the unit circle then it is clear that we have $\theta = \phi$, if we look at the angles from $(0, 2\pi)$. Thus we get that $\theta = \phi$. Therefore in the case of Enneper's surface, the Gauss map is one-to-one. ■

Theorem 4.4.2. *[Opr00, p. 95] The Gauss map* $\mathbf{N} : D \to S^2$ *for the catenoid is one-to-one.*

Proof. From (3.6) we have that the Gauss map for the catenoid is given by

$$\mathbf{N}(u, v) = \left(\frac{-\cos(v)}{\cosh(u)}, \frac{-\sin(v)}{\cosh(u)}, \frac{\sinh(u)}{\cosh(u)} \right)$$

Referring to figure 3.1b, lets look at the idea of the proof. Let us take a horizontal cross section of the catenoid. This horizontal cross section will take the shape of a circle which we will denote by C. Then at each point $p \in C$, we have a well defined normal vector $\mathbf{N}(p)$. Let us assume it is pointing in the outward direction. Furthermore the catenoid is a regular minimal surface and it never self intersects. At each point $p \in C$, we have a normal vector $\mathbf{N}(p)$, and each $\mathbf{N}(p)$ is mapped to a single unique point on the unit sphere S^2. This shows us that the Gauss map for the catenoid is one-to-one. ■

Thus the catenoid and Enneper's surface have a Gauss map that is one-to-one. Actually due to the result [Opr00, p. 95], the catenoid and Enneper's surface are the *only* minimal surfaces that have a one-to-one Gauss map. This means that we can use the formula (4.19) in the case of the catenoid and Enneper's surface. In the case of the catenoid and Enneper's surface, we have a theorem that tells us about the one-to-oneness of their Gauss maps.

Now Let us consider the Gauss map of the higher order Enneper surfaces. Let us check whether the higher order Enneper surfaces have a one-to-one Gauss map.

4.4.1 The Gauss map for the higher order Enneper surfaces

As a result of theorem 4.4.1 we know that Enneper's surface given in (3.24) has a one-to-one Gauss map. Let us look at the Gauss map of the higher order Enneper surfaces and investigate whether it is one-to-one, or not. It is convenient to consider the Gauss map for this surface in polar coordinates. From (3.43) we have that the Gauss map for Enneper's surface of order k is given by

$$\mathbf{N}(r, \theta) = \left(\frac{2r^k \cos(k\theta)}{1 + r^{2k}}, \frac{2r^k \sin(k\theta)}{1 + r^{2k}}, \frac{r^{2k} - 1}{1 + r^{2k}} \right)$$

Let us check whether this map is one-to-one or not. To check if the Gauss map is one-to-one, we want to check that if $\mathbf{N}(r, \theta) = \mathbf{N}(\rho, \phi)$, then does this imply that $r = \rho$ and $\theta = \phi$. Thus we begin by looking at $\mathbf{N}(r, \theta) = \mathbf{N}(\rho, \phi)$, which is given by

$$\left(\frac{2r^k \cos(k\theta)}{1 + r^{2k}}, \frac{2r^k \sin(k\theta)}{1 + r^{2k}}, \frac{r^{2k} - 1}{1 + r^{2k}} \right) = \left(\frac{2\rho^k \cos(k\phi)}{1 + \rho^{2k}}, \frac{2\rho^k \sin(k\phi)}{1 + \rho^{2k}}, \frac{\rho^{2k} - 1}{1 + \rho^{2k}} \right)$$

Here r and ρ are radiuses, thus $r, \rho > 0$. Then from the third components of $\mathbf{N}(r, \theta)$ and $\mathbf{N}(\rho, \phi)$, we get that,

$$\begin{aligned}\frac{r^{2k}-1}{1+r^{2k}} &= \frac{\rho^{2k}-1}{1+\rho^{2k}}\\ (r^{2k}-1)(1+\rho^{2k}) &= (\rho^{2k}-1)(1+r^{2k})\\ 2r^{2k}-2\rho^{2k} &= 0\\ r^{2k} &= \rho^{2k}\\ (r^{2k})^{1/2k} &= (\rho^{2k})^{1/2k} \Rightarrow r = \rho\end{aligned}$$

Hence from the third components we get that $r = \rho$. Let us see whether $\theta = \phi$. To check this, we look at the first and second components of $\mathbf{N}(r, \theta)$ and $\mathbf{N}(\rho, \phi)$. Since we know that $r = \rho$, we can now check if $\theta = \phi$ when,

$$\cos(k\theta) = \cos(k\phi) \text{ and } \sin(k\theta) = \sin(k\phi)$$

Let $\theta' = k\theta$ and $\phi' = k\phi$. Then the equation above can be rewritten as,

$$\cos(\theta') = \cos(\phi') \text{ and } \sin(\theta') = \sin(\phi')$$

Now we are in the same case as the Enneper's surface of order 1, θ' and ϕ' describe the same point on the unit circle up to some multiple of 2π. This gives us that,

$$\begin{aligned}\theta' &= \phi' + 2l\pi\\ k\theta &= k\phi + 2l\pi\\ \theta &= \phi + \frac{2l}{k}\pi\end{aligned}$$

where l is an integer, and k is the order of Enneper's surface. In this case we do not get that $\theta = \phi$. Therefore the Gauss map for higher order Enneper surfaces with $k > 1$, is not one-to-one.

We can restrict ourselves to a smaller domain D, in which the Gauss map is one-to-one. How small the domain can be depends on the value of k. For example if $k = 2$, then $\theta = \phi + l\pi$, there is more than one point that will be mapped to the same point on the unit circle. However if we restrict ourselves to the half unit circle, that is in the region $\theta \in [0, \pi)$, the map would be one-to-one.

4.5 Stability Properties

To look at the area-minimizing property for minimal surfaces we will look at the stability of the surface. The material presented here on the stability of minimal surfaces is based on [Fan96, chapter 20]. We would like to look at when a minimal surface has minimal surface area. Let $\mathbf{x}(u, v)$ be a minimal surface. Then the area functional $A(t)$ for $\mathbf{x}$ as mentioned in (4.2) is

$$A(t) = \int_D (B^t)^{1/2}\, du\, dv = \int_D \sqrt{E^t G^t - (F^t)^2}\, du\, dv$$

We had found the first derivative of the area functional at $t = 0$, $A'(0)$. From lemma 4.2.1 we found an expression for $A'(0)$ as

$$A'(0) = -2 \int_D B^{1/2} hH \; du\, dv = 0$$

Since $\mathbf{x}$ is a minimal surface, it has zero mean curvature. Due to the fact that $H = 0$, we get that $A'(0) = 0$. If $\mathbf{x}$ is a minimal surface then $A'(0) = 0$. For minimal surfaces we get that $A'(0) = 0$. If $\mathbf{x}$ is a minimal surface then $t = 0$ is a critical point for the area functional. We would like $t = 0$ to be a minimum point for $A(t)$, as this would imply that $A(0)$ is a local minimum for $A(t)$. This would then mean that $\mathbf{x}$ has minimal surface area. To see whether $t = 0$ is a maximum, minimum or a saddle point of $A(t)$, we looked at the second derivative of the area functional at $t = 0$, $A''(0)$. From lemma 4.3.1 we found an expression for $A''(0)$ as

$$A''(0) = \int_D \Big(2B^{1/2} K h^2 + B^{-1/2} (\nabla h)^T M (\nabla h) \Big) \; du\, dv$$

In order for $t = 0$ to be a minimum point for $A(t)$, we would like $A''(0) > 0$. Thus we would like to look at the positivity of the second derivative of the area functional. The condition that helps us in determining when $A''(0) > 0$ is the concept of stability. This leads us to the next definition.

Definition 4.5.1. Let $\mathbf{x}(u, v)$ be a minimal surface where $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Let $D \subset U$ be a bounded domain. Then $\mathbf{x}$ is said to be *stable* if $A''(0) > 0$. In addition $\mathbf{x}$ is said to be *unstable* if $A''(0) < 0$.

From the definition of the stability of a minimal surface mentioned above, we have that if $\mathbf{x}$ is stable, then $A''(0) > 0$. This then tells us that $A(0)$ is a local minimum for the area functional. This further tells us that in the region D, the surface area of $\mathbf{x}$ is a local minimum. Hence the minimal surface $\mathbf{x}$ is an area minimizing minimal surface in the region D. To summarize it all we have that if $\mathbf{x}$ is stable in a region D, then by the definition of stability we have that $A''(0) > 0$. This tells us that the surface area of $\mathbf{x}$ is a local minimum, making the minimal surface $\mathbf{x}$ area minimizing.

We would like to find local regions in which the minimal surface $\mathbf{x}$ is stable. Then in this local region the minimal surface $\mathbf{x}$ will have minimum surface area among all possible surfaces that share the same boundary. We mention that this is a *local* property of minimal surfaces, that is, minimal surfaces are locally area minimizing. If we can find a stable region D for $\mathbf{x}$, we will be able to find a locally area minimizing minimal surface $\mathbf{x}$ in D. Moreover if $\mathbf{x}$ is a minimal surface which has minimal surface area in a region D, then $\mathbf{x}$ is stable in the region D. In order to find the local regions in which a minimal surface $\mathbf{x}$ is stable or unstable, we will use the following theorem.

Theorem 4.5.2. *[Fan96, p. 96] Let $\mathbf{x}(u, v)$ be a minimal surface where $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Let $D \subset U$ be a bounded domain. If the Gauss map $\mathbf{N} : D \to S^2$ is one-to-one, and the area of $\mathbf{N}(D)$ is less than 2π, then $\mathbf{x} : D \to \mathbb{R}^3$ is stable.*

The theorem above tells us how to find the stable region of a minimal surface in the special case of the Gauss map being one-to-one. If the Gauss map of $\mathbf{x}$ is one-to-one, then from (4.19) we get that $\mathbf{x}$ is stable if

$$Area(\mathbf{N}(D)) = -\int_D KdA < 2\pi \tag{4.20}$$

The expression (4.20) shown above allows us to calculate stable regions of a minimal surface in the case that the Gauss map is one-to-one. There is another theorem that we can use to calculate stability, which does not require the Gauss map to be one-to-one. This theorem is as follows.

Theorem 4.5.3. *[Fan96, p. 96] Let $\mathbf{x}(u,v)$ be a minimal surface where $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Let $D \subset U$ be a bounded domain. If*

$$-\int_D K\ dA < 2\pi \tag{4.21}$$

then $\mathbf{x}$ is stable on D.

Theorem 4.5.3 tells us that if the negative of the total Gaussian curvature is less than 2π, then the minimal surface $\mathbf{x}$ is stable in the region D. This then tells us that $A''(0) > 0$. This theorem allows us to find when $A''(0)$. We look at some examples of theorem 4.5.3.

4.5.1 The catenoid

From the result of theorem 4.4.2, we found that the Gauss map of the catenoid is one-to-one. This means that both theorem 4.5.2 and theorem 4.5.3 are applicable for the catenoid. Let us find stable regions for the catenoid. We begin by computing the integral in theorem 4.5.3. To compute the total Gaussian curvature we can use the expression in (1.15). From (3.5) and (3.8) we have that

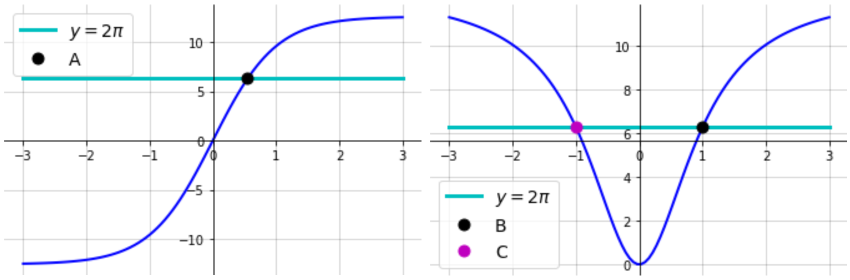
$$K = \frac{-1}{\cosh^4(u)}, \qquad \sqrt{EG - F^2} = \cosh^2(u)$$

Thus we compute the integral over the region D where

$$D = \{(u,v) \mid u \in (-R,R) \text{ and } v \in (0,2\pi)\}$$

The region D is a strip where $u \in (-R,R)$ and $v \in (0,2\pi)$. We get that

$$\begin{aligned}
-\int_D KdA &= -\int\int_D K\ \sqrt{EG-F^2}\ dudv \\
&= -\int_{-R}^{R}\int_0^{2\pi} \frac{-1}{\cosh^4(u)} \cosh^2(u)\ dvdu \\
&= \int_{-R}^{R}\int_0^{2\pi} \frac{1}{\cosh^2(u)}\ dvdu \\
&= 2\pi \int_{-R}^{R} \frac{1}{\cosh^2(u)}\ du \\
&= 2\pi\Big[\tanh(u)\Big]_{u=-R}^{u=R} = 2\pi\Big[\tanh(R) - \underbrace{\tanh(-R)}_{-\tanh(R)}\Big] \\
&= 2\pi[2\tanh(R)] = 4\pi\tanh(R)
\end{aligned}$$



(a) $f_1(R) = 4\pi \tanh(R)$ (b) $f_2(R) = 4\pi(1 - 1/(1 + R^2))$

Figure 4.1: Graph of $f_1(R)$ and $f_2(R)$. The points A, B and C have values given by $A = (0.5493, 6.2832)$, $B = (1,\ 6.2832)$ and $C = (-1,\ 6.2832)$.

To make the notation easier, we define

$$f_1(R) = 4\pi \tanh(R) \tag{4.22}$$

To find the stable regions we have to find the values for R such that $f_1(R) < 2\pi$. Thus we solve the inequality $f_1(R) < 2\pi$ for R. This leads to

$$\begin{aligned} f_1(R) < 2\pi \iff 4\pi \tanh(R) &< 2\pi \\ \tanh(R) &< \frac{1}{2} \\ R &< \operatorname{arctanh}\left(\frac{1}{2}\right) \approx 0.5493 \end{aligned}$$

Therefore $f_1(R) < 2\pi$ when we have $R < 0.5493$. We can further verify this by looking at the graph of $f_1(R)$, which is shown in figure 4.1a. The graph shows that $f_1(R) = 2\pi$ when $R = 0.5493$. So if we have $R < 0.5493$, then $f_1(R) < 0.5493$. Thus we find that

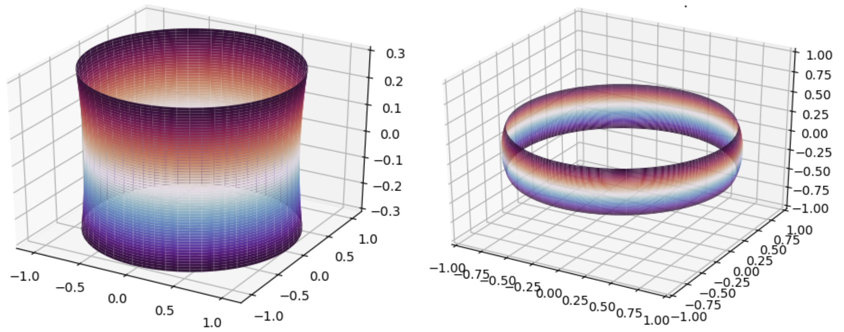
$$f_1(R) < 2\pi \text{ for } R < 0.5493$$

Therefore for the catenoid we find that

$$-\int_D K dA < 2\pi \text{ for } R < 0.5493 \tag{4.23}$$

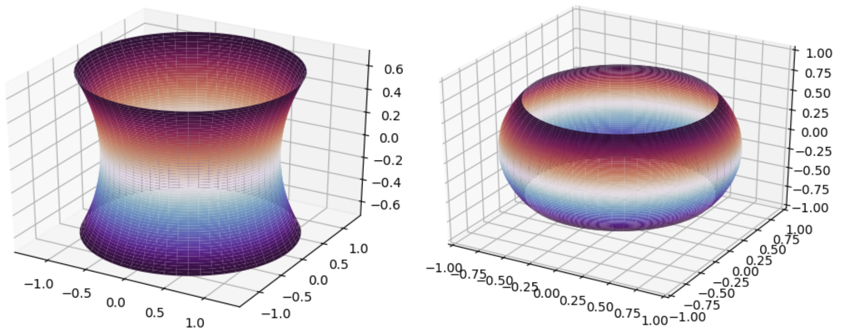
which further implies that $A''(0) > 0$ for $R < 0.5493$. The catenoid is stable in the region D with $u \in (-R, R)$ and $v \in (0, 2\pi)$ whenever we have $R < 0.5493$. In this region the catenoid is locally area minimizing and has the least surface area among all other surfaces that have the same boundary.

Let us see what the catenoid looks like in the stable region. Let $R = 0.3$. Then the region D is given in the parameter domain with $u \in (-0.3, 0.3)$ and $v \in (0, 2\pi)$. The catenoid in this region is shown in figure 4.2a. The image of the Gauss map of the catenoid in this region is shown in figure 4.2b. We can also find the area of the image of the Gauss map, it is given by the function $f_1(R)$ in (4.22). We find that $f_1(0.3) = 3.661$ which is less than 2π as $2\pi \approx 6.28318$. The area of the Gauss map for $R = 0.3$ is less than 2π. Thus for $R = 0.3$ we get that $-\int_D K dA < 2\pi$. Hence $A''(0) > 0$ and $\mathbf{x}(u, v)$ is stable, making the catenoid a local minimum in the region given by D.



(a) Catenoid (b) Gauss map

Figure 4.2: Catenoid with $R = 0.3$. This figure shows the catenoid in the region $u \in (-0.3, 0.3)$ and $v \in (0, 2\pi)$. The corresponding image of the Gauss map is also shown.



(a) Catenoid (b) Gauss map

Figure 4.3: Catenoid with $R = 0.7$. This figure shows the catenoid in the region $u \in (-0.7, 0.7)$ and $v \in (0, 2\pi)$. The corresponding image of the Gauss map is also shown.

Let us also look at the catenoid in the unstable region, that is, a region in which $R > 0.5439$. Let $R = 0.7$. Then D is the region with $u \in (-0.7, 0.7)$ and $v \in (0, 2\pi)$. The catenoid in this region is shown in figure 4.3a. The Gauss map of the catenoid in this region is shown in 4.3b. We can see from the figure that the surface area of the image of the Gauss map for $R = 0.7$ is larger than that for $R = 0.3$. From the function $f_1(R)$ we compute the surface area of the Gauss map to be $f_1(0.7) = 7.5947$, which is larger 2π. In addition the shape of the catenoid is narrower in the centre compared to the one for $R = 0.3$. In this case we do not get that $-\int_D K dA < 2\pi$.

4.5.2 Enneper's surface

Due to the result of theorem 4.4.1, the Gauss map of Enneper's surface is one-to-one. This means that both theorem 4.5.2 and 4.5.3 can be used to find stable regions for Enneper's surface. Just like the previous case, we begin by computing the integral in theorem 4.5.3. We will compute the integral in polar coordinates r and θ. From (3.31) and (3.32) we have that

$$K = \frac{-4}{(1+r^2)^4}, \qquad \sqrt{EG - F^2} = E = (1+r^2)^2$$

due to the isothermal parametrization. We compute the integral over the region D where

$$D = \{u^2 + v^2 < R^2\}$$

The region D is a disk of radius R with center $(0,0)$, where $r \in [0, R)$ and $\theta \in [0, 2\pi)$. Since the integral is in polar coordinates we have to multiply by the Jacobian determinant r. We get that

$$\begin{aligned}
-\int_D K dA &= -\int\int_D KE \cdot r \; dr d\theta \\
&= -\int_0^R \int_0^{2\pi} \frac{-4}{(1+r^2)^4}(1+r^2)^2 \; r \; d\theta dr \\
&= \int_0^R \int_0^{2\pi} \frac{4r}{(1+r^2)^2} \; d\theta dr \\
&= 4 \cdot 2\pi \int_0^R \frac{r}{(1+r^2)^2} \; dr = 8\pi \int_0^R \frac{r}{(1+r^2)^2} \; dr
\end{aligned}$$

We can compute this integral by using the substitution method. Let $m = 1+r^2$, then $dm/dr = 2r$ and $dr = dm/2r$. This leads to

$$\begin{aligned}
-\int_D K \; dA = 4\pi \int_0^R \frac{1}{m^2} \; dm = 4\pi[-m^{-1}]_{m=1}^{m=1+R^2} &= 4\pi\Big[\frac{-1}{1+r^2}\Big]_{r=0}^{r=R} \\
= 4\pi\Big[1 - \frac{1}{1+R^2}\Big]
\end{aligned}$$

To make the notation easier, we define

$$f_2(R) = 4\pi\Big(1 - \frac{1}{1+R^2}\Big) \tag{4.24}$$

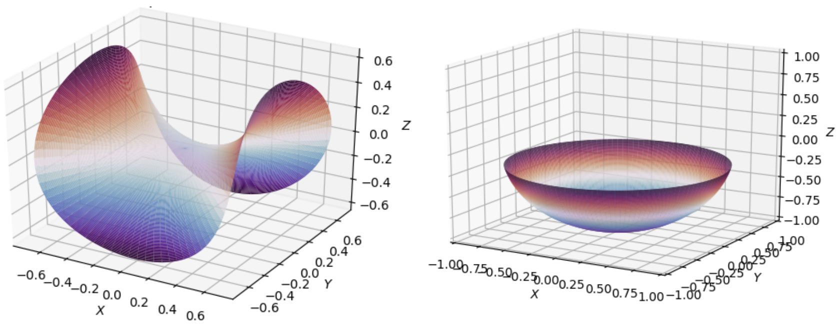
To find the stable regions we have to find the values for R such that $f_2(R) < 2\pi$. We have to solve the inequality $f_2(R) < 2\pi$ for R. This leads to

$$\begin{aligned}
f_2(R) < 2\pi \iff 4\pi(1 - \frac{1}{1+R^2}) &< 2\pi \\
1 - \frac{1}{1+R^2} &< \frac{1}{2} \\
\frac{1}{1+R^2} &> \frac{1}{2}
\end{aligned}$$

This is equivalent to

$$R^2 + 1 < 2 \iff R^2 < 1 \iff R < \pm\sqrt{1} \iff R < \pm 1$$

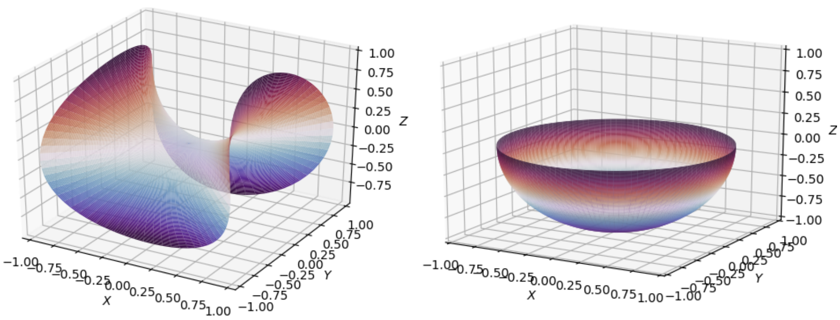
Since R is a radius, we will only be considering the positive values for R. On that note we get that $f_2(R) < 2\pi$ when we have $R < 1$. We can further verify this by looking at the graph of $f_2(R)$, which is shown in figure 4.1b. The graph



(a) Enneper's surface

(b) Gauss map

Figure 4.4: Enneper's surface with $R = 0.8$. This figure shows Enneper's surface in the region $r \in [0, 0.8)$ and $\theta \in [0, 2\pi)$. The corresponding image of the Gauss map is also shown.



(a) Enneper's surface

(b) Gauss map

Figure 4.5: Enneper's surface with $R = 1$. This figure shows Enneper's surface in the region $r \in [0, 1)$ and $\theta \in [0, 2\pi)$. The corresponding image of the Gauss map is also shown.

shows two intersection points of $f_2(R)$ with 2π, namely the points B and C. We are considering $R > 0$ so we are interested in point B. We see that $f_2(R) = 2\pi$ when $R = 1$. Thus we find that

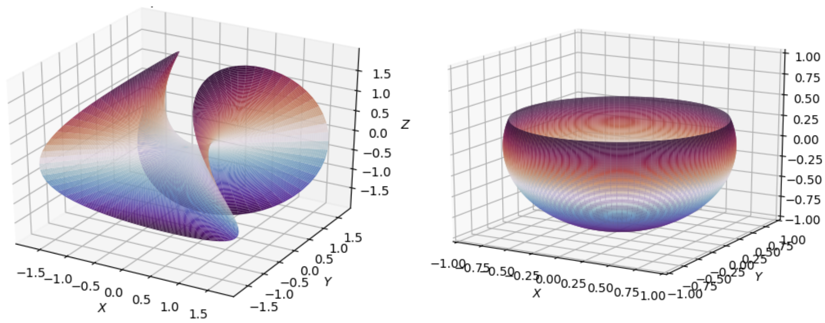
$$f_2(R) < 2\pi \text{ for } R < 1$$

Therefore for Enneper's surface we find that

$$-\int_D K dA < 2\pi \text{ for } R < 1 \tag{4.25}$$

which further implies that $A''(0) > 0$ for $R < 1$. Enneper's surface is stable in D where $r \in (0, R)$ and $\theta \in (0, 2\pi)$, whenever we have $R < 1$. In this particular region Enneper's surface is locally area minimizing and has the least surface area among all other surfaces that have the same boundary.

Let us see what Enneper's surface looks like in the stable region. Let $R = 0.8$. Then D is given by $r \in [0, 0.8)$ and $\theta \in [0, 2\pi)$. Enneper's surface in this region is shown in figure 4.4a. The corresponding image of the Gauss map is shown in figure 4.4b. We can find the surface area of the image of the Gauss map. It is



(a) Enneper's surface

(b) Gauss map

Figure 4.6: Enneper's surface with $R = 1.4$. This figure shows Enneper's surface in the region $r \in [0, 1.4)$ and $\theta \in [0, 2\pi)$. The corresponding image of the Gauss map is also shown.

given by the function $f_2(R)$ in (4.24). We find that $f_2(0.8) = 4.9039$, which is less than 2π. For $R = 0.8$ we get that $-\int_D K dA < 2\pi$, hence $A''(0) > 0$ and $\mathbf{x}(u, v)$ is stable. Enneper's surface is a local minimum of the area functional in the region given by D.

We know that $\mathbf{x}(u, v)$ is stable when $R < 1$. Let us look at the case when $R = 1$. Then D is given by $r \in [0, 1)$ and $\theta \in [0, 2\pi)$. Enneper's surface in this region is shown in figure 4.5. We can see that Gauss map has slightly larger area compared to when $R = 0.8$. The area of the Gauss map is $f_2(1) = 6.2832$, which is 2π.

Let us look at the Enneper's surface in the unstable region, that is, a region in which $R > 1$. Let $R = 1.4$. Then D is given by $r \in [0, 1.4)$ and $\theta \in [0, 2\pi)$. Enneper's surface in this region is shown in figure 4.6. We can see that the image of the Gauss map covers more than half of the unit sphere. In this case the surface area of the Gauss map is $f_2(1.4) = 8.3210$, which is greater than 2π. As the value of R increases, the Gauss map covers more surface area. In this case we do not get that $-\int_D K dA < 2\pi$.

4.5.3 Higher order Enneper Surfaces

In section 4.4 we looked at the Gauss map of the higher order Enneper surfaces. We found that the Gauss map is not one-to-one. In this case we can use theorem 4.5.3 to find stable regions for the higher order Enneper surfaces. To find the stable regions we begin by computing the integral in theorem 4.5.3. We will compute the integral in polar coordinates r and θ. From (3.41) and (3.42) we have

$$K = \frac{-4k^2\, r^{2k-2}}{(1 + r^{2k})^4}, \qquad \sqrt{EG - F^2} = E = (1 + r^{2k})^2$$

due to the isothermal parametrization. We compute the integral over the region D where

$$D = \{u^2 + v^2 < R^2\}$$

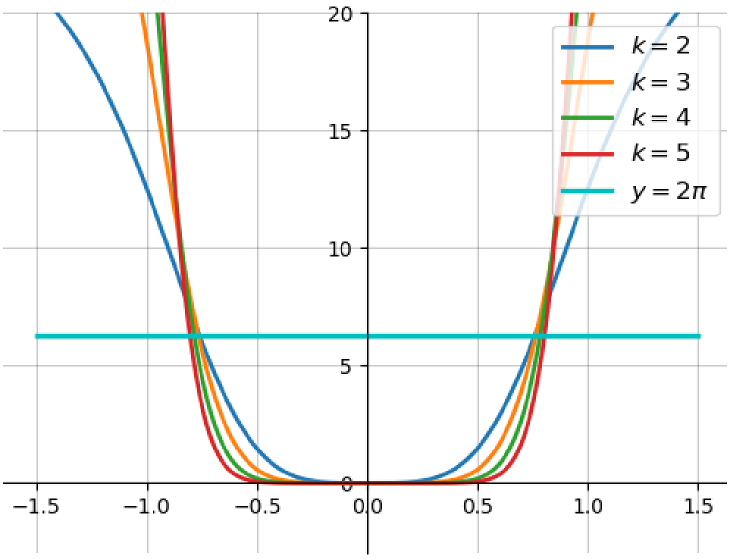


Figure 4.7: Graph of $f_3(R) = 4\pi k(1 - \frac{1}{1+R^{2k}})$ for $k = 2, 3, 4, 5$.

Thus D is a disk of radius R with center $(0, 0)$, where $r \in [0, R)$ and $\theta \in [0, 2\pi)$. We also have to multiply by the Jacobian determinant r. We get that

$$\begin{aligned}
-\int_D K\ dA &= -\int\int_D KE \cdot r\ drd\theta \\
&= -\int_0^R \int_0^{2\pi} -\frac{4k^2 r^{2k-2}}{(1+r^{2k})^4}(1+r^{2k})^2\ r\ d\theta dr \\
&= \int_0^R \int_0^{2\pi} \frac{4k^2 r^{2k-2+1}}{(1+r^{2k})^2}\ d\theta dr \\
&= \int_0^R \int_0^{2\pi} \frac{4k^2 r^{2k-1}}{(1+r^{2k})^2}\ d\theta dr \\
&= 8\pi \int_0^R \frac{k^2 r^{2k-1}}{(1+r^{2k})^2}\ dr
\end{aligned}$$

This can be solved using the substitution method. Let $m = 1 + r^{2k}$, then $dm/dr = 2kr^{2k-1}$ and $dr = dm/2kr^{2k-1}$. This leads to

$$\begin{aligned}
-\int_D K\ dA &= 4\pi k \int_0^R \frac{1}{m^2} dm = 4\pi k \int_0^R m^{-2} dm \\
&= 4\pi k[-m^{-1}]_{m=1}^{m=1+R^{2k}} = -4\pi k \left[\frac{1}{1+r^{2k}}\right]_{r=0}^{r=R} \\
&= 4\pi k \left[1 - \frac{1}{1+R^{2k}}\right]
\end{aligned}$$

To make the notation easier, we define

$$f_3(R) = 4\pi k \left(1 - \frac{1}{1+R^{2k}}\right) \tag{4.26}$$

The graph of $f_3(R)$ is shown in figure 4.7. The graph is shown for $k = 2, 3, 4, 5$. As the value for k increases, $f_3(R)$ becomes flatter at the origin. To find the stable region we have to find the value for R such that $f_3(R) < 2\pi$. We solve the inequality for R, which leads to

$$\begin{aligned} f_3(R) < 2\pi \iff 4\pi k \left(1 - \frac{1}{1+R^{2k}}\right) &< 2\pi \\ 1 - \frac{1}{1+R^{2k}} &< \frac{1}{2k} \\ \frac{1}{1+R^{2k}} &> 1 - \frac{1}{2k} \end{aligned}$$

This is equivalent to

$$1 + R^{2k} < \frac{1}{1-\frac{1}{2k}} \iff R^{2k} < \frac{1}{1-\frac{1}{2k}} - 1 \iff R < \left(\frac{1}{1-\frac{1}{2k}} - 1\right)^{1/2k}$$

We simplify this expression and we find that

$$f_3(R) < 2\pi \text{ for } R < \frac{1}{(2k-1)^{1/2k}}$$

Therefore for the higher order Enneper surfaces, we find that

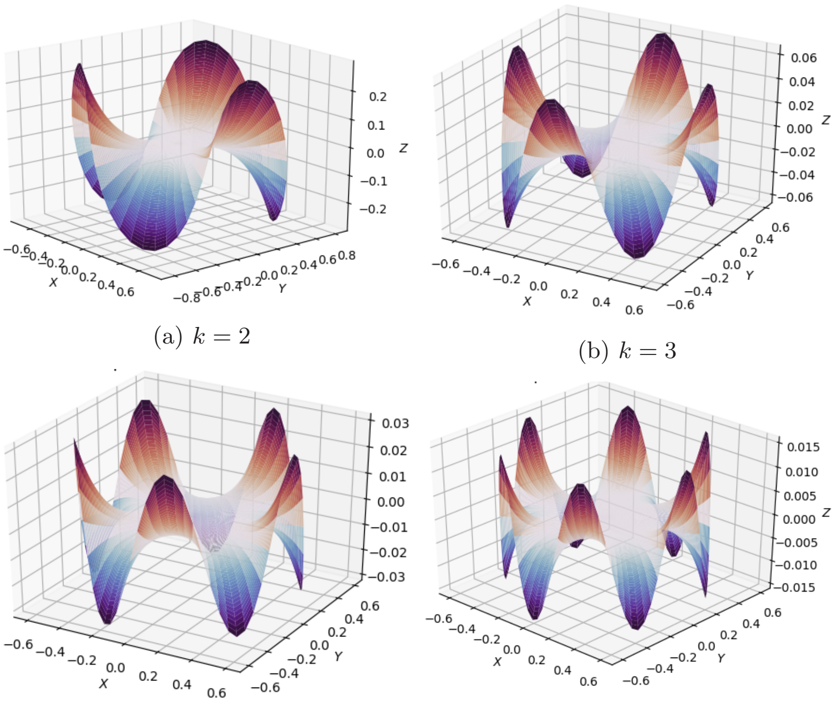
$$-\int_D K dA < 2\pi \text{ for } R < \frac{1}{(2k-1)^{1/2k}} \tag{4.27}$$

which further implies that $A''(0) > 0$ for $R < 1/(2k-1)^{1/2k}$. Thus the higher order Enneper surfaces of order k are stable in D, where $r \in [0, R)$ and $\theta \in [0, 2\pi)$, whenever we have $R < 1/(2k-1)^{1/2k}$. In this particular region the higher order Enneper surfaces are locally area minimizing among all other surfaces with the same boundary.

Enneper surface of order k	$\mathbf{R < 1/(2k-1)^{1/2k}}$
$k = 1$	$R < 1$
$k = 2$	$R < 0.7598$
$k = 3$	$R < 0.7647$
$k = 4$	$R < 0.7841$
$k = 5$	$R < 0.8027$

Table 4.1: Enneper surface of order k and the value for R such that the minimal surface is stable if R is less than the given value.

Let us look at some stable regions for the higher order Enneper surfaces. Table 4.1 shows the values for R such that in the region given by $r \in [0, R)$ and $\theta \in [0, 2\pi)$, the Enneper surface of order k is stable. The table shows values for $k = 1, 2, 3, 4$ and 5. Let $R = 0.6$. Then D is given by $r \in [0, 0.6)$ and $\theta \in [0, 2\pi)$. With this value of R, we see that Enneper surfaces of order $1, 2, 3, 4$ and 5 are all stable, as $R = 0.6$ is less than the value of R needed for each of them to be stable. Figure 4.8 shows these stable higher order Enneper surfaces for $k = 2, 3, 4$ and 5. Thus we can see what they look like in the stable regions.



(a) $k = 2$

(b) $k = 3$

(c) $k = 4$

(d) $k = 5$

Figure 4.8: Enneper surfaces of order k for $k = 2, 3, 4, 5$ with $R = 0.6$. This figure shows the higher order Enneper surfaces in the region $r \in [0, 0.6)$ and $\theta \in [0, 2\pi)$.

CHAPTER 5

Area Minimizing Property of Minimal Surfaces and Soap Bubbles

We look further into the area minimizing property of minimal surfaces. In this chapter we will consider the problem of having minimal surface area in relation to soap bubbles and soap films. We will make the connection between soap films and minimal surfaces. We then consider the area minimizing property of minimal surfaces in relation to soap films. The way we can relate minimal surfaces and soap films is by Plateau's problem. Let us begin by looking at this problem.

5.1 Plateau's Problem

The material presented in this section is based on [Opr00, chapter 3.3]. Soap bubbles come in different shapes and sizes. An enjoyable activity for pretty much everyone is playing with soap bubbles. It is not just the shapes of the soap bubbles that are fascinating, but the colors are delightful as well. When the sun shines on the surface of the soap bubbles, one can see the burst of colors that appear on the surface of the soap bubbles. This burst of rainbow like colors on the soap films is a sight to see.

One person that was fascinated by these soap bubbles was Joseph Plateau. He was interested in the various shapes of the soap bubbles. Plateau experimented with soap films. He was particularly involved with creating soap films that enclosed a given boundary curve. The way he performed his experiments is as follows; first he would take a wire frame that was of a specific shape. Then he would dip this wire frame into a soap solution. He would then carefully remove this wire frame from the soap solution and look at the resulting soap film that was created. The resulting soap film had the property that it would be spanned by the wire frame. In other words the resulting soap film has a form such that it encloses the wire frame. The soap film is bounded by the wire frame, which is a boundary curve for the soap film.

We have talked about soap films but we did not mention how it relates to minimal surfaces. The reason soap films are relevant to minimal surfaces and the reason Plateau's work is associated to minimal surfaces will be clear by the presentation of the next two theorems.

Theorem 5.1.1. *(The First Principle of Soap Films) [Opr00, p. 14] A soap film takes a shape such that the surface area is minimized.*

Theorem 5.1.2. *[Opr00, p. 59] Let a soap film be produced by Plateau's soap film experiments as mentioned above. That is, let the soap film be produced such that it is bounded by the wire frame. Then the resulting soap film is a physical model of a minimal surface.*

The theorems 5.1.1 and 5.1.2 allow us to make the connection between soap films and minimal surfaces. When we dip a closed wire frame into a soap solution, and slowly take it out, we get a soap film enclosing the given wire frame. This soap film that we see is actually a model of a minimal surface. The result of theorem 5.1.1 tells us that the soap film is formed in a way that minimizes its surface area. This is why soap film experiments done by Plateau are significant in the world of minimal surfaces. The resulting soap films allow us to get an idea of the shape of a minimal surface. Plateau wanted to see what kind of soap films could be produced using different wire frames as the boundary, which leads us to the problem of Plateau.

Definition 5.1.3. *Plateau's problem*: Let Γ be a given boundary curve in $\mathbb{R}^3$. Find a minimal surface $\mathbf{x}(u, v)$ that encloses the curve Γ, that is, the minimal surface $\mathbf{x}$ has the curve Γ as its boundary.

The idea of Plateau's problem is as follows. Let us assume that we have a curve Γ in $\mathbb{R}^3$. Then we want to find a minimal surface that has the curve Γ as its boundary. In the case of the soap film experiments, the curve Γ is given by the wire frame that we will dip into the soap solution. Once we dip and take out the wire frame from the soap solution, a soap film will be formed which has the wire frame Γ as its boundary. By the result of theorem 5.1.2 the soap film produced will be a minimal surface. This resulting soap film is a minimal surface $\mathbf{x}$ with the curve Γ as its boundary. Thus the minimal surface $\mathbf{x}$ is a solution to Plateau's problem. This simple soap film experiment lets us visualize a solution to Plateau's problem. Due to these soap film experiments, in the context of minimal surfaces, we usually say soap films rather than soap bubbles.

Let us look deeper into the solutions of Plateau's problem. Let us look at the area minimizing property of the solutions. In this case we wish to find a minimal surface that is the solution to Plateau's problem and it is also a minimal surface with least surface area among all surfaces that have the same boundary. From [Rad93, p. 38] we have that a minimal surface $\mathbf{x}$ that is a solution to Plateau's problem is not necessarily a solution that has minimum surface area. On the other hand a solution that has minimum surface area is not necessarily a solution to Plateau's problem. A natural question to ask ourselves is regarding the existence and uniqueness of the solutions to Plateau's problem, as well as considering when these solutions really have minimal surface area. Before we can move further we have to look at some definitions. Recall that a *Jordan curve* is a simple closed curve (see [Do 16, p. 32]). This means that a Jordan curve is a curve that does not have any self intersections, and that the starting point and ending point for the curve is the same. The next definition is that of a disk-like minimal surface.

Definition 5.1.4. Let $\mathbf{x}(u, v)$ be a minimal surface where $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$.

Then $\mathbf{x}$ is said to be *disk-like* if U is the unit disk, $U = \{(u,v)|u^2 + v^2 \leq 1\}$ and the boundary of the unit disk is mapped to the Jordan curve Γ (in relation to Plateau's problem given above).

Let us present a result that tells us about the existence of solutions to Plateau's problem such that the solutions also have minimal surface area.

Theorem 5.1.5. *[Opr00, p. 67] Let Γ be a given Jordan curve. Then there exists an area minimizing disk-like minimal surface $\mathbf{x}$ that spans any given Jordan curve.*

Moving further with the area minimizing property we mention the following. Due to [Rad93, p. 38, paragraph III.14] we have that,

Lemma 5.1.6. *Let Γ be a given Jordan curve. Let $\mathbf{x}$ be the minimal surface that has Γ as its boundary. Then $\mathbf{x}$ does not always possess minimal surface area.*

Having minimal surface area is not something that minimal surfaces always possess. A solution to Plateau's problem might be a minimal surface but it may not be minimal in the sense of having minimal surface area. To demonstrate these concepts we look at some examples of minimal surfaces that can be produced as a solution to Plateau's problem.

5.2 The Catenoid

We look at an example of Plateau's problem. A minimal surface that is the solution to Plateau's problem is the catenoid. The material presented here is based on [Opr00, section 5.6].

If we go back to our soap experiments then it is possible to produce a model of the catenoid by the soap film experiments[Opr00, p. 11]. The wire frame consists of two circular rings. To begin with we place these two circular rings on top of each other horizontally. This will be their starting position. Then we take these two wire frames and dip them into the soap solution. We take them out and begin to pull them apart. We will slowly start to pull the two wire frames apart in the vertical direction. The soap film that is produced will enclose the wire frame, it will have the two circular rings as its boundary. This soap film is a model of the catenoid. We will notice that the shape of the catenoid will slightly change as the distance between the two rings varies. If we keep pulling the two rings further apart, we will notice that at some point the soap film formed will simply break. The soap film will not be in the shape of a catenoid anymore. In fact the soap film will enclose the two circular rings themselves, the soap film will be in the shape of a circular disk. At this point we would not be able to form a catenoid, but the only solution that we can form will be that of the two circular disks. The soap film forms a catenoid for a specific distance between the two wire frames. At a distance greater than that, the soap film is a circular disk. Thus the catenoid is a solution to Plateau's problem. Here the boundary curve Γ consists of the two circular rings, and the minimal surface $\mathbf{x}(u, v)$ produced is the catenoid. We can visualize the formation of the catenoid by doing these soap film experiments. Let us now look at this phenomenon from a mathematical point of view. We will also look at the existence and uniqueness of the solutions.

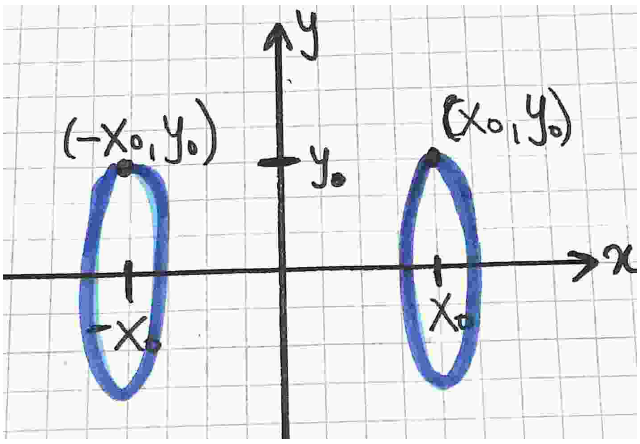


Figure 5.1: The blue circles are the boundary curve Γ that produce the catenoid in Plateau's problem.

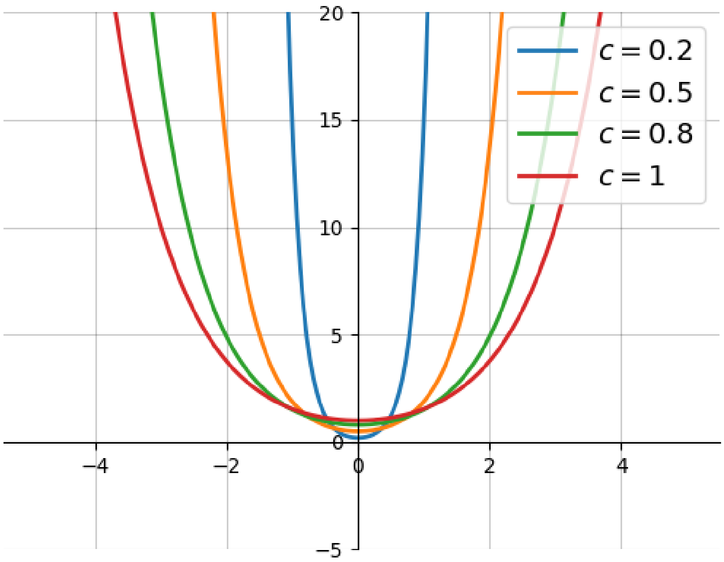


Figure 5.2: Graph of $y(x) = c\cosh(x/c)$ for different values of c. This is the catenary curve.

The catenoid is a minimal surface which is also a minimal surface of revolution. It is created by revolving the catenary curve around an axis. Let us look at the general case. If we take the x-axis to be the axis of revolution, then the catenary curve is given by the equation

$$y(x) = c\cosh\left(\frac{x}{c}\right) \tag{5.1}$$

and the corresponding surface of revolution that is generated by this catenary is parametrized as

$$\mathbf{x}(u,v) = \Big(u,\ c\cosh\left(\frac{u}{c}\right)\cos(v),\ c\cosh\left(\frac{u}{c}\right)\sin(v)\Big), \qquad -\infty < u < \infty,\ 0 \le v < 2\pi \tag{5.2}$$

If we take the z-axis to be the axis of revolution, then the catenary curve is given by the equation

$$y(z) = c\cosh\left(\frac{z}{c}\right) \tag{5.3}$$

and the corresponding surface of revolution that is generated by this catenary is parametrized as

$$\mathbf{x}(u,v) = \Big(c\cosh\left(\frac{u}{c}\right)\cos(v),\ c\cosh\left(\frac{u}{c}\right)\sin(v),\ u\Big), \qquad -\infty < u < \infty,\ 0 \le v < 2\pi \tag{5.4}$$

Let us consider the catenary curve in (5.1). Figure 5.2 shows the graph for this catenary curve as the value for c varies. The figure shows that as c increases from $c = 0.2$ to $c = 1$, the graph of $y(x)$ becomes wider and wider.

We will start by looking at the boundary curve Γ. The boundary curve Γ is given by the two circular disks. We let the x-axis be the axis of revolution. Then the catenary curve is given by (5.1). Let (x_0, y_0) and $(-x_0, y_0)$ be the centres of two disks of radius y_0. Here x_0 and $-x_0$ are points on the x-axis. This setup is illustrated in figure 5.1. Then these two disks are the boundary curve Γ. As the catenoid is a minimal surface of revolution, we have to find the equation of the catenary $y(x) = c\cosh(x/c)$ that passes through the points (x_0, y_0) and $(-x_0, y_0)$. Once we know the equation of the catenary curve, we will form the catenoid as the surface of revolution by rotating the catenary curve around the x-axis.

We want to look at the catenoid as the solution to Plateau's problem. We want to consider when the soap film forms a catenoid, and when the soap film forms the two circular disks. Moreover we want to investigate the surface area of the solutions. We would like to see when the minimal surface formed has minimal surface area, hence possessing the area minimizing property. For this reason we will look at the surface area of the catenoid compared to the surface area of the two disk solution. We will look at the ratio between x and y. This means that we will look at the problem as a function of x/y. This is the technique used in [Opr00, p. 181]. Here x is the distance from the origin $(0, 0)$ to the center of the disk in the x-coordinate, and y is the radius of the disk. We will be looking at the ratio between the distance from $(0, 0)$ to the center of the disk in the x-coordinate and the radius of the disk y. We will begin by looking at the catenary curve

$$y = c\cosh\left(\frac{x}{c}\right)$$

then solving for x

$$\begin{aligned}\cosh\left(\frac{x}{c}\right) &= \frac{y}{c}\\ \frac{x}{c} &= \operatorname{arccosh}\left(\frac{y}{c}\right)\end{aligned}$$

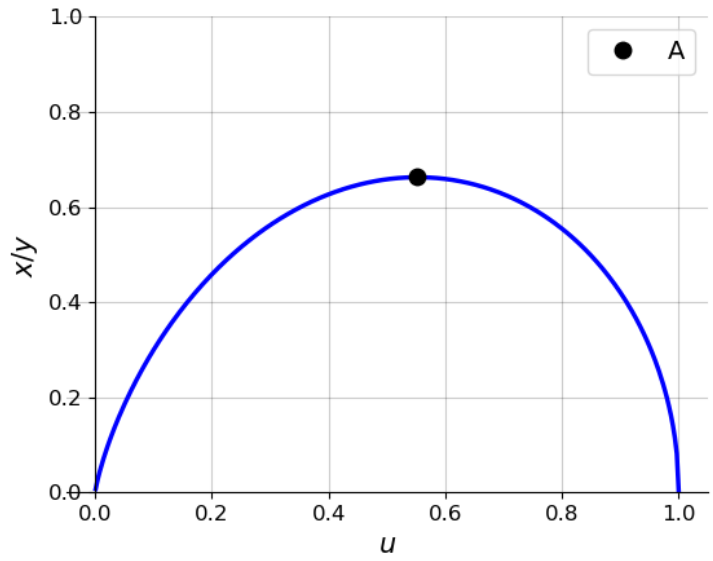


Figure 5.3: Graph of u versus x/y. The function $x/y = u\,\text{arccosh}(1/u)$. The point A is the maximum point. It is given by $A = (0.5524, 0.6627)$.

$$x = c\,\text{arccosh}\left(\frac{y}{c}\right) \tag{5.5}$$

Thus we will consider this function for expressing the ratio between x and y. We will look at this as a function of x/y. We let

$$u = \frac{c}{y} \tag{5.6}$$

then

$$\frac{1}{u} = \frac{y}{c} \tag{5.7}$$

Using the new variable u we can write x given in (5.5) in the form of x/y as

$$\begin{aligned}\frac{x}{y} &= \frac{c}{y}\,\text{arccosh}\left(\frac{y}{c}\right)\\ &= u\,\text{arccosh}\left(\frac{1}{u}\right)\end{aligned}$$

Thus the function x/y is given as

$$\frac{x}{y} = u\,\text{arccosh}\left(\frac{1}{u}\right) \tag{5.8}$$

Let us take a closer look at this function given by x/y. The graph of this function is given by figure 5.3. The x-axis is the u value, where $u = c/y$, and the y-axis is the value of x/y for the given value of u. So it is a plot of u versus the function $u\,\text{arccosh}(1/u)$. The graph of x/y allows us to see what is going on. The point A is the maximum point of $x/y = u\,\text{arccosh}(1/u)$. The maximum point occurs when $u = 0.5524$, at which the value for $x/y = 0.6627$. This tells

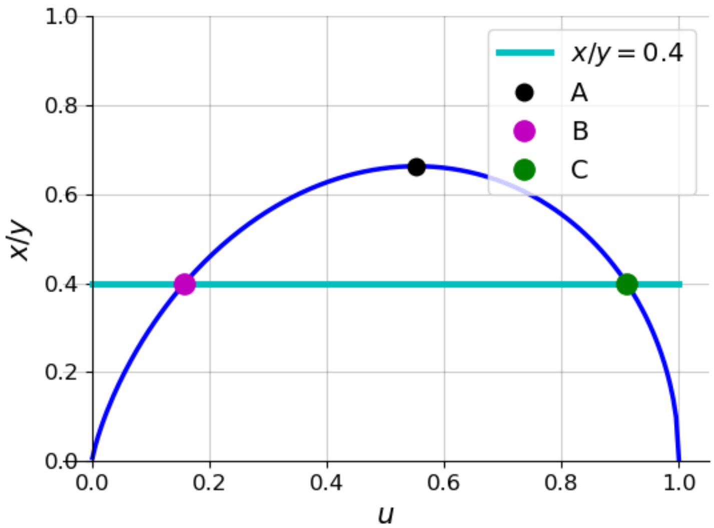


Figure 5.4: Graph of u versus x/y. The function $x/y = u\,\text{arccosh}(1/u)$. The point A is the maximum point, it is given by $A = (0.5524, 0.6627)$. The point $B = (0.1579, 0.4)$ and $C = (0.9107, 0.4)$.

us that for each value of $x/y < 0.6627$, there are two values of u that give the same x/y. This means that for each value of $x/y < 0.6627$, there are two possible catenary curves that pass through the points $(-x_0, y_0)$ and (x_0, y_0). Since two catenary curves exist at that point, this tells us that there are two possible catenoids that can be formed by rotating the catenary curve at that point.

Since the maximum value of $x/y = 0.6627$, we have that for values of $x/y > 0.6627$ no catenaries exist that pass through the points $(-x_0, y_0)$ and (x_0, y_0). This means that for each value of $x/y > 0.6627$, there are no catenoids that can be formed, and the catenoid solution does not exist.

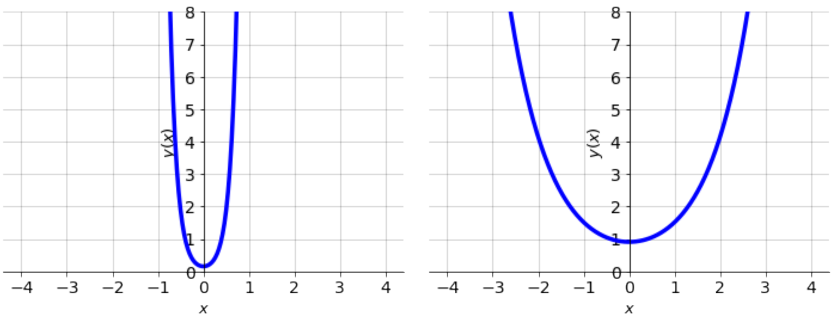
Let us demonstrate this phenomenon. Let

$$x_0 = 0.4, \text{ and } y_0 = 1,$$

Then $x/y = 0.4$. We have to find the equation of the catenary curve that passes through the points $(-0.4, 1)$ and $(0.4, 1)$. We have to find the value for c in the equation (5.1). We can find the value of c by finding the value of u. Therefore we want to find the value of u that gives us $x/y = 0.4$. Since $0.4 < 0.6627$, there are two values of u that have $x/y = 0.4$. We can find the value of u by finding the intersection of the curve $x/y = 0.4$ with $x/y = u\,\text{arccosh}(1/u)$. This is shown in figure 5.4. There are two points of intersection B and C. From the point B, we get that $u = 0.1579$, which is less than 0.5524. Let us denote this value of u by u_1. From the point C, we get that $u = 0.9107$, which is greater than 0.5524. Let us denote this value of u by u_2. The values of u_1 and u_2 give us the values of c_1 and c_2 respectively. Thus the values of c_1 and c_2 that give us $x/y = 0.4$ are

$$c_1 = 0.1579 \tag{5.9}$$

$$c_2 = 0.9107 \tag{5.10}$$



(a) $y_1(x)$ (b) $y_2(x)$

Figure 5.5: Catenary curves. This figure shows the graph of the catenary curves $y_1(x) = 0.1579\cosh(x/0.1579)$, and $y_2(x) = 0.9107\cosh(x/0.9107)$.

The corresponding catenary curves formed by c_1 and c_2 are given by the equations

$$y_1(x) = c_1 \cosh\left(\frac{x}{c_1}\right) = 0.1579\cosh\left(\frac{x}{0.1579}\right) \tag{5.11}$$

$$y_2(x) = c_2 \cosh\left(\frac{x}{c_2}\right) = 0.9107\cosh\left(\frac{x}{0.9107}\right) \tag{5.12}$$

The catenoid is formed by rotating the catenary curves $y_1(x)$ and $y_2(x)$ about the x-axis. The graph of $y_1(x)$ and $y_2(x)$ is shown in figure 5.5 a and b respectively. We can see that the shapes of the two curves are very different from each other. The curve $y_2(x)$ is much wider compared to $y_1(x)$. Let us choose two points $-x$ and x that pass through the curves. Then the distance between the two points on $y_1(x)$ is much less than the distance between the two points on $y_2(x)$. The catenoid formed has the parametrization given in (5.2). Let us write it in terms of x and y to make it more convenient

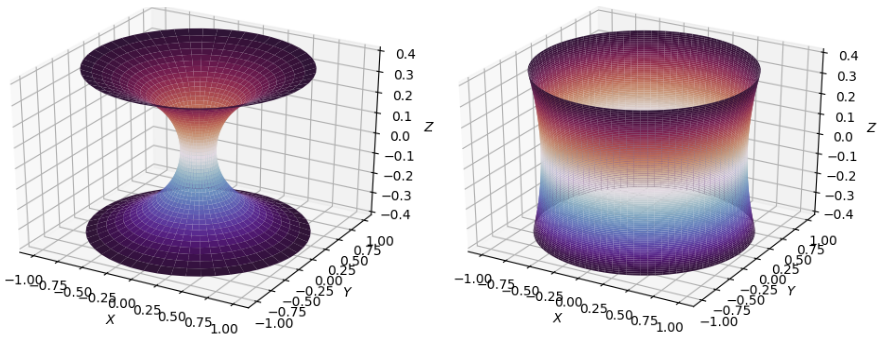
$$\mathbf{x}(x,y) = \Big(x,\ c\cosh\left(\frac{x}{c}\right)\cos(y),\ c\cosh\left(\frac{x}{c}\right)\sin(y)\Big), \tag{5.13}$$
$$-x_0 < x < x_0,\ 0 \leq y < 2\pi$$

We want to calculate the surface area of the catenoid that is formed. Thus we compute

$$\mathbf{x}_x = \Big(1,\ \sinh(x/c)\cos(y),\ \sinh(x/c)\sin(y)\Big)$$
$$\mathbf{x}_y = \Big(0,\ -c\cosh(x/c)\sin(y),\ c\cosh(x/c)\cos(y)\Big)$$

The coefficients of the first fundamental form E, F and G are given by,

$$E = \langle \mathbf{x}_x, \mathbf{x}_x \rangle = 1 + \sinh^2(x/c) = \cosh^2(x/c)$$
$$F = \langle \mathbf{x}_x, \mathbf{x}_y \rangle = 0$$
$$G = \langle \mathbf{x}_y, \mathbf{x}_y \rangle = c^2\cosh^2(x/c)$$



(a) $c_1 = 0.1579$ (b) $c_2 = 0.9107$

Figure 5.6: Catenoid. This figure shows the catenoid formed by rotating the catenary curve with the given value for c. The parameters are in the region $x \in (-0.4, 0.4)$ and $y \in [0, 2\pi)$.

From (1.8) and (1.9) we find the area element. We find the quantity $\sqrt{EG - F^2}$. This gives us

$$EG - F^2 = \cosh^2(x/c) \cdot c^2 \cosh^2(x/c) - 0^2 = c^2 \cosh^4(x/c)$$
$$\sqrt{EG - F^2} = \sqrt{c^2 \cosh^4(x/c)} = c \cosh^2(x/c)$$

From (1.10) we can compute the area of the catenoid. The area element dA is

$$dA = \sqrt{EG - F^2}\ dx\ dy = c \cosh^2(x/c)\ dx\ dy$$

The surface area of the catenoid is calculated by

$$\begin{aligned}
Area &= \int \int \sqrt{EG - F^2}\ dxdy \\
&= \int_{-x_0}^{x_0} \int_0^{2\pi} c \cosh^2(x/c)\ dydx = 2\pi c \int_{-x_0}^{x_0} \cosh^2(x/c)\ dx \\
&= \pi c \Big[c \cosh(x/c) \sinh(x/c) + x \Big]_{x=-x_0}^{x=x_0} \\
&= \pi c \Big[c \cosh(x_0/c) \sinh(x_0/c) + x_0 - c \underbrace{\cosh(-x_0/c)}_{\cosh(x_0/c)} \underbrace{\sinh(-x_0/c)}_{-\sinh(x_0/c)} + x_0 \Big] \\
&= \pi c \Big(2c \cosh(x_0/c) \sinh(x_0/c) + 2x_0 \Big) \\
&= 2\pi c \Big(c \cosh(x_0/c) \sinh(x_0/c) + x_0 \Big)
\end{aligned}$$

Thus the surface area of the catenoid $\mathbf{x}(x, y)$ with $x \in (-x_0, x_0)$ is given by

$$\text{Area of Catenoid} = AC = 2\pi c \Big(c \cosh(x_0/c) \sinh(x_0/c) + x_0 \Big) \tag{5.14}$$

Let us look at the catenoid that is formed by rotating the catenary curves $y_1(x)$ and $y_2(x)$ given by (5.11) and (5.12) respectively. These two catenoids are shown in figure 5.6. Figure 5.6a shows the catenoid that is formed by rotating the catenary curve $y_1(x)$, given by (5.11), around the x-axis. Figure 5.6b shows the catenoid that is formed by rotating the catenary curve $y_2(x)$, given by (5.12), around the x-axis. In figure 5.6 we have rotated the catenary curves around the x-axis, and then rotated the resulting catenoid by an angle of 90°. We have chosen to show the catenoid in this position along the z-axis, as this gives a better view of the shape of the catenoid.

The catenoid in 5.6a has a 'narrow neck', and the catenoid in 5.6b has a 'wide neck'. If we look at the shapes of the catenary curves in figure 5.5, we can see why the shape of the catenoid is the way that it is.

Let us see why the catenoid in 5.6a has a narrow neck. First of all this catenoid is formed by rotating the catenary curve $y_1(x) = 0.1579\cosh(x/0.1579)$ around the x-axis. From figure 5.5a we can see the shape of the curve $y_1(x)$. This curve is narrow (compared to the curve $y_2(x)$) in the x direction. If we revolve $y_1(x)$ around the x-axis, the surface formed is the narrow neck catenoid. Thus the narrow neck of the surface is due to the narrow shaped catenary curve $y_1(x)$. This gives us a justification for the shapes of the catenoids. In the same way, we can also see why the catenoid in 5.6b has a wide neck. From figure 5.5b, the catenary $y_2(x) = 0.9107\cosh(x/0.9107)$ is wide shaped. The catenoid formed by revolving $y_2(x)$ around the x-axis will form the corresponding wide neck catenoid. These are the two possible catenoids that have the same value of $x/y = 0.4$. Moving forward, we will use the following notation to denote the two catenoids, which will make it easier to refer to each of them.

- Consider the narrow neck catenoid in figure 5.6a, formed by rotating the catenary curve $y_1(x) = 0.1579\cosh(x/0.1579)$. We will denote this by

$$C_1 = \text{Narrow neck catenoid} \tag{5.15}$$

- Consider the wide neck catenoid in figure 5.6b, formed by rotating the catenary curve $y_2(x) = 0.9107\cosh(x/0.9107)$. We will denote this by

$$C_2 = \text{Wide neck catenoid} \tag{5.16}$$

Let us look at the area minimizing property of the catenoids. We consider the problem of having minimal surface area. From (5.14) we can calculate the surface area of the catenoids C_1 and C_2.

- The surface area of the catenoid C_1 with $c = c_1 = 0.1579$ and $x_0 = 0.4$ is

$$\begin{aligned} AC_1 &= 2\pi c_1\Big(c_1\cosh(x_0/c_1)\sinh(x_0/c_1) + x_0\Big) \\ &= 2\pi\cdot 0.1579\Big(0.1579\cosh(0.4/0.1579)\sinh(0.4/0.1579) + 0.4\Big) \\ &= 6.6086 \end{aligned} \tag{5.17}$$

- The surface area of the catenoid C_2 with $c = c_2 = 0.9107$ and $x_0 = 0.4$ is

$$\begin{aligned} AC_2 &= 2\pi c_2\Big(c_2\cosh(x_0/c_2)\sinh(x_0/c_2) + x_0\Big) \\ &= 2\pi\cdot 0.9107\Big(0.9107\cosh(0.4/0.9107)\sinh(0.4/0.9107) + 0.4\Big) \\ &= 4.8836 \end{aligned} \tag{5.18}$$

The catenoid C_2 has less surface area than the catenoid C_1. Thus the catenoid C_2 has minimum surface area. This means that the wide neck catenoid is the area minimizing minimal surface. We have found that the wide neck catenoid is the minimal surface that has the area minimizing property.

This result has an analogue in the soap film experiments that were discussed earlier. If we perform the soap film experiment, we will notice that the catenoid formed by the soap film will always take the shape of the wide neck catenoid. As we have seen from figure 5.6 that there are two possible catenoids that can be formed with the same value for $x/y = 0.4$. However the catenoid formed by the soap film, will always be in the shape of the wide neck catenoid. The reason for this is given by the first principle of soap films, in theorem 5.1.1. The theorem tells us that a soap film always takes the shape that will minimize its surface area. Since the wide neck catenoid is the area minimizing minimal surface, the soap film will take the shape of the wide neck catenoid, and not the narrow neck catenoid. Thus the catenoid that we will see from the soap film experiment will be the wide neck catenoid. This tells us that our findings of the catenoid with minimal surface area is consistent with theorem 5.1.1.

This result also has an analogue in the results that we have found by looking at the area functional $A(t)$ and the variational surfaces of a minimal surface. In order to make the connection we will have to go back to our stability analysis in section 4.5. Let us consider the definition of stability given by definition 4.5.1. We have that a minimal surface $\mathbf{x}$ has minimal surface area in a region D if $\mathbf{x}$ is stable in D, which by definition is that the area functional for $\mathbf{x}$ satisfies $A''(0) > 0$. Thus if $\mathbf{x}$ is stable in a region D, then it is area minimizing in D. In section 4.5.1 we have already looked at the stability for the catenoid and found the stable regions. We have found that the catenoid is stable in the region D with $u \in (-R, R)$ and $v \in (0, 2\pi)$ whenever we have $R < 0.5493$. We have also looked at the catenoid in a stable region with $R = 0.3$.

Figure 4.2a shows us the stable catenoid, so we can see the shape of the catenoid when it is area minimizing, that is, when it has minimal surface area. Thus we can see from figure 4.2a that the stable catenoid is the wide neck catenoid. The catenoid outside the stable region is shown in figure 4.3a for $R = 0.7$. We can see that as R gets larger than 0.54, that is, as R starts to go towards the unstable region, the shape of the catenoid starts to go towards the narrow neck catenoid, which is not an area minimizing catenoid. This tells us that the stable catenoid is the wide neck catenoid, which means that the shape of the area minimizing catenoid is given by the the wide neck catenoid. We can also see this by looking at the shapes of the catenoids from figure 5.6b and figure 4.2a. The catenoid in figure 5.6b is the catenoid C_2 that we have found to be area minimizing in this section, and the catenoid in figure 4.2a is the stable catenoid that we have found in section 4.5.1. They both are the wide neck catenoid. The catenoid C_2 which is the wide neck catenoid, is the minimal surface that is locally area minimizing.

Furthermore we can now classify the two catenoids C_1 and C_2 in terms of stability. Let D be the region given by $x \in (-0.4, 0.4)$ and $y \in [0, 2\pi)$. Since C_2 is the minimal surface with least surface area, from definition 4.5.1 we have that

- C_1 is the unstable solution in D
- C_2 is the stable solution in D

The catenoid C_2 is the area minimizing solution to Plateau's problem when the boundary curve Γ is given by the two disks with centre in $(-0.4, 1)$ and $(0.4, 1)$ and the radius is $r = 1$.

5.3 Existence and Uniqueness of The Catenoid

We have looked at an example of the catenoid being formed when $x_0 = 0.4$ and $y_0 = 1$. We also looked at the question of having minimal surface area. If we go back to the soap film experiments, we will notice that the soap film forms a catenoid as long as the two wire frames have a specific distance between them. At a distance greater than that, the soap film will form two circular disks. At this point the only possible solution we can get is that of the two disks.

Through the soap film experiments we can see the following. Given the wire frame which consists of two circular rings, the soap film formed is sometimes a catenoid, and sometimes the two disks. Let us consider when the solution to Plateau's problem is a catenoid, and when the solution is that of the two disks. We would like to consider the problem of the existence and uniqueness of the catenoid as a solution to Plateau's problem. We mention that we have been looking at the problem as a function of x/y, this was mentioned in the previous section. We are looking at the ratio between the distance from $(0, 0)$ to the centre of the disk in the x-coordinate, to the radius of the disk. We can interpret this in the context of the soap film experiments. Considering the two wire frames, this means that we are looking at the ratio of half of the distance between the two wire frames and the radius of the wire frame. We want to see how varying the distance between the wire frames, as well as the radius of the wire frame, affect the shape the soap film takes. This will allow us to look at when the soap film forms a catenoid, and when the catenoid solution does not exist.

Furthermore we would like to know when the surface area of the soap film solution formed is a minimum. To study the area minimizing property of the surface area, we will study the surface area of the catenoid versus the surface area of the two disk solution. This way, we will be able to figure out when we have an area minimizing catenoid solution. Since we are concerned with surface areas, we will divide the surface area of the catenoid with the surface area of the two disks. This technique was described in [Opr00, p. 186].

The surface area of a disk with centre in $(-x_0, y_0)$ and radius $r = y_0$ is

$$\pi r^2 = \pi(y_0^2) = \pi y_0^2$$

The surface area of a disk with centre in (x_0, y_0) and radius $r = y_0$ is

$$\pi r^2 = \pi(y_0^2) = \pi y_0^2$$

Hence the surface area of the two disks is given by

$$\text{Two Disks surface area} = \text{AD} = \pi y_0^2 + \pi y_0^2 = 2\pi y_0^2 \tag{5.19}$$

We have found an expression to calculate the surface area of the catenoid. From (5.14) we had found that the surface area of the catenoid, which is parametrized as in (5.13) with $x \in (-x_0, x_0)$ is given by

$$\text{Catenoid surface area} = \text{AC} = 2\pi c\Big(c \cosh(x_0/c) \sinh(x_0/c) + x_0\Big) \tag{5.20}$$

To find how the surface area of the catenoid solution varies with the surface area of the two disk solution, we divide the area of the catenoid AC, with the area of the two disks AD. From the expressions above we get that

$$\begin{aligned}\frac{\text{AC}}{\text{AD}} &= \frac{2\pi c(c\cosh(x_0/c)\sinh(x_0/c)+x_0)}{2\pi y_0^2}\\ &= \frac{c}{y_0^2}(c\cosh(x_0/c)\sinh(x_0/c)+x_0)\\ &= \frac{c^2}{y_0^2}\cosh(x_0/c)\sinh(x_0/c)+\frac{cx_0}{y_0^2}\\ &= \frac{c}{y_0}\left(\frac{x_0}{y_0}+\frac{c}{y_0}\cosh(x_0/c)\sinh(x_0/c)\right)\end{aligned}$$

We are looking at the problem as a function of x/y. We would like to make a graph of x/y versus the expression for catenoid area/disks area. To do this we would like to express the above expression AC/AD in terms of u, since x/y is also expressed in terms of u. In order to do this, we recall the way we had defined u and x/y. From (5.6), (5.7) we have that $u = c/y$ leading to $1/u = y/c$. Inserting this into the expression for x/y given in (5.8), we have that $x/y = (c/y)\operatorname{arccosh}(y/c)$. This gives us

$$\frac{x_0}{y_0} = \frac{c}{y_0}\operatorname{arccosh}\left(\frac{y_0}{c}\right) \tag{5.21}$$

Using the expression (5.21) and the definition of u, which is $u = c/y_0$ as well as $1/u = y_0/c$, we get that

$$\begin{aligned}\frac{\text{AC}}{\text{AD}} &= \frac{c}{y_0}(\frac{x_0}{y_0}+\frac{c}{y_0}\cosh(x_0/c)\sinh(x_0/c))\\ &= \frac{c}{y_0}(\frac{c}{y_0}\operatorname{arccosh}(y_0/c)+\frac{c}{y_0}\cosh(x_0/c)\sinh(x_0/c))\\ &= \frac{c^2}{y_0^2}(\operatorname{arccosh}(y_0/c)+\cosh(x_0/c)\sinh(x_0/c))\\ &= \frac{c^2}{y_0^2}\Big(\operatorname{arccosh}(y_0/c)+\underbrace{\cosh((x_0/y_0)\cdot(y_0/c))}_{a}\underbrace{\sinh((x_0/y_0)\cdot(y_0/c))}_{b}\Big)\end{aligned}$$

Let us simplify the expressions a and b by using the expression for x_0/y_0. This leads to

$$\begin{aligned}a = \cosh\left(\frac{x_0}{y_0}\cdot\frac{y_0}{c}\right) &= \cosh\left(\frac{c}{y_0}\operatorname{arccosh}\left(\frac{y_0}{c}\right)\cdot\frac{y_0}{c}\right)\\ &= \cosh\left(\operatorname{arccosh}\left(\frac{y_0}{c}\right)\right)\\ &= \frac{y_0}{c}\\ b = \sinh\left(\frac{x_0}{y_0}\cdot\frac{y_0}{c}\right) &= \sinh\left(\frac{c}{y_0}\operatorname{arccosh}\left(\frac{y_0}{c}\right)\cdot\frac{y_0}{c}\right)\\ &= \sinh\left(\operatorname{arccosh}\left(\frac{y_0}{c}\right)\right)\\ &= \sqrt{\left(\frac{y_0}{c}\right)^2-1}\end{aligned}$$

Now we insert the simplified expressions for a and b into the expression for AC/AD. Once we have done that, we use the fact that $u = c/y_0$ and $1/u = y_0/c$, as mentioned previously, to find an expression for AC/AD in terms of u. This leads to

$$\begin{aligned}\frac{\text{AC}}{\text{AD}} &= \frac{c^2}{y_0^2}\Big(\operatorname{arccosh}\left(\frac{y_0}{c}\right) + \frac{y_0}{c}\sqrt{\left(\frac{y_0}{c}\right)^2 - 1}\Big)\\ &= \left(\frac{c}{y_0}\right)^2\Big(\operatorname{arccosh}\left(\frac{y_0}{c}\right) + \frac{y_0}{c}\sqrt{\left(\frac{y_0}{c}\right)^2 - 1}\Big)\\ &= u^2\Big(\operatorname{arccosh}\left(\frac{1}{u}\right) + \frac{1}{u}\sqrt{\left(\frac{1}{u}\right)^2 - 1}\Big)\\ &= u^2\Big(\operatorname{arccosh}\left(\frac{1}{u}\right) + \frac{1}{u}\sqrt{\frac{1}{u^2} - 1}\Big)\end{aligned}$$

Hence an expression for AC/AD in terms of u is given by

$$\frac{\text{AC}}{\text{AD}} = u^2\left(\operatorname{arccosh}(1/u) + \frac{1}{u}\sqrt{\frac{1}{u^2} - 1}\right) \tag{5.22}$$

Now we are finally in a position to examine the surface area of the catenoid versus the surface area of the two disk solution. But first we make an observation from the previous section, and from figure 5.4. We found that $A = (0.5524, 0.6627)$ is the maximum for x/y. We have that for $x/y < 0.6627$, there will always be two values of u, and hence c, that give us the same x/y. In this case, we will always have one value of u that is less than 0.5524, and one value of u that is greater than 0.5524. The value of $u < 0.5524$ will give us the narrow neck catenoid when the catenoid is formed by revolving the catenary curve around the x-axis. The value of $u > 0.5524$ will give us the wide neck catenoid. Furthermore we also found that the wide neck catenoid is the area minimizing catenoid. For $x/y > 0.6627$ there are no values of u that exist to give the equation of the catenary. This means that for $x/y > 0.6627$ there are no catenoids that can be formed.

We are now in a position to interpret the plot shown in figure 5.7. This will allow us to examine the general case. In this figure we have plotted the function x/y given in (5.8), against the function AC/AD, given in (5.22). The x-axis shows

$$x\text{-axis: } \frac{x}{y} = u\operatorname{arccosh}\left(\frac{1}{u}\right)$$

and the y-axis shows

$$y\text{-axis: AC/AD} = u^2\Big(\operatorname{arccosh}(1/u) + \frac{1}{u}\sqrt{\frac{1}{u^2} - 1}\Big)$$

We have plotted x/y against AC/AD in two different intervals of u. We have plotted the graph in the *narrow neck catenoid interval*, which is given by

$$\text{Narrow neck catenoid interval: } u \in (0, 0.5524) \tag{5.23}$$

We have also plotted the graph in the *wide neck catenoid interval*, which is given by

$$\text{Wide neck catenoid interval: } u \in (0.5524, 1) \tag{5.24}$$

We mention that the solution that consists of two disks has a special name. It is called the *Goldschmidt solution*. We draw a straight line at AC/AD= 1 to represent the Goldschmidt solution. This is shown by the green line. The blue line shows us for which values of x/y the narrow neck catenoid solution exists. The yellow line shows for which values of x/y the wide neck catenoid solution exists. The function AC/AD gives us a measure of the surface area. From the plot we can see as x/y varies which solutions exists, as well as which solutions exist that give us the minimal surface area.

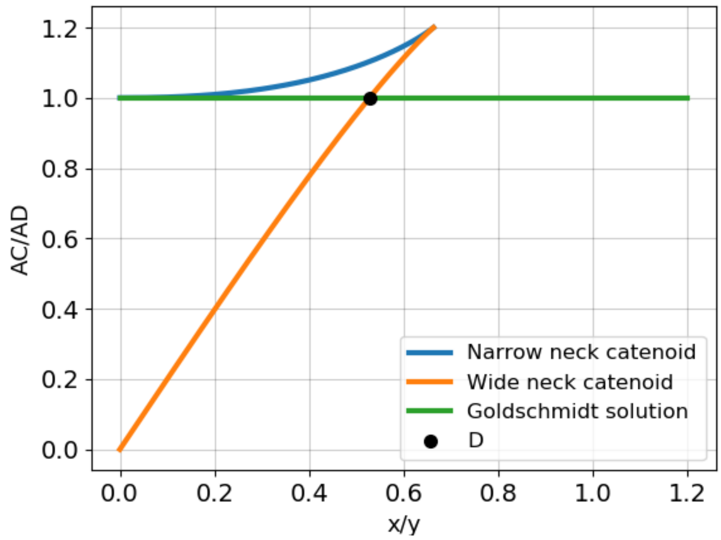


Figure 5.7: Graph of x/y against AC/AD. The function $x/y = u \operatorname{arccosh}(1/u)$. The point D is the intersection point between the wide neck catenoid solution and the Goldschmidt solution, it is given by $D = (0.5277, 1.0000)$.

Let us look at the existence of solutions. From the plot in figure 5.7 we can see that up to a certain value of x/y, all three solutions exist. This point is $x/y = 0.6627$. This means that for x/y up to the point $x/y = 0.6627$, the wide neck catenoid, narrow neck catenoid and the Goldschmidt solutions exist. This is consistent with what we found from figure 5.3. We can also see that for x/y beyond this point, only one solution exists. It is the Goldschmidt solution. We can see that for $x/y > 0.6627$, the catenoid solution does not exist and the only solution that exists is the Goldschmidt solution. This is what we found from figure 5.3. We found that 0.6627 is the maximum point of x/y. Thus for values of x/y beyond its maximum point, no catenoid solutions exist. We have also seen that for x/y less than its maximum value, there are two catenoids that can be formed.

Let us look at the solutions that exist and have minimal surface area. From the plot in figure 5.7 we can see that from 0 up to the point D, the wide neck catenoid has less surface area compared to the narrow neck catenoid and the

Goldschmidt solution. After the point D, the solution that has the least surface area is the Goldschmidt solution. From 0 up to the point $D = (0.5277, 1)$ the wide neck catenoid is the area minimizing solution. Beyond the point $D = (0.5277, 1)$ the Goldschmidt solution is the area minimizing solution. Note that beyond $x/y = 0.6627$, the only solution that exists is the Goldschmidt solution. Hence for $x/y > 0.6627$ the only solution that exists and minimizes the surface area is the Goldschmidt solution. Furthermore considering the narrow neck catenoid and the Goldschmidt solution, we can see that approximately after $x/y = 0.2$, the Goldschmidt solution always has less surface area compared to the narrow neck catenoid.

Let us consider the soap film experiment with the wire frame given by the two circular rings. Then we can interpret the plot in figure 5.7 in relation with the soap film experiments. We will need the result of theorem 5.1.1, which tells us about the first principle of soap films. Due to this theorem, we know that a soap film will always take a shape such that its surface area is minimized. The plot in figure 5.7 shows us that from the point $(0, 0)$ to $D = (0.5277, 1)$, the soap film will take the shape of the wide neck catenoid. This is because the wide neck catenoid is the solution with the least surface area, hence satisfying the condition for least surface area for soap films from theorem 5.1.1. Beyond the point D, the soap film will take the shape of the two disks, which is the Goldschmidt solution.

We can see this in relation to the soap film experiments. To begin with, the soap film will form the shape of the catenoid. We slowly increase the distance between the two wire frames. At a certain point the distance will increase so much that the catenoid soap film will break and form the two disk solution. This is reflected by the discontinuity at point D. Initially the solution is the yellow curve, at point D it changes to the green curve. The solution changes abruptly as the catenoid soap film breaks and forms the two disk solution at point D. Looking at the plot in figure 5.7 we can now explain why the soap film formed will never take the shape of the narrow neck catenoid. The wide neck catenoid always has less surface area than the narrow neck catenoid. Thus the soap film will take the shape of the catenoid that has least surface area, which is the wide neck catenoid, that is when the catenoid solution does exist.

We can say more about least surface area solutions. Looking at the plot in figure 5.7 we can see that in the interval $x/y \in (0, 0.5277)$, the catenoid gives us an absolute minimum for surface area, while the Goldschmidt solution gives us a local minimum for surface area. Note that when we talk about the catenoid solution with minimum surface area, we are referring to the wide neck catenoid. Thus the two solutions that can exist are the catenoid solution and the Goldschmidt solution. We can classify these two solutions as an absolute minimum for surface area, or as a local minimum for surface area.

This classification is given by table 5.1. Table 5.1 shows us which solution is the area minimizing solution given a value for x/y. The table gives us a summary of what we have found by looking at the plot in figure 5.7. If we are given some value of x/y, we can find out which solution will be an absolute minimum for surface area by looking at table 5.1. Thus the table gives an overview of the general case. We note that for $x/y > 0.6627$, no catenoid solution exist.

In the previous section, we have looked at an example where we had $x_0 = 0.4$ and $y_0 = 1$, so we had $x/y = 0.4$. Let us verify what we had found in the

Interval for x/y	Solution to Plateau's problem	
	Absolute Minimum for Surface Area	**Local Minimum for Surface Area**
$0 < x/y < 0.5277$	Catenoid solution	Goldschmidt solution
$0.5277 < x/y < 0.6627$	Goldschmidt solution	Catenoid solution
$x/y > 0.6627$	Goldschmidt solution	—

Table 5.1: Solution to Plateau's problem when the boundary curve Γ is given by the two circular rings. We classify the two solutions as being absolute minimum or local minimum in terms of the surface area, in the given interval for x/y.

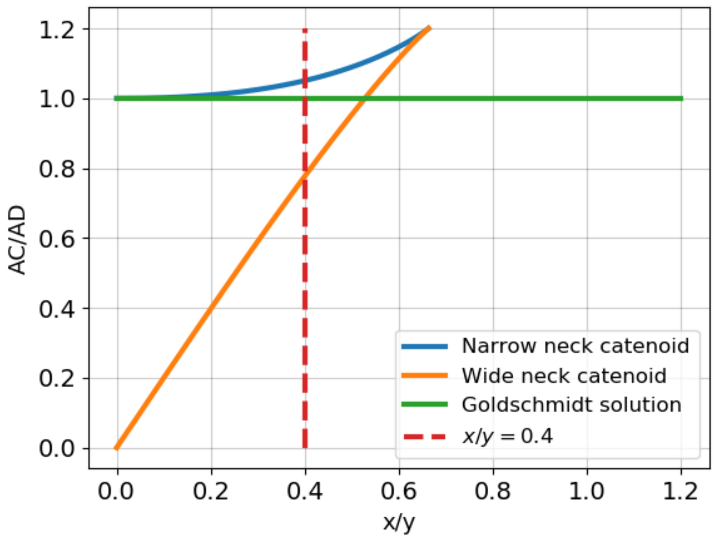


Figure 5.8: Graph of x/y against AC/AD. The function $x/y = u \operatorname{arccosh}(1/u)$. We draw a line at $x/y = 0.4$. This is given by the red dotted line.

previous section by looking at table 5.1 and the results found from the plot in figure 5.7. To verify our results for $x/y = 0.4$, we draw a straight line at the point $x/y = 0.4$ on the plot that we had in figure 5.7. This plot with the line $x/y = 0.4$ is given in figure 5.8.

From the plot in figure 5.8, we can see that at $x/y = 0.4$ three solutions exist. These three solutions are the narrow neck catenoid, the wide neck catenoid and the Goldschmidt solution. The solution that is area minimizing is the wide neck catenoid solution. We can see this on the plot as well. It is given by the intersection of the red dotted line with the yellow line. This is the point which gives the least surface area. This solution is given by the wide neck catenoid. This is what we would expect as we have found that the wide neck catenoid is the area minimizing catenoid. Let us calculate the surface area of each of the solutions;

- From (5.19) we calculate the area of the two disks. The radius of the disk is $r = 1$, so

  $$\text{Area of the two disks } = 2\pi(1)^2 = 2\pi \approx 6.2832$$

  To compare it with the plot in figure 5.8, we recall that we were looking at the ratios of the surface areas of the two solutions. This means we divide the area of the disk by itself to get the area of the Goldschmidt solution, which is

  $$\textit{Area of Goldschmidt solution } = 2\pi/2\pi = 1 \tag{5.25}$$

- We have found the area of the narrow neck catenoid in (5.17). We have to divide by the area of the disks to compare it to the plot in figure 5.8.

  $$\textit{Area of Narrow neck catenoid solution } = 6.6086/2\pi \approx 1.0518 \tag{5.26}$$

- Similarly we have found the area of the wide neck catenoid in (5.18). We divide by the area of the two disks and get

  $$\textit{Area of Wide neck catenoid solution } = 4.8836/2\pi \approx 0.7772 \tag{5.27}$$

From doing the calculations we get that the solution with the least surface area is the wide neck catenoid, with a surface area of 0.7772. This is consistent with the plot in figure 5.8. We get that the surface area of the narrow neck catenoid, which is 1.0518, is greater than that of the Goldschmidt solution. This is what we can see by looking at the plot in figure 5.8. To read the plot, the surface area of the narrow neck catenoid is given by the intersection of the red dotted line and the blue line. We can see that they both intersect at approximately 1.05. Thus our calculations are consistent with what the plot shows in figure 5.8.

For $x/y = 0.4$, the wide neck catenoid solution is the absolute area minimizing solution. On the other hand, the Goldschmidt solution is the local area minimizing solution. By using the concept of stability from definition 4.5.1, the catenoid is the stable solution in the region D given by $x \in (-0.4, 0.4)$ and $y \in [0, 2\pi)$, while the Goldschmidt solution is the unstable solution in D. This is what we get by referring to table 5.1 with $x/y = 0.4$, and this is what we get by explicitly calculating their surface areas.

5.4 Further Examples of The Catenoid as a Solution to Plateau's Problem

We have been looking at the solutions to Plateau's problem where the boundary curve Γ is given by the two disks with centres in $(-x_0, y_0)$ and (x_0, y_0). The disks have a radius of y_0. We found in the previous section, that the solutions to the problem with this given Γ is either a catenoid or the two disks. Table 5.1 gives us a summary of the solutions that can exist. We can also see which solution will give us an absolute minimum for surface area hence giving us an area minimizing solution.

Let us consider the results of table 5.1. In the previous section we looked at the case of $x/y = 0.4$. This falls in the interval $0 < x/y < 0.5277$. We have found that the catenoid gives us the absolute minimum for surface area. We mention that whenever we talk about the catenoid with minimal surface area, we are referring to the wide neck catenoid. This is due to the fact that the wide neck catenoid always has less surface area compared to the narrow neck catenoid. Our result is consistent with what is shown by the results in table 5.1. Thus we have looked at an example of the case $0 < x/y < 0.5277$, with $x/y = 0.4$.

Let us also look at the other two cases that are given by table 5.1. Let us look at an example for a value of x/y in the interval $0.5277 < x/y < 0.6627$, and an example for a value for $x/y > 0.6627$. We would like look at an example of these two cases in order to check if the solutions we get will be consistent with the solutions given by table 5.1, just like we had looked at the example with $x/y = 0.4$.

5.4.1 Example with $x/y = 0.5941$

We look at an example of the case where x/y is in the interval $0.5277 < x/y < 0.6627$. Let

$$x_0 = 0.5941 \text{ and } y_0 = 1$$

Then

$$\frac{x}{y} = \frac{x_0}{y_0} = \frac{0.5941}{1} = 0.5941$$

Thus $x/y = 0.5941$. The boundary curve Γ is given by the two circles with centre in $(-0.5941, 1)$ and $(0.5941, 1)$. Both of the circles have radius $r = 1$. In the previous section, we talked about the existence and uniqueness of the solutions. We consider the existence of solutions. For our value of $x/y = 0.5941$, we have that $x/y < 0.6627$. From section 5.3, we know that for $x/y < 0.6627$, all three solutions exist. The three solutions are the wide neck catenoid, the narrow neck catenoid, and the Goldschmidt solution. We can also see this directly by looking at the plot in figure 5.7. This means that we expect to get a catenoid solution, wide neck and narrow neck, as well as the Goldschmidt solution.

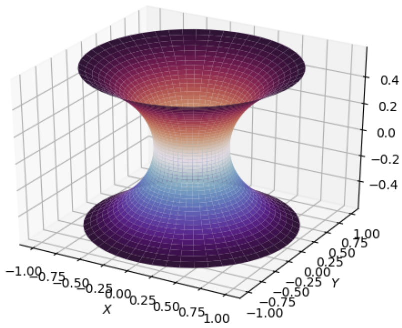
We find the catenoid solutions first. In order to find the catenoid solutions, we do the same thing here as we had done in section 5.2 for the case of $x/y = 0.4$. We omit the details here as everything done here is based on the same approach that was taken in section 5.2 for $x/y = 0.4$. We start by finding the equation of the catenary curve

$$y(x) = c\cosh(x/c)$$

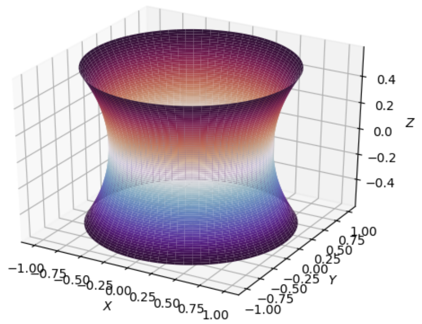
that passes through the points

$$(-0.5941, 1) \text{ and } (0.5941, 1) \tag{5.28}$$

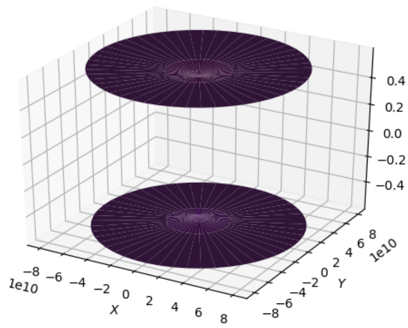
That is, we have to find the value for c in $y(x)$ such that $y(x)$ will pass through the two points given in (5.28). We can find the value for c by finding the value of u defined in (5.6), as $u = c/y$. We find the value for u by finding the intersection of the curve $x/y = 0.5941$ and the curve $x/y = u\,\text{arccosh}(1/u)$. Since $x/y < 0.6627$ we know that that there are two values of u that give us $x/y = 0.5941$. This means that there are two points of intersection, one point that is less than 0.5524, and one point that is greater than 0.5524. Let us



(a) $c_1 = 0.3428$



(b) $c_2 = 0.7533$



(c) Goldschmidt solution

Figure 5.9: Solutions to Plateau's problem with $x/y = 0.5941$. Figure a and b show the catenoid formed by rotating the catenary curve with the given value for c. The parameters are in the region $x \in (-0.5941, 0.5941)$ and $y \in (0, 2\pi)$. Figure c shows the Goldschmidt solution.

denote the intersection points by u_1 and u_2. The values of u give us the values for c, so we denote the corresponding values for c in $y(x)$ by c_1 and c_2. We find that the values for c_1 and c_2 are

$$c_1 = 0.3428 \tag{5.29}$$
$$c_2 = 0.7533 \tag{5.30}$$

Let us denote the corresponding catenary curves formed by c_1 and c_2 by $y_1(x)$ and $y_2(x)$. Then these catenary curves have the equations

$$y_1(x) = c_1 \cosh\left(\frac{x}{c_1}\right) = 0.3428 \cosh\left(\frac{x}{0.3428}\right) \tag{5.31}$$
$$y_2(x) = c_2 \cosh\left(\frac{x}{c_2}\right) = 0.7533 \cosh\left(\frac{x}{0.7533}\right) \tag{5.32}$$

The catenoid is formed by rotating the catenary curves $y_1(x)$ and $y_2(x)$ around the x-axis. These catenoids are shown in figure 5.9 a and b. Figure 5.9a shows the catenoid formed by rotating the catenary curve $y_1(x) = 0.3428 \cosh(x/0.3428)$ around the x-axis. This is the narrow neck catenoid. Let us denote this narrow neck catenoid by C_1. Figure 5.9b shows the catenoid formed by rotating the catenary curve $y_2(x) = 0.7533 \cosh(x/0.7533)$ around the x-axis. This is the wide neck catenoid. Let us denote this wide neck catenoid by C_2. Once again we mention that the catenoids shown in figure 5.9a and b have been rotated by an angle of 90°. We have chosen to show the catenoids in this position as it gives a better view of their shapes.

Besides the catenoid solution, the Goldschmidt solution also exists. Let us denote this Goldschmidt solution by G. The solution G is the two disk solution. It is shown in figure 5.9c.

Now that we have found all three solutions that exist, we calculate their surface areas to find the solution that gives us an absolute minimum for surface area. We wish to find the area minimizing solution. According to table 5.1, we expect the Goldschmidt solution to give us the absolute minimum for surface area, and we expect the catenoid to be the local minimum for surface area. Let us compute their surface areas to verify this.

- Consider the narrow neck catenoid C_1. From (5.14) we can calculate the surface area. The surface area of the catenoid C_1 with $c = c_1 = 0.3428$ and $x_0 = 0.5941$ is

$$AC_1 = 7.1832 \tag{5.33}$$

- Consider the wide neck catenoid C_2. Similarly the surface area of the catenoid C_2 with $c = c_2 = 0.7533$ and $x_0 = 0.5941$ is

$$AC_2 = 6.9438 \tag{5.34}$$

- Consider the Goldschmidt solution G. Then from (5.19) we can calculate the surface area of the two disks. Both of the disks have a radius of $r = 1$. The surface are of the two disks is $2\pi(1)^2 = 2\pi \approx 6.2832$. The surface area of the Goldschmidt solution G is

$$AG = 6.2832 \tag{5.35}$$

By computing the surface areas of the three solutions, we can see that the solution with the least surface area is G, with a surface area of 6.2832. The catenoid C_2 has a surface area of 6.9438, which is less then the surface area of C_1, which is 7.1832. This is what we had expect as the wide neck catenoid is the area minimizing catenoid.

Thus we can see that the Goldschmidt solution is the absolute minimum for surface area, with a surface area of 6.2832. The catenoid C_2 is the local minimum for surface area, with a surface area of 6.9438. We have verified the results given by table 5.1. By the definition of stability, the solution G is the stable solution in the region D given by $x \in (-0.5941, 0.5941)$ and $y \in (0, 2\pi)$, while the catenoid is the unstable solution in D.

The solution G is the absolute area minimizing solution to Plateau's problem when the boundary curve Γ consists of the two disks with centres in $(-0.5941, 1)$ and $(0.5941, 1)$, and the radius is $r = 1$.

5.4.2 Example with $x/y = 0.7$

We look at an example of the case where x/y has a value such that $x/y > 0.6627$. Let

$$x_0 = 0.7 \text{ and } y_0 = 1$$

Then

$$\frac{x}{y} = \frac{x_0}{y_0} = \frac{0.7}{1} = 0.7$$

Thus $x/y = 0.7$. The boundary curve Γ is given by the two circles with centre in $(-0.7, 1)$ and $(0.7, 1)$. Both of the circles have radius $r = 1$. In the previous section, we talked about the existence and uniqueness of solutions. Let us consider the existence of solutions. From section 5.3 we have that for $x/y > 0.6627$, there exists only one solution. This only solution is the Goldschmidt solution. For the value of $x/y = 0.7$, we have that $x/y > 0.6627$. We expect to get the only solution, the Goldschmidt solution.

Let us start by finding the catenoid solutions. We start by finding the equation of the catenary curve

$$y(x) = c\cosh(x/c)$$

that passes through the points

$$(-0.7, 1) \text{ and } (0.7, 1) \tag{5.36}$$

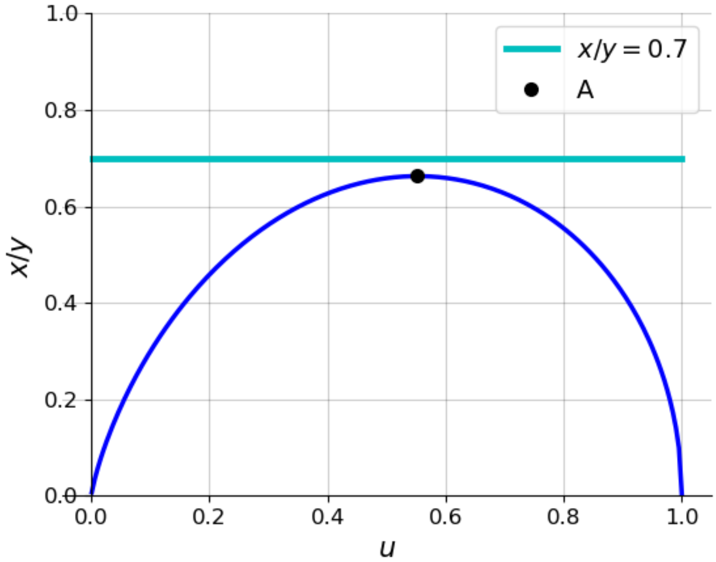


Figure 5.10: Graph of u against x/y. The function $x/y = u\,\text{arccosh}(1/u)$. The point A is the maximum point, it is given by $A = (0.5524, 0.6627)$.

We have to find the value for c in $y(x)$. As we have done in the previous example, we can find the value for c by finding the value of u in (5.6), as $u = c/y$. We find the value for u by finding the intersection of the curve $x/y = 0.7$ and the curve $x/y = u\,\text{arccosh}(1/u)$. However let us take a look at the plot of $x/y = u\,\text{arccosh}(1/u)$ and the line $x/y = 0.7$. This is shown in figure 5.10. From the figure we can see that the two curves never intersect. This is what we had expected as 0.6627 is the maximum value for x/y, and $x/y = 0.7$ exceeds the maximum value. Thus there are no catenary curves that pass through the points $(-0.7, 1)$ and $(0.7, 1)$, and hence no catenoid solutions can be formed.

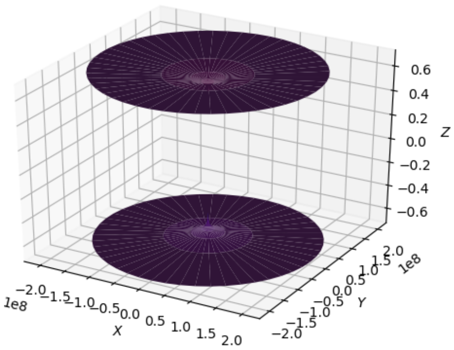


Figure 5.11: Solution to Plateau's problem with $x/y = 0.7$. There is only one solution that exists, namely the Goldschmidt solution.

Therefore the only solution that exists is the Goldschmidt solution. We denote this solution by G. It is shown in figure 5.11. We can see that for $x/y = 0.7$, G is the only solution that exists. Hence G is the unique solution that is also the area minimizing solution. That is, G gives us an absolute minimum for surface area. Let us compute the surface area for G.

- We can compute surface area of the two disks from (5.19). Both of the disks have a radius of $r = 1$. The surface area of the Goldschmidt solution G is

$$AG = 6.2832 \tag{5.37}$$

The solution G is the unique area minimizing solution with a surface area of 6.2832. By the results of table 5.1, we expect to find the Goldschmidt solution as the absolute minimum for surface area. We have verified the results of table 5.1.

By the definition of stability. the solution G is the stable solution in the region D given by $x \in (-0.7, 0.7)$ and $y \in (0, 2\pi)$. The solution G is the absolute area minimizing solution to Plateau's problem when the boundary curve Γ consists of the two disks with centres in $(-0.7, 1)$ and $(0.7, 1)$, and the radius is $r = 1$.

5.5 Enneper's surface

We look at another example of Plateau's problem. Another minimal surface that is a solution to Plateau's problem is the Enneper surface [Opr00, pp. 113–114].

It is possible to produce a model the Enneper's surface by the soap film experiments. Let us consider the wire frame that we have to dip in the soap solution to produce a model of the Enneper surface. From section 3.4 we have that the Enneper surface is parameterized in u, v coordinates by

$$\mathbf{x}(u,v) = (u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2), \quad -\infty < u < \infty,\ -\infty < v < \infty$$

where $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Consider a circle of radius r in u, v coordinates given by $u^2 + v^2 = r^2$. Then the wire frame is obtained by evaluating the parametrization $\mathbf{x}(u, v)$ on the circle $u^2+v^2 = r^2$. This means that the boundary curve Γ is given by

$$\Gamma = \mathbf{x}(u, v) \text{ where } \mathbf{x} : \underbrace{\{(u, v)|u^2 + v^2 = r^2\}}_{U} \subset \mathbb{R}^2 \to \mathbb{R}^3$$

The boundary curve Γ is obtained by evaluating the Enneper parametrization $\mathbf{x}(u, v)$ on the circle of radius r, thus $U = \{(u, v)|u^2 + v^2 = r^2\}$. We note that r is a radius so $r \in [0, \infty)$. We can simplify the expression for Γ above by considering the parametrization for Enneper's surface in polar coordinates. From (3.30) the parametrization in polar coordinates is given by

$$\mathbf{x}(r, \theta) = \Big(r\cos(\theta) - \frac{r^3}{3}\cos(3\theta),\ -r\sin(\theta) - \frac{r^3}{3}\sin(3\theta),\ r^2\cos(2\theta)\Big)$$
$$0 \le r < \infty,\ 0 \le \theta < 2\pi$$

Fix some value for $R \in (0, \infty)$ to be the radius in the parametrization given above. Then the boundary curve Γ can be written as

$$\Gamma = \mathbf{x}(R, \theta) \text{ with } \theta \in [0, 2\pi)$$

The main result for Enneper's surface as a solution to Plateau's problem is given by the following theorem.

Theorem 5.5.1. *[Opr00, p. 114] Let R be the radius. Let the boundary curve Γ be given as*

$$\Gamma = \mathbf{x}(R, \theta) = \Big(R\cos(\theta) - \frac{R^3}{3}\cos(3\theta),\ -R\sin(\theta) - \frac{R^3}{3}\sin(3\theta),\ R^2\cos(2\theta)\Big) \tag{5.38}$$

for $\theta \in [0, 2\pi)$. Then the Enneper's surface $\mathbf{x}(r, \theta)$, which is given by

$$\mathbf{x}(r, \theta) = \Big(r\cos(\theta) - \frac{r^3}{3}\cos(3\theta),\ -r\sin(\theta) - \frac{r^3}{3}\sin(3\theta),\ r^2\cos(2\theta)\Big) \tag{5.39}$$
$$0 \le r \le R,\ 0 \le \theta < 2\pi$$

is the unique solution to Plateau's problem for

$$0 < R \le 1$$

Let us take a closer look at the result of theorem 5.5.1. This theorem tells us that as long as the radius R has values in the interval $0 < R \le 1$, the Enneper surface that is formed by $\mathbf{x}(r, \theta)$ with the parameters $0 \le r \le R$ and $0 \le \theta < 2\pi$, is the unique solution to Plateau's problem, when the boundary curve Γ is given by $\mathbf{x}(R, \theta)$ as in (5.38).

Let the boundary curve Γ be given by (5.38). Let us look at the existence of solutions. Firstly we mention that for $0 < R \le 1$, the Enneper surface has no self intersections. We know this because of the result of theorem 3.4.1. Theorem 3.4.1 tells us that Enneper's surface has no self intersections for $0 \le r < \sqrt{3}$. In this case we have $0 \le r \le 1$, which is less than $\sqrt{3}$. Thus for $0 \le r \le 1$, the Enneper surface has no self intersections. This further tells us that the boundary curve $\Gamma = \mathbf{x}(R, \theta)$ has no self intersections for $0 < R \leq 1$. The curve Γ is a Jordan curve as it is a simple closed curve for $0 < R \leq 1$. Then by lemma 5.1.5 we know that there exists an area minimizing disk like minimal surface that encloses the curve Γ.

Let the boundary curve Γ be given by (5.38). Then from theorem 5.5.1 we know that the Enneper surface is the unique solution to Plateau's problem for $0 < R \leq 1$. Let us look at the area minimizing property of the solutions with the boundary curve Γ given as in (5.38). We would like to see if the solution to Plateau's problem has minimal surface area. To look at this we revisit our stability analysis. We have already looked at the stability of Enneper's surface in section 4.5.2. We found the stable region D for Enneper's surface. From (4.25) we have that the Enneper surface is stable for $R < 1$. The stable region D is given by $r \in [0, 1)$ and $\theta \in [0, 2\pi)$. This tells us that the Enneper surface $\mathbf{x}(r, \theta)$, given by the parametrization (5.39), is the area minimizing solution to Plateau's problem for $0 < R < 1$. This Enneper surface is given by the parametrization (5.39) where $0 \leq r < 1$ and $0 \leq \theta < 2\pi$.

We can see what the area minimizing Enneper's surface looks like. In figure 4.4a we show the Enneper surface for $R = 0.8$. This value of R is less than 1, and therefore figure 4.4a shows the area minimizing solution to Plateau's problem when the boundary curve $\Gamma = \mathbf{x}(0.8, \theta)$ as defined by (5.38). We mention that this is the area minimizing minimal surface that encloses $\Gamma = \mathbf{x}(0.8, \theta)$ among all surfaces that have Γ as their boundary.

CHAPTER 6

Conclusion

In this thesis we have looked at the theory of minimal surfaces in $\mathbb{R}^3$. We started by recalling some concepts from differential geometry. We then looked at the theory and definition of a minimal surface. To get a better understanding of the theory, we looked at some examples of minimal surfaces in $\mathbb{R}^3$. We also looked at some interesting properties of these surfaces.

The next property we looked at is the area minimizing property of minimal surfaces. The question we wanted to answer was 'When does a minimal surface have minimal surface area?'. We looked at three examples of minimal surfaces and answered this question for each of them. The minimal surfaces we looked at are the catenoid, the Enneper surface, and the higher order Enneper surfaces.

Let us consider the catenoid. From the stability analysis we found that the catenoid is stable in the region D where $u \in (-R, R)$ and $v \in (0, 2\pi)$, whenever we have $R < 0.5493$. In this region the catenoid is an area minimizing minimal surface. Furthermore by looking at the catenoid as a solution to Plateau's problem, we found that the catenoid gives an absolute minimum for surface area when $0 < x/y < 0.5277$, with x/y given in (5.8). Figure 4.2a and figure 5.6b show the shape of the catenoid when it is area minimizing. From these figures we can see that the area minimizing catenoid has the 'wide neck', as discussed in section 5.2. The wide neck catenoid always has less surface area compared to the narrow neck catenoid. This can be seen from the graph in figure 5.7. Therefore the catenoid has minimal surface area when it is the wide neck catenoid.

Let us consider Enneper's surface. From the stability analysis we found that the Enneper surface is stable in the region D where $r \in [0, R)$ and $\theta \in [0, 2\pi)$, whenever we have $R < 1$. In this region the Enneper surface is an area minimizing minimal surface. Let us fix some value for the radius R. Then we consider Enneper's surface as a solution to Plateau's problem. The boundary curve Γ is given by $\mathbf{x}(R, \theta)$ in (5.38). The Enneper surface has minimal surface area when $0 < R < 1$. It is parametrized by $\mathbf{x}(r, \theta)$ in (5.39), where $0 \leq r < 1$ and $0 \leq \theta < 2\pi$. Figure 4.4a shows the Enneper surface when it is area minimizing with the value of R chosen to be $R = 0.8$.

Let us consider the higher order Enneper surfaces. Let k be the order of the Enneper surface. For this example of minimal surfaces, we looked at the stability condition in section 4.5.3. We found that the higher order Enneper surfaces are stable in the region D where $r \in [0, R)$ and $\theta \in [0, 2\pi)$, whenever we have $R < 1/(2k-1)^{1/2k}$. In this region, the Enneper surface of order k is an area minimizing minimal surface. Table 4.1 shows the value for R for

$k = 1, 2, 3, 4, 5$. Furthermore figure 4.8 shows the higher order Enneper surfaces when they are area minimizing, for $k = 2, 3, 4$ and 5 with $R = 0.6$.

Thus we have looked at some examples where we where able to find when a minimal surface has minimal surface area.

Bibliography

[DHS10] Dierkes, U., Hildebrandt, S. and Sauvigny, F. *Minimal Surfaces*. 2nd ed. Vol. 339. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 2010.

[Do 16] Do Carmo, M. *Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition*. Dover Publications, 2016.

[Fan96] Fang, Y. *Lectures on minimal surfaces in R^3*. Proceedings of the Centre for Mathematics and its Applications, Volume 35. Centre for Mathematics and its Applications, Australian National University, 1996.

[Nit89] Nitsche, J. C. C. *Lectures on Minimal Surfaces: Volume 1, Introduction, fundamentals, geometry and basic boundary value problems*. Cambridge University Press, 1989.

[ONe06] O'Neill, B. *Elementary Differential Geometry*. Revised Second Edition. Elsevier Academic Press, 2006.

[Opr00] Oprea, J. *The Mathematics of Soap Films: Explorations with Maple*. American Mathematical Society, 2000.

[Pér17] Pérez, J. 'A New Golden Age of Minimal Surfaces'. In: *Notices of the American Mathematical Society* vol. 64 (2017), pp. 347–358.

[Rad93] Radó, T. *On the Problem of Plateau*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge. Springer Berlin, Heidelberg, 1993.

[Tap16] Tapp, K. *Differential Geometry of Curves and Surfaces*. Springer, 2016.

[Web18] Weber, M. *Higher Order Enneper Surfaces*. en. https://minimalsurfaces.blog/home/repository/symmetrizations/higher-order-enneper-surfaces/. Oct. 2018.