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Optimizing bivariate reinsurance with dependent risks

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

In the optimal risk model, people usually are concerned about the dependent risks to explore how the optimal reinsurance contracts vary with the degree of dependence. In this thesis, we investigate this problem in bivariate case using value-at-risk as risk measure further. There is no bundling of the two risks, and each risk is insured under a separate reinsurance contract. It is possible to formulate the optimization problem as an optimization task with two variables, subject to a single constraint. Specifically, we present an efficient method for estimating optimal contracts using importance sampling. The dependence is modeled using a Gaussian copula. The optimal solution is evaluated by the constraint curves and iso-curves of the objective function. The methods will be illustrated on a suitable set of examples, including symmetric and asymmetric cases as well as mixtures of distributions from Pareto, lognormal, truncated normal and gamma distributions. The optimal reinsurance contract relies on the correlation coefficient and the hazard rates of the risk distributions. With the increase in correlation coefficient, the optimal solution for symmetric risks will eventually be the balanced solution which means the insurance layer contracts should be chosen. However, the optimal solution is usually unbalanced for asymmetric risks for changing correlation coefficients. Furthermore, the more asymmetric the risks are, the closer the optimal solution is to the boundary and, therefore, the better the lighter-tailed risk should be covered by a stop-loss contract.

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CHAPTER 1

Introduction

Reinsurance plays a vital role in the insurance business. Insurance companies sign reinsurance contracts with reinsurance companies. The essence of the contracts is to provide financial support to the less stable insurance companies since it transfers the risks of the primary insurer to a reinsurance company. The insurance companies pay out premiums to the reinsurance companies to get corresponding protection from losses that may arise from natural disasters or that the customer may face in the long run while using its products that are deemed to have been expensively obtained.

In Actuarial Science, an optimal insurance contract is usually studied from the perspective of the insurer by considering different contract types and risk measures. Cheung et al.[3] considered the univariate optimal reinsurance contract under four risk measures. It was shown that if average value-at-risk or more generally, a law-invariant convex risk measure is used, a stop-loss contract is optimal. On the other hand, an insurance layer contract is optimal if value-at-risk or conditional tail expectation is used. Huseby and Christensen [5] showed that the optimal reinsurance in the multivariate case must satisfy certain conditions under value-at-risk.

In this thesis, we consider an insurance company with two business lines or two policyholders. There are two types of bivariate reinsurances; reinsurance with independent and dependent risks. Independent risks mean that the occurrence of insurance risk events is regarded as independent events, resulting in independent compensation. In dependent risk reinsurance, there can be different types of dependency between risks. For example, the occurrence of a single risk triggers the compensation of all the other events that are deemed to be related and dependent on the risk. A case to point out of the dependent risks is that if the insured goods are damaged due to the burning of the house caused by natural disasters, the insurance company can compensate the customer for the insured goods and any other insurance, such as house insurance.

This thesis mainly focuses on dependent bivariate reinsurance, and the optimal reinsurance model is closely related to Huseby [4]. Firstly, we all have the exact formulation of the optimal reinsurance model under value-at-risk as a risk measure, and the objective function and constraint are the same. Secondly, importance sampling is used for simulation. Finally, the concavity or convexity of the objective function is determined by hazard rate. The difference is that we study the bivariate case instead of its multivariate case. More importantly, the correlation of risks will be modeled using a Gaussian copula.

The thesis is organized as follows. Chapter 2 expounds some concepts about reinsurance and risk measures. Methodology for optimizing reinsurance, both in the univariate and multivariate case is reviewed in Chapter 3. Chapter 4 generalizes the methods for optimizing bivariate reinsurance proposed in Huseby and Christensen [5] to cases with dependent risks. More specifically, we develop Gaussian copula methods for quantifying the effect of dependence in relation to reinsurance optimization. Chapter 5 uses numerical examples to describe how the optimal reinsurance contracts vary with the degree of dependence on symmetric and asymmetric risks. In particular, we present the preliminary results. Chapter 6 concludes the thesis.

CHAPTER 2

Notation and Theory

Insurance is the act of compensating for the loss of others. *Reinsurance* can be defined as an act of contracting an insurance policy in order to protect an insurer from loss. As opposed to insurance, cession simply refers to the transfer of liability from an insurer to a reinsurer. This chapter will go through the basic notations and definitions used when talking about spread risk and reinsurance contracts.

2.1 Reinsurance Contracts

In the most basic sense, reinsurance is the insurance of insurance companies. The insurance company transfers the risk of high claims to the reinsurance company by paying a premium. Therefore, reinsurance contracts can be regarded as a protective measure to avoid the bankruptcy of insurance companies. Insurance companies are known as *cedents*, whereas reinsurance companies are known as $reinsurers$.

Let $R(x)$ be a reinsurance function if x is the realized loss. Here, we introduce Cheung et al.[3]'s hypothesis, in order to avoid ethical issues and maintain fairness, any viable reinsurance contract should meet certain characteristics. First, the loss of one additional unit cannot result in more than one incremental unit claim. Second, the reinsurance company's compensation will not decrease with the increase in loss. Mathematically, the hypothesis of the theory on viable reinsurance contract R is as follows:

- $R\left(x\right)-R\left(y\right)\leqslant x-y\ \ \ for\ any\ x\geqslant y\geqslant 0,$
- $R\left(0\right)=0\ and\ R\ is\ non-decreasing.$

Moreover, we will mention two branches of reinsurance contracts, insurance layer contracts and stop-loss insurance contracts.

2.1.1 Stop-loss Contracts

The $stop-loss\ insurance$ coverage is insurance that protects the insurers from having large claims being made by the insured clients. To put it crudely, Stop-loss reinsurance means that the risk is borne by the insurance company within a certain range, and the reinsurance company bears the risk once the risk is greater than this range. For instance, a reinsurance company only assumes a risk that is more than value a which is a positive constant, but claims for compensation amount (risk) is less than a, assumed by the insurance company. Therefore stop-loss reinsurance can be defined as:

$$R(X) = \begin{cases} 0 & for\ X < a \\ X - a & otherwise \end{cases}$$

It could also be written as:

$$R(x) = (x - a)^{+} = max\{0, x - a\}$$

2.1.2 Insurance Layer Contracts

Insurance layer contracts refer to the fact that the reinsurance company bears the risks within a specific range, below which the Insurance company bears the risks, and above which the insurer bears the risks. In other words, it is to point to the former insurer and reinsurer agreement, to every dangerous unit regulation assumes responsibility by former insurer leave the specified amount oneself, exceed the part that leaves the specified amount above oneself to call surplus, the agreement that surplus share gives reinsurer to assume. The insurance layer contracts can be expressed as follow:

$$R(X) = \begin{cases} 0 & for\ X < a \\ X - a & for\ a \leqslant X \leqslant b \\ b - a & otherwise \end{cases}$$

or

$$R(X) = (X - a)^{+} - (X - b)^{+} = \max(\min(X, b) - a, 0),\ 0 \leqslant a \leqslant b$$

. It is important to note that the stop-loss contracts are the special case of insurance layer contracts.

2.2 Pricing Reinsurance Contracts

A Reinsurance contract is a contract that is signed between underwriter and underwriter. One party is the cedent, the other party is the reinsurer. The cedent is an insurer who, according to the reinsurance contract, must pay a certain premium to the reinsurer and has the right to obtain compensation from the reinsurer for the indemnity cost and other related expenses caused by the original insurance contract; the reinsurer is the basis reinsurance contract, have right to collect certain reinsurance costs from the cedent, have an obligation to give the cedent at the same time by reinsurance contract place pay cost.

To measure the fee charged for transferring insurance risk, here introduce the *pure reinsurance premium* concept where a break-even condition is implied:

$$\mathbb{E}[R(X)]$$

For reinsurance company that receives the premium for bearing the risk, the expected cost is $\mathbb{E}[R(X)]$ (this excludes the impact of operation cost and financial earnings). Consequently, risk loading θ needs to be added to the pure reinsurance premium. The value of $\theta > 0$ is determined by the actuarial pricing principle. Under such assumption, the *reinsurance premium* covered by insurers is calculated as:

$$(1+\theta)\,\mathbb{E}\left[R\left(X\right)\right]$$

$\theta\,\mathbb{E}\left[R\left(X\right)\right]$ is considered as the cost of risk covered by the reinsurer. The pricing assumptions are also affected by the fluctuation in market conditions[Bølviken [2] p. 5].

2.3 Risk for Insurer

Let risk X be a nonnegative integrable random variable on a probability space. The *survival function* will be defined and introduced to the model as:

$$S_X\left(x\right) \stackrel{d}{=} \mathbb{P}\left(X > x\right) = 1 - F_X\left(x\right)$$

where F_X is the cumulative distribution function of X and is strictly increasing. We assume S_X is strictly decreasing on $[0, \sup X]$. In this case, $supX$ represents the essential supremum of X.

2.3.1 Retained Risk

During the reinsurance practice, the risk is transferred to companies. The companies receiving the risk are also known as cedent. The logic behind the practice needs to be considered, which is, in essence, transferring risk from one relatively weaker entity to a stronger agent. The Original risk has been analyzed in the previous section, and the reinsurer and the cedent share the risk. We define $I\left(X\right) \stackrel{d}{=} X - R\left(X\right)$, where $I\left(X\right)$ represents the retained risk covered by the cedent. As a result, the loss borne by the cedent in stop-loss contract is:

$$I\left(X\right) = \begin{cases} X & for\ X < a \\ a & otherwise \end{cases} = min\left\{X, a\right\}$$

For a layer insurance the retained risk is given by:

$$\begin{aligned} I\left(X\right) &= \begin{cases} X & for\ X < a \\ a & for\ a \leqslant X \leqslant b \\ X-(b-a) & otherwise \end{cases} \\ &= min\left\{X, a\right\} + min\left\{X - b, 0\right\} \end{aligned}$$

The total retained risk for the cedent after risk exchange is given by:

$$I\left(X\right) + (1+\theta)\,\mathbb{E}\left[R\left(X\right)\right]$$

2.4 Risk Measures

In insurance, the statistical analysis of the risks will make the companies that offer insurance and reinsurance determine the risks they can insure against. If a risk is potentially determined to be a high-risk one, then the company can decide to insure a portion of it or none. Otherwise, the insurance companies will be more than willing to offer insurance cover on low-risk factors. The high risk factors increase the loss of the insurance companies and make the companies vulnerable financially. Thus the insurance companies make a reliable decision following the feedback from results under different risk measures.

2.4.1 Value-at-risk

$Value-at-risk(VaR)$ is the statistic that is used to quantify the extent of the possible financial losses that a firm can incur in a portfolio or a given position over a specified period of time. It is used in the public institutional services to determine the amount and the extent of loss that can be incurred by the firm over the given portfolio of the institutions see Bladt at al.[1]. The risk managers use VaR to analyze the level of the risk exposure and implement strategies that will be effective in controlling the level of the risk exposure. The calculations of the VaR can be used to measure the wide risks of the firms.

The α-level VaR is defined as follows: Fix some level $\alpha \in (0,1)$. For any random variable X,

$$VaR_{\alpha}(X) \stackrel{d}{=} S_X^{-1}(\alpha) \stackrel{d}{=} \inf\{x \ : \ P(X > x) \leqslant \alpha\}$$

VaR has some significant properties which are widely used in this thesis as a risk measure:

(i) Since S_X is strictly decreasing, $S_X^{-1}(\alpha) = r$ if only if

$$\mathbb{P}(X > r) \leqslant \alpha \leqslant \mathbb{P}(X \geqslant r) \tag{2.1}$$

In particular, since S_X is strictly decreasing, $S_X^{-1}(\alpha) = r$ if:

$$\mathbb{P}(X > r) = \alpha \tag{2.2}$$

(ii) For any strictly increasing continuous function ϕ, we have

$$VaR_{\alpha}(\phi(X)) = S_{\phi(X)}^{-1}(\alpha) = \phi\left(S_X^{-1}(\alpha)\right)$$

2.4.2 Average Value-at-risk

$Average\ value - at - risk(AVaR)$, also called $conditional\ value - at - risk(CVaR)$, is considered to be a special case of spectral risk measures as it is superior to the VaR, as it works to satisfy all the variables and the properties of the coherent risk measures. Therefore, it was created to address the shortcomings of VaR.

For $\alpha \in (0,1)$, the average value-at-risk at level $1-\alpha$ is defined by:

$$AVaR_{\alpha}(X) \stackrel{d}{=} \frac{1}{\alpha}\int_0^{\alpha} VaR_{\lambda}(X)\,d\lambda \stackrel{d}{=} \frac{1}{\alpha}\int_0^{\alpha} S_X^{-1}(\lambda)\,d\lambda$$

As a result, AVaR can be thought of as the expected loss in a given percentage of worst cases.

2.4.3 General Law-invariant Convex Risk Measure

$Law - invariant\ convex$ risk measures are risk measures that assign the same value to multiple risky positions that have the same distribution in relation to the measure of the probability of the risk positions. The law convex invariant risk is considered as law invariant because it violates the laws of risk variation between the risk positions. It is considered as invariant due to the values duplication that it has for the risks with the same probability measure.

Definition 2.4.1. ($Law - invariant\ convex\ risk\ measure$) A convex risk measure is a function $\rho\colon\ L^{\infty} \to \mathbb{R}$ which satisfies the following for each $X, Y \in L^{\infty}$:

(i) ($Convexity$) $\rho(\lambda X + (1-\lambda) Y) \leqslant \lambda \rho(X) + (1-\lambda)\rho(Y)$ for $\lambda \in [0,1]$

(ii) ($Monotonicity$) If $X \leqslant Y$, then $\rho(X) \leqslant \rho(Y)$

(iii) ($Translation\ Invariance$) If $m \in \mathbb{R}$, then $\rho(X+m) = \rho(X) + m$

(iv) ($Law\ Invariance$) If Y_1 and Y_2 have the same distribution under $\mathbb{P}$,then $\rho(Y_1) = \rho(Y_2)$

Any law-invariant convex risk measure can be expressed as

$$\rho(X) \stackrel{d}{=} \sup_{\mu \in M} \left(\int_0^1 AVaR_{\alpha}(X)\,\mu(d\alpha) - \beta(\mu) \right)$$

where M is the set of probability measures on $(0,1]$, and β is a function defined on M.

2.4.4 Conditional Tail Expectation

$Conditional\ tail\ expectation(CTE)$, also known as $Tail\ value - at - risk(TVaR)$, is used in the analysis of the risk exposure to describe the amount of risk which can be experienced, provided that the potential risk is beyond the threshold in excess value. The CTE is mostly used in the multivariate financial analysis method to analyze the extent of the risk exposure of the firms using the equations and models that will use the calculations in the analysis of the risks. The analysis can then be used to make concrete decisions on how to minimize the risks and their extent, before they occur to cause great damage.

Fix an level $\alpha \in (0,1)$. CTE of any random variable X at level $1-\alpha$ is defined by:

$$CTE_{\alpha}(X) \stackrel{d}{=} \mathbb{E}\left[X \mid X \geqslant VaR_{\alpha}(X)\right]$$

In other words, CTE represents the expected value of the loss in the event that an event outside of a given level of probability occurred.

CHAPTER 3

Optimizing Reinsurance

It is well-known that actuarial science has been extensively concerned with determining what reinsurance contracts are optimal based on some reasonable criteria. This chapter will review the methodology for optimizing reinsurance, both in the univariate and multivariate cases. Most of the theory in this chapter is obtained from Cheung et al.[3], Huseby and Christensen [5] and Huseby [4].

3.1 The Univariate Case

Univariate cases focus on the analysis of a single risk, which contributes to the financial contribution of the regular operation of the insurance company. Therefore, there is only one non-negative random variable X, which is reinsured. Next, we will study the optimal reinsurance under different risk measures. In the rest of this section, we denote stop-loss contract and insurance layers by R_a and $R_{a,b}$, respectively. Additionally, the corresponding retained risks of these two contracts are I_{R_a} and $I_{R_{a,b}}$.

3.1.1 Under VaR

The total retained loss under α-level VaR is denoted by:

$$V_\alpha (R) \stackrel{d}{=} VaR_\alpha (I (X) + (1 + \theta) \mathbb{E} [R (X)])$$

By the translation invariant property of VaR, we have:

$$V_\alpha (R) = VaR_\alpha (I (X)) + (1 + \theta) \mathbb{E} [R (X)]$$

We start out by considering which reinsurance is optimal to minimize the V_α. If $S_X (0) \leqslant \alpha \leqslant 1$, then the optimal contract is $R^* (X) = 0$. That is, the risk X should not be reinsured.

Now we turn to consider the case where $0 < \alpha < S_X (0)$. It is important to note that the insurance layers are always smaller than other reinsurance contracts, such that $R_{a,b} \leqslant R$, where R is any feasible reinsurance contract. By making a difference between $V_\alpha (R)$ and $V_\alpha (R_{a,b})$, we then obtain an insurance layer that is the best reinsurance contract under VaR. The optimization model can be formulated as

$$Minimize: \ V_\alpha (R_{a,b})$$

$$Subject\ to: \ 0 \leqslant a \leqslant b$$

Ready to find optimal contract parameter values a* and b*.

Theorem 3.1.1 (Theorem 4.3 in Cheung et al.[3]). *The optimal reinsurance contract under VaR satisfies:*

(i) $R^*(x) = \left(x - S_X^{-1}\left(\frac{1}{1+\theta}\right)\right)^+ - \left(x - S_X^{-1}(\alpha)\right)^+$ *, if* $\alpha < \frac{1}{1+\theta}$.

(ii) $R^*(x) = 0$*, otherwise.*

3.1.2 Under AVaR

The mathematical expression of curve equation of the optimal reinsurance under α-level AVaR can be given as

$$T_\alpha(R) \stackrel{d}{=} AVaR_\alpha\left[I(X) + (1+\theta)\mathbb{E}[R(X)]\right]$$

By the translation invariant property of $AVaR$ and I is continuous increasing function, we get

$$T_\alpha(R) = \frac{1}{\alpha}\int_0^\alpha I\left(S_X^{-1}(\lambda)\right) d\lambda + (1+\theta) E[R(X)]$$

We first consider the case where $S_X(0) \leqslant \alpha \leqslant 1$. If $\alpha < \frac{1}{1+\theta}$, then the optimal contract is $R^*(X) = X$. That is, the risk X should be fully reinsured. If $\alpha > \frac{1}{1+\theta}$, then the optimal contract is $R^*(X) = 0$. Refer to Cheung et al.[3] for details.

We then consider the case where $0 < \alpha < S_X(0)$ and make the substitution $x = S_X^{-1}(\lambda)$ which implies that $\lambda = S_X(x)$ and $d\lambda = -dF_X(x)$. The upper and lower integral limits become respectively $S_X^{-1}(\alpha) = b$ and $S_X^{-1}(0) = \sup X$. Hence, we get:

$$\begin{aligned}
& T_\alpha(R) - T_\alpha(R_a) \\
= \ & \frac{1}{\alpha}\int_b^{\sup X} \left[I_R(x) - I_{R_a}(x)\right] dF_X(x) + (1+\theta)\mathbb{E}\left[R(X) - R_a(X)\right] \\
= \ & \frac{1}{\alpha}\mathbb{E}\left[I_R(X) - I_{R_a}(X)\right] - \frac{1}{\alpha}\int_0^b \left[I_R(x) - I_{R_a}(x)\right] dF_X(x) \\
& + (1+\theta)\mathbb{E}\left[R(X) - R_a(X)\right] \\
= \ & \frac{1}{\alpha}\mathbb{E}\left[R_a(X) - R(X)\right] - \frac{1}{\alpha}\int_0^b \left[I_R(x) - I_{R_a}(x)\right] dF_X(x) \\
& + (1+\theta)\mathbb{E}\left[R(X) - R_a(X)\right]
\end{aligned}$$

Assume there exists a unique $a \in [0, supX]$, such that $\mathbb{E}[R(X)] = \mathbb{E}[R_a(X)]$. It follows that

$$\begin{aligned}
T_\alpha(R) - T_\alpha(R_a) &= \frac{1}{\alpha}\int_0^b \left[I_{R_a}(x) - I_R(x)\right] dF_X(x) \\
&= \frac{1}{\alpha}\int_0^b \left[R(x) - R_a(x)\right] dF_X(x) \geqslant 0
\end{aligned}$$

It shows that an optimal reinsurance contract can be found in the stop-loss contracts R_a, where $a \in [0, \sup X]$. Thus the optimal reinsurance under α-level AVaR is defined by:

$$\begin{aligned} Minimize: &\ T_\alpha(R_a) \\ Subject\ to: &\ 0 \leqslant a \leqslant supX \end{aligned}$$

Theorem 3.1.2 (Theorem 3.3 in Cheung et al.[3]). *The optimal retention level a^* satisfies*

(i) $a^* = S_X^{-1}\left(\frac{1}{1+\theta}\right)$ *for* $\alpha < \frac{1}{1+\theta}$

(ii) $a^* = supX$ *for* $\alpha > \frac{1}{1+\theta}$

(iii) *Any* $a^* \geqslant S_X^{-1}\left(\frac{1}{1+\theta}\right)$ *for* $\alpha = \frac{1}{1+\theta}$

3.1.3 Under Law-invariant Convex Risk Measure

The corresponding optimal reinsurance under general law-invariant convex risk measures can be solved as

$$\rho(I(X) + (1+\theta)\mathbb{E}[R(X)])$$

Based on the definition of ρ, the above formula can be simplified as

$$\sup_{\mu\in M}\left(\int_0^1 AVaR_\alpha(I(X) + (1+\theta)\mathbb{E}[R(X)])\,\mu(d\alpha) - \beta(\mu)\right)$$

To find the optimal reinsurance under general law-invariant convex risk measures, we consider the following optimization problem

$$G_\alpha \stackrel{d}{=} \sup_{\mu\in M}\left(\int_0^1 T_\alpha(R)\,\mu(d\alpha) - \beta(\mu)\right)$$

Note that exists $a \in [0, \sup X]$ such that $T_\alpha(R) \geqslant T_\alpha(R_a)$ for any $\alpha \in [0,1]$. We see that

$$G_\alpha(R) - G_\alpha(R_a) = \sup_{\mu\in M}\int_0^1 [T_\alpha(R) - T_\alpha(R_a)]\,\mu(d\alpha) \geqslant 0$$

which means the stop-loss contract is the optimal reinsurance under this risk measure. In order to find the optimal solution a^*, we assume Φ be the supremum probability measure on $(0,1]$ with density ψ. The law-invariant convex risk measure can be formulated by:

$$\rho(I_{R_a}(X)) = \int_0^1 AVaR_\alpha(I_{R_a}(X))\,\Phi(d\alpha) - \beta(\Phi)$$

Theorem 3.1.3 (Proposition 3.8 in Cheung et al.[3]). *There is an optimal solution a^* of the following minization problem:*

$$Minimize: \ \rho(I_{R_a}(X) + (1+\theta)\mathbb{E}[R_a(X)])$$

$$Subject\ to: \ 0 \leqslant a \leqslant supX$$

where Φ *is a probability measure on* $(0,1]$ *with density* ψ. *If* $\int_{\alpha_0}^{1} \frac{1}{(1+\theta)\alpha} \Phi(d\alpha) > 1$ *for* $\alpha_0 \in [0,1]$, *then the optimal value choosed in interval of* a^* *lies in* $\left[S_X^{-1}\left(\frac{1}{1+\theta}\right), sup\ X\right)$. *Otherwise* $a^* = supX$.

3.1.4 Under CTE

CTE of the insurer's total risk at the confidence level α is defined by

$$C_\alpha(R) \ \overset{d}{=} \ CTE_\alpha\left(I(X) + (1+\theta)\mathbb{E}[R(X)]\right)$$

Using the translation invariance property of CTE and I is increasing and continuous, we may write

$$\begin{aligned} C_\alpha(R) &= \mathbb{E}\left[I(X) + (1+\theta)\mathbb{E}[R(X)] \mid I(X) \geqslant VaR_\alpha(I(x))\right] \\ &= b - R(b) + (1+\theta)\mathbb{E}[R(X)] + \frac{\mathbb{E}[I(X) - I(b)]^+}{\mathbb{P}(I(X) \geqslant I(b))} \end{aligned}$$

If $S_X(0) \leqslant \alpha \leqslant 1$, then the optimal contract is $R^*(X) = 0$. That is, the risk X should not be reinsured. In the case where $0 < \alpha < S_X(0)$, we define an insurance layer contract

$$\hat{R}(x) \ \overset{d}{=} \ \begin{cases} (x - b + R(b))^+ & for\ x \in [0,b] \\ R(x) & for\ x \in [b,\infty) \end{cases}$$

By making a difference between $C_\alpha(R)$ and $C_\alpha\left(\hat{R}\right)$, it can be shown that the optimal reinsurance contract under CTE is a layer insurance contract. Hence we have

$$\begin{aligned} C_\alpha(R_{a,b}) &= a + (1+\theta)\mathbb{E}[R_{a,b}(X)] + \frac{\mathbb{E}[I(X) - I(b)]^+}{\mathbb{P}(I(X) \geqslant I(b))} \\ &= a + (1+\theta)\mathbb{E}[R_{a,b}(X)] + \frac{\mathbb{E}[X - b]^+}{\mathbb{P}(X \geqslant a)} \end{aligned}$$

The mathematical model we use to study optimal reinsurance is to minimize the CTE of an insurer's total risk:

$$\begin{aligned} &Minimize: \ C_\alpha(R_{a,b}) \\ &Subject\ to: \ 0 \leqslant a \leqslant b \end{aligned}$$

Theorem 3.1.4 (Theorem 5.3 in Cheung et al.[3]). *Assume that* a^* *and* b^* *are the optimal contract parameter values. The chosen of* a^* *should meet the below requirement*

$$0 \leqslant a^* \leqslant \min\left\{b, \ S_X^{-1}\left(\frac{1}{1+\theta}\right)\right\}$$

We denote $S_X^{-1}(\alpha)$ *as* b. *The optimal solution can be further explained as*

(i) *If* $a^* \neq S_X^{-1}\left(\frac{1}{1+\theta}\right)$, *then the optimal contract is* $R^*(x) = (x-a^*)^+ - (x-b^*)^+$ *where* $b^* = b$.

(ii) *If* $a^* = S_X^{-1}\left(\frac{1}{1+\theta}\right)$, *then the optimal contract is* $R^*(x) = (x-a^*)^+$ *on* $[0, b]$.

3.2 The Multivariate Case

The multivariate situation where the cedent has different risks cannot be packaged together. We only solve the case where VaR is used as a risk measure. Because an insurance layer contract is optimal from section 3.1.1, we concentrate on such contracts in the multivariate case as well. The risks covered by the reinsurer are as follows: intervals are used to characterize. This implies that we have two parameters in each contract that correspond to the bounds of these intervals. The main conclusion of this case is that any optimal solution must meet certain conditions. The material in this chapter is obtained from Huseby and Christensen [5] and Huseby [4].

3.2.1 The Model

Assume the insurer has n lines of business. We then let the random variable X_i denote the loss in line i, with $i = 1, \ldots, n$. Moreover, when we consider the individual risk model, in which the total loss of the insurer is given by

$$X_1 + \ldots + X_n$$

The insurer has to apply for a reinsurance strategy $R(x)$ to transfer risks of big losses to the reinsurer. $R_i(x)$ is called the reinsurance strategy of line i. Because an insurance layer contract is optimal from section 3.1.1, we concentrate on such contracts in the multivariate case as well. Similar to the univariate case, the function R_i is given by:

$$R_i(X_i) = \begin{cases} 0 & for\ X_i < a_i \\ X_i - a_i & for\ a_i \leqslant X_i \leqslant b_i \\ b_i - a_i & otherwise \end{cases}$$

Where $0 \leqslant a_i < b_i \leqslant \sup X_i$. Moreover, the insurer retains the loss $I_i(X_i)$ for risk i, denoted $I_i(X_i) = X_i - R_i(X_i)$, $i = 1, \ldots, n$. This is given by:

$$I_i(X_i) = \begin{cases} X_i & for\ X_i < a_i \\ a_i & for\ a_i \leqslant X_i \leqslant b_i \\ X_i - (b_i - a_i) & otherwise \end{cases} \tag{3.1}$$

We also let $\mathbf{X} = (X_1, \ldots, X_n)$. The total premium paid by the insurer which is so called *premium term*, is denoted by $\pi_{\mathbf{X}}$, is given by:

$$\pi_{\mathbf{X}} = (1+\theta)\sum_{i=1}^{n} E\left[R_i(X_i)\right]$$

where the risk loading $\theta \geqslant 0$. The total retained loss of insurer is then:

$$\sum_{i=1}^{n} I_i(X_i) + (1+\theta)\sum_{i=1}^{n} \mathbb{E}\left[R_i(X_i)\right]$$

The optimal reinsurance under α-VaR is defined by:

$$V_\alpha = VaR_\alpha\left(\sum_{i=1}^{n} I_i(X_i)\right) + (1+\theta)\sum_{i=1}^{n} \mathbb{E}\left[R_i(X_i)\right]$$

where the first term is the *retained risk term*, the rest is the *premium term*.

The objective is to solve following VaR-minization problem:

$$Minimize\ \ V_\alpha$$

3.2.2 Optimal Reinsurance under VaR

The main objective now is to find $\mathbf{a}^* = (a_1^*, \ldots, a_n^*)$ and $\mathbf{b}^* = (b_1^*, \ldots, b_n^*)$ so that V_α is minimized. We start out by considering the retained risk term. By Eq.(3.1), the ith risk retained by the cedent is less than, equal to and greater than a_i in different intervals. Therefore, according to the contracts parameters $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (a_1, \ldots, a_n)$, we define:

$$\mathcal{A} = \left\{\mathbf{X}:\ \sum_{i=1}^{n} I_i(X_i) < \sum_{i=1}^{n} a_i\right\}$$
$$\mathcal{B} = \left\{\mathbf{X}:\ \sum_{i=1}^{n} I_i(X_i) = \sum_{i=1}^{n} a_i\right\}$$
$$\mathcal{C} = \left\{\mathbf{X}:\ \sum_{i=1}^{n} I_i(X_i) > \sum_{i=1}^{n} a_i\right\}$$

where $\mathbf{X} = (X_1, \ldots, X_n)$.

Since S_{X_i} are strictly decreasing for all i, it follows that $\mathbb{P}(\mathbf{X} \in \mathcal{C} \cup \mathcal{B})$ and $\mathbb{P}(\mathbf{X} \in \mathcal{C})$ are strictly decreasing in a_i for all i. From Eq.(2.1) (2.2), if the contracts parameters $\mathbf{a}$ and $\mathbf{b}$ are selected as follows:

$$\mathbb{P}\left(\sum_{i=1}^{n} I_i(X_i) \geqslant \sum_{i=1}^{n} a_i\right) = \mathbb{P}(\mathbf{X} \in \mathcal{C} \cup \mathcal{B}) \geqslant \alpha$$
$$\mathbb{P}\left(\sum_{i=1}^{n} I_i(X_i) > \sum_{i=1}^{n} a_i\right) = \mathbb{P}(\mathbf{X} \in \mathcal{C}) \leqslant \alpha$$

or

$$\mathbb{P}\left(\sum_{i=1}^{n} I_i(X_i) > \sum_{i=1}^{n} a_i\right) = \mathbb{P}(\mathbf{X} \in \mathcal{C}) = \alpha$$

then we have:

$$VaR_\alpha\left(\sum_{i=1}^{n} I_i(X_i)\right) = S^{-1}_{\sum_{i=1}^{n} I_i(X_i)}(\alpha) = \sum_{i=1}^{n} a_i$$

The following important result has been demonstrated by Huseby and Christensen [5].

Theorem 3.2.1 (Theorem 2.1 in Huseby [4]). *Let $a_1^*, b_1^*, \ldots, a_n^*, b_n^*$ be optimal contract parameter values for line i, and let*

$$\mathbb{P}\left(\bigcap_{i=1}^{n}(X_i > a_i^*)\right) \geqslant \alpha \tag{3.2}$$

Then the following conditions must be met:

$$a_i^* = S_{X_i}^{-1}\left(\frac{1}{1+\theta}\right), i = 1, \ldots, n \tag{3.3}$$

and:

$$\mathbb{P}\left(\mathbf{X} \in \mathcal{C}\right) = \alpha \tag{3.4}$$

Note that X_is' are *positively upper orthant dependent*(see Shaked [6]) which contains a special case of independence, the assumption follows that:

$$\mathbb{P}\left(\bigcap_{i=1}^{n}(X_i > a_i^*)\right) \geqslant \prod_{i=1}^{n}\mathbb{P}\left(X_i > a_i^*\right) = (1+\theta)^{-n}$$

Thus, in this case, the following is a sufficient condition for Eq.(3.2) to hold:

$$(1+\theta)^{-n} \geqslant \alpha \tag{3.5}$$

In general, the condition Eq.(3.5) is satisfied whenever the risk premium θ which is charged by the reinsurance company, is not too large. Without this condition Eq.(3.5) being satisfied, the cedent has little or nothing to gain, and therefore should not purchase reinsurance contracts. Thus we let $\theta = 0.2$, when we later deal with the optimal solution in the bivariate case.

Theorem 3.2.2 (Theorem 2.2 in Huseby [4]). *Base on Eq.(3.3), assume that the optimal values $a_1^*, \ldots, a_n^*$ satisfy Eq.(3.2). Then $b_1^*, \ldots, b_n^*$ can be determined by solving the following minimization problem:*

$$Minimize: \quad \sum_{i=1}^{n}\mathbb{E}\left[R_i\left(X_i\right)\right]. \tag{3.6}$$

$$Subject\ to: \quad \mathbb{P}\left(\mathbf{X} \in \mathcal{C}\right) = \alpha \tag{3.7}$$

Proof. The constraint Eq.(3.7) is the same as Eq.(3.4) in Theorem 3.2.1. Moreover, we know the constraint Eq.(3.7) can be writen as:

$$\mathbb{P}\left(\sum_{i=1}^{n} I_i\left(X_i\right) > \sum_{i=1}^{n} a_i^*\right) = \alpha$$

which means the retained risk term is given by:

$$S^{-1}_{\sum_{i=1}^{n} I_i(X_i)}\left(\alpha\right) = \sum_{i=1}^{n} a_i^*$$

Thus the total α-level VaR becomes:

$$V_\alpha = \sum_{i=1}^{n} a_i^* + (1+\theta) \sum_{i=1}^{n} \mathbb{E}\left[R_i\left(X_i\right)\right]$$

This implies that minimizing V_α is equivalent to minimizing $\sum_{i=1}^{n} \mathbb{E}\left[R_i\left(X_i\right)\right]$ with respect to $b_1, \ldots, b_n$. ■

According to Theorem 3.2.2, we can find that the optimal value $\mathbf{a}^*$ is fixed and unique, but there are countless values of $\mathbf{b}$ satisfying Eq.(3.7).

3.2.2.1 Unrestricted Solutions

It is convenient to denote the common value of $P\left(X_i > a_i^*\right)$ by A:

$$A = S_{X_i}\left(a_i^*\right) = (1+\theta)^{-1},\ i = 1, \ldots, n$$

We search for solutions where the B_i does not have to be equal to B_j for $i \neq j$ and $1 \leqslant i, j \leqslant n$. We then introduce:

$$B_i = S_{X_i}\left(b_i\right),\ i = 1, \ldots, n$$

where $B_i \in [0,1]$. Note that:

$$\mathcal{C} \subseteq \bigcup_{i=1}^{n} \left(X_i > b_i\right)$$

Then it is easy to get that:

$$A^n - \prod_{i=1}^{n} \left(A - B_i\right) \leqslant \mathbb{P}\left(\mathbf{X} \in \mathcal{C}\right) \leqslant 1 - \prod_{i=1}^{n} \left(1 - B_i\right) \tag{3.8}$$

Thus, given $B_j\ for\ j \neq i$, the lower bound B_i^L for the correct value of B_i is then:

$$B_i^L = 1 - \frac{1-\alpha}{\prod_{j \neq i} \left(1 - B_j\right)}$$

Given $B_j\ for\ j \neq i$, the upper bound B_i^U for the correct value of B_i is then:

$$B_i^U = A - \frac{A^n - \alpha}{\prod_{j \neq i} \left(A - B_j\right)}$$

In the high dimensional case, the global optimal solution could be a complicated task because we have many B_i that should be determined. In the bivariate case, we choose a suitable B_1 value. Then it is simple to compute B_2 given B_1 by using the well-known *bisection method*, so that the fraction of simulations belonging to $\mathcal{C}$ is about equal to α. That is the reason we use upper and lower bounded. We then iterate the value B_1 to find the optimal reinsurance contract that minimizes V_α.

CHAPTER 4

Reinsurance Simulation in the Dependent Bivariate Case

The thesis will focus on how reinsurance optimization is affected by dependence between the two risks. Only the bivariate case, i.e., the case where $n = 2$ will be considered. This chapter will first generalize the methods for optimizing bivariate reinsurance proposed in Huseby [5]. Then we will develop methods for quantifying the effect of dependence in relation to reinsurance optimization.

4.1 Problem Description

Assume the insurer has $n = 2$ lines of business. We let the random variable X_i denote the loss in line i, for $i = 1, 2$. The reinsurance contract of line i is $R_i(X_i)$. Therefore, the insurer retains the loss is calculated by $I_1(X_1) + I_2(X_2)$ and the total risk covered by the insurer is then:

$$\sum_{i=1}^{2} I_i(X_i) + (1+\theta)\sum_{i=1}^{2} \mathbb{E}\left[R_i(X_i)\right]$$

In the 2-dimensional case, the total reinsurance risk under α-VaR can be formulated by:

$$V_\alpha(\mathbf{a}, \mathbf{b}) = VaR_\alpha[\sum_{i=1}^{2} I_i(X_i)] + (1+\theta)\sum_{i=1}^{2} \mathbb{E}\left[R_i(X_i)\right]$$

where $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$. Our goal is to find $\mathbf{a}^* = (a_1^*, a_2^*)$ and $\mathbf{b}^* = (b_1^*, b_2^*)$ so that $V_\alpha(\mathbf{a}, \mathbf{b})$ is minimized. As in the multivariate case, we focus on the retained risk term $VaR_\alpha[\sum_{i=1}^{2} I_i(X_i)]$ and define:

$$\mathcal{A} = \left\{\mathbf{X} \colon \sum_{i=1}^{2} I_i(X_i) < \sum_{i=1}^{2} a_i\right\}$$

$$\mathcal{B} = \left\{\mathbf{X} \colon \sum_{i=1}^{2} I_i(X_i) = \sum_{i=1}^{2} a_i\right\}$$

$$\mathcal{C} = \left\{\mathbf{X} \colon \sum_{i=1}^{2} I_i(X_i) > \sum_{i=1}^{2} a_i\right\}$$

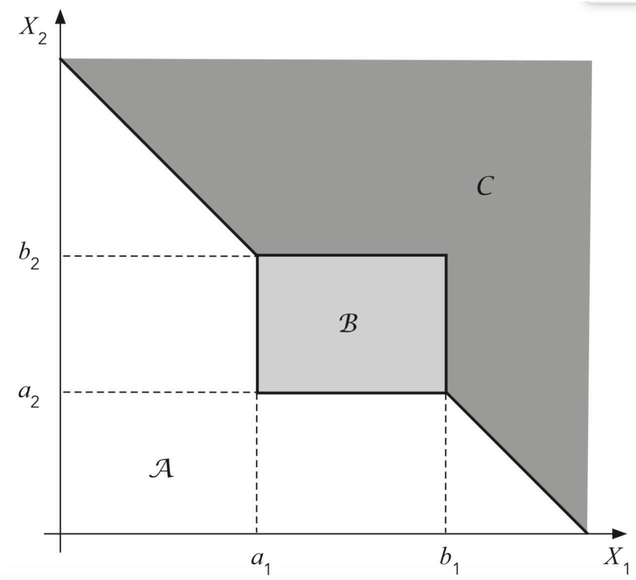


Figure 4.1: The sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

Figure 4.1 illustrates their distribution. Furthermore, the white area represents set $\mathcal{A}$. The dark gray area represents set $\mathcal{C}$. Set $\mathcal{B}$ is represented by the light gray area and the solid line between $\mathcal{A}$ and $\mathcal{C}$. See Theorem 3.2.1 3.2.2 that in order to minimize $V_\alpha(\mathbf{a}, \mathbf{b})$, $\mathbf{b}^*$ should be chosen so that:

$$\mathbb{P}\left(I_1\left(X_1\right)+I_2\left(X_2\right)>a_1+a_2\right)=\alpha \tag{4.1}$$

The optimal value $a_i^* = S_{X_i}^{-1}\left(\frac{1}{1+\theta}\right)$, $i = 1, 2$. In conclusion, the optimization problem is equivalent to:

$$\begin{aligned} &Minimize: \ \mathbb{E}\left[R_1\left(X_1\right)\right]+\mathbb{E}\left[R_2\left(X_2\right)\right] \\ &Subject\ to: \ \mathbb{P}\left(\sum_{i=1}^{2} I_i\left(X_i\right)>\sum_{i=1}^{2} a_i^*\right)=\alpha \end{aligned}$$

To solve the problem above, we should

(i) determine $\mathbf{b}$ so that $\mathbb{P}\left(\sum_{i=1}^{2} I_i\left(X_i\right)>\sum_{i=1}^{2} a_i^*\right)=\alpha$ holds.

(ii) minimize $\mathbb{E}\left[R_1\left(X_1\right)\right]+\mathbb{E}\left[R_2\left(X_2\right)\right]$ with respect to $\mathbf{b}$.

4.2 Plan for the Simulation Study

Let X be a random variable with values in $\mathcal{X}$ which is a set of all possible values of X, and distribution F_X , and let $\phi = \phi\left(b, X\right)$ be a known function of X and a deterministic variable b. We also introduce:

$$h\left(b\right) = \mathbb{E}[\phi\left(b, X\right)]$$

It is important to note that both X and b may be scalars or vectors. Since we study bivariate risks, these two variables are replaced by 2-dimensional vectors in the rest of the thesis. We then consider two problems related to h:

- **Problem 1**. Determine b so that $h(b) = c$, for some constant c.
- **Problem 2**. Minimize $h(b)$ with respect to b.

Both problems can in principle be solved simply by calculating:

$$h\,(b) = \int_{\mathcal{X}} \phi\,(b, X)\, dF_X\,(\mathrm{x})$$

for each value of b. However, unless the integral is easy to calculate, this may not be the best approach.

4.2.1 Monte Carlo Simulation

Monte Carlo method is an available tool in the field of insurance. It relies on repeated random sampling to study the results of uncertain events. Thus, we can use Monte Carlo simulation to estimate $h(b)$.

Let $X_1, \ldots, X_N$ be a sequence of independent variables from the distribution F_X on the set $\mathcal{X}$, and estimate $h(b)$ by the standard (crude) Monte Carlo estimate:

$$\hat{h}\,(b) = \frac{1}{N}\sum_{j=1}^{N} \phi\,(b, X_j) \tag{4.2}$$

Then it is easy to see that:

$$\mathbb{E}[\hat{h}\,(b)] = \frac{1}{N}\sum_{j=1}^{N} \mathbb{E}[\phi\,(b, X_j)] = \frac{1}{N}\sum_{j=1}^{N} h\,(b) = h\,(b)$$

and that:

$$Var[\hat{h}\,(b)] = \frac{1}{N^2}\sum_{j=1}^{N} Var[\phi\,(b, X_j)] = \frac{1}{N} Var\,[\phi\,(b, X)]$$

When N approaches to infinity, $Var[\hat{h}\,(b)]$ is equal to 0. Thus, $\hat{h}\,(b)$ is an *unbiased* and *consistent* estimator for $h(b)$. Still, the estimate $\hat{h}\,(b)$ may be too unstable for solving the two problems due to sampling uncertainty. Then we use importance sampling to solve the two problems.

4.2.2 Importance Sampling

Importance sampling is used to estimate the expectation based on some known and easily sampled distribution to avoid Monte Carlo's defect. Assume that there exists a subset $D \subset \mathcal{X}$ and $E = \mathcal{X} \backslash D$ such that:

- $p_D = \mathbb{P}\,(X \in D)$ and $p_E = \mathbb{P}\,(X \in E) = 1 - p_D$ are known quantities.
- It is possible to sample efficiently from the conditional distributions of X given the events $\{X \in D\}$ and $\{X \in E\}$.

We then generate a sample $X_1, \ldots, X_{N_1}$ from the conditional distribution of X given the event $\{X \in D\}$ and another sample $Y_1, \ldots, Y_{N_2}$ from the conditional distribution of X given the event $\{X \in E\}$, where $N_1 + N_2 = N$. The function $h(b)$ can then be estimated by:

$$\tilde{h}\,(b) = p_D \frac{1}{N_1} \sum_{j=1}^{N_1} \phi\,(b, X_j) + p_E \frac{1}{N_2} \sum_{j=1}^{N_2} \phi\,(b, Y_j) \tag{4.3}$$

We then get:

$$\begin{aligned}
\mathbb{E}[\tilde{h}\,(b)] &= p_D \frac{1}{N_1} \sum_{j=1}^{N_1} \mathbb{E}[\phi\,(b, X_j)] + p_E \frac{1}{N_2} \sum_{j=1}^{N_2} \mathbb{E}[\phi\,(b, Y_j)] \\
&= p_D \mathbb{E}[\phi\,(b, X) \mid X \in D] + p_E \mathbb{E}[\phi\,(b, X) \mid X \in E] \\
&= \mathbb{E}\,[\phi\,(b, X)] \\
&= h(b)
\end{aligned}$$

and that:

$$\begin{aligned}
Var[\tilde{h}\,(b)] &= p_D^2 \frac{1}{N_1^2} \sum_{j=1}^{N_1} Var[\phi\,(b, X_j)] + p_E^2 \frac{1}{N_2^2} \sum_{j=1}^{N_2} Var[\phi\,(b, Y_j)] \\
&= \frac{p_D^2}{N_1} Var[\phi\,(b, X)\,|X \in D] + \frac{p_E^2}{N_2} Var[\phi\,(b, X)\,|X \in E]
\end{aligned}$$

$Var[\tilde{h}\,(b)]$ will approach 0 as N_1 and N_2 increase. Thus, $\tilde{h}\,(b)$ is an unbiased and consistent estimator for $h(b)$ as well.

4.2.2.1 Special Cases

We now consider a case which is related to Problem 2 in 4.2, where it is known that:

$$\phi\,(b, \mathrm{x}) = c_0, \;\; for\ all\ \mathrm{x} \in E$$

where c_0 is a known constant. This obviously implies that:

$$\begin{aligned}
\mathbb{E}[\phi\,(b, X) \mid X \in E] &= c_0 \\
Var[\phi\,(b, X)\,|X \in E] &= 0
\end{aligned}$$

In this case we may let $N_1 = N$, and estimate $h(b)$ by a simplified version of $\tilde{h}\,(b)$ given by:

$$\tilde{h}\,(b) = p_D \frac{1}{N} \sum_{j=1}^{N} \phi\,(b, X_j) + p_E c_0$$

We still get that $E\left[\tilde{h}\,(b)\right] = h\,(b)$. In order to calculate the variance, it is convenient to introduce $I_D = \mathbb{1}(X \in D)$. That is, $I_D = 1$ if $X \in D$ and zero otherwise. Since $Var[\phi(b,X)|X \in E] = Var[\phi(b,X)|I_D = 0] = 0$, we may write:

$$\begin{aligned} Var[\tilde{h}(b)] &= \frac{p_D^2}{N} Var[\phi(b,X)|I_D] \\ &= \frac{p_D}{N}(p_D Var[\phi(b,X)|I_D = 1] + p_E Var[\phi(b,X)|I_D = 0]) \\ &= \frac{p_D}{N}\mathbb{E}[Var[\phi(b,X)|I_D]] \\ &= \frac{p_D}{N}(Var[\phi(b,X)] - Var[\mathbb{E}[\phi(b,X)|I_D]]) \\ &\leqslant \frac{p_D}{N} Var[\phi(b,X)] = p_D Var[\hat{h}(b)] \end{aligned}$$

If p_D is small, then $Var[\tilde{h}(b)]$ will typically be much smaller than $Var[\hat{h}(b)]$.

We now turn our attention to another special case related to Problem 1 in 4.2, where we let $\mathcal{C} = C(b) \subseteq D \subset \mathcal{X}$. That is, $C(b)$ is a subset of $\mathcal{X}$ which in some way depends on the deterministic variable b. Moreover, define $\phi(b, \mathrm{x})$ as an indicator function as before:

$$\phi(b,X) = \mathbb{1}(X \in C(b))$$

Then it follows that:

$$h(b) = \mathbb{E}[\phi(b,X)] = \mathbb{E}[\mathbb{1}(X \in C(b))] = \mathbb{P}(X \in C(b))$$

The objective is to determine b so that:

$$h(b) = \mathbb{P}(X \in C(b)) = \alpha$$

Furthermore, we let $E = \mathcal{X} \backslash D$. Since $C(b) \subseteq D$, it follows that $C(b) \cap E = \emptyset$, and hence:

$$\phi(b,X) = \mathbb{1}(X \in C(b)) = 0, \ for\ all\ X \in E$$

This obviously implies that:

$$\begin{aligned} \mathbb{E}[\phi(b,X) \mid X \in E] &= 0 \\ Var[\phi(b,X)|X \in E] &= 0 \end{aligned}$$

In this case we generate $X_1, \ldots, X_N$ from the conditional distribution of X given that $X \in D$, and estimate $h(b)$ by:

$$\begin{aligned} \tilde{h}(b) &= p_D \frac{1}{N}\sum_{j=1}^{N}\phi(b,X_j) + p_E \cdot 0 \\ &= p_D \frac{1}{N}\sum_{j=1}^{N}\mathbb{1}(X_j \in C(b)) \end{aligned}$$

On the other hand, we still get that $E\left[\tilde{h}\left(b\right)\right] = h\left(b\right)$ and $Var[\tilde{h}\left(b\right)]$ is samller than $Var[\hat{h}\left(b\right)]$ when p_D is small. The solution to the equation $h\left(b\right) = \mathbb{P}\left(X \in C\left(b\right)\right) = \alpha$ can then be estimated by instead solving the equation:

$$\tilde{h}\left(b\right) = p_D \frac{1}{N} \sum_{j=1}^{N} \mathbb{1}\left(X_j \in C\left(b\right)\right) = \alpha$$

This is typically done using some iterative method like the bisection method.

4.2.3 Random Number Generator

Let $\mathbf{X} = \left(X_1, X_2\right)$ be a bivariate absolutely continuously distributed random variable, and where $\mathcal{X} = \mathbb{R}^+ \times \mathbb{R}^+$. Assume that X_1 and X_2 are independent and given its cumulative distribution function that $X_i \sim F_{X_i}$, for $i = 1, 2$. Then the independent vectors $\left(X_{1,1}, X_{2,1}\right), \ldots, \left(X_{1,N}, X_{2,N}\right)$ have the same joint distribution as $\left(X_1, X_2\right)$. can be generated as follows:

Let $\left(U_{1,1}, U_{2,1}\right), \ldots, \left(U_{1,N}, U_{2,N}\right)$ be independent vectors such that $U_{1,j}, U_{2,j}$ are independent and uniformly distributed on $[0, 1]$, $j = 1, \ldots, N$. We then let:

$$\mathbf{X}_j = \left(X_{1,j}, X_{2,j}\right) = \left(F_{X_1}^{-1}\left(U_{1,j}\right), F_{X_2}^{-1}\left(U_{2,j}\right)\right) \;\; for\ j = 1, \ldots, N.$$

It is then easy to verify that the samples $\left(X_{1,1}, X_{2,1}\right), \ldots, \left(X_{1,N}, X_{2,N}\right)$ get the correct joint distribution. To show this, we first note that if $U \sim R\left[0, 1\right]$ which means the uniform distribution on the interval $[0, 1]$, then:

$$\mathbb{P}\left(U < u\right) = \int_0^u 1 \cdot dv = u \quad for\ all\ u \in [0, 1]$$

We then consider $Y = F_X^{-1}\left(U\right)$ where $U \sim R\left[0, 1\right]$, where F_X is the cumulative distribution function of an absolute continuously distributed random variable X. We then have:

$$\begin{aligned}
\mathbb{P}\left(Y \leqslant y\right) &= \mathbb{P}\left(F_X^{-1}(U) \leqslant y\right) \\
&= \mathbb{P}\left(F_X\left(F_X^{-1}\left(U\right)\right) \leqslant F_X\left(y\right)\right) \\
&= \mathbb{P}\left(U \leqslant F_X\left(y\right)\right) \\
&= F_X\left(y\right)
\end{aligned}$$

which proves that the random variable $Y = F_X^{-1}\left(U\right)$ has the cumulative distribution function F_X, i.e.:

$$Y \stackrel{d}{=} X$$

Note that since X is assumed to be absolutely continuously distributed, the mapping from a bivariate uniformly distributed $\mathbf{U} = \left(U_1, U_2\right)$ to $\mathbf{X} = \left(X_1, X_2\right)$ is $one - to - one$. Thus, the mapping has a $unique\ well - defined\ inverse$:

$$\mathbf{u} = \left(u_1, u_2\right) = \left(F_{X_1}\left(x_1\right), F_{X_2}\left(x_2\right)\right)$$

Hence, for any set $S \subseteq \mathcal{X}$ and corresponding set:

$$S' = \left\{\left(u_1, u_2\right) = \left(F_{X_1}\left(x_1\right), F_{X_2}\left(x_2\right)\right) : \left(x_1, x_2\right) \in S\right\}$$

the events $\{\mathbf{X} \in S\}$ and $\{\mathbf{U} \in S'\}$ are equivalent, and thus, we have:

$$\mathbb{P}(\mathbf{X} \in S) = \mathbb{P}(\mathbf{U} \in S')$$

Moreover, note that we also have:

$$S = \{(x_1, x_2) = \left(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2)\right) : (u_1, u_2) \in S'\}$$

If $\mathbf{U} = (U_1, U_2)$ is uniformly distributed on $[0,1] \times [0,1]$, then $\mathbf{U}$ has a density, $f_{\mathbf{U}}(u_1, u_2) = 1$ for all $(u_1, u_2) \in [0,1] \times [0,1]$. This implies that for any $A \in [0,1] \times [0,1]$, we have:

$$\mathbb{P}(\mathbf{U} \in A) = \int_A \int f_{\mathbf{U}}(u_1, u_2)\, du_1 du_2 = m(A) = The\ area\ of\ A$$

Moreover, for any $B \subseteq A$, the conditional probability that $\mathbf{U} \in B$ given that $\mathbf{U} \in A$ becomes:

$$\mathbb{P}(\mathbf{U} \in B | \mathbf{U} \in A) = \frac{\mathbb{P}(\mathbf{U} \in A \cap B)}{\mathbb{P}(\mathbf{U} \in A)} = \frac{\mathbb{P}(\mathbf{U} \in B)}{\mathbb{P}(\mathbf{U} \in A)} = \frac{m(B)}{m(A)}$$

Hence, $(\mathbf{U}|\mathbf{U} \in A)$ is uniformly distributed on the set A.

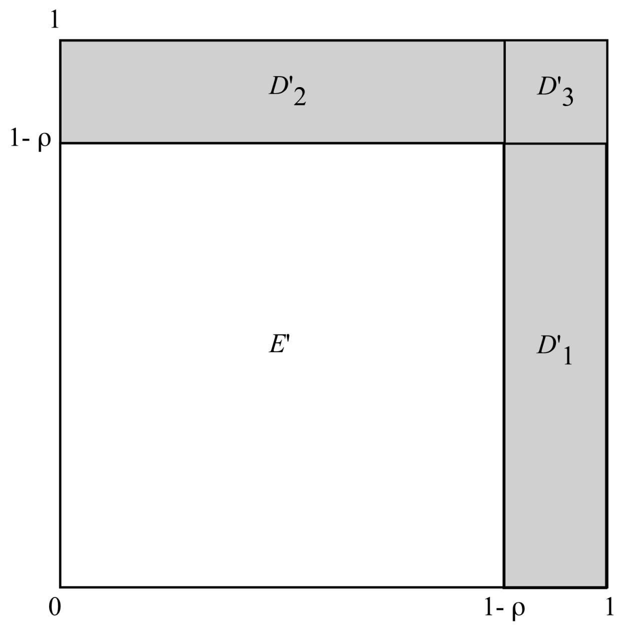


Figure 4.2: The sets D' and E'.

We then choose $\rho \in [0,1]$, and define:

$$\begin{aligned} D' &= \{(u_1, u_2) : 1 - \rho \leqslant u_i \leqslant 1,\ i = 1, 2\} \\ E' &= \{(u_1, u_2) : 0 \leqslant u_i \leqslant 1 - \rho,\ i = 1, 2\} \end{aligned}$$

Since $\mathbf{U}$ is uniformly distributed, two sets D' and E' have probabilities:

$$\mathbb{P}(\mathbf{U} \in D') = m(D') = 1 - (1 - \rho)^2$$

$$\mathbb{P}(\mathbf{U} \in E') \quad = \quad m(E') = (1-\rho)^2$$

As is shown in Figure 4.2, the set $D' = \{(u_1, u_2):\ 1-\rho \leqslant u_i \leqslant 1,\ i = 1, 2\}$ is partitioned into three subsets:

$$\begin{aligned} D'_1 &= \{(u_1, u_2):\ 1-\rho \leqslant u_1 \leqslant 1,\ 0 \leqslant u_2 \leqslant 1-\rho\} \\ D'_2 &= \{(u_1, u_2):\ 0 \leqslant u_1 \leqslant 1-\rho,\ 1-\rho \leqslant u_2 \leqslant 1\} \\ D'_3 &= \{(u_1, u_2):\ 1-\rho \leqslant u_1 \leqslant 1,\ 1-\rho \leqslant u_2 \leqslant 1\} \end{aligned}$$

We have already shown that $(\mathbf{U}|\mathbf{U} \in D')$ is uniformly distributed on the set D'. Hence, it follows that:

$$\begin{aligned} \mathbb{P}(\mathbf{U} \in D'_1|\mathbf{U} \in D') &= \frac{m(D'_1)}{m(D')} = \frac{\rho \cdot (1-\rho)}{1-(1-\rho)^2} = \frac{\rho \cdot (1-\rho)}{2\rho - \rho^2} = \frac{1-\rho}{2-\rho} \\ \mathbb{P}(\mathbf{U} \in D'_2|\mathbf{U} \in D') &= \frac{m(D'_2)}{m(D')} = \frac{\rho \cdot (1-\rho)}{1-(1-\rho)^2} = \frac{\rho \cdot (1-\rho)}{2\rho - \rho^2} = \frac{1-\rho}{2-\rho} \\ \mathbb{P}(\mathbf{U} \in D'_3|\mathbf{U} \in D') &= \frac{m(D'_3)}{m(D')} = \frac{\rho^2}{1-(1-\rho)^2} = \frac{\rho^2}{2\rho - \rho^2} = \frac{\rho}{2-\rho} \end{aligned}$$

We now describe how to generate samples from the conditional distribution of a bivariate uniformly distributed variable $\mathbf{U} = (U_1, U_2)$ given that $\mathbf{U} \in D'$. We start out by sampling from which of three sets D'_1, D'_2, D'_3 the vector $\mathbf{U}$ should be sampled. This is done by sampling an auxiliary variable U_0 which is uniformly distributed on the interval $[0, 2-\rho]$.

- If $0 \leqslant U_0 < 1-\rho$, then $\mathbf{U}$ is sampled from D'_1
- If $1-\rho \leqslant U_0 < 2(1-\rho)$, then $\mathbf{U}$ is sampled from D'_2
- If $2(1-\rho) \leqslant U_0 < 2-\rho$, then $\mathbf{U}$ is sampled from D'_3

As a result, the three subsets D'_1, D'_2, D'_3 get the correct probabilities.

Note that for $k = 1, 2, 3,$ $(\mathbf{U}|\mathbf{U} \in D'_k)$ is uniformly distributed on the set D'_k. After the subset D'_k has been determined, the variable $\mathbf{U}$ is sampled as follows:

- If $\mathbf{U} \in D'_1$, then $U_1 \sim R[1-\rho, 1]$ and $U_2 \sim R[0, 1-\rho]$
- If $\mathbf{U} \in D'_2$, then $U_1 \sim R[0, 1-\rho]$ and $U_2 \sim R[1-\rho, 1]$
- If $\mathbf{U} \in D'_3$, then $U_1 \sim R[1-\rho, 1]$ and $U_2 \sim R[1-\rho, 1]$

where $R[\gamma, \delta]$ denotes the uniform distribution on the interval $[\gamma, \delta]$.

We then transform the sets D' and E' using the mapping from $\mathbf{U}$ to $\mathbf{X}$:

$$\begin{aligned} D &= \{(x_1, x_2) = \left(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2)\right):\ (u_1, u_2) \in D\prime\} \\ E &= \{(x_1, x_2) = \left(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2)\right):\ (u_1, u_2) \in E\prime\} \end{aligned}$$

By the one-to-one correspondence between the sets D' and E' and the transformed sets D and E, it follows that:

$$\begin{aligned} p_D &= \mathbb{P}(\mathbf{X} \in D) = \mathbb{P}(\mathbf{U} \in D') = m(D') = 1-(1-\rho)^2 \\ p_E &= \mathbb{P}(\mathbf{X} \in E) = \mathbb{P}(\mathbf{U} \in E') = m(E') = (1-\rho)^2 \end{aligned}$$

Thus, both p_D and p_E are known quantities. By choosing ρ sufficiently large, e.g., $\rho \geqslant 2\alpha$, we can ensure that:

$$C(\mathbf{b}) \subseteq D, \; for \; all \; \mathbf{b}$$

where $\mathbf{b} = (b_1, b_2)$ in bivariate case.

We assume that $\mathbf{U}_j = (U_{1,j}, U_{2,j}), \; j = 1, \ldots, N$ are N independent vectors sampled from the conditional distribution of $\mathbf{U}$ given that $\mathbf{U} \in D'$ as previously described, and let:

$$\mathbf{X}_j = (X_{1,j}, X_{2,j}) = \left(F_{X_1}^{-1}(U_{1,j}), F_{X_2}^{-1}(U_{2,j})\right) \; j = 1, \ldots, N$$

We then let $S \subseteq D$ be arbitrarily chosen, and define:

$$S' = \{(u_1, u_2) = \left(F_{X_1}(x_1), F_{X_2}(x_2)\right) : \; (x_1, x_2) \in S\}$$

By the one-to-one correspondence between the sets S' and S it follows that:

$$\mathbb{P}(\mathbf{X}_j \in S) = \mathbb{P}(\mathbf{U}_j \in S') = \frac{\mathbb{P}(\mathbf{U} \in S')}{\mathbb{P}(\mathbf{U} \in D')} = \frac{\mathbb{P}(\mathbf{X} \in S)}{\mathbb{P}(\mathbf{X} \in D)} \; , j = 1, \ldots, N$$

Since this holds for any subset S of D, we conclude that $\mathbf{X}_j$ has the same distribution as $(\mathbf{X}|\mathbf{X} \in D)$ for $j = 1, \ldots, N$.

4.2.4 Gaussian Copula

The dependence between X_1 and X_2 is modeled using a *Gaussian Copula*. Let U_1 and U_2 be independent and $U_i \sim R[0,1], \; i = 1, 2$, and introduce:

$$G_i = \Phi^{-1}(U_i), \; i = 1, 2$$

where Φ denotes the cumulative distribution function of a standard Gaussian distribution. This implies that G_1 and G_2 are independent and $G_i \sim N(0,1), \; i = 1, 2$.

We introduce the following *linear transformation* of G_1 and G_2:

$$\begin{aligned} H_1 &= c_1 G_1 + c_2 G_2 \\ H_2 &= c_2 G_1 + c_1 G_2 \end{aligned}$$

where c_1 and c_2 are chosen so that:

$$\mathbb{E}[H_i] = 0, \; Var[H_i] = 1, \; i = 1, 2 \; and \; Cov[H_1, H_2] = k$$

Since normal distributions are preserved under linear transformations, it follows that H_1 and H_2 are both $N(0,1)$-distributed marginally. This implies that we always have $\mathbb{E}[H_i] = 0$ for this linear transformation. Thus we focus on the variances and the covariance. In order to get the correct variances, c_1 and c_2 must satisfy:

$$\begin{aligned} Var[H_1] = c_1^2 Var[G_1] + c_2^2 Var[G_2] = c_1^2 + c_2^2 = 1 \\ Var[H_2] = c_2^2 Var[G_1] + c_1^2 Var[G_2] = c_2^2 + c_1^2 = 1 \end{aligned}$$

where we have used that $Var[G_1] = Var[G_2] = 1$. Furthermore, in order to get the *correct covariance*, c_1 and c_2 must satisfy:

$$\begin{aligned} & Cov[H_1, H_2] \\ = & Cov[c_1G_1 + c_2G_2, c_2G_1 + c_1G_2] \\ = & c_1c_2Cov[G_1, G_1] + c_1^2Cov[G_1, G_2] + c_2^2Cov[G_2, G_1] + c_1c_2Cov[G_2, G_2] \\ = & 2c_1c_2 = k \end{aligned}$$

where we have used that $Cov[G_i, G_i] = Var[G_i] = 1$ for $i = 1, 2$, and that $Cov[G_1, G_2] = 0$. Thus, we have the following equations:

$$\begin{aligned} c_1^2 + c_2^2 &= 1 \\ 2c_1c_2 &= k \end{aligned}$$

Adding and subtracting these two equations yield:

$$(c_1 + c_2)^2 = 1 + k$$
$$(c_1 - c_2)^2 = 1 - k$$

It is then easy to verify that one possible solution is:

$$c_1 = \frac{1}{2}\left(\sqrt{1+k} + \sqrt{1-k}\right)$$
$$c_2 = \frac{1}{2}\left(\sqrt{1+k} - \sqrt{1-k}\right)$$

We then transform H_1 and H_2 over to the variables V_1 and V_2 given by:

$$V_i = \Phi(H_i), \ i = 1, 2$$

Since H_1 and H_2 are both $N(0,1)$-distributed marginally, it follows that V_1 and V_2 are both $R[0,1]$-distributed marginally. However, since H_1 and H_2 are correlated, this implies that V_1 and V_2 are correlated as well. Finally, we transform V_1 and V_2 over to the variables X_1 and X_2 given by:

$$X_i = F_{X_i}^{-1}(V_i), \ i = 1, 2$$

Since V_1 and V_2 are both $R[0,1]$-distributed marginally, it follows that X_1 and X_2 get the correct marginal distributions. However, since V_1 and V_2 are correlated, this implies that X_1 and X_2 are correlated as well. If we sample the initial random variables U_1 and U_2 from the conditional distribution given that $(U_1, U_2) \in D'$, then it is easy to verify that X_1 and X_2 get the corresponding the conditional distribution given that $(X_1, X_2) \in D$. This follows since there still is a one-to-one mapping between (U_1, U_2) and (X_1, X_2) (even though this mapping is much more complicated in the case of correlation).

4.2.5 Risk Distributions

In the insurance business, different distributions are often used to model insurance losses X. When the potential claims of insurance companies are very high, we can use the heavy tail distributions for modelling, such as Pareto distribution and lognormal distribution. For the simulation of extreme losses, especially in more risky insurance types, heavy tailing is essential. When the high claim risk does not occur frequently, we use light tails such as truncated normal and gamma for modelling.

Definition 4.2.1. Pareto Distribution
If X is a positive random variable with a Pareto distribution, then the probability density function of X is given by:

$$f_X(x) = \begin{cases} \frac{\tau x_m^{\tau}}{x^{\tau+1}} & for\ x \geqslant x_m \\ 0 & otherwise \end{cases}$$

where the parameter τ is positive and x_m is the lower bound value of X.

Cumulative distribution function of X is given by

$$F_X(x) = \begin{cases} 1 - \left(\frac{x_m}{x}\right)^{\tau} & for\ x \geqslant x_m \\ 0 & otherwise \end{cases}$$

The mean and variance of X are:

$$\begin{aligned} \mathbb{E}(X) &= \begin{cases} \infty & for\ \tau \leqslant 1 \\ \frac{\tau x_m}{\tau - 1} & for\ \tau > 1 \end{cases} \\ Var(X) &= \begin{cases} \infty & for\ 1 < \tau \leqslant 2 \\ \left(\frac{x_m}{\tau-1}\right)^2 \frac{\tau}{\tau-2} & for\ \tau > 2 \end{cases} \end{aligned}$$

Note that when $\tau \leqslant 1$, the variance does not exist.

We can estimate the Pareto parameters x_m and τ by the following method. Assuming that we are given that $\mathbb{E}(X) = m > 0$ and $SD(X) = s > 0$, we can find the corresponding parameter values, x_m and τ by solving the equations:

$$\begin{aligned} m &= \frac{x_m \tau}{\tau - 1} \\ s^2 &= \frac{x_m^2 \tau}{(\tau - 1)^2 (\tau - 2)} \end{aligned}$$

We square both sides of the first equation and divide the resulting equation by the second equation. As a result we get:

$$\frac{m^2}{s^2} = \tau(\tau - 2)$$

or equivalently:

$$\tau^2 - 2\tau + 1 = (\tau - 1)^2 = \frac{m^2}{s^2} + 1$$

Hence, we get:

$$\tau = 1 \pm \sqrt{\frac{m^2}{s^2} + 1}$$

Since τ is assumed to be positive, the negative root can be neglected, and we get the unique solution:

$$\tau = 1 + \sqrt{\frac{m^2}{s^2} + 1}$$

Having determined τ, it also follows that:

$$x_m = \frac{m(\tau - 1)}{\tau}$$

Note that the parameter formulas are valid for all possible combinations of m and s as long as $m, s > 0$. Moreover, since $m^2/s^2 + 1 > 1$, it follows that $\tau > 2$. Thus, $Var[X]$ exists.

Definition 4.2.2. Lognormal Distribution
If the random variable $Y = ln(X)$ is a normal distribution with mean μ and variance σ^2, then the positive random variable X is said to be log-normally distributed. The probability density function of X is given by:

$$f_X(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} exp\left(-\frac{(lnx-\mu)^2}{2\sigma^2}\right) & for\ x \geqslant 0 \\ 0 & otherwise \end{cases}$$

The cumulative distribution function is

$$F_X(x) = \begin{cases} \Phi\left(\frac{(lnx)-\mu}{\sigma}\right) & for\ x \geqslant 0 \\ 0 & otherwise \end{cases}$$

where Φ is the cumulative distribution function of the standard normal distribution. The mean and variance of X are:

$$\mathbb{E}(X) = e^{\mu + \frac{1}{2}\sigma^2} \qquad Var(X) = (\mathbb{E}(X))^2 \left(e^{\sigma^2} - 1\right)$$

The next step is to determine the parameters μ and σ such that $E(X) = m$ and $Var(X) = s^2$. We get the fellow equations:

$$\begin{aligned} m &= e^{\mu + \frac{1}{2}\sigma^2} \\ s^2 &= (\mathbb{E}(X))^2 \left(e^{\sigma^2} - 1\right) \end{aligned}$$

A short calculation revealed that

$$\begin{aligned} \sigma &= \sqrt{log\left(\frac{s^2}{m^2} + 1\right)} \\ \mu &= log(m) - \frac{1}{2} log\left(\frac{s^2}{m^2} + 1\right) \end{aligned}$$

Definition 4.2.3. Truncated Normal distribution
Assume Y has a normal distribution with mean μ and variance σ^2, and falls inside the interval (a, b), with $-\infty \leqslant a < b \leqslant \infty$. Then Y in $[a, b]$ is a truncated normal distribution with probability density function:

$$f_X(x) = \begin{cases} \frac{1}{\sigma} \frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} & for\ a \leqslant x \leqslant b \\ 0 & otherwise \end{cases}$$

where Φ and ϕ are the cumulative distribution function and probability density function of standard normal distribution respectively. In this thesis, we only consider the case where $a = 0$ and $b = \infty$.

The inverse cumulative distribution function for the truncated normal distribution, truncated to the interval $[0, \infty)$, is given by:

$$X = \mu + \sigma \cdot \Phi^{-1}\left[\Phi\left(-\frac{\mu}{\sigma}\right) + U \cdot \left(1 - \Phi\left(-\frac{\mu}{\sigma}\right)\right)\right]$$

It is easy to see that X is an increasing function of U. Moreover, if $U = 0$, then:

$$\begin{aligned} X &= \mu + \sigma \cdot \Phi^{-1}\left[\Phi\left(-\frac{\mu}{\sigma}\right)\right] \\ &= \mu + \sigma \cdot \left(-\frac{\mu}{\sigma}\right) = \mu - \mu = 0 \end{aligned}$$

Hence, X is non-negative with probability 1. In order to verify the distribution X, we let $X \geqslant 0$ and consider the probability $\mathbb{P}(X > x)$. We then get:

$$\begin{aligned} \mathbb{P}(X > x) &= \mathbb{P}\left(\frac{X-\mu}{\sigma} > -\frac{x-\mu}{\sigma}\right) \\ &= \mathbb{P}\left(\Phi^{-1}\left[\Phi\left(-\frac{\mu}{\sigma}\right) + U \cdot \left(1 - \Phi\left(-\frac{\mu}{\sigma}\right)\right)\right] > -\frac{x-\mu}{\sigma}\right) \\ &= \mathbb{P}(\Phi\left(-\frac{\mu}{\sigma}\right) + U \cdot (1 - \Phi\left(-\frac{\mu}{\sigma}\right)) > \Phi\left(\frac{x-\mu}{\sigma}\right)) \\ &= \mathbb{P}(U > \frac{\Phi\left(\frac{x-\mu}{\sigma}\right) - \Phi\left(\frac{-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{-\mu}{\sigma}\right)}) \\ &= 1 - \frac{\Phi\left(\frac{x-\mu}{\sigma}\right) - \Phi\left(\frac{-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{-\mu}{\sigma}\right)} \\ &= \frac{1 - \Phi\left(\frac{x-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{-\mu}{\sigma}\right)} = \frac{\mathbb{P}(Y > x)}{\mathbb{P}(Y > 0)} \end{aligned}$$

Moreover, we let $\tau = -\mu/\sigma$. The mean and variance of truncated normal variable X can be shown that:

$$\mathbb{E}(X) = \mu + \sigma \cdot \frac{\phi(\tau)}{1 - \Phi(\tau)} \tag{4.4}$$

$$Var(X) = \sigma^2 \cdot \left[1 + \tau\left(\frac{\phi(\tau)}{1 - \Phi(\tau)}\right) - \left(\frac{\phi(\tau)}{1 - \Phi(\tau)}\right)^2\right] \tag{4.5}$$

We now want to determine μ and σ such that $E(X) = m$ and $Var(X) = s^2$. Inserting m and s^2 into Eq.(4.4) and Eq.(4.5), these equations may be rewritten as:

$$\frac{m}{\sigma} = -\tau + \frac{\phi(\tau)}{1 - \Phi(\tau)} = \frac{\phi(\tau) - \tau(1 - \Phi(\tau))}{1 - \Phi(\tau)} \tag{4.6}$$

$$\frac{s^2}{\sigma^2} = \left[1 + \tau\left(\frac{\phi(\tau)}{1 - \Phi(\tau)}\right) - \left(\frac{\phi(\tau)}{1 - \Phi(\tau)}\right)^2\right] \tag{4.7}$$

We then divide the left-hand side of Eq.(4.7) by the square of the left-hand side of Eq.(4.6). Similarly, we divide the right-hand side of Eq.(4.7) by the square of the right-hand side of Eq.(4.6). This yields the following equation:

$$\frac{s^2}{m^2} = \frac{\left[1 + \tau\left(\frac{\phi(\tau)}{1-\Phi(\tau)}\right) - \left(\frac{\phi(\tau)}{1-\Phi(\tau)}\right)^2\right]}{\left[\frac{\phi(\tau)}{1-\Phi(\tau)} - \tau\right]^2} \tag{4.8}$$

We observe that Eq.(4.8) contains only one unknown quantity, τ. Moreover, τ can easily be determined numerically from this equation. Having determined τ we may insert this into Eq.(4.6) and obtain:

$$\sigma = m \cdot \frac{1 - \Phi(\tau)}{\phi(\tau) - \tau\left(1 - \Phi(\tau)\right)}$$

Finally, the parameter μ is given by:

$$\mu = \frac{\mu}{\sigma} \cdot \sigma = -\tau \cdot \sigma$$

Definition 4.2.4. Gamma Distribution
If X is a positive random variable with a gamma distribution, then the probability density function of X is given by:

$$f_X(x) = \begin{cases} \frac{\beta^\tau}{\Gamma(\tau)} x^{\tau-1} e^{-x\beta} & for\ x > 0 \\ 0 & otherwise \end{cases}$$

where $\Gamma(\tau) = \int_0^\infty x^{\tau-1} e^{-x} dx$ and τ is the shape parameter and β is the scale parameter.

Mean and variance of gamma variables are

$$E(X) = \tau/\beta \qquad Var(X) = \tau/\beta^2$$

Let $\mathbb{E}(X) = m > 0$ and $SD(X) = s > 0$, we can find the corresponding parameter values, τ and β are given by:

$$\begin{aligned} \tau &= \frac{m^2}{s^2} \\ \beta &= \frac{\tau}{m} = \frac{m}{s^2} \end{aligned}$$

4.3 The Model

We now return to the main problem where $\mathbf{X} = (X_1, X_2)$ is a bivariate absolutely continuously distributed random variable, and where $\mathcal{X} = \mathbb{R}^+ \times \mathbb{R}^+$. We start out by considering the constraint. The retained risk covered by the cedent for line i, denoted:

$$I_i(X_i) = \min(X_i, a_i) + \min(X_i - b_i, 0), \ i = 1, 2$$

where $0 \leqslant a_i \leqslant b_i$ for $i = 1, 2$. By subsection 4.2.1, $C(\mathbf{b})$ is given by:

$$C(\mathbf{b}) = C(b_1, b_2) = \{(x_1, x_2) : \ I_1(x_1) + I_2(x_2) > a_1 + a_2\}$$

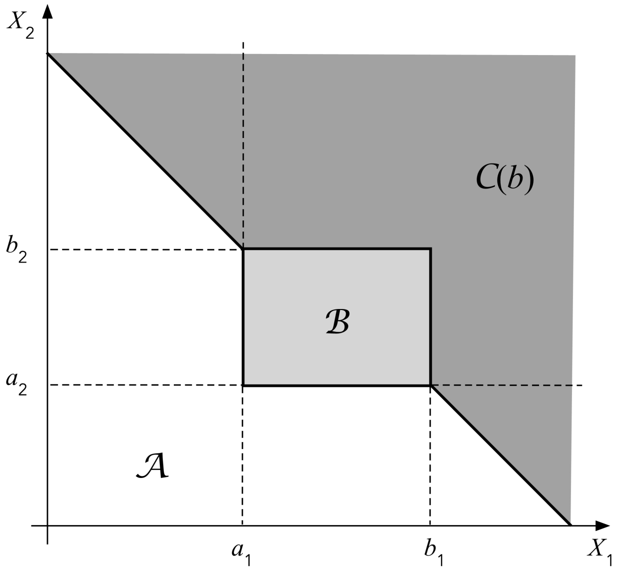


Figure 4.3: The sets $\mathcal{A}$, $\mathcal{B}$ and $C(b)$.

which is illustrated in Figure 4.3. For any given $\mathbf{b} = (b_1, b_2)$ we estimate:

$$h(\mathbf{b}) = \mathbb{E}[\phi(\mathbf{b}, \mathbf{X})] = \mathbb{E}[\mathbb{1}(\mathbf{X} \in C(\mathbf{b}))] = \mathbb{P}(\mathbf{X} \in C(\mathbf{b}))$$

based on a sample $\mathbf{X}_j = (X_{1,j}, X_{2,j})$ for $j = 1, ..., N$, by:

$$\begin{aligned} \tilde{h}(\mathbf{b}) &= p_D \frac{1}{N} \sum_{j=1}^{N} \mathbb{1}(\mathbf{X}_j \in C(\mathbf{b})) \\ &= p_D \frac{1}{N} \sum_{j=1}^{N} \mathbb{1}(I_1(X_{1,j}) + I_2(X_{2,j}) > a_1 + a_2) \end{aligned}$$

We now introduce $\mathbf{B} = (B_1, B_2)$ where:

$$B_i = 1 - F_{X_i}(b_i) = S_{X_i}(b_i), \ i = 1, 2$$

where $S_{X_i}(x) = 1 - F_{X_i}(x)$ denotes the survival probability function of X_i. We then express $\tilde{h}$ as function of $\mathbf{B}$ instead, and our equivalent objective is to determine $\mathbf{B}$ so that:

$$\tilde{h}(\mathbf{B}) = \mathbb{P}(\mathbf{X} \in C(\mathbf{B})) = \alpha \tag{4.9}$$

Note that there is a one-to-one correspondence between $\mathbf{b}$ and $\mathbf{B}$:

$$b_i = S_{X_i}^{-1}(B_i), \ i = 1, 2$$

Thus, solving the problem for $\mathbf{B}$ immediately gives us a solution for $\mathbf{b}$ as well. Typically, there exists infinitely many combinations of B_1 and B_2 which satisfy Eq.4.9. Thus, we instead consider the problem where B_1 is given, and we want to determine the corresponding value of B_2 such that Eq.4.9 holds.

We finish by considering the objective function, where we let:

$$R_i(X_i) = X_i - I_i(X_i) = \max(\min(X_i, b_i) - a_i, 0), \;\; i = 1, 2$$

and where $0 \leqslant a_i \leqslant b_i$ for $i = 1, 2$, and introduce:

$$\psi(\mathbf{b}, \mathbf{X}) = R_1(X_1) + R_2(X_2)$$

As before, a_1 , a_2 are considered to be fixed and known, while $\mathbf{b} = (b_1, b_2)$ is the solution to the optimization problem:

$$\begin{aligned} &Minimize: \;\; g(\mathbf{b}) = \mathbb{E}[\psi(\mathbf{b}, \mathbf{X})] \\ &Subject\ to: \;\; h(\mathbf{b}) = \mathbb{E}[\phi(\mathbf{b}, \mathbf{X})] = \mathbb{P}(\mathbf{X} \in C(\mathbf{b})) = \alpha \end{aligned}$$

We still let:

$$\begin{aligned} D' &= \{(u_1, u_2): \;\; 1 - \rho \leqslant u_i \leqslant 1, \;\; i = 1, 2\} \\ E' &= \{(u_1, u_2): \;\; 0 \leqslant u_i \leqslant 1 - \rho, \;\; i = 1, 2\} \end{aligned}$$

$$\begin{aligned} D &= \{(x_1, x_2) = \left(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2)\right): \; (u_1, u_2) \in D'\} \\ E &= \{(x_1, x_2) = \left(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2)\right): \; (u_1, u_2) \in E'\} \end{aligned}$$

Moreover, we assume that ρ is chosen sufficiently large so that:

$$C(\mathbf{b}) = \{(x_1, x_2): \;\; I_1(x_1) + I_2(x_2) > a_1 + a_2\} \subseteq D \;\; for\ all\ \mathbf{b}$$

The blue region in Figure 4.4 below represents the set E , while $D = X \setminus E$. Here $C(\mathbf{b}) \subseteq D$, at least for all values of $\mathbf{b}$ we need to consider given the

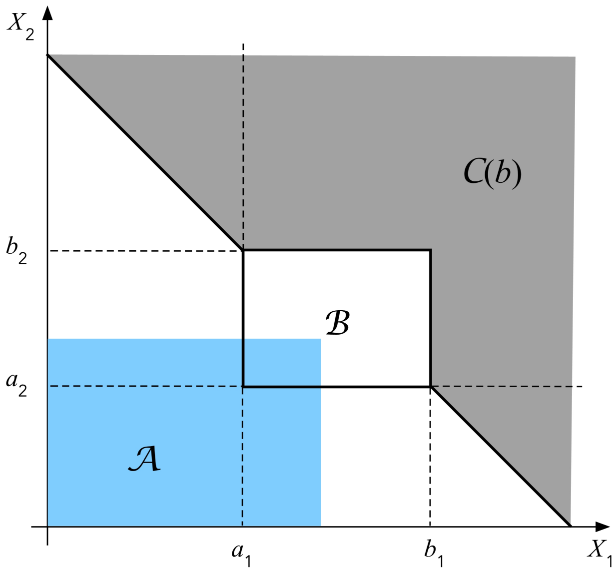


Figure 4.4: Blue Region and C(b).

constraint that $\mathbb{P}(\mathbf{X} \in C(\mathbf{b})) = \alpha$.

Remark 4.3.1. The blue region is not rectangular if risks X_1 and X_2 are dependent. Based on the premise that the blue region does not interfere with the gray region, the blue region needs to be as large as possible to get much importance sampling.

In order to estimate $g(\mathbf{b})$ we generate a sample $\mathbf{X}_1, \ldots, \mathbf{X}_N$ from the conditional distribution of X given the event $X \in D$ and another sample $\mathbf{Y}_1, \ldots, \mathbf{Y}_M$ from the conditional distribution of X given the event $X \in E$, and estimate $g(\mathbf{b})$ by:

$$\tilde{g}\left(\mathbf{b}\right) = p_D \frac{1}{N} \sum_{j=1}^{N} \psi\left(\mathbf{b}, \mathbf{X}_j\right) + p_E \frac{1}{M} \sum_{j=1}^{M} \psi\left(\mathbf{b}, \mathbf{Y}_j\right)$$

Note, $\mathbf{X}_j = (X_{1,j}, X_{2,j})$ for $j = 1, ..., N$ and $\mathbf{Y}_j = (Y_{1,j}, Y_{2,j})$ for $j = 1, ..., M$, however, that if $X \in E$, then

$$\psi\left(\mathbf{b}, \mathbf{X}\right) = \sum_{i=1}^{2} \max\left(\min\left(X_i, b_i\right) - a_i, 0\right) = \sum_{i=1}^{2} \max\left(X_i - a_i, 0\right)$$

Hence, $\psi\left(\mathbf{b}, \mathbf{X}\right)$ is constant given distributions of X_i for $i = 1, 2$ and does not depend on $\mathbf{b}$ for all $\mathbf{X} \in E$. Thus, in order to minimize $g(\mathbf{b})$, we can focus on minimizing the term:

$$\frac{1}{N} \sum_{j=1}^{N} \psi\left(\mathbf{b}, \mathbf{X}_j\right)$$

This implies that simulation errors related to the second term of $\tilde{g}\left(\mathbf{b}\right)$ will not affect the minimization. As a result, we may choose $M \ll N$ without losing any precision. In fact, if we are only interested in the minimization of $g(\mathbf{b})$, and not the resulting minimal value of $g(\mathbf{b})$, we may even let $M = 0$.

We now have all the needed tools available to solve the optimization problem. We assume that we have determined an interval $[B_1^L, B_1^U]$ where we know that the optimal value of B_1 must be located.

- We run an iteration where we let B_1 go through a suitable sequence of values in $[B_1^L, B_1^U]$.
- For each B_1-value we calculate the corresponding B_2-value such that $\tilde{h}\left(\mathbf{b}\right) = \alpha$. Furthermore, Eq.(3.8) indicates that for a given B_1 the lower and upper bounds of B_2 are:

$$B_2^L = \frac{\alpha - B_1}{1 - B_1},\ B_2^U = \frac{\alpha - A \cdot B_1}{A - B_1}$$

- For each B_1-value and B_2-value we calculate $\tilde{g}\left(\mathbf{b}\right)$.
- Having calculated $\tilde{g}\left(\mathbf{b}\right)$ for all combinations of B_1 and B_2, we choose the combination which minimizes $\tilde{g}\left(\mathbf{b}\right)$.
- The total reinsurance risk under α-VaR is:

$$V_\alpha(\mathbf{a}^*, \mathbf{b}) = a_1^* + a_2^* + (1 + \theta)\, \tilde{g}\left(\mathbf{b}\right)$$

In order to see the optimal solution more intuitively, we will plot the iso-curves for the objective function $g(\mathbf{B})$ and constraint curve $h(\mathbf{B})$. It is worth noting that the shape of the iso-curves depends on the hazard rates of risks from Huseby [4]. The corresponding theorem is as follows.

Theorem 4.3.2 (Theorem 2.3 in Huseby [4])**.** *Risks X_1 and X_2 with decreasing hazard rates, g is known as quasiconvex functions of $\mathbf{B}$. Risks X_1 and X_2 with increasing hazard rates, g is known as quasiconcave functions of $\mathbf{B}$.*

CHAPTER 5

Numerical Examples

This chapter will estimate the optimal reinsurance in the bivariant case for different risk distributions by using importance sampling. It is worth noting that the two risks random variables X_1 and X_2 are dependent on the same type or different types of distributions. We use Gaussian cupola to construct correlation. More specifically, we would like to describe how the optimal reinsurance contracts vary with the degree of dependence. In all of these tasks, numerical examples will be selected and investigated in detail. Let $\theta = 0.2$ and the values of correlation coefficient $Cov\,[H_1, H_2] = k$ are taken as -0.8, -0.5, 0, 0.5 and 0.8 respectively. To visualize the location of the optimal solution, we will plot the iso-curves of expected insured risk $\mathbb{E}(R_1\,(X_1) + R_2\,(X_2))$ and the constraint curve under different k values.

5.1 Symmetric Risks

Two risks are chosen from heavy-tailed distributions such as Pareto, lognormal distribution, and light-tailed distributions such as truncated normal and gamma distribution. Due to the fact that X_1 and X_2 are symmetric risks, the optimal values a_1^* and a_2^* will be equal. We can calculate the $\mathbf{a}^*$ by using

$$a_1^* = a_2^* = S_{X_1}^{-1}\left(\frac{1}{1+\theta}\right) = S_{X_2}^{-1}\left(\frac{1}{1+\theta}\right)$$

5.1.1 Under Pareto Distribution

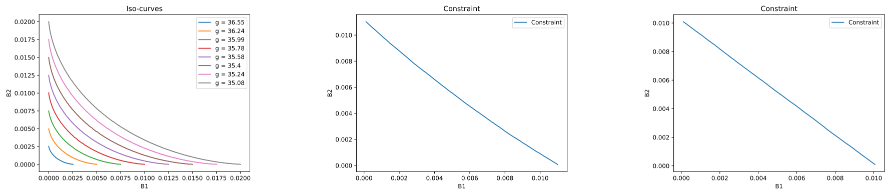
First we check the following risks:

$$X_1 \sim Pareto\,(m = 50,\ s = 50)$$
$$X_2 \sim Pareto\,(m = 50,\ s = 50)$$

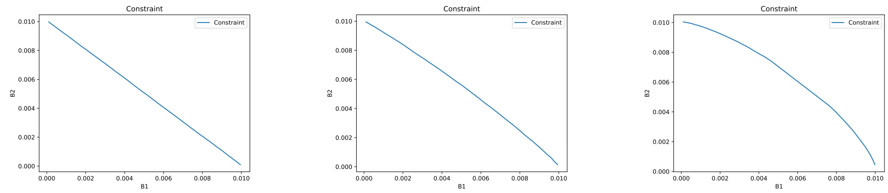
The optimal results are given in Table 5.1 below. In Table 5.1, all optimal solutions are balanced solutions. When $k = 0$, the optimal VaR equals 105.60, the maximum value compared with the other four values. This means that the risk of optimal reinsurance is the highest when the symmetric Pareto risks are independent under VaR.

k	a^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	31.59	255.71	258.41	0.005	0.005	105.52	21.18	21.17
-0.5	31.59	253.84	269.00	0.005	0.005	105.57	21.16	21.23
0	31.59	262.49	261.90	0.005	0.005	105.60	21.22	21.21
0.5	31.59	254.94	258.12	0.005	0.005	105.53	21.17	21.18
0.8	31.59	244.43	242.45	0.006	0.006	105.35	21.10	21.08

Table 5.1: Results summary where symmetric risks are Pareto distribution with mean 50 and standard deviation 50.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$



(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.1: Iso-curves and constraint where symmetric risks are Pareto distribution with mean 50 and standard deviation 50.

Now we decrease the standard deviation of X_i, and calculate the optimal values of following risks:

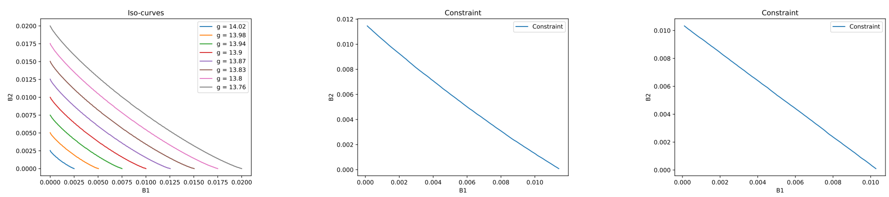
$$X_1 \sim Pareto\,(m = 50,\ s = 10)$$
$$X_2 \sim Pareto\,(m = 50,\ s = 10)$$

The results are presented in Table 5.2. We conclude that all the optimal solutions are balanced, and the V_α is highest if the Pareto risks are uncorrelated.

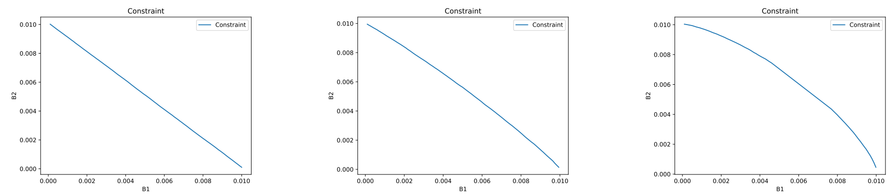
k	a^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	43.07	97.96	98.16	0.006	0.006	102.77	8.32	8.31
-0.5	43.07	97.92	100.14	0.005	0.005	102.78	8.32	8.32
0	43.07	99.51	99.42	0.005	0.005	102.80	8.33	8.32
0.5	43.07	98.44	98.92	0.005	0.005	102.79	8.32	8.32
0.8	43.07	96.81	96.50	0.006	0.006	102.76	8.31	8.31

Table 5.2: Results summary where symmetric risks are Pareto distribution with mean 50 and standard deviation 10.

In Figure 5.1 and Figure 5.2, iso-curves are locus of different combinations of two factors B_1 and B_2 giving the same level of reinsurance premiums. The characteristic of the iso-curves is that they are slope downwards and convex to the origin since two risks have decreasing hazard rates. In case of different iso-curves the level of reinsurance premiums differs. Higher the iso-curve, lower the level of premiums. Moreover, iso-curves cannot intersect each other. With the decrease of standard deviations for both risks, the curvature of iso-curves becomes smaller.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$



(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.2: Iso-curves and constraint where symmetric risks are Pareto distribution with mean 50 and standard deviation 10.

In Figure 5.1 and Figure 5.2, the difference in k value leads to a slight difference in the constraint curves. As a result, it can be shown that the set of B_1 and B_2 is almost a straight line when $k = 0$. By increasing the k value, the constraint curve changes from being slightly convex to the origin to being concave to the origin. If we put the constraint curves and iso-curves in the same plot, it is not difficult to find the constraint curves always bend significantly from iso-curves. Thus, the optimal solutions for any correlation should be in the middle, which is balanced.

On the premise that k is negative, if the value of k is increased, the amount of total α-level VaR will increase. At the same time, when the correlation has a high negative value, the constraint curve itself is further away from the origin. Thus, the value of the objective function still becomes smaller compared to the independent case. On the premise that k is positive, if we increase the value of k, then the constraint curve bends more away from the origin. As a result, the value of B is increasing and the total risk covered by reinsurer is decreasing. Moreover, for $k \leqslant 0$, the range of V_α is small. This shows that there is little difference between choosing balanced solution and unbalanced solution.

5.1.2 Under Lognormal Distribution

Assume that we have two random risks:

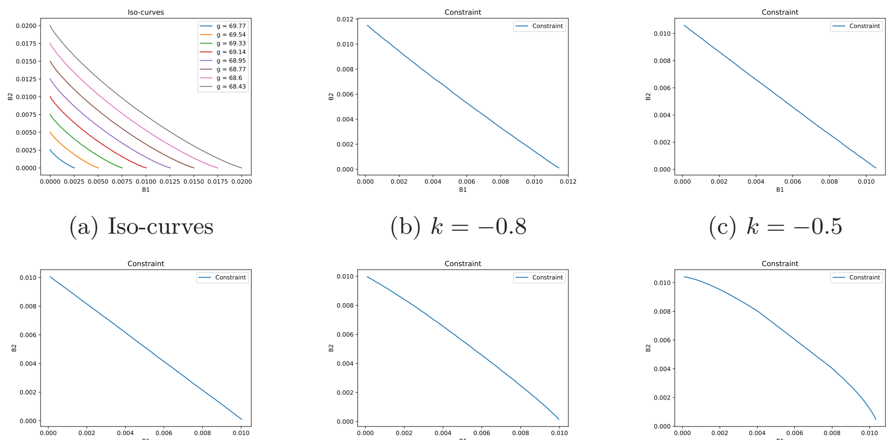
$$X_1 \sim Lognormal\,(m = 50,\ s = 50)$$
$$X_2 \sim Lognormal\,(m = 50,\ s = 50)$$

Table 5.3 lists the optimal values under different k values. Based on the fifth and sixth columns of Table 5.3, we can see that the balanced solutions are optimal.

k	a^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	15.80	299.67	283.65	0.006	0.006	114.26	41.41	41.25
-0.5	15.80	296.69	297.46	0.005	0.005	114.35	41.39	41.35
0	15.80	300.95	300.48	0.005	0.005	114.41	41.42	41.39
0.5	15.80	298.55	295.76	0.005	0.005	114.38	41.41	41.37
0.8	15.80	286.46	285.69	0.006	0.006	114.24	41.33	41.31

Table 5.3: Results summary where symmetric risks are lognormal distribution with mean 50 and standard deviation 50.

Figure 5.3 illustrates how the optimal solution approaches balanced solutions for different k values. In this example, the risks have decreasing hazard rates. Thus, iso-curves in the isoplane are quasiconvex. When the correlation has a high negative value, the constraint curve bends slightly against the origin. The constraint curve clearly bends away from the origin when the correlation has a high positive value. With the increase of k, the constraint curve change from slightly convex to the origin to straight line and then to concave to the origin. As long as the constraint curves either bend away from the origin or the curvatures of convex constraint curves are not as significant as that of iso-curves, the optimal solution is always balanced.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.3: Iso-curves and constraint where symmetric risks are lognormal distribution with mean 50 and standard deviation 50.

Now we turn to solve two risks as follow:

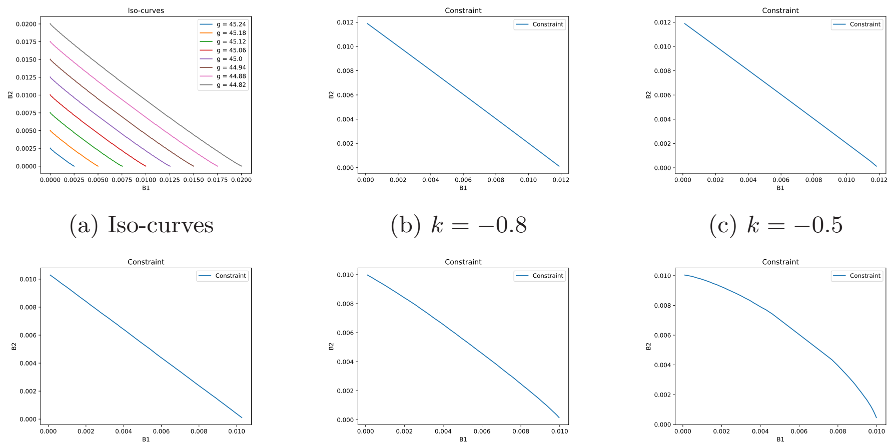
$$X_1 \sim Lognormal\,(m = 50,\ s = 25)$$
$$X_2 \sim Lognormal\,(m = 50,\ s = 25)$$

Table 5.4 displays the optimal values under various k values. We can see that all results point out that the optimal solutions are balanced.

k	a^*	b_1^*	b_2^*	B_1^*	B_2^*	min V_α	π_{X_1}	π_{X_2}
-0.8	28.32	145.41	147.55	0.006	0.006	110.615	27.00	26.98
-0.5	28.32	145.81	147.13	0.006	0.006	110.62	27.00	26.98
0	28.32	150.14	149.99	0.005	0.005	110.677	27.03	27.01
0.5	28.32	150.04	149.25	0.005	0.005	110.676	27.03	27.01
0.8	28.32	147.48	145.48	0.006	0.006	110.63	27.01	26.98

Table 5.4: Results summary where symmetric risks are lognormal distribution with mean 50 and standard deviation 25.

From Figure 5.4, we can see the iso-curves are slightly quasiconvex. When k is -0.8, -0.5 and 0, the constraint curves are approximately linear, and the corresponding optimal values are close to balanced solutions. By the way, when the correlation coefficient k is negative, the constraint curve gradually approaches the origin as k goes 0, leading to an increase in the value of V_α. Additionally, the constraint curves bend away from the origin for k is positive, and When the correlation has a high positive value, the constraint curve clearly bends. Therefore, balanced solutions are always optimal. A future novel finding is that the constraint curve gradually shifts toward $x = y = 0.01$ while changing the curvature. This implies that the maximum value of V_α is in the correlation k between 0 and 0.5.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.4: Iso-curves and constraint where symmetric risks are lognormal distribution with mean 50 and standard deviation 25.

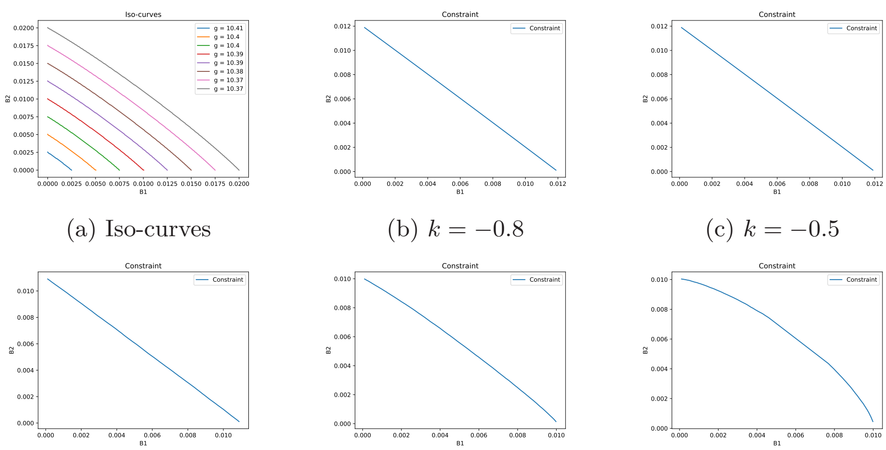
Now we continue to reduce the standard deviations of risks, and calculate the optimal values of following risks:

$$X_1 \sim Lognormal\,(m = 50,\ s = 5)$$
$$X_2 \sim Lognormal\,(m = 50,\ s = 5)$$

Table 5.5 shows the results. According to the results, it is not difficult to find that if k = -0.8, -0.5, 0, the optimal solution is at the boundary, and in other cases, the optimal solutions could lie in the middle.

k	a^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	45.18	inf	62.32	0	0.012	102.8131	6.25	6.21
-0.5	45.18	inf	62.32	0	0.012	102.8135	6.25	6.21
0	45.18	62.53	inf	0.011	0	102.817	6.22	6.24
0.5	45.18	62.94	67.55	0.010	0.001	102.82	6.23	6.24
0.8	45.18	63.39	64.59	0.008	0.004	102.816	6.23	6.23

Table 5.5: Results summary where symmetric risks are lognormal distribution with mean 50 and standard deviation 5.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.5: Iso-curves and constraint where symmetric risks are lognormal distribution with mean 50 and standard deviation 5.

Figure 5.5 shows the expected reinsured risk iso-curves of the sets of B_1 and B_2. X_1 and X_2 have increasing hazard rates. According to Theorem 4.3.2, the iso-curves are concave towards the origin. It is notable that the set of B_1 and B_2 satisfying the constraint is almost a straight line when k is equal to -0.8, -0.5 and 0. This implies that the best solution can be determined at the boundary. When k is equal to 0.5 and 0.8, the constraint curves are concave to the origin. In addition, the constraint curve for k=0.5 and iso-curves almost coincide. Therefore, all solutions are optimal when we keep the result to two decimal places. Moreover, the constraint curve with k = 0.8 is bending more away from the origin than the iso-curves. This will allow the best solution to be determined in the middle. The result is that the optimal solution changes from boundary to balanced as k goes from negative to positive.

5.1.3 Under Truncated Normal Distribution

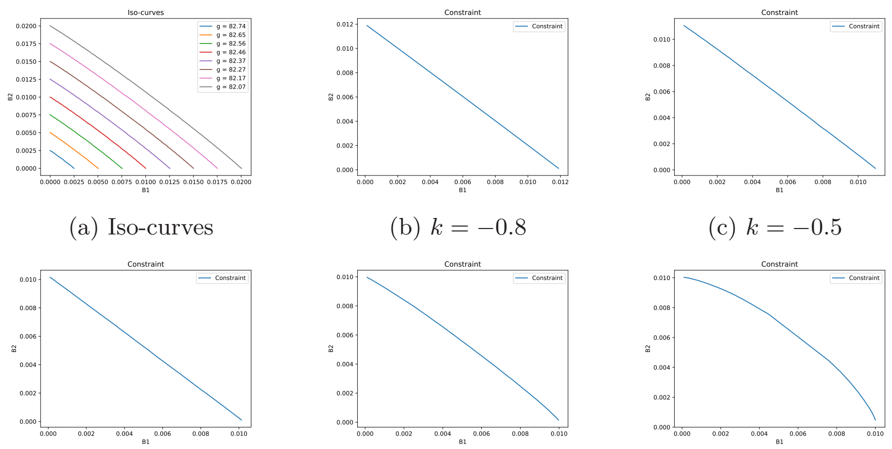
Let the symmetric risks be as follows:

$$X_1 \sim Truncnormal\,(m = 50,\ s = 50)$$
$$X_2 \sim Truncnormal\,(m = 50,\ s = 50)$$

The optimal results are shown in Table 5.6. We found that the optimal solutions are close to balanced solutions when $k = 0.5,\ 0.8$. For the remaining cases, the optimal solutions are at the boundary. To find out the reason, we draw iso-curves and constraint curves in Figure 5.6.

k	a^*	b_1^*	b_2^*	B_1^*	B_2^*	min V_α	π_{X_1}	π_{X_2}
-0.8	10.03	inf	202.50	0	0.012	119.01	49.76	49.19
-0.5	10.03	inf	205.38	0	0.011	119.05	49.77	49.22
0	10.03	208.67	inf	0.010	0	119.09	49.32	49.72
0.5	10.03	234.15	232.96	0.005	0.005	119.11	49.55	49.50
0.8	10.03	238.20	221.10	0.006	0.006	119.04	49.57	49.42

Table 5.6: Results summary where symmetric risks are truncated normal distribution with mean 50 and standard deviation 50.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.6: Iso-curves and constraint where symmetric risks are truncated normal distribution with mean 50 and standard deviation 50.

The risks have increasing hazard rates. Thus, iso-curves in Figure 5.6 are quasiconcave and slope downward from left to right. When the iso-curve is higher, the expected reinsured risk level is lower. The sets of B_1 and B_2 are almost straight lines when k takes -0.8, -0.5 and 0, so the corresponding optimal solutions are unbalanced and located at the boundary. Although the slopes of the constraint curve are almost the same, the closer the constraint curve is to the origin with k going to 0, which causes increased V_α. On the other hand, the sets of B_1 and B_2 are bending away from the origin when k is 0.5 and 0.8.

However, the constraint curves have greater bend than iso-curves, implying that the optimal solutions are balanced.

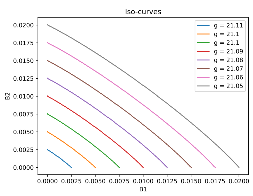
After reducing the standard deviation of X_i, we calculate the optimal values of the following risks:

$$X_1 \sim Truncnormal\,(m = 50,\ s = 10)$$
$$X_2 \sim Truncnormal\,(m = 50,\ s = 10)$$

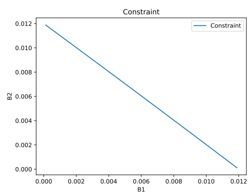
Table 5.7 shows the optimal solutions under different k. All solutions except k=0.8 correspond to unbalanced boundary solutions. When k is 0.8, the balanced solution is optimal.

k	a^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	40.33	inf	72.58	0	0.012	105.945	12.68	12.61
-0.5	40.33	inf	72.58	0	0.012	105.946	12.68	12.61
0	40.33	inf	72.84	0	0.011	105.951	12.68	12.62
0.5	40.33	73.25	inf	0.01	0	105.957	12.64	12.66
0.8	40.33	74.26	76.20	0.006	0.006	105.95	12.65	12.65

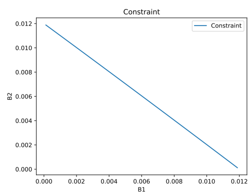
Table 5.7: Results summary where symmetric risks are truncated normal distribution with mean 50 and standard deviation 10.



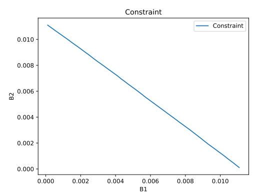
(a) Iso-curves



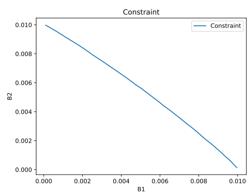
(b) $k = -0.8$



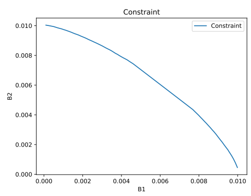
(c) $k = -0.5$



(d) $k = 0$



(e) $k = 0.5$



(f) $k = 0.8$

Figure 5.7: Iso-curves and constraint where symmetric risks are truncated normal distribution with mean 50 and standard deviation 10.

As shown in Figure 5.7, the iso-curves are clearly quasiconcave and descend from left to right. When k is non-positive, B_1 and B_2 are approximately linearly related. Therefore, their corresponding optimal solutions are at the boundary. Other times, the constraints bend away from the origin. When k is 0.5, the constraint curve and the iso-curves nearly coincide. Thus, the difference between choosing a balanced solution and an unbalanced solution is negligible. When k is 0.8, the constraint curve has a greater curvature than iso-curves. In other words, the optimal solution occurs in the middle.

In the case of truncated normal distributions, the hazard rates always increase. By Theorem 4.3.2, the iso-curves are always quasiconcave. The curvature of the iso-curves increase with the decrease of standard deviation. Furthermore, as the k value increases, the optimal solution moves from boundary to balanced.

On the other hand, both examples above show that the constraint curve gradually shifts to $x = y = 0.01$ as k increases when the correlation is slightly greater than 0. Thus, the value of the objective function with a small negative correlation still becomes larger then the independent case.

5.1.4 Under Gamma Distribution

Assume we have the following risks:

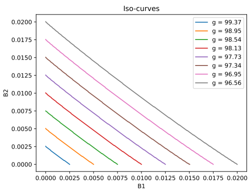
$$X_1 \sim Gamma\,(m = 50,\ s = 100)$$
$$X_2 \sim Gamma\,(m = 50,\ s = 100)$$

The optimal results are presented in Table 5.8. In this case, we can see that the value of a^* is very small. Most of the results indicate that the optimal solutions are close to balanced solutions, except for the case where k = -0.8.

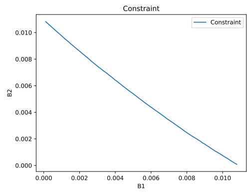
k	a^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	0.10	inf	471.81	0	0.011	117.74	59.85	57.68
-0.5	0.10	589.93	608.57	0.005	0.005	117.83	58.78	58.84
0	0.10	596.73	602.31	0.005	0.005	117.89	58.83	58.85
0.5	0.10	588.01	592.96	0.005	0.005	117.78	58.78	58.80
0.8	0.10	576.42	562.96	0.006	0.006	117.51	58.71	58.60

Table 5.8: Results summary where symmetric risks are gamma distribution with mean 50 and standard deviation 100.

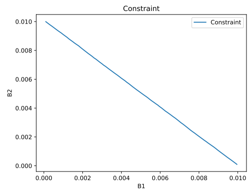
An illustration of how the optimal solution approaches balanced or unbalanced solutions for different k values is provided in Figure 5.8. In this example, the symmetric risks have decreasing hazard rates. Then the iso-curves are quasiconvex. However, increasing the correlation coefficient causes the constraint curve to change from slightly convex to the origin to concave to the origin. In particular, the constraint curve of k = -0.8 bends more against the origin than iso-curves which is why the optimal solution is at the boundary. The constraint for other k values is either close to linear or concave towards the origin. Thus the corresponding optimal solutions are balanced.



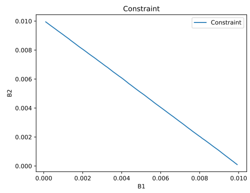
(a) Iso-curves



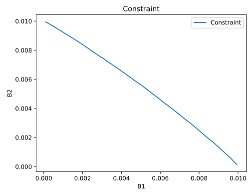
(b) $k = -0.8$



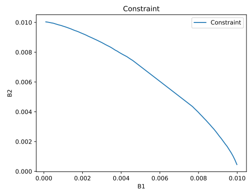
(c) $k = -0.5$



(d) $k = 0$



(e) $k = 0.5$



(f) $k = 0.8$

Figure 5.8: Iso-curves and constraint where symmetric risks are gamma distribution with mean 50 and standard deviation 100.

Now we decrease the standard deviation and assume the following risks:

$$X_1 \sim Gamma\,(m = 50,\ s = 45)$$
$$X_2 \sim Gamma\,(m = 50,\ s = 45)$$

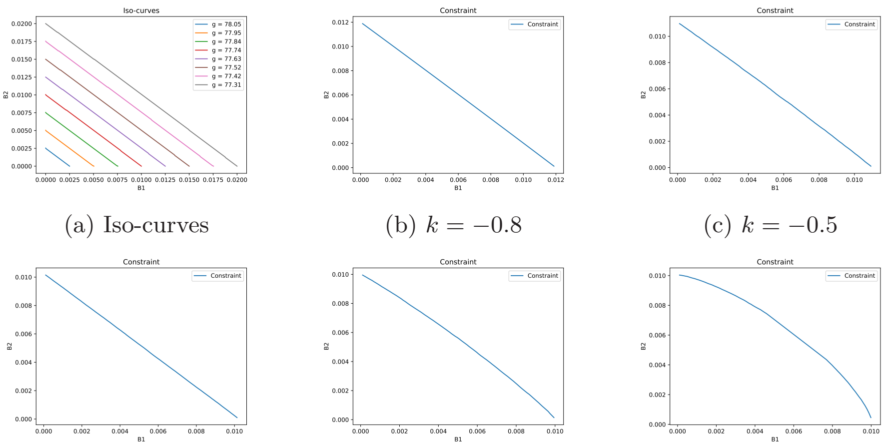
The optimal results are presented in Table 5.9. We can determine that the balanced solution is optimal for any k.

k	a^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	11.85	226.20	231.60	0.006	0.006	116.85	46.59	46.56
-0.5	11.85	239.29	225.33	0.006	0.006	116.90	46.67	46.52
0	11.85	208.95	325.87	0.010	0.0006	116.96	46.43	46.83
0.5	11.85	229.34	239.41	0.006	0.006	116.94	46.61	46.64
0.8	11.85	237.94	221.32	0.006	0.006	116.87	46.66	46.52

Table 5.9: Results summary where symmetric risks are gamma distribution with mean 50 and standard deviation 45.

An illustration of how the optimal solution approaches balanced or unbalanced solutions for different k values is provided in Figure 5.9. Since the risks have increasing hazard rates, the iso-curves are qusiconcave. Moreover, the iso-curves are approximately linear. The constrain curves, which bend away from the origin, correspond to the optimal balanced solution. When k is 0.5 and 0.8, there will be one optimal solution in the middle of the constraint curve. The constraint curve for k=-0.8, -0.5 and 0 tend to linearity. There is a very slight gap between all the solutions. When we keep 2 decimal places for all solutions, the errors are negligible. This means that all solutions are optimal, with k being non-positive.

Moreover, when k is slightly greater than 0 the constraint curve bends away from the origin while gradually moving to the position of $x = y = 0.01$. This



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.9: Iso-curves and constraint where symmetric risks are gamma distribution with mean 50 and standard deviation 45.

leads to the fact that the V_α value corresponding to the independent risks is not the maximum.

Now we continue to reduce the standard deviation s and calculate the optimal values of following risks:

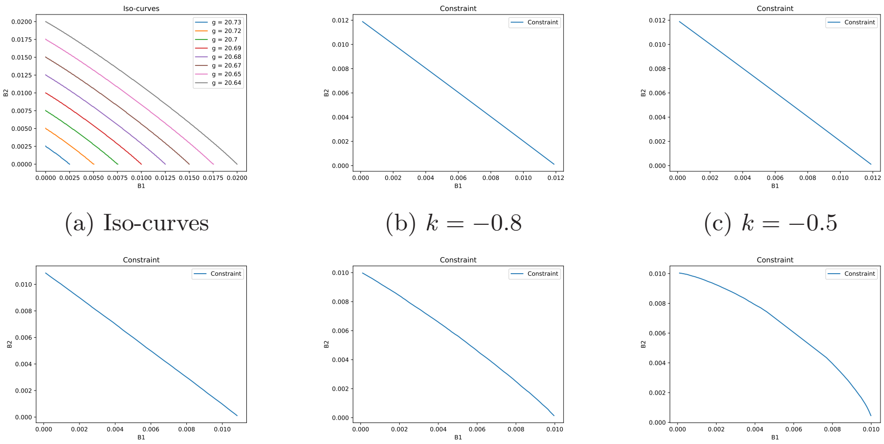
$$X_1 \sim Gamma\,(m = 50,\ s = 10)$$
$$X_2 \sim Gamma\,(m = 50,\ s = 10)$$

According to the results in Table 5.10, it is not difficult to find that if k= -0.8, -0.5, 0, the optimal solution is then at the boundary. When k is increasing from 0 to 1, the optimal value shifts from the boundary to the balanced solution.

k	a^*	b_1^*	b_2^*	B_1^*	B_2^*	min V_α	π_{X_1}	π_{X_2}
-0.8	40.35	inf	75.24	0	0.012	105.51	12.45	12.37
-0.5	40.35	inf	75.24	0	0.012	105.51	12.45	12.37
0	40.35	75.72	inf	0.010	0	105.523	12.39	12.44
0.5	40.35	78.67	79.79	0.006	0.006	105.526	12.42	12.41
0.8	40.35	77.50	80.04	0.006	0.006	105.518	12.41	12.42

Table 5.10: Results summary where symmetric risks are gamma distribution with mean 50 and standard deviation 10.

It should be noted that the iso-curves are quasiconcave in Figure 5.10. When k is -0.8, -0.5, 0, the constraints approach the linear functions, which lead to the boundary optimal solutions, and increasing k will not reduce the optimal V_α value. When k is positive, the constraint curve becomes concave to the origin. Furthermore, the larger the positive correlation is, the more concave the constraint curve is. When k is 0.5, the constraint curve and the iso-curves



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.10: Iso-curves and constraint where symmetric risks are gamma distribution with mean 50 and standard deviation 10.

basically overlap but the middle of the constraint curve is slightly bending more away than the iso-curves. Thus, balanced solutions could be optimal. However, the constraint curve for k=0.8 has a more significant bend than iso-curves. In other words, when k is equal to 0.8, it is best to choose a solution in the middle.

As correlation increases to 0, the constraint curve gradually moves toward the origin. More specifically, when k is slightly greater than 0, the constraint curve bends from the origin while continuing to move closer to the origin until the curve stops moving at the position $x = y = 0.01$. Thus, the optimal value V_α has a process of increasing and then decreasing in the interval from 0 to 0.5 for k. Alternately, it could simply mean that the risks with a small positive correlation correspond to the largest value of V_α.

5.2 Asymmetric Risks

Since X_1 and X_2 are asymmetric risks, the optimal values a_1^* and a_2^* are not equal and can be calculated from

$$\begin{aligned} a_1^* &= S_{X_1}^{-1}\left(\frac{1}{1+\theta}\right) \\ a_2^* &= S_{X_2}^{-1}\left(\frac{1}{1+\theta}\right) \end{aligned}$$

5.2.1 From Same Type of Distributions

To ensure the asymmetry of risks, we will change the expected value, standard deviation and both of them for each of the risks X_2 from the four distributions to observe the optimal reinsurance contract.

5.2.1.1 Under Pareto Distribution

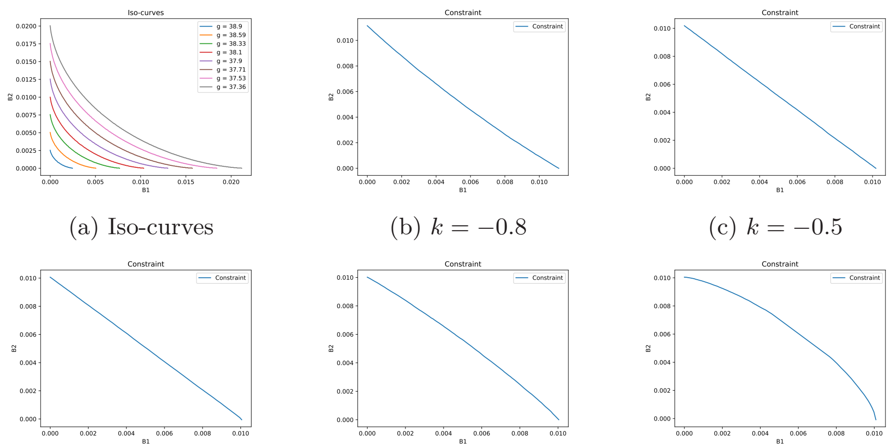
First we check the following risks:

$$X_1 \sim Pareto\,(m = 50,\ s = 50)$$
$$X_2 \sim Pareto\,(m = 60,\ s = 50)$$

The optimal results are given in Table 5.11, which shows that all optimal solutions are unbalanced solutions. The risk with a higher expected value is that with higher a^* and premium. Moreover, the value of π_{X_2}, and in particular, a_2^*, has increased from Table 5.1 following the increase of expectation to X_2. This has resulted in an increase in the premium and retained risk compared to Table 5.1. Therefore, the optimal V_α values are larger than those in Table 5.1.

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	31.59	39.28	264.27	250.23	0.0049	0.0056	116.03	21.23	23.74
-0.5	31.59	39.28	263.49	258.81	0.0049	0.0051	116.07	21.22	23.80
0	31.59	39.28	269.40	255.57	0.0047	0.0053	116.10	21.26	23.79
0.5	31.59	39.28	260.40	252.88	0.0051	0.0055	116.03	21.20	23.77
0.8	31.59	39.28	248.15	239.02	0.0057	0.0063	115.85	21.12	23.64

Table 5.11: Results summary where X_1 is Pareto distribution with mean 50 and standard deviation 50, while X_2 is Pareto distribution with mean 60 and standard deviation 50.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.11: Iso-curves and constraint where X_1 is Pareto distribution with mean 50 and standard deviation 50, while X_2 is Pareto distribution with mean 60 and standard deviation 50.

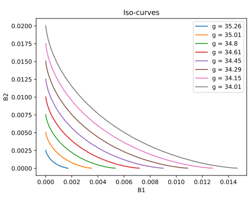
Figure 5.11 shows iso-curves and the constraint curves for variant value k. The Pareto distributions always have decreasing hazard rates. Thus, the iso-curves are quasiconvex. With the increase of k, the constraint curve changes from slightly convex towards the origin to concave to the origin. Although the risks are asymmetric, the expected value change keeps the iso-curves appear to be approximately symmetric around line $y = x$, making the optimal solution unbalanced and laying close to the middle. Additionally, as k increases, the optimal solution V_α first increases and then decreases.

Now we decrease the standard deviation of X_2, and calculate the optimal values of following risks:

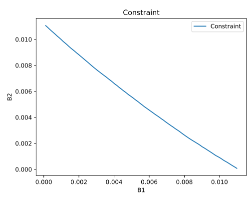
$$X_1 \sim Pareto\,(m = 50,\ s = 50)$$
$$X_2 \sim Pareto\,(m = 50,\ s = 40)$$

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	min V_α	π_{X_1}	π_{X_2}
-0.8	31.59	33.01	235.70	286.06	0.007	0.004	105.46	21.04	19.94
-0.5	31.59	33.01	240.45	240.45	0.006	0.004	105.50	21.07	19.96
0	31.59	33.01	241.38	290.88	0.006	0.004	105.53	21.08	19.98
0.5	31.59	33.01	244.88	270.15	0.006	0.005	105.46	21.10	19.90
0.8	31.59	33.01	223.62	270.97	0.007	0.005	105.30	20.93	19.90

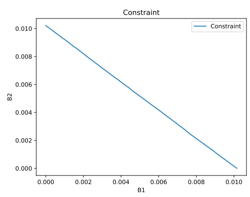
Table 5.12: Results summary where X_1 is Pareto distribution with mean 50 and standard deviation 50, while X_2 is Pareto distribution with mean 50 and standard deviation 40.



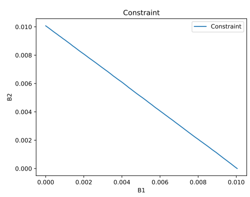
(a) Iso-curves



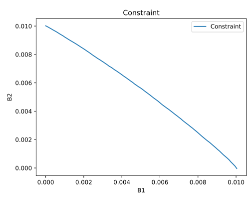
(b) $k = -0.8$



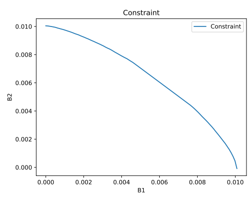
(c) $k = -0.5$



(d) $k = 0$



(e) $k = 0.5$



(f) $k = 0.8$

Figure 5.12: Iso-curves and constraint where X_1 is Pareto distribution with mean 50 and standard deviation 50, while X_2 is Pareto distribution with mean 50 and standard deviation 40.

The results are presented in Table 5.12. We conclude that balanced solutions are no longer optimal. The risk with the lower standard deviation is the one with lower premiums and higher a^* and b^* values. In addition, the retained risk term has been raised, and the premium term has been decreased compared to

Table 5.1. The decreasing amount is larger than the increasing amount which leads to the value of V_α being slightly less than that in Table 5.1.

Although iso-curves are quasiconvex and constraint curves change from convex to the origin to concave to the origin with increasing k, we can find that the iso-curves are asymmetric around line $y = x$ in Figure 5.12. Thus, no matter what value k takes, the optimal solution is always the unbalanced solution. When $k \leqslant 0$, the increase of k will cause the constraint curve to move to the origin. Thus the optimal solution V_α gradually increases. When $k > 0$, the rise of k will make the constraint curve more concave to the origin, which leads to the reduction of V_α.

Now we increase the expectation and decrease the standard deviation of X_2, and calculate the optimal values of following risks:

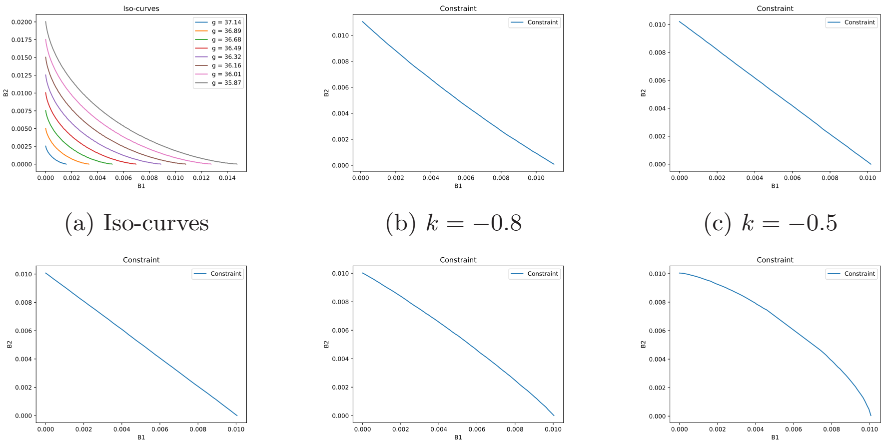
$$X_1 \sim Pareto\,(m = 50,\ s = 50)$$
$$X_2 \sim Pareto\,(m = 60,\ s = 40)$$

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	31.59	41.19	235.70	285.55	0.007	0.004	115.89	21.04	22.12
-0.5	31.59	41.19	240.29	289.54	0.006	0.004	115.93	21.07	22.14
0	31.59	41.19	241.34	290.89	0.006	0.004	115.96	21.08	22.16
0.5	31.59	41.19	244.88	270.15	0.006	0.005	115.90	21.10	22.07
0.8	31.59	41.19	223.70	270.81	0.007	0.005	115.72	20.93	22.07

Table 5.13: Results summary where X_1 is Pareto distribution with mean 50 and standard deviation 50, while X_2 is Pareto distribution with mean 60 and standard deviation 40.

The results are presented in Table 5.13. We conclude that all the optimal solutions are unbalanced and lie between the middle and the boundary. Changes in expected value and standard deviation act on the retained risk term. Moreover, the results of b^* are very similar to what we saw in Table 5.12. As a result, the standard deviation appears to be the most significant determinant of the outcomes of a^* and b^*. Thus, the expected value dominates the value of a^*. Because of the increase of a_2^* and π_{X_2}, the optimal solution V_α is significantly higher than its value in Table 5.1.

Figure 5.13 illustrates that, while iso-curves are convex and constraint curves change from a bend against the origin to a bend away from the origin as k increases, the range of iso-curves on the y-axis is greater than that on the x-axis which has the almost same iso-curves as changing standard deviation. So, standard deviation plays a leading role in changing the range of iso-curves. Therefore, there are no longer optimal balanced solutions, regardless of the value of k.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.13: Iso-curves and constraint where X_1 is Pareto distribution with mean 50 and standard deviation 50, while X_2 is Pareto distribution with mean 60 and standard deviation 40.

5.2.1.2 Under Lognormal Distribution

Assume that we have two random risks:

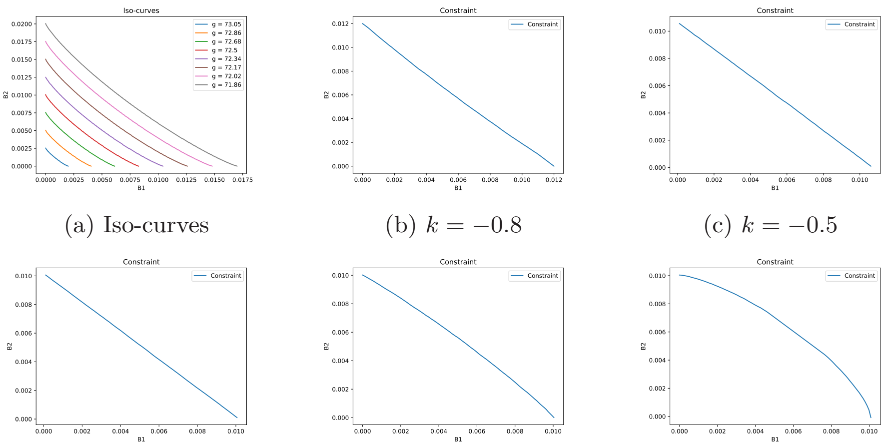
$$X_1 \sim Lognormal\,(m = 50,\ s = 50)$$
$$X_2 \sim Lognormal\,(m = 60,\ s = 50)$$

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	15.80	22.83	257.70	348.67	0.009	0.003	125.24	41.08	45.56
-0.5	15.80	22.83	268.83	339.56	0.007	0.003	125.32	41.18	45.54
0	15.80	22.83	268.43	357.87	0.007	0.003	125.40	41.18	45.62
0.5	15.80	22.83	279.03	320.82	0.007	0.004	125.37	41.27	45.49
0.8	15.80	22.83	266.94	312.34	0.008	0.004	125.23	41.16	45.45

Table 5.14: Results summary where X_1 is lognormal distribution with mean 50 and standard deviation 50, while X_2 is lognormal distribution with mean 60 and standard deviation 50.

Table 5.14 lists the optimal values under different k values. We can find that the unbalanced solutions are optimal. The higher the expected value, the higher the a^*, b^* values and premium π. This results in a significant improvement in V_α compared to Table 5.3.

Two risks X_1 and X_2 have decreasing hazard rates. Thus, iso-curves are quasiconvex as shown in Figure 5.14. The constraint curves are change from convex towards the origin to concave towards the origin for increasing k. However, the optimal solution is the point farthest from the origin where constraint curve and iso-curves intersect. It is evident that the optimal solution is between the middle and the boundary. Furthermore, with the increase of the k value, the changing trend of V_α increases first and then decreases.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.14: Iso-curves and constraint where X_1 is lognormal distribution with mean 50 and standard deviation 50, while X_2 is lognormal distribution with mean 60 and standard deviation 50.

Now we turn to solve two risks as follow:

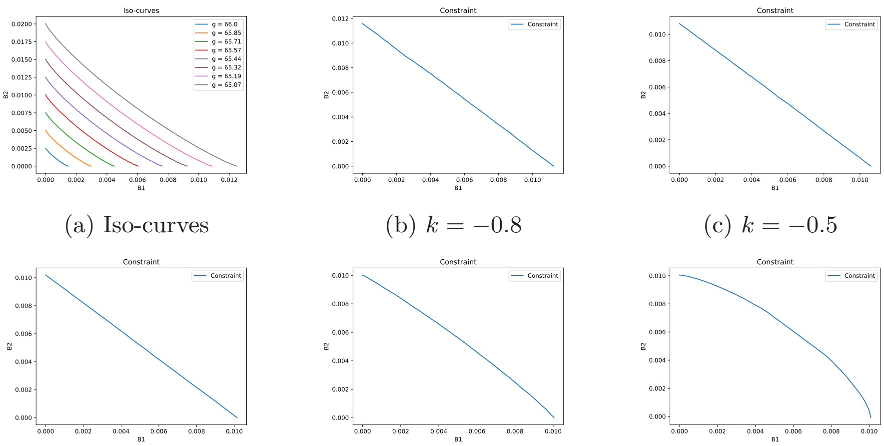
$$X_1 \sim Lognormal\,(m = 50,\ s = 50)$$
$$X_2 \sim Lognormal\,(m = 50,\ s = 40)$$

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	min V_α	π_{X_1}	π_{X_2}
-0.8	15.80	19.77	242.88	470.32	0.010	0.001	113.73	40.91	37.31
-0.5	15.80	19.77	245.77	508.95	0.010	0.001	113.80	40.95	37.33
0	15.80	19.77	249.21	514.82	0.010	0.001	113.86	40.99	37.35
0.5	15.80	19.77	255.22	412.42	0.009	0.002	113.85	41.05	37.32
0.8	15.80	19.77	262.71	323.08	0.008	0.004	113.75	41.13	37.25

Table 5.15: Results summary where X_1 is lognormal distribution with mean 50 and standard deviation 50, while X_2 is lognormal distribution with mean 50 and standard deviation 40.

Table 5.15 displays the optimal values under various k values. We can see that most results point out that the optimal solutions are close to the boundary. Since the standard deviation of X_2 has been decreased to 40, the value of π_{X_2} is less than π_{X_1} and a_2^*, b_2^* are larger than a_1^*, b_1^* in Table 5.15. Compared to Table 5.3, the retained risk has been increased, and the premium has decreased, resulting in a slightly smaller value of V_α. .

Figure 5.15 indicates that the iso-curves are quasiconvex, and constraint curves change from approximately linear to concave to the origin. If k is not positive, the constraint curves are approximately linear, and the curve gradually moves closer to the origin with increasing k. If k is positive, the constraint curves become more concave to the origin with increasing k. Moreover, when k is slightly greater than 0, the constraint curve changes the curvature while



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.15: Iso-curves and constraint where X_1 is lognormal distribution with mean 50 and standard deviation 50, while X_2 is lognormal distribution with mean 50 and standard deviation 40.

gradually moving towards $x = y = 0.01$. Thus, as the k value goes from negative to positive, the changing trend of V_α first increases and then decreases.

Now we change the mean and the standard deviation of X_2, and calculate the optimal values of following risks:

$$X_1 \sim Lognormal\,(m = 50,\ s = 50)$$
$$X_2 \sim Lognormal\,(m = 60,\ s = 40)$$

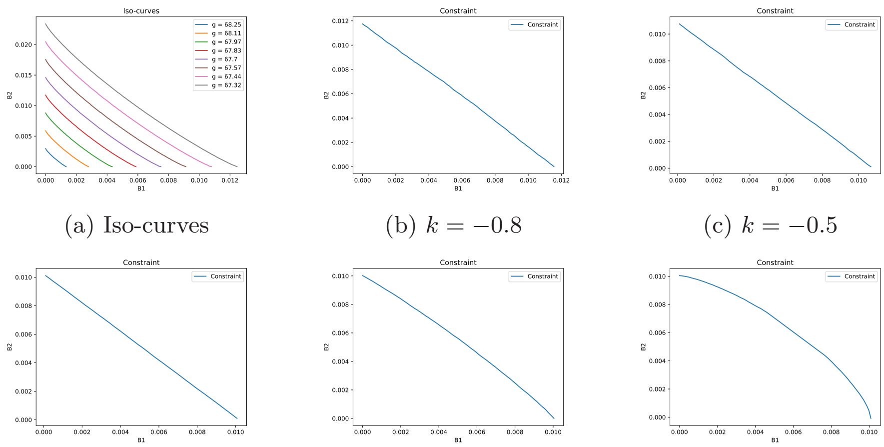
Table 5.16 shows the results. According to the results, it is not difficult to find that the optimal solutions are unbalanced. Although the premium π_{X_2} is smaller than π_{X_1} and b_2^* value is larger than b_1^* , we obtain the result is more similar to Table 5.15 than Table 5.14. The standard deviation seems to be the more important factor in this regard. However, the value of V_α is much larger than that in Table 5.15, mainly due to the difference of a_2^* value. Thus, a change in the expected value and standard deviation acts on the value of a^*, which affects the optimal V_α value.

It can be seen from Figure 5.16 that the iso-curves have a greater range on the y-axis than on the x-axis, unlike constraint curves which always have the symmetric ranges. If we put the constraint curve and iso-curves in the same plot, it is not difficult to find that the constraint curves are more flat than the iso-curves for any k. Thus, the point where the constraint curve touches an iso-curve will typically be close to the right boundary.

Note that the lognormal risk with the higher standard deviation has a heavier tail. The heavier tail acts on a larger premium, which means that the corresponding reinsurance contract is more expensive.

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	15.80	27.77	235.16	457.49	0.011	0.0001	124.43	40.80	40.06
-0.5	15.80	27.77	240.30	468.53	0.011	0.0001	124.50	40.87	40.03
0	15.80	27.77	244.79	476.09	0.010	0.0001	124.560	40.93	40.06
0.5	15.80	27.77	248.12	359.54	0.010	0.0006	124.562	40.97	40.03
0.8	15.80	27.77	255.37	267.04	0.009	0.003	124.49	41.05	39.87

Table 5.16: Results summary where X_1 is lognormal distribution with mean 50 and standard deviation 50, while X_2 is lognormal distribution with mean 60 and standard deviation 40.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.16: Iso-curves and constraint where X_1 is lognormal distribution with mean 50 and standard deviation 50, while X_2 is lognormal distribution with mean 60 and standard deviation 40.

5.2.1.3 Under Truncated Normal Distribution

Let the asymmetric risks be as follows:

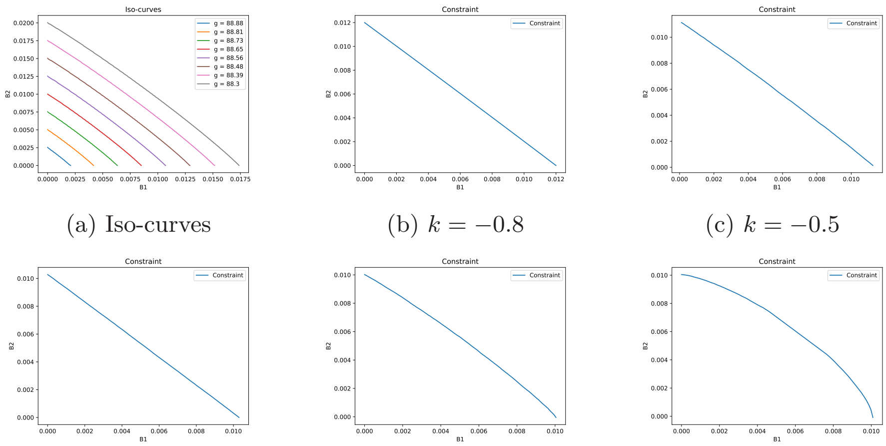
$$X_1 \sim Truncnormal\,(m = 50,\ s = 50)$$
$$X_2 \sim Truncnormal\,(m = 60,\ s = 50)$$

The optimal results are shown in Table 5.17. It is found that the optimal solutions are unbalanced. All of them lie at the boundary except k=0.8. Moreover, a risk with a higher expected value is the one with higher a^*, b^* values, as well as higher premiums. In comparison with Table 5.6, this results in a significant improvement in V_α.

According to Figure 5.17, iso-curves are quasiconcave since the hazard rates of truncated normal distributions are always increasing. The constraint curves are approximately linearly and move to the origin gradually when k is equal to -0.8, -0.5 and 0 in that order. However, constraint curves are concave to the origin with k being positive. The optimal solution is where the constraint curve intersects the farthest iso-curve. Thus, the optimal solution V_α increases at first but decreases after that by increasing k.

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	10.03	13.62	205.87	292.22	0.011	0.001	129.91	49.29	56.98
-0.5	10.03	13.62	205.87	322.98	0.011	0.0004	129.93	49.29	57.01
0	10.03	13.62	208.44	516.24	0.010	5E-07	130.00	49.32	57.03
0.5	10.03	13.62	209.39	inf	0.010	0	130.01	49.33	57.04
0.8	10.03	13.62	218.99	242.02	0.008	0.004	129.97	49.42	56.89

Table 5.17: Results summary where X_1 is truncated normal distribution with mean 50 and standard deviation 50, while X_2 is truncated normal distribution with mean 60 and standard deviation 50.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.17: Iso-curves and constraint where X_1 is truncated normal distribution with mean 50 and standard deviation 50, while X_2 is truncated normal distribution with mean 60 and standard deviation 50.

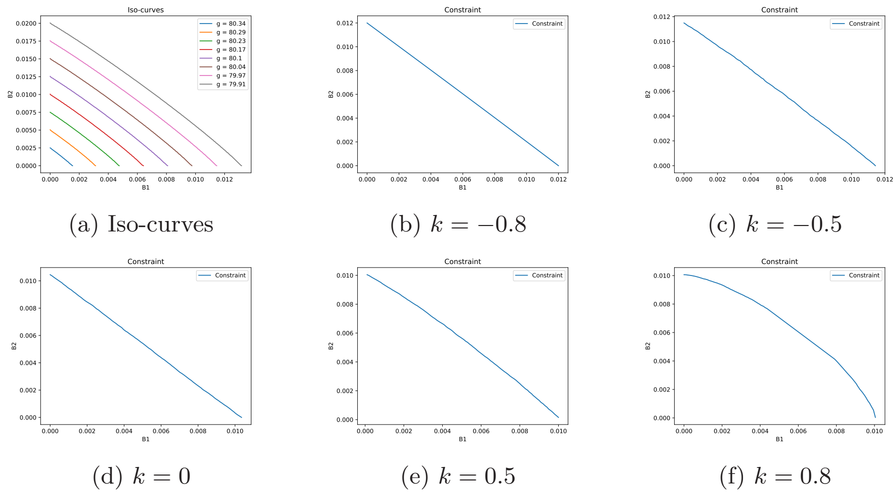
After reducing the standard deviation of X_2, we calculate the optimal values of the following risks:

$$X_1 \sim Truncnormal\,(m = 50,\ s = 50)$$
$$X_2 \sim Truncnormal\,(m = 50,\ s = 40)$$

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	10.03	12.03	205.87	292.22	0.011	0.001	118.05	49.27	46.73
-0.5	10.03	12.03	205.87	327.61	0.011	0.0004	118.07	49.29	46.72
0	10.03	12.03	208.40	515.43	0.010	5E-07	118.11	49.31	46.73
0.5	10.03	12.03	209.53	667.55	0.010	1E-09	118.12	49.31	46.74
0.8	10.03	12.03	213.62	261.01	0.009	0.002	118.10	49.35	46.74

Table 5.18: Results summary where X_1 is truncated normal distribution with mean 50 and standard deviation 50, while X_2 is truncated normal distribution with mean 50 and standard deviation 40.

Table 5.18 shows that the optimal solutions under different k are close to the right boundary. The risk with the lower standard deviation is the one with the higher a^* and b^* value, it is still the risk with the lower premium. On the other hand, when the standard deviation of X_2 is decreased to 40, it can be seen that the value of a_2^* has increased and π_2 has decreased from Table 5.6. This has resulted in a reduction in the premium term, and a substantial increase in the retained risk term. Consequently, there has been little change from Table 5.6 to the optimal solution V_α.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.18: Iso-curves and constraint where X_1 is truncated normal distribution with mean 50 and standard deviation 50, while X_2 is truncated normal distribution with mean 50 and standard deviation 40.

When the standard deviation is changed, the range of the x-axis of the iso-curve is much smaller than the y-axis. See Figure 5.18 for details. When k is -0.8, -0.5 and 0, the constraint curves are approximately linear. However, the constraint curve gradually becomes more concave to the origin for increasing k. Together, the present findings confirm that the optimal solution is unique and unbalanced.

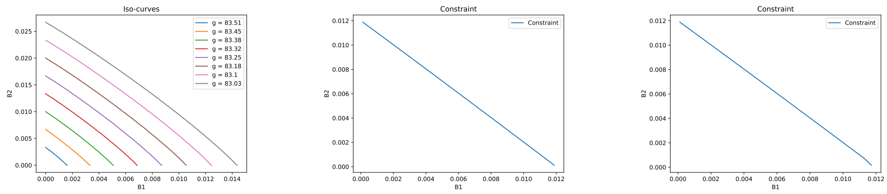
After reducing the standard deviation and increasing the expectation of X_2, we calculate the optimal values of the following risks:

$$X_1 \sim Truncnormal\,(m = 50,\ s = 50)$$
$$X_2 \sim Truncnormal\,(m = 60,\ s = 40)$$

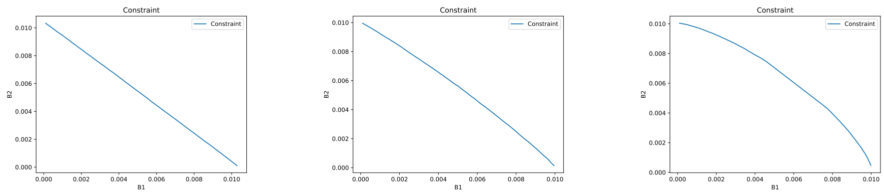
Table 5.19 shows that the optimal solutions under different k are close to the boundary. a_2^* is much larger than the previous two examples, which means that both the expected value and standard deviation act on the retained risk term. We also get the higher value of π_{X_2} which is closer to the value of decreasing the standard deviation. Therefore, standard deviation plays a leading role in the results of the premium term.

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	10.03	19.43	205.87	292.22	0.011	0.001	129.25	49.29	50.52
-0.5	10.03	19.43	205.87	292.22	0.011	0.001	129.26	49.29	50.53
0	10.03	19.43	208.12	300.24	0.010	1E-06	129.28	49.32	50.55
0.5	10.03	19.43	209.38	564.72	0.010	8E-08	129.329	49.33	50.55
0.8	10.03	19.43	210.09	299.92	0.010	0.001	129.327	49.33	50.55

Table 5.19: Results summary where X_1 is truncated normal distribution with mean 50 and standard deviation 50, while X_2 is truncated normal distribution with mean 60 and standard deviation 40.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$



(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.19: Iso-curves and constraint where X_1 is truncated normal distribution with mean 50 and standard deviation 50, while X_2 is truncated normal distribution with mean 60 and standard deviation 40.

When we change the mean and standard deviation, Figure 5.19 indicates that the y-axis of the iso-curve has a larger range than the x-axis. Although the constraint curves change from linear to concave to the origin with the increase of k value, the optimal solutions are always unbalanced.

The asymmetric risks from the truncated normal distributions show that the constraint curve gradually shifts toward $x = y = 0.01$ while changing the curvature when k is slightly greater than 0. Consequently, when the correlation is positive, V_α increases and then decreases with increasing k.

5.2.1.4 Under Gamma Distribution

Assume we have the following risks:

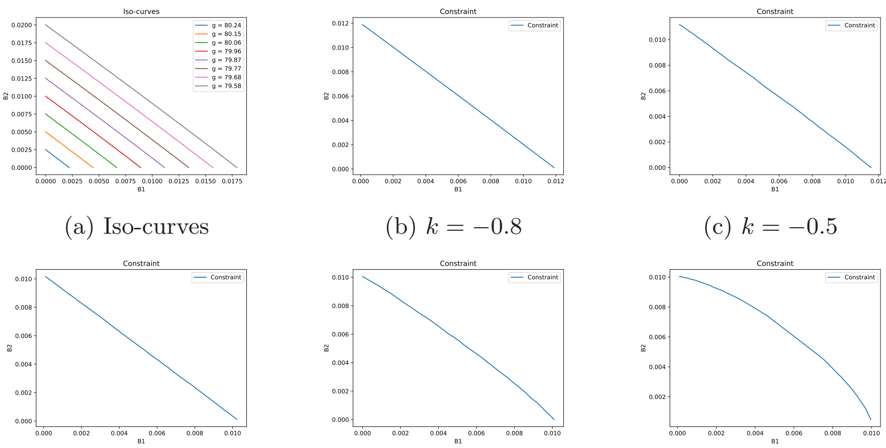
$$X_1 \sim Gamma\,(m = 50,\ s = 45)$$
$$X_2 \sim Gamma\,(m = 60,\ s = 45)$$

The optimal results are presented in Table 5.20. It is not difficult to find that the insurance layer contracts are optimal. From Table 5.20, where the expected

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	min V_α	π_{X_1}	π_{X_2}
-0.8	11.85	20.07	203.45	303.33	0.011	0.001	127.68	46.36	49.41
-0.5	11.85	20.07	203.45	333.61	0.011	0.0005	127.72	46.36	49.45
0	11.85	20.07	206.18	613.79	0.010	6E-07	127.78	46.39	49.47
0.5	11.85	20.07	214.74	273.32	0.008	0.002	127.79	46.48	49.40
0.8	11.85	20.07	219.08	241.52	0.0076	0.004	127.73	46.52	49.30

Table 5.20: Results summary where X_1 is gamma distribution with mean 50 and standard deviation 45, while X_2 is gamma distribution with mean 60 and standard deviation 45.

of X_2 is increased to 60, it can be seen that the value of a_2^* and specifically b_2^* has increased from Table 5.9. Moreover, the risk with a higher expected value is the one with a higher premium. In the end, the optimal V_α has increased greatly from Table 5.9.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.20: Iso-curves and constraint where X_1 is gamma distribution with mean 50 and standard deviation 45, while X_2 is gamma distribution with mean 60 and standard deviation 45.

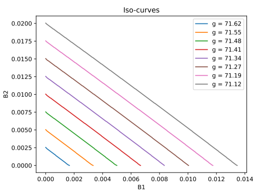
Figure 5.20 shows that iso-curves are slightly quasiconcave since both risks have increasing hazard rates, whereas constraint curve changes from linear to concave to the origin with increasing k. when the correlation coefficient goes from negative to positive, the optimal value V_α first increases and then decreases. However, we also find that the range of constraint curves are between 0 to around 0.01, but the iso-curves have a longer range on the y-axis. That means if we draw the curves in the same plot, the point where the constraint curve touches an iso-curve will typically be unbalanced and skewed to the right.

Now we continue to reduce the standard deviation and calculate the optimal values of following risks:

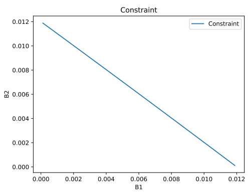
$$X_1 \sim Gamma\,(m = 50,\ s = 45)$$
$$X_2 \sim Gamma\,(m = 50,\ s = 35)$$

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	11.85	18.54	203.45	303.33	0.011	0.001	115.80	46.36	39.08
-0.5	11.85	18.54	203.45	349.83	0.011	0.0003	115.83	46.36	39.09
0	11.85	18.54	206.37	619.46	0.010	5E-07	115.89	46.40	39.11
0.5	11.85	18.54	207.32	inf	0.010	0	115.90	46.40	39.11
0.8	11.85	18.54	212.72	261.32	0.009	0.003	115.88	46.46	39.11

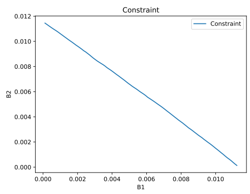
Table 5.21: Results summary where X_1 is gamma distribution with mean 50 and standard deviation 45, while X_2 is gamma distribution with mean 50 and standard deviation 35.



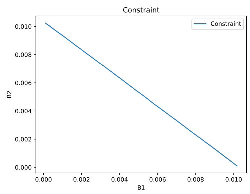
(a) Iso-curves



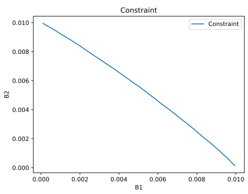
(b) $k = -0.8$



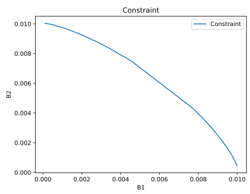
(c) $k = -0.5$



(d) $k = 0$



(e) $k = 0.5$



(f) $k = 0.8$

Figure 5.21: Iso-curves and constraint where X_1 is gamma distribution with mean 50 and standard deviation 45, while X_2 is gamma distribution with mean 50 and standard deviation 35.

According to Table 5.21, it is not difficult to find that a balanced solution is no longer optimal. A risk with lower standard deviation can contribute to higher a^*, b^* and lower premium. As a result, the difference between the optimal solution V_α and that in Table 5.9 is very small.

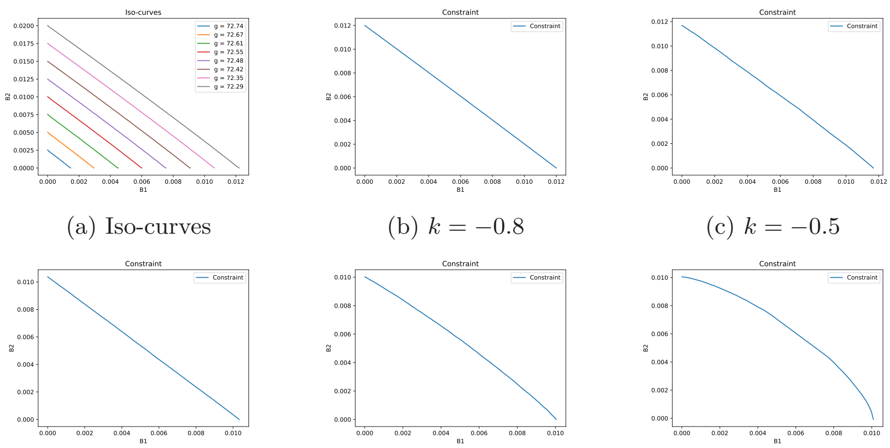
Due to the increasing hazard rates of two risks, the iso-curves are quasiconcave. In Figure 5.21, as the standard deviation is decreased, the range of iso-curves on the y-axis is much higher than that of the constraint curves forces the optimal solutions are unique and much closer to the x-axis than the y-axis. Furthermore, the value of V_α is related to the correlation coefficient. It is pointed out that the optimal solution first increases and then decreases with the increase of k.

Now we continue to change the expectation and the standard deviation of X_2, then calculate the optimal values of the following risks:

$$X_1 \sim Gamma\,(m = 50,\ s = 45)$$
$$X_2 \sim Gamma\,(m = 60,\ s = 35)$$

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	min V_α	π_{X_1}	π_{X_2}
-0.8	11.85	27.72	203.45	303.33	0.011	0.001	126.32	46.36	40.41
-0.5	11.85	27.72	203.45	318.68	0.011	0.001	126.33	46.36	40.42
0	11.85	27.72	206.15	400.45	0.010	6E-07	126.40	46.39	40.44
0.5	11.85	27.72	207.32	697.57	0.010	7E-08	126.41	46.40	40.44
0.8	11.85	27.72	209.88	281.17	0.009	0.002	126.40	46.43	40.45

Table 5.22: Results summary where X_1 is gamma distribution with mean 50 and standard deviation 45, while X_2 is gamma distribution with mean 60 and standard deviation 35.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.22: Iso-curves and constraint where X_1 is gamma distribution with mean 50 and standard deviation 45, while X_2 is gamma distribution with mean 60 and standard deviation 35.

According to the results, it is not difficult to find that the optimal solutions are unbalanced. Most of them are at the boundary except that k is 0.8. Table 5.22 shows that it is cheaper to reinsurance X_2 than X_1, which points out the same result as only changing the standard deviation of X_2. As a consequence, the standard deviation dominates the premium term. Nevertheless, V_α has a much larger value than in Table 5.21, primarily due to the larger a_2^* value. From Table 5.20 and Table 5.21, we know that the risk with a higher expected value or lower standard deviation has the higher a^* values. In this case, we both increase the expected value and decrease the standard deviation for X_2, which results in the higher a_2^* values.

The iso-curves are quasiconcave as two risks have increasing hazard rates. Observe in Figure 5.22 that the range of iso-curves on the y-axis is much broader than that of the constraint curves resulting in corresponding iso-curves being much steeper than the constraint curves for any k. Thus, the optimal solutions are closer to the extreme solution.

The above three examples all point out that if k is non-positive, the constraint curve is approximately linear, and the less negative the value of k, the constraint curve closer to the origin. On the other hand, constraint curves are concave to the origin for k is positive. When k is slightly greater than 0, the constraint curve changes the curvature while gradually moving towards $x = y = 0.01$. Therefore, the optimal solution decreases initially, and then increases when k changes from negative to positive.

5.2.2 From Different Types of Distributions

Assume we have following risks:

$$X_1 \sim Pareto\,(m = 50,\ s = 50)$$
$$X_2 \sim Lognormal\,(m = 50,\ s = 50)$$

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	$\min V_\alpha$	π_{X_1}	π_{X_2}
-0.8	31.59	15.80	212.93	331.21	0.008	0.003	109.84	20.85	41.52
-0.5	31.59	15.80	226.15	315.85	0.007	0.003	109.91	20.95	41.45
0	31.59	15.80	227.20	324.46	0.007	0.003	109.96	20.97	41.52
0.5	31.59	15.80	230.45	297.84	0.007	0.004	109.92	20.99	41.38
0.8	31.59	15.80	220.68	277.16	0.008	0.004	109.76	20.91	41.23

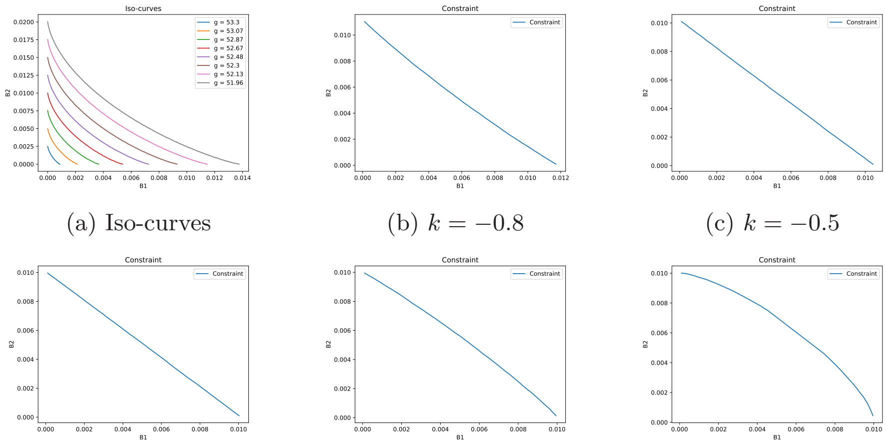
Table 5.23: Results summary where X_1 is Pareto distribution with mean 50 and standard deviation 50, while X_2 is lognormal distribution with mean 50 and standard deviation 50.

The results are presented in Table 5.23. Since two risks have different distributions, we conclude that all the optimal solutions are unbalanced. Although X_1 and X_2 are from the light tail distributions, X_2 has a lighter tail and the difference between the a_2^* and b_2^* is significantly larger than the difference between the a_1^* and b_1^*, which means the reinsurance contract for X_2 covers more risk. Moreover, the premium for X_2 is higher.

Figure 5.23 shows iso-curves and constraint curves for different correlation k. The risks X_1 and X_2 have decreasing hazard rates. Thus the iso-curves are quasiconvex. The constraints curve bends slightly against the origin when k is less than 0 and bends away from the origin when k is larger than 0. Nevertheless, we may observe by looking at the x and y-axis that a negative value of k results in the constraint curve approaching the origin as k increases. Overall, increasing k increases the optimal value V_α at first and decreases after that.

Now we turn to solve two risks as follow:

$$X_1 \sim Gamma\,(m = 50,\ s = 10)$$
$$X_2 \sim Truncnormal\,(m = 50,\ s = 10)$$



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

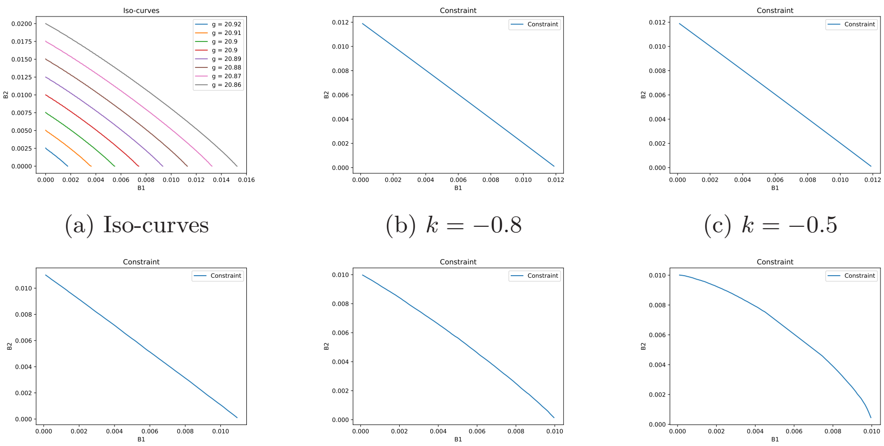
Figure 5.23: Iso-curves and constraint where X_1 is Pareto distribution with mean 50 and standard deviation 50, while X_2 is lognormal distribution with mean 50 and standard deviation 50.

Table 5.24 displays the optimal values under various k values. We can see that all results indicate that the optimal solutions are unbalanced and close to the boundary. Although X_1 is from the heavier tail distributions than X_2 and the values of a_1^* and a_2^* are very close, b_2^* is significantly larger than b_1^* which means that the reinsurance contract for X_2 allows more risk to be transferred from the insurer to the reinsurer. Since the difference between premiums is tiny, the risk X_2 gets a better reinsurance coverage than the risk X_1.

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	min V_α	π_{X_1}	π_{X_2}
-0.8	40.35	40.33	75.29	86.72	0.012	0.0001	105.723	12.38	12.67
-0.5	40.35	40.33	75.29	86.72	0.012	0.0001	105.723	12.38	12.67
0	40.35	40.33	75.71	86.96	0.011	1E-04	105.729	12.38	12.67
0.5	40.35	40.33	76.17	86.27	0.010	1E-04	105.734	12.39	12.67
0.8	40.35	40.33	76.70	78.02	0.009	0.003	105.732	12.39	12.66

Table 5.24: Results summary where X_1 is gamma distribution with mean 50 and standard deviation 10, while X_2 is truncated normal distribution with mean 50 and standard deviation 10.

There are increasing hazard rates associated with risks X_1 and X_2. Thus, it follows from Theorem 4.3.2 that the iso-curves are quasiconcave. According to Figure 5.24, it is not difficult to determine that when k is non-positive, the constraint curves are approximately linear. In particular, the constraint curve moves toward the origin as k gradually increases to 0. In other cases, the constraint curve is bending away from the origin. It is worth noting that the constraint curve gradually shifts toward $x = y = 0.01$ while changing the curvature. Thus when the correlation is positive, V_α increases and then decreases with increasing k. Due to the fact that the two risks derive from different distributions, the optimal solution is unique and unbalanced.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.24: Iso-curves and constraint where X_1 is gamma distribution with mean 50 and standard deviation 10, while X_2 is truncated normal distribution with mean 50 and standard deviation 10.

It is also essential to observe that the range of the y-axis representing V_α in this case is negligible. The optimal value for varied k is quite small. Thus, there is no significant amount to gain from choosing an unbalanced solution over a balanced solution for all correlation k.

In the final example, we let the asymmetric risks be as follows:

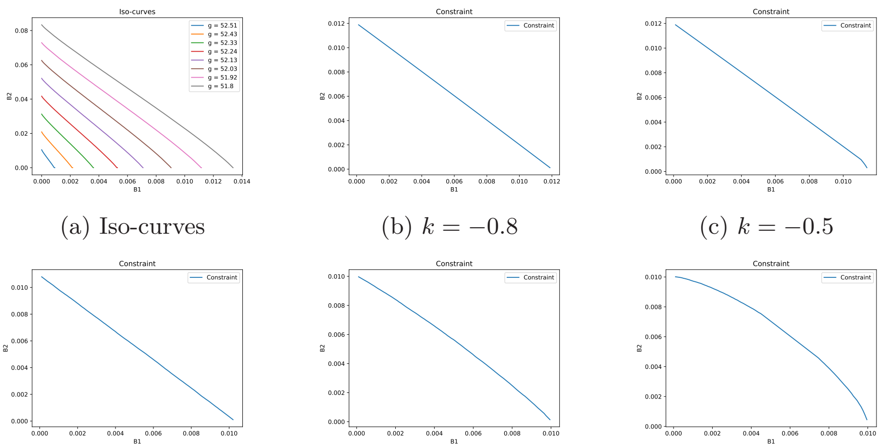
$$X_1 \sim \mathit{Lognormal}\,(m = 50,\ s = 40)$$
$$X_2 \sim \mathit{Truncnormal}\,(m = 50,\ s = 20)$$

The optimal results are shown in Table 5.25. It is found that the optimal solution lies at the right boundary. In this case, the reinsurance contract for X_2 covers more risk and is cheaper to buy.

k	a_1^*	a_2^*	b_1^*	b_2^*	B_1^*	B_2^*	min V_α	π_{X_1}	π_{X_2}
-0.8	19.77	30.20	195.57	342.27	0.011	0.001	112.278	36.62	25.70
-0.5	19.77	30.20	195.57	342.27	0.011	0.001	112.270	36.60	25.70
0	19.77	30.20	199.10	1214.03	0.010	5E-07	112.327	36.65	25.70
0.5	19.77	30.20	200.58	inf	0.010	0	112.342	36.66	25.71
0.8	19.77	30.20	200.58	inf	0.010	0	112.341	36.66	25.71

Table 5.25: Results summary where X_1 is lognormal distribution with mean 50 and standard deviation 40, while X_2 is truncated normal distribution with mean 50 and standard deviation 20.

Note that the risk X_1 has a decreasing hazard rate and the risk X_2 has an increasing hazard rate. The iso-curves in Figure 5.25 indicate that the sublevel sets are neither convex nor concave for this particular combination of risks. Although constraint curves vary from linear to concave to the origin with increasing k, we can find that the range of iso-curves is significantly larger than that of constraint curves from the y-axis in Figure 5.25. Still, it is straightforward to see that no matter what value k takes, the optimal solution is a boundary solution in this case.



(a) Iso-curves (b) $k = -0.8$ (c) $k = -0.5$

(d) $k = 0$ (e) $k = 0.5$ (f) $k = 0.8$

Figure 5.25: Iso-curves and constraint where X_1 is lognormal distribution with mean 50 and standard deviation 40, while X_2 is truncated normal distribution with mean 50 and standard deviation 20.

Obviously, when k is slightly greater than 0, the constraint curve bends from the origin while gradually moving towards $x = y = 0.01$. This leads to the V_α value corresponding to the independent risks is not the maximum value. Consequently, when the correlation is positive, V_α increases and then decreases with increasing k.

5.3 Summary

This chapter presents the optimal reinsurance contract by constructing dependent bivariate risks from combinations of four different distributions. In the symmetric risks situation, iso-curves and constraint curves are symmetric around the line $y = x$. Their concavity and convexity will affect the position and size of the optimal solution. Due to symmetry, when the optimal solution is at the boundary, there will be two optimal unbalanced solutions. When the optimal solution is in the middle, there is only one optimal balanced solution.

The quasiconvex iso-curves where risks have decreasing hazard rates correspond to Pareto distributions, lognormal distribution with a large standard deviation and gamma distributions with a large standard deviation. As long as the constraint curves are either concave to the origin or approximately linear, the optimal solution is always the balanced solution. This suggests that the balanced solution will always be optimal when the correlation is non-negative.

In the case of negative correlation, the constraint curve, in addition to always being close to linear, may also be slightly convex to the origin and gradually approach a straight line as k goes to 0. If the negatively correlated constraint curve is close to linear, the balanced solution is of course the optimal solution. If the constraint curve bends more against the origin than iso-curves, the optimal solution is at the boundary. In this case, the optimal solution changes from boundary to balanced as the correlation coefficient goes from negative to 0, e.g., the symmetric risks are gamma distribution with mean 50 and standard deviation 100.

The quasiconcave iso-curves where risks have increasing hazard rates correspond to truncated normal distributions, lognormal distribution with a small standard deviation and gamma distributions with a small standard deviation. As long as the constraint curves are either slightly convex to the origin or approximately linear, the optimal solutions are always the unbalanced boundary solutions. This implies that the boundary solution will always be optimal when the correlation is negative and 0. Furthermore, the small positive correlation corresponds to a constraint curve that is slightly concave toward the origin. If the constraint curve bends less away from the origin than the iso-curves, the optimal solution is still at the boundary. However, the balanced solution is easier to be determined as optimal when k is large because the constraint curve for large positive correlation bends significantly away from the origin than the iso-curves.

Note that typically when two risks are symmetric, the intersection of the constraint curves passes through fewer iso-curves, which means that the range of the y-axis representing V_α is tiny. Therefore, for any correlation between these two risks, especially a non-positive correlation, there is no significant benefit in choosing a balanced or unbalanced solution. However, when the iso-curves are quite quasiconvex, such as the symmetric risks come from a heavy-tailed distribution with large variance, which leads to a more extensive range of V_α, then the balanced solution is a better choice.

There are two cases in the asymmetric risks situation, such as two risks from the same type of distribution and different types of distributions. In either case, the unbalance solution is optimal. For the first case, no matter what kind of distribution the risks come from, we have found the following rules. If only change the expected value, the risk with a higher expectation is the one with higher a a^* and premium. If only change standard deviation, the risk with a lower standard deviation is the one with higher a^*, b^* and lower premium. Because changes in the standard deviation will affect the tails. The risk with a smaller standard deviation has a lighter tail and is therefore cheaper to reinsure. Additionally, its corresponding reinsurance contract has better coverage. If change both standard deviation and expectation, then the risk with a lower standard deviation and higher expectation is the one also with higher a^*, b^* values, and its corresponding reinsurance contract can cover more risks.

For asymmetric risks, the constraint curve approximates symmetry about $y = x$. However, the iso-curves are asymmetric about $y = x$. When the two risks are more asymmetric, the difference between the x-axis range and the y-axis of iso-curves is more significant. Therefore, the optimal solutions are always unbalanced, and for most of k, they lie at the boundary.

On the other hand, if the risks are more asymmetric, the range of the y-axis representing V_α is more extensive. The reason for this is that the constraint curve will cross more iso-curves. Consequently, regardless of the correlation coefficient, the unbalanced solution is preferred.

We also have an interesting finding of how the constraint curve moves with variant correlation coefficients. The constraint curves for negatively dependent risks get closer to the origin, with the correlation coefficient going to 0. Therefore, the value of V_α does not decrease with the increase of negative correlation. On the other hand, less negative correlation coefficients correspond to the constraint curves closer to the origin, which leads to smaller values of B^* and correspondingly larger values of b^*, implying that the reinsurance contracts cover more risks and the cost of reinsurance is higher and are more expensive to reinsure. It is worth noting that if the left and right boundaries of the constraint curve are greater than 0.01 at $k = 0$, then as the positive correlation coefficient increases, the constraint curve becomes more concave to the origin while gradually moving toward the origin until the boundary of the constraint curve is at 0.01, after which the constraint curve only changes curvature with further increase in k. Therefore, in this case, the optimal value V_α first increases and then decreases with the increase of positive correlation. If not, the constraint curve does not move but only bends more away from the origin as k increases. Therefore, the optimal value V_α does not increase with an increase in positive correlation. In conclusion, the optimal value of V_α first increases and then decreases when the correlation coefficient turns from negative to positive.

CHAPTER 6

Conclusion

This thesis investigates how reinsurance optimization is affected by dependence between the two risks under the value-at-risk measure. In particular, the bivariate risks are typically modeled through heavy and light-tailed distributions such as the Pareto, lognormal, truncated normal and gamma distributions. By the lead of Huseby [4], we have laid out the importance sampling, which is an efficient simulation method to solve the numerical part of the problem. Additionally, the Gaussian copula models the dependence of risks. We also use unrestricted solutions from Huseby and Christensen [5] to solve the general optimization problem. To intuitively see the optimal solution of different risks, we plot the relevant iso-curves of reinsured expected risks and constraint curves for varying correlation coefficient k. Optimal reinsurance contracts can be found by Lagrange optimization or a simple search along the constraint curve.

It is worth mentioning that the optimal reinsurance contract relies on the hazard rates of the risk distributions and the correlation coefficient k. In other words, the optimal solution depends on the shapes of iso-curves and constraints curves. On the premise that the risks are positively correlated, the greater the correlation coefficient is, the constraint curve bends more away from the origin. Therefore, the optimal solution for the symmetric risks will eventually be the balanced solution when the correlation has a high positive value, no matter what iso-curves look like. Under the premise of negative risk correlation, the constraint curve has changing patterns. In the first case, with the correlation increase of 0, the curve gradually changes from slightly convex to the origin to approximately linear. In the second case, the curve is always close to a straight line. The specific performance can be analyzed according to different risk distributions. In either of these cases, the constraint curve gradually approaches the origin as k increases which indicates that the optimal value V_α does not decrease.

Whether the risks are symmetric, the general trend of the optimal value V_α increases and then decreases as the correlation increases. This result is directly related to the change in the constraint curve. More specifically, more negative and positive correlation coefficients lead to the more considerable corresponding B^* and smaller b^* values, which means the reinsurance contracts cover less risk and are cheaper to reinsure.

The insurance layer contracts are optimal for any correlation coefficient when the symmetric risks come from heavy-tailed distributions, such as the Pareto and lognormal distributions with large standard deviations. The corresponding iso-curves are quasiconvex because of the decreasing hazard rates, and the constraint curves always bend away from the iso-curves, so the optimal solution must be balanced. Furthermore, for each correlation coefficient, especially when k is not positive, the difference between the balanced and the unbalanced solutions is relatively significant. Therefore, the insurance layer contracts corresponding to the balanced solution should be chosen when an insurance company encounters such a situation.

In the case of asymmetric risks, the optimal solutions are all unbalanced. The reinsurance contract corresponding to the risk with a lighter tail covers more risk. It should be noted that the more asymmetric the risks are, the optimal solution is at the boundary. Together, the present findings confirm that when the two risks are obviously asymmetric, the risk with a heavier tail should be covered by an insurance layer contract, and the other risk should be covered by a stop-loss contract, regardless of the correlation between the two risks.

Future work in this area includes the optimal contract of bivariate correlated risk under other risk measures, and how to use other types of copulas to deal with dependent risks. As well as how to optimize reinsurance in a multivariate case under different risk measures.

Appendices

APPENDIX A

Python Codes

A.1 Distributions List

```python
from random import *
from math import *
from scipy.optimize import minimize
from scipy.stats import gamma
from cmath import pi

import numpy as np



SQRTPI: float = 1.772453850905516027298167483341     # sqrt(pi)
SQRT2PI: float = 2.50662827463100029                 # sqrt(2*pi)

EULER_MASCHERONI: float = 0.57721566490153286060



def gaussian_pdf(x) -> float:
    return exp(-x * x / 2.0) / SQRT2PI


def gaussian_cdf(x) -> float:
    if x >= 0.0:
        t = 1.0/(1.0 + 0.33267 * x)
        return 1.0 - gaussian_pdf(x) * (0.4361836*t - 0.1201676*t*t +
    0.9372980*t*t*t)
    else:
        t = 1.0/(1.0 - 0.33267 * x);
        return gaussian_pdf(x) * (0.4361836*t - 0.1201676*t*t + 0.9372980*t*t*
    t)


def gaussian_sdf(x) -> float:
    if x >= 0.0:
        t = 1.0/(1.0 + 0.33267 * x)
        return gaussian_pdf(x) * (0.4361836*t - 0.1201676*t*t + 0.9372980*t*t*
    t)
    else:
        t = 1.0/(1.0 - 0.33267 * x);
        return 1.0 - gaussian_pdf(x) * (0.4361836*t - 0.1201676*t*t +
    0.9372980*t*t*t)


```

```
def gaussian_invcdf(u) -> float:
    if u < 0.5:
        t = sqrt(log(1.0/(u*u)))
        return -t + (2.515517 + 0.802853*t + 0.010328*t*t) / (1.0 + 1.432788*t + 0.189269*t*t + 0.001308*t*t*t)
    elif u == 0.5:
        return 0.0
    else:
        t = sqrt(log(1.0/((1.0 - u)*(1.0 - u))))
        return t - (2.515517 + 0.802853*t + 0.010328*t*t) / (1.0 + 1.432788*t + 0.189269*t*t + 0.001308*t*t*t)




################################################################
#    Generate a univariate uniform variable restricted to      #
#    the importance sample region:                             #
#        D = [1-r, 1]                                          #
################################################################
def uni_uniform_D(r):
    return uniform(1-r,1)


################################################################
#    Generate a univariate uniform variable restricted to      #
#    the importance sample region:                             #
#        E = [0, 1-r]                                          #
################################################################
def uni_uniform_E(r):
    return uniform(0,1-r)


################################################################
#    Generate bivariate uniform variables restricted to        #
#    the importance sample region:                             #
#        D = ([1-r, 1] x [0, 1]) union ([0, 1] x [1-r, 1])     #
################################################################
def bi_uniform_D(r):
    u0 = uniform(0,1) * (2-r)
    u1 = uniform(0,1)
    u2 = uniform(0,1)
    v1 = 0
    v2 = 0
    if u0 < (1-r):
        v1 = 1-(r * u1)
        v2 = (1-r) * u2
    elif u0 < 2*(1-r):
        v1 = (1-r) * u1
        v2 = 1-(r * u2)
    else:
        v1 = 1-(r * u1)
        v2 = 1-(r * u2)
    return v1, v2

################################################################
#     Generate bivariate uniform variables restricted to       #
#     the complement of importance sample region:              #
#         E = [0, 1-r] x [0, 1-r]                              #
################################################################
def bi_uniform_E(r):
    u1 = uniform(0,1-r)
```

```
    u2 = uniform(0,1-r)
    return u1, u2


################################################################
#                                                              #
#      The base Distribution class                             #
#                                                              #
################################################################

class Distribution:
    def __init__(self, name, mean, stdev):
        self.name: str = name
        self.mean: float = mean
        self.stdev: float = stdev
        self.min: float = -np.inf
        self.max: float = np.inf

    def print_name(self):
        print(self.name, "(", self.mean, ",", self.stdev, ")")

    def get_name(self):
        namestr = self.name + "({0}, {1})"
        return namestr.format(self.mean, self.stdev)

    def getMean(self) -> float:
        return self.mean

    def getStdev(self) -> float:
        return self.stdev

    def getMin(self) -> float:
        return self.min

    def getMax(self) -> float:
        return self.max

    def getPDF(self, x) -> float:
        pass

    def getCDF(self, x) -> float:
        pass

    def getSDF(self, x) -> float:
        pass

    def getHazardRate(self, x) -> float:
        f = self.getPDF(x)
        s = self.getSDF(x)
        if s > 0:
            return f/s
        else:
            return 0

    def lowerPercentile(self, p) -> float:
        pass

    def getLowerPercentile(self, p) -> float:
        if p <= 0:
            return self.min
        elif p >= 1:
            return self.max
```

```
        else:
            return self.lowerPercentile(p)

    def getUpperPercentile(self, p) -> float:
        if p <= 0:
            return self.max
        elif p >= 1:
            return self.min
        else:
            return self.lowerPercentile(1-p)

    def getStochasticValue(self) -> float:
        u = uniform(0,1)
        return self.getLowerPercentile(u)


################################################################
#                                                              #
#      The Pareto distribution class                           #
#                                                              #
#      For details see:                                        #
#        Huseby (2021)  Pareto distributions                   #
#      or alternatively:                                       #
#        https://en.wikipedia.org/wiki/Pareto_distribution     #
#                                                              #
################################################################

class Pareto(Distribution):
    def __init__(self, mean, stdev):
        super().__init__("PARETO", mean, stdev)
        self.tau: float = sqrt((mean * mean) / (stdev * stdev) + 1) + 1
        self.Xm: float = mean * (self.tau - 1) / self.tau
        self.min = self.Xm
        print(self.name, "(mean = ", self.mean, ", stdev = ", self.stdev, ",
 tau = ", self.tau, ", Xm = ", self.Xm, ")")

    def getPDF(self, x) -> float:
        if x > self.min:
            return self.tau * (self.min**self.tau) / (x**(self.tau+1))
        else:
            return 0

    def getCDF(self, x) -> float:
        if x > self.min:
            return 1 - (self.min / x)**self.tau
        else:
            return 0

    def getSDF(self, x) -> float:
        if x > self.min:
            return (self.min / x)**self.tau
        else:
            return 1

    def lowerPercentile(self, p) -> float:
        return self.Xm * ((1-p)**(-1/self.tau))

################################################################
#                                                              #
#     The Lognormal distribution class                         #
#                                                              #
################################################################
```

```

class Lognormal(Distribution):
    def __init__(self, mean, stdev):
        super().__init__("LOGNORMAL", mean, stdev)
        self.h = 1.0 + (stdev * stdev) / (mean * mean)
        self.sigma = sqrt(log(self.h))
        self.mu = log(self.mean) - 0.5 * log(self.h)
        self.min = 0
        print(self.name, "(mean = ", self.mean, ", stdev = ", self.stdev, ", logmean = ", self.mu, ", logstdev = ", self.sigma, ")")

    def getPDF(self, x) -> float:
        if x > self.min:
            return gaussian_pdf((log(x)-self.mu) / self.sigma) / (self.sigma * x)
        else:
            return 0

    def getCDF(self, x) -> float:
        if x > self.min:
            return gaussian_cdf((log(x)-self.mu) / self.sigma)
        else:
            return 0

    def getSDF(self, x) -> float:
        if x > self.min:
            return gaussian_sdf((log(x)-self.mu) / self.sigma)
        else:
            return 1

    def lowerPercentile(self, p) -> float:
        return self.mean * exp(self.sigma * gaussian_invcdf(p)) / sqrt(self.h)


################################################################
#                                                              #
#     The Truncnormal distribution class                       #
#                                                              #
################################################################

class Truncnormal(Distribution):
    def __init__(self, mean, stdev):
        super().__init__("TNORMAL", mean, stdev)
        alpha = find_alpha(mean, stdev)
        self.sigma = mean * (1 - gaussian_cdf(alpha)) / (gaussian_pdf(alpha) - alpha*(1 - gaussian_cdf(alpha)))
        self.mu = -alpha * self.sigma
        self.omega = gaussian_cdf(-self.mu/self.sigma)
        self.min = 0
        print(self.name, "(mean = ", self.mean, ", stdev = ", self.stdev, ", sigma = ", self.sigma, ", mu = ", self.mu, ")")

    def getPDF(self, x) -> float:
        if x > self.min:
            return (gaussian_pdf((x-self.mu) / self.sigma) / self.sigma) / (1 - self.omega)
        else:
            return 0

    def getCDF(self, x) -> float:
        return 1 - self.getSDF(x)

```

```python
    def getSDF(self, x) -> float:
        if x > self.min:
            return gaussian_sdf((x-self.mu) / self.sigma) / (1 - self.omega)
        else:
            return 1

    def lowerPercentile(self, p) -> float:
        return self.mu + self.sigma * gaussian_invcdf(self.omega + p * (1 - self.omega))


#################################################################
#                                                               #
#     The Gamma distribution class                              #
#                                                               #
#################################################################

class Gamma(Distribution):
    def __init__(self, mean, stdev):
        super().__init__("GAMMA", mean, stdev)
        self.tau = (mean * mean) / (stdev * stdev)
        self.beta = self.tau / mean
        self.min = 0
        print(self.name, "(mean = ", self.mean, ", stdev = ", self.stdev, ", tau = ", self.tau,  ", beta = ", self.beta, ")")

    def getPDF(self, x) -> float:
        if x > self.min:
            return gamma.pdf(self.beta * x, self.tau) * self.beta
        else:
            return 0

    def getCDF(self, x) -> float:
        if x > self.min:
            return gamma.cdf(self.beta * x, self.tau)
        else:
            return 0

    def getSDF(self, x) -> float:
        if x > self.min:
            return 1 - gamma.cdf(self.beta * x, self.tau)
        else:
            return 1

    def lowerPercentile(self, p) -> float:
        return  gamma.ppf(p, self.tau) / self.beta
```

A.2 Reinsurance Functions List

```python


#############################################
#         RETAINED AND INSURED RISK         #
#############################################
def get_retained_risk(x, a, b) -> float:
    if x < a:
        return x
    elif x < b:
        return a
    else:
```

```
        return x - (b - a)


def get_insured_risk(x, a, b) -> float:
    if x < a:
        return 0
    elif x < b:
        return x - a
    else:
        return b - a


def get_expected_risk(xx) -> float:
    s: float = 0
    for i in range(len(xx)):
        s += xx[i]
    return s / len(xx)


def get_expected_retained_risk(xx, a, b) -> float:
    s: float = 0
    for i in range(len(xx)):
        s += get_retained_risk(xx[i], a, b)
    return s / len(xx)


def get_expected_insured_risk(xx, a, b) -> float:
    s = 0
    for i in range(len(xx)):
        s += get_insured_risk(xx[i], a, b)
    return s / len(xx)


# The set C is the set of points where retained risk is greater than (aa1 +
    aa2)
# NOTE: We assume that len(xx1) = len(xx2)
def getCCount(xx1, aa1, bb1, xx2, aa2, bb2) -> float:
    count = 0
    for i in range(len(xx1)):
        rr: float = get_retained_risk(xx1[i], aa1, bb1) + get_retained_risk(
    xx2[i], aa2, bb2)
        if rr > aa1 + aa2:
            count += 1
    return count

def getCFraction(xx1, aa1, bb1, xx2, aa2, bb2) -> float:
    count = 0
    for i in range(len(xx1)):
        rr: float = get_retained_risk(xx1[i], aa1, bb1) + get_retained_risk(
    xx2[i], aa2, bb2)
        if rr > aa1 + aa2:
            count += 1
    return count / len(xx1)
```

A.3 Optimal Reinsurance Contracts

```
############################################################################
#                                                                          #
#    Optimizing bivariate reinsurance contracts in the unbalanced,         #
#    bivariate case using total value at risk as objective                 #
```

```
#    function.                                                                 #
#                                                                              #
#                                                                              #
################################################################################


from math import *
from dist import *
from insureutils import *

import matplotlib.pyplot as plt
import numpy as np


seed_num = 135246780
num_points = 1000000
num_iter = 50
epsilon = 0.000001

print_results = False
print_counter = True
plot_scatter = True

alpha = 0.01
gamma = 0.10
theta = 0.20
delta = 0.00

corr = 0.8
c1 = 0.5 *(sqrt(1+corr) + sqrt(1-corr))
c2 = 0.5 *(sqrt(1+corr) - sqrt(1-corr))


rho = 0.05      # Determines the size of the importance sample region


mean1: float = 50
stdev1: float = 50
mean2: float = 50
stdev2: float = 50

dist1 = Lognormal(mean1, stdev1)
dist2 = Lognormal(mean2, stdev2)


# Alternative distributions found in dist
#   Truncnormal(mean, stdev)
#   Lognormal(mean, stdev)
#   Gamma(mean, stdev)
#   Pareto(mean, stdev)



probD = 1 - (1-rho)*(1-rho)
probE = (1-rho)*(1-rho)

seed(seed_num)

A = 1 / (1+theta)
a1 = dist1.getUpperPercentile(A)
a2 = dist2.getUpperPercentile(A)
```

```
print("--------")
print("a1 = ", a1, ", a2 = ", a2)


##############################################################
#    Generate risks sampling from the set D                  #
##############################################################

print("--------")
print("Generate risks sampling from the set D")

x1 = np.zeros(num_points)
x2 = np.zeros(num_points)

for i in range(num_points):
    u =  bi_uniform_D(rho)
    g1 = gaussian_invcdf(u[0])
    g2 = gaussian_invcdf(u[1])
    h1 = c1 * g1 + c2 * g2
    h2 = c1 * g2 + c2 * g1
    v1 = gaussian_cdf(h1)
    v2 = gaussian_cdf(h2)
    x1[i] = dist1.getLowerPercentile(v1)
    x2[i] = dist2.getLowerPercentile(v2)

expectedRiskD1 = probD * (get_expected_risk(x1))
expectedRiskD2 = probD * (get_expected_risk(x2))
expectedRiskD = expectedRiskD1+expectedRiskD2
print("--------")
print("expectedRiskD = ", expectedRiskD)


##############################################################
#    Generate risks sampling from the set E                  #
##############################################################

print("--------")
print("Generate risks sampling from the set E")

y1 = np.zeros(num_points)
y2 = np.zeros(num_points)

for i in range(num_points):
    u =  bi_uniform_E(rho)
    g1 = gaussian_invcdf(u[0])
    g2 = gaussian_invcdf(u[1])
    h1 = c1 * g1 + c2 * g2
    h2 = c1 * g2 + c2 * g1
    v1 = gaussian_cdf(h1)
    v2 = gaussian_cdf(h2)
    y1[i] = dist1.getLowerPercentile(v1)
    y2[i] = dist2.getLowerPercentile(v2)

insuredRiskE1 =probE * (get_expected_insured_risk(y1, a1, np.inf))
insuredRiskE2 =probE * (get_expected_insured_risk(y2, a2, np.inf))
insuredRiskE = insuredRiskE1 +  insuredRiskE2


expectedRiskE = probE * (get_expected_risk(y1)) +probE * (get_expected_risk(y2
    ))

print("--------")
```

```
print("expectedRiskE = ", expectedRiskE, ",  expectedRiskDE = ", expectedRiskD
      + expectedRiskE,)


#####################################
#    Determine B1_max and b1_min     #
#####################################

print("--------")
print("Determine B1_max and b1_min")

b1 = dist1.getUpperPercentile(A/2)
b2 = dist2.getUpperPercentile(0)

B1_L = alpha            # B1_L = 0
B1_U = alpha / A        # B1_U = A
B1 = (B1_L + B1_U) / 2

while B1_U - B1_L > epsilon:
    probC = getCFraction(x1, a1, b1, x2, a2, b2) * probD
    if probC > alpha:
        B1_U = B1
    else:
        B1_L = B1
    B1 = (B1_L + B1_U) / 2
    b1 = dist1.getUpperPercentile(B1)

B1_max = max(B1, alpha)
b1_min = dist1.getUpperPercentile(B1_max)

countC = getCCount(x1, a1, b1_min, x2, a2, b2)

probC_given_D = countC / num_points
probC = probC_given_D * probD

print("--------")
print("P(D) = ", probD, ", P(E) = ", probE)
print("p(C | D) = ", probC_given_D, ", P(C) = ", probC)

print("--------")
print("b2 = ", b2, ", b1_min = ", b1_min, ", B1_max = ", B1_max)


# Create scatter plot
if plot_scatter:
    C1 = np.zeros(countC)
    C2 = np.zeros(countC)
    AB1 = np.zeros(num_points - countC)
    AB2 = np.zeros(num_points - countC)

    j: int = 0
    k: int = 0

    for i in range(num_points):
        rr: float = get_retained_risk(x1[i], a1, b1_min) + get_retained_risk(
     x2[i], a2, b2)
        if rr > a1 + a2:
            C1[j] = x1[i]
            C2[j] = x2[i]
            j += 1
        else:
            AB1[k] = x1[i]
```

```
            AB2[k] = x2[i]
            k += 1

    plt.scatter(AB1, AB2, c ="blue")
    plt.scatter(C1, C2, c ="red")
    plt.show()


######################################
#    Determine B2_max and b2_min     #
######################################

print("--------")
print("Determine B2_max and b2_min")

b1 = dist1.getUpperPercentile(0)
b2 = dist2.getUpperPercentile(A/2)

B2_L = alpha            # B2_L = 0
B2_U = alpha / A        # B2_U = A
B2 = (B2_L + B2_U) / 2

while B2_U - B2_L > epsilon:
    probC = getCFraction(x1, a1, b1, x2, a2, b2) * probD
    if probC > alpha:
        B2_U = B2
    else:
        B2_L = B2
    B2 = (B2_L + B2_U) / 2
    b2 = dist2.getUpperPercentile(B2)

B2_max = max(B2, alpha)
b2_min = dist2.getUpperPercentile(B2_max)

countC = getCCount(x1, a1, b1, x2, a2, b2_min)

probC_given_D = countC / num_points
probC = probC_given_D * probD

print("--------")
print("P(D) = ", probD, ", P(E) = ", probE)
print("p(C | D) = ", probC_given_D, ", P(C) = ", probC)

print("--------")
print("b1 = ", b1, ", b2_min = ", b2_min, ", B2_max = ", B2_max)


# Create scatter plot
if plot_scatter:
    C1 = np.zeros(countC)
    C2 = np.zeros(countC)
    AB1 = np.zeros(num_points - countC)
    AB2 = np.zeros(num_points - countC)

    j: int = 0
    k: int = 0

    for i in range(num_points):
        rr: float = get_retained_risk(x1[i], a1, b1) + get_retained_risk(x2[i
    ], a2, b2_min)
        if rr > a1 + a2:
            C1[j] = x1[i]
```

```
            C2[j] = x2[i]
            j += 1
        else:
            AB1[k] = x1[i]
            AB2[k] = x2[i]
            k += 1

    plt.scatter(AB1, AB2, c ="blue")
    plt.scatter(C1, C2, c ="red")
    plt.show()


#########################################
#    Determine optimal b2 given b1      #
#########################################

print("--------")
print("Determine optimal b2 given b1")

BB1 = np.zeros(num_iter)
BB2 = np.zeros(num_iter)


PC = np.zeros(num_iter)
VAR = np.zeros(num_iter)

opt_B1 = 0
opt_B2 = 0
minVAR = np.inf


for ii in range(num_iter):
    if print_counter:
        print(".", end = "")
    B1 = B1_max * ii / (num_iter - 1)
    b1 = dist1.getUpperPercentile(B1)

    B2_L = (alpha - B1) / (1 - B1)
    B2_U = (alpha - A * B1) / (A - B1)
    B2 = (B2_L + B2_U) / 2
    b2 = dist2.getUpperPercentile(B2)

    while B2_U - B2_L > epsilon:
        probC = getCFraction(x1, a1, b1, x2, a2, b2) * probD
        if probC > alpha:
            B2_U = B2
        else:
            B2_L = B2
        B2 = (B2_L + B2_U) / 2
        b2 = dist2.getUpperPercentile(B2)

    BB1[ii] = B1
    BB2[ii] = B2

    PC[ii] = getCFraction(x1, a1, b1, x2, a2, b2) * probD

    insuredRiskD = probD * (get_expected_insured_risk(x1, a1, b1)  +
     get_expected_insured_risk(x2, a2, b2))

    VAR[ii] = a1 + a2 + (1 + theta) * (insuredRiskD + insuredRiskE)

    if VAR[ii] < minVAR:
```

```
            opt_B1 = B1
            opt_B2 = B2
            minVAR = VAR[ii]

    if print_results:
        print("B1 = ", BB1[ii], ", B2 = ", BB2[ii], ", P(C) = ",PC[ii],  ",
     VAR = ", VAR[ii])


opt_b1 = dist1.getUpperPercentile(opt_B1)
opt_b2 = dist1.getUpperPercentile(opt_B2)
insuredRiskD1=probD * (get_expected_insured_risk(x1, a1, opt_b1))
insuredRiskD2=probD * (get_expected_insured_risk(x2, a2, opt_b2))

print("opt_b1 = ", opt_b1, ", opt_b2 = ", opt_b2, ", opt_B1 = ", opt_B1,  ",
     opt_B2 = ", opt_B2,   ", minVAR = ", minVAR
    ,", PI1 = ",(1 + theta) * (insuredRiskD1 + insuredRiskE1),
    ", PI2 = ",(1 + theta) * (insuredRiskD2 + insuredRiskE2))




plt.plot(BB1, BB2, label='Constraint')
plt.xlabel('B1')
plt.ylabel('B2')
plt.title("Constraint")
plt.legend()
plt.show()

plt.plot(BB1, PC, label='P(C)')
plt.xlabel('B1')
plt.ylabel('P(C)')
plt.title("P(C) versus B1")
plt.legend()
plt.show()

plt.plot(BB1, VAR, label='Value-at-risk')
plt.xlabel('B1')
plt.ylabel('V@R')
plt.title("V@R as a function of B1")
plt.legend()
plt.show()
```

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