UiO : Department of Mathematics
University of Oslo

Markovianity and Time Changed Lévy Processes

Simon Kudsk Skoffer

Master's Thesis, Spring 2022

This master's thesis is submitted under the master's programme *Stochastic Modelling, Statistics and Risk Analysis*, with programme option *Statistics*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 60 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

Replacing the index parameter of a Lévy process (called a base process) with an increasing positive stochastic process (called a time process) gives rise to new stochastic processes. Such processes are called time changed Lévy processes. We examine the properties of such processes with a special focus on the Markov property. A variety of time processes are under study. A time changed Lévy process of particular interest in this thesis, is the sort where the time process is a Lévy process (the concept of subordination). A proof is presented showing that the resulting process is a Lévy process, and therefore it possesses the Markov property. We are interested in filtrations with respect to which the time changed Lévy process is measurable and with respect to which the Markov property can be expressed. We arrive at a filtration which can be expressed in terms of the natural filtration of the time process and the base process, respectively, and with respect to which the time changed Lévy process is adapted. With this filtration we obtain a result, that is relevant in the investigation of the Markov properties of time changed Lévy processes.
Concurrently, we investigate whether solutions of stochastic differential equations driven by time changed Brownian motion possess the Markov property. Cases of interest in the sequel are Brownian motion time changed with a subordinator, and cases where the time process is continuous.

Acknowledgements

I would like to thank my supervisor Giulia Di Nunno for inspiration, mathematical counseling, and for providing me with appropriate literature. Moreover I would like to thank her for supporting me in the writing process, making it educational and enjoyable.

Additionally, I would like to thank Cecilia and Jakob for valuable comments on my English writing.

Contents

CHAPTER 1

Introduction

Let $(L_t)_{t\geq 0}$ be a Lévy process and $(T_\theta)_{\theta\geq 0}$ an increasing positive stochastic process. We study the properties of the time changed process $(Y_\theta)_{\theta\geq 0}$, defined by $Y_\theta := L_{T(\theta)}$. Our main focus is to show in which cases, processes of this type possess the Markov property, that is, the property

$$\mathbb{E}[f(Y_{\theta_2})|\mathscr{F}_{\theta_1}] = \mathbb{E}[f(Y_{\theta_2})|Y_{\theta_1}],$$

for $0 \leq \theta_1 \leq \theta_2$ and f is a bounded and measurable mapping from $\mathbb{R}^d$ to $\mathbb{R}$. Here $(Y_\theta)_{\theta\geq 0}$ is adapted to $(\mathscr{F}_\theta)_{\theta\geq 0}$. We show the well established result that a Lévy process is a Markov process, so the question can be reformulated to: What properties of the process T preserves the Markov property of the resulting time changed Lévy process? In the cases where T is deterministic, a Lévy process, or a process with independent increments, the time changed process is a Markov process. The case where T is a Lévy process (i.e. a subordinator) is examined thoroughly.

In order to learn about the filtration generated by the time changed process, we exploit filtrations expressed in terms of $\mathbb{F}^T$ and $\mathbb{F}^L$ with respect to which the time changed Lévy process is measurable. The Markov property of Y with respect to such a filtration will immediately secure the Markov property with respect to the filtration generated by the time changed process.

Solutions to specific types of stochastic differential equations can be shown to be Markov processes. In the cases where it is possible to integrate with respect to a time changed Brownian motion, we replace the Brownian motion by the time changed Brownian motion in the stochastic differential equation. We investigate if there are solutions of such types of processes and whether the Markov property can be demonstrated in these instances.

In order to make a theoretical basis for examining the Markovianity for the time changed Lévy processes, sections on Lévy processes and Markovianity, respectively, are included.

The main advanced literature that I have followed in the project is [BS15], [Sat13], [Bal17], and [App09]. Also, I have used [Ped20] (lecture notes from Aarhus University) extensively for the study of both Lévy processes and the Markov property. [BS15] has been used as inspiration to motivate the concept of time change and see some of the possibilities it gives. [Bal17] has been useful for both the study of stochastic differential equations, and for general results about filtrations and stochastic processes and the Markov property. [Sat13] has been the main reference in the study of Lévy processes. Additionally, the proof for the fact that a subordination process is a Lévy process comes from this

book. [App09] has primarily been used to study the Lévy-Itô decomposition and integration with respect to Lévy processes. For general results in probability theory, measure theory and the theory of stochastic processes, I have primarily consulted [Tho14], [Tho19], and [Ped20]. [Tho19] and [Ped20] are lecture notes from Aarhus University. The results that I have used from these books are long-established results from probability theory and can be found in other material as well. The book [Kal21] has been consulted for results in measure theory. [Gra76] was used to introduce the concept of random measures, in order to be able to introduce Poisson processes and Cox processes. The book [Rei93] was included briefly in this presentation.
A lot of attention has been put into the arguments of the proofs, generally making them more extensive than the ones found in the referenced literature. A prime example is the proof of the uniqueness of Lévy-Khintchine representation Proposition 4.3.7, where several arguments are left out in the original material. The rule of thumb regarding the level of detail has been to give arguments that would be useful for a peer reader. There are exceptions for this high degree of detail. In the sections 2.7, 2.8, 4.5, and 5.5 I rely heavily on results and proofs from [Bal17], [App09], and [IW89]. These sections are the ones concerning the theoretical basis for stochastic integration and stochastic differential equations. The most original material in the thesis is contained in Section 5.4. Here I work with finding filtrations with respect to which the time changed Lévy processes are measurable. By combining theory on stopping times, Lévy processes and the *Freezing Lemma* we reach an interesting result, that is interesting in the study of Markovianity of time changed Lévy processes. The idea comes from working with the measurability of time changed progressively measurable stochastic processes and from the assumption in [BS15] that the elements of the time process are stopping times.
In most cases, the results, examples, and definitions (numbered entities) are inspired by or taken from the literature to which I refer. The reference will either be written in text above, or at the top of, the numbered entity. It is not always clear from the proofs to what extend the ideas are my own or whether they come from the literature in reference (although there is often an explanatory text before a result that comments on the literature in reference). In order to clarify this, there is a rating mark before each of the proofs and examples, either (§), (§§) or (§§§), ranging from low to high contribution. To clarify:
(§): The ideas and arguments comes from the literature in reference.
(§§): I have contributed with substantial arguments or I have made it myself following the ideas put forward in the referenced literature.
(§§§): I have made the proof or example independently.
Note that this system says nothing about the originality of the result or the complexity of the proof.

1.1 Outline of the Thesis

The chapters are organized as follows:

Chapter 2 contains preliminary material, which are relevant for the rest of the thesis. In this chapter, we give assumptions that will be standing throughout the thesis. Some practical remarks on the notation are given. The chapter contains a couple of practical definitions regarding stochastic processes and general probability theory. Moreover, we give proofs of certain lemmas and theorems, which are important for proofs in subsequent chapters. For example, we give some measurability results, e.g. that a right continuous stochastic process is progressively measurable. Moreover, in this chapter we introduce the concepts of point processes and elements of stochastic calculus.

Chapter 3 contains material on the Markov property of stochastic processes that takes values in $\mathbb{R}^d$. We introduce Markov transition functions, i.e. mappings defined on the space $\mathbb{R}^d \times \mathscr{B}(\mathbb{R}^d)$. Markov semigroups will allow us to define a Markov process uniquely from its finite dimensional distribution. The chapter also contains a full proof of the *Freezing Lemma*. This lemma comes in handy several times in the sequel. In the last part of the chapter, we present a proof showing that a diffusion process (a class of solutions to a stochastic differential equation) is in fact a Markov process.

Chapter 4 deals with Lévy processes. This class of stochastic processes contains many important processes including the Brownian motion and the Poisson process. We present a proof showing that Lévy processes are time homogeneous Markov processes. Moreover we establish a correspondence between Lévy processes and infinitely divisible distributions. In this context we present the form of the characteristic function of a Lévy process, which can be represented by a generating triplet (the Lévy-Khintchine representation). We study the Markovianity of stochastic processes with independent increments. In the last part we present and discuss the Lévy-Itô decomposition and use this to define integration with respect to Lévy processes.

Chapter 5 contains material on time changed stochastic processes, that is, stochastic processes in continuous time, where we instead of time (t) insert a new stochastic process. We will aim to define such a process in a meaningful way. We need to make sure that the elements are definable and measurable. The Markov property is expressed in terms of conditional expectation. We only need to be certain that the process is adapted to the filtration we take the conditional expectation with respect to. We work with filtrations that can be expressed in terms of the filtrations generated by the time process and the base process. This examination makes us able to retrieve a structure that is relevant in the study of Markovianity of time changed Lévy processes. The chapter also contains a proof of the well known result that a Lévy process time changed with a subordinator is a Lévy process, that is, a Markov process.

In the appendices I included important results and definitions, to which I refer throughout the thesis, but that need no commenting or elaboration. Some of the results are well-known and can be referenced by name. An example of this is *Fubini's Theorem*. Moreover I have included results, where several hypothesis need to be checked in order to draw a conclusion. Reporting such results makes it more transparent for the reader to see we arrive at the conclusions.

PART I

Markovianity and Time Changed Lévy Processes

CHAPTER 2

Preliminaries

The purpose of this chapter is to lay the mathematical foundation for the thesis. By not having to introduce new notions as they appear in the later chapters, we can focus on the subject under examination. The current chapter contains a mix of explanations of notation, definitions, results and presentations of advanced topics. The presented material will either be referred to or simply serve as basis for the mathematics throughout the thesis.

2.1 Remarks on Notation

Denote vectors in $\mathbb{R}^d$ as x or y (not in boldface), also in the case where $d = 1$. The i'th coordinate of x is denoted x_i. Let $x, y \in \mathbb{R}^d$. Equip $\mathbb{R}^d$ with the inner product, $\langle \cdot, \cdot \rangle$, defined by

$$\langle x, y \rangle := \sqrt{\sum_{i=1}^{d} x_i y_i}.$$

$| \cdot |$ denotes the Euclidean norm induced by the inner product above, and is defined as

$$|x| := \sqrt{\sum_{i=1}^{d} x_i^2},$$

for $x \in \mathbb{R}^d$. $Mat(d, m)$ denotes the space of real $d \times m$ matrices.

Define $\mathscr{M}(\mathbb{R}^d, \mathbb{R})$ to be the space of measurable functions from $\mathbb{R}^d$ to $\mathbb{R}$. Let $\mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$ and $\mathscr{M}_c(\mathbb{R}^d, \mathbb{R})$ be the space of functions from $\mathscr{M}(\mathbb{R}^d, \mathbb{R})$ that are bounded and continuous, respectively.

Let $(E, \mathscr{E})$ and $(G, \mathscr{G})$ be measurable spaces. Define the mapping $\Phi : E \to G$, and let $B \in \mathscr{G}$. The preimage of B through Φ is the set $\{x \in E : \Phi(x) \in B\}$ and it is denoted $\Phi^{-1}(B)$.

$\mathbb{F} := (\mathscr{F}_t)_{t \geq 0}$, where $(\mathscr{F}_t)_{t \geq 0}$ is a filtration on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. $\mathbb{F}^Y$ is the filtration generated by the stochastic process $(Y_t)_{t \geq 0}$.

Y will in general denote the stochastic process $(Y_t)_{t \geq 0}$, but will in some cases denote a stochastic variable or a stochastic vector. What Y denotes will be clear from the context. The same holds true for X, T and L.

$\mathbb{N}$ denotes the natural numbers including 0; that is $\{0, 1, 2, ...\}$.

Let $\mathscr{C}$ be a collection of subsets of a space E. $\sigma(\mathscr{C})$ denotes the σ-algebra generated by $\mathscr{C}$ (i.e. the smallest σ-algebra containing $\mathscr{C}$). Let $\mathscr{F}$ and $\mathscr{G}$ be two σ-algebras on E. $\mathscr{F} \vee \mathscr{G}$ is defined as $\sigma(\mathscr{F} \cup \mathscr{G})$. If $\mathscr{F}$ and $\mathscr{G}$ are σ-algebras on different spaces, we write $\mathscr{F} \otimes \mathscr{G}$ for $\sigma(\mathscr{F} \times \mathscr{G})$, where $\mathscr{F} \times \mathscr{G} := \{A \times B : A \in \mathscr{F}, B \in \mathscr{G}\}$. Moreover, if we let μ be a measure on $(E, \mathscr{E})$ and ν be a measure on $(G, \mathscr{G})$. Then $\mu \otimes \nu$ denotes the unique measure on the space $(E \times G, \mathscr{E} \otimes \mathscr{G})$ such that $\mu \otimes \nu(A \times B) = \mu(A)\nu(B)$ for $A \in \mathscr{E}$ and $B \in \mathscr{G}$ (see e.g. [Tho14, Theorem 6.3.3]).

$\mathscr{B}(\mathbb{R}^d)$ is the Borel σ-algebra on $\mathbb{R}^d$. That is, the σ-algebra generated by the open subsets of $\mathbb{R}^d$. In short we write $\mathscr{B}^d$.

Let $t \geq 1$, and denote by $L^t(\mu)$ the spaces of functions, f, such that $\int |f|^t d\mu < \infty$. We say that a stochastic vector is *integrable* if it is in $L^1(\mathbb{P})$ and square integrable if it is $L^2(\mathbb{P})$.

$M^2[0, T]$ denotes the space of stochastic processes in $L^2(\lambda_{[0,T]} \otimes \mathbb{P})$, that are progressively measurable.

SDEs are an abbreviation for 'Stochastic differential equations'.

2.2 Probability Spaces

Throughout the thesis $(\Omega, \mathscr{F}, \mathbb{P})$ denotes the underlying probability space on which the random elements are defined. The following definitions are general and many of them can be found in, for example, [Bal17, Section 2.1].

Definition 2.2.1. A *stochastic vector* is a measurable mapping from $(\Omega, \mathscr{F}, \mathbb{P})$ to $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. A *stochastic process* on $\mathbb{R}^d$, (denoted $(Y_t)_{t\geq 0}$) is a family of indexed stochastic vectors.

□

Definition 2.2.2. Let $Y := (Y_t)_{t\geq 0}$ and $Y' := (Y'_t)_{t\geq 0}$ be stochastic processes defined on $(\Omega, \mathscr{F}, \mathbb{P})$ and $(\Omega', \mathscr{F}', \mathbb{P}')$ respectively. Then we say that Y and Y' are *equivalent* if for all $n \in \mathbb{N}$ and $0 \leq t_1 < ... < t_n$ and $B_1, ..., B_n \in \mathscr{B}(\mathbb{R}^d)$,

$$\mathbb{P}(Y_{t_1} \in B_1, ..., Y_{t_n} \in B_n) = \mathbb{P}(Y'_{t_1} \in B_1, ..., Y'_{t_n} \in B_n).$$

Or in short

$$(Y_{t_1}, ..., Y_{t_n}) \stackrel{d}{=} (Y'_{t_1}, ..., Y'_{t_n}).$$

□

Definition 2.2.3. [Bal17, p. 32] Two stochastic processes, X and Y defined on the same probability space are *modifications* if $\mathbb{P}(X_t = Y_t) = 1$ for all $t \geq 0$.

□

Definition 2.2.4. A *filtration*, $(\mathscr{F}_t)_{t\geq 0}$, is an increasing collection of sub σ-algebras of $\mathscr{F}$, indexed by $t \geq 0$. A stochastic process, $(Y_t)_{t\geq 0}$ is said to be *adapted* to the filtration if Y_t is $\mathscr{F}_t$-measurable for all $t \geq 0$. A filtration is *right continuous* if $\mathscr{F}_t = \cap_{s>t}\mathscr{F}_s$.

□

Definition 2.2.5. Let Y be s stochastic process. $\mathbb{F}^Y := (\mathscr{F}_t^Y)_{t\geq 0}$ denotes the *natural filtration* or the filtration generated by Y. It is defined as

$$\mathscr{F}_t^Y := \bigvee_{u\leq t} \sigma(Y_u),$$

where

$$\sigma(Y_u) = \{Y_u^{-1}(B) : B \in \mathscr{B}(\mathbb{R}^d)\}.$$

Remark 2.2.6. $\{Y_u^{-1}(B) : B \in \mathscr{B}(\mathbb{R}^d)\}$ is a σ-algebra by [Kal21, Lemma 1.3].

□

Definition 2.2.7. A probability space, $(\Omega, \mathscr{F}, \mathbb{P})$, is *complete* if $A \in \mathscr{F}$ for all $A \subseteq B$, where $B \in \mathscr{F}$ and $\mathbb{P}(B) = 0$. In words, the probability space contains all null sets.

□

Definition 2.2.8. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. Let $\mathscr{N} := \{A \in \mathscr{F} : \mathbb{P}(A) = 0\}$. The filtration $(\mathscr{F}_t)_{t\geq 0}$ is said to be $\mathbb{P}$-*augmented* if for all $t \geq 0$; $\mathscr{F}_t = \mathscr{F}_t \vee \mathscr{N}$.

□

The assumption that the probability space is complete is useful in many situations. For instance, the following lemma that will be used throughout the thesis depends on this assumption. The theorem is an extension of a result from [Kal21] (Theorem A.2.1).

Lemma 2.2.9. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of stochastic vectors (on $\mathbb{R}^d$) defined on a complete probability space, $(\Omega, \mathscr{F}, \mathbb{P})$, and assume that f is a mapping from Ω to $\mathbb{R}^d$. If f_n converges to f almost surely then f is a stochastic vector.*

Proof. (§§) Let Ω_0 be the subset of Ω such that $f_n(\omega)$ converges to $f(\omega)$ for $\omega \in \Omega_0$. As the probability space is assumed to be complete, we get that Ω_0 is in $\mathscr{F}$ because Ω_0^C is a null set. Let $B \in \mathbb{R}^d$

$$f^{-1}(B) = (\Omega_0 \cap \{f \in B\}) \cup (\Omega_0^C \cap \{f \in B\}).$$

$\Omega_0 \cap \{f \in B\} \in \mathscr{F}$ by Theorem A.2.1 and $\Omega_0^C \cap \{f \in B\}$ is a null set and thereby it is contained in $\mathscr{F}$. ■

It will be a standing assumption that the probability space, on which the random elements are defined, is complete, and that the filtrations that are given are $\mathbb{P}$-augmented and right continuous. Moreover, we assume that the stochastic processes take values in the measurable space $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$.

2.3 Independence

Let $\mathscr{F}_1, ..., \mathscr{F}_n$ be sub σ-algebras of $\mathscr{F}$. By [Bal17, p. 6] they are independent if for all $A_1 \in \mathscr{F}_1, ..., A_n \in \mathscr{F}_n$

$$\mathbb{P}(A_1, ..., A_n) = \prod_{i=1}^{n} \mathbb{P}(A_i).$$

This means that if we let $X_1, X_2, ..., X_n$ be random vectors that takes values in $\mathbb{R}^{d_i}$ for $i \in \{1, 2, ..., n\}$. They are independent if $\sigma(X_1), ..., \sigma(X_n)$ are independent. Remark that all sets from $\sigma(X_i)$ can be written as $\{X_i \in B\}$ for some $B \in \mathscr{B}(\mathbb{R}^{d_i})$.
The following consideration comes from [Bal17, Remark 1.1] and is crucial throughout the thesis. Let X and Y denote stochastic processes $(\mathscr{F}_t^X)_{t \geq 0}$ and $(\mathscr{F}_t^Y)_{t \geq 0}$ denote their natural filtrations. Write $\mathscr{F}_\infty^X$ for $\vee_{t \geq 0} \mathscr{F}_t^X$ and note that it is generated by sets of the form $\{X_t \in B\}$ for $t \geq 0$ and $B \in \mathscr{B}(\mathbb{R}^d)$. Sets of this form is contained in the intersection stable system of sets

$$\mathscr{C} := \{\{X_{t_1} \in B_1, ..., X_{t_n} \in B_n\} : n \in \mathbb{N}, 0 \leq t_1 < ... < t_n, B_1, ..., B_n \in \mathscr{B}^d\}.$$

Clearly $\mathscr{C}$ is contained in $\mathscr{F}_\infty^X$. We say that X and Y are independent if $\mathscr{F}_\infty^X$ and $\mathscr{F}_\infty^Y$ are independent, which they are if and only if $(X_{t_1}, ..., X_{t_n})$ and $(Y_{s_1}, ..., Y_{s_m})$ are independent for all $0 \leq t_1 < ... < t_n$ and $0 \leq s_1 < ... < s_m$ for $n, m \in \mathbb{N}$.

It is possible to construct independent elements by using the properties of the product space (see e.g. [Tho19, Exercise 1.11]). In general, stochastic elements can be chosen in a way, such that are independent. This is justified in the two following lemmas.

Lemma 2.3.1. *[Tho19, Exercise 1.11] Let X and Y be stochastic elements, taking values in the measurable space $(E, \mathscr{E})$. Let them be defined on the two probability spaces $(\Omega_1, \mathscr{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathscr{F}_2, \mathbb{P}_2)$ respectively (might be the same probability space). Then there exist $X' \sim X$ and $Y' \sim Y$ such that X' is independent of Y'.*

Proof. (§) Define the probability space $(\Omega, \mathscr{F}, \mathbb{P}) := (\Omega_1 \times \Omega_2, \mathscr{F}_1 \otimes \mathscr{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$. Let $(\omega_1, \omega_2) = \omega \in \Omega$ and define $X'(\omega) = X'(\omega_1, \omega_2) := X(\omega_1)$ and $Y'(\omega) = Y'(\omega_1, \omega_2) := Y(\omega_2)$. Let $A_1, A_2 \in \mathscr{E}$. Then

$$\mathbb{P}(X \in A_1) = \mathbb{P}(X^{-1}(A_1) \times \Omega_2) = \mathbb{P}_1(X^{-1}(A_1))\mathbb{P}_2(\Omega_2) = \mathbb{P}_1(X^{-1}(A_1)),$$

so $X \sim X'$. The analogous arguments shows that $Y' \sim Y$. Also

$$\begin{aligned}\mathbb{P}(X' \in A_1, Y' \in A_2) &= \mathbb{P}((X^{-1}(A_1) \times \Omega_2) \cap (\Omega_1 \times Y^{-1}(A_2))) \\ &= \mathbb{P}_1 \otimes \mathbb{P}_2(X^{-1}(A_1) \times Y^{-1}(A_2)) \\ &= \mathbb{P}_1(X \in A_1)\mathbb{P}_2(Y \in A_2),\end{aligned}$$

which shows they are independent. ∎

If we let X and Y be stochastic processes, one can make the same construction. Then one obtain that for all $n, m \in \mathbb{N}$, $s_1, ..., s_n, t_1, ..., t_m \in [0, \infty)$,

where $s_i < s_{i+1}$ and $t_i < t_{i+1}$ for all $i \in \{1, ..., n-1\}$, $(X_{s_1}, ..., X_{s_n})$ and $(Y_{t_1}, ..., Y_{t_m})$ are independent.
The following lemma is a direct consequence of [FG97, Corollary 17] and the proof is omitted here.

Lemma 2.3.2. *Let Z be a random variable. Then there exist a sequence of independent stochastic variables, $(Z_n)_{n\in\mathbb{N}}$, on some probability space, $(\Omega, \mathscr{F}, \mathbb{P})$, such that $Z_i \sim Z$.*

□

2.4 Properties of Stochastic Processes

We give definitions and various results regarding the measurability of stochastic processes.
In this section Y denotes an $\mathbb{F}$-adapted stochastic process taking values in $\mathbb{R}^d$.

Definition 2.4.1. [Bal17, p. 33] We say that Y is *measurable* if the function

$$\begin{aligned}\Phi : [0,\infty) \times \Omega &\to \mathbb{R}^d \\ (s,\omega) &\mapsto Y_s(\omega).\end{aligned}$$

is $(\mathscr{B}([0,\infty)) \otimes \mathscr{F}, \mathscr{B}^d)$-measurable.

□

Definition 2.4.2. [Bal17, p. 33] Let $T > 0$, and define the function

$$\begin{aligned}\Phi_T : [0,T] \times \Omega &\to \mathbb{R}^d \\ (s,\omega) &\mapsto Y_s(\omega).\end{aligned} \tag{2.1}$$

We say that Y is progressively measurable if Φ_T is $(\mathscr{B}([0,T]) \otimes \mathscr{F}_T, \mathscr{B}(\mathbb{R}^d))$-measurable for all $T > 0$.

□

Lemma 2.4.3. *[Bal17, Proposition 2.1] A right continuous process is progressively measurable.*

Proof. (§) Let Y be right continuous and define Φ_T for a $T > 0$ as in Equation (2.1). Let $n \in \mathbb{N}$ and define the function $a_n(s) : [0,T] \to [0,T]$ as

$$a_n(s) = \sum_{k=1}^{n} \frac{kT}{n} \mathbb{1}_{[\frac{k-1}{n}T, \frac{k}{n}T)}(s) + T\mathbb{1}_{\{s=T\}}(s).$$

Set $\Phi_T^{(n)}(s,\omega) := Y_{a_n(s)}(\omega)$ and let $B \in \mathscr{B}^d$. Then

$$(\Phi_T^{(n)})^{-1}(B) = (\cup_{k=1}^{n} [\frac{k-1}{n}T, \frac{k}{n}T) \times Y_{\frac{k}{n}}^{-1}(B)) \cup (\{T\} \times Y_T^{-1}(B)),$$

which is a set from $\mathscr{B}([0,T]) \otimes \mathscr{F}_T$. Let $(s,\omega) \in [0,T) \times \Omega$. Then because of right continuity, we get that

$$\Phi_T^{(n)}(s,\omega) = Y_{a_n(s)}(\omega) \to Y_s(\omega),$$

for $n \to \infty$, as $a_n(s) > s$ and $a_n(s) \to s$. If $s = T$, then $\Phi_T^{(n)}(T, \omega) = Y_T(\omega)$ for all $n \in \mathbb{N}$. As $Y : [0, \infty) \times \Omega \to \mathbb{R}^d$ is a limit of measurable functions, we get that it is measurable by Theorem A.2.1. ∎

The idea for the proof of the following lemma is similar to the one behind the proof of the previous lemma.

Lemma 2.4.4. *Let $d, m \in \mathbb{N}$ and define the function*

$$\Phi : \mathbb{R}^m \times \Omega \to \mathbb{R}^d$$

be a continuous function for all fixed $\omega \in \Omega$ and let it be a stochastic vector for fixed $x \in \mathbb{R}^m$. Then Φ is $(\mathscr{B}^m \otimes \mathscr{F}, \mathscr{B}^d)$-measurable.

Proof. (§§) Define the function for $n \in \mathbb{N}$:

$$\Phi^{(n)} : [-n, n)^m \times \Omega \to \mathbb{R}^d$$

$$(s, \omega) \mapsto \sum_{i_1,\dots,i_m=-n^2}^{n^2-1} \mathbb{1}_{[\frac{i_1}{n}, \frac{i_1+1}{n}) \times \cdots \times [\frac{i_m}{n}, \frac{i_m+1}{n})}(s) \Phi((\frac{i_1}{n}, \dots, \frac{i_m}{n}), \omega).$$

As $\Phi^{(n)}$ is a sum of measurable functions, it is measurable. As Φ is continuous for fixed ω, it is not hard to see that $\Phi^{(n)}(s, \omega) \to \Phi(s, \omega)$ for $n \to \infty$ for all $(s, \omega) \in \mathbb{R}^m \times \Omega$. By Theorem A.2.1 Φ is $(\mathscr{B}^m \otimes \mathscr{F}, \mathscr{B}^d)$-measurable. ∎

The following two results are claimed in [Bal17, pp. 33–34]. Here, we give detailed arguments for the assertions.

Lemma 2.4.5. *A progressively measurable stochastic process is measurable.*

Proof. (§§§) Let $(Y_t)_{t \geq 0}$ be a progressively measurable stochastic process on $(\mathbb{R}^d, \mathscr{B}^d)$ adapted to the filtration $(\mathscr{F}_t)_{t \geq 0}$. Let Y be progressively measurable and define for $N \in \mathbb{N}$ the function:

$$\Phi_N : [0, N] \times \Omega \to \mathbb{R}^d$$
$$(t, \omega) \mapsto Y_t(\omega).$$

By assumption this function is $(\mathscr{F}_N \otimes \mathscr{B}([0, N]), \mathscr{B}(\mathbb{R}^d))$. Let $B \in \mathscr{B}(\mathbb{R}^d)$ and consider the set $\cup_{N \in \mathbb{N}} \Phi_N^{-1}(B)$, and note that it is a set in $\mathscr{F} \otimes \mathscr{B}([0, \infty))$. We show that $\cup_{N \in \mathbb{N}} \Phi_N^{-1}(B) = Y^{-1}(B)$.
We show that $Y^{-1}(B) \subseteq \cup_{N \in \mathbb{N}} \Phi_N^{-1}(B)$: Let $(t, \omega) \in Y^{-1}(B)$, that is $Y_t(\omega) \in B$. Let $K \in \mathbb{N}$ be such that $t \leq K$. Then $\Phi_K(t, \omega) = Y_t(\omega) \in B$. The inverse inclusion is obvious. ∎

2.5 Random Measures

The introduction of Poisson processes and Cox processes will be based on the theory of random measures, including point processes. The introduction given here will be based mainly on the one given in [Gra76], but for some of the notions we consult [Rei93].
Let $(E, \mathscr{E})$ be a measurable space, here denoted *the state space*. Let M be the space of all measures on $(E, \mathscr{E})$ that are finite on compact sets. Let $\mathscr{M}$ be a σ-algebra on M.

Definition 2.5.1. [Gra76, Definition I.2] A *random measure*, N, is a measurable mapping from Ω to M. That is, for $A \in \mathscr{M}$, $N^{-1}(A) \in \mathscr{F}$.

Remark 2.5.2. We write $N_\omega := N(\omega)$.

□

Definition 2.5.3. [Gra76, p. 4] Let π_N be a probability measure on $(M, \mathscr{M})$. It is called the distribution of N if $\pi_N(B) = \mathbb{P}(\{\omega \in \Omega : N(\omega) \in B\})$ for $B \in \mathscr{M}$.

□

Definition 2.5.4. [Gra76, Definition I.2] Let $M_n \subseteq M$ be the set of measures on $(E, \mathscr{E})$, where for $m \in M_n$ and $A \in \mathscr{E}$, $m(A) \in \mathbb{N} \cup \{\infty\}$. Let N be a random measure with distribution π_N. If $\pi_N(M_n) = 1$, then N is a *point process*.

□

From now on, we let N be a point process. By [Rei93, p. 6], we assume that $\mathscr{M}_n$ is the σ-algebra generated by sets of the form $\{m \in M : m(A) \in B\}$, where $A \in \mathscr{E}$ and $B \subseteq \mathbb{N} \cup \{\infty\}$. This way of constructing the $\mathscr{M}_n$ allows us to introduce the following lemma, which is presented as Criterion 1.1.1 in [Rei93]. This lemma connects the abstract definition of a point process given above to a more intuitive definition.

Lemma 2.5.5. *N is a point process if the mapping $N(B)$ is measurable for all $B \subseteq \mathscr{E}$ and if N_ω is a $\mathbb{N} \cup \{\infty\}$-valued measure for almost all $\omega \in \Omega$.*

Proof. (§) The condition that N_ω is a $\mathbb{N} \cup \{\infty\}$-valued measure for almost all $\omega \in \Omega$ is needed in order for $\pi_N(M_n) = 1$. Assume that $N(B)^{-1}(A) \in \mathscr{F}$ for all $A \subseteq \mathbb{N} \cup \{\infty\}$. Then

$$N(B)^{-1}(A) = \{\omega \in \Omega : N_\omega(B) \in A\} = N^{-1}(\{m \in M : m(B) \in A\}).$$

As $\{m \in M : m(B) \in A\}$ generates $\mathscr{M}_n$, $N^{-1}(A) \in \mathscr{F}$ for all $A \in \mathscr{M}_n$. ■

The concept of Poisson random measures are central in the theory of point processes. Late we introduce the Poisson process based on this definition

Definition 2.5.6. [Gra76, Definition I.3] A random measure is called *completely random* if for $n \in \mathbb{N}$ and disjoint sets $B_1, ..., B_n \in \mathscr{E}$, $(N(B_1), ..., N(B_n))$ is a collection of independent random variables.

□

Definition 2.5.7. [Rei93, p. 46] Let μ be a σ-finite measure on $(E, \mathscr{E})$, which are finite on compact sets. N is a Poisson random measure if it is completely random and for all $k \in \mathbb{N}$

$$\mathbb{P}(N(A) = k) = e^{-\mu(A)} \frac{(\mu(A))^k}{k!} \mathbb{1}_{\{\mu(A) < \infty\}},$$

for $A \in \mathscr{E}$.

□

We present the result and proof from [Sat13, Proposition 19.4]. He shows the statement for a general σ-finite measure. We only present the proof in the case where μ is a finite measure.

Lemma 2.5.8. *Let μ be a σ-finite measure on a measurable space $(E, \mathscr{E})$. Then there exist a Poisson random measure with intensity measure μ on some probability space.*

Proof. (§) Assume that $\mu(E) < \infty$. Let $(Z_n)_{n\in\mathbb{N}}$ be independent stochastic variables with distribution $\mu/\mu(E)$ and let Y be a Poisson random variable with parameter $\mu(E)$ (this can be constructed on some probability space by Lemma 2.3.2). Define for $B \in \mathscr{E}$, $N(B) := \sum_{i=1}^{Y} \mathbb{1}_B(Z_i)$, then for each $\omega \in \Omega$, $N_\omega(\cdot) = \sum_{i=1}^{Y(\omega)} \mathbb{1}_\cdot(Z_i(\omega))$ is a $\mathbb{N} \cup \{\infty\}$-valued measure. Moreover for for fixed $B \in \mathscr{E}$, $N(B)$ is a stochastic variable. We need to show that the N is completely independent. Let $k, n \in \mathbb{N}$ and $B_1, ..., B_k \in \mathscr{E}$ be disjoint such that $\cup_{i=1}^{k} B_i = E$. Then $N(B_1) + ... + N(B_k) = N(E) = Y$. Let $n_1, ..., n_k \in \mathbb{N}$ such that $n_1 + ... + n_k = n$. Remark that for fixed $n \in \mathbb{N}$, $(\sum_{i=1}^{n} \mathbb{1}_{B_1}(Z_i), ..., \sum_{i=1}^{n} \mathbb{1}_{B_k}(Z_i))$ is multinomial distributed with parameter $(\frac{\mu(B_1)}{\mu(E)}, ..., \frac{\mu(B_k)}{\mu(E)})$.

$$
\begin{aligned}
&\mathbb{P}(N(B_1) = n_1, ..., N(B_k) = n_k) \\
&\quad = \mathbb{P}(\sum_{i=1}^{n} \mathbb{1}_{B_1}(Z_i) = n_1, ..., \sum_{i=1}^{n} \mathbb{1}_{B_k}(Z_i) = n_k | Y = n)\mathbb{P}(Y = n) \\
&\quad = \mathbb{P}(\sum_{i=1}^{n} \mathbb{1}_{B_1}(Z_i) = n_1, ..., \sum_{i=1}^{n} \mathbb{1}_{B_k}(Z_i) = n_k)\mathbb{P}(Y = n) \\
&\quad = \frac{n!}{n_1! \cdots n_k!}(\frac{\mu(B_1)}{\mu(E)})^{n_1} \cdots (\frac{\mu(B_k)}{\mu(E)})^{n_k} e^{-\mu(E)} \frac{\mu(E)^n}{n!}
\end{aligned}
$$

As $e^{-\mu(E)} = \prod_{i=1}^{k} e^{-\mu(B_i)}$, one can reduce the expression above to obtain

$$
\mathbb{P}(N(B_1) = n_1, ..., N(B_k) = n_k) = \prod_{i=1}^{k} e^{-\mu(B_i)} \frac{\mu(B_i)^{n_i}}{n_i!}.
$$

Assume that $\cup_{i=1}^{k} B_k \neq \mathscr{E}$, and define $B_{k+1} := \mathscr{E} \setminus \cup_{i=1}^{k} B_k$. Then

$$
\mathbb{P}(N(B_1) = n_1, ..., N(B_k) = n_k, N(B_{k+1}) = n_{k+1}) = \prod_{i=1}^{k+1} e^{-\mu(B_i)} \frac{\mu(B_i)^{n_i}}{n_i!}.
$$

Letting $n_1, ..., n_k$ be fixed and summing over n_{k+1} on both sides, we obtain

$$
\mathbb{P}(N(B_1) = n_1, ..., N(B_k) = n_k) = \prod_{i=1}^{k} e^{-\mu(B_i)} \frac{\mu(B_i)^{n_i}}{n_i!}.
$$

This also gives us that $\mathbb{P}(N(B) = n) = e^{-\mu(B)} \frac{\mu(B)^n}{n!}$, which shows the complete randomness and Poisson distribution of N. ■

2.6 The Freezing Lemma

Several times throughout the thesis, we refer to the *Freezing Lemma* in order to show that certain processes possess the Markov property. The result is given in [Bal17, Lemma 4.1] for general measurable spaces $(E, \mathscr{E})$, but only a sketch of a proof is presented. Because of the importance of the theorem, we will here give a full proof of it. We do it a little differently than the method he has in mind.

Lemma 2.6.1. *[Bal17, Lemma 4.1] Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $\mathscr{G}$ and $\mathscr{D}$ independent σ-algebras contained in $\mathscr{F}$. Let Z be a $\mathscr{D}$-measurable random variable that takes values in a measurable space $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. Now let $\psi : \mathbb{R}^d \times \Omega \to \mathbb{R}$ be an $\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{G}$-measurable function such that the function $\omega \mapsto \psi(Z(\omega), \omega)$ is integrable. Then we have that*

$$\mathbb{E}[\psi(Z, \cdot)|\mathscr{D}] = \Phi(Z),$$

where $\Phi(z) = \mathbb{E}[\psi(z, \cdot)]$. $\Phi(z)$ is a $\mathscr{E}$-measurable function.

Remark 2.6.2. The lemma also holds in the case where X takes its values in the complex plane. Then $\psi(Z, \cdot) = \psi_1(Z, \cdot) + i\psi_2(Z, \cdot)$ where ψ_1 and ψ_2 are real random functions. Using the linearity of the conditional expectation and the result of the lemma, one can obtain

$$\mathbb{E}[\psi(Z, \cdot)|\mathscr{D}] = \mathbb{E}[\psi_1(Z, \cdot)|\mathscr{D}] + i\mathbb{E}[\psi_2(Z, \cdot)|\mathscr{D}] = \Phi_1(Z) + i\Phi_2(Z).$$

Where $\Phi_1(z) = \mathbb{E}[\psi_1(z, \cdot)]$ and $\Phi_2(z) = \mathbb{E}[\psi_2(z, \cdot)]$. As

$$\mathbb{E}[\psi_1(z, \cdot)] + i\mathbb{E}[\psi_2(z, \cdot)] = \mathbb{E}[\psi(z, \cdot)] = \Phi(z),$$

we have it. This will come in handy when we shall use the lemma in the setting of characteristic functions.

Proof. (§§) Define $\mathscr{H}$ to be the space of bounded measurable mappings from $\mathbb{R}^d \times \Omega$ to $\mathbb{R}$ for which the theorem holds true. Define the class of sets

$$\mathscr{C} := \{A \times B : A \in \mathscr{B}(\mathbb{R}^d), B \in \mathscr{G}\}.$$

We show that the conditions in the *Monotone Class Theorem* (Theorem A.2.4) is fulfilled in this situation. Let $A \times B \in \mathscr{C}$ and define the function

$$\psi(z, \omega) := \mathbb{1}_{A \times B}(z, \omega) = \mathbb{1}_A(z)\mathbb{1}_B(\omega).$$

Remark that $\mathbb{1}_A(Z)$ is $\mathscr{D}$-measurable and $\mathbb{1}_B(\omega)$ is independent of $\mathscr{D}$.

$$\mathbb{E}[\mathbb{1}_{A \times B}(Z, \omega)|\mathscr{D}) = \mathbb{1}_A(Z)\mathbb{E}(\mathbb{1}_B(\omega)|\mathscr{D}) = \mathbb{1}_A(Z)\mathbb{P}(B) =: \Phi(Z),$$

and for $z \in \mathbb{R}^d$:

$$\mathbb{E}[\mathbb{1}_{A \times B}(z, \omega)] = \mathbb{1}_A(z)\mathbb{E}[\mathbb{1}_B(\omega)] = \mathbb{1}_A(z)\mathbb{P}(B) = \Phi(z),$$

as $\mathbb{1}_A(z)$ is a constant. Remark by the linearity of the conditional expectation and the expectation that $\mathscr{H}$ is a vector space. Let $(\psi_n)_{n \in \mathbb{N}} \in \mathscr{H}$ be positive and increasing such that $\lim_{n \to \infty} \psi_n$ is bounded. Then for all $n \in \mathbb{N}$

$$\mathbb{E}[\psi_n(Z, \cdot)|\mathscr{D}] = \psi_n(Z), \text{ where } \Phi_n(z) = \mathbb{E}[\psi_n(z, \cdot)].$$

As ψ_n is $\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{G}$-measurable, $\lim_{n\to\mathbb{N}} \psi_n$ is $\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{G}$-measurable as well. Taking the limit on both sides using bounded convergence for conditional expectations ([Bal17, Proposition 4.2(c)]) and *Bounded Convergence* (Theorem A.2.2), we obtain

$$\mathbb{E}[\lim_{n\to\infty} \psi_n(Z,\cdot)|\mathscr{D}] \overset{a.s.}{=} \lim_{n\to\infty} \psi_n(Z),$$

where

$$\lim_{n\to\infty} \psi_n(z) = \mathbb{E}[\lim_{n\to\infty} \psi_n(z,\cdot)].$$

That the equality holds almost surely is sufficient in this case, as we are working with conditional expectations. We conclude that the lemma holds true for all bounded $\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{G}$-measurable functions.
Let ψ be a $(\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{G})$-measurable function, such that $\psi(Z(\omega),\omega)$ is integrable. Define $\psi^+(z,\omega) := 0 \vee \psi(z,\omega)$ and $\psi^-(z,\omega) = -(\psi(z,\omega) \wedge 0)$, which are both $(\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{G})$-measurable functions and $\psi^+(Z(\omega,),\omega)$ and $\psi^-(Z(\omega),\omega)$ are integrable. Remark that for $K \in \mathbb{N}$, $\psi^+(z,\omega) \wedge K \in \mathscr{H}$ as it is bounded and $(\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{G})$-measurable. Therefore the lemma holds true for $\psi^+(z,\omega) \wedge K$. As $\psi^+(z,\omega) \wedge K$ goes to $\psi^+(z,\omega)$, and $\psi^+(Z(\omega),\omega)$ is integrable, we refer to [Bal17, Proposition 4.2(a)] to conclude that

$$\mathbb{E}[\psi^+(Z,\cdot)|\mathscr{D}] \overset{a.s.}{=} \lim_{K\to\infty} \Phi_K^+(Z),$$

where

$$\lim_{K\to\infty} \Phi_K^+(z) = \lim_{K\to\infty} \mathbb{E}[\psi^+(z,\cdot) \wedge K] = \mathbb{E}[\psi^+(z,\cdot)],$$

by *Monotone Convergence* (Theorem A.2.3). The same argument applies for ψ^-, so by linearity of conditional expectation and expectation, we obtain the theorem for ψ. ■

2.7 Integration With Respect to Martingales

introduce integration of *predictable* processes with respect to square integrable martingales. The introduction relies heavily on [IW89, section II.2]. A lot of details will be omitted. In order to make an introduction we will need to be familiar with the following definitions:

Definition 2.7.1. [Bal17, Definition 3.3] A *stopping time* with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ is a mapping

$$\tau : \Omega \to [0,\infty],$$

such that for all $t \geq 0$, $\{\tau \leq t\} \in \mathscr{F}_t$. The *optional σ-algebra* for τ is defined as

$$\mathscr{F}_\tau = \{A \in \bigvee_{t\geq 0} \mathscr{F}_t : A \cap \{\tau \leq t\} \in \mathscr{F}_t \text{ for all } t \geq 0\}.$$

We say that τ is finite if $\mathbb{P}(\tau < \infty) = 1$.

□

Definition 2.7.2. [Bal17, Definition 5.1 and Definition 7.3] Let $M := (M_t)_{t\geq 0}$ be a stochastic process adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$. M is a *martingale* if M_t is integrable for all $t \geq 0$ and that for all $0 \leq s \leq t$

$$\mathbb{E}[M_t|\mathscr{F}_s] = M_s.$$

M is a *local martingale* if there exist a sequence of stopping times, $(\tau_n)_{n\in\mathbb{N}}$, such that for all $\omega \in \Omega$, $\tau_n(\omega) \to \infty$ for $n \to \infty$, and such that for all $n \in \mathbb{N}$, $(M_{t\wedge\tau_n})_{t\geq 0}$ is a martingale. Such a sequence is called a *reducing sequence* of M. Using the wording of [IW89, Definition II.1.7] a locally square integrable martingale is a local martingale such that for all $n \in \mathbb{N}$ and $t \geq 0$, $\mathbb{E}[M^2_{\tau_n\wedge t}] < \infty$.

□

The following definition comes from a lecture in the course MAT4720 (autumn semester 21/22).

Definition 2.7.3. A stochastic process $(\phi_t)_{t\geq 0}$ that is adapted to the filtration $\mathbb{F} := (\mathscr{F}_t)_{t\geq 0}$ is *predictable* if it is measurable with respect to the σ-algebra

$$\mathscr{P}^{\mathbb{F}} := \sigma\{(s,t] \times F : s \leq t, F \in \mathscr{F}_s\}.$$

□

Let $M := (M_t)_{t\geq 0}$ be a right continuous square integrable martingale, adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$, such that $M_0 = 0$ almost surely. Let $\mathscr{M}_2$ be the space of martingales of this type. This space is complete with the metric

$$\parallel M \parallel_{\mathscr{M}_2} = \sum_{n=1}^{\infty} 2^{-n}(\mathbb{E}[M_n^2]^{1/2} \wedge 1),$$

for $M \in \mathscr{M}_2$. There exists a process $(A := (\langle M\rangle_t)_{t\geq 0})$ that almost surely is increasing, right continuous and starting at 0. This process is integrable and $M_t^2 - A_t$ is a martingale. Let $\mathscr{L}_2(M)$ be the space of predictable processes $((\phi_t)_{t\geq 0})$ such that for all $T > 0$,

$$\mathbb{E}[\int_0^T \phi_t^2 dA_t] < \infty.$$

This integration can be done pathwise, as A has finite variation. Define the metric on $\mathscr{L}_2(M)$ by

$$\parallel \phi \parallel_{\mathscr{L}_2(M)} := \sum_{n=1}^{\infty} 2^{-n}(\int_0^n \phi_t^2 dA_t \wedge 1)$$

Identity two elements ϕ and ϕ' in $\mathscr{L}_2(M)$ if $\parallel \phi - \phi' \parallel_{\mathscr{L}_2(M)} = 0$.
Define elementary processes as processes in $\mathscr{L}_2(M)$ of the form

$$\phi_E(t) = \varphi_0 \mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^{\infty} \varphi_{t_i} \mathbb{1}_{(t_i,t_{i+1}]}(t)$$

for $0 = t_0 < t_1 < t_2 < \ldots$ and φ'_{t_i}s that are $\mathscr{F}_{t_i}$-measurable. Remark that such functions are measurable with respect to $\mathscr{P}^{\mathbb{F}}$ and that all left continuous adapted processes from $\mathscr{L}_2(M)$ can be approximated by functions of this type. Let $\phi \in \mathscr{L}_2(M)$ be left continuous. Then the sequence

$$\phi^{(k)}(t) = \phi_0 \mathbb{1}_{\{t=0\}}(t) + \sum_{i=0}^{\infty} \phi_{\frac{i}{n}} \mathbb{1}_{(\frac{i}{n}, \frac{i+1}{n}]}(t)$$

is of the desired form and approximates ϕ pointwise. By [IW89] the space of elementary processes from $\mathscr{L}_2(M)$ is dense in $\mathscr{L}_2(M)$ with respect to the metric given. This means that if we let a stochastic process from $\mathscr{L}_2(M)$ be given, there exist a sequence of functions $(\phi_E^{(k)})_{k\in\mathbb{N}}$ on the form above such that $\| \phi_E^{(k)} - \phi \|_{\mathscr{L}_2(M)} \to 0$ for $k \to \infty$. For elementary functions from $\mathscr{L}_2(M)$, we define the integral with respect to M as

$$\int_0^T \phi_E(t) dM(t) = \sum_{i=0}^{n-1} \varphi_{t_i}(M(t_{i+1}) - M(t_i)) + \varphi_{t_n}(M(t_{n+1} \wedge T) - M(t_n)),$$

where $T \in (t_n, t_{n+1}]$.

Definition 2.7.4. We denote the integral of ϕ_E with respect to M from 0 to T as $I(T)$.

□

In what follows, it will be implied that we integrate an elementary function. The computations of the proof of the following lemma are similar to the ones I delivered as a mandatory exercise in the course MAT4720 (autumn semester 21/22).

Lemma 2.7.5. *$I(T)$ is a martingale and* $\mathbb{E}[(I(T))^2] = E[\int_0^T \phi_E^2(t) dA_t]$.

Proof. (§) Let $s \leq T$ and remark that if $t_i \geq s$ then by the use of the *Tower Property* (Proposition A.1.11)

$$\begin{aligned}\mathbb{E}[\varphi(t_i)(M(t_{i+1}) - M(t_i))|\mathscr{F}_s] &= \mathbb{E}[\mathbb{E}[\varphi(t_i)(M(t_{i+1}) - M(t_i))|\mathscr{F}_{t_i}]|\mathscr{F}_s] \\ &= \mathbb{E}[\varphi(t_i)\mathbb{E}[M(t_{i+1}) - M(t_i)|\mathscr{F}_{t_i}]|\mathscr{F}_s] = 0,\end{aligned}$$

and if $t_i \leq s \leq t_{i+1}$, then

$$\mathbb{E}[\varphi(t_i)(M(t_{i+1}) - M(t_i))|\mathscr{F}_s] = \varphi(t_i)(M(s) - M(t_i))$$

We take a look at the expression $\mathbb{E}[I(T)^2]$. Squaring the integral expression above $(\int_0^T \phi_E(t) dM(t))$ gives rise to a sum of quadratic terms of the form $\varphi^2(t_i)(M(t_{i+1}) - M(t_i))^2$ and mixed terms of the form

$$\varphi(t_k)\varphi(t_l)(M(t_{k+1}) - M(t_k))(M(t_{l+1}) - M(t_l)).$$

It is not hard to show (again by the use of the *Tower Property*) that the expectation of the mixed terms are 0. This means we just need to take the expectation of the sum

$$\sum_{i=0}^{n-1} \varphi(t_i)^2(M(t_{i+1}) - M(t_i))^2 + \varphi(t_n)^2(M(t_{n+1} \wedge T) - M(t_n))^2.$$

By utilizing the fact that $M_t^2 - A_t$ is a martingale we obtain for $t \geq s$

$$\mathbb{E}[M_t^2 - A_t|\mathscr{F}_s] = M_s^2 - A_s = \mathbb{E}[M_s^2 - A_s|\mathscr{F}_s],$$

which implies that

$$\mathbb{E}[M_t^2 - M_s^2|\mathscr{F}_s] = \mathbb{E}[A_t - A_s|\mathscr{F}_s].$$

Moreover, remark that (again using the *Tower Property*)

$$\begin{aligned}\mathbb{E}[\varphi_{t_i}^2(M_{t_{i+1}} - M_{t_i})^2] &= \mathbb{E}[\varphi_{t_i}^2(M_{t_{i+1}}^2 + M_{t_i}^2 - 2M_{t_i}M_{t_{i+1}})] \\ &= \mathbb{E}[\varphi_{t_i}^2(\mathbb{E}[M_{t_{i+1}}^2|\mathscr{F}_{t_i}] + M_{t_i}^2 - 2M_{t_i}\mathbb{E}[M_{t_{i+1}}|\mathscr{F}_{t_i})]] \\ &= \mathbb{E}[\varphi_{t_i}^2(\mathbb{E}[M_{t_{i+1}}^2|\mathscr{F}_{t_i}] + M_{t_i}^2 - 2M_{t_i}^2)]] \\ &= \mathbb{E}[\varphi_{t_i}^2(M_{t_{i+1}}^2 - M_{t_i}^2)].\end{aligned}$$

This means that

$$\mathbb{E}[\varphi_{t_i}^2(M_{t_{i+1}}^2 - M_{t_i}^2)] = \mathbb{E}[\varphi_{t_i}^2(A_{t_{i+1}} - A_{t_i})] = \mathbb{E}[\int_{t_i}^{t_{i+1}} \varphi_{t_i}^2 dA_s].$$

Conclude that

$$\mathbb{E}[I(T)^2] = \mathbb{E}[\int_0^T \phi_E^2(t)dA_t] < \infty.$$

This relies on the fact that such integrals can be split up on disjoint time intervals. ■

This secures that $\| I \|_{\mathscr{M}_2} = \| \phi_E^2 \|_{\mathscr{L}_2(M)}$. Now let $\phi \in \mathscr{L}_2(M)$ and choose a sequence of elementary processes $(\phi^{(k)})_{k\in\mathbb{N}}$ that approximates ϕ. Remark that subtraction of one elementary process from another leads to a new elementary process. Also, we have that the integral operation of elementary processes is linear. Denote by I_n the stochastic process $\int_0^{\cdot} \phi^{(n)}(s)dM_s$ We obtain that

$$\| I_n - I_m \|_{\mathscr{M}_2} = \| \phi^{(n)} - \phi^{(m)} \|_{\mathscr{L}_2(M)} \to 0 \quad \text{for } m, n \to \infty.$$

This makes $(I_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $\mathscr{M}_2$ space. Consequently, as $\mathscr{M}_2$ is complete [IW89, Lemma II.1.2], there is an element $I \in \mathscr{M}_2$ such that $\| I_n - I \| \to 0$ for $n \to \infty$. We will just state that such a limit is unique (independent of the choice of approximating sequence), and we define the integral of ϕ with respect to M as this limit I.

Still following the presentation from [IW89] closely, the theory can be expanded to integration with respect to local martingales. Let $\mathscr{M}_2^{loc}$ be the space of $(\mathscr{F}_t)_{t\geq 0}$-adapted locally square integrable martingales that are 0 at time 0 almost surely. There exist a process locally square integrable process $(A_t)_{t\geq 0}$ that is $\mathbb{F}$-adapted, right continuous, increasing and 0 at 0 almost surely such that $(M_t^2 - A_t)_{t\geq 0}$ is a local martingale. Let $M \in \mathscr{M}_2^{loc}$. Define $\mathscr{L}_2^{loc}(M)$ to be the space of predictable stochastic processes, ϕ such that there exists a sequence of stopping times, $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n \to \infty$ almost surely for $n \to \infty$ and

$$\int_0^{\sigma_n \wedge T} \phi_t^2 dA_t < \infty$$

for all $T > 0$ and $n \in \mathbb{N}$. If we $(\gamma_n)_{n\in\mathbb{N}}$ be a reducing sequence of M. Choosing the sequence $(\tau_n)_{n\in\mathbb{N}}$ to be $\tau_n := \sigma_n \wedge \gamma_n$ this sequence is both a reducing sequence of M by [Bal17, Proposition 5.6] and $\int_0^{\tau_n \wedge T} \phi_t^2 dA_t < \infty$ for all $n \in \mathbb{N}$ and $T > 0$ as the integral is increasing. Therefore for $\phi \in \mathscr{L}_2^{loc}$ it is possible to define the integral of $\mathbb{1}_{\{t\le\tau_n\}}(t)\phi_t$ with respect to $M^{\tau_n} := (M_{t\wedge\tau_n})_{t\ge 0}$ for all $n \in \mathbb{N}$. It is possible to obtain that there exists a stochastic process $I^{loc}(t)$ (see [IW89, p. 57]), such that $I^{loc}(t\wedge\tau_n)$ is the stochastic integral of $\mathbb{1}_{\{t\le\tau_n\}}(t)\phi_t$ with respect to M^{τ_n} (so it is a martingale). Thereby, I^{loc} is a locally square integrable martingale.

2.8 Stochastic Differential Equations

This section relies on [Bal17] and [Øks03]. Let B be a standard Brownian motion as defined in Definition A.3.2. As this is a martingale with $A_t = t$ (one can show that $B_t^2 - t$ is a martingale), it is possible to define an integral of $(v_t)_{t\ge 0}$ with respect to the Brownian motion if $(v_t)_{t\ge 0}$ is a predictable stochastic process such that $\mathbb{E}[\int_0^T v_s^2 ds] < \infty$ (see Section 2.7). It can be shown that the integral is an element in $L^2(\mathbb{P})$. In fact the integration can be defined for a broader class of stochastic processes, namely processes that are progressively measurable and for which it holds that $\mathbb{E}[\int_0^T v_t dB_t] < \infty$. This is proved in [Bal17, Chapter 7].

Definition 2.8.1. [Øks03, Definition 4.1.1] Let B be a standard Brownian motion. $(Y_t)_{t\in[0,T]}$ is a real Itô process if it can be written on the form

$$Y_t = Y_0 + \int_0^t u_s ds + \int_0^t v_s dB_s, \qquad t \in [0, T].$$

Here $(u_t)_{t\in[0,T]}$ and $(v_t)_{t\in[0,T]}$ are real stochastic processes, where $\int_0^t |u_s| ds < \infty$ for all $t \in [0, T]$ almost surely and $\int_0^t u_s^2 ds < \infty$ for all $t \in [0, T]$ almost surely.

□

It can be proved that the space of Itô processes is closed under smooth mappings, a result better known as the Itô formula. We state it here without a proof:

Theorem 2.8.2. *[Øks03, Equation (4.1.9)] Let Y be a Itô process as defined in Definition 2.8.1. Let*

$$g : [0, \infty) \times \mathbb{R} \to \mathbb{R}$$

be a function, that is two times continuously differentiable. That is

$$\frac{\partial g(t,x)}{\partial t}, \frac{\partial g(t,x)}{\partial x}, \frac{\partial^2 g(t,x)}{\partial x \partial t}, \frac{\partial^2 g(t,x)}{\partial x^2}, \frac{\partial^2 g(t,x)}{\partial t^2}$$

are all continuous functions in both variables. Then $(g(t, Y_t))_{t\in[0,T]}$ is an Itô process and

$$\begin{aligned} g(t, Y_t) = g(0, Y_0) + &\int_0^t \left(\frac{\partial g}{\partial s}(s, Y_s) + u_s \frac{\partial g}{\partial x}(s, Y_s) + \frac{1}{2} v_s \frac{\partial^2 g}{\partial x^2}\right) ds \\ &+ \int_0^t v_s \frac{\partial g}{\partial x}(s, Y_s) dB_s, \qquad t \in [0, T]. \end{aligned} \tag{2.2}$$

□

It is also possible to define the multi dimensional stochastic integral with respect to a Brownian motion. We introduce it as done in [Bal17, Section 8.4]). Let $(B_t)_{t\geq 0}$ be an m-dimensional Brownian motion and let $(\mathscr{F}_t)_{t\geq 0}$ be the sigma-algebra generated by B. Let $(\Sigma(t))_{t\in[0,T]}$ be a $Mat(d,m)$-valued stochastic process. Let $(i,j) \in \{1,...,d\} \times \{1,...,m\}$ and assume that $(\Sigma_{i,j}(t))_{t\in[0,T]}$ are progressively measurable and that $\mathbb{E}[\int_0^T \Sigma_{i,j}^2(s)ds] < \infty$ (In this case we also write $\Sigma \in M^2[0,T]$). We can define the stochastic integral of Σ with respect to B, as a d-dimensional vector, where for $i \in \{1,...,d\}$:

$$(\int_0^T \Sigma(s)dB(s))_i = \sum_{j=1}^m \int_0^T \Sigma_{i,j}(s)dB_j(s).$$

We introduce the concept of stochastic differential equations, which is a differential equation containing a path integral part and a stochastic integral part.

Definition 2.8.3. [Bal17, Definition 9.1] A *stochastic differential equation* is an expression on the form

$$\begin{aligned} d\zeta_t &= b(t,\zeta_t)dt + \sigma(t,\zeta_t)dB_t, \qquad t \in [u,T], \\ \zeta_u &= \eta. \end{aligned} \tag{2.3}$$

Here b and σ are measurable mappings, that are defined on $[u,T] \times \mathbb{R}^d$. b takes values in $\mathbb{R}^d$ and σ takes values in $Mat(d,m)$. We assume that η is $\mathscr{F}_u$-measurable. A solution to Equation (2.3) is a stochastic process $(\xi_t)_{s\in[u,T]}$, where

$$\xi_t = \eta + \int_u^t b(s,\xi_s)ds + \int_u^t \sigma(s,\xi_s)dB_s,$$

for all $t \in [u,T]$.

□

For a full proof of the following theorem, see [Bal17, Theorem 9.2]. We will just give the main idea for the proof, because it is constructive and we will use it in Section 3.5 to prove other results.

Theorem 2.8.4. *Assume there exist an $M > 0$ such that for $t \geq 0$ and $x, y \in \mathbb{R}^d$:*

(i) $|\sigma(t,x)|, |b(t,x)| \leq M(1 + |x|)$ (Global linear growth).

(ii) $|\sigma(t,x) - \sigma(t,y)|, |b(t,x) - b(t,y)| \leq M|x - y|$ (Lipschitz condition).

Then there exists a stochastic process from $M^2[u,T]$, which is a solution to Equation (2.3). The solution is unique in the sense, that if there exist two processes $(\zeta_t)_{t\in[u,T]}$ and $(\zeta'_t)_{t\in[u,T]}$ from $M^2[u,T]$ that are solutions to Equation (2.3), then $\zeta_t = \zeta'_t$ for all $t \geq 0$ almost surely. We call such a solution a diffusion.

Proof. (§) The idea of the proof is to follow an iteration process (Picard iteration). Let $\zeta_t^{(0)} := \eta$ for $t \in [u, T]$. Define for $m \in \{1, 2, ...\}$:

$$\zeta_t^{(m)} := \eta + \int_u^t b(s, \zeta_s^{(m-1)}) ds + \int_u^t \sigma(s, \zeta_s^{(m-1)}) dB_s,$$

for $t \in [u, T]$. It can then be shown that $\lim_{m \to \infty} \zeta^{(m)}$ is a continuous element in $M^2[u, T]$, and that it is a solution to the stochastic differential equation Equation (2.3). ■

CHAPTER 3

Markovianity

In this chapter we let $Y := (Y_t)_{t\geq 0}$ be an adapted stochastic process defined on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Moreover $\mathbb{F}^Y := (\mathscr{F}_t^Y)_{t\geq 0}$ denotes the filtration generated by Y. Assume that Y takes values in the measurable space $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$.
The aim of this chapter is to introduce the Markov property for stochastic processes in continuous time.

3.1 Conditional Distributions

In the following section we define the concept of *conditional laws*.

Definition 3.1.1. [Bal17, page 98] Let X and Y be stochastic vectors on a probability space, $(\Omega, \mathscr{F}, \mathbb{P})$, that takes values in $(\mathbb{R}^m, \mathscr{B}(\mathbb{R}^m))$ and $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ respectively. The family of probability measures $(\eta(y, dx))_{y\in\mathbb{R}^d}$ is the conditional law of X given Y if the following two conditions are met:

(a) For every $A \in \mathscr{B}(\mathbb{R}^m)$, $\eta(y, A)$ is a $\mathscr{B}(\mathbb{R}^d)$-measurable function.

(b) For every $A \in \mathscr{B}(\mathbb{R}^m)$ and $B \in \mathscr{B}(\mathbb{R}^d)$,

$$\mathbb{P}(X \in A, Y \in B) = \int_B \eta(y, A)\mathbb{P}_Y(dy).$$

□

The following lemma and proof is inspired by the arguments presented in [Bal17, Section 4.3] and creates the connection between the conditional distribution with respect to Y and the conditional expectation with respect to $\sigma(Y)$.

Lemma 3.1.2. *Let X and Y be defined as in Definition 3.1.1. Define*

$$\eta : \mathbb{R}^d \times \mathscr{B}(\mathbb{R}^m) \to [0, 1]$$

and assume that $\eta(y, A)$ is measurable for fixed $A \in \mathscr{B}(\mathbb{R}^m)$. $(\eta(y, dx))_{y\in\mathbb{R}^d}$ is the conditional law of X given Y if and only if

$$\mathbb{E}[f(X)|Y] = \int_E f(x)\eta(Y, dx)$$

for all $f \in \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$.

Proof. (§) We start off showing the 'only if' part of the proof. Let $\eta(y,dx)_{y\in\mathbb{R}^d}$ be the conditional distribution of X given Y. We show that $\int_{\mathbb{R}^m} f(x)\eta(Y,dx)$ is $\sigma(Y)$-measurable and that

$$\mathbb{E}[\mathbb{E}[f(X)|Y]\mathbb{1}_D] = \mathbb{E}[\int_{\mathbb{R}^m} f(x)\eta(Y,dx)\mathbb{1}_D]$$

for $D \in \sigma(Y)$. This is sufficient by [Bal17, Remark 4.2]. It is possible to rewrite condition (b) to

$$\mathbb{E}[\mathbb{1}_{\{X\in A\}}\mathbb{1}_{\{Y\in B\}}] = \int_B \int_{\mathbb{R}^m} \mathbb{1}_A(x)\eta(y,dx)\mathbb{P}_Y(dy).$$

By *Monotone Class Theorem* (Theorem A.2.4) we obtain

$$\mathbb{E}[f(X)\mathbb{1}_{\{Y\in B\}}] = \int_B \int_{\mathbb{R}^m} f(x)\eta(y,dx)\mathbb{P}_Y(dy),$$

for $f \in \mathscr{M}_b(\mathbb{R}^m,\mathbb{R})$, because $\int_B \eta(y,\cdot)\mathbb{P}_Y(dy)$ is a measure on $\mathscr{B}(\mathbb{R}^m)$. Remark that as $\int_{\mathbb{R}^m} f(x)\eta(y,dx)$ is $\mathscr{B}(\mathbb{R}^d)$-measurable, then $\int_{\mathbb{R}^m} f(x)\eta(Y,dx)$ is $\sigma(Y)$-measurable. Also we have

$$\mathbb{E}[f(X)\mathbb{1}_{\{Y\in B\}}] = \mathbb{E}[\mathbb{E}[f(X)|Y]\mathbb{1}_{\{Y\in B\}}]$$

because $\mathbb{1}_{\{Y\in B\}}$ is $\sigma(Y)$-measurable and that

$$\int_B \int_{\mathbb{R}^m} f(x)\eta(y,dx)\mathbb{P}_Y(dy) = \mathbb{E}[\int_{\mathbb{R}^m} f(x)\eta(Y,dx)\mathbb{1}_{\{Y\in B\}}].$$

As all sets from $\sigma(Y)$ can be written as $\{Y \in B\}$, this is sufficient.
On the other hand, assume that

$$\mathbb{E}[f(X)|Y] = \int_{\mathbb{R}^m} f(x)\eta(Y,dx)$$

for all f from $\mathscr{M}_b(\mathbb{R}^d,\mathbb{R})$. Let $B \in \mathscr{B}(\mathbb{R}^d)$.

$$\begin{aligned}\mathbb{P}(X\in A, Y\in B) &= \mathbb{E}[\mathbb{1}_A(X)\mathbb{1}_B(Y)]\\ &= \mathbb{E}[\mathbb{1}_B(Y)\mathbb{E}[\mathbb{1}_A(X)|Y]]\\ &= \mathbb{E}[\mathbb{1}_B(Y)\eta(Y,A)]\\ &= \int_B \eta(y,A)\mathbb{P}_Y(dy).\end{aligned}$$

■

3.2 The Markov Property

In both [Bal17] and [Sat13] the definition of the Markov property is expressed in terms of Markov transition functions. Here we give a well known alternative definition. Later in the chapter, we show that the definitions are equivalent.

Definition 3.2.1. Let Y be a stochastic process that takes values in $\mathbb{R}^d$ and assume it is adapted to the filtration $\mathbb{F}$. It is said to be an $\mathbb{F}$-Markov process if

$$\mathbb{E}[f(Y_t)|\mathscr{F}_s] = \mathbb{E}[f(Y_t)|Y_s], \tag{3.1}$$

for all $t \geq s \geq 0$ and all $f \in \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$. In this situation it is also said that Y possess *the Markov property*.

Remark 3.2.2. If Y is a Markov process with respect to the filtration $\mathbb{F}$ it is also a Markov process with respect to its natural filtration, $\mathbb{F}^Y$. This comes from taking the conditional expectation on both sides with respect to $\mathscr{F}_s^Y$ and using the *Tower Property* (Proposition A.1.11).

$$\mathbb{E}[\mathbb{E}[f(Y_t)|\mathscr{F}_s]|\mathscr{F}_s^Y] = \mathbb{E}[f(Y_t)|\mathscr{F}_s^Y]$$

because $\mathscr{F}_s^Y \subseteq \mathscr{F}_s$. Also

$$\mathbb{E}[\mathbb{E}[f(Y_t)|Y_s]|\mathscr{F}_s^Y] = \mathbb{E}[f(Y_t)|Y_s]$$

because $\sigma(Y_s) \subseteq \mathscr{F}_s^Y$.

Remark 3.2.3. If for all $t \geq s \geq 0$,

$$\mathbb{E}[f(Y_t)|\mathscr{F}_s] = g_{s,t} \circ f(Y_s)$$

for real measurable functions $g_{s,t}$, then Y is a Markov process with respect to $\mathbb{F}$. This is true because the equality implies that $\mathbb{E}[f(Y_t)|\mathscr{F}_s]$ is $\sigma(Y_s)$-measurable, which secures Equation (3.1).

□

3.3 Transition Kernels

As mentioned above, Markov transition functions are often used to express the right hand side in equation Equation (3.1). Markov transition functions are based on the concept of *transition kernels*, which we start off defining.

Definition 3.3.1. [Ped20, Definition 4.1] A mapping

$$P : \mathbb{R}^d \times \mathscr{B}(\mathbb{R}^d) \to [0, 1]$$

is a transition kernel if the following conditions hold:

(i) $A \mapsto P(x, A)$ is a probability measure for $x \in \mathbb{R}^d$.

(ii) $x \mapsto P(x, A)$ is a measurable function for $A \in \mathscr{B}(\mathbb{R}^d)$.

□

The following lemma and proof are based on [Ped20, Remark 4.4].

Lemma 3.3.2. *Let P be a transition kernel and let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded and measurable function. For $x \in \mathbb{R}^d$ it is possible to integrate f with respect to the measure $P(x, \cdot)$. We write*

$$Pf(x) := \int_{\mathbb{R}^d} f(y)P(x, dy). \tag{3.2}$$

$Pf(x)$ is measurable.

Proof. (§§) Let $x \in \mathbb{R}^d$. We argue that $Pf(x)$ is measurable function by *Monotone Class Theorem* (Theorem A.2.4). $\int \mathbb{1}_A(y)P(x,dy) = P(x,A)$ is by the definition above a measurable function. Assume that f and g are bounded and measurable and that $Pf(x)$ and $Pg(x)$ are measurable, then $P(f(x)+g(x)) = Pf(x)+Pg(x)$ are measurable and $P(cf(x)) = cPf(x)$ are measurable by the properties of integrals. Let $(f_n(x))_{n\in\mathbb{N}}$ be an increasing sequence of bounded and measurable functions such that $Pf_n(x)$ is measurable and such that $\lim_{n\to\infty} f_n(x)$ is bounded. $Pf_n(x)$ is an increasing measurable function. Then we have, that $\lim_{n\to\infty} Pf_n(x) = Pf(x)$ is measurable and bounded by *Bounded Convergence* (Theorem A.2.2). Thus $Pf(x)$ is a measurable function for all measurable and bounded functions, f. ∎

Define for $(x,A) \in (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ $PQ(x,A) := \int Q(y,A)P(x,dy)$ as in [Ped20, Definition 5.1]. We show that this is in fact a transition semigroup. As $Q(y,A)$ is a measurable and bounded function, we must have that $PQ(x,A) = \int Q(y,A)P(x,dy)$ is a measurable and bounded function with respect to x by the previous lemma. Also $PQ(x,A)$ is a probability measure because

$$PQ(x,\mathbb{R}^d) = \int Q(y,\mathbb{R}^d)P(x,dy) = \int \mathbb{1}P(x,dy) = P(x,\mathbb{R}^d) = 1.$$

We also have by an analogous calculation that $PQ(x,\emptyset) = 0$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of disjoint sets from $\mathscr{B}(\mathbb{R}^d)$.

$$\begin{aligned} PQ(x,\cup_{n\in\mathbb{N}}A_n) &= \int Q(y,\cup_{n\in\mathbb{N}}A_n)P(x,dy) \\ &= \int \lim_{k\to\infty}\sum_{i=0}^{k} Q(y,A_i)P(x,dy) \\ &= \lim_{k\to\infty}\int \sum_{i=1}^{k} Q(y,A_i)P(x,dy) \\ &= \sum_{i=1}^{\infty} PQ(x,A_i), \end{aligned}$$

where *Monotone Convergence* (Theorem A.2.3) is applied. By the definition of measures (see e.g. Definition A.1.2) $PQ(x,\cdot)$ is a probability measure. It is not hard to see that if R is also a transition kernel, we have that $(RP)Q = R(PQ)$ (associativity).

Definition 3.3.3. [Sat13, Definition 10.1] A family of transition kernels $(P_{s,t})_{t\geq s\geq 0}$ is called an *Markov transition function* if it fulfills the following two properties.

(i) $P_{s,s}(x,A) = \delta_x(A)$ for all $s \geq 0$.

(ii) $P_{u,s}P_{s,t} = P_{u,t}$ for all $t \geq s \geq u \geq 0$.

Remark 3.3.4. For $t \geq s \geq u \geq 0$ and $f \in \mathscr{M}_b(\mathbb{R}^d,\mathbb{R})$:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(z)P_{s,t}(y,dz)P_{u,s}(x,dy) = P_{u,s}P_{s,t}f(x)$$

$$= P_{u,t}f(x) = \int_{\mathbb{R}^d} f(y)P_{u,t}(x, dy).$$

Remark 3.3.5. In the case where $P_{t,t+s} = P_{t',t'+s}$ for all $s, t, t' \geq 0$, the Markov semigroup can be written more compactly as $(P_s)_{s\geq 0}$, where $P_s = P_{t,t+s}$ for $t, s \geq 0$. In this case the group is called a *time homogeneous Markov transition function.*

□

The following definition of the Markov property is most common and is the content of for example [Bal17, Definition 6.1].

Definition 3.3.6. Y is an $\mathbb{F}$-Markov process with respect to the Markov transition function $(P_{s,t})_{t\geq s\geq 0}$ if for any bounded measurable functions $f : \mathbb{R}^d \to \mathbb{R}$, we have that

$$\mathbb{E}[f(Y_t)|\mathscr{F}_s] = P_{s,t}(f(Y_s)), \tag{3.3}$$

for $t \geq s \geq 0$.

Remark 3.3.7. By Lemma 3.3.2 $P_{s,t}(f(x))$ is a measurable function, so by Remark 3.2.3, Equation (3.3) secures that Y is indeed a $\mathbb{F}$-Markov process.

□

We show now that the Markov property that are given in Equation (3.1) and Equation (3.3) are equivalent.

Lemma 3.3.8. *Let $(Y_t)_{t\geq 0}$ be a an $\mathbb{F}$-Markov process. Then there exist a Markov transition function $(P_{s,t})_{t\geq s\geq 0}$, such that Y is a Markov process with respect Markov transition function $(P_{s,t})_{t\geq s\geq 0}$.*

Proof. (§§§) Let $(\eta_{s,t}(y, dx))_{y\in\mathbb{R}^d}$ be the conditional distribution of Y_t given Y_s. We show that this is the Markov transition function we are looking for. Then by Lemma 3.1.2 for $f \in \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$,

$$\mathbb{E}[f(Y_t)|Y_s] = \int_{\mathbb{R}^d} f(x)\eta_{s,t}(Y_s, dx).$$

If we use the notation $\eta_{s,t}(f(Y_s)) := \int_{\mathbb{R}^d} f(x)\eta_{s,t}(Y_s, dx)$ we see that it fits Equation (3.3). We just need to show that $(\eta_{s,t})_{t\geq s\geq 0}$ lives up to Definition 3.3.3.Let $0 \leq u \leq s \leq t$.

$$\begin{aligned}
\mathbb{E}[f(Y_t)|Y_u] &= \mathbb{E}[\mathbb{E}[f(Y_t)|\mathscr{F}_s]|Y_u] \\
&= \mathbb{E}[\mathbb{E}[f(Y_t)|Y_s]|Y_u] \\
&= \mathbb{E}[\int f(x)\eta_{s,t}(Y_s, dx)|Y_u] \\
&= \int\int f(x)\eta_{s,t}(y, dx)\eta_{u,s}(Y_u, dy)
\end{aligned}$$

Thus $\eta_{u,s}\eta_{s,t} = \eta_{u,t}$, because $\eta_{u,t}(f(Y_u)) = \mathbb{E}[f(Y_t)|Y_u]$. Moreover

$$\mathbb{E}[f(Y_u)|Y_u] = f(Y_u) = \int_{\mathbb{R}^d} f(x)\delta_{Y_u}(dx).$$

We conclude that $(\eta_{s,t})_{t\geq s\geq 0}$ is the Markov transition function associated to Y. ■

A third equivalent way of expressing the Markov property is the one given in the following lemma. It is for example noted in [Bal17, Remark 6.1]. The proof is a straight forward application of the *Monotone Class Theorem*, but for completeness we give the full proof.

Lemma 3.3.9. *Let $(P_{s,t})_{t\geq s\geq 0}$ be a Markov transition function. Y is a $\mathbb{F}$-Markov process associated with $(P_{s,t})_{t\geq s\geq 0}$ if and only if for all $A \in \mathscr{B}(\mathbb{R}^d)$*

$$\mathbb{P}(Y_t \in A|\mathscr{F}_s) = P_{s,t}(Y_s, A). \tag{3.4}$$

Proof. (§§) Let $A \in \mathscr{B}(\mathbb{R}^d)$ and $f(x) := \mathbb{1}_A(x)$, which is obviously measurable and bounded. If we set $f(x)$ into (3.3) we obtain 3.3.
On the other hand, we assume that Equation (3.4) holds true. We prove it by using the Monotone Class Theorem (Theorem A.2.4). We choose $\mathscr{H}$ to be the set of functions for which Equation (3.3) hold true. As

$$P_{s,t}(Y_s, A) = \int \mathbb{1}_A(y)P_{s,t}(Y_s, dy)$$

and

$$\mathbb{E}(\mathbb{1}_A(Y_t)|\mathscr{F}_s) = \mathbb{P}(Y_t \in A|\mathscr{F}_s),$$

we get that $1_A \in \mathscr{H}$ for all $A \in \mathscr{B}(\mathbb{R}^d)$. We need to verify (2) and (3) from Theorem A.2.4 holds true. Property (2) is obvious by the linearity of conditional expectation. Assume that $(f_n)_{n\in\mathbb{N}} \in \mathscr{H}$ is a sequence of positive, non decreasing functions and define $f := \lim_{n\to\infty} f_n$, which is bounded. Then by [Bal17, Proposition 4.2]

$$\lim_{n\to\infty} \mathbb{E}[f_n(Y_t)|\mathscr{F}_s] = \mathbb{E}[f(Y_t)|\mathscr{F}_s]$$

almost surely, and by *Monotone Convergence*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)P_{s,t}(Y_s, dx) = \int_{\mathbb{R}^d} f(x)P_{s,t}(Y_s, dx).$$

This validates property (3). ■

3.4 Distribution of Markov Processes

Definition 3.4.1. Let Y be a Markov process. The measure $\mu := \mathbb{P}_{Y_0}$ is called the *initial distribution* of Y.

□

The following theorem shows how to express the finite dimensional distribution of a Markov process in terms of the corresponding Markov transition function. The result is taken from [Ped20, p. 48] and the proof is generalized from the proof of the similar result for Markov processes in discrete time ([Ped20, Theorem 4.17]).

Lemma 3.4.2. *$Y := (Y_t)_{t\geq 0}$ is a Markov process associated to the Markov transition function $(P_{s,t})_{t\geq s\geq 0}$ if and only if for all $0 \leq t_1 < t_2 < t_3 < ... < t_n$ and $A_1, A_2, ..., A_n \in \mathscr{B}(\mathbb{R}^d)$*

$$\begin{aligned}\mathbb{P}(Y_{t_1} \in A_1, ..., Y_{t_n} \in A_n) = \int_{\mathbb{R}^d} \int_{A_1} \cdots \int_{A_n} \mathbb{1} \\ P_{t_{n-1},t_n}(y_{n-1}, dy_n) \cdots P_{0,t_1}(y_0, dy_1)\mathbb{P}_{Y_0}(dy_0).\end{aligned} \tag{3.5}$$

Proof. (§§) Assume that $\mathbb{E}[f(Y_t)|\mathscr{F}_s] = P_{s,t}(f(Y_s))$ for all $f \in \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$ and $t \geq s \geq 0$. We prove the 'only if' part of the statement by induction in n. Let $n = 1$, $t > 0$ and $A \in \mathscr{B}(\mathbb{R}^d)$. Then we have that

$$\begin{aligned}\mathbb{P}(Y_t \in A) &= \mathbb{E}[\mathbb{1}_A(Y_t)] = \mathbb{E}[\mathbb{E}[\mathbb{1}_A(Y_t)|\mathscr{F}_0]] = \mathbb{E}[P_{0,t}(\mathbb{1}_A(Y_0))] \\ &= \int_{\mathbb{R}^d} \int_A \mathbb{1} P_{0,t}(y, dx)\mathbb{P}_{Y_0}(dy).\end{aligned}$$

We start off proving that if for $A_1, ..., A_n \in \mathscr{B}(\mathbb{R}^d)$:

$$\mathbb{E}(\mathbb{1}_{A_1\times\cdots\times A_n}(Y_{t_1}, ..., Y_{t_n})) = \int_{\mathbb{R}^d} \int \mathbb{1}_{A_1\times\cdots\times A_n}(y_1, ..., y_n)P_{(t_1,...,t_n)}(dy)\mu(dy_0),$$

where

$$P_{(t_1,...,t_n)}(dy) := P_{t_{n-1},t_n}(y_{n-1}, dy_n) \cdots P_{0,t_1}(y_0, dy_1)$$

then we have that

$$\mathbb{E}[f(Y_{t_1}, ..., Y_{t_n})] = \int_{\mathbb{R}^d} \int f(y_1, ..., y_n)P_{(t_1,...,t_n)}(dy)\mu(dy_0) \tag{3.6}$$

for all $f \in \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$.
Define $\mathscr{A} := \{A_1 \times \cdots \times A_n : A_1, ..., A_n \in \mathscr{B}(\mathbb{R}^d)\}$ and let $\mathscr{H}$ be the collection of measurable functions fulfilling (3.6). We will show that (1)-(3) in Theorem A.2.4 holds true for this choice of $\mathscr{A}$ and $\mathscr{H}$.
(1): A set from $\mathscr{A}$ has the form $A_1 \times \cdots \times A_n$, and we have assumed that functions of the form $\mathbb{1}_{A_1\times\cdots\times A_n}$ are in $\mathscr{H}$.
(2): By the general properties of integrals, we have that if f, g is bounded and measurable and fulfills equation (3.6), then we have that

$$\begin{aligned}\mathbb{E}[f(Y_{t_1}, ..., Y_{t_n}) + g(Y_{t_1}, ..., Y_{t_n})] &= \mathbb{E}[f(Y_{t_1}, ..., Y_{t_n})] + \mathbb{E}[g(Y_{t_1}, ..., Y_{t_n})] \\ &= \int_{\mathbb{R}^d} \int f(y_1, ..., y_n)P_{(t_1,...,t_n)}(dy)\mu(dy_0) \\ &\quad + \int_{\mathbb{R}^d} \int g(y_1, ..., y_n)P_{(t_1,...,t_n)}(dy)\mu(dy_0) \\ &= \int_{\mathbb{R}^d} \int (f(y_1, ..., y_n) + g(y_1, ..., y_n))P_{(t_1,...,t_n)}(dy)\mu(dy_0).\end{aligned}$$

The same principle applies for cf.
(3): We let $(f_k)_{k\in\mathbb{N}}$ be an increasing sequence in $\mathscr{H}$. By *Monotone Convergence* (Theorem A.2.3)

$$\begin{aligned}\mathbb{E}[\lim_{n\to\infty} f_k(Y_{t_1}, ..., Y_{t_n})] &= \lim_{k\to\infty} \mathbb{E}[f_k(Y_{t_1}, ..., Y_{t_n})] \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^d} \int f_k(y_1, ..., y_n)P_{(t_1,...,t_n)}(dy)\mu(dy_0) \\ &= \int_{\mathbb{R}^d} \int \lim_{k\to\infty} f_k(y_1, ..., y_n)P_{(t_1,...,t_n)}(dy)\mu(dy_0).\end{aligned}$$

We conclude that $\mathscr{H}$ contains all bounded functions that are measurable with respect to $\sigma(\mathscr{A}) = \mathscr{B}(\mathbb{R}^d)^n$. Now assume that the statement holds true for $n \geq 1$. Let $0 < t_1 < t_2 \cdots < t_n < t_{n+1}$ and $A_1, ..., A_n, A_{n+1} \in \mathscr{B}(\mathbb{R}^d)$.

$$\begin{aligned}
&\mathbb{P}(Y_{t_1} \in A_1, ..., Y_{t_n} \in A_n, Y_{t_{n+1}} \in A_{n+1}) \\
&\qquad = \mathbb{E}[\mathbb{E}[\mathbb{1}_{A_1\times\cdots\times A_n\times A_{n+1}}(Y_{t_1}, ..., Y_{t_n}, Y_{t_{n+1}})|\mathscr{F}_{t_n}]] \\
&\qquad = \mathbb{E}[\mathbb{1}_{A_1\times\cdots\times A_n}(Y_{t_1}, ..., Y_{t_n})\mathbb{E}[\mathbb{1}_{A_{n+1}}(Y_{t_{n+1}})|\mathscr{F}_{t_n}]] \\
&\qquad = \mathbb{E}[\mathbb{1}_{A_1\times\cdots\times A_n}(Y_{t_1}, ..., Y_{t_n})P_{t_n,t_{n+1}-t_n}(\mathbb{1}_{A_{n+1}}(Y_{t_n}))].
\end{aligned}$$

Here we used the *Tower Property*, the Markov property of Y and that $\mathbb{1}_{A_1\times\cdots\times A_n} = \mathbb{1}_{A_1\times\cdots\times A_{n-1}}\mathbb{1}_{A_n}$. By the induction assumption and the fact that

$$f(Y_{t_1}, ..., Y_{t_n}) := \mathbb{1}_{A_1\times\cdots\times A_n}(Y_{t_1}, ..., Y_{t_n})P_{t_n,t_{n+1}-t_n}(\mathbb{1}_{A_{n+1}}(Y_{t_n}))$$

is bounded and measurable we can write

$$\begin{aligned}
&\mathbb{E}[\mathbb{1}_{A_1\times\cdots\times A_n}(Y_{t_1}, ..., Y_{t_n})P_{t_n,t_{n+1}-t_n}(\mathbb{1}_{A_{n+1}}(Y_{t_n}))] \\
&\quad = \int_{\mathbb{R}^d}\int\cdots\int \mathbb{1}_{A_1\times\cdots\times A_n}(y_1, ..., y_n)P_{t_n,t_n}(\mathbb{1}_{A_{n+1}}(y_n)) \\
&\qquad\qquad P_{t_{n-1},t_n}(y_{n-1}, dy_n)\cdots P_{0,t_1}(y_0, dy_1)\mathbb{P}_{Y_0}(dy_0),
\end{aligned}$$

which can be rewritten to the desired form.
We show the 'if part' of the statement. We assume that we can write the finite dimensional distribution of $(Y_t)_{t\geq 0}$ as in Equation (3.5). We want to show that

$$\mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{Y_t\in B\}}] = \mathbb{E}[\mathbb{1}_A P_{s,t}(Y_s, B)]$$

for all $A \in \mathscr{F}_s^Y$ and $t \geq s$. We start of showing the identity for a intersection stable generator system for $\mathscr{F}_s^Y$, namely $\mathscr{G} := \{\{Y_{u_1} \in R_1, ..., Y_{u_n} \in R_n\} : n \in \mathbb{N}, 0 \leq u_1 < \cdots < u_n \leq s, R_1, ..., R_n \in \mathscr{B}(\mathbb{R}^d)\}$ (see Section 2.3). Now we choose a set $A \in \mathscr{G}$. Remark we can always assume that $u_n = s$. We write $\mathbb{1}_A(\omega) = \mathbb{1}_{R_1\times\cdots\times R_n}(Y_{u_1}, ..., Y_{u_n})$. For $h > 0$, we have that

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{Y_t\in B\}}] &= \mathbb{E}[\mathbb{1}_{R_1\times\cdots\times R_n\times B}(Y_{u_1}, ..., Y_{u_{n-1}}, Y_s, Y_t)] \\
&= \int_{\mathbb{R}^d}\int_{R_1}\cdots\int_{R_n}\int_B \mathbb{1}P_{s,t}(y_n, dy_{n+1})P_{u_{n-1},s}(y_{n-1}, dy_n) \\
&\qquad \cdots P_{0,u_1}(y_0, dy_1)\mu_0(dy_0) \\
&= \int_{\mathbb{R}^d}\int_{R_1}\cdots\int_{R_n} P_{s,t}(y_n, B)P_{u_{n-1},s}(y_{n-1}, dy_n) \\
&\qquad \cdots P_{0,u_1}(y_0, dy_1)\mu_0(dy_0) \\
&= \mathbb{E}[\mathbb{1}_A P_{s,t}(Y_s, B)].
\end{aligned}$$

This shows the Markov property ($\mathbb{P}(Y_t \in B|\mathscr{F}_s^Y) = P_{s,t}(Y_s, B)$) by in [Bal17, Remark 4.4] because for $A \in \mathscr{F}_s^Y$

$$\mathbb{E}[\mathbb{1}_A\mathbb{E}[1_{\{Y_t\in B\}}|\mathscr{F}_s^Y]] = \mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{Y_t\in B\}}]$$

and that $P_{s,t}(Y_s, B)$ is $\mathscr{F}_s^Y$-measurable. ■

The proof of the following theorem is by and large the same as given in [Ped20, Lemma 7.3].

Theorem 3.4.3. *Let $(P_{s,t})_{t\geq s\geq 0}$ be an Markov transition group on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. Then there exist a stochastic process, $(Y_t)_{t\geq 0}$, that meets the condition that*

$$\mathbb{E}[f(Y_t)|\mathscr{F}_s^Y] = P_{s,t}(f(Y_s))$$

for all $t \geq s \geq 0$ and for all measurable functions, f. Here $(\mathscr{F}_t^Y)_{t\geq 0}$ is the natural filtration for $(Y_t)_{t\geq 0}$.

Proof. (§) Define the measure on $\mathbb{R}^{dn}$

$$\begin{aligned}&\nu_{t_{\sigma(1)},\dots,t_{\sigma(n)}}(B_{\sigma(1)} \times \cdots \times B_{\sigma(n)})\\ &\quad= \int_{\mathbb{R}^d}\int_{B_1}\cdots\int_{B_n} 1 P_{t_{n-1},t_n}(y_{n-1}, dy_n)\cdots P_{0,t_1}(y_0, dy_1)\mu_0(dy_0).\end{aligned}$$

for all permutations, σ, on $\{1, ..., n\}$, where $0 \leq t_1 < t_2 < \cdots < t_n$. Let $(s_1, ..., s_m) \in [0, \infty)^m$. If $s_1 = s_2$ we define

$$\begin{aligned}&\nu_{s_1,s_2,s_3,\dots,s_m}(B_1 \times B_2 \times B_3 \times \cdots \times B_m)\\ &\qquad\qquad= \nu_{s_1,s_3,\dots,s_m}(B_1 \cap B_2 \times B_3 \times \cdots \times B_m),\end{aligned}$$

for all $B_1, ..., B_m \in \mathscr{B}(\mathbb{R}^d)$.
We want to show that this measure fulfills the two conditions in Theorem A.1.15. We see right away that this measure fulfills (A.1).
We want to show that it also fulfills (A.2). It will be sufficient to show that

$$\begin{aligned}&\nu_{t_1,\dots,t_i,\dots,t_n}(B_1 \times \cdots \times \mathbb{R}^d \times \cdots \times B_n)\\ &\qquad= \nu_{t_1,\dots,t_{i-1},t_{i+1},\dots,t_n}(B_1 \times \cdots B_{i-1} \times B_{i+1} \times \cdots \times B_n),\end{aligned}$$

for all $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \cdots < t_n$. We make the following calculation:

$$\begin{aligned}&\int_{\mathbb{R}^d}\cdots\int_{B_{i-1}}\int_{\mathbb{R}^d}\int_{B_{i+1}}\cdots\int_{B_n} \mathbb{1}P_{t_{n-1},t_n}(y_{n-1,dy_n})\\ &\qquad\cdots P_{t_i,t_{i+1}}(y_i, dy_{i+1})P_{t_{i-1},t_i}(y_{i-1}, dy_i)\cdots P_{0,t_1}(y_0, dy_1)\mu_0(dy_0)\\ &= \int_{\mathbb{R}^d}\cdots\int_{B_{i-1}}\int_{B_{i+1}}\cdots\int_{B_n} \mathbb{1}P_{t_{n-1},t_n}(y_{n-1,dy_n})\\ &\qquad\cdots P_{t_{i-1},t_{i+1}}(y_{i-1}, dy_{i+1})P_{t_{i-2},t_{i-1}}(y_{i-2}, dy_{i-1})\\ &\qquad\quad\cdots P_{0,t_1}(y_0, dy_1)\mu_0(dy_0),\end{aligned}$$

which gives us that

$$\begin{aligned}&\nu_{t_1,\dots,t_{i-1},t_i,t_{i+1},\dots,t_n}(B_1 \times \cdots \times B_{i-1} \times \mathbb{R}^d \times B_{i+1} \times \cdots \times B_n)\\ &\qquad= \nu_{t_1,\dots,t_{i-1},t_{i+1},\dots,t_n}(B_1 \times \cdots \times B_{i-1} \times B_{i+1} \times \cdots \times B_n).\end{aligned}$$

By Theorem A.1.15 we have that there exist a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a stochastic process $(Y_t)_{t\geq 0}$ on the space such that for all $n \in \mathbb{N}, 0 \leq t_1 < t_2 < \cdots < t_n$ and $B_1, ..., B_n \in \mathscr{B}(\mathbb{R}^d)$:

$$\mathbb{P}(Y_{t_1} \in B_1, ..., Y_{t_n} \in B_n) = \nu_{t_1,\dots,t_n}(B_1 \times \cdots \times B_n).$$

By Lemma 3.4.2 we have that $(Y_t)_{t\geq 0}$ is a Markov process with the Markov transition group $(P_{s,t})_{t\geq s\geq 0}$. ∎

Remark 3.4.4. We notice here, that we can choose the distribution of Y_0 freely.

In [Bal17, Example 6.1] the example of a Brownian motion as a Markov process is given. This is shown by specifying the Markov transition function of the Brownian motion and then showing that Equation (3.3) holds true in this case. Instead, in the following example, we show that the finite dimensional distribution of a Brownian motion can be expressed as in Lemma 3.4.2, with the Markov transition function given in [Bal17, Example 6.1].

Example 3.4.5. (§§) Define a transition kernel $(P_t)_{t\geq 0}$ as

$$P_t(x, A) := \int_A \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-x)^2}{2t})dy,$$

for $t > 0, x \in \mathbb{R}$ and $A \in \mathscr{B}(\mathbb{R})$. In [Bal17, Example 6.1] they show it is a Markov transition function.
Let $B = (B_t)_{t\geq 0}$ be a one-dimensional Brownian motion and let $(\mathscr{F}_t)_{t\geq 0}$ be the filtration generated B. We show that the finite dimensional distribution of the Brownian motion has the distribution described in Equation (3.5). Let $A_1, ..., A_n \in \mathscr{B}(\mathbb{R})$, $t_1 < ... < t_n$ and let $\mu_{X|Y}$ be the conditional law of X given Y. By using theory for conditional Gaussian laws from [Bal17, Section 4.4], we find that

$$B_{t_n}|(B_{t_{n-1}}, ..., B_1) \sim N(B_{t_{n-1}}, t_n - t_{n-1}) = P_{t_n - t_{n-1}}(B_{t_{n-1}}, \cdot).$$

This gives us that

$$B_{t_n}|(B_{t_{n-1}}, ..., B_1) \sim B_{t_n}|B_{t_{n-1}}$$

So we have that

$$\mathbb{E}[\mathbb{1}_{\{B_{t_n}\in A_n\}}|(B_{t_{n-1}}, ..., B_{t_1})] = P_{t_n - t_{n-1}}(B_{t_{n-1}}, A_n),$$

and that

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{B_{t_{n-1}}\in A_{n-1}\}} P_{t_n - t_{n-1}}(B_{t_{n-1}}, A_n)|(B_{t_{n-2}}, ..., B_{t_1})]\\ = \int_{A_{n-1}} P_{t_n - t_{n-1}}(y, A_n) P_{t_{n-1} - t_{n-2}}(B_{t_{n-2}}, dy),\end{aligned}$$

and then we can take the conditional expectation of the expression

$$\mathbb{1}_{\{B_{t_{n-2}}\in A_{n-2}\}} \int_{A_{n-1}} P_{t_n - t_{n-1}}(y, A_n) P_{t_{n-1} - t_{n-2}}(B_{t_{n-2}}, dy)$$

with respect to $(B_{t_{n-3}}, ..., B_{t_1})$ to arrive at

$$\int_{A_{n-2}} \int_{A_{n-1}} P_{t_n - t_{n-1}}(y_{n-1}, A_n) P_{t_{n-1} - t_{n-2}}(y_{n-2}, dy_{n-1}) P_{t_{n-2} - t_{n-3}}(B_{t_{n-3}}, dy_{n-2}).$$

By successive use of the *Tower Property* and by using the system we just derived, we obtain:

$$\begin{aligned}\mathbb{P}(B_{t_1} \in A_1, ..., B_{t_1} \in A_n) = \mathbb{E}[\mathbb{1}_{\{B_{t_n}\in A_n, ..., B_{t_1}\in A_{t_1}\}}]\\ = \int_{A_1} \cdots \int_{A_n} \mathbb{1} P_{t_n - t_{n-1}}(x_{n-1}, dx_n) \cdots P_{t_1}(0, dx_1).\end{aligned}$$

The 0 comes from the fact that B_0 almost surely. By Lemma 3.4.2 we obtain that the Brownian motion is a Markov process.

□

3.5 Example: Diffusion

We argue that a diffusion (defined in Theorem 2.8.4) is a Markov process, following the presentation from [Bal17, Section 9.7].
Let $B := (B_t)_{t\geq 0}$ be an m-dimensional Brownian motion and let $(\mathscr{F}_t)_{t\geq 0}$ be the filtration generated by B. Let σ and b be defined as in Section 2.8 fulfilling condition (i) and (ii) from Theorem 2.8.4. By Theorem 2.8.4 there exists a stochastic process $\xi := (\xi_s)_{s\in[u,T]} \in M^2[u,T]$, such that

$$\xi_t = \eta + \int_u^t b(s,\xi_s)ds + \int_u^t \sigma(s,\xi_s)dB_s, \qquad t\in[u,T]. \tag{3.7}$$

We show that the multidimensional stochastic integral can be approximated almost surely by a multidimensional integral of elementary processes (as it is the case for the one dimensional stochastic integral).

Lemma 3.5.1. *Let $(\Sigma(t))_{t\in[0,T]} \in M^2[0,T]$ be a stochastic process, that takes values in $Mat(d,m)$. Then there exist a sequence of elementary stochastic processes $(\Sigma^{(k)})_{k\in\mathbb{N}}$, that takes values in $Mat(d,m)$, such that $\int_0^T \Sigma^{(k)}(s)dB_s \to \int_0^T \Sigma(s)dB_s$ almost surely for $k\to\infty$.*

Proof. (§§§) Let $(i,j) \in \{1,...,m\}\times\{1,...,d\}$. As $\Sigma_{i,j}$ is in $M^2[0,T]$, the stochastic integral $\int_0^T \Sigma_{i,j}(s)dB_j(s)$ is defined as the L^2 limit of the sequence $\int_0^T \Sigma_{i,j}^{(n)}(s)dB_j(s)$, where $(\Sigma_{i,j}^{(n)})_{n\in\mathbb{N}}$ is an approximating sequence of elementary processes (see [Bal17, Lemma 7.2] for existence of such a sequence of processes). By [Bal17, Proposition 1.5] this implies that there exist a subsequence $(n_k^{i,j})_{k\in\mathbb{N}}$ such that the convergence is almost sure, that is

$$\int_0^T \Sigma_{i,j}^{(n_k^{i,j})}(s)dB_j(s) \overset{a.s.}{\to} \int_0^T \Sigma_{i,j}(s)dB_j(s), \tag{3.8}$$

for $k\to\infty$. Recall that the multidimensional stochastic integral is a vector of sums of one dimensional stochastic integrals. This leads to the conclusion that if we let $(\Sigma^{(k)})_{i,j} := \Sigma_{i,j}^{(n_k^{i,j})}$ for all $(i,j)\in\{1,...,d\}\times\{1,...,m\}$, then

$$\int_0^T \Sigma_s^{(k)} dB_s \to \int_0^T \Sigma_s dB_s$$

almost surely for $k\to\infty$. ■

Lemma 3.5.2. *[Bal17, Lemma 9.3] Let $(B_t)_{t\geq 0}$ be an m-dimensional Brownian motion. Let $\mathscr{N}$ be the null sets of $\mathscr{F}$. Let $\mathscr{G}^u := \sigma(\mathscr{N}, \{B_t - B_u : t \geq u\})$ for $u\geq 0$ be the sigma algebra generated by the increments of B after u. Let $\Sigma := (\Sigma_t)_{t\geq 0} \in M^2[0,T]$ be a stochastic process that takes values in $Mat(d,m)$. Assume for all $s\geq u$, that Σ_s is $\mathscr{G}^u$ measurable. Then $\int_u^t \Sigma_s dB_s$ is $\mathscr{G}^u$-measurable.*

Proof. (§) Assume that $(\Sigma_s)_{u\leq s\leq T}$ is of the form $\sum_{i=1}^n \varphi_i \mathbb{1}_{[t_i,t_{i+1})}$ (elementary process), where φ_i is $\mathscr{G}^u$-measurable. This gives us that

$$\int_u^t \Sigma_s dB_s = \sum_{i=1}^n \varphi_i (B_{t_{i+1}} - B_{t_i}),$$

which is again $\mathscr{G}^u$-measurable because $B_{t_{i+1}} - B_{t_i}$ can be written as $(B_{t_{i+1}} - B_u) - (B_{t_i} - B_u)$. Let $(\Sigma^{(n)})_{n\in\mathbb{N}}$ be a sequence of elementary processes such that $\int_u^t \Sigma_s^{(n)} dB_s$ converges almost surely to $\int_u^t \Sigma_s dB_s$. Such a sequence exist by Lemma 3.5.1. By Lemma 2.2.9 we conclude that $\int_u^t \Sigma_s dB_s$ is $\mathscr{G}^u$-measurable. ∎

The following consideration and calculation is taken directly from [Bal17, pp. 275–276]: We can write $\xi_t^{\eta,u}$ for the solution of Equation (3.7). Let $s \in [u,t]$. By utilizing the linearity of integrals, we obtain:

$$\begin{aligned}\xi_t^{u,\eta} &= \eta + \int_u^s b(v,\xi_t^{u,\eta})dv + \int_u^s \sigma(v,\xi_t^{u,\eta})dB_v \\ &\quad + \int_s^t b(v,\xi_t^{u,\eta})dv + \int_s^t \sigma(v,\xi_t^{u,\eta})dB_v \\ &= \xi_s^{u,\eta} + \int_s^t b(v,\xi_t^{u,\eta})dv + \int_s^t \sigma(v,\xi_t^{u,\eta})dB_v = \xi_t^{s,\xi_s^{u,\eta}},\end{aligned}$$

making $\xi_t^{s,\xi_s^{u,\eta}}$ a solution to the stochastic differential equation

$$\begin{aligned}d\zeta_t &= b(t,\zeta_t)dt + \sigma(t,\zeta_t)dB_t, \qquad t \in [s,T], \\ \zeta_s &= \xi_s^{u,\eta}.\end{aligned}$$

The next result is a consequence of Theorem 2.8.4, as a process from $M^2[u,T]$ is adapted. But as we have not shown this result, it is still relevant to give a proof here.

Lemma 3.5.3. *$\xi_s^{u,\eta}$ is $\mathscr{F}_s$-measurable.*

Proof. (§§) We argue that if $(\zeta_t)_{t\in[u,T]}$ is progressively measurable, then $\int_u^\cdot b(v,\zeta_v)dv$ and $\int_u^\cdot \sigma(v,\zeta_v)dB_v$ are progressively measurable as well. $\int_u^r \sigma(v,\zeta_v)dB_v$ is the L^2-limit of $\mathscr{F}_r$-measurable stochastic variables, so there exist a subsequence, such that it is the almost sure limit of $\mathscr{F}_r$-measurable stochastic variables. By Lemma 2.2.9 $\int_u^r \sigma(v,\zeta_v)dB_v$ is $\mathscr{F}_r$-measurable. This holds for all $r \in [u,T]$, so it is adapted to $(\mathscr{F}_t)_{t\geq 0}$. By $\int_u^\cdot \sigma(v,\zeta_v)dB_v$ we understand the continuous modification of the process ([Bal17, page 195]). This implies by Lemma 2.4.3 it is progressively measurable. The following considerations on the Lebesgue integral of stochastic processes is given in [Bal17, Example 2.2]. As the integral $\int_u^r b(v,\zeta_v)dv$ is defined ω by ω it is continuous in r. As

$$\begin{aligned}\Phi_s : [u,s] \times \Omega &\to \mathbb{R}^d \\ (v,\omega) &\mapsto b(v,\zeta_v(\omega))\end{aligned}$$

is $(\mathscr{B}([u,s]) \otimes \mathscr{F}_s, \mathscr{B}^d)$-measurable ($b(t,x)$ is measurable). By *Fubini's Theorem*, the mapping

$$\omega \mapsto \int_u^s \Phi_s(v,\omega)dv = \int_u^s \sigma(v,\zeta_v)dv$$

is $\mathscr{F}_s$-measurable. So it is progressively measurable. As the first element in the Picard iteration $\zeta_t^{(0)} = \eta$ is progressively measurable, the statement is true for all $n \in \mathbb{N}$, and therefore also holds true for the limit. ∎

The relation between a solution to a stochastic differential equation and its starting value will now be examined. We take a look at the stochastic differential equation with the starting value being a constant:

$$\begin{aligned} d\zeta_t &= b(t,\zeta_t)dt + \sigma(t,\zeta_t)dB_t, & t &\in [s,T], \\ \zeta_s &= x & x &\in \mathbb{R}^d \end{aligned} \tag{3.9}$$

A solution to this differential equation can be written as $(\xi_t^{s,x})_{t\in[s,T]}$.

The idea for the proof of the following lemma is given in [Bal17, p. 276]. It is written in full length below.

Lemma 3.5.4. *$\xi_t^{s,x}$ is independent of $\mathscr{F}_s$ for all $t \in [s,T]$.*

Proof. (§) Let $\mathscr{N}$ be the null sets of the complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Remark that $\mathscr{G}^s := \sigma(\mathscr{N}, \{B_t - B_s : t \geq s\})$ is independent of $\mathscr{F}_s$. Again we prove this by using the Picard iteration. We show by induction, that $\zeta_t^{(n)}$ is $\mathscr{G}^s$-measurable for $t \in [s,T]$ and $n \in \mathbb{N}$. Set $\zeta_t^{(0)} := x$ for all $t \in [s,T]$. Assume for an $n \in \mathbb{N}$, that $\zeta_t^{(n)}$ is $\mathscr{G}^s$-measurable for all $t \in [s,T]$. By Lemma 3.5.2 we have that $\int_s^t \sigma(v,\zeta_v^{(n)})dB_v$ is $\mathscr{G}^s$-measurable. Also x and $\int_s^t b(v,\zeta_v^{(n)})dv$ are $\mathscr{G}^s$-measurable, which makes $\zeta_t^{(n+1)}$ $\mathscr{G}^s$-measurable. As $\mathscr{G}^s$ is a σ-algebra, $\xi_t := \lim_{n\to\infty} \zeta_n^{(n)}$ is $\mathscr{G}^s$-measurable and a solution to Equation (3.9). ∎

In [Bal17, Theorem 9.9] it is shown that for every $\omega \in \Omega$; $\zeta_t^{s,x}$ is continuous in all three variables (t,s,x). Especially it is continuous in x. Define $\psi(x,\omega) := \zeta_t^{s,x}(\omega)$. Conclude that the function

$$\begin{aligned} \psi : \mathbb{R}^d \times \Omega &\to \mathbb{R}^d \\ (x,\omega) &\mapsto \psi(x,\omega), \end{aligned}$$

is $(\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{F}, \mathscr{B}(\mathbb{R}^d))$-measurable (see Lemma 2.4.4). Conclude that $\psi(X(\cdot),\cdot) : \Omega \to \mathbb{R}^d$ is also measurable by Lemma 5.1.3, as it is the composition of two measurable functions (Lemma A.1.8(ii)).

Theorem 3.5.5. *[Bal17, p. 276] $(\xi_t^{u,\eta})_{t\in[u,T]}$ is an $(\mathscr{F}_t)_{t\geq 0}$ Markov process.*

Proof. (§) Let $u \leq s \leq t \leq T$ and remark that $\xi_t^{u,\eta} = \xi_t^{s,\xi_s^{u,\eta}}$. By the previous argument, we have that

$$\psi(\xi_s^{u,\eta}(\cdot),\cdot) := \xi_t^{s,\xi_s^{u,\eta}}$$

is measurable. By Lemma 3.5.4 and Lemma 3.5.3 we have that $\psi(x,\omega)$ is independent of $\mathscr{F}_s$ and that $\xi_s^{u,\eta}$ is $\mathscr{F}_s$-measurable. Let $f \in \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$ and use Lemma 2.6.1 to conclude that

$$\mathbb{E}[\xi_t^{u,\eta}|\mathscr{F}_s] = \mathbb{E}[\xi_t^{s,\xi_s^{u,\eta}}|\mathscr{F}_s] = \Phi(\xi_s^{u,\eta}),$$

where $\Phi(x) = \mathbb{E}[\phi(x,\cdot)]$, which is a measurable function by *Fubini's Theorem* (Theorem A.1.10). Thus $\mathbb{E}[\xi_t^{u,\eta}|\mathscr{F}_s]$ is $\sigma(\xi_s^{u,\eta})$-measurable. Hence it is a Markov process with respect to $(\mathscr{F}_t)_{t\geq 0}$. ∎

The following example of the *geometric Brownian motion* as the solution to a differential equation is examined in many textbooks on stochastic calculus (see [Bal17, Example 9.2]). In the following example we will not solve a stochastic differential equation, but instead turn it around and show that the geometric Brownian motion can be written as the solution to a differential equation using the Itô formula.

Example 3.5.6. (§§) Let B be a one dimensional Brownian motion and let $x_0, a, b \in \mathbb{R}$. Define the process

$$\xi_t := x_0 \exp(at + bB_t),$$

for $t \in [0, \infty)$. Now define the function

$$g(t, x) = x_0 \exp(at + bx)$$

for $(t, x) \in [0, \infty) \times \mathbb{R}$. This function is continuously differentiable infinitely often in both variables. Remark also that $B_t = \int_0^t dB_s$. So it is a Itô process with $u = 0$ and $v = 1$. By Theorem 2.8.2 we get that

$$\xi_t = x_0 + \int_0^t (a\xi_s + \frac{1}{2}b^2\xi_s)ds + \int_0^t b\xi_s dB_s.$$

This makes $(\xi_t)_{t\geq 0}$ a solution to the differential equation

$$\begin{aligned} d\zeta_t &= (a + \frac{1}{2}b^2)\zeta_t dt + b\zeta_t dB_t, \qquad t \in [0, \infty) \\ \zeta_0 &= x_0. \end{aligned}$$

Remark that the functions $y \mapsto (a + \frac{1}{2}b^2)y$ and $y \mapsto by$ satisfies the conditions from Theorem 2.8.4, namely global linear growth and the Lipschitz condition. We show that $\xi_t \in M^2[0, T]$ for all $T > 0$. Obviously ξ_t is right continuous and thereby progressively measurable. As ξ_t is either positive or negative everywhere, we can utilize *Fubini's Theorem* (Theorem A.1.10) to conclude that

$$\begin{aligned} \mathbb{E}[\int_0^T |\xi_s|^2 ds] &= \int_0^T \mathbb{E}[|x_0|^2 \exp(2at + 2bB_t)]ds \\ &= |x_0|^2 \int_0^T \exp(2(b + a)t)ds < \infty. \end{aligned}$$

We conclude that $(\xi_t)_{t\geq 0}$ is a diffusion. Especially it is an example of a Markov process.

□

As we shall see later, a sufficient condition for a process to be a Markov process is the one of independent increments. It is easy, by the *Freezing Lemma* to check that this is the case. The following example serves as an example to show that the condition is not necessary. The example is the one of the Brownian bridge. As an exercise we show that the Brownian bridge is a diffusion and that it does not have independent increments.

Example 3.5.7. [Bal17, Example 8.5](§§) The Brownian bridge (see for example [Bal17, Exercise 4.15]) is a well known example of a Gaussian process. We let B be a standard Brownian motion. We define the Brownian bridge for $t \in [0,1]$ as

$$b_t = B_t - tB_1.$$

We look at the stochastic differential equation

$$\begin{aligned} d\zeta_t &= -\frac{\zeta_t}{(1-t)}dt + dB_t \\ \zeta_0 &= 0 \end{aligned} \tag{3.10}$$

for $t < 1$. We start off solving the homogeneous differential equation; $d\zeta_t = -\frac{\zeta_t}{(1-t)}dt$. By the chain rule of differentiation and the fundamental rule of fundamental rule of calculus, $\xi_1(t) := e^{-\int_0^t \frac{1}{(1-s)}ds}$ can be shown to solve it. We can compute that $\xi_1(t) = (1-t)$. Let $\xi_2(t)$ be another stochastic process and define $\xi_t = \xi_1(t)\xi_2(t)$. As in [Bal17, Example] we will choose $\xi_2(t)$ such that ξ_t is a solution to the stochastic differential equation in 3.10. By [Bal17, Proposition 8.1] (with $\langle \xi_1, \xi_2 \rangle_t$), we get that

$$d\xi_t = \xi_1(t)d\xi_2(t) + \xi_2(t)d\xi_1(t) = \xi_1(t)d\xi_2(t) - \frac{\xi_2(t)\xi_1(t)}{(1-t)}dt.$$

. Now assume that $d\xi_2(t) = \frac{1}{(1-t)}dB_t$. Then $\xi_2(t) = \int_0^t \frac{1}{(1-s)}dB_s$, and $\xi_t = \xi_1(t)\xi_2(t) = (1-t)\int_0^t \frac{1}{(1-s)}dB_s$ is a solution to Equation (3.10). Remark that $\xi_0 = 0$. We conclude that the Brownian bridge is a diffusion, and therefore a Markov process. We show that b and Y are both Gaussian and that their finite dimensional distribution have the same covariance matrix. As $\frac{1}{(1-s)}$ are a deterministic function, by [Bal17, Proposition 7.1] Y_t is Gaussian with mean 0. Let $0 \le s \le t < 1$.

$$\begin{aligned} \mathbb{E}[Y_sY_t] &= (1-t)(1-s)\mathbb{E}[(\int_0^s \frac{1}{1-v}dB_v)^2] \\ &= (1-t)(1-s)\int_0^s \frac{1}{(1-v)^2}dv = s(1-t). \end{aligned}$$

Here we used the Itô isometry and that $\mathbb{E}[(\int_0^s \frac{1}{1-v}dB_v)(\int_s^t \frac{1}{(1-v)}dB_v)] = 0$ because $\int_0^s \frac{1}{1-v}dB_v$ and $\int_s^t \frac{1}{(1-v)}dB_v$ are independent. Moreover we calculate

$$\mathbb{E}[b_tb_s] = \mathbb{E}[(B_t - tB_1)(B_s - sB_1)] = s - st - ts + ts = s(1-t).$$

We conclude that b and Y are equivalent.
At last we show that the Brownian bridge does not have independent increments. Let $0 \le u < s < t < 1$:

$$\begin{aligned} \mathbb{E}[(b_t - b_s)(b_s - b_u)] &= \mathbb{E}[(B_t - B_s - (t-s)B_1)(B_s - B_u - (s-u)B_1)] \\ &= (t-s)(s-u) - (s-u)\mathbb{E}[(B_t - B_s)B_1] - (t-s)\mathbb{E}[B_1(B_s - B_u)] \\ &= (t-s)(s-u) - 2(s-u)(t-s) = -(t-s)(s-u), \end{aligned}$$

which is different from 0.

□

CHAPTER 4

Lévy Processes

Lévy processes are stochastic processes enjoying certain properties. Examples of such processes are *Brownian motions* and *Poisson processes*. Throughout the thesis, Lévy processes will play the role as *base process* in the time changed process under examination. Also choosing the *time process* to be a Lévy process serves as an interesting example, and is the concept known as *subordination*.
In this chapter, let again $Y := (Y_t)_{t\geq 0}$ be a stochastic process taking values in $(\mathbb{R}^d, \mathscr{B}^d)$ defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$. We let Y be $\mathbb{F}$-adapted.

4.1 Definition of Lévy Processes

In the following we define Lévy processes, $\mathbb{F}$-Lévy processes and Lévy processes in law.

Definition 4.1.1. [Sat13, Definition 1.6] and [Ped20, Definition 9.5].
Consider the following conditions:

(1) Independent increments: Let $n \in \mathbb{N}$. For $0 \leq t_0 < t_1 < t_2 < ... < t_n$, $L_{t_0}, L_{t_0} - L_{t_1}, ..., L_{t_n} - L_{t_{n-1}}$ are independent.

(1') For $t, s \geq 0$. $L_{t+s} - L_t$ is independent of $\mathscr{F}_t$.

(2) $L_0 = 0$ almost surely.

(3) Stationary increments: $L_{t+s} - L_s \overset{d}{=} L_t$ for $s, t \geq 0$.

(4) Stochastic continuity: Let $\epsilon > 0$ and $t_0 \geq 0$. Then

$$\lim_{t\to t_0} \mathbb{P}(|L_{t_0} - L_t| > \epsilon) = 0.$$

(5) $\mathbb{F}$-adapted.

(6) The càdlàg property: There exist $\Omega_0 \in \mathscr{F}$ such that $\mathbb{P}(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, $L_t(\omega)$ has left limits and is right continuous.

If L meets (1), (2), (3) and (4) it is a Lévy process in law.

If L meets (1'), (2), (3), (5) and (6), we call it an $\mathbb{F}$-Lévy process.
If L meets (1), (2), (3) and (6) it is called a Lévy process.

Remark 4.1.2. An $(\mathscr{F}_t)_{t\geq 0}$-Lévy process is a Lévy process, and a Lévy process is an $\mathbb{F}^L$-Lévy process, where $\mathbb{F}^L$ is its natural filtration (see e.g. [Ped20, Proposition 9.6]).

Remark 4.1.3. If L is a Lévy process in law, there exist another process L' that is a Lévy process and such that the two processes are modifications. This is stated and proved in [Sat13, Theorem 11.5]

□

In [Sat13] condition (4) is included in the definition of a Lévy process but it is not included in the definition in [Ped20]. The following argument shows that one can omit this condition, when one has the condition of càdlàg paths. This gives us one less condition to check when demonstrating that a given process is a Lévy process. In [Sat13, p. 4] he claims that if a stochastic process meets condition (1)-(3), condition (4) can be reduced to

$$\lim_{t\downarrow 0} \mathbb{P}(|L_t| > \epsilon) = 0,$$

for all $\epsilon > 0$. This seems reasonable, but in the following lemma we proof it and use it.

Lemma 4.1.4. *A Lévy process is a Lévy process in law.*

Proof. (§§) The statement is equivalent to the statement that a Lévy process fulfills condition (4). We deduce it by using the condition of stationary increments and the càdlàg property. The condition of stochastic continuity can be reformulated to this: For all $\delta, \epsilon > 0$ and $t_0 \geq 0$, there exist a $\gamma > 0$ such that

$$\mathbb{P}(|L_{t_0} - L_t| > \epsilon) < \delta \text{ for } |t_0 - t| < \gamma.$$

Let $\epsilon, \delta > 0$ be given. Let Ω_0 be the space where L is càdlàg and where $L_0 = 0$. Then

$$\mathbb{P}(|L_{1/k} - L_0| > \epsilon) = \mathbb{P}(|L_{1/k}| > \epsilon).$$

Define the sets $A_k = \Omega_0 \cap \{|L_t| \leq \epsilon \text{ for } 0 \leq t \leq \frac{1}{k}\}$ for all $k \in \mathbb{N}$. Remark that $A_k \subseteq A_{k+1}$ for all $k \in \mathbb{N}$. For $\omega \in \Omega_0$, there exists a $k \in \mathbb{N}$ such that $\omega \in A_k$ by the right continuity of the paths. Consequently $\cup_{k\in\mathbb{N}} A_k = \Omega_0$. By properties of measures (see [Tho14, Theorem 1.3.4(v)])

$$1 = \mathbb{P}(\Omega_0) = \lim_{k\to\infty} \mathbb{P}(\cup_{i=1}^k A_i).$$

This ensures the existence of a $K \in \mathbb{N}$ such that $\mathbb{P}(A_K) > 1-\delta$. Thus $\mathbb{P}(A_K^C) < \delta$. $A_K^C = \Omega_0^C \cup \{|L_t| > \epsilon \text{ for some } 0 \leq t \leq \frac{1}{K}\}$. As $\{|L_t| > \epsilon\}$ is contained in this set for $t \leq \frac{1}{K}$, clearly $\mathbb{P}(|L_t| > \epsilon) < \delta$ for $0 \leq t \leq \frac{1}{K}$.
Now let $t_0 \geq 0$ be given and let t be such that $\sigma := |t_0 - t| < \frac{1}{K}$. Then as $|L_{t_0} - L_t| = |L_t - L_{t_0}|$ and by stationary increments, we obtain:

$$\mathbb{P}(|L_{t_0} - L_t| > \epsilon) = \mathbb{P}(|L_\sigma - L_0| > \epsilon) = \mathbb{P}(|L_\sigma| > \epsilon) < \delta.$$

■

By the definition of an $\mathbb{F}$-Lévy process we have that $L_{t+s}-L_s$ is independent of $\mathscr{F}_s$ and that $L_{t+s}-L_s$ is equal in distribution to L_t. In the following lemma we show that the definition of an $\mathbb{F}$-Lévy process actually secures us that the whole process $(L_{t+s}-L_s)_{t\geq 0}$ is independent of $\mathscr{F}_s$. Also we show that the process $(L_{t+s}-L_s)_{t\geq 0}$ is equivalent to $(L_t)_{t\geq 0}$. The following lemma is a special case of [Ped20, Theorem 9.15], we give a different proof here. The idea of the matrix transformation comes from [Ped20] and will also be used in later proofs.

Lemma 4.1.5. *Let L be an $\mathbb{F}$-Lévy process and s be a positive real number. Then $(L_{t+s}-L_s)_{t\geq 0}$ is independent of $\mathscr{F}_s$ and is equivalent to L.*

Proof. (§§) Assume that L is an $\mathbb{F}$-Lévy process. Let $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \cdots < t_n$. By stationary increments of L, we have that

$$L_{t_i+s} - L_{t_{i-1}+s} \overset{d}{=} L_{t_i-t_{i-1}} \overset{d}{=} L_{t_i} - L_{t_{i-1}},$$

which by the property of independent increments allows us to conclude that

$$(L_{t_1+s}-L_s, L_{t_2+s}-L_{t_1+s}, ..., L_{t_n+s}-L_{t_{n-1}+s}) \overset{d}{=} (L_{t_1}, L_{t_2}-L_{t_1}, ..., L_{t_n}-L_{t_{n-1}}).$$

There exist a $1-1$ matrix transformation sending the left hand side to $(L_{t_1+s}-L_s, L_{t_2+s}-L_s, ..., L_{t_n+s}-L_s)$ and the right hand side to $(L_{t_1}, ..., L_{t_n})$. This matrix (A) is $n \times n$ with $A_{i,j} = 1$ when $i \leq j$ and 0 else. This is an invertible matrix, so by the properties of characteristic functions of transformed random vectors (see Lemma A.1.4(3)), we conclude that

$$(L_{t_1+s}-L_s, L_{t_2+s}-L_s, ..., L_{t_n+s}-L_s) \overset{d}{=} (L_{t_1}, ..., L_{t_n}).$$

We conclude that L is equivalent to the processes $(L_{s+t}-L_s)_{t\geq 0}$.
We show that $(L_{s+t}-L_s)_{t\geq 0}$ is independent of $\mathscr{F}_s$. Let X be an $\mathscr{F}_s$-measurable stochastic vector. We look at the characteristic function of the vector $(L_{t_1+s}-L_s, L_{t_2+s}-L_{t_1+s}, ..., L_{t_n+s}-L_{t_{n-1}+s}, X)$. Let $\theta_1, ..., \theta_n, \theta_{n+1} \in \mathbb{R}^d$.

$$\mathbb{E}[e^{i\langle\theta_{n+1},X\rangle}\prod_{i=1}^{n} e^{i\langle\theta_i,(L_{t_i+s}-L_{t_{i-1}+s})\rangle}],$$

where $t_0 := 0$. By use of the *Tower Property* (Proposition A.1.11) with $\mathscr{F}_{t_{n-1}+s}$, we obtain

$$\begin{aligned}&\mathbb{E}[e^{i\langle\theta_{n+1},X\rangle}\prod_{i=1}^{n} e^{i\langle\theta_i,(L_{t_i+s}-L_{t_{i-1}+s})\rangle}]\\ &\quad= \mathbb{E}[e^{i\langle\theta_{n+1},X\rangle}\prod_{i=1}^{n-1} e^{i\langle\theta_i,(L_{t_i+s}-L_{t_{i-1}+s})\rangle}]\mathbb{E}[e^{i\langle\theta_i,(L_{t_n+s}-L_{t_{n-1}+s})\rangle}].\end{aligned}$$

Doing this successively (exactly as it is done in the proof [Ped20, Proposition 9.6]) with $\mathscr{F}_{t_{n-2}+s}, ..., \mathscr{F}_s$ we obtain that $(L_{t_1+s}-L_s, L_{t_2+s}-L_{t_1+s}, ..., L_{t_n+s}-L_{t_{n-1}+s})$ is independent of X, as the characteristic function can be split into a product of their characteristic functions. As $(L_{t_1+s}-L_s, ..., L_{t_n+s}-L_s)$ is a measurable transformation of $(L_{t_1+s}-L_s, L_{t_2+s}-L_{t_1+s}, ..., L_{t_n+s}-L_{t_{n-1}+s})$ it is also independent of X (see [Tho14, Corollary 13.5.5]). ∎

In the following example we introduce the Poisson process building on the theory of random measures, as is the case in [Gra76]. It will be verified that the Poisson process can be defined as a Poisson random measure on $((0,\infty), \mathscr{B}((0,\infty)))$ with intensity measure being the Lebesgue measure. Although this example is well studied, it is an exercise to show that all the properties of the Lévy process are in fact fulfilled.

Example 4.1.6. (§§) Let $(E, \mathscr{E}) = ((0,\infty), \mathscr{B}(0,\infty))$. Let μ be a σ-finite measure on $((0,\infty), \mathscr{B}(0,\infty))$. Then by Lemma 2.5.8, there exist a completely random point process, N, such that for $B \in \mathscr{B}((0,\infty))$,

$$\mathbb{P}(N(B) = k) = e^{-\mu(B)} \frac{\mu(B)^k}{k!},$$

for $k \in \mathbb{N}$. We call μ the *intensity measure* of N. Denote $N(t) := N((0,t])$. Let $\omega \in \Omega$ be fixed. Denote by $N_\omega(A)$ the measure $N(\omega)$ utilized on the set $A \in \mathscr{B}(0,\infty)$. We have that

$$N_\omega(t) - N_\omega(s) = N_\omega((0,s]) + N_\omega((s,t]) - N_\omega((0,s]) = N_\omega((s,t]),$$

for $s \leq t$. This ensures that

$$\mathbb{P}(N(t) - N(s) = k) = e^{-\mu((s,t])} \frac{\mu((s,t])^k}{k!},$$

for $k \in \mathbb{N}$. Moreover

$$N_\omega(0) = N_\omega((0,0]) = N_\omega(\emptyset) = 0.$$

Let $t > 0$ and $(t_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $t_i > t$ for $i \in \mathbb{N}$ and $t_n \to t$ for $n \to \infty$. By [Tho14, Theorem 1.3.4(vi)]

$$\lim_{n\to\infty} N_\omega(t_n) = N_\omega(\cap_{n\in\mathbb{N}}(0,t_n]) = N_\omega((0,t]) = N_\omega(t).$$

This shows right continuity of the process. Let $(s_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive numbers such that $0 < s_i < t$ and $s_n \to t$. Then by [Tho14, Theorem 1.3.4(v)]

$$\lim_{n\to\infty} N_\omega(s_n) = N_\omega(\cup_{n\in\mathbb{N}}(0,s_n]) = N_\omega((0,t)).$$

This shows that the process has left limits. Let $n \in \mathbb{N}$ and $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq ... \leq s_n < t_n$. Let $A_1, ..., A_n \subseteq \mathbb{N}$. Then

$$\begin{aligned}
&\mathbb{P}(N(t_1) - N(s_1) \in A_1, ..., N(t_n) - N(s_n) \in A_n) \\
&\quad = \mathbb{P}(N((s_1,t_1]) \in A_1, ..., N((s_n - t_n]) \in A_n) \\
&\quad = \mathbb{P}(N((s_1,t_1]) \in A_1) \cdots \mathbb{P}(N((s_n - t_n]) \in A_n),
\end{aligned}$$

because N is completely random. This shows that the process has independent increments.

If we choose μ to be the Lebesgue measure (denoted λ), then for $s \leq t$

$$\begin{aligned}
\mathbb{P}(N(t) - N(s) = k) &= e^{-\mu((s,t])} \frac{\mu((s,t])^k}{k!} \\
&= e^{-(t-s)} \frac{(t-s)^k}{k!} = \mathbb{P}(N(t-s) = k),
\end{aligned}$$

for $k \in \mathbb{N}$. Thus the process has stationary increments.
To sum up, the process N with Lebesgue measure as intensity measure is a Lévy process, because it fulfills $(1), (2), (3)$ and (6) of Definition 4.1.1. As it is a Lévy process, it holds true that if another process has the same properties, it is equivalent to N.

□

We can generalize the notion of a Brownian motion, making us able to vary the mean and the variance. This construction is very general and can be used to model financial asset prices (see for example [BS15, p. 9]).

Example 4.1.7. (§§) Let $(B_t)_{t\geq 0}$ be a d-dimensional Brownian motion. Let $\theta \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}$. We define a Brownian motion with *drift* θ and *volatility* σ as $B^{\theta,\sigma} := (B_t^{\theta,\sigma})_{t\geq 0}$, where

$$B_t^{\theta,\sigma} := \theta t + \sigma B_t.$$

We show that this is a Lévy process. $B_0^{\theta,\sigma}$ is zero whenever B_0 is zero. The stationary increments comes from the fact that $B_t^{\theta,\sigma} - B_s^{\theta,\sigma} \sim N_d(\theta(t-s), \sigma^2(t-s)I)$. For all the ω, where $B_t(\omega)$ is càdlág, $B_t^{\theta,\sigma}(\omega)$ is as well. We see that $\theta\cdot(t-s)+\sigma\cdot(B_t-B_s)$ is independent of $\mathscr{F}_s^B$ whereas $\theta s+\sigma B_s$ is $\mathscr{F}_s^B$-measurable, which secures the independent increments of the process.

□

We introduce the *inverse Gaussian distribution.* This is an infinitely divisible distribution, and therefore it gives rise to a Lévy process. We call this an inverse Gaussian process. In [BS15, p. 13], the distribution and the Laplace transform of the distribution is given. In [Car+03, p. 349] the Laplace transform and an interpretation of the process are given. We connect the two presentations (i.e. find common parameters) in order to connect interpretation of the parameters with the distribution parameters.
The inverse Gaussian process can be shown to be a subordinator. Later we will time change a Brownian motion with positive drift and volatility σ with an inverse Gaussian distribution. We will then obtain a *Normal inverse Gaussian process*, which is a Lévy process.

Example 4.1.8. (§§) An inverse Gaussian variable, X with parameter $a, b > 0$ (written $\mathrm{IG}(a, b)$) has the density (given in [BS15, p. 13])

$$f_X(x) = \sqrt{\frac{b}{2\pi x^3}}\exp(-\sqrt{ab}(\frac{ax + b/x}{2})), \qquad x \in (0, \infty).$$

Let $(B_t^{\theta,1})_{t\geq 0}$ be a one-dimensional Brownian motion with drift θ and volatility 1. Define the process $(I_\alpha)_{\alpha\geq 0}$, where $I_\alpha = \inf\{t : B_t^{\theta,1} = \alpha\}$. Then $I_\alpha \sim \mathrm{IG}(\theta^2, \alpha^2)$ for all $\alpha > 0$. We call this an inverse Gaussian process with parameter $\theta > 0$. By [Car+03, Section 2.1] the Laplace transform is of the form

$$\mathscr{L}_{I(\alpha)}(\lambda) = \mathbb{E}[\exp(-\lambda I_\alpha)] = \exp(-\alpha(\sqrt{2\lambda + \theta^2} - \theta)),$$

for $\lambda > 0$. We show this directly:

$$\mathscr{L}_{I(\alpha)}(\lambda) = \int_0^\infty \frac{\alpha}{\sqrt{2\pi x^3}}\exp(\theta\alpha)\exp(-\frac{\theta^2 x + \alpha^2/x}{2})\exp(-\lambda x)dx$$

$$= \int_0^\infty \frac{\alpha}{\sqrt{2\pi x^3}} \exp(\theta\alpha)\exp(-\frac{(\theta^2+2\lambda)x+\alpha^2/x}{2})dx$$

Abbreviate by γ the term $\theta - \sqrt{\theta^2+2\lambda}$ and calculate:

$$\mathscr{L}_{I(\alpha)}(\lambda) = \int_0^\infty \frac{\alpha}{\sqrt{2\pi x^3}} \exp((\gamma+\sqrt{\theta^2+2\lambda})\alpha)\exp(-\frac{(\theta^2+2\lambda)x+\alpha^2/x}{2})dx$$
$$= \exp(\gamma\alpha)\int_0^\infty \frac{\alpha}{\sqrt{2\pi x^3}} \exp((\sqrt{\theta^2+2\lambda})\alpha)\exp(-\frac{(\theta^2+2\lambda)x+\alpha^2/x}{2})dx$$
$$= \exp(\gamma\alpha) = \exp(-\alpha(\sqrt{\theta^2+2\lambda}-\theta)).$$

Inside the integral on the second line we have an inverse Gaussian density with parameters $\theta^2+2\lambda > 0$ and $\alpha^2 > 0$.
Remark that I is an increasing process almost surely. Let Ω_0 be a subset of Ω such that $\mathbb{P}(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, $B_0^{\theta,1}(\omega) = 0$ and $B_t^{\theta,1}(\omega)$ is continuous. Then for all $\omega \in \Omega_0$; $\inf\{t : B_t^{\theta,1}(\omega) = \alpha_1\} \leq \inf\{t : B_t^{\theta,1}(\omega) = \alpha_2\}$, for $0 \leq \alpha_1 \leq \alpha_2$.

□

4.2 Regular Convolution Semigroups

A *regular convolution semigroup* is a family of probability measures enjoying certain properties. There is a one-to-one correspondence between Lévy processes and regular convolution semigroups, which will be established in the current section. In order to introduce regular convolution semigroups, we introduce the concept of convolution between probability measures. In this section we follow the introduction in [Ped20] closely.

Definition 4.2.1. [Ped20, p. 106] Let μ and ν be probability measures on $\mathbb{R}^d$. *The convolution between μ and ν* ($\mu * \nu$) is defined as follows: Let $B \in \mathscr{B}(\mathbb{R}^d)$:

$$\mu * \nu(B) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{1}_B(x+y)\mu(dx)\nu(dy). \tag{4.1}$$

□

We will need certain properties for the convolution operation between measures. The following properties are mentioned without proof on [Sat13, p. 8]. It is a good exercise to verify the assertions.

Proposition 4.2.2. *Let μ and ν be probability measures on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$.*

(1) $\mu * \nu$ is a probability measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$.

(2) $\mu * \nu = \nu * \mu$.

(3) Let X and Y be independent stochastic vectors, that takes values in $\mathbb{R}^d$. Then $\mathbb{P}_{X+Y} = \mathbb{P}_X * \mathbb{P}_Y$.

Proof. (§§§) of (1):

$$\mu * \nu(\emptyset) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{1}_\emptyset(x+y)\mu(dx)\nu(dy) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 0\mu(dx)\nu(dy) = 0.$$

Also

$$\mu * \nu(\mathbb{R}^d) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{\mathbb{R}^d}(x+y)\mu(dx)\nu(dy) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1\mu(dx)\nu(dy) = 1.$$

Let $(A_i)_{i\in\mathbb{N}} \in \mathscr{B}(\mathbb{R}^d)$ be a sequence disjoint sets.

$$\begin{aligned}
\mu * \nu(\cup_{i\in\mathbb{N}} A_i) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{\cup_{i\in\mathbb{N}} A_i}(x+y)\mu(dx)\nu(dy) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \lim_{k\to\infty} \sum_{i=1}^{k} \mathbb{1}_{A_i}(x+y)\mu(dx)\nu(dy) \\
&= \lim_{k\to\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{i=1}^{k} \mathbb{1}_{A_i}(x+y)\mu(dx)\nu(dy) \\
&= \sum_{i=1}^{\infty} \mu * \nu(A_i),
\end{aligned}$$

where we used *Monotone Convergence* (Theorem A.2.3) and linearity of integrals.
of (2) : Let $B \in \mathscr{B}(\mathbb{R}^d)$, then by *Fubini's Theorem* (Theorem A.1.10), we obtain

$$\begin{aligned}
\mu * \nu(B) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_B(x+y)\mu(dx)\nu(dy) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_B(x+y)\nu(dx)\mu(dy) = \nu * \mu(B).
\end{aligned}$$

of (3) : As X and Z are independent, it holds true for $A, B \in \mathscr{B}(\mathbb{R}^d)$ that $\mathbb{P}_{(X,Y)}(A,B) = \mathbb{P}_X(A)\mathbb{P}_Y(B)$. Thus for $f(x,y) := x+y$ and $B \in \mathscr{B}(\mathbb{R}^d)$ we obtain:

$$\begin{aligned}
\mathbb{P}(X+Y \in B) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbb{1}_B(f(x,y))\mathbb{P}_{(X,Y)}(dx,dy) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_B(x+y)\mathbb{P}_X(dx)\mathbb{P}_Y(dy) = \mathbb{P}_X * P_Y(B).
\end{aligned}$$

■

Definition 4.2.3. [Ped20, p. 62] Let $d \in \mathbb{N}$. A family of probability measures $(\nu_t)_{t\geq 0}$ on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ is a *regular convolution semigroup* if it meets the following conditions:

(a) $\nu_0 = \delta_0$.

(b) $\nu_{t+s} = \nu_t * \nu_s$ for $t, s \geq 0$, and if $t_n \geq 0$ and $t_n \to 0$, then $\nu_{t_n} \overset{w}{\to} \delta_0$.

□

The proof of Lemma 4.2.4 follows the one given in [Ped20, p. 63] closely.

Lemma 4.2.4. *Let L be a Lévy process and define $\nu_t := \mathbb{P}_{L_t}$ for $t \geq 0$, then $(\nu_t)_{t\geq 0}$ is a convolution semigroup.*

Proof. (§) We need to show that $(\nu_t)_{t\geq 0}$ fulfills the conditions above. As $L_0 = 0$ almost surely, then for $A \in \mathscr{B}(\mathbb{R}^d))$, $\mathbb{P}_{L_0}(A) = \delta_0(A)$. Also we have for $s, t \geq 0$, that

$$\begin{aligned}\nu_{t+s} = \mathbb{P}_{L_{t+s}} &= \mathbb{P}_{L_{t+s}-L_s+L_s} \\ &= \mathbb{P}_{L_{t+s}-L_s} * \mathbb{P}_{L_s} \\ &= \mathbb{P}_{L_{t+s-s}} * \mathbb{P}_{L_s} \\ &= \mathbb{P}_{L_t} * \mathbb{P}_{L_s} = \nu_t * \nu_s.\end{aligned}$$

Here we used that $L_{t+s} - L_s$ is independent of L_s, and that $L_{t+s} - L_s \overset{\mathscr{L}}{=} L_t$. Let $t_n \geq 0$ and $t_n \to 0$ for $n \to \infty$. Then $L_{t_n}(\omega) \to L_0(\omega)$ for all $\omega \in \Omega$, because it has right continuous paths. By [Bal17, Proposition 1.5], this gives that

$$\nu_{t_n} = \mathbb{P}_{L_{t_n}} \overset{w}{\to} \mathbb{P}_{L_0} = \delta_0 = \nu_0,$$

which we wanted. ■

We prove that a Lévy process is a Markov process and go forward as in [Ped20, pp. 63–64]. Let $(L_t)_{t\geq 0}$ be a Lévy process, and define $(\nu_t)_{t\geq 0}$ to be the regular convolution semigroup where $\nu_t := \mathbb{P}_{L_t}$ for all $t \geq 0$. Now define $(P_t)_{t\geq 0}$, as:

$$P_t(x, A) := \nu_t(A - x) \tag{4.2}$$

for $A \in \mathscr{B}(\mathbb{R}^d), t \geq 0$ and $x \in \mathbb{R}^d$.

Lemma 4.2.5. *[Ped20, theorem 9.10(1)] Let $(\nu_t)_{t\geq 0}$ be a regular convolution semigroup. Define $(P_t)_{t\geq 0}$ as in (4.2), then it is a time homogeneous Markov transition function.*

Before proceeding, we must make sure that the construction made in Equation (4.2) has the properties that makes it a Markov transition function.

Lemma 4.2.6. *Let Z be a stochastic variable that takes values in $\mathbb{R}^d$. Let $A \in \mathscr{B}(\mathbb{R}^d)$. Then the function*

$$\begin{aligned}P_Z : \mathbb{R}^d \times \mathscr{B}(\mathbb{R}^d) &\to [0, 1] \\ (x, A) &\mapsto \mathbb{P}_Z(A - x)\end{aligned}$$

is a transition kernel.

Proof. (§§§) Fix an x in $\mathbb{R}^d$ and let $A \in \mathscr{B}(\mathbb{R}^d)$. $A - x = \{y - x \in \mathbb{R}^d : y \in A\}$. This is just a translation of A by $-x$, so $A - x$ is a set in $\mathscr{B}(\mathbb{R}^d)$. Thus $\mathbb{P}_Z^x(\cdot) := \mathbb{P}_Z(\cdot - x)$ is a measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$.
Fix an A in $\mathscr{B}(\mathbb{R}^d)$. Consider the function

$$\begin{aligned}f : \mathbb{R}^d \times \mathbb{R}^d &\to \mathbb{R} \\ (x, y) &\mapsto \mathbb{1}_A(x + y).\end{aligned}$$

This is a $\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R}^d)$-measurable function. By *Fubini's Theorem* (Theorem A.1.10)

$$x \mapsto \int_{\mathbb{R}^d} \mathbb{1}_A(x + y)\mathbb{P}_Z(dy)$$

is a $\mathscr{B}(\mathbb{R}^d)$-measurable function. As

$$\int_{\mathbb{R}^d} \mathbb{1}_A(x+y)\mathbb{P}_Z(dy) = \int_{\mathbb{R}^d} \mathbb{1}_{A-x}(y)\mathbb{P}_Z(dy) = \mathbb{P}_Z(A-x),$$

we have the desired properties in order for P_Z to be a transition kernel on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. ■

Now we are equipped to prove Lemma 4.2.5 as in [Ped20, p. 64].

Proof of Lemma 4.2.5. (§) Let $\nu_t^x(A) := \nu_t(A-x)$. By an application of the *Monotone Class Theorem* (Theorem A.2.4) it holds for every $f \in \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$ that

$$\int_{\mathbb{R}^d} f(y)P_t(x,dy) = \int_{\mathbb{R}^d} f(y)\nu_t^x(dy) = \int_{\mathbb{R}^d} f(x+y)\nu_t(dy). \tag{4.3}$$

We prove that $P_tP_s = P_{s+t}$ by using (4.3). Let $t, s \geq 0$, $A \in \mathscr{B}(\mathbb{R}^d)$ and $x \in \mathbb{R}$. Then

$$P_tP_s(x,A) = \int_{\mathbb{R}^d} P_s(y,A)P_t(x,dy) = \int_{\mathbb{R}^d} \nu_s(A-y)P_t(x,dy)$$

By properties of the transition kernels. By the use of (4.3) where we set $f(y) := \nu_s(A-y)$, which is a bounded and measurable function from $\mathbb{R}^d \to \mathbb{R}$ by Lemma 4.2.6, then we obtain that $f(x+y) = \nu_s(A-x-y) = \nu_s((A-x)-y)$. This gives us

$$\int_{\mathbb{R}^d} \nu_s(A-y)P_t(x,dy) = \int_{\mathbb{R}^d} \nu_s((A-x)-y)\nu_t(dy) = \nu_t * \nu_s(A-x)$$

As $\nu_t * \nu_s = \nu_{t+s}$ because it is a regular convolution semigroup, we arrive at $P_tP_s(x,A) = P_{t+s}(x,A)$. For the second property we consider that

$$P_0(x,A) = \nu_0(A-x) = \delta_0(A-x) = \delta_x(A),$$

as we wished. ■

Now we show that a Lévy process has the Markov property. We use the ideas an calculations put forward in [Ped20, p. 64].

Theorem 4.2.7. *A Lévy process is a Markov process.*

Proof. (§) Let L and Y be stochastic processes adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$. Assume that $(L_t)_{t\geq 0}$ is a Lévy process on $\mathbb{R}^d$ and define $\nu_t := \mathbb{P}_{L_t}$ for $t \geq 0$. Let $(Y_t)_{t\geq 0}$ be a Markov process on $\mathbb{R}^d$ with the Markov transition function $(P_t)_{t\geq 0}$ defined in (4.2) and let $\mathbb{P}(Y_0 = 0) = 1$. Remark that by Theorem 3.4.3 such a process exist. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable and bounded function and $t \geq s \geq 0$. Then

$$\mathbb{E}[f(Y_t)|\mathscr{F}_s] = P_{t-s}(f(Y_s)) = \int f(y)P_{t-s}(Y_s,dy) = \int f(y+Y_s)\nu_{t-s}(dy), \tag{4.4}$$

by Equation (4.3). We can easily expand this equality to hold true for functions taking values in the complex plane by linearity of conditional expectation and of the integral. We let $f(\theta) := e^{i\langle\theta, Y_t - Y_s\rangle}$. We can now use this identity in the following calculation:

$$\begin{aligned}\mathbb{E}(e^{i\langle\theta,Y_t-Y_s\rangle}|\mathscr{F}_s) &= e^{-i\langle\theta,Y_s\rangle}\mathbb{E}(e^{i\langle\theta,Y_t\rangle}|\mathscr{F}_s)\\ &= e^{-i\langle\theta,Y_s\rangle}\int e^{i\langle\theta,z+Y_s\rangle}\nu_{t-s}(dz)\\ &= e^{i\langle\theta,Y_s-Y_s\rangle}\int e^{i\langle\theta,z\rangle}\nu_{t-s}(dz)\\ &= \int e^{i\langle\theta,z\rangle}\nu_{t-s}(dz)\end{aligned}$$

We take the expectation on both sides:

$$\mathbb{E}(e^{i\langle\theta,Y_t-Y_s\rangle}) = \int e^{i\langle\theta,z\rangle}\nu_{t-s}(dz), \tag{4.5}$$

which by Lemma A.1.4(1) shows that $Y_t - Y_s \sim \nu_t$. The following calculation shows that $Y_t - Y_s$ is independent of $\mathscr{F}_s$ and presents a solution to [Ped20, Exercise 46]. Let X be a real $\mathscr{F}_s$-measurable random variable and let $\theta_1 \in \mathbb{R}^d$ and $\theta_2 \in \mathbb{R}$. Then we have, that

$$\begin{aligned}\mathbb{E}[e^{i\langle(\theta_1,\theta_2),(Y_t-Y_s,X)\rangle}] &= \mathbb{E}[e^{i\langle\theta_1,Y_t-Y_s\rangle}e^{i\langle\theta_2,X\rangle}]\\ &= \mathbb{E}[\mathbb{E}[e^{i\langle\theta_1,Y_t-Y_s\rangle}e^{i\langle\theta_2,X\rangle}|\mathscr{F}_s]]\\ &= \mathbb{E}[e^{i\langle\theta_2,X\rangle}\mathbb{E}[e^{i\langle\theta_1,Y_t-Y_s\rangle}|\mathscr{F}_s]]\\ &= \mathbb{E}[e^{i\langle\theta_2,X\rangle}]\int e^{i\langle\theta,z\rangle}\nu_{t-s}(dz).\end{aligned}$$

As $\int e^{i\langle\theta,z\rangle}\nu_{t-s}(dz)$ is the characteristic function of $Y_t - Y_s$, we conclude by Lemma A.1.4(2) that $Y_t - Y_s$ is independent of every random variable, that are $\mathscr{F}_s$-measurable, that is $Y_t - Y_s$ is independent of $\mathscr{F}_s$, hence $(Y_t)_{t\geq 0}$ has independent increments. We set $s = 0$ and obtain that $Y_{t-s} \overset{d}{=} Y_t - Y_s \sim \nu_{t-s}$. Conclude that $Y_t \overset{d}{=} L_t$ for all $t \geq 0$.
Let $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < ... < t_n$. The aim of the following considerations is to show that

$$Y_{(t_1,t_2,...,t_n)} := (Y_{t_1}, Y_{t_2}, ..., Y_{t_n})^T \overset{d}{=} (L_{t_1}, L_{t_2}, ..., L_{t_n})^T =: L_{(t_1,t_2,...,t_n)}.$$

By the previous considerations

$$(Y_{t_1}, Y_{t_2} - Y_{t_1}, ..., Y_{t_n} - Y_{t_{n-1}})^T \overset{d}{=} (L_{t_1}, L_{t_2} - L_{t_1}, ..., L_{t_n} - L_{t_{n-1}})^T,$$

because of the independent increments of the processes. It is not hard to see that the $dn \times dn$ matrix A, satisfying that

$$A \cdot (Y_{t_1}, Y_{t_2}, ..., Y_{t_n})^T = (Y_{t_1}, Y_{t_2} - Y_{t_1}, ..., Y_{t_n} - Y_{t_{n-1}})^T$$

is invertible. Lemma A.1.4(3) secures that

$$\hat{\mathbb{P}}_{Y_{(t_1,t_2,...,t_n)}}(A^T \cdot s) = \hat{\mathbb{P}}_{L_{(t_1,t_2,...,t_n)}}(A^T \cdot s)$$

for $s \in \mathbb{R}^{dn}$. Let $s' = (A^T)^{-1} \cdot s$, and set s' instead of s we get that

$$\hat{\mathbb{P}}_{Y_{(t_1,t_2,\ldots,t_n)}}(s) = \hat{\mathbb{P}}_{L_{(t_1,t_2,\ldots,t_n)}}(s).$$

By Lemma A.1.4(1) we obtain that $Y_{(t_1,\ldots,t_n)} \overset{d}{=} L_{(t_1,\ldots,t_n)}$ for all choices of $t_1, ..., t_n$.
By Lemma 3.4.2 we conclude that for $A_1, A_2, ..., A_n \in \mathbb{R}^d$:

$$\begin{aligned}\mathbb{P}(L_{t_1} \in A_1, ..., L_{t_n} \in A_n) &= \mathbb{P}(Y_{t_1} \in A_1, ..., Y_{t_n} \in A_n) \\ &= \int_{\mathbb{R}^d}\int_{A_1}\cdots\int_{A_n} \mathbb{1} P_{t_n - t_{n-1}}(y_{n-1}, dy_n)\cdots P_{t_1-0}(y_0, dy_1)\mu(dy_0),\end{aligned}$$

which again by Lemma 3.4.2 makes L a Markov process with Markov transition function $(P_t)_{t\geq 0}$. ■

Remark 4.2.8. From the proof it is clear that a Lévy process $(L_t)_{t\geq 0}$ is a Markov process with the time homogeneous Markov transition function, $(P_t)_{t\geq 0}$, defined as

$$P_t(x, A) = \mathbb{P}_{L_t}(A - x)$$

for $x \in \mathbb{R}^d$ and $A \in \mathscr{B}(\mathbb{R}^d)$.

The following example is inspired by [Ped20, Example 9.13]

Example 4.2.9. (§§) Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda > 0$. In Example 4.1.6 we showed that this is a Lévy process. Define $P_t(k, A) := \mathbb{P}_{N_t}(A - k)$ for $k \in \mathbb{N}$ and $A \subseteq \mathbb{N}$. By Remark 4.2.8 $(N_t)_{t\geq 0}$ is a Markov process with Markov transition function $(P_t)_{t\geq 0}$. We calculate:

$$\begin{aligned}P_t(k, A) &= \sum_{l \in (A-k)\cap\mathbb{N}} e^{-l\lambda t}\frac{(\lambda t)^l}{l!} \\ &= \sum_{l\in A, l\geq k} e^{-(l-k)\lambda t}\frac{(\lambda t)^{(l-k)}}{(l-k)!}.\end{aligned}$$

Let $(\mathscr{F}_t^N)_{t\geq 0}$ be the filter generated by the process $(N_t)_{t\geq 0}$. For $s, t \geq 0$ and $j \in \mathbb{N}$;

$$\mathbb{P}(N_{t+s} = j|\mathscr{F}_t^N) = P_s(N_t, j) = e^{-(j-N(t))\lambda s}\frac{(\lambda s)^{(j-N(t))}}{(j-N(t))!}\mathbb{1}_{\{j\geq N(t)\}}.$$

Especially

$$\mathbb{P}(N_{t+s} = j|N_t = k) = P_s(k, j) = e^{-(j-k)\lambda s}\frac{(\lambda s)^{(j-k)}}{(j-k)!}\mathbb{1}_{\{k\geq j\}}.$$

That is $\mathbb{P}(N_{t+s} = j|N_t = k) = \mathbb{P}(X = j - k)$, , where $X \sim Pois(\lambda s)$.

□

4.3 Infinitely Divisible Distributions

Infinitely divisible distributions are a class of distributions, that can be written as a convolution between n identical measures for any $n \in \mathbb{N}$. We will show that the distribution L_t, where L is a Lévy process, is infinitely divisible. Key references in this section is [Sat13] and [Ped20].

Definition 4.3.1. [Sat13, Definition 7.1] The probability measure μ is called an infinitely divisible distribution if for all $n \in \mathbb{N}$ there exists a probability measure μ_n such that $\mu = \mu_n^n$ (μ_n convoluted with itself n times).

□

In [Ped20, Proposition 9.20] we see that if L is a Lévy process then $\mathbb{P}_{L_t}$ is infinitely divisible. Set $\nu_t := \mathbb{P}_{L_t}$ for all $t \geq 0$ and note by Lemma 4.2.4 that $(\nu_t)_{t\geq 0}$ is a regular convolution semigroup. Hence $\nu_t = (\nu_{t/n})^n$.

We will need some properties of the characteristic function in conjunction with the concept of convolution. We state them here without a proof. For proofs of the statements see [Sat13, Proposition 2.5 (i), (iii), (vii), (viii), page 34 and Lemma 7.8].

Lemma 4.3.2. *[Sat13] Let $\nu, \mu, (\mu_k)_{k\in\mathbb{N}}$ be probability measures on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$:*

(a) $\widehat{\mu * \nu}(z) = \hat{\mu}(z) \cdot \hat{\nu}(z)$ for all $z \in \mathbb{R}^d$.

(b) Assume that $\hat{\mu}_n(z) \to \hat{\mu}(z)$ for $n \to \infty$ for all $z \in \mathbb{R}^d$. Then $\mu_n \xrightarrow{w} \mu$ for $n \to \infty$.

(c) Let ψ be a complex function that is continuous in 0. Assume that $\hat{\mu}_n(z) \to \psi(z)$ for $n \to \infty$ for all $z \in \mathbb{R}^d$. Then $\psi(z)$ is a characteristic function of some distribution on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$.

(d) If μ is infinitely divisible and $\mu = \mu_n^n$. Then μ_n is unique and $\hat{\mu}_n(z) = \hat{\mu}(z)^{1/n}$.

(e) If μ and ν are infinitely divisible $\mu * \nu$ is infinitely divisible as well.

(f) (Bochner's Theorem) Let $\psi(z)$ be a complex valued function on $\mathbb{R}^d$, continuous in 0 and $\psi(0) = 1$. Further assume that for all $n \in \mathbb{N}$:

$$\sum_{l,k=1}^{n} \psi(z_l - z_k)\zeta_l\overline{\zeta}_k \geq 0, \text{ for } z_1, ..., z_n \in \mathbb{R}^d \text{ and } \zeta_1, ..., \zeta_n \in \mathbb{C}. \tag{4.6}$$

Then $\psi(z)$ is a characteristic function of some distribution on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. Also if μ is a distribution on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$, $\hat{\mu}$ is 1 in 0, continuous in 0 and meets Equation (4.6).

(g) If $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of infinitely divisible probability measures and $\mu_n \xrightarrow{w} \mu$ for $n \to \infty$, then μ is infinitely divisible.

□

The following lemma is a combination of [Sat13, Lemma 7.9 and Lemma 7.10(i)]. The proof presented here is an elaboration of the ones given in [Sat13].

Lemma 4.3.3. *Let μ be an infinitely divisible distribution on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. The definition of the measure μ^n for $n \in \mathbb{N}$ can be naturally expanded to μ^t for $t \in [0, \infty)$ and μ^t is infinitely divisible. Moreover let $(L_t)_{t \geq 0}$ be a Lévy process and define $\mu := \mathbb{P}_{L_1}$, then $\mathbb{P}_{L_t} = \mu^t$, and the characteristic function $\hat{\mathbb{P}}_{L_t}(z) = \hat{\mu}(z)^t$ for all $z \in \mathbb{R}^d$.*

Proof. (§) We let $\mu^{1/n}$ be the measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ such that $\mu = (\mu^{1/n})^n$, then this measure will also be infinitely divisible. This comes from the fact that for $k \in \mathbb{N}$, there exist a measure, $\mu^{1/kn}$, such that $(\mu^{1/kn})^{kn} = \mu$. We then have that $((\mu^{1/kn})^k)^n = \mu$ by associativity of convolution. Thus $(\mu^{1/kn})^k$ has the same property as $\mu^{1/n}$. By Lemma 4.3.2 (d), $\mu^{1/n}$ is unique. Thus

$$(\mu^{1/kn})^k = \mu^{1/n}.$$

This shows that $\mu^{1/n}$ is infinitely divisible. By Lemma 4.3.2 (e), $\mu^{m/n} := (\mu^{1/n})^m$ is infinitely divisible. Let $t \in \mathbb{R}$ and $(r_n)_{n\in\mathbb{N}} \in \mathbb{Q}$ be a sequence of rational numbers such that $r_n \to t$ for $n \to \infty$. We have by Lemma 4.3.2 (a) and (d) that $\widehat{\mu^{m/n}}(z) = \hat{\mu}(z)^{m/n}$ and that $\hat{\mu}(z)^{r_n} \to \hat{\mu}(z)^t$. As $\hat{\mu}(z)^{r_n}$ is continuous in 0 by Lemma 4.3.2 (f), then $\hat{\mu}^t$ is continuous in 0 as well. By Lemma 4.3.2 (c) $\hat{\mu}^t$ is a characteristic function of some distribution on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. We can define μ^t to be the measure having the characteristic function $\hat{\mu}^t$. Thus by Lemma 4.3.2 (b) and (g) μ^t is infinitely divisible.
By stationary and independent increments of Lévy processes, we can write $\mu = (\mathbb{P}_{L_{1/n}})^n$. That makes $\mathbb{P}_{L_{1/n}} = \mu^{1/n}$ and $\mathbb{P}_{L_{m/n}} = \mu^{m/n}$. Now we let $(r_n)_{n\in\mathbb{N}} \in \mathbb{Q}$ such that $t \leq r_n$ and $r_n \to t$ for $n \to \infty$. As L has right continuous paths, $L_{r_n} \to L_t$ almost surely, and by [Bal17, Proposition 1.5] the convergence also holds true in law. As a consequence

$$\lim_{n\to\infty} \hat{\mathbb{P}}_{L_{r_n}}(z) = \hat{\mathbb{P}}_{L_t}(z)$$

for every $z \in \mathbb{R}^d$. As $\mathbb{P}_{L_{r_n}} = \mu^{r_n}$ and thus for all $z \in \mathbb{R}^d$, $\hat{\mathbb{P}}_{L_{r_n}}(z) = \hat{\mu}^{r_n}(z)$. Taking the limit on both sides gives us $\hat{\mathbb{P}}_{L_t}(z) = \hat{\mu}^t(z)$. Again by properties of the characteristic function $\mathbb{P}_{L_t} = \mu^t$. ■

Here we present the Lévy-Khintchine representation of infinitely divisible distributions. The theorem will allow us to characterize Lévy processes. We state the theorem without giving a proof. Later we will give a full proof of the uniqueness of the representation (Proposition 4.3.7).

Theorem 4.3.4. *[Sat13, Theorem 8.1] Let μ be an infinitely divisible distribution on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$, then we can write the characteristic function of μ as*

$$\begin{aligned}\hat{\mu}(z) = \exp(-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \\ \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbb{1}_{\{x:|x|\leq 1\}}(x))\nu(dx)),\end{aligned} \tag{4.7}$$

for $z \in \mathbb{R}^d$. A is a $d \times d$ positive semi-definite matrix, $\gamma \in \mathbb{R}^d$ and ν is a Lévy measure, i.e. $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \min(1, x^2)\nu(dx) < \infty. \tag{4.8}$$

This representation is known as the Lévy-Khintchine representation, and (A, ν, γ) is called the generating triplet of μ.

Remark 4.3.5. If $\int_{\{|x|\leq 1\}} |x|\nu(dx) < \infty$, we can write

$$\hat{\mu}(z) = \exp[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_0, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx)],$$

where A and ν is as in Equation (4.7), but γ_0 is different that γ. We call γ_0 the *drift*, and we can write the alternative generating triple $(A, \nu, \gamma_0)_0$, where the zero in the subscript in the triple indicates that we have changed the $\mathbb{1}_{\{x:|x|\leq 1\}}$ to 0 in the representation.

Remark 4.3.6. By [Sat13, Remark 8.4] there exists a broad class of functions, $\mathscr{C}$, that we can set instead of the function $\mathbb{1}_{\{x:|x|\leq 1\}}(x)$ in the representation of the characteristic function of μ. If we set $f \in \mathscr{C}$ instead of $\mathbb{1}_{\{x:|x|\leq 1\}}(x)$, we can find a new generating triplet for μ, $(A_f, \gamma_f, \nu_f)_f$, where $A_f = A$ and $\nu_f = \nu$. It will be useful in the proof of the theorem that the family of functions, $\{\mathbb{1}_{D(\epsilon)}(x)\}_{\epsilon\in(0,1)}$, where $D(\epsilon) = \{x : |x| \leq \epsilon\}$, is contained in $\mathscr{C}$.

□

Proposition 4.3.7. *[Sat13, Theorem 8.1(ii)] Let μ be an infinitely divisible distribution on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. Then the generating triplet, (A, γ, ν), from the representation Equation* (4.7) *is unique.*

□

For making a proof of the proposition we will need some technical results. The results are either given as lemmas in [Sat13] or it is taken as given in the proof of [Sat13, Theorem 8.1(ii)].

Lemma 4.3.8. *[Sat13, pp. 40–41] Let $x \in \mathbb{R}^d$. Then we have the following equality:*

$$\int_{[-1,1]^d} (1 - e^{i\langle x,w\rangle})dw = 2^d(1 - \prod_{i=1}^{d} \frac{sin(x_i)}{x_i}),$$

if $x_i \neq 0$ for all $i \in \{1, ..., d\}$. If $x_i = 0$ for an $i \in \{1, ..., d\}$, then set 1 in the expression instead of $\sin(x_i)/x_i$.

Proof. (§§§) By [Pou15, Definition 2.25] we have that $e^{i\langle x,w\rangle} = \cos(\langle x, w\rangle) + i\sin(\langle x, w\rangle)$. So

$$\begin{aligned}
&\int_{[-1,1]^d} (1 - e^{i\cdot\langle x,w\rangle})dw \\
&\quad= \int_{[-1,1]^d} 1dw - \int_{[-1,1]^d} [\cos(\langle x, w\rangle) + i\sin(\langle x, w\rangle)]dw \\
&\quad= 2^d - \int_{[-1,1]^d} \cos(\langle x, w\rangle)dw - i\int_{[-1,1]^d} \sin(\langle x, w\rangle)dw.
\end{aligned}$$

We can write $\langle x, w\rangle = \sum_{i=1}^{d} x_i w_i$. From [Pou15, Theorem 2.17] (the trigonometric addition formulas) we have that

$$\sin(s+t) = \sin(s)\cos(t) + \sin(t)\cos(s)$$

and

$$\cos(s+t) = \cos(s)\cos(t) - \sin(s)\sin(t)$$

for $s, t \in \mathbb{R}$. We also recall the following properties of cosine and sine: $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ for a real number x. Thus

$$\int_{[-x,x]} \sin(t)dt = \int_{[0,x]} \sin(t)dt - \int_{[0,x]} \sin(t)dt = 0$$

and

$$\int_{[-x,x]} \cos(t)dt = 2\int_{[0,x]} \cos(t)dt = 2 \cdot \sin(x).$$

Keeping these properties in mind we can proceed. First we want to show that

$$\int_{[-1,1]^d} \cos(\langle x, w\rangle)dw = 2^d \prod_{i=1}^{d} \frac{\sin(x_i)}{x_i}.$$

We prove this by induction. Let $d = 1$ and assume $x \neq 0$. We use integration by substitution to calculate the expression $\int_{[-1,1]} \cos(xw)dw$. See for example [Pou15, Theorem 8.22(a)]. Let $g(w) = xw$. Then we have $g'(w) = x$.

$$\begin{aligned}
\int_{[-1,1]} \cos(xw)dw &= \frac{1}{x}\int_{[-1,1]} x\cos(xw)dw \\
&= \frac{1}{x}\int_{[g(-1),g(1)]} \cos(g(w))g'(w)dg(w) \\
&= \frac{1}{x}[\sin(u)]_{-x}^{x} = 2 \cdot \frac{\sin(x)}{x}.
\end{aligned}$$

If $x = 0$ then

$$\int_{[-1,1]} \cos(xw)dw = \int_{[-1,1]} \cos(0)dw = 2.$$

We note that by doing a calculation like the one above we obtain that $\int_{[-1,1]} \sin(xw)dw = 0$ for $x \in \mathbb{R}$.
Assume the expression holds true for $d - 1 \geq 1$:

$$\begin{aligned}
\int_{[-1,1]^d} \cos(\langle x, w\rangle)dw &= \int_{[-1,1]^d} \cos(\sum_{i=1}^{d} x_i w_i)dw_1 \cdots dw_d \\
&= \int_{[-1,1]^d} \cos(x_1 w_1)\cos(\sum_{i=2}^{d} x_i w_i)dw_1 \cdots dw_d \\
&\quad - \int_{[-1,1]^d} \sin(x_1 w_1)\sin(\sum_{i=2}^{d} x_i w_i)dw_1 \cdots dw_d \\
&= \int_{[-1,1]} \cos(x_1 w_1)dw_1 \int_{[-1,1]^{d-1}} \cos(\sum_{i=2}^{d} x_i w_i)dw_2 \cdots dw_d
\end{aligned}$$

$$
\begin{aligned}
&- \int_{[-1,1]} \sin(x_1 w_1) dw_1 \int_{[-1,1]^{d-1}} \sin(\sum_{i=2}^{d} x_i w_i) dw_2 \cdots dw_d \\
&= 2 \cdot \frac{\sin(x_1)}{x_1} \int_{[-1,1]^{d-1}} \cos(\sum_{i=2}^{d} x_i w_i) dw_2 \cdots dw_d - 0 \\
&= 2^d \prod_{i=1}^{d} \frac{\sin(x_i)}{x_i}.
\end{aligned}
$$

In the last equality we used the induction assumption.
By induction we can show that $i \int_{[-1,1]^d} \sin(\langle x, w\rangle) dw = 0$ again using the trigonometric addition formulas. We leave out this part.
By putting the calculations together, we obtain the expression stated in the lemma. ∎

In [Sat13] it is asserted that $\rho(dx) = 2^d(1 - \prod_{i=1}^{d} \frac{\sin(x_i)}{x_i})dx$ is a finite measure. The following lemma is a part of showing that.

Lemma 4.3.9. *Let $d \in \mathbb{N} \setminus \{0\}$ and $x \in \mathbb{R}^d$ be given such that $|x|^2 < 1$. Then*

$$(1 - \prod_{i=1}^{d} \frac{\sin(x_i)}{x_i}) \leq |x|^2,$$

where it is assumed that $\frac{\sin(x_i)}{x_i} = 1$ if $x_i = 0$.

Proof. (§§§) Assume that $|x| = 0$. Then $x_i = 0$ for all i, and both the right hand and the left hand side becomes 0. As $|x|^2 = \sum_{i=1}^{d} x_i^2$, the assumption that $|x|^2 < 1$ implies that $x_i^2 < 1$ for all $i \in \{1, ..., d\}$. The Taylor series of the sine function is given by

$$\sin(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}.$$

This immediately gives us that $\sin(x) < x$ and $\sin(x) > x - x^3/3!$ for $x \in (0, 1)$.
We show the lemma by induction in d.
Let $d = 1$ and $x \in (0, 1)$ such that $0 < |x|^2 < 1$.

$$(1 - \frac{\sin(x)}{x}) < (1 - \frac{x - x^3/3!}{x}) < \frac{x^2}{3!} < x^2 = |x|^2.$$

This calculation also holds true for $x \in (-1, 0)$ because both $x \mapsto sin(x)/x$ and $x \mapsto (x - x^3/3!)/x$ are symmetric around zero.
Now assume that $d > 1$ and assume that the claim holds true for $d - 1$. Also assume that $|x|^2 > 0$. If $x_d = 0$, the statement is obvious. Otherwise we

calculate.

$$
\begin{aligned}
(1-\prod_{i=1}^{d}\frac{\sin(x_i)}{x_i}) &< (1-\prod_{i=1}^{d-1}\frac{\sin(x_i)}{x_i}(\frac{x_d - x_d^3/3!}{x_d})) \\
&= (1-\prod_{i=1}^{d-1}\frac{\sin(x_i)}{x_i}) + \frac{x_d^2}{3!} \\
&\leq \sum_{i=1}^{d-1} x_i^2 + \frac{x_d^2}{3!} < |x|^2,
\end{aligned}
$$

where we used the induction assumption in the last line. ■

Lemma 4.3.10. *Let $x \in \mathbb{R}^d$. Then*

$$\int_{[-1,1]^d} \langle x, w\rangle dw = 0.$$

Proof. (§§§) We compute

$$
\begin{aligned}
\int_{[-1,1]^d} \langle x, w\rangle dw &= \int_{[-1,1]^d} \sum_{k=1}^{d} x_k w_k dw_1 \cdots dw_d \\
&= \sum_{k=1}^{d} \int_{[-1,1]} x_k w_k dw_k = 0.
\end{aligned}
$$

■

Lemma 4.3.11. *[Sat13, Lemma 8.6]*

$$e^{iu} = \sum_{k=0}^{n-1} \frac{(iu)^k}{k!} + \theta \cdot \frac{|u|^n}{n!},$$

where $u \in \mathbb{R}$, $n \in \mathbb{N}$ and $\theta \in \mathbb{C}$, where $|\theta| \leq 1$.

Proof. (§§) Prove first that for each $u \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$\frac{i^n}{(n-1)!}\int_0^u (u-v)^{n-1} e^{iv} dv = \theta \cdot \frac{|u|^n}{n!}.$$

$$
\begin{aligned}
|\frac{i^n}{(n-1)!}\int_0^u (u-v)^{n-1} e^{iv} dv| &\leq |\frac{i^n}{(n-1)!}| \int_0^u |(u-v)^{n-1} e^{iv}| dv \\
&\leq \frac{1}{(n-1)!}\int_0^u |u-v|^{n-1} dv \\
&= \frac{1}{(n-1)!}\int_0^u |v|^{n-1} dv = \frac{|u|^n}{n!}.
\end{aligned}
$$

As the expression on the left hand side inside the norm is a complex number, it can be written on the form: $a + ib$, where a, b are real numbers. If $u = 0$, then

$a, b = 0$. If $u \neq 0$ there exist real numbers c, d such that $a + ib = (c + id)\frac{|u|^n}{n!}$. It follows that

$$|(c + id)\frac{|u|^n}{n!}| = |a + ib| \leq \frac{|u|^n}{n!},$$

making $|c + id| \leq 1$, thus we have the integral expression on the desired form. The only thing left to show now is that

$$e^{iu} = \sum_{k=0}^{n-1} \frac{(iu)^k}{k!} + \frac{i^n}{(n-1)!} \int_0^u (u - v)^{n-1} e^{iv} dv.$$

Fix $n \in \mathbb{N}$ and $u \in \mathbb{R}$. First we show that the integral part on the right hand side can be split up into a new sum:

$$\frac{i^n}{(n-1)!} \int_0^u (u - v)^{n-1} e^{iv} dv = \sum_{j=n}^{n+k} \frac{(iu)^j}{j!} + \frac{i^{n+k}}{(n+k-1)!} \int_0^u (u - v)^{n+k-1} e^{iv} dv.$$

It will be sufficient to show the property for $k = 1$ because n is arbitrary. By *integration by parts*, we obtain:

$$\begin{aligned}
&\frac{i^n}{(n-1)!} \int_0^u (u - v)^{n-1} e^{iv} dv \\
&\qquad = \frac{i^n}{(n-1)!}[-\frac{1}{n}(u - v)^n e^{iv}]_0^u + \frac{i^{n+1}}{n!} \int_0^u (u - v)^n e^{iv} dv \\
&\qquad = \frac{(iu)^n}{n!} + \frac{i^{n+1}}{n!} \int_0^u (u - v)^n e^{iv} dv.
\end{aligned}$$

Fix $l, n \in \mathbb{N}$. Using the triangular inequality for complex numbers and the first property and the equality just shown, we obtain:

$$\begin{aligned}
|e^{iu} - (&\sum_{k=0}^{n-1} \frac{(iu)^k}{k!} + \frac{i^n}{(n-1)!} \int_0^u (u - v)^{n-1} e^{iv} dv)| \\
&= |e^{iu} - \sum_{k=0}^{n+l-1} \frac{(iu)^k}{k!} + \frac{i^{n+l}}{(n+l-1)!} \int_0^u (u - v)^{n+l-1} e^{iv} dv| \\
&\leq |e^{iu} - \sum_{k=0}^{n+l-1} \frac{(iu)^k}{k!}| + |\frac{i^{n+l}}{(n+l-1)!} \int_0^u (u - v)^{n+l-1} e^{iv} dv| \\
&= |e^{iu} - \sum_{k=0}^{n+l-1} \frac{(iu)^k}{k!}| + \frac{|u|^{n+l}}{(n+l-1)!}.
\end{aligned}$$

As $e^{iu} = \sum_{k=0}^{\infty} \frac{(iu)^k}{k!}$ both parts on the right hand side goes to zero as l goes to infinity. As l can be chosen arbitrarily big, the desired equality must hold true. ■

Now we are ready to prove the uniqueness of the Lévy-Khintchine representation. The following is an elaboration of the proof given in [Sat13, Theorem 8.1(ii)].

Proof of Proposition 4.3.7. (§§) We start off proving that $e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle\mathbb{1}_D(x)$ is integrable with respect to ν. Here D denotes $\{x : |x| \leq 1\}$ We assume that $|x| \leq 1$. We use *Lemma* 4.3.11 with $n = 2$ to write

$$e^{i\langle z,x\rangle} = 1 + i\langle z,x\rangle + \frac{\theta}{2}|\langle z,x\rangle|^2.$$

Thus

$$\begin{aligned}|e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle| &= |1 + i\langle z,x\rangle + \frac{\theta}{2}|\langle z,x\rangle|^2 - 1 - i\langle z,x\rangle| \\ &\leq \frac{1}{2}|\langle z,x\rangle|^2 \leq \frac{1}{2}|x|^2|z|^2.\end{aligned}$$

The last inequality follows from Cauchy-Schwartz' inequality for inner products (see e.g. [Tho14, Theorem 9.1.4(iii)]). For $x > 1$ we can just use the fact that $|e^{iu}| \leq 1$ for $u \in \mathbb{R}$. Thus

$$|e^{i\langle z,x\rangle} - 1| \leq |e^{i\langle z,x\rangle}| + 1 \leq 2,$$

using the triangular inequality for norms. Now we can write

$$\begin{aligned}|\int_{\mathbb{R}^d} e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle\mathbb{1}_D(x)\nu(dx)| &\leq \int_{\mathbb{R}^d} |e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle\mathbb{1}_D(x)|\nu(dx) \\ &= \int_D |e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle\mathbb{1}_D(x)|\nu(dx) \\ &\qquad + \int_{D^c} |e^{i\langle z,x\rangle} - 1|\nu(dx) \\ &\leq \int_D \frac{1}{2}|x|^2|z|^2\nu(dx) + \int_{D^c} 2\nu(dx) \\ &= \frac{1}{2}|z|^2 \int_D |x|^2\nu(dx) + 2\int_{D^c} 1\nu(dx) \\ &\leq \max(\frac{1}{2}|z|^2, 2) \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty.\end{aligned}$$

In the first inequality we use [Tho14, Theorem 5.4.6(iv)] and at last we use the property of Lévy measures.
If we let $0 \leq \epsilon < 1$, we can let $D(\epsilon) := \{x : |x| \leq \epsilon\}$ we can obtain the more general inequality:

$$\begin{aligned}&|\int_{\mathbb{R}^d} e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle\mathbb{1}_{D(\epsilon)}(x)\nu(dx)| \\ &\qquad \leq \frac{1}{2}|z|^2 \int_{D(\epsilon)} |x|^2\nu(dx) + 2\int_{D(\epsilon)^c} 1\nu(dx) < \infty.\end{aligned}$$

The fact that it is less than infinity needs an elaboration. The first part of the expression is obviously less that infinity. We show that the latter part of the expression is less than infinity as well:

$$\begin{aligned}2\int_{D(\epsilon)^c} 1\nu(dx) &= \frac{2}{\epsilon^2}\int_{D(\epsilon)^c} \epsilon^2\nu(dx) \\ &\leq \frac{2}{\epsilon^2}\int_{D(\epsilon)^c} (|x|^2 \wedge 1)\nu(dx) < \infty.\end{aligned}$$

Now let $s > 0$, then

$$\log(\hat{\mu}(sz)) = -\frac{1}{2}s^2\langle z, Az\rangle + is\langle \gamma, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle sz,x\rangle} - 1 - i\langle sz, x\rangle \mathbb{1}_D(x))\nu(dx)$$

In general we can not be sure that the logarithm of a complex number is unique, but [Sat13, Lemma 7.6] secures in this case that the logarithm is unique. We just need to observe that $\hat{\mu}(z)$ is continuous, $\log(\hat{\mu}(0)) = 0$ and $\exp(\log(\hat{\mu}(z))) = \hat{\mu}(z)$.
We want to show that

$$\lim_{s\to\infty} |s^{-2}\log(\hat{\mu}(sz)) + \frac{1}{2}\langle z, Az\rangle| = 0.$$

If we take an $\epsilon \in (0, 1)$, we can obtain the generating triplet, $(A, \gamma_\epsilon, \nu)_\epsilon$ that fits the representation where we use the function $\mathbb{1}_{D(\epsilon)}(x)$. We write

$$\begin{aligned}
&|s^{-2}\log(\hat{\mu}(sz)) + \frac{1}{2}\langle z, Az\rangle| \\
&\quad = |s^{-1}i\langle \gamma_\epsilon, z\rangle + s^{-2}\int_{\mathbb{R}^d} (e^{i\langle sz,x\rangle} - 1 - i\langle sz, x\rangle \mathbb{1}_{D(\epsilon)}(x))\nu(dx)| \\
&\quad \leq s^{-1}|i\langle \gamma_\epsilon, z\rangle| + s^{-2}|\int_{\mathbb{R}^d} (e^{i\langle sz,x\rangle} - 1 - i\langle sz, x\rangle \mathbb{1}_{D(\epsilon)}(x))\nu(dx)|.
\end{aligned}$$

We note that $\lim_{s\to\infty} s^{-1}|i\langle \gamma_\epsilon, z\rangle| = 0$, so we can turn the attention to the second part of the upper bound:

$$\begin{aligned}
&s^{-2}|\int_{\mathbb{R}^d} (e^{i\langle sz,x\rangle} - 1 - i\langle sz, x\rangle \mathbb{1}_{D(\epsilon)}(x))\nu(dx)| \\
&\qquad \leq s^{-2}\frac{1}{2}|sz|^2 \int_{D(\epsilon)} |x|^2\nu(dx) + s^{-2}[2\int_{D(\epsilon)^c} 1\nu(dx)] \\
&\qquad = \frac{1}{2}|z|^2 \int_{D(\epsilon)} |x|^2\nu(dx) + s^{-2}[2\int_{D(\epsilon)^c} 1\nu(dx)].
\end{aligned}$$

Again we can note that $\lim_{s\to\infty} s^{-2}[2\int_{D(\epsilon)^c} 1\nu(dx)] = 0$. To sum up we get that

$$\lim_{s\to\infty} |s^{-2}\log(\hat{\mu}(sz)) + \frac{1}{2}\langle z, Az\rangle| \leq \frac{1}{2}|z|^2 \int_{D(\epsilon)} |x|^2\nu(dx), \qquad \forall\epsilon \in (0, 1).$$

As $\nu(\{0\}) = 0$ and

$$\int_{D(\epsilon)} |x|^2\nu(dx) \leq \int_D |x|^2\nu(dx) < \infty,$$

we must have that

$$\lim_{\epsilon\to 0+} \int_{D(\epsilon)} |x|^2\nu(dx) = 0.$$

To sum it up we get

$$\lim_{s\to\infty} s^{-2}\log(\hat{\mu}(sz)) = \frac{1}{2}\langle z, Az\rangle$$

for all $z \in \mathbb{R}^d$. Thus A is uniquely determined by μ.
Now we define the function

$$\psi(z) := \log(\hat{\mu}(z)) + \frac{1}{2}\langle z, Az\rangle,$$

and note that

$$\psi(z) - \psi(z+w) = \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - e^{i\langle z+w,x\rangle} + i\langle w,x\rangle \mathbb{1}_D(x))\nu(dx) - i\langle \gamma, w\rangle.$$

We check that the integrand is integrable. We only have to consider the case where $|x| \leq 1$.

$$\begin{aligned}
|e^{i\langle z,x\rangle} - &e^{i\langle z+w,x\rangle} + i\langle w,x\rangle| \\
&= |e^{i\langle z,x\rangle}(1 - e^{i\langle w,x\rangle}) + e^{i\langle z,x\rangle} i\langle w,x\rangle + (1 - e^{i\langle z,x\rangle}) i\langle w,x\rangle| \\
&\leq |1 - e^{i\langle w,x\rangle} + i\langle w,x\rangle| + |\langle w,x\rangle||1 - e^{i\langle z,x\rangle}| \\
&\leq \frac{1}{2}|w|^2|x|^2 + |z||x|^2|w| \\
&\leq \max(\frac{1}{2}|w|^2, |w||z|) \cdot |x|^2.
\end{aligned}$$

Here we have used previous calculations and Lemma 4.3.11 with $n = 1$. From this we can conclude that the integral is well defined and finite. I integrate with respect to w:

$$\begin{aligned}
\int_{[-1,1]^d} &\psi(z) - \psi(z+w)dw \\
&= \int_{[-1,1]^d} [\int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - e^{i\langle z+w,x\rangle} + i\langle w,x\rangle \mathbb{1}_D(x))\nu(dx) - i\langle \gamma, w\rangle]dw \\
&= \int_{[-1,1]^d} \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - e^{i\langle z+w,x\rangle} + i\langle w,x\rangle \mathbb{1}_D(x))\nu(dx)dw \\
&\qquad\qquad - i\int_{[-1,1]^d} \langle \gamma, w\rangle dw \\
&= \int_{\mathbb{R}^d} [\int_{[-1,1]^d} e^{i\langle z,x\rangle}(1 - e^{i\langle w,x\rangle})dw + i\int_{[-1,1]^d} \langle w,x\rangle \mathbb{1}_D(x)dw]\nu(dx) \\
&= \int_{\mathbb{R}^d} [e^{i\langle z,x\rangle} \int_{[-1,1]^d} (1 - e^{i\langle w,x\rangle})dw]\nu(dx) \\
&= 2^d \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}(1 - \prod_{i=1}^d \frac{sin(x_i)}{x_i})\nu(dx).
\end{aligned}$$

During this calculation, both Lemma 4.3.10 and Lemma 4.3.8 was used.
The expression can be recognized as the Fourier transform of the measure

$$\rho(dx) := 2^d(1 - \prod_{i=1}^d \frac{sin(x_i)}{x_i})\nu(dx). \tag{4.9}$$

We will show that this is a finite measure, because then we know it is uniquely determined by it's Fourier transform (see Lemma A.1.4(1)). It is understood by Lemma 4.3.8 that if $x_i = 0$ for some $i \in \{1, ...d\}$, then set $\sin(x_i)/x_i = 1$. Remark that the density of ρ with respect to ν is positive for all $x \neq 0$. For $x = 0$ it is 0. This means that ν is uniquely determined by ρ because we know already that $\nu(\{0\}) = 0$. Generally we have that

$$2^d(1 - \prod_{i=1}^{d} \frac{sin(x_i)}{x_i}) \leq 2^{d+1},$$

for all $x \in \mathbb{R}^d$. By the properties of a Lévy process, this means we only need to concentrate on the part where $0 < |x|^2 < 1$. In Lemma 4.3.9 it is shown that

$$(1 - \prod_{i=1}^{d} \frac{\sin(x_i)}{x_i}) \leq |x|^2,$$

which is enough because we know that ν is a Lévy measure.
As ψ is completely determined from A and μ, we get that ν is unique for μ. As both ν and A is uniquely determined, γ can be determined as well. ■

As L_t is infinitely divisible for all $t \geq 0$, $\mathbb{P}_{L_t}$ will have a generating triplet. we call this the generating triplet of L.

Remark 4.3.12. [EK19, Section 2.2] For a Lévy process L, the characteristic function of L_1 can be written as $\exp(\psi(u))$, so by Lemma 4.3.3 we have that the characteristic function of L_t is $\exp(t\psi(u))$ for all $t \geq 0$.

The following Lévy-Khintchine representation of the Poisson process is given in [EK19, Section 2.4]. We show that the representation can be found by calculating the characteristic function directly and recognizing the different parts.

Example 4.3.13. (§§) Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda > 0$. We calculate the characteristic function of N_1:

$$\begin{aligned}\mathbb{E}[\exp(izN_1)] &= \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \exp(izk) \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda \exp(iz))^k}{k!} \\ &= \exp(\lambda e^{iz} - \lambda) \\ &= \exp(\lambda \cdot (e^{iz} - 1)),\end{aligned}$$

for $z \in \mathbb{R}$.
Consider the Lévy-Khintchine representation $(0, \lambda, \lambda\delta_1)$. This gives rise to the characteristic function:

$$\begin{aligned}&\exp(iz\lambda + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbb{1}_{\{x:|x|\leq 1\}})\lambda\delta_1(dx)) \\ &\qquad = \exp(iz\lambda + \lambda(e^{iz} - 1 - iz)) \\ &\qquad = \exp(\lambda(e^{iz} - 1)).\end{aligned}$$

Thus $(N_t)_{t\geq 0}$ has $(0, \lambda, \lambda\delta_1)$ as its Lévy-Khintchine representation.

□

4.4 Processes with Independent Increments

As we have seen, a Lévy process is a time homogeneous Markov process. It turns out that we can loosen up on the Lévy conditions and still keep the Markov property. Actually, we can be content with the condition of independent increments.

Theorem 4.4.1. *Let X be a stochastic process that takes values on $\mathbb{R}^d$ defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that X has independent increments. Then it is a Markov process.*

The Markov transition function associated with this Markov process will be specified here, before we prove that the process is in fact a Markov process. Some of the calculation are similar to the ones int he proof of Lemma 4.2.5.

Lemma 4.4.2. *Let X be a stochastic process with independent increments. Define for $(x, A) \in (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ the group $(P_{s,t})_{t\geq s\geq 0}$ as*

$$P_{s,t}(x, A) := \mathbb{P}_{X_t - X_s}(A - x).$$

Then $(P_{s,t})_{t\geq s\geq 0}$ is a Markov transition function.

Proof. (§§) First remark that by Lemma 4.2.6 $(P_{s,t})_{t\geq s\geq 0}$ is a family of transition kernels. We calculate

$$P_{s,s}(x, A) = \mathbb{P}_{X_s - X_s}(A - x) = \mathbb{P}_0(A - x) = \delta_0(A - x) = \delta_x(A).$$

Also we have that for $t \geq s \geq u \geq 0$

$$\begin{aligned} P_{u,s}P_{s,t}(x, A) &= \int_{\mathbb{R}^d} P_{s,t}(y, A)P_{u,s}(x, dy) \\ &= \int_{\mathbb{R}^d} \mathbb{P}_{X_t - X_s}(A - y)P_{u,s}(x, dy) \\ &= \int_{\mathbb{R}^d} \mathbb{P}_{X_t - X_s}(A - y - x)\mathbb{P}_{X_s - X_u}(dy) \\ &= \mathbb{P}_{X_s - X_u} * \mathbb{P}_{X_t - X_s}(A - x) \\ &= \mathbb{P}_{X_t - X_s + X_s - X_u}(A - x) \\ &= \mathbb{P}_{X_t - X_u}(A - x) = P_{u,t}(x, A). \end{aligned}$$

We used that X has independent increments, which gives us that $(X_t - X_s)$ is independent of $(X_s - X_u)$. So we have shown what we set out to show. ■

Now we show that X is in fact the Markov process associated with the Markov transition function that was just defined. The arguments utilized below are in essence the same as the ones used to show that a Lévy process is a Markov process (Theorem 4.2.7). Thus, the ideas again come from [Ped20].

Proof of Theorem 4.4.1. (§) Let $(X_t)_{t\geq 0}$ be a stochastic process with independent increments and define the group $(P_{s,t})_{t\geq s\geq 0}$ by $P_{s,t}(x, A) := \mathbb{P}_{X_t - X_s}(A - x)$ for all $t \geq s \geq 0$ and $(x, A) \in (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. Then by Lemma 4.4.2 this is a Markov transition function. By Theorem 3.4.3 we have that there exist a Markov process $(Y_t)_{t\geq 0}$ such that for all $t \geq s \geq 0$ and $f \in \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$

$$\mathbb{E}[f(Y_t)|\mathscr{F}_s^Y] = P_{s,t}(f(Y_s)).$$

We choose the process $(Y_t)_{t\geq 0}$ such that $Y_0 \overset{d}{=} X_0$. We want to show that $(Y_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ are equivalent. We notice that

$$\mathbb{E}[f(Y_t)|\mathscr{F}_s^Y] = \int f(x + Y_s)\mathbb{P}_{X_t - X_s}(dx).$$

By letting $f(\theta) = e^{i\langle \theta, x\rangle}$ for a $\theta \in \mathbb{R}^d$ we get that

$$\mathbb{E}[e^{\langle y, Y_t - Y_s\rangle}|\mathscr{F}_s^Y] = e^{-i\langle \theta, Y_s\rangle}\mathbb{E}[e^{i\langle \theta, Y_t\rangle}|\mathscr{F}_s^Y] = \int e^{i\langle \theta, x\rangle}\mathbb{P}_{X_t - X_s}(dx).$$

As shown in the proof for Theorem 4.2.7, this gives us that $(Y_t)_{t\geq 0}$ has independent increments and that $Y_t - Y_s \sim X_t - X_s$. Conclude that for all $n \in \mathbb{N}$ and $0 < t_1 < t_2 < \cdots < t_n$

$$(X_0, X_{t_1} - X_0, ..., X_{t_n} - X_{t_{n-1}})^T \overset{d}{=} (Y_0, Y_{t_1} - Y_0, ..., Y_{t_n} - Y_{t_{n-1}})^T,$$

which secures that

$$(X_0, X_{t_1}, ..., X_{t_n})^T \overset{d}{=} (Y_0, Y_{t_1}, ..., Y_{t_n})^T.$$

By Lemma 3.4.2 we have that $(Y_t)_{t\geq 0}$ is a Markov process with Markov transition group $(P_{s,t})_{t\geq s\geq 0}$. ■

4.5 The Lévy-Itô Decomposition and Lévy Integration

In the following section, define $D(I) := \{x \in \mathbb{R}^m : |x| \in I\}$, where I is a interval in $[0, \infty]$. In this section we aim to make sense of integration with respect to Lévy processes. In order to do this properly, we start of with a presentation of the Lévy-Itô decomposition. The Lévy-Itô decomposition shows how a Lévy process can be decomposed into two independent processes, namely a *jump process* and a *continuous process*. We state and discuss the result without giving a proof. The result is presented and proved both in [Sat13] and [App09]. In [Sat13] he make use of the Lévy-Khintchine representation to prove it, whereas in [App09], he does not. We use the result to make sense of integration with respect to a Lévy process.

Let $(L_t)_{t\geq 0}$ be a fixed m-dimensional Lévy process with generating triplet (A, ν, γ). Denote by $(H, \mathscr{H})$ the space $H := (0, \infty)\times(\mathbb{R}^m\setminus\{0\})$ and $\mathscr{H} := \mathscr{B}(H)$. We let an $A \in \mathscr{H}$ be given and define the random element:

$$N(A) := \#\{s : (s, \Delta L(s)) \in A\}\mathbb{1}_{\Omega_0},$$

where $\Delta L_t := L(t) - L(t-)$ and Ω_0 is the space where L is càdlàg. As L is a Lévy process, the characteristic function function of L_t can be represented by the triplet $(A_t, \nu_t, \gamma_t) = (tA, t\nu, t\gamma)$ by Remark 4.3.12. In [Sat13, Theorem 19.2(i)] he claims that N is a Poisson random measure on $(H, \mathscr{H})$, with an intensity measure, μ, for which it holds that $\mu((0, t] \times A) = \nu_t(A)$, for $A \in \mathscr{B}(\mathbb{R}^m \setminus \{0\})$.

In [Sat13, Remark 9.9] he claims that such an extension exists in the general case (for additive processes) and he touches on how it can be proved. We are only interested in the case where $\nu_t = t\nu$ (the case of a Lévy process). We prove that such a measure exist by showing that the measure coincides with the product measure ($\lambda \otimes \nu$, where λ is the Lebesgue measure).

Lemma 4.5.1. *There exist a unique measure μ on $(H, \mathscr{H})$ such that $\mu((0,t] \times B) = t\nu(B)$ for $B \in \mathscr{B}(\mathbb{R}^m \setminus \{0\})$.*

Proof. (§§) Assume there exists a measure, μ, on $(H, \mathscr{H})$ such that for all $t \geq 0$, $\mu((0,t] \times B) = t\nu(B)$, where $B \in \mathscr{B}(\mathbb{R}^m \setminus \{0\})$. Remark that such a measure must fulfill for $t \geq s$ that

$$\mu((s,t] \times B) = \mu((0,t] \times B) - \mu((0,s] \times B) = (t-s)\nu(B).$$

Define the product measure on $\mathscr{B}((0,\infty)) \otimes \mathscr{B}(\mathbb{R}^m \setminus \{0\})$, $\lambda \otimes \nu$. In the proof of Proposition 4.3.7 we showed that ν is a finite measure outside an arbitrarily small neighbourhood around 0. By letting $A_n := D(\frac{1}{n}, \infty)$ for all $n \in \mathbb{N}$, we have a sequence of sets, such that $\nu(A_n) < \infty$ and such that $\cup_{n\in\mathbb{N}} A_n = \mathbb{R}^d \setminus \{0\}$. We conclude that both λ and ν are σ-finite measures on $\mathscr{B}((0,\infty))$ and $\mathscr{B}(\mathbb{R}^m \setminus \{0\})$, respectively. This secures by [Tho14, Theorem 6.3.3] that $\lambda \otimes \nu$ exists and is unique.
Define the system of sets

$$\mathscr{D} := \{(s,t] \times B : 0 \leq s \leq t, B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})\}.$$

We have that for a $(s,t] \times B \in \mathscr{D}$ that $\lambda \otimes \nu((s,t] \times B) = (t-s)\mu(B)$. We show properties (a)-(d) of Theorem A.1.14 in order to conclude that $\mu = \lambda \otimes \nu$. (a) $\mathscr{D}$ is stable to intersection. (b) $\mathscr{D}$ generates $\mathscr{H}$. This is secured by [Tho14, Theorem 6.1.6] as the system $\{(s,t] : 0 \leq s \leq t < \infty\}$ generates $\mathscr{B}((0,\infty))$ and $\cup_{n\in\mathbb{N}}(0,n] = (0,\infty)$ and $\cup_{n\in\mathbb{N}}[-n,n]^d \setminus \{0\} = \mathbb{R}^d \setminus \{0\}$. (c) We have shown that μ and $\lambda \otimes \nu$ coincide on $\mathscr{D}$. As for condition (d) let $A_n = (0,n] \times D(\frac{1}{n}, \infty)$. We have that $\cup_{n\in\mathbb{N}} A_n = H$, and that $\lambda \otimes \nu(A_n) = n\nu((\frac{1}{n}, \infty)) = \mu(A_n) < \infty$ for all $n \in \mathbb{N}$.
Together $(a) - (d)$ secures that $\lambda \otimes \nu$ and μ are identical on $(H, \mathscr{H})$. ∎

The following statements come from [Sat13, Theorem 19.2]. For almost all $\omega \in \Omega$ the limit

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \int_{(0,t] \times D(\epsilon,1]} & x(N(d(s,x),\omega) - \mu(d(s,x))) \\ + \int_{(0,t] \times D(1,\infty)} & xN(d(s,x),\omega), \end{aligned} \tag{4.10}$$

exists and is a Lévy process with generating triplet $(0, \nu, 0)$. We can call this process $(L_t^{(J)})_{t\geq 0}$. Moreover the process $(L_t^{(C)})_{t\geq 0}$, where $L_t^{(C)} = L_t - L_t^{(J)}$ is a continuous Lévy process with the Lévy-Khintchine representation $(A, 0, \gamma)$ and $L^{(C)}$ and $L^{(J)}$ are independent processes.
In [App09] he defines for fixed $t \geq 0$ the measure on $\mathscr{B}(\mathbb{R}^m \setminus \{0\})$, $\tilde{N}(t,B) := N((0,t] \times B) - \mu((0,t] \times B)$. This is a martingale values measure for $B \subseteq D(\epsilon, 1)$ for $\epsilon > 0$ by [App09, p. 105]. Moreover he defines $N(t,B) := N((0,t] \times B)$ for

$B \subseteq D[1,\infty)$ and he writes the decomposition of the i'th coordinate of the m-dimensional Lévy process as

$$\begin{aligned} L_i(t) &= \lambda_i t + \sum_{k=1}^{r} \theta_{i,k} B_k(t) \\ &\quad + \int_0^t \int_{|x|<1} x_i \tilde{N}(ds,dx) + \int_0^t \int_{|x|\geq 1} x_i N(ds,dx), \end{aligned} \tag{4.11}$$

where $\lambda \in \mathbb{R}^m$, $\theta \in Mat(m,r)$ and B is an r-dimensional Brownian motion. Here the first two terms corresponds to $L^{(C)}$ and the two last terms to $L^{(J)}$. The term $\int_0^t \int_{|x|<1} x_i \tilde{N}(ds,dx)$ is the limit written in Equation (4.10). By [App09, p. 121] this convergence takes place in the $\mathscr{M}_2$ space. Remark that $\int_0^t \int_{|x|<1} x_i \tilde{N}(ds,dx)$ can be written as $\int_{|x|<1} x_i \tilde{N}(t,dx)$.

By [App09] it is possible to define Lévy type integrals. That is integrals of the form

$$\begin{aligned} Y_i(t) = Y_i(0) &+ \int_0^t G_i(s)ds + \sum_{j=1}^{r} \int_0^t F_{i,j}(s)(dB_s)_j \\ &+ \int_0^t \int_{|x|<1} H_i(s,x)\tilde{N}(ds,dx) + \int_0^t \int_{|x|\geq 1} K_i(s,x)N(ds,dx), \end{aligned}$$

for all $i \in \{1,...,d\}$ $K_i, G_i, F_{i,j}$ and H_i is predictable and $\mathbb{P}(\int |G_i|ds < \infty) = 1$, $\mathbb{P}(\int |F_{i,j}(s)|^2 ds < \infty) = 1$ and $\mathbb{P}(\int_0^t \int_{|x|<1} |H_i(s,x)|^2 \nu(dx)ds < \infty) = 1$.

In this thesis we are only interested making sense of integration in the $'t'$ parameter with respect to a Lévy process. In practice this means we will only consider the case where $G = M(t)\lambda$, $F = M(t)\theta$ (takes values in $Mat(d,r)$), $K(s,x) = H(t,x) = M(t)x$ and $H(t,x) = M(t)x$ (see [App09, Section 6.3]). Here $M(t)$ is a $Mat(d,m)$-valued predictable matrix and x is m-dimensional vector.
We discuss integration with respect to each of the elements in the decomposition individually. The first two parts corresponds to Lebesgue integration and integration with respect to Brownian motion, which we worked with in Section 2.8. We can define this integration with respect to progressively measurable processes from $L^2(\lambda_{[0,T]} \otimes \mathbb{P})$ for $T > 0$.
As $\int_{|x|<1} x_i \tilde{N}(t,dx)$ is an element in $\mathscr{M}_2$, by Section 2.7 we can define an integration with respect to this element. This holds for predictable stochastic matrices, $(M_t)_{t\geq 0}$, taking values in $Mat(d,m)$ and such that for $T > 0$

$$\mathbb{E}[\int_0^T \int_{|x|<1} |(M(t)x)_i|^2 dt\nu(dx)] < \infty$$

for $i \in \{1,...,d\}$ by [App09, Theorem 4.3.4(2)].
By [App09, 105 and 106], when $A \subseteq D[1,\infty)$ then $N(t,A)$ is a Poisson process and $P(t) := \int_A xN(t,dx)$ is a compound Poisson process. For predictable mappings K defined on $[0,\infty) \times \mathbb{R}^d \times \Omega$ it is possible to define for $T \geq 0$

$$\int_0^T \int_{|x|\geq 1} K(t,x)N(dt,dx) = \sum_{0\leq u\leq T} K(u,\Delta P(u))\mathbb{1}_A(\Delta P(u)).$$

This is a finite random sum because $P(t)$ is a compound Poisson process with associated Poisson process $N(t, \{|x| \geq 1\})$, which is finite (see the proof of [App09, Theorem 2.3.9]). These considerations leads us to conclude that under the right assumptions, we can define integration with respect to a Lévy process.

Corollary 4.5.2. *Let $(M_t)_{t\geq 0}$ be an $Mat(d,m)$-valued predictable process such that*

$$\mathbb{E}[\int_0^T \int_{|x|<1} |(M_t x)_i|^2 dt\nu(dx)] < \infty.$$

for $i \in \{1, ..., d\}$. Moreover assume that $\mathbb{E}[\int_0^T M_{i,j}(t)^2 dt] < \infty$ for all $i \in \{1, ..., d\}$ and $j \in \{1, ..., m\}$. We can integrate this process with respect to a Lévy process. This gives rise to a d-dimensional process $X(t)$, where for $i \in \{1, ..., d\}$

$$dX_i = \sum_{k=1}^{m} M_{i,k}(t) dL_k(t).$$

For a fixed $l \in \{1, ..., m\}$

$$M_{i,l}(t)dL_l = M_{i,l}(t)\lambda_l dt + \sum_{k=1}^{r} M_{i,l}(t)\theta_{l,k} dB_k(t)$$
$$+ \int_{|x|<1} M_{i,l}(t)x_l \tilde{N}(dt, dx) + \int_{|x|<1} M_{i,l}(t)x_l \tilde{N}(dt, dx).$$

□

CHAPTER 5

Time Changed Processes

5.1 Introduction to Time Change

In the current chapter, we work with a collection of indexed elements of the form $Y := (L_{T(\theta)})_{\theta\geq 0}$, where $(L_t)_{t\geq 0}$ is a Lévy process (we call it the *base process*), and $(T(\theta))_{\theta\geq 0}$ is a real stochastic process, that meets certain properties to be defined. It will be shown that if T is a *time change* process, then Y is a stochastic process. In this case we call Y a *time changed process*. The aim of this chapter is to examine which conditions on T that are sufficient to ensure that the time changed process is a Markov process. An important aspect of our examination of the Markov property of time changed Lévy processes is a study of filtrations with respect to which the time changed Lévy processes is adapted. The time change processes of interest are Lévy processes, deterministic processes, and processes with independent increments.
There are different ways of defining time change processes. In [BS15] they demand that the process takes values in the space $[0, \infty]$ and that it for all indices is a stopping time with respect to $(\mathscr{F}_t)_{t\geq 0}$ (a filtration with respect to which, the base process is adapted). We will return to a discussion of their definition, but for now, we give a more general definition of a time change process.

Definition 5.1.1. Let $(T_\theta)_{\theta\geq 0}$ be a stochastic process. We call it a *time change process* if for almost all $\omega \in \Omega$ the function $\theta \mapsto T_\theta(\omega)$ is non-decreasing and $[0, \infty)$-valued.

Remark 5.1.2. Later in the thesis we shall add two more conditions on the time change process, namely right-continuity and that $T(0) = 0$.

□

We start off examining the general situation, where the base process is not necessarily a Lévy process. Let Ω_0 be the space on which the function $\theta \mapsto T_\theta(\omega)$ is non-decreasing and $[0, \infty)$-valued. Remark that $\mathbb{P}(\Omega_0) = 1$, and that Ω_0^c is measurable, because $(\Omega, \mathscr{F}, \mathbb{P})$ is complete. A simple useful lemma when making the time change composition is the following one.

Lemma 5.1.3. *Let X be a stochastic vector, that takes values in $\mathbb{R}^d$. Then the process*

$$\begin{aligned} T : \Omega &\to \mathbb{R}^d \times \Omega \\ \omega &\mapsto (X(\omega), \omega) \end{aligned}$$

is $(\mathscr{F}, \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{F})$-measurable.

Proof. (§§§) $\mathscr{B}(\mathbb{R}^d) \times \mathscr{F}$ is a generator system for $\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{F}$, so let $R \times A \in \mathscr{B}(\mathbb{R}^d) \times \mathscr{F}$.

$$\{\omega \in \Omega : (X(\omega), \omega) \in R \times A\} = \{X \in R\} \cap A \in \mathscr{F},$$

by the properties of σ-algebras. This is sufficient (see Lemma A.1.8(i)). ■

We show that a time changed progressively measurable stochastic process is measurable.

Lemma 5.1.4. *Let $(X_t)_{t\geq 0}$ be a progressively measurable stochastic process that takes values on $\mathbb{R}^d$, and let $(T_\theta)_{\theta\geq 0}$ be a time change process. We define for all $\theta \geq 0$ and $\omega \in \Omega$; $Y_\theta(\omega) := X_{T_\theta(\omega)}(\omega)\mathbb{1}_{\Omega_0}(\omega)$. Y_θ is a stochastic variable for all $\theta \geq 0$.*

Proof. (§§§) Let $(X_t)_{t\geq 0}$, $(T_\theta)_{\theta\geq 0}$ and $(Y_\theta)_{\theta\geq 0}$ be defined as above. Now let $\theta \geq 0$ be given. We want to show that Y_θ is a stochastic variable, meaning that we need to show that the following function is measurable:

$$\begin{aligned} Y_\theta : \Omega &\to \mathbb{R}^d \\ \omega &\mapsto X_{T_\theta(\omega)}(\omega)\mathbb{1}_{\Omega_0}(\omega) \end{aligned}$$

We define the functions

$$\begin{aligned} \Lambda : \Omega &\to [0, \infty) \times \Omega \\ \omega &\mapsto (T_\theta(\omega), \omega), \end{aligned}$$

and

$$\begin{aligned} \Theta : [0, \infty) \times \Omega &\to \mathbb{R}^d \\ (s, \omega) &\mapsto X_s(\omega)\mathbb{1}_{\Omega_0}(\omega). \end{aligned}$$

As X is progressively measurable, then it is also measurable by Lemma 2.4.5 and as the probability space is complete, $\mathbb{1}_{\Omega_0}$ is measurable as well, so Θ is measurable. By Lemma 5.1.3 Λ is measurable as well. As the composition of two measurable mappings are measurable, we obtain the desired result. ■

We start off by studying the simplest time change processes, namely the deterministic ones. A simple example of such a process is given below.

Example 5.1.5. (§§§) We consider an $(\mathscr{F}_t^N)_{t\geq 0}$-Poisson process N with parameter 1. Define for a positive λ the deterministic process $\Lambda(t) = \lambda \cdot t$. Then the time changed process $N^\lambda(t) := N(\Lambda(t))$ is an $(\mathscr{F}_{\lambda t}^N)_{t\geq 0}$-Poisson process with parameter λ. Remark $N^\lambda(0) = N(0) = 0$. As $\lambda \cdot t$ is a continuous, increasing function, $N^\lambda(t)$ preserves the properties càdlàg, non-increasing and $\mathbb{N}_0$-valued for fixed $\omega \in \Omega$. Let $s \leq t$. $N^\lambda(t) - N^\lambda(t) = N(\lambda \cdot t) - N(\lambda \cdot s) \sim Pois(\lambda \cdot t - \lambda \cdot s) =$

$Pois(\lambda \cdot (t-s))$. $\{N(\Lambda(s)) : 0 \leq s \leq t\} = \{N(s) : 0 \leq s \leq \lambda \cdot t\}$, thus we easily obtain that $N(\Lambda(t))$ is $\mathscr{F}^N_{\lambda \cdot t}$-measurable and $N(\Lambda(t)) - N(\Lambda(s))$ is independent of $\mathscr{F}^N_{\lambda \cdot s}$.
If $\lambda > 1$ we have an acceleration of time and if $\lambda < 1$ it is a deceleration of time.

We show that a Lévy process time changed by a deterministic process is a Markov process. Actually, the weaker assumption that the base process is a Markov process is sufficient to show that the time changed process is a Markov process. Before giving the argument for this assertion, we need the following lemma for conditional expectation.

Lemma 5.1.6. *Let X be a bounded stochastic variable and let $\mathbb{E}[X|\mathscr{F}_1] = \mathbb{E}[X|\mathscr{F}_2]$, where $\mathscr{F}_1$ and $\mathscr{F}_2$ are sigma algebras where $\mathscr{F}_1 \subseteq \mathscr{F}_2$. Then we have that $\mathbb{E}[X|\mathscr{F}_1] = \mathbb{E}[X|\mathscr{G}]$ if $\mathscr{F}_1 \subseteq \mathscr{G} \subseteq \mathscr{F}_2$.*

Proof. (§§§) We take the conditional expectation with respect to $\mathscr{G}$ on both sides and using the *Tower Property* (Proposition A.1.11).

$$\mathbb{E}[X|\mathscr{F}_1] = \mathbb{E}[\mathbb{E}[X|\mathscr{F}_1]|\mathscr{G}] = \mathbb{E}[\mathbb{E}[X|\mathscr{F}_2]|\mathscr{G}] = \mathbb{E}[X|\mathscr{G}].$$

■

Theorem 5.1.7. *Let $(Y_t)_{t\geq 0}$ be a Markov process. Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an increasing function. Then $(Y_{g(t)})_{t\geq 0}$ is a Markov process with respect to the filtration $(\mathscr{F}^{Y(g)}_{g(t)})_{t\geq 0}$ where $\mathscr{F}^{Y(g)}_{g(t)} := \sigma\{Y_{g(s)} : s \leq t\}$ for all $t \geq 0$.*

Proof. (§§§) We show that $\mathbb{E}[f(Y_{g(t)})|\mathscr{F}^{Y(g)}_{g(s)}] = \mathbb{E}[f(Y_{g(t)}|Y_{g(s)}]$ for $t \geq s \geq 0$. Let $k_1 := g(s)$ and $k_2 := g(t)$. We know that $\mathbb{E}[f(Y_{k_2})|\mathscr{F}_{k_1}] = \mathbb{E}[f(Y_{k_2})|Y_{k_1}]$. Now we remark that $\sigma(Y_{g(s)}) \subseteq \mathscr{F}^{Y(g)}_{g(t)} \subseteq \mathscr{F}^{Y}_{g(t)}$. Thus from the lemma above we have that $\mathbb{E}[f(Y_{g(t)})|\mathscr{F}^{Y(g)}_{g(s)}] = \mathbb{E}[f(Y_{g(t)})|Y_{g(s)}]$, which gives us the Markov property. ■

5.2 Conditional Stationary Independent Increments and the Cox Process

We show that a time changed Lévy process has conditional stationary independent increments and that a process with conditional stationary independent increments can be written as a time changed Lévy process. We use this characterisation to show the equivalence of two different definitions of the Cox process.
In [Ser72, Section 2] the definition of conditional stationary independent increments is given for stochastic variables. The definition given below is for stochastic vectors, which is the relevant case in this thesis.

Definition 5.2.1. Let $(T_\theta)_{\theta\geq 0}$ be a non-negative real valued stochastic process with non-decreasing right continuous paths, where $T_0 = 0$ almost surely. Let $\mathscr{F}^T = \sigma(T_\theta : \theta \geq 0)$ and $(Y_\theta)_{\theta\geq 0}$ be an $\mathbb{R}^d$-valued stochastic process, for which the following two assertions hold true:

(a) For any $n \in \mathbb{N}$, $0 \leq \alpha_1 < \beta_1 \leq ... \leq \alpha_n < \beta_n$ and $B_1, ..., B_n \in \mathscr{B}(\mathbb{R}^d)$:

$$\mathbb{P}[Y_{\beta_1} - Y_{\alpha_1} \in B_1, ..., Y_{\beta_n} - Y_{\alpha_n} \in B_n | \mathscr{F}^T] = \prod_{i=1}^{n} \mathbb{P}[Y_{\beta_i} - Y_{\alpha_i} \in B_i | \mathscr{F}^T].$$

(b) For any $0 \leq \alpha \leq \beta$ and $\zeta \in \mathbb{R}^d$.

$$\mathbb{E}[\exp[i\langle \zeta, (Y_\beta - Y_\alpha)\rangle] | \mathscr{F}^T] = \phi(\zeta)^{T_\beta - T_\alpha},$$

where ϕ is the characteristic function of an infinitely divisible distribution. Then we say that $(Y_\theta)_{\theta \geq 0}$ has conditionally stationary independent increments with respect to T.

□

Together Lemma 5.2.2 and Lemma 5.2.3 makes a characterisation of time changed Lévy processes. In [Ser72, Section 2] he states the two following assertions as one. He presents ideas for a proof of the statements, but does not make the actual calculation. We give a full proof for the assertions.

Lemma 5.2.2. *Let $Y := (Y_\theta)_{\theta \geq 0}$ be a time changed process with base process $L := (L_t)_{t \geq 0}$ and time process $T := (T_\theta)_{\theta \geq 0}$. Let L and T be independent and assume that L is a Lévy process. Then Y has conditional stationary independent increments with respect to T.*

Proof. (§§) We show that Y fulfills condition (a) and (b) of Definition 5.2.1. Remark that the time change process of Y fulfills the restrictions on the T process in the definition. Let $\mathscr{F}^T$ be defined as in Definition 5.2.1.
of (a): Define for $0 \leq s_1 \leq t_1 \leq \cdots \leq s_n \leq t_n$ the following functions:

$$\psi((s_1, t_1, ..., s_n, t_n), \cdot) := \mathbb{1}_{\{L_{t_1} - L_{s_1} \in B_1\}} \cdots \mathbb{1}_{\{L_{t_n} - L_{s_n} \in B_n\}},$$

and

$$\Phi(s_1, t_1, ..., s_n, t_n) := \mathbb{E}[\psi((s_1, t_1, ..., s_n, t_n), \cdot)]$$

Define also for $i \in \{1, ..., n\}$:

$$\psi_i((s_1, t_1, ..., s_n, t_n), \cdot) := \mathbb{1}_{\{L_{t_i} - L_{s_i} \in B_i\}}.$$

and

$$\Phi_i(s_1, t_1, ..., s_n, t_n) := \mathbb{E}[\psi_i((s_1, t_1, ..., s_n, t_n), \cdot)].$$

As L has independent increments, it holds true that

$$\begin{aligned}\prod_{i=1}^{n} \Phi_i(s_1, t_1, ..., s_n, t_n) &= \prod_{i=1}^{n} \mathbb{P}(L_{t_i} - L_{s_i} \in B_i)\\ &= \mathbb{P}(L_{t_1} - L_{s_1} \in B_1, ..., L_{t_n} - L_{s_n} \in B_n)\\ &= \Phi(s_1, t_1, ..., s_n, t_n).\end{aligned}$$

As T is an increasing process, we have that

$$\prod_{i=1}^{n} \Phi_i(T_{\alpha_1}, T_{\beta_1}, ..., T_{\alpha_n}, T_{\beta_n}) = \Phi(T_{\alpha_1}, T_{\beta_1}, ..., T_{\alpha_n}, T_{\beta_n}).$$

Using the *Freezing Lemma* (Lemma 2.6.1) to calculate the left hand side of the Definition 5.2.1 (a),

$$\begin{aligned}\mathbb{P}(Y_{\beta_1} - Y_{\alpha_1} \in B_1, ..., Y_{\beta_n} - Y_{\alpha_n} \in B_n | \mathscr{F}^T) \\ = \mathbb{E}[\psi((T_{\alpha_1}, T_{\beta_1}, ..., T_{\alpha_n}, T_{\beta_n}), \cdot) | \mathscr{F}^T] \\ = \Phi(T_{\alpha_1}, T_{\beta_1}, ..., T_{\alpha_n}, T_{\beta_n})\end{aligned}$$

and the right hand side

$$\begin{aligned}\prod_{i=1}^{n} \mathbb{P}(Y_{\beta_i} - Y_{\alpha_i} \in B_i | \mathscr{F}^T) &= \prod_{i=1}^{n} \mathbb{E}[\psi_i((T_{\alpha_1}, T_{\beta_1}, ..., T_{\alpha_n}, T_{\beta_n}), \cdot) | \mathscr{F}^T] \\ &= \prod_{i=1}^{n} \Phi_i(T_{\alpha_1}, T_{\beta_1}, ..., T_{\alpha_n}, T_{\beta_n}) \\ &= \Phi(T_{\alpha_1}, T_{\beta_1}, ..., T_{\alpha_n}, T_{\beta_n}).\end{aligned}$$

of (b): As L is a Lévy process, the characteristic function of L_1 is infinitely divisible. Define the characteristic function $\phi(\zeta) := \mathbb{E}[\exp(i\langle \zeta, L_1 \rangle)]$. Then by Lemma 4.3.3 we have that the characteristic function of L_t is ϕ^t. Again using Lemma 2.6.1, and the fact that it also holds for complex functions we obtain for $\zeta \in \mathbb{R}^d$ and $0 \leq \alpha \leq \beta$:

$$\mathbb{E}[\exp(i\langle \zeta, L_{T_\beta} - L_{T_\alpha} \rangle) | \mathscr{F}^T] = \Psi(T_\alpha, T_\beta),$$

where

$$\Psi(s,t) = \mathbb{E}[exp(i\langle \zeta, L_t - L_s \rangle)] = \mathbb{E}[exp(i\langle \zeta, L_{t-s} \rangle)] = \phi^{t-s}(\zeta).$$

This gives us the desired equality:

$$\mathbb{E}[exp(i\langle \zeta, L_{T_\beta} - L_{T_\alpha} \rangle) | \mathscr{F}^T] = \phi^{T_\beta - T_\alpha}(\zeta).$$

■

Lemma 5.2.3. *Let Y' be a stochastic process with conditional stationary independent increments with respect to the stochastic process $(T_\theta)_{\theta \geq 0}$. Let ϕ be the infinitely divisible characteristic function given in Definition 5.2.1. Then there exist a stochastic process L that is independent of T, such that $(Y_\theta)_{\theta \geq 0} := (L_{T(\theta)})_{\theta \geq 0}$ is equivalent to $(Y'_\theta - Y'_0)_{\theta \geq 0}$.*

Proof. (§§) Let Y' be a stochastic process with independent stationary increments. As ϕ is the characteristic function of an infinitely divisible distribution, there is a Lévy process L, such that L_1 has the characteristic function ϕ. Chose this Lévy process such that it is independent of T. We define $Y := (Y_\theta)_{\theta \geq 0}$ to be $(L_{T(\theta)})_{\theta \geq 0}$. Consequently, by Lemma 5.2.2, Y satisfies Definition 5.2.1. By taking the expectation in condition (b), we get for $\beta \geq \alpha$ and $\zeta \in \mathbb{R}^d$ that

$$\mathbb{E}[\exp(i\langle \zeta, (Y_\beta - Y_\alpha) \rangle)] = \mathbb{E}[\phi(\zeta)^{T_\beta - T_\alpha}] = \mathbb{E}[\exp(i\langle \zeta, (Y'_\beta - Y'_\alpha) \rangle)].$$

That is $Y_\beta - Y_\alpha \stackrel{d}{=} Y'_\beta - Y'_\alpha$. By the definition of conditional expectation, it holds true for $B \in \mathscr{B}^d$ that

$$\mathbb{P}(Y_\beta - Y_\alpha \in B | \mathscr{F}^T) = \mathbb{P}(Y'_\beta - Y'_\alpha \in B | \mathscr{F}^T).$$

Let $n \in \mathbb{N}$, $0 \leq \alpha_1 < \beta_1 \leq ... \leq \alpha_n < \beta_n$ and $B_1, ..., B_n \in \mathscr{B}^d$:

$$\begin{aligned}\mathbb{P}[Y'_{\beta_1} - Y'_{\alpha_1} \in B_1, ..., Y'_{\beta_n} - Y'_{\alpha_n} \in B_n | \mathscr{F}^T] &= \prod_{i=1}^{n} \mathbb{P}[Y'_{\beta_i} - Y'_{\alpha_i} \in B_i | \mathscr{F}^T] \\ &= \prod_{i=1}^{n} \mathbb{P}[Y_{\beta_i} - Y_{\alpha_i} \in B_i | \mathscr{F}^T] \\ &= \mathbb{P}[Y_{\beta_1} - Y_{\alpha_1} \in B_1, ..., Y_{\beta_n} - Y_{\alpha_n} \in B_n | \mathscr{F}^T].\end{aligned}$$

By taking the expectation on both sides, we obtain

$$(Y'_{\beta_1} - Y'_{\alpha_1}, ..., Y'_{\beta_n} - Y'_{\alpha_n}) \stackrel{d}{=} (Y_{\beta_1} - Y_{\alpha_1}, ..., Y_{\beta_n} - Y_{\alpha_n}).$$

Let $\alpha_1 = 0$ and $\alpha_i = \beta_{i-1}$ for $i \in \{2, ..., n\}$. Now transform the two vectors of stochastic variables by the quadratic matrix A, defined as $A_{i,j} = 1$ if $i = j$ or $i + 1 = j$. Let it be 0 in other cases. As this matrix is invertible, using the same argument as in the proof for Theorem 4.2.7, gives us that

$$\begin{aligned}(Y'_{\beta_1} - Y'_0, Y'_{\beta_2} - Y'_0, ..., Y'_{\beta_n} - Y_0) &\stackrel{d}{=} (Y_{\beta_1} - Y_0, Y_{\beta_2} - Y_0, ..., Y_{\beta_n} - Y_0) \\ &= (Y_{\beta_1}, Y_{\beta_2}, ..., Y_{\beta_n}).\end{aligned}$$

■

Before we define the Cox process, we give an example of a Poisson process time changed with a deterministic time change process.

Example 5.2.4. (§§§) Let $(N(t))_{t \geq 0}$ be a Poisson process with parameter 1. As seen in Example 5.1.5 the process $N^\lambda(\theta) := N(\lambda\theta)$ is a Poisson process with parameter λ. This can also be written as $N(\int_0^\theta \lambda du)$. Let now $\lambda : [0, \infty) \to \mathbb{R}_+$ be a deterministic function. Then $\Lambda(\theta) = \int_0^\theta \lambda(u)du$ is an increasing continuous function. Define a new process $N^\Lambda(\theta) := N(\Lambda(\theta))$. Arguing as in Example 5.1.5, we obtain that the new process fulfills all the conditions for it to be a Poisson process, except it does not have stationary increments. For $\alpha \leq \beta$, $N^\Lambda(\beta) - N^\Lambda(\alpha) \sim Poi(\int_\alpha^\beta \lambda(u)du)$.

□

Let N^μ be a Poisson random measure with intensity measure μ, and let π_μ be the distribution of N^μ. Let $(M, \mathscr{M})$ be the space of measures on $((0, \infty), \mathscr{B}((0, \infty)))$, that are finite on compact sets. Let $(M_n, \mathscr{M}_n) \subseteq (M, \mathscr{M})$ be the measures that are $\mathbb{N} \cup \{\infty\}$-valued. Fix an $A \in \mathscr{M}_n$. By [Gra76, Lemma I.1], the function $\mu \mapsto \pi_\mu(A)$ is measurable. Remark that $\pi_\mu(M_n) = 1$. A Cox process, also called a *doubly stochastic Poisson process*, is in [Gra76, Definition I.5] defined as follows:

Definition 5.2.5. Let $\Lambda : \Omega \to M$ be a random measure with distribution π_Λ. A random measure Π is called a Cox process corresponding to Λ if it has the distribution $\int_M \pi_\mu(\cdot)\pi_\Lambda(d\mu)$. That is, for all $A \in \mathscr{M}_n$,

$$\mathbb{P}(\Pi \in A) = \int_M \pi_\mu(A)\pi_\Lambda(d\mu).$$

Remark 5.2.6. Write Π_t for $\Pi((0,t])$ for $t \geq 0$. Then $(\Pi_t)_{t\geq 0}$ is a stochastic process. Remark that $\Pi_0 = \Pi(\emptyset)$.

□

We give an alternative definition of the Cox process as a Poisson process with parameter 1, time changed with an increasing process.

Definition 5.2.7. [Gra76, Definition I.5'] Let $N := (N(t))_{t\geq 0}$ be a Poisson process with parameter 1. Let $(\Lambda(\theta))_{\theta\geq 0}$ be an increasing, non-negative, right continuous stochastic process independent of N. Let also $\Lambda(0) = 0$. A Cox process can be defined as $N^{\Lambda}(\theta) := N(\Lambda(\theta))$.

□

We will show that the two definitions are equivalent. In both definitions we use the symbol Λ. In Definition 5.2.5 it is used for a random measure and in Definition 5.2.7 we use it for a stochastic process. In the following lemma we show that there can be established a one-to-one correspondence between these elements, thereby justifying the choice of notation.

Lemma 5.2.8. *Let $(T(\theta))_{\theta\geq 0}$ be a non-decreasing, right continuous stochastic process with $T(0) = 0$. Then there exist a random measure, Λ on $((0,\infty), \mathscr{B}((0,\infty)))$, such that for all $0 \leq \alpha \leq \beta$, $\Lambda((\alpha,\beta]) = T(\beta) - T(\alpha)$.*

Proof. (§§§) First notice that by Lebesgue Stieltjes measures (see [Kal21, Theorem 2.14]) for each $\omega \in \Omega$, there exists a unique measure ν on $((0,\infty), \mathscr{B}((0,\infty)))$, such that $\nu((\alpha,\beta]) = T_\beta(\omega) - T_\alpha(\omega)$ for all $0 < \alpha \leq \beta$. This means that we only have to show that $\Lambda(A)$ is measurable for all $A \in \mathscr{B}((0,\infty))$. In order to show this, we use the setting of Definition A.1.12 and Theorem A.1.13. We show that the class of sets, $A \in \mathscr{B}((0,\infty))$ where $\Lambda(A)$ is measurable is a λ-system in $(0,\infty)$. Denote this class $\mathscr{D}$. If we can also show that the system of sets $\mathscr{C} := \{(\alpha,\beta] : 0 \leq \alpha \leq \beta\}$ is a π-system in $(0,\infty)$, we are satisfied, as we know that $\mathscr{C}$ generates $\mathscr{B}((0,\infty))$.
We start off showing that $(0,\infty) \in \mathscr{D}$.

$$\Lambda((0,\infty)) = \Lambda(\cup_{n\in\mathbb{N}}(0,n]) = \lim_{n\to\infty} \Lambda((0,n]).$$

This is the limit of measurable functions, and by Theorem A.2.1 this is measurable. Secondly show that for $A, B \in \mathscr{D}$, where $A \subseteq B$, $B \setminus A \in \mathscr{D}$: $\Lambda(B \setminus A) = \Lambda(B) - \Lambda(A)$, by properties of measures. The third thing is to show that if $A_1 \subseteq A_2 \subseteq ... \in \mathscr{D}$, then $\cup_{n\in\mathbb{N}} A_n \in \mathscr{D}$. Again by the properties of measures $\Lambda(\cup_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} \Lambda(A_n)$. As it is the limit of measurable functions, it is again measurable.
Now we show that $\mathscr{C}$ is a π-system, which is the case as it is obviously closed under intersection. We get that $\mathscr{B}((0,\infty)) \subseteq \mathscr{D}$, which we were to show. ■

Lemma 5.2.9. Π *defined in Definition 5.2.5 and N^{Λ} defined in Definition 5.2.7 are equivalent.*

Proof. (§§§) Let $n \in \mathbb{N}$ and $\alpha_0 := 0 < \alpha_1 < ... < \alpha_n$. We start off giving the finite dimensional distribution of N^Λ. Let $k_1, k_2, ..., k_n \in \mathbb{N}$. Let $\mathscr{F}^\Lambda$ be the filtration generated by the whole process $(\Lambda(\theta))_{\theta \geq 0}$.

$$\begin{aligned}&\mathbb{P}(N^\Lambda(\alpha_1) - N^\Lambda(0) = k_1, ..., N^\Lambda(\alpha_n) - N^\Lambda(\alpha_{n-1}) = k_n) \\ &\quad = \mathbb{E}[\mathbb{P}(N^\Lambda(\alpha_1) - N^\Lambda(0) = k_1, ..., N^\Lambda(\alpha_n) - N^\Lambda(\alpha_{n-1}) = k_n | \mathscr{F}^\Lambda)] \\ &\quad = \mathbb{E}[\prod_{i=1}^{n} \mathbb{P}(N^\Lambda(\alpha_i) - N^\Lambda(\alpha_{i-1}) = k_i | \mathscr{F}^\Lambda)].\end{aligned}$$

Remark that $\Lambda(\beta) \geq \Lambda(\alpha)$ for $\beta \geq \alpha$. Then by Lemma 2.6.1

$$\mathbb{P}(N(\Lambda(\alpha_i)) - N(\Lambda(\alpha_{i-1})) = k_i | \mathscr{F}^\Lambda) = \Phi(\Lambda(\alpha_{i-1}), \Lambda(\alpha_i)),$$

where

$$\begin{aligned}\Phi(z_1, z_2) &= \mathbb{P}(N(z_2) - N(z_1) = k_i) \\ &= \mathbb{P}(N(z_2 - z_1) = k_i) = e^{-(z_2 - z_1)} \frac{(z_2 - z_1)^{k_i}}{k_i!}.\end{aligned}$$

That is

$$\begin{aligned}&\mathbb{E}[\prod_{i=1}^{n} \mathbb{P}(N(\Lambda(\alpha_i)) - N(\Lambda(\alpha_{i-1})) = k_i | \mathscr{F}^\Lambda)] \\ &\qquad = \mathbb{E}[\prod_{i=1}^{n} e^{-(\Lambda(\alpha_i) - \Lambda(\alpha_{i-1}))} \frac{(\Lambda(\alpha_i) - \Lambda(\alpha_{i-1}))^{k_i}}{k_i!}].\end{aligned}$$

Now we look at the process defined in Definition 5.2.5.

$$\begin{aligned}&\mathbb{P}(\Pi_{t_1} - \Pi_0 = k_1, ..., \Pi_{t_n} - \Pi_{t_{n-1}} = k_n) \\ &\quad = \mathbb{P}(\Pi((0, t_1]) = k_1, ..., \Pi((t_n, t_{n-1}]) = k_n) \\ &\quad = \int_M \pi_\mu(\{\nu : \nu((0, t_1]) = k_1, ..., \nu((t_{n-1}, t_n]) = k_n\}) \pi_\Lambda(d\mu).\end{aligned}$$

Remark that as π_μ is the distribution of a Poisson process with intensity measure μ, then

$$\begin{aligned}&\pi_\mu(\{\nu : \nu((0, t_1]) = k_1, ..., \nu((t_{n-1}, t_n]) = k_n\}) \\ &\qquad = \prod_{i=1}^{n} e^{-\mu((t_{i-1}, t_i])} \frac{\mu((t_{i-1}, t_i])^{k_i}}{k_i!}.\end{aligned}$$

So we obtain

$$\begin{aligned}&\int_M \pi_\mu(\{\nu : \nu((0, t_1]) = k_1, ..., \nu((t_{n-1}, t_n]) = k_n\}) \pi_\Lambda(d\mu) \\ &\qquad = \int_M \prod_{i=1}^{n} e^{-\mu((t_{i-1}, t_i])} \frac{\mu((t_{i-1}, t_i])^{k_i}}{k_i!} \pi_\Lambda(d\mu) \\ &\qquad = \mathbb{E}[\prod_{i=1}^{n} e^{-\Lambda((t_{i-1}, t_i])} \frac{\Lambda((t_{i-1}, t_i])^{k_i}}{k_i!}]\end{aligned}$$

$$= \mathbb{E}[\prod_{i=1}^{n} e^{-(\Lambda(t_i)-\Lambda(t_{i-1}))} \frac{(\Lambda(t_i) - \Lambda(t_{i-1}))^{k_i}}{k_i!}],$$

which was required. ■

The following lemma shows how a Cox process is distributed using Definition 5.2.7. In [Gra76, p. 7] it is shown by using Definition 5.2.5.

Lemma 5.2.10. *Let $(N^{\Lambda}(\theta))_{\theta \geq 0}$ be the Cox process defined in Definition 5.2.7. Let $n \in \mathbb{N}$. Then for $\theta \geq 0$*

$$\mathbb{P}(N^{\Lambda}(\theta) = n) = \mathbb{E}[e^{-\Lambda(\theta)} \frac{\Lambda(\theta)^n}{n!}],$$

and the characteristic function of the process is of the form

$$\mathbb{E}[\exp(i\zeta N^{\Lambda}(\theta))] = \mathbb{E}[(1 - e^{i\zeta})\Lambda(\theta)],$$

for $\zeta \in \mathbb{R}$.

Proof. (§§) In both cases we use the *Tower Property* followed by the *Freezing Lemma* as the processes are independent of each other.

$$\begin{aligned} \mathbb{P}(N^{\Lambda}(\theta) = n) &= \mathbb{E}[\mathbb{P}(N^{\Lambda}(\theta) = n | \Lambda(\theta))] \\ &= \mathbb{E}[\phi(\Lambda(\theta))], \end{aligned}$$

where $\phi(z) = \mathbb{P}(N(z) = n) = e^{-z} \frac{z^n}{n!}$, because $N(z)$ is Poisson distributed with parameter z.
Let $\theta \geq 0$.

$$\mathbb{E}[\exp(i\zeta N^{\Lambda}(\theta))] = \mathbb{E}[\mathbb{E}[\exp(i\zeta N^{\Lambda}(\theta))|\Lambda(\theta)]] = \Phi(\Lambda(\theta)),$$

where $\Phi(z) = \mathbb{E}[\exp(i\zeta N(z))] = \exp((1 - e^{i\zeta})z)$. So

$$\mathbb{E}[\exp(i\zeta N^{\Lambda}(\theta))] = \mathbb{E}[(1 - e^{i\zeta})\Lambda(\theta)].$$

■

5.3 Subordination

Subordination is the concept of letting both the base process and the time process be Lévy processes. In this case the time process is called a subordinator. The time changed process is called a subordination process. These types of processes are in fact Markov processes, which we will show in this section.
In the following section we let both the base process $L := (L_t)_{t \geq 0}$ and the time process $T := (T_\theta)_{\theta \geq 0}$ be Lévy processes. In order to simplify the considerations define the processes on the probability space $(\Omega_0, \mathscr{F}_0, \mathbb{P}_0)$, where

$$\Omega_0 = \{L_0 = 0\} \cap \{T_0 = 0\} \cap \{\omega : L_t(\omega) \text{ is càdlàg}\} \cap \{\omega : T_\theta(\omega) \text{ is càdlàg}\}.$$

As unions of null sets are null sets, we have that $\mathbb{P}(\Omega_0) = 1$. As $(\Omega, \mathscr{F}, \mathbb{P})$ is complete, $\Omega_0 \in \mathscr{F}$. Define the σ-algebra $\mathscr{F}_0 := \{A \cap \Omega_0 : A \in \mathscr{F}\}$ and define the measure $\mathbb{P}_0$ on $(\Omega_0, \mathscr{F}_0)$ to be $\mathbb{P}_0(A) = \mathbb{P}(A)$ for all $A \in \mathscr{F}_0$. In this section we take $(\Omega_0, \mathscr{F}_0, \mathbb{P}_0)$ to be the given probability measure. The consideration above comes from [Sat13, Chapter 6].

Definition 5.3.1. [Sat13, Definition 21.4, Definition 30.2] Assume that L and T are given as above and define the process $Y := (Y_\theta)_{\theta\geq 0}$ on $(\Omega_0, \mathscr{F}_0, \mathbb{P}_0)$, where $Y_\theta := L_{T(\theta)}$ for all $\theta \geq 0$. We call Y a *subordination process* and we call T a *subordinator*.

□

In the sequel we will need the following lemma.

Lemma 5.3.2. *Let $n \in \mathbb{N}$ and let $f, f_1, ..., f_n \in \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$. Let L be a d-dimensional Lévy process and T be a subordinator. Let $\mathscr{F}^T$ be the σ-algebra generated by T. Let $\theta \geq 0$ and $0 \leq \theta_1 < \cdots < \theta_n < \theta_{n+1}$. Then the following three assertions hold true:*

(1) $\mathbb{E}[f(L_{T(\theta)})] = \mathbb{E}[g(T_\theta)]$, where $g(s) = \mathbb{E}[f(L_s)]$, for $s \geq 0$.

(2) $\mathbb{E}[f(L_{T(\theta_2)} - L_{T(\theta_1)})] = \mathbb{E}[h(T_{\theta_1}, T_{\theta_2})]$, where $h(s_1, s_2) = \mathbb{E}[f(L_{s_2} - L_{s_1})]$, for $s_2 \geq s_1$.

(3) $\mathbb{E}[\prod_{i=1}^n f_i(L_{T(\theta_{i+1})} - L_{T(\theta_i)})] = \mathbb{E}[G(T_{\theta_1}, ..., T_{\theta_{n+1}})]$, where

$$G(s_1, ..., s_{n+1}) = \mathbb{E}[\prod_{i=1}^n f_i(L_{s_{i+1}} - L_{s_i})],$$

for $s_1 \leq s_2 \leq \cdots \leq s_{n+1}$.

Proof. (§§) The lemma is a consequence of the *Freezing Lemma* (Lemma 2.6.1).

of (1): Let $\psi(s, \cdot) = f(L_s)$ for $s \geq 0$ and observe that

$$\mathbb{E}[\psi(T_\theta, \cdot)|\mathscr{F}^T] = g(T_\theta),$$

where $g(s) = \mathbb{E}[\psi(s, \cdot)]$. Taking the expectation reveal $\mathbb{E}[\psi(T_\theta, \cdot)] = \mathbb{E}[g(T_\theta)]$.

of (2): Let $\psi((s_1, s_2), \cdot) = f(L_{s_2} - L_{s_1})$ for $s_1 \leq s_2$ and observe that

$$\mathbb{E}[\psi((T_{\theta_1}, T_{\theta_2}), \cdot)|\mathscr{F}^T] = h(T_{\theta_1}, T_{\theta_2}),$$

where $h(s_1, s_2) = \mathbb{E}[\psi((s_1, s_2), \cdot)]$. Again taking the expectation on both sides, we obtain

$$\mathbb{E}[\psi((T_{\theta_1}, T_{\theta_2}), \cdot)] = \mathbb{E}[h(T_{\theta_1}, T_{\theta_2})].$$

Remark that we can be certain that $T_{\theta_1} \leq T_{\theta_2}$.

of (3): Let $\psi((s_1, ..., s_n, s_{n+1}), \cdot) = \prod_{i=1}^n f_i(L_{s_{i+1}} - L_{s_i})$, where $s_1 \leq \cdots \leq s_n \leq s_{n+1}$ and observe that

$$\mathbb{E}[\psi((T_{\theta_1}, ..., T_{\theta_n}, T_{\theta_{n+1}}), \cdot)|\mathscr{F}^T] = G(T_{\theta_1}, ..., T_{\theta_n}, T_{\theta_{n+1}}),$$

where $G(s_1, ..., s_n, s_{n+1}) = \mathbb{E}[\prod_{i=1}^n f_i(L_{s_{i+1}} - L_{s_i})]$. Taking the expectation on both sides reveal that

$$\mathbb{E}[\psi((T_{\theta_1}, ..., T_{\theta_n}, T_{\theta_{n+1}}), \cdot)] = \mathbb{E}[G(T_{\theta_1}, ..., T_{\theta_n}, T_{\theta_{n+1}})].$$

■

The following theorem shows how one can express the distribution of Y_θ in terms of the distribution of L and T. The theorem is a part of [Sat13, Theorem 30.1] and in the proof we go forward as he does with some elaborations. Our presentation is though based on Lemma 5.3.2 instead of a similar result [Sat13, Proposition 1.16]. This is in practise a significant simplification of the proof.

Theorem 5.3.3. *[Sat13, Theorem 30.1] Let Y be a subordination process taking values in $\mathbb{R}^d$ consisting of the base Lévy process L and the subordinator T. Assume that L and T are independent. Letting $\mu := \mathbb{P}_{L_1}$ and $\nu := \mathbb{P}_{T_1}$ we have for $t \geq 0$ and $B \in \mathscr{B}(\mathbb{R}^d)$ that*

$$\mathbb{P}(Y_\theta \in B) = \int_{[0,\infty)} \mu^s(B)\nu^\theta(ds). \tag{5.1}$$

Proof. (§) Let $t \geq 0$ be fixed and assume that f is a real bounded function defined on $\mathbb{R}^d$. From Lemma 5.3.2(1) we have

$$\mathbb{E}[f(L_{T_\theta})] = \mathbb{E}[g(T_\theta)], \quad \text{where} \quad g(s) = \mathbb{E}[f(L_s)]. \tag{5.2}$$

Letting $f(x) = \mathbb{1}_B(x)$ for $B \in \mathscr{B}^d$, $g(s) = \mathbb{E}[\mathbb{1}_B(L_s)] = \mathbb{P}(L_s \in B) = \mu^s(B)$. $\mu^s(B)$ is measurable as a function of s. This is due to the fact that L_s is progressively measurable and by Lemma 2.4.5 $(\mathscr{F} \otimes \mathscr{B}([0,\infty)), \mathscr{B}^d)$-measurable. $\mathbb{1}_B$ is $(\mathscr{B}^d, \mathscr{B}(\mathbb{R}))$-measurable. By *Fubini's Theorem* (Theorem A.1.10), we obtain that the function

$$s \mapsto \mu^s(B) = \mathbb{P}(L_s \in B) = \mathbb{E}[\mathbb{1}_B(L_s)]$$

is $(\mathscr{B}([0,\infty)), \mathscr{B}(\mathbb{R}))$-measurable. We obtain

$$\mathbb{P}(Y_\theta \in B) = \mathbb{E}[\mathbb{1}_B(Y_\theta)] = \mathbb{E}[\mathbb{1}_B(L_{T_\theta})] = \mathbb{E}[\mu^{T_\theta}(B)] = \int_{[0,\infty)} \mu^s(B)\nu^\theta(ds).$$

■

The following simple example shows how the theorem can be utilized to calculate the distribution of a subordination process.

Example 5.3.4. (§§§) Let $(B_t)_{t\geq 0}$ be a one dimensional Brownian motion and let $(N_\theta)_{\theta\geq 0}$ be a Poisson process with intensity 1. $\mu := \mathbb{P}_{B_1} = N(0,1)$, then μ^n is distributed as the sum of n independent standard normally distributed random variables. That is $\mu^n = N(0,n)$. Moreover N_θ is a Poisson stochastic variable with parameter θ. So for a $\theta \geq 0$ and $A \in \mathscr{B}(\mathbb{R})$:

$$\mathbb{P}(B_{N_\theta} \in A) = \int_{[0,\infty)} \mu^s(A)\mathbb{P}_{N_\theta}(ds) = \sum_{n=0}^{\infty} \mu^n(A)e^{-\theta}\frac{\theta^n}{n!}.$$

Let $n \in \mathbb{N}$ and let $Z_1, ..., Z_n \sim N(0,1)$ be independent.

$$\mu^n(A) = \mathbb{P}(\sum_{i=1}^{n} Z_i \in A) = \mathbb{P}(\sqrt{n}(\sum_{i=1}^{\infty} Z_i/\sqrt{n}) \in A).$$

If $A = [-a, a]$ for $a \geq 0$, then

$$\mu^n(A) = \mathbb{P}(Z_1 \in [\frac{-a}{\sqrt{n}}, \frac{a}{\sqrt{n}}]).$$

So if we let $Z \sim N(0,1)$, we get

$$\mathbb{P}(B_{N_t} \in [-a,a]) = e^{-t} \sum_{n=1}^{\infty} \frac{t^n}{n!} \mathbb{P}_Z([\frac{-a}{\sqrt{n}}, \frac{a}{\sqrt{n}}])$$

□

The claim that a Lévy process time changed with a subordinator is again a Lévy process is stated in [Sat13, Theorem 30.1]. The proof given below is an elaboration of the one he gives.

Theorem 5.3.5. *A subordination process is a Lévy process.*

Proof. (§) Let $Y := (Y_\theta)_{\theta \geq 0}$ be the subordination process described in Theorem 5.3.3. We show that Y fulfills the conditions required in Definition 4.1.1 in order for it to be a Lévy process.

$Y_0 = 0$: $Y_0 = L_{T_0} = L_0 = 0$, because L and T are Lévy processes.

Y is càdlàg: Let $\omega \in \Omega$ and $\theta \in (0,\infty)$. Now we take the left limit. We know that the left limit $l_T := \lim_{\theta' \uparrow \theta} T_{\theta'}(\omega)$ exists. As $T_{\theta'}(\omega)$ is increasing we have that when θ' goes to θ from the left, then $T_{\theta'}(\omega)$ goes to l_T from the left. We take the limit:

$$\lim_{\theta' \uparrow \theta} Y_{\theta'}(\omega) = \lim_{\theta' \uparrow \theta} L_{T_{\theta'}(\omega)}(\omega) = \lim_{s \uparrow l_T} L_s(\omega),$$

which again exists because L is a Lévy process. Let $\theta \geq 0$. Then by the right continuity of T and L, and as T is increasing:

$$\lim_{\theta' \downarrow \theta} Y_{\theta'}(\omega) = \lim_{\theta' \downarrow \theta} L_{T_{\theta'}(\omega)}(\omega) = L_{T_\theta(\omega)}(\omega).$$

for all $\omega \in \Omega$.

Y has stationary increments: As in the proof of Theorem 5.3.3 we let $g(s) := \mathbb{E}[f(L_s)]$ for $s \geq 0$ for an $f \in \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$. By Lemma 5.3.2 (2), as the processes L and T are independent we can write for $0 \leq \theta_1 < \theta_2$

$$\mathbb{E}[f(Y_{\theta_2} - Y_{\theta_1})] = \mathbb{E}[h(T_{\theta_1}, T_{\theta_2})], \text{ where } h(s_1, s_2) := \mathbb{E}[f(L_{s_2} - L_{s_1})]$$

for $s_1 \leq s_2$, because $T_{\theta_1}(\omega) \leq T_{\theta_2}(\omega)$ as it is an increasing process. We note that

$$h(s_1, s_2) = \mathbb{E}[f(L_{s_2} - L_{s_1})] = \mathbb{E}[f(L_{s_2 - s_1})] = g(s_2 - s_1),$$

where we used that $(L_t)_{t \geq 0}$ has stationary increments. We can write

$$\mathbb{E}[f(Y_{\theta_2} - Y_{\theta_1})] = \mathbb{E}[h(T_{\theta_1}, T_{\theta_2})] = \mathbb{E}[g(T_{\theta_2} - T_{\theta_1})] = \mathbb{E}[g(T_{\theta_2 - \theta_1})] = \mathbb{E}[f(Y_{\theta_2 - \theta_1})],$$

by the stationary increments of $(T_\theta)_{\theta \geq 0}$. Thus $Y_{\theta_2} - Y_{\theta_1} \sim Y_{\theta_2 - \theta_1}$, and thus we have the stationary increments.

Y has independent increments: We define for an $n \in \mathbb{N}$: $f_1, f_2, ..., f_n \in \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$. Let $0 = \theta_1 < \theta_2 < ... < \theta_n < \theta_{n+1}$ and define for $i \in \{1, 2, ..., n\}$:

$h_i(s_i, s_{i+1}) := \mathbb{E}[f_i(L_{s_{i+1}} - L_{s_i})]$ for $s_{i+1} \geq s_i$ and $g_i(s) := \mathbb{E}[f_i(L_s)]$. Thus we have $g_i(s_{i+1} - s_i) = h_i(s_i, s_{i+1})$. Because of the independence of L and T we can write (by Lemma 5.3.2 (3)):

$$\mathbb{E}[\prod_{i=1}^{n} f_i(Y_{\theta_{i+1}} - Y_{\theta_i})] = \mathbb{E}[G(T_{\theta_1}, ..., T_{\theta_n}, T_{\theta_{n+1}})], \tag{5.3}$$

where $G(s_1, ..., s_n, s_{n+1}) = \mathbb{E}[\prod_{i=1}^{n} f_i(L_{s_{i+1}} - L_{s_i})]$, where $s_1 = 0$. By utilizing the independent increments of L, we obtain:

$$\mathbb{E}[\prod_{i=1}^{n} f_i(L_{s_{i+1}} - L_{s_i})] = \prod_{i=1}^{n} \mathbb{E}[f_i(L_{s_{i+1}} - L_{s_i})] = \prod_{i=1}^{n} h_i(s_i, s_{i+1}).$$

Thus

$$\begin{aligned}\mathbb{E}[\prod_{i=1}^{n} f_i(Y_{\theta_{i+1}} - Y_{\theta_i})] &= \mathbb{E}[\prod_{i=1}^{n} h_i(T_{\theta_i}, T_{\theta_{i+1}})] = \mathbb{E}[\prod_{i=1}^{n} g_i(T_{\theta_{i+1}} - T_{\theta_i})] \\ &= \prod_{i=1}^{n} \mathbb{E}[g_i(T_{\theta_{i+1}} - T_{\theta_i})] = \prod_{i=1}^{n} \mathbb{E}[f_i(Y_{\theta_{i+1}} - Y_{\theta_i})],\end{aligned}$$

where we used that that the process T has independent increments. This gives us independent increments of the subordination process Y. ■

We can now draw the conclusion that a subordinated process possess the Markov property.

Corollary 5.3.6. *The process $(Y_\theta)_{\theta\geq 0}$ described in Theorem 5.3.3 is a Markov process.*

Proof. (§§§) As $(Y_\theta)_{\theta\geq 0}$ is a Lévy process, by Theorem 4.2.7 it is also a Markov process with transition semigroup, $(P_\theta)_{\theta\geq 0}$ defined as:

$$P_\theta(x, A) = \mathbb{P}_{Y_\theta}(A - x), \tag{5.4}$$

for $(x, A) \in \mathbb{R}^d \times \mathscr{B}(\mathbb{R}^d)$. ■

Remark 5.3.7. We note here that $\mathbb{P}_{Y_\theta}(A) = \int_{[0,\infty)} \lambda^s(A)\nu^\theta(ds)$, where $\lambda = \mathbb{P}_{L_1}$ and $\nu = \mathbb{P}_{T_1}$.

Actually, it is possible to leave out some of the conditions of the Lévy process. The only assumption that is needed is the one of independent increments.

Theorem 5.3.8. *Let $(L_t)_{t\geq 0}$ be a Lévy process and let $(T_\theta)_{\theta\geq 0}$ be a time process with independent increments. Let L and T be independent processes. Then we have that $(Y_\theta)_{\theta\geq 0}$, where $Y_\theta := L_{T(\theta)}$ for all $\theta \geq 0$, is a Markov process with respect to its natural filtration.*

Proof. (§§§) We can mimic the proof for Theorem 5.3.5. Here we showed that the subordination process has independent increments, only using the independent increments of T. As $(Y_\theta)_{\theta\geq 0}$ has independent increments, we refer to Theorem 4.4.1 to conclude that $(Y_\theta)_{\theta\geq 0}$ is a Markov process. ■

The following example of a subordination process can be found in both [Car+03], [BS15] and in [EK19]. Here we give all the details of the calculations, which are not given in the literature mentioned.

Example 5.3.9. [Car+03, Section 2.1](§§) Let $I := (I_\alpha)_{\alpha\geq 0}$ be an inverse Gaussian process with parameter $\theta > 0$. This process is described in Example 4.1.8. Let $B^{\nu,\sigma}$ be a one dimensional Brownian motion with drift $\nu \in \mathbb{R}$ and volatility $\sigma > 0$ independent of I. Then $(B^{\nu,\sigma}(I_\alpha))_{\alpha\geq 0}$ is a Normal inverse Gaussian process. We show this by calculating the characteristic function of $B(I_1)$. We use the *Freezing Lemma* to obtain

$$\mathbb{E}[e^{iuB^{\nu,\sigma}(I_1)}] = \mathbb{E}[\mathbb{E}[e^{iuB^{\nu,\sigma}(I_1)}]|\mathscr{F}_1^I]] = \mathbb{E}[\Phi(I_1)],$$

where

$$\Phi(z) = \mathbb{E}[e^{iu(\nu z+\sigma B_z)}] = e^{iu\nu z-u^2\sigma^2 z/2} = e^{(iu\nu-u^2\sigma^2/2)z},$$

as $(\nu z + \sigma B_z)$ is normally distributed. Therefore

$$\mathbb{E}[e^{iuB^{\nu,\sigma}(I_1)}] = \mathbb{E}[e^{(iu\nu-u^2\sigma^2/2)I_1}] = \mathbb{E}[e^{-(u^2\sigma^2/2-iu\nu)I_1}]$$

This is close to the Laplace transform of I_1 but instead of a positive λ, we have a complex number, where the real part is positive. By [BS15, Remark 8.1], we can set $u^2\sigma^2/2 - 2iu\nu$ instead of λ in the Laplace transform in Example 4.1.8. By doing this, we obtain:

$$\mathbb{E}[e^{iuB^{\nu,\sigma}(I_1)}] = \exp(-1(\sqrt{2(u^2\sigma^2/2 - iu\nu) + \theta^2} - \theta)). \tag{5.5}$$

By [EK19, Equation (2.63)] the characteristic function of an Normal inverse Gaussian distributed variable, X, takes the form

$$\mathbb{E}[e^{iuX}] = \exp(i\mu u + \delta(\sqrt{\gamma^2 - \beta^2} - \sqrt{\gamma^2 - (\beta + iu)^2})), \tag{5.6}$$

where $\gamma, \delta > 0$, $\beta \in (-\gamma, \gamma)$ and $\mu \in \mathbb{R}$. We make computations on (5.5) with the purpose of getting it on the form in (5.6):

$$\begin{aligned}\exp(-(\sqrt{2(u^2\sigma^2/2 - iu\nu) + \theta^2} - \theta)) &= \exp(\theta - \sqrt{\theta^2 + (u^2\sigma^2 - 2iu\nu)}) \\ &= \exp(\theta - \sqrt{\theta^2 + \frac{\nu^2}{\sigma^2} - (\frac{\nu}{\sigma} + iu\sigma)^2}) \\ &= \exp(\sigma(\frac{\theta}{\sigma} - \sqrt{\frac{\theta^2}{\sigma^2} + \frac{\nu^2}{\sigma^4} - (\frac{\nu}{\sigma^2} + iu)^2})),\end{aligned}$$

so we have arrived at the desired form with $\delta = \sigma$, $\gamma^2 = \frac{\theta^2}{\sigma^2} + \frac{\nu^2}{\sigma^4}$ and $\beta = \frac{\nu}{\sigma^2}$. We conclude that the process $(\nu I_\alpha + \sigma B_{I_\alpha})_{\alpha\geq 0}$ is a Normal inverse Gaussian process.

□

5.4 The Filtrations of Time Changed Processes

In the section above we made a construction on the probability space, such that both the time process and the base process was 0 at time 0 and that they were càdlàg on the whole space. In Lemma 5.1.4 the time process was defined as

$L_{T(\theta)}\mathbb{1}_{\Omega_0}$, where Ω_0 is the space, where T_θ has the desired properties. Instead of this construction, that complicates the notation, the time changed process will only be defined on the space Ω_0. On this space we assume that T_θ has the desired properties. As the probability space on which the processes are defined is complete it is straight forward to reduce the probability space. We still call this probability space $(\Omega, \mathscr{F}, \mathbb{P})$. This means that the assumptions on T will hold true for all $\omega \in \Omega$. Remark that this will not be the case for the base process. We are content with working on a probability space on which certain assumptions on the base process only holds almost surely (for a Lévy process these assumptions are $L_0 = 0$ and càdlàg paths).
Let $\mathbb{F}^T := (\mathscr{F}^T_\theta)_{\theta\geq 0}$ and $\mathbb{F}^X := (\mathscr{F}^X_t)_{t\geq 0}$ be the filtrations generated by $(T_\theta)_{\theta\geq 0}$ and $(X_t)_{t\geq 0}$ respectively. Preferably, the filtration generated by the time changed process can be written in terms of the filtrations $\mathbb{F}^X$ and $\mathbb{F}^T$. We define a filtration with respect to which the time changed process is adapted.
The filtration below is inspired by a direct proof that a time changed process is measurable, if one does not use the knowledge that a progressively measurable process is measurable.

Definition 5.4.1. Let $(X_t)_{t\geq 0}$ be a progressively measurable stochastic process taking values on $\mathbb{R}^d$, and let $(T_\theta)_{\theta\geq 0}$ be a right continuous time process. Define the set

$$\mathscr{H}_\theta := \{A \in \mathscr{F}^T_\theta \vee \mathscr{F}^X_\infty : A \cap \{T_\theta \leq t\} \in \mathscr{F}^T_\theta \vee \mathscr{F}^X_t \quad \text{for all } t \geq 0\}.$$

□

Lemma 5.4.2. *$(\mathscr{H}_\theta)_{\theta\geq 0}$ is a filtration.*

Proof. (§§§) We show that $\mathscr{H}_\theta$ is a σ-algebra, that $\mathscr{H}_{\theta_1} \subseteq \mathscr{H}_{\theta_2}$ for $\theta_1 \leq \theta_2$, and that it is right continuous. We show that $\mathscr{H}_\theta$ meets the three conditions of Definition A.1.1. Let $t, \theta \geq 0$ be fixed values.

(i) $\Omega \in \mathscr{H}_\theta$: Obviously, $\Omega \in \mathscr{F}^T_\theta \vee \mathscr{F}^X_\infty$ and

$$\Omega \cap \{T_\theta \leq t\} = \{T_\theta \leq t\} \in \mathscr{F}^T_\theta \subseteq \mathscr{F}^T_\theta \vee \mathscr{F}^X_t.$$

(ii) $A \in \mathscr{H}_\theta \Rightarrow A^C \in \mathscr{H}_\theta$: Assume that $A \in \mathscr{H}_\theta$. We have that $\Omega \cap \{T_\theta \leq t\} \in \mathscr{F}^T_\theta \vee \mathscr{F}^X_t$, and

$$\Omega \cap \{T_\theta \leq t\} = (A \cup A^C) \cap \{T_\theta \leq t\} = (A \cap \{T_\theta \leq t\}) \cup (A^C \cap \{T_\theta \leq t\}).$$

As $\mathscr{F}^T_\theta \vee \mathscr{F}^X_t$ is a sigma-algebra we have that

$$(A^C \cap \{T_\theta \leq t\}) = ((A \cap \{T_\theta \leq \theta\}) \cup (A^C \cap \{T_\theta \leq t\})) \setminus (A \cap \{T_\theta \leq t\}),$$

which is an element in $\mathscr{F}^T_\theta \vee \mathscr{F}^X_t$.

(iii) $(A_i)_{i\in\mathbb{N}} \in \mathscr{H}_\theta \Rightarrow \cup_{i\in\mathbb{N}} A_i \in \mathscr{H}_\theta$:

$$(\cup_{i\in\mathbb{N}} A_i) \cap \{T_\theta \leq t\} = \cup_{i\in\mathbb{N}}(A_i \cap \{T_\theta \leq t\}) \in \mathscr{F}^T_\theta \vee \mathscr{F}^X_t$$

because $A_i \cap \{T_\theta \leq s\} \in \mathscr{F}^T_\theta \vee \mathscr{F}^X_t$ and $\mathscr{F}^Z_\theta \vee \mathscr{F}^X_t$ is a sigma-algebra.

We show that $(\mathscr{H}_\theta)_{\theta\geq 0}$ is an increasing sequence of σ-algebras. Let $\theta_1 \leq \theta_2$ and $A \in \mathscr{H}_{\theta_1}$. Let Ω_0 be the space, where T is increasing and such that $\mathbb{P}(\Omega_0) = 1$. Remark that $\Omega_0 \cap \{T_{\theta_2} \leq t\} \subseteq \Omega_0 \cap \{T_{\theta_1} \leq t\}$, as T is increasing. Then we have that

$$A \cap \{T_{\theta_2} \leq t\} \cap \Omega_0 = A \cap \{T_{\theta_1} \leq t\} \cap \{T_{\theta_2} \leq t\} \cap \Omega_0 \in \mathscr{F}^T_{\theta_2} \vee \mathscr{F}^X_t, \quad (5.7)$$

because $A \cap \{T_{\theta_1} \leq t\} \in \mathscr{F}^T_{\theta_1} \vee \mathscr{F}^X_\gamma$ which is obviously contained in $\mathscr{F}^T_{\theta_2} \vee \mathscr{F}^X_t$ and $\{T_{\theta_2} \leq t\} \in \mathscr{F}^T_{\theta_2} \subseteq \mathscr{F}^T_{\theta_2} \vee \mathscr{F}^X_t$. Ω_0 is also contained in $\mathscr{F}^T_{\theta_2} \vee \mathscr{F}^X_t$, because we work with augmented filtrations of complete probability spaces. Thus $A \in \mathscr{H}_{\theta_2}$, as we just need to add a null set in Equation (5.7) to obtain that $A \cup \{T_{\theta_2} \leq t\} \in \mathscr{F}^T_{\theta_2} \vee \mathscr{F}^X_t$.

It is clear that $\mathscr{H}_\theta$ is contained in $\cap_{\theta'>\theta}\mathscr{H}_{\theta'}$. Let $A \in \cap_{\theta'>\theta}\mathscr{H}_{\theta'}$. This means that $A \cap \{T_{\theta+\frac{1}{n}} \leq t\} \in \mathscr{F}^T_{\theta+\frac{1}{n}} \vee \mathscr{F}^X_t$ for all $n \in \mathbb{N}$ and $t \geq 0$. As T is increasing $A \cap \{T_{\theta+\frac{1}{n_1}} \leq t\} \subseteq A \cap \{T_{\theta+\frac{1}{n_2}} \leq t\}$ for $n_1 \leq n_2$. This means that

$$\bigcup_{k=1}^n A \cap \{T_{\theta+\frac{1}{k}} \leq t\} \in \bigcap_{k=1}^n \mathscr{F}^T_{\theta+\frac{1}{k}} \vee \mathscr{F}^X_t.$$

Taking the limit on both sides and using the right continuity of $(\mathscr{F}^T_\theta \vee \mathscr{F}^X_t)_{\theta\geq 0}$ and T, we get that $A \cap \{T_\theta \leq t\} \in \mathscr{F}^T_\theta \vee \mathscr{F}^X_t$. ∎

Lemma 5.4.3. *Let $(X_t)_{t\geq 0}$ be a progressively measurable stochastic process, that takes values in $\mathbb{R}^d$, and let $(T_\theta)_{\theta\geq 0}$ be a time process. Let Y be the time changed process with X as the base process and T as the time process. Then Y_θ is $\mathscr{H}_\theta$-measurable for all $\theta \geq 0$.*

Proof. (§§§) Let $(X_t)_{t\geq 0}$, $(T_\theta)_{\theta\geq 0}$ and $(Y_\theta)_{\theta\geq 0}$ be as described above. Now let $\theta \geq 0$ be given. We want to show that Y_θ is a stochastic variable, meaning that we need to show that the following function:

$$\begin{aligned} Y_\theta : \Omega &\to \mathbb{R}^d \\ \omega &\mapsto X_{T_\theta(\omega)}(\omega) \end{aligned}$$

is measurable. We define for all $\lambda \geq 0$ the following two functions:

$$\begin{aligned} \Lambda^\lambda : \Omega &\to [0,\lambda] \times \Omega \\ \omega &\mapsto (T_\theta(\omega)\mathbb{1}_{\{T_\theta \leq \lambda\}}(\omega), \omega), \end{aligned}$$

and

$$\begin{aligned} \Theta^\lambda : [0,\lambda] \times \Omega &\to \mathbb{R}^d \\ (s,\omega) &\mapsto X_s(\omega)\mathbb{1}_{\Omega_0}(\omega) \end{aligned}$$

We show that both of these functions are measurable. As we know that $(X_t)_{t\geq 0}$ is progressively measurable, we conclude that Θ^λ is $(\mathscr{B}([0,\lambda]) \otimes \mathscr{F}^X_\lambda, \mathscr{B}^d)$-measurable.
Now we prove that Λ^λ is $(\mathscr{F}^T_\theta \vee \mathscr{F}^X_\lambda, \mathscr{B}([0,\lambda]) \otimes \mathscr{F}^X_\lambda)$-measurable. By Lemma A.1.8(i), we just need to show that $(\Lambda^\lambda)^{-1}(D) \in \mathscr{F}^T_\theta \vee \mathscr{F}^X_\lambda$ for all $D \in \mathscr{B}([0,\lambda]) \times \mathscr{F}^X_\lambda$ because $\mathscr{B}([0,\lambda]) \times \mathscr{F}^X_\lambda$ is a generator system for the

σ-algebra $\mathscr{B}([0,\lambda]) \otimes \mathscr{F}_\lambda^X$. We write D as a product set $D = R \times A$ where $R \in \mathscr{B}([0,\lambda])$ and $A \in \mathscr{F}_\lambda^X$. We write

$$\begin{aligned}(\Lambda^\lambda)^{-1}(R \times A) &= \{\omega \in \Omega : (T_\theta(\omega)\mathbb{1}_{\{T_\theta \le \lambda\}}(\omega), \omega) \in (R \times A)\} \\ &= \{\omega \in \Omega : T_\theta(\omega)\mathbb{1}_{\{T_\theta \le \lambda\}}(\omega) \in R\} \cap A \\ &= \{T_\theta \mathbb{1}_{\{T_\theta \le \lambda\}} \in R\} \cap A \in \mathscr{F}_\theta^T \vee \mathscr{F}_\lambda^X.\end{aligned}$$

By Lemma A.1.7 we have that $\Theta^\lambda \circ \Lambda^\lambda$ is $(\mathscr{F}_\theta^T \vee \mathscr{F}_\lambda^X, \mathscr{B}^d)$-measurable.
Remark that if $\omega \in \{T_\theta \le \lambda\}$, then $X_{T_\theta(\omega)}(\omega) = \Phi^\lambda \circ \Lambda^\lambda(\omega)$. Let $B \in \mathscr{B}^d$. So for all $\lambda \ge 0$

$$\begin{aligned}\{T_\theta \le \lambda\} \cap \{Y_\theta \in B\} &= \{T_\theta \le \lambda\} \cap \{X_{T_\theta} \in B, T_\theta \le \lambda\} \\ &= \{T_\theta \le \lambda\} \cap (\Phi^\lambda \circ \Lambda^\lambda)^{-1}(B) \in \mathscr{F}_\theta^T \vee \mathscr{F}_\lambda^X,\end{aligned}$$

The only thing left to show is that $(Y_\theta)^{-1}(B) \in \mathscr{F}_\theta^T \vee \mathscr{F}_\infty^X$. Define for $n \in \mathbb{N}$:

$$Q_n := \{T_\theta \le n\} \cap (\Theta^n \circ \Lambda^n)^{-1}(B) \in \mathscr{F}_\theta^T \vee \mathscr{F}_n^X \subseteq \mathscr{F}_\theta^T \vee \mathscr{F}_\infty^X.$$

Thus we have that $\cup_{k\in\mathbb{N}} Q_k \in \mathscr{F}_\theta^T \vee \mathscr{F}_\infty^X$ as $\mathscr{F}_\theta^T \vee \mathscr{F}_\infty^X$ is a sigma algebra. We show that $(Y_\theta)^{-1}(B) = \cup_{k\in\mathbb{N}} Q_k$. Let $\omega \in (Y_\theta)^{-1}(B)$. Then there exist an $n \in \mathbb{N}$ such that $T_\theta(\omega) \le n$. This secures that $\omega \in Q_n$, because $\Theta^n \circ \Lambda^n(\omega) = Y_\theta(\omega) \in B$. Assume that $\omega \in \cup_{k\in\mathbb{N}} Q_k$. Then there exist an $n \in \mathbb{N}$ such that $\omega \in Q_n$. This means that $Y_\theta(\omega) = \Theta^n \circ \Lambda^n(\omega) \in B$. Conclude that $(Y_\theta)^{-1}(B) \in \mathscr{F}_\theta^T \vee \mathscr{F}_\infty^X$. ■

In [BS15] they introduce the setup to time changed processes a little differently than it is done in this thesis. They introduce a progressively measurable stochastic process $(X_t)_{t\ge0}$ that is adapted to the filtration $(\mathscr{F}_\theta)_{t\ge0}$. The time process $(T_\theta)_{\theta\ge0}$ is defined as a non-decreasing, right continuous, $[0,\infty]$-valued, stochastic process, that is adapted to the filtration $(\mathscr{G}_\theta)_{\theta\ge0}$. Moreover, they demand that for all $\theta \ge 0$, $T(\theta)$ is a stopping time with respect to the filtration $(\mathscr{F}_t)_{t\ge0}$. These assumptions secure that $X_{T(\theta)}$ is $\mathscr{F}_{T(\theta)}$-measurable ([BS15, p. 4]). $\mathscr{F}_{T(\theta)}$ is the optional σ-algebra

$$\mathscr{F}_{T(\theta)} := \{A \in \mathscr{F}_\infty : A \cap \{T(\theta) \le t\} \in \mathscr{F}_t \text{ for all } t \ge 0\}.$$

Often we will assume that the base process and the time process are independent processes. In this case we can say, that $T(\theta)$ is *not* a stopping time with respect to the σ-algebra generated by the base process, $(\mathscr{F}_t^X)_{t\ge0}$. So if one will demand this assumption, an expansion of the filtration might be in place.
An easy way to secure that the assumption is always fulfilled is to define the filtration $(\mathscr{F}_t^{X,T})_{t\ge0}$, where

$$\mathscr{F}_t^{X,T} := \mathscr{F}_t^X \vee \mathscr{F}_\infty^T. \tag{5.8}$$

Remark that $(X_t)_{t\ge0}$ is adapted to that σ-algebra and that $T(\theta)$ is a stopping time with respect to it as $\{T(\theta) \le t\} \in \mathscr{F}_\infty^T$ for all $t \ge 0$. This filter gives rise to the optional σ-algebra

$$\mathscr{F}_{T(\theta)} := \{A \in \mathscr{F}_\infty^{X,T} : A \cap \{T(\theta) \le t\} \in \mathscr{F}_t^X \vee \mathscr{F}_\infty^T \text{ for all } t \ge 0\}.$$

We will show that this σ-algebra makes sense to use in some settings. First we show a useful lemma, which is posed as an exercise in [Bal17].

Lemma 5.4.4. *[Bal17, Exercise 4.1] Let X be a real stochastic variable on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\mathscr{G}, \mathscr{H} \subseteq \mathscr{F}$ be sub σ-algebras of $\mathscr{F}$. Assume that $\sigma(X) \vee \mathscr{G}$ is independent of $\mathscr{H}$. Then*

$$\mathbb{E}[X|\mathscr{G} \vee \mathscr{H}] = \mathbb{E}[X|\mathscr{G}].$$

Remark 5.4.5. This Lemma can be expanded to hold for complex random variables as well by linearity of the conditional expectation.

Proof. (§) By [Bal17, Remark 4.2] we need to make sure that $\mathbb{E}[X|\mathscr{G}]$ is $\mathscr{G} \vee \mathscr{H}$-measurable, which is by definition the case. We show that

$$\mathbb{E}[\mathbb{E}[X|\mathscr{G} \vee \mathscr{H}]\mathbb{1}_D] = \mathbb{E}[\mathbb{E}[X|\mathscr{G}]\mathbb{1}_D]$$

for all $D \in \mathscr{G} \vee \mathscr{H}$. We start off showing the identity for sets of the form $A \cap B$ for $A \in \mathscr{G}$ and $B \in \mathscr{H}$. We calculate

$$\mathbb{E}[\mathbb{E}[X|\mathscr{G} \vee \mathscr{H}]\mathbb{1}_A\mathbb{1}_B] = \mathbb{E}[X\mathbb{1}_A\mathbb{1}_B] = \mathbb{E}[X\mathbb{1}_A]\mathbb{P}(B),$$

because $X\mathbb{1}_A$ is $\sigma(X) \vee \mathscr{G}$-measurable, and this σ-algebra is independent of $\mathscr{H}$. As $\mathbb{E}[X\mathbb{1}_A|\mathscr{G}]$ is $\mathscr{G}$-measurable, we get that

$$\mathbb{E}[\mathbb{E}[X|\mathscr{G}]\mathbb{1}_A\mathbb{1}_B] = \mathbb{E}[\mathbb{E}[X\mathbb{1}_A|\mathscr{G}]\mathbb{1}_B] = \mathbb{E}[X\mathbb{1}_A]\mathbb{P}(B).$$

Again by [Bal17, Remark 4.2] it has to be shown that $\mathscr{D} := \{A \cap B : A \in \mathscr{G}, B \in \mathscr{H}\}$ (i) generates $\mathscr{G} \vee \mathscr{H}$, (ii) is stable to intersection and (iii) contains Ω. (i) and (iii) is satisfied. As for (ii), let $(A_1 \cap B_1), (A_2 \cap B_2) \in \mathscr{D}$;

$$(A_1 \cap B_1) \cap (A_2 \cap B_2) = (A_1 \cap A_2) \cap (B_1 \cap B_2),$$

which is in $\mathscr{D}$ as $\mathscr{G}$ and $\mathscr{H}$ are both σ-algebras. ∎

In [Bal17, Theorem 3.3] he proves that for an $(\mathscr{F}_t)_{t \geq 0}$-stopping time, τ, and a d-dimensional $(\mathscr{F}_t)_{t \geq 0}$-Brownian motion B the process $(B_{\tau+t} - B_\tau)_{t \geq 0}$ is again a Brownian motion. In the proof of this theorem, he does not use other properties than the ones of a Lévy process, so in this section the theorem is reproved with general Lévy processes instead of Brownian motion.
As in [Bal17, Lemma 3.3] we show that a finite stopping time can be approximated from the right by a sequence of discrete valued stopping times. Define for all $n \in \mathbb{N}$ and $s \geq 0$ the function

$$\tau_n(s) = \sum_{i=0}^{\infty} \frac{i+1}{2^n} \mathbb{1}_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}(s).$$

For each $\omega \in \Omega$, $\tau_n(\tau(\omega)) \geq \tau(\omega)$ and $\tau_n(\tau(\omega)) \to \tau(\omega)$ for $n \to \infty$. We just need to show that $\tau_n(\tau)$ is a stopping time. Let $t \geq 0$ and let $M := \max\{i \in \mathbb{N} : t > \frac{i}{2^n}\}$

$$\{\tau_n(\tau) \leq t\} = \{\tau \leq \frac{M}{2^n}\} \in \mathscr{F}_{M/2^n} \subseteq \mathscr{F}_t.$$

Lemma 5.4.6. *[Bal17, Theorem 3.3] Let L be a d-dimensional $(\mathscr{F}_t)_{t\geq 0}$-Lévy process and let τ be a finite $(\mathscr{F}_t)_{t\geq 0}$ stopping time. Then $(L_{\tau+t} - L_\tau)_{t\geq 0}$ is a Lévy process equivalent to L and it is independent of $\mathscr{F}_\tau$.*

Proof. (§) As above let τ_n be a decreasing sequence of discrete valued stopping times approximating τ. Let $k, n \in \mathbb{N}$, $C \in \mathscr{F}_\tau \subseteq \mathscr{F}_{\tau_n}$ (see [Bal17, Proposition 3.5(c)]) and remark that

$$\{\tau_n = (i+1)/2^n\} \cap C = \{\tau_n \leq (i+1)/2^n\} \cap C \setminus (\{\tau_n \leq i/2^n\} \cap C) \in \mathscr{F}_{(i+1)/2^n}.$$

Let $t_1, ..., t_k$ and $A := A_1 \times \cdots \times A_k$ for $A_i \in \mathscr{B}^d$.

$$\begin{aligned}
&\mathbb{E}[\mathbb{1}_A(L_{\tau_n+t_1} - L_{\tau_n}, ..., L_{\tau_n+t_k} - L_{\tau_n})\mathbb{1}_C] \\
&\quad = \mathbb{P}(L_{\tau_n+t_1} - L_{\tau_1} \in A_1, ..., L_{\tau_n+t_k} - L_{\tau_n} \in A_k, C) \\
&\quad = \sum_{i=0}^{\infty} \mathbb{P}(L_{\frac{i}{2^n}+t_1} - L_{\frac{i}{2^n}} \in A_1, ..., L_{\frac{i}{2^n}+t_k} - L_{\frac{i}{2^n}} \in A_k, C \cap \{\tau_n = \frac{i}{2^n}\}).
\end{aligned}$$

We have from Remark 4.1.2 and Lemma 4.1.5 that $(L_{\frac{i}{2^n}+t_1} - L_{\frac{i}{2^n}}, ..., L_{\frac{i}{2^n}+t_k} - L_{\frac{i}{2^n}})$ is independent of $\mathscr{F}_{\frac{i}{2^n}}$ and distributed as $(L_{t_1}, ..., L_{t_k})$. Moreover $C \cap \{\tau_n = \frac{i}{2^n}\}$ is $\mathscr{F}_{\frac{i}{2^n}}$-measurable. Applying these observations to the expression above

$$\begin{aligned}
&\mathbb{E}[\mathbb{1}_A(L_{\tau_n+t_1} - L_{\tau_n}, ..., L_{\tau_n+t_k} - L_{\tau_n})\mathbb{1}_C] \\
&\quad = \sum_{i=0}^{\infty} \mathbb{P}(L_{\frac{i}{2^n}+t_1} - L_{\frac{i}{2^n}} \in A_1, ..., L_{\frac{i}{2^n}+t_k} - L_{\frac{i}{2^n}} \in A_k)\mathbb{P}(C \cap \{\tau_n = \frac{i}{2^n}\}) \\
&\quad = \mathbb{P}(L_{t_1} \in A_1, ..., L_{t_k} \in A_k) \sum_{i=0}^{\infty} \mathbb{P}(C \cap \{\tau_n = \frac{i}{2^n}\}) \\
&\quad = \mathbb{E}[\mathbb{1}_A(L_{t_1}, ..., L_{t_k})]\mathbb{P}(C).
\end{aligned}$$

By setting $C = \Omega$ we see that

$$(L_{\tau_n+t_1} - L_{\tau_n}, ..., L_{\tau_n+t_k} - L_{\tau_n}) \overset{d}{=} (L_{t_1}, ..., L_{t_k}),$$

and therefore we conclude that $(L_{\tau_n+t_1} - L_{\tau_n}, ..., L_{\tau_n+t_k} - L_{\tau_n})$ is independent of $\mathscr{F}_\tau$. Let $f \in \mathscr{M}_c(\mathbb{R}^d, \mathbb{R}) \cap \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$ be positive. As Lévy processes are right continuous we obtain by *Bounded Convergence*, that

$$\begin{aligned}
&\mathbb{E}[f(L_{\tau+t_1} - L_\tau, ..., L_{\tau+t_k} - L_\tau)] \\
&\quad = \lim_{n\to\infty} \mathbb{E}[f(L_{\tau_n+t_1} - L_{\tau_n}, ..., L_{\tau_n+t_k} - L_{\tau_n})] = \mathbb{E}[f(L_{t_1}, ..., L_{t_k})].
\end{aligned}$$

By [Tho19, Theorem 5.2.3] (independence is preserved for weak convergence limits). $(L_{\tau+t_1} - L_\tau, ..., L_{\tau+t_k} - L_\tau)$ is independent of $\mathscr{F}_\tau$, and by [Tho19, Theorem 5.1.4] (convergence in distribution) that $(L_{\tau+t} - L_\tau)_{t\geq 0}$ is equivalent to $(L_t)_{t\geq 0}$. ■

Theorem 5.4.7. *Let $(L_t)_{t\geq 0}$ be a d-dimensional Lévy process and let $(\mathscr{F}^L_t)_{t\geq 0}$ be its natural filtration. Let $(T_\theta)_{\theta\geq 0}$ be a time process, and let $(\mathscr{F}^T_\theta)_{\theta\geq 0}$ be the* *filtration generated by T. Assume that T is independent of L. Then for all $f \in \mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$ and $t \geq 0$:*

$$\mathbb{E}[f(L_{T(\theta)+t})|\mathscr{F}_{T(\theta)}] = \Phi(L_{T(\theta)}),$$

where Φ is a measurable function and

$$\mathscr{F}_{T(\theta)} := \{A \in \mathscr{F}^L_\infty \vee \mathscr{F}^T_\infty : A \cap \{T(\theta) \leq t\} \in \mathscr{F}^L_t \vee \mathscr{F}^T_\infty \text{ for all } t \geq 0\}.$$

Proof. (§§§) We define the σ-algebra $\mathbb{F}^{L,T}$ where $\mathscr{F}^{L,T}_t := \mathscr{F}^L_t \vee \mathscr{F}^T_\infty$. We show that L is a Lévy process with respect to $(\mathscr{F}^{L,T}_t)_{t\geq 0}$. Obviously we have that it is adapted to it. We only need to show that $(L_{t+s} - L_t)$ is independent of $\mathscr{F}^{L,T}_t$. We take a look at the function $e^{i\langle (L_{t+s}-L_t),u\rangle}$ for $u \in \mathbb{R}^d$ which is $\sigma(L_{t+s} - L_t)$-measurable. Consequently, $\sigma(e^{i\langle (L_{t+s}-L_t),u\rangle}) \vee \mathscr{F}^L_t$ is independent of $\mathscr{F}^T_\infty$. Therefore by Lemma 5.4.4 (for complex random elements)

$$\mathbb{E}[e^{i\langle (L_{t+s}-L_t),u\rangle}|\mathscr{F}^L_t \vee \mathscr{F}^T_\infty] = \mathbb{E}[e^{i\langle (L_{t+s}-L_t),u\rangle}|\mathscr{F}^L_t].$$

As $L_{t+s} - L_s$ is independent of $\mathscr{F}^L_t$, we obtain that

$$\mathbb{E}[e^{i\langle (L_{t+s}-L_t),u\rangle}|\mathscr{F}^L_t \vee \mathscr{F}^T_\infty] = \mathbb{E}[e^{i\langle (L_{t+s}-L_t),u\rangle}].$$

By [Bal17, Exercise 4.5] we obtain that $(L_{t+s} - L_t)$ is independent of $\mathscr{F}^L_t \vee \mathscr{F}^T_\infty$. So $(L_t)_{t\geq 0}$ is an $(\mathscr{F}^L_t \vee \mathscr{F}^T_\infty)_{t\geq 0}$-Lévy process.
Let $\theta \geq 0$, then T_θ is obviously a $\mathscr{F}^{L,T}_t$-stopping time. By Lemma 5.4.6 $(L_{T(\theta)+t} - L_{T(\theta)})_{t\geq 0}$ is a Lévy process independent of $\mathscr{F}_{T(\theta)}$. Let f be a function from $\mathscr{M}_b(\mathbb{R}^d, \mathbb{R})$. By the *Freezing Lemma*

$$\mathbb{E}[f(L_{T(\theta)+t})|\mathscr{F}_{T(\theta)}] = \mathbb{E}[f(L_{T(\theta)+t} - L_{T(\theta)} + L_{T(\theta)})|\mathscr{F}_{T(\theta)}] = \Phi(L_{T(\theta)}),$$

for all $t \geq 0$. ∎

Remark 5.4.8. Let $(\mathscr{G}_\theta)_{\theta\geq 0}$ be a filtration generated by the process $(L_{T(\theta)})_{\theta\geq 0}$. Then

$$\mathbb{E}[f(L_{T(\theta)+t})|\mathscr{G}_\theta] = \Phi(L_{T(\theta)}).$$

5.5 SDEs Driven by Time Changed Brownian Motion

In Section 2.8 we considered stochastic differential equations driven by Brownian motion. In this section we are interested in stochastic differential equations driven by time changed Brownian motion, that is, expressions of the type

$$dY_\theta = F(T(\theta))d\theta + M(Y(\theta))dB_{T(\theta)}.$$

We start considering the case where the time process is a Lévy process. As Brownian motion is a Lévy process B_T is a subordination process.

Let B be an m-dimensional Brownian motion and let T be an increasing Lévy process (a subordinator). By Theorem 5.3.5, $B_{T(\theta)}$ is an m-dimensional Lévy process. We write it on the form of Equation (4.11). Let $M(x)$ be a measurable mapping from $\mathbb{R}^m$ to $Mat(d,m)$. We seek càdlàg solutions to the stochastic differential equation

$$dY_\theta = M(Y(\theta-))dB_{T(\theta)}.$$

The integration on the right hand side is defined in Section 4.5. Define $Q^1(x,y) := M(x)\lambda\lambda^T M(y)^T$ and $Q^2(x,y) = M(x)^T M(y)$ for $x, y \in \mathbb{R}^d$ and λ is the $\mathbb{R}^m$ vector from Equation (4.11). These are $Mat(d,d)$-valued and $Mat(m,m)$-valued, respectively. By [App09, section 6.3], this differential equation has an adapted càdlàg solution under the following conditions: There exist $D_1, D_2 > 0$ such that for all $x, y \in \mathbb{R}^d$:

(1) $\| Q^1(x,x) - 2Q^1(x,y) + Q^1(y,y) \| \leq D_1|x-y|^2$

(2) $\max_{i,j\in\{1,\dots,m\}} |Q^2_{i,j}(x,x) - 2Q^2_{i,j}(x,y) + Q^2_{i,j}(y,y)| \leq D_2|x-y|^2$.

Here

$$\| A \| = \sum_{i=1}^m |A_{i,i}| \quad \text{(the absolute value of } A_{i,i})$$

for $A \in Mat(d,d)$. Remark that as the solution Y is càdlàg, the left limit exists for all $t \geq 0$. Consequently, it makes sense to define the process $Y(\theta-)$, which is left continuous. As M is measurable, this secures that the integrand is predictable.
By [App09, Theorem 6.4.5], the solution to the stochastic differential equation above is in fact a Markov process.
We move on to consider another situation, namely the case where the time process is a continuous process.

Let B be an m-dimensional Brownian motion and T be a continuous time process. As we are interested in stochastic differential equations driven by B_T, we need to make sure that it makes sense to integrate with respect to B_T. In [BS15, Section 8.2] it is shown (without details) that B_T is a local martingale with the localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ where

$$\sigma_n := \inf\{\theta \geq 0 : |B_{T(\theta)}| \geq n\}.$$

We show the same statement but with another localizing sequence.
As in the section above we let $(\mathscr{F}_t^{T,B})_{t\geq 0}$ be the filtration with respect to which we define the optional σ-algebras $\mathscr{F}_{T(\theta)}$ for $\theta \geq 0$. By the stopping theorem ([Bal17, Theorem 5.13]) we obtain for all $K \in \mathbb{N}$ and $\theta_2 \geq \theta_1$ that

$$\mathbb{E}[B_{T(\theta_2)\wedge K} | \mathscr{F}_{T(\theta_1)\wedge K}] = B_{T(\theta_1)\wedge K}, \tag{5.9}$$

as $T(\theta_1) \wedge K \leq T(\theta_2) \wedge K$ and $T(\theta_2) \wedge K$ is a bounded stopping time.
Moreover by [IW89, Proposition 5.5 (5.4)] if σ and τ are stopping times and X an integrable variable. Then

$$\mathbb{E}[\mathbb{1}_{\{\sigma>\tau\}} X | \mathscr{F}_\tau] = \mathbb{1}_{\{\sigma>\tau\}} \mathbb{E}[X|\mathscr{F}_{\sigma\wedge\tau}]. \tag{5.10}$$

These statements will be used to show that B_T is a local martingales.

Lemma 5.5.1. *Let B be a Brownian motion and T be a continuous time process, where $T(0) = 0$, that goes to infinity when θ goes to infinity and define the filtration $(\mathscr{F}_{T(\theta)})_{\theta \geq 0}$ as above. Then B_T is a local martingale with the localizing sequence $(\tau_K)_{K \in \mathbb{N}}$ where $\tau_K := \inf\{\theta \geq 0 : T(\theta) = K\}$.*

Proof. (§§) Define the stopping time $\tau_K := \inf\{\theta : T(\theta) = K\}$ and observe that $T(\theta \wedge \tau_K) = T(\theta) \wedge K$ by the continuity of T. We show that $B_T(\theta)$ is a local martingale with respect to the filtration $(\mathscr{F}_{T(\theta)})_{\theta \geq 0}$ with the localizing sequence $(\tau_K)_{K \in \mathbb{N}}$. By [Bal17, Exercise 5.6] $B_{T(\theta) \wedge K}$ is integrable for all $K \in \mathbb{N}$. We can thereby define the conditional expectation $\mathbb{E}[B_{T(\theta_2 \wedge K)} | \mathscr{F}_{T(\theta)}]$. We use equation (5.10) to show that

$$\begin{aligned}
&\mathbb{E}[\mathbb{1}_{\{T(\theta_2) \wedge K > T(\theta_1)\}} B_{T(\theta_2) \wedge K} | \mathscr{F}_{T(\theta_1)}] \\
&\quad = \mathbb{1}_{\{T(\theta_2) \wedge K > T(\theta_1)\}} \mathbb{E}[B_{T(\theta_2) \wedge K} | \mathscr{F}_{T(\theta_1) \wedge T(\theta_2) \wedge K}] \\
&\quad = \mathbb{1}_{\{T(\theta_2) \wedge K > T(\theta_1)\}} \mathbb{E}[B_{T(\theta_2) \wedge K} | \mathscr{F}_{T(\theta_1) \wedge K}] \\
&\quad = \mathbb{1}_{\{T(\theta_2) \wedge K > T(\theta_1)\}} B_{T(\theta_1) \wedge K}.
\end{aligned}$$

In the second equality we use that $T(\theta_1) \wedge T(\theta_2) = T(\theta_1)$ and in the last equality we used equation (5.9).
Define the new stopping time $\tau := (T(\theta_2) \wedge K)\mathbb{1}_{\{T(\theta_2) \wedge K \leq T(\theta_1)\}}$. This stopping time is less that $T(\theta_1)$ securing that B_τ is $\mathscr{F}_{T(\theta_1)}$-measurable. Remark that $B_0 = 0$ almost surely and therefore

$$B_\tau = B_{T(\theta_2) \wedge K} \mathbb{1}_{\{T(\theta_2) \wedge K \leq T(\theta_1)\}}$$

almost surely. It follows then that

$$\mathbb{E}[B_{T(\theta_2) \wedge K} \mathbb{1}_{\{T(\theta_2) \wedge K \leq T(\theta_1)\}} | \mathscr{F}_{T(\theta_1)}] = B_{T(\theta_2) \wedge K} \mathbb{1}_{\{T(\theta_2) \wedge K \leq T(\theta_1)\}}.$$

Remark that

$$B_{T(\theta_2) \wedge K} \mathbb{1}_{\{T(\theta_2) \wedge K \leq T(\theta_1)\}} = B_{T(\theta_1) \wedge K} \mathbb{1}_{\{T(\theta_2) \wedge K \leq T(\theta_1)\}}$$

as $T(\theta_1) \leq T(\theta_2)$. Putting the two conditional expectations together, we obtain

$$\mathbb{E}[B_{T(\theta_2) \wedge K} | \mathscr{F}_{T(\theta_1)}] = B_{T(\theta_1) \wedge K}.$$

Consequently B_T is a local Martingale with the localizing sequence $(\tau_K)_{K \in \mathbb{N}}$. Remark that $\tau_K \to \infty$ as $K \to \infty$ because $T(\theta)$ goes to ∞ for $\theta \to \infty$. ■

By Section 2.7 this secures that we can define integration of functions from $\mathscr{L}_2^{loc}(B_T)$ with respect to B_T, and that the integral is a local martingale.
Define the mapping

$$M(\theta, x) : [0, \infty) \times \mathbb{R}^d \to Mat(d, m).$$

Let M be measurable and assume for all $x \in \mathbb{R}^d$ that $\theta \mapsto M(\theta, x)$ is right continuous and has left limits. Moreover assume that there exist a $k > 0$ such that for all $i \in \{1, ..., d\}$, $j \in \{1, ..., m\}$ and $\theta \geq 0$

$$|M(\theta, x) - M(\theta, y)| \leq k|x - y| \text{ for } x, y \in \mathbb{R}^d.$$

By [Pro90, Theorem V.7] the stochastic differential equation

$$dY_\theta = M(\theta, Y_{\theta-})dB_{T(\theta)}$$

has a unique adapted càdlàg solution, that is a semimartingale. That is, there exist a d-dimensional semimartingale $(Y_\theta)_{\theta\geq 0}$ such that for $i \in \{1, ..., d\}$

$$Y_i(\theta) = Y_i(0) + \sum_{k=1}^{m} \int_0^\theta M_{i,k}(v, Y_{v-})d(B_T)_k(v).$$

It is not obvious, that such processes possess the Markov property in general. Some subcases that could be investigated is the case where T is deterministic and continuous, and the case where T is on the form $\int_0^t S_v dv$, where S is a positive stochastic process. Remark that if T is deterministic, B_T has independent increments. In Section 4.4 it was shown that this is sufficient to secure Markovianity of the process. It is then a natural thought, that a solution to a stochastic differential equation driven by this type of time changed Brownian motion is a Markov process.

PART II

Appendices

Appendices

APPENDIX A

Appendix

A.1 Definitions and Theorems

Definition A.1.1. [Øks03, Definition 2.1.1] Definition of a sigma algebra] Let Ω be a given set. A sigma algebra $\mathscr{F}$ in Ω is a family of subsets of Ω with the following properties:

(i) $\emptyset \in \mathscr{F}$.

(ii) $F \in \mathscr{F} \Rightarrow F^C \in \mathscr{F}$.

(iii) $(A_i)_{i\in\mathbb{N}} \in \mathscr{F} \Rightarrow \cup_{i\in\mathbb{N}} A_i \in \mathscr{F}$.

□

Definition A.1.2. [Tho14, Definition 1.3.2] Let $(\Omega, \mathscr{F})$ be a measurable space. A measure μ on $(\Omega, \mathscr{F})$ is a mapping $\mu : \mathscr{F} \to [0, \infty]$ such that:

(1) $\mu(\emptyset) = 0$.

(2) If $(A_n)_{n\in\mathbb{N}}$ is a disjoint sequence of sets from $\mathscr{F}$, then $\mu(\cup_{n\in\mathbb{N}} A_n) = \sum_{i=1}^{n} \mu(A_i)$.

□

Definition A.1.3. [Sat13, Definition 2.1] Let μ be a probability measure on $(\mathbb{R}^d, \mathscr{B}^d)$. The characteristic function of μ is defined as:

$$\hat{\mu}(\theta) = \int_{\mathbb{R}^d} e^{i\langle\theta,x\rangle}\mu(dx), \qquad \theta \in \mathbb{R}^d.$$

□

Lemma A.1.4. *[Tho19, 1.2.5(ii), Corollary 1.2.6 and Corollary 1.1.7(iv)]*

(1) Let μ and ν be two measures on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. If $\hat{\mu}(\theta) = \hat{\nu}(\theta)$ for all $\theta \in \mathbb{R}^d$. Then $\mu = \nu$.

(2) Let Y and X be stochastic vectors on $\mathbb{R}^m$ and $\mathbb{R}^d$ respectively. If $\hat{\mathbb{P}}_{(X,Y)}(\theta_1, \theta_2) = \hat{\mathbb{P}}_X(\theta_1)\hat{\mathbb{P}}_Y(\theta_2)$, for all $\theta_1 \in \mathbb{R}^m$ and $\theta_2 \in \mathbb{R}^d$. Then X

and Y are independent.

(3) Let X be a stochastic vector on $\mathbb{R}^d$ and let $A \in Mat(m,d)$. Then $\hat{P}_{AX}(\theta) = \hat{P}_X(A^T\theta)$ for all $\theta \in \mathbb{R}^m$.

□

Definition A.1.5. [Sat13, p. 10] Let μ be a probability measure on $[0,\infty)$. Then the Laplace transform of μ is defined as

$$\mathscr{L}_\mu(u) = \int_{[0,\infty)} e^{-ux}\mu(dx),$$

for $u \geq 0$.

□

Lemma A.1.6. *[Sat13, p. 10] Let μ_1 and μ_2 be probability measures on $[0,\infty)$. Assume that $\mathscr{L}_{\mu_1}(u) = \mathscr{L}_{\mu_2}(u)$ for all $u \geq 0$. Then $\mu_1 = \mu_2$.*

□

Lemma A.1.7. *[Tho14, Theorem 4.2.4(ii)] Let $(E\mathscr{E})$ be a measurable space and $f, g : E \to \mathbb{R}$ be measurable mappings and $c \in \mathbb{R}$. Then $cf, f+g, fg, f \wedge g, f \vee g$ are measurable mappings.*

□

Lemma A.1.8. *[Tho14, Theorem 4.1.6(iv) and (v)] Let $f : E \to G$ and $g : G \to H$ be mappings where $(E,\mathscr{E})$, $(G,\mathscr{G})$ and $(H,\mathscr{H})$ are measurable spaces. Then the following assertions holds true.*

(i) Let $\mathscr{D}$ be a generator set of a $\mathscr{E}$ and assume that $f^{-1}(D) \in \mathscr{E}$ for all $D \in \mathscr{D}$. Then f is measurable.

(ii) If f and g are measurable mappings, then $g \circ f$ is $(\mathscr{E},\mathscr{H})$-measurable.

□

Lemma A.1.9. *[Øks03, Lemma 2.1.2] If $X, Y : \Omega \to \mathbb{R}^n$ are two given functions, then Y is $\sigma(X)$-measurable if and only if there exist a measurable function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $Y = g(X)$.*

□

Theorem A.1.10 ([Bal17, Theorem 1.2] Fubini)**.** *Let μ_1 and μ_2 be measures on the measurable spaces $(E_1,\mathscr{E}_1)$ and $(E_2,\mathscr{E}_2)$ respectively. Let $f : E_1 \times E_2 \to \mathbb{R}$ be a $\mathscr{E}_1 \otimes \mathscr{E}_2$-measurable function such that at least one of the two conditions is true:*

(1) *f is integrable with respect to $\mu_1 \otimes \mu_2$.*

(2) *f is positive. Then*

$$x_1 \mapsto \int_{E_2} f(x_1,z)\mu_2(dz)$$

is $\mathscr{E}_1$-measurable and analogously

$$x_2 \mapsto \int_{E_1} f(z, x_2)\mu_2(dz)$$

is $\mathscr{E}_2$-measurable.
Moreover

$$\int_{E_1 \times E_2} f d\mu_1 \otimes \mu_2 = \int_{E_1}\int_{E_2} f d\mu_2 d\mu_1 = \int_{E_2}\int_{E_1} f d\mu_1 d\mu_2.$$

□

Proposition A.1.11 ([Bal17, Proposition 4.1] Tower property)**.** *Let X be integrable and let $\mathscr{D}$ and $\mathscr{D}'$ be sigma algebras such that $\mathscr{D} \subseteq \mathscr{D}'$. Then*

$$\mathbb{E}[\mathbb{E}[X|\mathscr{D}]|\mathscr{D}'] = \mathbb{E}[\mathbb{E}[X|\mathscr{D}']|\mathscr{D}] = \mathbb{E}[X|\mathscr{D}].$$

□

Definition A.1.12. [Kal21, p. 10] Let S be a space. A π-system in S is a collection of subsets of S, that is closed under intersection.
A λ-system, $\mathscr{D}$, in S is a collection of subsets of S that meets the following conditions:

(i) $S \in \mathscr{D}$.

(ii) $A, B \in \mathscr{D}, A \subseteq B \Rightarrow B \setminus A \in \mathscr{D}$.

(iii) $A_1, A_2, ... \in \mathscr{D}, A_n \uparrow A \Rightarrow A \in \mathscr{D}$.

□

Theorem A.1.13. *[Kal21, Theorem 1.1] Let S be a space, and assume that $\mathscr{C}$ is a π-system in S and that $\mathscr{D}$ is a λ-system in S. Then if $\mathscr{C} \subseteq \mathscr{D}$, it follows that $\sigma(\mathscr{C}) \subseteq \mathscr{D}$.*

□

Theorem A.1.14. *[Tho14, Theorem 2.2.2] Let $(H, \mathscr{H})$ be a measurable space and let μ and ν be measures on the space. Let $\mathscr{D}$ be a system of subsets such that:*

(a) $\mathscr{D}$ is stable to finite intersection.

(b) $\sigma(\mathscr{D}) = \mathscr{H}$.

(c) $\mu(A) = \nu(A)$ for all $A \in \mathscr{D}$.

(d) There exist a sequence of sets of $\mathscr{D}$ such that $\cup_{i\in\mathbb{N}} A_i = H$ and such that $\mu(A_i) = \nu(A_i) < \infty$.

Then $\mu(A) = \nu(A)$ for all $A \in \mathscr{H}$.

□

Theorem A.1.15. *[Øks03, Theorem 2.1.5] Let $n \in \mathbb{N}$ and $t_1, t_2, ..., t_n \in [0, \infty)$. Define the $\nu_{t_1,t_2,...,t_n}$ on $\mathbb{R}^{dn}$ such that*

$$\nu_{t_{\sigma(1)},...,t_{\sigma(n)}}(F_1 \times \cdots \times F_n) = \nu_{t_1,...,t_n}(F_{\sigma^{-1}(1)} \times \cdots \times F_{\sigma^{-1}(n)}) \tag{A.1}$$

for all $F_1, ..., F_n \in \mathscr{B}(\mathbb{R}^d)$ and all permutations on $\{1, 2, ..., n\}$ and that

$$\nu_{t_1,...,t_n}(F_1 \times \cdots \times F_n) = \nu_{t_1,...,t_n,t_{n+1},...,t_{n+m}}(F_1 \times \cdots \times F_n \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d), \tag{A.2}$$

for all $m \in \mathbb{N}$. Then there exist a stochastic process $(Y_t)_{t \geq 0}$ on a probability measure $(\Omega, \mathscr{F}, \mathbb{P})$, that takes values in $\mathbb{R}^d$ such that

$$\nu_{t_1,...,t_n}(F_1 \times \cdots \times F_n) = \mathbb{P}(Y_{t_1} \in F_1, ..., Y_{t_n} \in F_n),$$

for all $n \in \mathbb{N}$ and $t_1, ..., t_n \in [0, \infty)$.

□

A.2 Limiting Results

Theorem A.2.1. *[Kal21, Lemma 1.11] Let E be a measurable space and S be a metric space. Let for all $n \in \mathbb{N}$*

$$f_n : E \to S$$

be measurable functions. Assume that $f_n(x)$ converges for all $x \in E$. Then $\lim_{n\to\infty} f_n$ is a measurable function.

□

Theorem A.2.2. *(Bounded convergence Theorem)[Tho14, p. 5.5.3] Assume that $g, f, (f_n)_{n\in\mathbb{N}} \in \mathscr{M}(\mathbb{R}^d, \mathbb{R})$ and assume that $\lim_{n\to\infty} f_n = f$ almost surely. Let μ be a measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d)$ and assume that*

(a) $|f_n| \leq g$ for all $n \in \mathbb{N}$.

(b) $\int g d\mu < \infty$.

Then $f, (f_n)_{n\in\mathbb{N}} \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

□

Theorem A.2.3. *[Tho14, Theorem 5.2.4] Let μ be a measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $(\mathscr{B}(\mathbb{R}^d), \mathscr{B}(\mathbb{R}))$-measurable, positive and increasing functions. Then $\lim_{n\to\infty} f_n$ is $(\mathscr{B}(\mathbb{R}^d), \mathscr{B}(\mathbb{R} \cup \{\infty\}))$-measurable and*

$$\int_{\mathbb{R}^d} \lim_{n\to\infty} f_n(x)\mu(dx) = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\mu(dx).$$

□

Theorem A.2.4. *([Ped20, Appendix E] Monotone Class Theorem) Let $\mathscr{A}$ be a collection of subsets of Ω which is stable under intersection. We also assumes that $\mathscr{A}$ contains Ω. Now let $\mathscr{H}$ be a collection of real valued functions that satisfies the following three criteria:*

(1) If $A \in \mathscr{A}$ then $\mathbb{1}_A \in \mathscr{H}$.

(2) If $f, g \in \mathscr{H}$ and $c \in \mathbb{R}$ then $f + g, cf \in \mathscr{H}$.

(3) If $(f_n)_{n\in\mathbb{N}}$ is a non decreasing sequence of positive functions from $\mathscr{H}$ and $f := \lim_{n\to\infty} f_n$ is bounded, then $f \in \mathscr{H}$.

Then we have that $\mathscr{H}$ contains all bounded functions that are measurable with respect to $\sigma(\mathscr{A})$.

□

A.3 Brownian Motion and Compound Poisson Processes

Definition A.3.1. A stochastic vector, X, is d-dimensional normally distributed with covariance matrix A ($d \times d$) and mean vector μ (d-dimensional) if the following properties holds true:

(i) $\mathbb{E}[X_i] = \mu_i$ and $\mathrm{Cov}(X_i, X_j) = A_{i,j}$ for all $i, j \in \{1, ..., d\}$.

(ii) $t_1 X_1 + ... + t_d X_d$ is normally distributed for all $t_1, ..., t_d \in \mathbb{R}$.

We write $X \sim N_d(\mu, A)$.

□

Definition A.3.2. [Sat13, Definition 5.1] An $\mathbb{R}^d$-valued process $(B_t)_{t\geq 0}$ adapted to the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is a Brownian motion if it meets the following conditions:

(i) $(B_t)_{t\geq 0}$ is a Lévy process.

(ii) $t \mapsto B_t(\omega)$ is continuous for almost all $\omega \in \Omega$

(iii) For all $0 \leq s \leq t$, $B_t - B_s$ is $N_d(0, (t-s)I)$ distributed.

Remark A.3.3. If B is one dimensional, we sometimes call it a *standard Brownian motion.*

□

Definition A.3.4. [Sat13] Let N be a Poisson process with intensity $c > 0$ and let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of independently identically distributed stochastic vectors on $\mathbb{R}^d$. Then the process $\sum_{k=1}^{N_t} Z_k$ is called a *compound Poisson process* with associated Poisson process N.

□

Bibliography

[App09] Applebaum, D. *Lévy Processes and Stochastic Calculus (Second Edition)*. Cambridge University Press, 2009, pp. xxx + 460.

[Bal17] Baldi, P. *Stochastic Calculus - An Introduction Through Theory and Exercises*. Springer International Publishing AG, 2017, pp. xiv + 627.

[BS15] Barndorff-Nielsen, O. E. and Shiryaev, A. *Change of time and change of measure*. World Scientific Publishing Co. Pte. Ltd., 2015, p. 229.

[Car+03] Carr, P. et al. 'Stochastic Volatility for Lévy Processes'. In: *Mathematical Finance, Vol. 13, No. 3* (2003), pp. 345–382.

[EK19] Eberlein, E. and Kallsen, J. *Mathematical Finance*. Springer Nature Schwitzerland AG, 2019, pp. xvii + 772.

[FG97] Fristedt, B. and Gray, L. *A Modern Approach to Probability Theory*. Birkhäuser Boston, 1997, pp. xx+756.

[Gra76] Grandell, J. *Doubly stochastic Poisson processes, Lecture notes in mathematics; 529*. Springer-Verlag Berlin, 1976, pp. x+234.

[IW89] Ikeda, N. and Watanabe, S. *Stochastic Differential Equations and Diffusion Processes, Second Edition*. North-Holland Publishing Company, 1989, pp. xvi + 555.

[Kal21] Kallenberg, O. *Foundations of Modern Probability (Third Edition)*. Springer, 2021, pp. xii + 946.

[Ped20] Pedersen, J. *Stochastic Processes*. Department of Mathematics, Aarhus University, 2020, pp. ix+128.

[Pou15] Poulsen, E. T. *Funktioner af en of flere vairable, Indledning til matematisk analyse*. Department of Mathematics, Aarhus University, 2015, pp. x+407.

[Pro90] Protter, P. *Stochastic Integration and Differential Equations, A New Approach (Second, Corrected Printing)*. Springer-Verlag Berlin Heidelberg, 1990, pp. x+ 302.

[Rei93] Reiss, R.-D. *A Course on Point Processes*. Springer-Verlag New York, Inc., 1993, pp. x+254.

[Sat13] Sato, K.-I. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 2013, pp. ix+128.

[Ser72] Serfozo, R. F. 'Processes with Conditional Stationary Independent Increments'. In: *Journal of Applied Probability, Vol. 9, No. 2* (1972), pp. 303–315.

[Tho14] Thorbjørnsen, S. *Grundlæggende mål- og integralteori.* Aarhus Universitetsforlag, 2014, pp. xii+425.

[Tho19] Thorbjørnsen, S. *Forelæsningsnoter i Videregående Sansynlighedsteori.* Institut for Matematik, Aarhus Universitet, 2019, pp. iv+223.

[Øks03] Øksendal, B. *Stochastic Differential Equations: An Introduction with Applications (Sixth Edition).* Springer-Verlag Berlin Heidelberg, 2003, pp. xxiii + 360.