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Node Polynomials for Curves on Surfaces

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Abstract. We complete the proof of a theorem we announced and partly proved in [*Math. Nachr.* **271** (2004), 69–90, math.AG/0111299]. The theorem concerns a family of curves on a family of surfaces. It has two parts. The first was proved in that paper. It describes a natural cycle that enumerates the curves in the family with precisely r ordinary nodes. The second part is proved here. It asserts that, for $r \leq 8$, the class of this cycle is given by a computable universal polynomial in the pushdowns to the parameter space of products of the Chern classes of the family.

Key words: enumerative geometry; nodal curves; nodal polynomials; Bell polynomials; Enriques diagrams; Hilbert schemes

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Happy 60th, Lothar

1 Introduction

This paper is the fourth in a series about enumerating nodal curves on smooth complex surfaces. Here we complete the proof of Theorem 2.5 on p. 74 in [16]. It has two parts. The first was proved in [16]. It describes a natural cycle $U(r)$ on the parameter space of a family of pairs of a surface and of a curve on it; $U(r)$ enumerates the curves with precisely r ordinary nodes. The second part is proved here. It asserts that, for $r \leq 8$, the class $[U(r)]$ is given by a computable universal polynomial in the pushdowns of products of the Chern classes of the family.

The second part was not proved in [16], because we believed our approach, inspired by Vainsencher's paper [27], would eventually yield an algorithm for computing the entire polynomial for $[U(r)]$ not only for $r \leq 7$, but also for $r = 8$ and perhaps for all r. So we chose to publish only the construction of $U(r)$ and to postpone the rest. Unfortunately, we were too optimistic. Thus here we work out an ad hoc determination of the polynomial for $r = 8$; specifically, we show that the "correction term" is independent of the family, and so can be found by working out a particular example, such as we did in [16, Example 3.8, p. 80].

In [16, Remark 2.7, p. 74] we conjectured that the class $[U(r)]$ is given for all r by a universal polynomial in certain classes $y(a, b, c)$, defined here in Section 4.4, which are pushdowns of products of the (relative) Chern classes of the family. Moreover, this polynomial should be of the form $P_r(a_1, \dots, a_r)/r!$, where $P_r(a_1, \dots, a_r)$ is the rth (complete) Bell polynomial and the a_i are linear polynomials in the $y(a, b, c)$.

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Göttsche [9] had already conjectured the special case where the pairs consist of a fixed surface and of the divisors in a linear system. This celebrated conjecture was proved independently by Tzeng [26] and Kool–Shende–Thomas [18]. For the history of the case of plane curves, see [16, Remark 3.7, p. 78] and the more recent [4].

A part of our conjecture has now been proved by Laarakker [19, Theorem A, p. 4921]. He defined a cycle $\gamma(r)$ which, under suitable genericity assumptions on the family, is supported on $U(r)$, and its class $[\gamma(r)]$ is given by a universal polynomial in the $y(a,b,c)$. Although he did not prove that the polynomial is Bell (see his footnote [19, p. 4918]), he did prove that $[\gamma(r)]$ is "multiplicative" when the family of surfaces is a direct sum of families over the same base (see [19, Lemma 5.5 and Remark 5.6, p. 4936]). When the family is trivial, Göttsche had observed that this multiplicative property implies the polynomial is Bell. However, when the family is nontrivial, the multiplicative property is insufficient.

In [16] we applied our theorem in several enumerations involving nontrivial families of surfaces, including the family of all planes in $\mathbb{P}^4$. In [19, Theorem B, p. 4922] Laarakker proved that the number of r-nodal plane curves of degree d in $\mathbb{P}^3$ meeting the appropriate number of general lines, is given by a universal polynomial in d of degree $\leq 9+2r$. Moreover, he explicitly computed the polynomial for $r \leq 12$. In [21] Mukherjee, Paul, and Singh did the same; they obtained a recursive formula, and verified that their results agree with Laarakker's. In [5] Das and Mukherjee treated the case where the curves may have one additional nonnodal singularity. In [22] Mukherjee and Singh did the same for rational curves.

In [16, Remark 2.7, p. 74] we conjectured that universal polynomials also enumerate curves with any given equisingularity type. In [13, Theorem 10.1, p. 713] Kazaryan gave a "topological justification" of our conjecture, but gave no algebraic proof. He worked with a linear system on a fixed surface, and found several explicit formulas for curves with singularities of codimension ≤ 7. A few of these formulas had been given in [15, Theorem 1.2, p. 210]. In [2] Basu and Mukherjee gave recursive formulas for the number of curves in a linear system on a fixed surface that have r nodes and one additional singularity of codimension $\leq 8-r$. In particular, their formula for 8-nodal curves recovers ours in this case; see [15, Theorem 1.1, p. 210].

In [20] Li and Tzeng and, independently in [23], Rennemo proved the existence of universal polynomials enumerating divisors with isolated singularities of given topological or analytical types in a trivial family of varieties of arbitrary dimension.

In short, we work here over an algebraically closed field of characteristic 0 with pairs $(F/Y, D)$, where Y is a Cohen–Macaulay algebraic scheme, F/Y is a smooth projective family of surfaces, and D is a relative, or Y-flat, effective divisor on F. We let $\pi\colon F \to Y$ denote the structure map.

In Section 2, given a pair $(F/Y, D)$, we recall from [15, pp. 226–227] the construction and elementary properties of its induced pairs $(F_i/X_i, D_i)$. Then we prove some further properties. Intuitively, $(F_i/X_i, D_i)$ represents a family of curves that sit on blowups of the surfaces of F/Y and that have one less i-fold point.

In Section 3, the main results are Lemmas 3.2 and 3.3, which concern properties of certain subschemes of the relative Hilbert scheme $\mathrm{Hilb}^{3r}_{D/Y}$. In Section 4, we develop some results of bivariant intersection theory for use in the subsequent sections. Our treatment here generalizes and improves our shorter one in [15].

In Section 5, we state the main theorem, Theorem 5.4. Then we prove a key recursion relation; we prove the theorem for $r \leq 7$; and we explain what more is needed for $r = 8$. The difficulty is that the induced pair $(F_2/X_2, D_2)$ does not satisfy the hypotheses of the theorem, as D_2/X_2 has nonreduced fibers in codimension 7 above the relative quadruple-point locus X_4 of D/Y.

Therefore, the recursion that works for $r \leq 7$ must be corrected accordingly. In Section 6 we find an expression for the correction term, and in Section 7 we prove that the correction term is equal to $C[X_4]$ for some integer C that is independent of the given $(F/Y, D)$. Our proof illustrates the advantage of developing intersection theory over any universally catenary Noetherian base. Thus, to complete the proof of the theorem, it suffices to compute the integer C in a particular case, such as that of 8-nodal quintic plane curves, which we did in [16, Example 3.8, p. 80].

However, our proof requires an additional genericity hypothesis: the analytic type of a fiber of D at an ordinary quadruple point must not remain constant along any irreducible component of X_4. This hypothesis comes into play at just one spot in the proof of Lemma 7.4 to ensure a certain map is flat. We believe that Lemma 7.4 and Theorem 5.4 hold without this hypothesis. At any rate, the hypothesis is usually fulfilled in practice.

2 The induced pairs

The induced pairs $(F_i/X_i, D_i)$ of a given pair $(F/Y, D)$ play a central role in the present work. So, in this section, we recall the theory and develop it further. Here F and Y need only be Noetherian, and F/Y need only be of finite type.

2.1. The induced pairs. From [15, pp. 226–227], let's recall the construction and elementary properties of the induced pairs, but make a few minor changes appropriate for the present work.

Denote by $p_j\colon F \times_Y F \to F$ the jth projection, by $\Delta \subset F \times_Y F$ the diagonal subscheme, and by $\mathcal{I}_\Delta$ its ideal. Say D is defined by the global section σ of the invertible sheaf $\mathcal{O}_F(D)$. Then σ induces a section σ_i of the sheaf of relative twisted principal parts,

$$\mathcal{P}^{i-1}_{F/Y}(D) := p_{1*}\big(p_2^*\mathcal{O}_F(D)/\mathcal{I}^i_\Delta\big) \qquad \text{for} \quad i \geq 1. \tag{2.1}$$

Take the scheme of zeros of σ_i to be X_i, and set $X_0 := F$.

Then $X_1 = D$. Further, a geometric point of X_i, that is, a map $\xi\colon \operatorname{Spec}(K) \to X_i$, where K is an algebraically closed field, is just a geometric point ξ of F at which the fiber $D_{\pi(\xi)}$ has multiplicity at least i. Also, as i varies, the X_i form a descending chain of closed subschemes.

The sheaf $\mathcal{P}^{i-1}_{F/Y}(D)$ fits into the exact sequence,

$$0 \to \operatorname{Sym}^{i-1}\Omega^1_{F/Y}(D) \to \mathcal{P}^{i-1}_{F/Y}(D) \to \mathcal{P}^{i-2}_{F/Y}(D) \to 0,$$

where the first term is the symmetric power of the sheaf of relative differentials, twisted by $\mathcal{O}_F(D)$. Hence $\mathcal{P}^{i-1}_{F/Y}(D)$ is locally free of rank $\binom{i+1}{2}$ by induction on i. Therefore, at each scheme point $x \in X_i$, we have

$$\operatorname{cod}_x(X_i, F) \leq \binom{i+1}{2}, \tag{2.2}$$

where, as usual, $\operatorname{cod}_x(X_i, F)$ stands for the minimum $\min(\dim \mathcal{O}_{F,\eta})$ as η ranges over the generizations of x in X_i. If $\operatorname{cod}_x X_i = \binom{i+1}{2}$ and if Y is Cohen–Macaulay at $\pi(x)$, then, since F/Y is smooth, X_i is a local complete intersection in F at x, and is Cohen–Macaulay at x.

Denote by $\beta\colon F' \to F \times_Y F$ the blowup along Δ, and by E the exceptional divisor. Set $\varphi' := p_1\beta$ and $\pi' := p_2\beta$. Then $\pi'\colon F' \to F$ is again a smooth family of surfaces, and projective if π is; in fact, over a point ξ of F, the fiber $F'_\xi := \pi'^{-1}(\xi)$ is just the blowup (via p_1) of the fiber $F_{\pi(\xi)} := \pi^{-1}\pi(\xi)$ at ξ. For each i, set $F_i := \pi'^{-1}(X_i)$, and denote by $\pi_i\colon F_i \to X_i$ the restriction of π'. In sum, we have this diagram:

$$\begin{array}{ccccccc}
F & \xleftarrow{p_1} & F \times_Y F & \xleftarrow{\beta} & F' & \xleftarrow{\beta'_i} & F_i \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi_i} \\
Y & \xleftarrow{\pi} & F & = & F & \xleftarrow{\beta_i} & X_i.
\end{array}$$

In addition, given $r \geq 1$, set

$$r_i := r - \binom{i+1}{2} + 2, \qquad D'_i := {\varphi'}^{-1}D - iE \qquad \text{and} \qquad D_i := D'_i\big|_{F_i}. \tag{2.3}$$

As F' has no associated points on E, the subscheme ${\varphi'}^{-1}D$ is an effective divisor; so D'_i is a divisor on F'. If $i \geq 1$, then

$$D'_i = D'_{i-1} - E.$$

In [15, p. 227], we proved the second assertion of the next lemma. Taking a little more care, we now prove the first too. Later, in Lemma 2.8, we relate X_i and r_i.

Lemma 2.2. *For each $i \geq 1$, the subscheme X_i of F is the largest subscheme over which D'_i is effective. Furthermore, $D_i := D'_i\big|_{F_i}$ is relative effective on F_i/X_i.*

Proof. By definition of X_i, a Y-map $t\colon T \to F$ factors through X_i iff $t^*\sigma_i = 0$. Now, $\mathcal{P}^{i-1}_{F/Y}(D)$ is locally free on F, so flat over Y; hence, $(1\times t)^*\mathcal{I}^i_\Delta \mathcal{O}_{F\times_Y F}(p_1^*D)$ is a subsheaf of $(1\times t)^*\mathcal{O}_{F\times_Y F}(p_1^*D)$ owing to display (2.1). Therefore, $t^*\sigma_i = 0$ iff $(1\times t)^*p_1^*\sigma\colon \mathcal{O}_{F\times_Y T} \to (1\times t)^*\mathcal{O}_{F\times_Y F}(p_1^*D)$ factors through that subsheaf.

Let $q\colon F\times_Y T \to F$ denote the projection. Then $q = p_1(1\times t)$. So

$$(1\times t)^*\mathcal{O}_{F\times_Y F}(p_1^*D) = \mathcal{O}_{F\times_Y T}(q^*D).$$

Let $\Gamma \subset F\times_Y T$ be the graph subscheme of t, and $\mathcal{I}_\Gamma$ its ideal. Then $(1\times t)^{-1}\Delta = \Gamma$. Hence $(1\times t)^*\mathcal{I}^i_\Delta = \mathcal{I}^i_\Gamma$. Therefore, $t^*\sigma_i = 0$ iff $q^*\sigma\colon \mathcal{O}_{F\times_Y T} \to \mathcal{O}_{F\times_Y T}(q^*D)$ factors through $\mathcal{I}^i_\Gamma\mathcal{O}_{F\times_Y T}(q^*D)$.

Set $F'_T := F'\times_F T$ and $\beta_T := \beta\times_F T$. Then $\beta_T\colon F'_T \to F\times_Y T$ is the blowup of $F\times_Y T$ along Γ as $(1\times t)^*\mathcal{I}^i_\Delta = \mathcal{I}^i_\Gamma$. Set $E_T := E\times_F T$. Then E_T is the exceptional divisor. Trivially, $I^i_\Gamma\mathcal{O}_{F'_T} = \mathcal{O}_{F'_T}(-iE_T)$. However, $I^i_\Gamma \xrightarrow{\sim} (\beta_T)_*\mathcal{O}_{F'_T}(-iE_T)$ since Γ is a local complete intersection; see [7, display (6), p. 601]; so the projection formula yields

$$I^i_\Gamma\mathcal{O}_{F\times_Y T}(q^*D) = \beta_{T*}\mathcal{O}_{F'_T}(\beta_T^*q^*D - iE_T).$$

Set $\varphi'_T := q\beta_T$. Then, therefore, $t^*\sigma_i = 0$ iff $\varphi'^*_T\sigma\colon \mathcal{O}_{F'_T} \to \mathcal{O}_{F'_T}(\varphi'^*_T D)$ factors through $\mathcal{O}_{F'_T}(\varphi'^*_T D - iE_T)$.

Let $\tau\colon F'_T \to F'$ denote the projection. Then $\varphi'^*_T D - iE_T = \tau^*D'_i$. Therefore, $t^*\sigma_i = 0$ iff $\tau^*D'_i$ is effective. Thus X_i is the largest subscheme of F over which D'_i is effective.

In particular, on every fiber of π_i, the restriction of D_i is effective. Furthermore, π_i is flat. Hence, D_i is relative effective. Thus the lemma holds. ∎

Lemma 2.3. *Let $(F/Y, D)$ be a pair. Then forming all of the induced pairs $(F_i/X_i, D_i)$ commutes with arbitrary base change $g\colon Y' \to Y$.*

Proof. It follows from [17, Proposition 3.4, p. 422] that the formation of $F^{(1)}$ and $E^{(1)}$ commutes with base change. Set $g'\colon F\times_Y Y' \to F$. By [11, Proposition 16.4.5, p. 19], we have $g'^*\mathcal{P}^{i-1}_{F/Y}(D) = \mathcal{P}^{i-1}_{F\times_Y Y'/Y'}\big(g'^{-1}(D)\big)$, and the section σ_i pulls back to the corresponding section σ'_i. Hence the zero scheme of σ'_i is equal to $X_i\times_Y Y'$. ∎

Definition 2.4. Let $Y(\infty)$ denote the subset of Y whose geometric points are those η of Y whose fiber D_η is not reduced.

Fix a minimal Enriques diagram $\mathbf{D}$; see [15, Section 2, p. 213]. Denote by $Y(\mathbf{D})$ the subset of Y whose geometric points are those η whose fiber D_η has diagram $\mathbf{D}$.

2.5. Arbitrarily near points. Recall the following notions, notation, and results. First, as in [17, Definition 3.1, p. 421], for $j \ge 0$, iterate the construction of $\pi' \colon F' \to F$ from $\pi \colon F \to Y$ to obtain $\pi^{(j)} \colon F^{(j)} \to F^{(j-1)}$ with $\pi^{(0)} := \pi$, with $\pi^{(1)} := \pi'$, and so forth. By [17, Proposition 3.4, p. 422], the Y-schemes $F^{(j)}$ represent the functors of arbitrarily near points of F/Y; the latter are defined in [17, Definition 3.3, p. 422]. As in [17, Definition 3.1, p. 421], we denote by $\varphi^{(j)} \colon F^{(j)} \to F^{(j-1)}$ the map equal to the composition of the blowup and the first projection, and by $E^{(j)} \subset F^{(j)}$ the exceptional divisor.

Given a minimal Enriques diagram $\mathbf{D}$ on $j+1$ vertices, fix an ordering θ of these vertices. Also, let $\mathbf{U}$ be the unweighted diagram underlying $\mathbf{D}$. By [17, Theorem 3.10, p. 425], the functor of arbitrarily near points with $(\mathbf{U}, \theta)$ as associated diagram is representable by a Y-smooth subscheme $F(\mathbf{U}, \theta)$ of $F^{(j)}$.

By [17, Corollary 4.4, p. 430], the group of automorphisms $\operatorname{Aut}(\mathbf{U})$ acts freely on $F(\mathbf{U}, \theta)$. So its subgroup $\operatorname{Aut}(\mathbf{D})$, of automorphisms of $\mathbf{D}$, does too. Set

$$Q(\mathbf{D}) := F(\mathbf{U}, \theta)/\operatorname{Aut}(\mathbf{D});$$

it is independent of the choice of θ by [17, Theorem 5.7, p. 438]. Set $d := \deg \mathbf{D}$.

Form the structure map and the universal injection of [17, Theorem 5.7, p. 438]:

$$q\colon\ Q(\mathbf{D}) \to Y \qquad \text{and} \qquad \Psi\colon\ Q(\mathbf{D}) \to \operatorname{Hilb}^d_{F/Y};$$

in fact, Ψ is a an embedding in characteristic 0. The construction and study of Ψ is based on the modern theory of complete ideals. Finally, set

$$G(\mathbf{D}) := \operatorname{Hilb}^d_{D/Y} \times_{\operatorname{Hilb}^d_{F/Y}} Q(\mathbf{D}). \tag{2.4}$$

Lemma 2.6. *The sets $Y(\mathbf{D})$ and $Y(\infty)$ are constructible; in fact, $Y(\infty)$ is closed if F/Y is proper. Furthermore, for all $z \in G(\mathbf{D})$ and $y \in Y(\mathbf{D})$, we have*

$$\operatorname{cod}_z(G(\mathbf{D}), Q(\mathbf{D})) \le d \qquad \textit{and} \qquad \operatorname{cod}_y(Y(\mathbf{D}), Y) \le \operatorname{cod} \mathbf{D}. \tag{2.5}$$

Finally, for only finitely many $\mathbf{D}$, is either $G(\mathbf{D}) \smallsetminus q^{-1}Y(\infty)$ or $Y(\mathbf{D})$ nonempty.

Proof. Note that $Y(\infty)$ is just the image in Y of the set of $x \in X_2$ at which the fiber of X_2/Y is of dimension at least 1. This set is closed in X_2, so in F. Hence $Y(\infty)$ is constructible; in fact, $Y(\infty)$ is closed if π is proper.

Only finitely many $\mathbf{D}$ arise from the fibers of D/Y; indeed, this statement is proved in [16, Lemma 2.4, p. 73] without making use of its blanket hypothesis that Y is Cohen–Macaulay and of finite type over the complex numbers; that proof just requires Y to be Noetherian. Thus there are only finitely many $\mathbf{D}$ such that $Y(\mathbf{D})$ is nonempty; denote the set of these $\mathbf{D}$ by Σ.

The subscheme $\operatorname{Hilb}^d_{D/Y}$ of $\operatorname{Hilb}^d_{F/Y}$ is locally cut out by d equations by [1, Proposition 4, p. 5]. Therefore, the first bound holds in (2.5).

The definitions yield $q(G(\mathbf{D})) \supset Y(\mathbf{D})$. Further, take any $y \in q(G(\mathbf{D})) \smallsetminus Y(\infty)$, and let $\mathbf{D}'$ be the diagram of D_K, where K is the algebraic closure of $k(y)$. Then the definitions yield a natural injection $\alpha\colon \mathbf{D} \hookrightarrow \mathbf{D}'$ such that each $V \in \mathbf{D}$ has weight at most that of $\alpha(V)$. So $\deg \mathbf{D}' > d$ if $y \notin Y(\mathbf{D})$. Hence

$$Y(\mathbf{D}) = q(G(\mathbf{D})) \smallsetminus \big(Y(\infty) \cup \big(\textstyle\bigcup\{q(G(\mathbf{D}')) \mid \mathbf{D}' \in \Sigma \text{ and } \deg \mathbf{D}' > d\}\big)\big).$$

But $G(\mathbf{D})$ and the $G(\mathbf{D}')$ are locally closed. Thus $Y(\mathbf{D})$ is constructible.

To prove the second bound in (2.5), note that $G(\mathbf{D})$ has a unique point, z say, lying over the given y. Now, $Q(\mathbf{D})/Y$ is smooth of relative dimension $\dim \mathbf{D}$ by [17, Theorem 3.10, p. 425]. Thus, as desired,

$$\operatorname{cod}_y(Y(\mathbf{D}), Y) = \operatorname{cod}_z(G(\mathbf{D}), Q(\mathbf{D})) - \dim \mathbf{D} \le d - \dim \mathbf{D} = \operatorname{cod} \mathbf{D}.$$

Finally, suppose $G(\mathbf{D}) \smallsetminus h^{-1}Y(\infty)$ is nonempty. Then, as we have just seen, there is an injection $\alpha\colon \mathbf{D} \hookrightarrow \mathbf{D}'$, where $\mathbf{D}' \in \Sigma$, and each $V \in \mathbf{D}$ has weight at most that of $\alpha(V)$. But there are only finitely many such $\mathbf{D}$, as desired. ∎

Definition 2.7. We say that $(F/Y, D)$ is *r-generic* if for every minimal Enriques diagram $\mathbf{D}$ and for every $y \in Y(\mathbf{D})$, we have

$$\operatorname{cod}_y(Y(\mathbf{D}), Y) \geq \min(r+1, \operatorname{cod}\mathbf{D}). \tag{2.6}$$

We say that $(F/Y, D)$ is *strongly 8-generic* if it is 8-generic and if the analytic type of $D_{\pi(x)}$ at an ordinary quadruple point $x \in X_4$ is not constant along any irreducible component Z of X_4; that is, the cross ratio of the four tangents at x is not the same for all $x \in Z$.

Proposition 2.8. *Fix r. Assume that Y is universally catenary and that $(F/Y, D)$ is r-generic. Then, for each $i \geq 2$, the induced pair $(F_i/X_i, D_i)$ is r_i-generic.*

Proof. Fix i. Let $\mathbf{D}'$ be a minimal Enriques diagram. Let x be a generic point of the closure of $X_i(\mathbf{D}')$. Then $x \in X_i(\mathbf{D}')$ as $X_i(\mathbf{D}')$ is constructible by Lemma 2.6 applied with $(F_i/X_i, D_i)$ and $\mathbf{D}'$ for $(F/Y, D)$ and $\mathbf{D}$. Set $y := \pi(x)$. Let K be an algebraically closed field containing $k(x)$; then K contains $k(y)$ too.

Consider the curves D_K and $(D_i)_K$. Note $x \in X_i(\mathbf{D}')$. So the curve $(D_i)_K$ is reduced, and is obtained from D_K as follows: blow up F_K, at the K-point, x_K say, defined by x; take the preimage of D_K; and subtract i times the exceptional divisor. Hence D_K is reduced and of multiplicity either i or $i+1$ at x_K. In the latter case, $(D_i)_K$ contains the exceptional divisor; in the former, it doesn't. In either case, let $\mathbf{D}$ be the diagram of D_K. Then [17, Proposition 2.8, p. 420] yields

$$\operatorname{cod}(\mathbf{D}) \geq \operatorname{cod}(\mathbf{D}') + \binom{i+1}{2} - 2. \tag{2.7}$$

Since F/Y is flat, the dimension formula yields

$$\dim \mathcal{O}_{F,x} = \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{F_y,x}.$$

However, x is the generic point of a component, X say, of the closure of $X_i(\mathbf{D}')$; hence, $\dim \mathcal{O}_{F,x} = \operatorname{cod}_x(X, F)$. So $y = \pi(x)$. So $\dim \mathcal{O}_{Y,y} = \operatorname{cod}_y(\pi(X), Y)$. Further, F/Y is of relative dimension 2; so $\dim \mathcal{O}_{F_y,x} = 2$. Thus

$$\operatorname{cod}_x(X, F) - 2 = \operatorname{cod}_y(\pi(X), Y). \tag{2.8}$$

However, $y := \pi(x) \in Y(\mathbf{D})$. Hence,

$$\operatorname{cod}_y(\pi(X), Y) \geq \operatorname{cod}_y(Y(\mathbf{D}), Y).$$

Combine the last two displays; then (2.6) yields

$$\operatorname{cod}_x(X, F) - 2 \geq \min(r+1, \operatorname{cod}\mathbf{D}). \tag{2.9}$$

Since Y is universally catenary, F is catenary; hence,

$$\operatorname{cod}_x(X, X_i) = \operatorname{cod}_x(X, F) - \operatorname{cod}_x(X_i, F). \tag{2.10}$$

Hence (2.9) and (2.2) yield

$$\operatorname{cod}_x(X, X_i) \geq \min(r+1, \operatorname{cod}\mathbf{D}) + 2 - \binom{i+1}{2}.$$

Therefore, (2.3) and (2.7) yield the desired lower bound:

$$\operatorname{cod}_x(X, X_i) \geq \min(r_i+1, \operatorname{cod}\mathbf{D}').$$ ∎

Corollary 2.9. *Fix r. Assume $(F/Y, D)$ is r-generic. Fix $i \geq 2$, let X be a component of X_i, take $x \in X \smallsetminus Y(\infty)$, and set $y := \pi(x)$. Then*

$$\operatorname{cod}_x(X,F) = \binom{i+1}{2}, \qquad \operatorname{cod}_y(\pi(X),Y) = \binom{i+1}{2} - 2 \qquad \text{if} \quad r_i \geq -1, \tag{2.11}$$

$$\operatorname{cod}_x(X,F) \geq r+3, \qquad \operatorname{cod}_y(\pi(X),Y) \geq r+1 \qquad \text{if} \quad r_i \leq -1. \tag{2.12}$$

Proof. Plainly we may assume x is the generic point of X. Let K be an algebraically closed field containing $k(x)$, so $k(y)$. Then D_K is reduced as $x \notin Y(\infty)$. Let $\mathbf{D}$ be the diagram of D_K, and $\mathbf{D}'$ that of D_i. Then X is a component of the closure of $X_i(\mathbf{D}')$. So we may appeal to the proof of Proposition 2.8. Note that equation (2.10) is trivial here, and we do not need Y to be universally catenary.

Since $x \in X_i$, at the corresponding K-point, D_K is of multiplicity at least i. Hence $\mathbf{D}$ has a root of weight at least i. So $\operatorname{cod}\mathbf{D} \geq \binom{i+1}{2} - 2$.

Suppose $r_i \geq -1$. Then (2.3) yields $r+1 \geq \binom{i+1}{2} - 2$. So (2.9) yields

$$\operatorname{cod}_x(X,F) \geq \binom{i+1}{2}.$$

But the opposite inequality is (2.2), which always holds. So equality holds. Thus, in (2.11), the first equation holds. The second follows from it and (2.8).

Suppose $r_i \leq -1$ instead. Then $\binom{i+1}{2} - 2 \geq r+1$. So $\operatorname{cod}\mathbf{D} \geq r+1$. Hence (2.9) and (2.8) yield (2.12). Thus the corollary is proved. ∎

3 Virtual double points

The minimal Enriques diagram $r\mathbf{A}_1$ consists of r roots of weight 2 and no other vertices. The corresponding scheme $G(r\mathbf{A}_1)$ is particularly important, as it is equal to the subscheme of the Hilbert scheme $\operatorname{Hilb}^{3r}_{D/Y}$ associated to the geometric fibers of D/Y with at least r distinct singular points. Moreover, we need to consider it for various $(F/Y, D)$ and r. So, for clarity, we set $G(F/Y, D; r) := G(r\mathbf{A}_1)$.

In this section, we first recall the basic properties of $G(F/Y, D; r)$, which were treated in [17, Proposition 5.9, p. 439]. Then we fix $r \geq 1$, and assume $(F/Y, D)$ is r-generic. For each $i \geq 1$, we find a natural large open subscheme of

$$H_i := G(F_i/X_i, D_i; r_i)$$

such that the associated geometric fibers of D_i/X_i have exactly r_i nodes. None lies on the exceptional divisor of a fiber of F_i. Further, adding the exceptional divisor to the fiber of D_i/X_i yields a fiber of D_{i-1}/X_{i-1}, and thus establishes an isomorphism from the preceding open subscheme to a natural open subscheme of H_{i-1}, which is dense in the preimage of X_i. These results are treated in Lemmas 3.2 and 3.3 below for later use.

3.1. Subschemes of the Hilbert scheme. Fix r. If $r \geq 1$, let $H(r)$ denote the open subscheme of $\operatorname{Hilb}^r_{F/Y}$ over which the universal family is smooth; in other words, $H(r)$ parameterizes the unions of r distinct reduced points in the geometric fibers of F/Y. By convention, if $r = 0$, then $H(r)$ and $\operatorname{Hilb}^r_{F/Y}$ are both equal to Y; if $r \leq -1$, then both are empty.

If $r \geq 1$, then Proposition 5.9 on p. 439 in [17] asserts that $H(r) = Q(r\mathbf{A}_1)$ and that the map $\Psi \colon H(r) \to \operatorname{Hilb}^{3r}_{F/Y}$ is given on T-points, where T is a Y-scheme, by sending a subscheme W of F_T, say with ideal $\mathcal{I}$, to the subscheme W' with ideal $\mathcal{I}^2$ (note that W' is flat, because the standard sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_{W'} \to \mathcal{O}_W \to 0$$

is exact and because $\mathcal{I}/\mathcal{I}^2$ and $\mathcal{O}_W$ are flat); furthermore, Ψ is always an embedding.

Consequently, we may view $G(r\mathbf{A}_1)$ as a subscheme of $\mathrm{Hilb}^{3r}_{D/Y}$. Set

$$G(F/Y, D; r) := G(r\mathbf{A}_1) \subset \mathrm{Hilb}^{3r}_{D/Y}$$

to avoid confusion. Furthermore, set $G(F/Y, D; 0) := Y$, and for $r \le -1$, set $G(F/Y, D; r) := \varnothing$. Finally, for an arbitrary fixed r and for $i \ge 0$, set

$$H_i := G(F_i/X_i, D_i; r_i).$$

Lemma 3.2. *Fix $r \ge 1$. Assume that $(F/Y, D)$ is r-generic and that $Y(\infty)$ is empty. Then there is an open subscheme $U \subset Y$ such that* (1) *for every $y \in Y \setminus U$,*

$$\mathrm{cod}_y(Y \setminus U, Y) \ge r + 1 \tag{3.1}$$

and (2) *for every $i \ge 1$ with $r_i \ge 0$, if we set*

$$U_i := \big(\pi^{-1}U\big) \cap (X_i \setminus X_{i+1}) \qquad \textit{and} \qquad V_i := \big(\pi^{-1}U\big) \cap (X_{i-1} \setminus X_{i+1}),$$

then U_i is dense in X_i, and there is a natural isomorphism of F-schemes

$$\gamma_i\colon\ H_i \times_F U_i \xrightarrow{\sim} H_{i-1} \times_F U_i.$$

Proof. Let U be the complement in Y of the union of the closures of those $Y(\mathbf{D})$ with $\mathrm{cod}\,\mathbf{D} \ge r+1$. Then U is open since there are only finitely many nonempty $Y(\mathbf{D})$ by Lemma 2.6. By the same token, (2.6) implies (3.1). Thus (1) holds.

Fix $i \ge 1$ such that $r_i \ge 0$. Then $r + 1 > \binom{i+1}{2} - 2$. Let X be a component of X_i; let $x \in X$, set $y := \pi(x)$. Then (2.11) yields that $r + 1 > \mathrm{cod}_y(\pi(X), Y)$; moreover, if $x \in X_{i+1}$, then $\mathrm{cod}_y(\pi(X_{i+1}), Y) > \mathrm{cod}_y(\pi(X), Y)$. So (3.1) implies $\pi(X) \setminus \pi(X_{i+1}) \not\subset Y \setminus U$. Hence $\pi(X_i \setminus X_{i+1})$ meets U. Thus U_i is dense in $X_i \setminus X_{i+1}$.

Let $z \in H_i \times_F U_i$; let x be its image in U_i, and set $y := \pi(x)$. Let K be an algebraically closed field containing $k(z)$, so $k(x)$ and $k(y)$ too. Then D_K is reduced since $Y(\infty)$ is empty, and D_K has multiplicity exactly i at the K-point x_K defined by x since $x \in U_i$, so $x \notin X_{i+1}$. Hence $(D_i)_K$ is reduced, and does not contain the exceptional divisor E_K.

Let $\mathbf{D}$ be the diagram of D_K, and $\mathbf{D}'$ that of $(D_i)_K$. By [17, Proposition 2.8, p. 420], we have

$$\mathrm{cod}(\mathbf{D}) \ge \mathrm{cod}(\mathbf{D}') + \binom{i+1}{2} - 2, \tag{3.2}$$

with equality if and only if D_K has an ordinary i-fold point at x_K. Now, $(D_i)_K$ has at least r_i singular points since $z \in H_i$; hence, formula (2.6.2) in [17, p. 419] yields $\mathrm{cod}(\mathbf{D}') \ge r_i$ since $\mathbf{D}'$ has at least r_i roots, each root has multiplicity at least 2, and the summands in that formula corresponding to the other vertices of $\mathbf{D}'$ are nonnegative. So the right-hand side of (3.2) is at least r. However, $r \ge \mathrm{cod}(\mathbf{D})$ since $y \in U$. So equality obtains everywhere. Hence D_K has an ordinary i-fold point at x_K. Furthermore, $(D_i)_K$ has exactly r_i singular points, each is an ordinary double point, and none lies on E_K; also, $(D_i)_K$ and E_K meet transversally in i points.

We define γ_i as follows. A T-point of its source $H_i \times_F U_i$ is given by a map $T \to U_i$ and a T-smooth subscheme $W \subset F'_T$ of relative length r_i whose squared ideal defines a subscheme $W' \subset F'_T$ contained in $(D_i)_T$. Owing to the discussion above, in every geometric fiber of F'_T/T, the fibers of $(D_i)_T$ and E_T meet transversally in i points. Hence, since $(D_i)_T$ and E_T are relative effective divisors, their intersection is a T-smooth subscheme $Z \subset F'_T$ of relative length i.

Let Z' be the subscheme of F'_T defined by the squared ideal of Z. Then Z' is contained in the sum $(D_i)_T + E_T$, which is equal to $(D_{i-1})_T$. So $W \cup Z$ is a T-smooth subscheme of F'_T of relative length $r_i + i$, or r_{i-1}. And its squared ideal defines a subscheme of F'_T, namely $W' \cup Z'$, which is contained in $(D_{i-1})_T$. So $W \cup Z$ determines a T-point of $H_{i-1} \times_F U_i$, and the latter scheme is to be the target of γ_i. We define γ_i by sending W to $W \cup Z$. Plainly, γ_i is injective on T-points since W is determined by $W \cup Z$ as the part off E_T.

To prove γ_i is surjective on T-points, fix a T-point of $H_{i-1} \times_F U_i$. It is given by a map $T \to U_i$ and a T-smooth subscheme $S \subset F'_T$ of relative length r_{i-1} such that its squared ideal defines a subscheme $S' \subset F'_T$ contained in $(D_{i-1})_T$. Then $(D_{i-1})_T = (D_i)_T + E_T$ by (2.3), and $(D_i)_T$ is relative effective by Lemma 2.2 since $U_i \subset X_i$. Let W be the part of S off E_T. Plainly, W is a T-smooth subscheme of F'_T, and its squared ideal defines a subscheme contained in $(D_i)_T$.

Consider a geometric point of T, say with (algebraically closed) field K. Then D_K is reduced since $Y(\infty)$ is empty, and D_K has multiplicity exactly i at the center of K since T maps into U_i. Hence $(D_i)_K$ is reduced, and $(D_i)_K \cap E_K$ is a scheme of length i. Now, S_K is K-smooth of length r_{i-1}. Hence S_K consists of r_{i-1} distinct reduced points, of which at most i lie on E_K. So W_K consists of at least $r_{i-1} - i$, or r_i, distinct reduced points. By choosing any r_i of them, we obtain a K-point of $H_i \times_F U_i$. But then, by the discussion of such points right after (3.2), there was no choice: $(D_i)_K$ has exactly r_i singular points, and all are ordinary nodes. Hence W_K consists exactly of r_i distinct reduced points. Thus W is of relative length r_i.

Therefore, W defines a T-point of $H_i \times_F U_i$. According to the discussion above, this T-point is carried by γ_i to the T-point of $H_{i-1} \times_F U_i$ that is given by R, where $R := W \cup Z$ and $Z := (D_i)_T \cap E_T$. To prove that γ_i is surjective on T-points, so bijective on T-points, so an isomorphism, it remains to prove that $R = S$.

The equation $R = S$ may be checked locally over T and locally on F. So we may replace T and F by affine open subsets Spec(A) and Spec(B). Then B is étale over a polynomial subring $A[x, y]$. Let $I \subset B$ denote the ideal of S. Shrinking F further if necessary, we can find an $f \in B$ that generates the ideal of $(D_{i-1})_T$. Then $f \in I^2$ as R determines a T-point of H_{i-1}. Hence f, $\partial f/\partial x$, $\partial f/\partial y \in I$. But those three elements generate the ideal of $Z := (D_i)_T \cap E_T$ on a neighborhood N of E_T. Hence $Z \supset S \cap N$. But both Z and $S \cap N$ are T-flat of relative length i. Hence $Z = S \cap N$. But R and S are equal off E_T. Thus $R = S$, as desired. ■

Lemma 3.3. *Under the conditions of Lemma 3.2, the closed subscheme $H_{i-1} \times_F U_i$ of $H_{i-1} \times_F V_i$ is also open.*

Proof. Consider any T-point of $H_{i-1} \times_F V_i$. Let T' be the preimage of $H_{i-1} \times_F U_i$. It suffices to prove T' is an open subscheme, as we may take $T = H_{i-1} \times_F V_i$.

Let $\mathcal{I} \subset \mathcal{O}_T$ denote the ideal of T'. Then it suffices to prove that the stalk $\mathcal{I}_t$ vanishes for all $t \in T'$ for the following reason. Since $\mathcal{I}$ is coherent, the $t \in T$, where I_t vanishes form an open subset T''. By hypothesis, $T' \subset T''$. But, if $t \notin T'$, then $I_t = \mathcal{O}_{T,t}$; whence, $T' \supset T''$. So $T' = T''$. Give T'' the induced structure as an open subscheme of T. Then T' is the closed subscheme of T'' with ideal $\mathcal{I}\mid_{T''}$. But $\mathcal{I}\mid_{T''} = 0$. Thus T' is equal to the open subscheme T''.

Given $t \in T'$, to check if $\mathcal{I}_t$ vanishes, we may replace T by Spec$(\mathcal{O}_{T,t})$. Thus we may assume that T is of the form Spec(A), where A is local and that T' is nonempty. Then it suffices to prove the ideal $I \subset A$ of T' vanishes, or equivalently, $T = T'$.

There exists a flat local homomorphism $A \to B$ such that B is complete and its residue class field is algebraically closed. Then $A \to B$ is faithfully flat. So I vanishes if $I \otimes_A B$ does. Thus we may replace A by B, and so assume that A is complete and its residue class field is algebraically closed.

Consider the composition $T \to H_{i-1} \times_F V_i \to V_i$. Via it, T' is the preimage of U_i. Hence $T = T'$ if and only if $(D'_i)_T$ is effective, owing to Lemma 2.2. By the same token, $(D'_{i-1})_T$ is effective.

Consider the local ring C of F_T at the closed point of the center of the blowing up $F'_T \to F_T$. Let $\widehat{C}$ be its completion. Since $C \to \widehat{C}$ is faithfully flat, $(D'_i)_T$ is effective if and only if $(D'_i)_T \otimes_C \widehat{C}$ is effective.

As F/Y is a smooth family of surfaces and as $\widehat{C}$ is complete with algebraically closed residue class field, $\widehat{C}$ is a power series ring; say $\widehat{C} = A[[u,v]]$. Say that the section $T \to F_T$ is defined by mapping u, v to $a, b \in A$. Replacing u, v by $u-a$, $v-b$, we may assume that $T \to F$ is defined by mapping u, v to 0, 0.

Let f in $A[[u,v]]$ define the pullback of D. Write $f = f_1 + f_2 + \cdots$, where f_j is homogeneous of degree j in u and v. Then $f_j = 0$ for $1 \le j \le i-2$ since $(D'_{i-1})_T \otimes_C \widehat{C}$ is effective. It remains to prove that $f_{i-1} = 0$.

To prove that $f_{i-1} = 0$, denote the maximal ideal of A by $\mathbf{m}$, and write

$$f_{i-1}(u,v) = a_1 u^{i-1} + a_2 u^{i-2} v + \cdots + a_i v^{i-1} \qquad \text{with} \quad a_j \in A. \tag{3.3}$$

Then it suffices to prove that $a_j \in \mathbf{m}^n$ for all j and $n \ge 0$.

Since T maps into H_{i-1}, there is a T-smooth subscheme $S \subset F'_T$ of relative length r_{i-1} whose squared ideal defines a subscheme $S' \subset F'_T$ contained in $(D_{i-1})_T$. Since A has an algebraically closed residue class field K, the fiber S_K consists of r_{i-1} distinct points. Of them, exactly i lie on E_K according to our discussion above. Further, the fiber S_K is the singular locus of $(D_{i-1})_K$, which consists of r_{i-1} ordinary double points, and i of them constitute $(D_i)_K \cap E_K$, which is a transverse intersection.

By replacing u and v with suitable linear combinations of themselves, we may assume that the u-axis is not tangent to D_K at the center of the blowing up. Now, A is complete; so by Hensel's lemma, S decomposes into the disjoint sum of r_{i-1} sections. Of them, i sections meet E_T. Hence, they correspond to A-algebra maps

$$s_j\colon\ A[[u,v]][w]/(u - vw) \to A \qquad \text{for} \quad 1 \le j \le i.$$

Set $b_j := s_j(v)$ and $c_j := s_j(w)$. Then, for all j, let's check that

$$b_j \in \mathbf{m} \qquad \text{and} \qquad f_i(c_j, 1) \in \mathbf{m}.$$

The first relation holds as the closed points of the sections lie on E_K. The second holds because these same points lie on $(D_i)_K \cap E_K$. Further, the points are distinct; so mod $\mathbf{m}$, the c_j are distinct elements of K.

Proceeding by induction on $n \ge 1$, suppose that $b_j \in \mathbf{m}^n$ for all j. Set

$$\bar{f}(v,w) := f(vw, v)/v^{i-1}.$$

Then $\bar{f}$ defines the pullback of $(D_{i-1})_T$. Now, $S \subset (D_{i-1})_T$. Hence, for each j,

$$0 = \bar{f}(b_j, c_j) = f_{i-1}(c_j, 1) + b_j f_i(c_j, 1) + b_j^2 d_j$$

for some $d_j \in A$. But $f_i(c_j, 1) \in \mathbf{m}$. Thus $f_{i-1}(c_j, 1) \in \mathbf{m}^{n+1}$.

From (3.3), we obtain the following linear system of equations for the a_j:

$$f_{i-1}(c_j, 1) = a_1 c_j^{i-1} + a_2 c_j^{i-2} + \cdots + a_i \qquad \text{for} \quad 1 \le j \le i.$$

The coefficient matrix is Vandermonde. Its determinant is invertible in A, as the c_j are distinct mod $\mathbf{m}$. As $f_{i-1}(c_j, 1) \in \mathbf{m}^{n+1}$ for all j, solving yields $a_j \in \mathbf{m}^{n+1}$.

To complete the proof, we must show $b_j \in \mathbf{m}^{n+1}$ for each j. Set $I_j := \operatorname{Ker}(s_j)$. Then $\bar{f} \in I_j^2$ as $S' \subset (D_{i-1})_T$. Hence $\partial\bar{f}/\partial w \in I_j$. Therefore,

$$0 = (\partial\bar{f}/\partial w)(b_j, c_j) = (\partial f_{i-1}/\partial u)(c_j, 1) + b_j(\partial f_i/\partial u)(c_j, 1) + b_j^2 d'_j \tag{3.4}$$

for some $d'_j \in A$. Now, $a_j \in \mathbf{m}^{n+1}$; so (3.3) yields

$$(\partial f_{i-1}/\partial u)(c_j, 1) = (i-1)a_1 c_j^{i-2} + (i-2)a_2 c_j^{i-3} + \cdots + a_{i-1} \in \mathbf{m}^{n+1}.$$

But $b_j \in \mathbf{m}^n$. So (3.4) yields $b_j(\partial f_i/\partial u)(c_j, 1) \in \mathbf{m}^{n+1}$. But $(c_j, 1)$ is, mod $\mathbf{m}$, a simple root of f_i; so $(\partial f_i/\partial u)(c_j, 1) \notin \mathbf{m}$. Thus $b_j \in \mathbf{m}^{n+1}$, as desired. ■

4 Intersection theory

For use in the remaining sections, we extend the intersection theory of bivariant classes developed in [8, Chapter 17] and generalized over any universally catenary base in [14, Sections 2 and 3], in [25], and in [24, Chapter 42]. However, only [8] is cited below.

4.1. Push down. Assume that $f\colon X \to Y$ is a map of schemes such that its orientation class $[f]$ is defined [8, Section 17.4, p. 326]. If f is also proper, define an additive map

$$f_{\#}\colon\; A^*(X) \to A^*(Y) \qquad \text{by} \qquad f_{\#}a := f_*(a\cdot[f]),$$

where $f_*\colon A^*(f) \to A^*(Y)$ is the proper push-forward operation discussed in $(\mathbf{P}_2)$ on p. 322 of [8].

Proposition 4.2. *Let $a, b \in A^*(X)$. Assume that a is a polynomial in Chern classes of vector bundles on X. Then $f_{\#}a \cdot f_{\#}b = f_{\#}b \cdot f_{\#}a$.*

Proof. By $(\mathbf{A}_{12})$ on p. 323 of [8], product and push-forward commute; so

$$f_{\#}a\cdot f_{\#}b = f_*(a\cdot[f]\cdot f_*(b\cdot[f])).$$

Let $p_i\colon X\times_Y X \to X$ denote the ith projection. Form the diagram

$$\begin{array}{ccccccc}
X\times_Y X & \xrightarrow{\;1\;} & X\times_Y X & \xrightarrow{\;p_2\;} & X & & \\
\big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle f} & & \\
X & \xrightarrow{\;1\;} & X & \xrightarrow{\;f\;} & Y & \xrightarrow{\;1\;} & Y.
\end{array}$$

Apply the projection formula of $(\mathbf{A}_{123})$ of [8, p. 323] with $f := f$, with $g := f$, with $h := 1_Y$, with $c := [f]$ and with $d := b\cdot[f]$. The result is

$$[f]\cdot f_*(b\cdot[f]) = p_{1*}\big(f^*([f])\cdot b\cdot[f]\big).$$

The definitions of f^*, of $[f]$, and of p_2 yield $f^*([f]) = [p_2]$. But Axiom $(\mathbf{C}_2)$ on p. 320 of [8] yields $[p_2]\cdot b = p_2^*(b)\cdot[p_2]$. Thus $[f]\cdot f_*(b\cdot[f]) = p_{1*}(p_2^*(b)\cdot[p_2]\cdot[f])$.

Apply this projection formula again, but now with $f := 1_X$, with $g := p_1$, with $h := f$, with $c := a$ and with $d := p_2^*(b)\cdot[p_2]\cdot[f]$. The result is

$$a\cdot p_{1*}\big(p_2^*(b)\cdot[p_2]\cdot[f]\big) = p_{1*}\big(p_1^*(a)\cdot(p_2^*(b)\cdot[p_2]\cdot[f])\big).$$

The formula just before Proposition 17.4.1 on p. 327 of [8] yields $[p_2]\cdot[f] = [fp_2]$. So the functoriality of pushforwards, stated in $(\mathbf{A}_2)$ on p. 323 of [8], yields

$$f_*\big(p_{1*}\big(p_1^*(a)\cdot p_2^*(b)\cdot[p_2]\cdot[f]\big)\big) = (fp_1)_*\big(p_1^*(a)\cdot p_2^*(b)\cdot[fp_2]\big).$$

Putting it all together yields

$$f_{\#}a\cdot f_{\#}b = (fp_1)_*\big(p_1^*(a)\cdot p_2^*(b)\cdot[fp_2]\big). \tag{4.1}$$

By hypothesis, a is a polynomial in Chern classes of vector bundles on F. But product and pullback commute by property $(\mathbf{A}_{13})$ on p. 323 of [8]. So $p_1^*(a)$ is the same polynomial in the same Chern classes of the pullbacks under p_1 of those vector bundles. But, as stated just before Proposition 17.3.2 on p. 325 of [8], Chern classes commute with all bivariant classes. Thus $p_1^*(a)\cdot p_2^*(b) = p_2^*(b)\cdot p_1^*(a)$.

Of course, $fp_1 = fp_2$. Thus

$$f_\# a \cdot f_\# b = (fp_2)_*\big(p_2^*(b) \cdot p_1^*(a) \cdot [fp_1]\big).$$

But a and b are arbitrary in (4.1); moreover, p_1 and p_2 may be interchanged. So

$$(fp_2)_*\big(p_2^*(b) \cdot p_1^*(a) \cdot [fp_1]\big) = f_\# b \cdot f_\# a.$$

Thus $f_\# a \cdot f_\# b = f_\# b \cdot f_\# a$, as asserted. ■

Assume $g\colon Y' \to Y$, and consider $f'\colon X' := X \times_Y Y' \to Y'$. Then,

$$g^* f_\# = f'_\# g^*\colon\; A^*(X) \to A^*(Y'). \tag{4.2}$$

This property results from property ($\mathbf{A}_{23}$) on p. 323 of [8] as follows:

$$\begin{aligned} g^* f_\#(a) &= g^* f_*(a \cdot [f]) = f'_* g^*(a \cdot [f]) = f'_*(g^*(a) \cdot g^*[f]) \\ &= f'_*(g^*(a) \cdot [f']) = f'_\#(g^*(a)). \end{aligned}$$

Assume that $f\colon X \to Y$ and $g\colon Y \to Z$ are proper and that $[f]$, $[g]$, and $[gf]$ exist. Then

$$(gf)_\# = g_\# f_\#,$$

since, by [8, Section 17.4, p. 327],

$$(gf)_\#(a) = g_* f_*(a \cdot [gf]) = g_* f_*(a \cdot [f] \cdot [g]) = g_*(f_*(a \cdot [f]) \cdot [g]) = g_\#(f_\#(a)).$$

4.3. Blowups. Let $\iota\colon W \to V$ be a closed, regular embedding of codimension d. Then the orientation class $[\iota]$ is defined. Let V' denote the blowup of V along W, with exceptional divisor E. Then $E = \mathbb{P}\big(\nu^\vee\big)$, where ν is the normal bundle of W in V. Set $\xi := c_1(\mathcal{O}_{V'}(-E))$. The map $f\colon E \to W$ is flat, hence has an orientation class $[f]$. Then, by [8, Corollary 4.2.2, p. 75], for $k \geq 1$,

$$f_\# \xi^k = -s_{k-d}\big(\nu^\vee\big), \tag{4.3}$$

where $s\big(\nu^\vee\big) = c(\nu)^{-1}$ denotes the Segre class of $\nu^\vee$.

4.4. Derived classes. Consider the setup of Section 2.1. We have the Chern classes $v := c_1(\mathcal{O}_F(D))$, $w_j := c_j\big(\Omega^1_{F/Y}\big)$ (for $j = 1, 2$), $e := c_1(\mathcal{O}_{F'}(E)) \in A^*(F')$, where E is the exceptional divisor on F'. Let $\beta\colon E \to \Delta \cong F$ also denote the restriction of $\beta\colon F' \to F \times_Y F$. Recall that $\beta_i\colon X_i \to F$ and $\beta'_i\colon F_i \to F'$ denote the inclusions. Let $v, w_j, e \in A^*(F')$ also denote their own pullbacks via $\varphi' = p_1\beta$. As in [15, 16], we have

$$c_1(\mathcal{O}_{F_i}(D_i)) = (\beta'_i)^*(v - ie), \qquad c_1\big(\Omega^1_{F'/F}\big) = w_1 + e, \qquad \text{and} \qquad c_2\big(\Omega^1_{F'/F}\big) = w_2 - e^2.$$

We also have the following relations:

$$e^3 + w_1 e^2 + w_2 e = 0, \qquad \beta_\# e = 0, \qquad \text{and} \qquad \beta_\# e^2 = -s_0\big(\nu^\vee\big).$$

The first relation results from the relation $e^2 + w_1 e + w_2 = 0$ on E, cf. [8, Remark 3.2.4, p. 55]; the second and third, from equation (4.3).

Let $\iota\colon \Delta \to F \times_Y F$ denote the embedding, which is regular of codimension 2, since $\pi\colon F \to Y$ is smooth of relative dimension 2. As a class on $F \times_Y F$, we can write $\beta_\# e^2 = -\iota_\#\, s_0\big(\nu^\vee\big)$.

Set $y(a,b,c) := \pi_{\#}\big(v^a w_1^b w_2^c\big) \in A^*(Y)$. (Note, by Proposition 4.2, multiplying these $y(a,b,c)$ is commutative.) The corresponding classes for the map $\pi_i\colon F_i \to X_i$ (defined in Section 2.1) are

$$y_i(a,b,c) := \pi_{i\#}(\beta_i')^*\big((v-ie)^a(w_1+e)^b\big(w_2-e^2\big)^c\big).$$

By (4.2), $\beta_i^*\pi'_{\#} = \pi_{i\#}\beta_i'^*$; hence,

$$y_i(a,b,c) = \beta_i^*\pi'_{\#}\big((v-ie)^a(w_1+e)^b\big(w_2-e^2\big)^c\big).$$

We have $\pi' = p_2\beta$, and

$$\begin{aligned}\beta_{\#}\big((v-ie)^a(w_1+e)^b\big(w_2-e^2\big)^c\big) &= \beta_{\#}\big(v^a w_1^b w_2^c - Q(i;a,b,c)e^2\big)\\ &= p_1^*\big(v^a w_1^b w_2^c\big) + p_1^*Q(i;a,b,c)\cdot \iota_{\#}s_0\big(\nu^\vee\big),\end{aligned}$$

where $Q(i;a,b,c)$ is the (weighted homogeneous, degree $a+b+2c-2$) polynomial in v, w_1, w_2 returned by the function in Algorithm 2.3 in [16, p. 72]. Hence, since $[\pi'] = [p_2\beta] = [\beta]\cdot[p_2]$ by [8, Section 17.4, p. 327]), we get

$$\pi'_{\#}\big((v-ie)^a(w_1+e)^b\big(w_2-e^2\big)^c\big) = p_{2\#}p_1^*\big(v^a w_1^b w_2^c\big) + p_{2\#}\big(p_1^*Q(i;a,b,c)\cdot\iota_{\#}s_0\big(\nu^\vee\big)\big).$$

Since $p_1\iota = p_2\iota$, we have

$$p_1^*Q(i;a,b,c)\cdot\iota_{\#}s_0\big(\nu^\vee\big) = \iota_{\#}\big(\iota^*p_1^*Q(i;a,b,c)\cdot s_0\big(\nu^\vee\big)\big) = p_2^*Q(i;a,b,c)\cdot\iota_{\#}s_0\big(\nu^\vee\big).$$

Since $s_0(\nu^\vee) = c_0(\nu) =: 1_\Delta \in A^*(\Delta)$ is the class that acts as identy on $A_*(\Delta)$ and since $p_2\iota$ is an isomorphism, we get $p_{2\#}\iota_{\#}s_0\big(\nu^\vee\big) = 1_F$. Since $p_{2\#}p_1^* = \pi^*\pi_{\#}$, we obtain

$$\pi'_{\#}\big((v-ie)^a(w_1+e)^b\big(w_2-e^2\big)^c\big) = \pi^*\pi_{\#}v^a w_1^b w_2^c + Q(i;a,b,c).$$

Thus we have

$$y_i(a,b,c) = \beta_i^*(\pi^*y(a,b,c) + Q(i;a,b,c)). \tag{4.4}$$

5 The main theorem

Fix a smooth projective family of surfaces $\pi\colon F \to Y$, and a relative effective divisor D on F/Y. For each $r \geq 1$, we introduce a natural cycle $U(D,r)$ on Y that enumerates the fibers D_y with r nodes. Our first goal is to prove Proposition 5.3, which gives a recursive relation for the class $u(D,r) := [U(D,r)]$ in terms of the classes $u(D_i,r_i)$ of the induced pairs $(F_i/X_i, D_i)$ introduced in Section 2.1. This relation is the key to the proof of our main theorem, Theorem 5.4.

Definition 5.1. Fix $r \geq 1$. Form the direct image on Y of the fundamental cycle $[G(F/Y,D;r)]$, remove the part supported in $Y(\infty)$, and denote the result by $U(D,r)$. In other words, $U(D,r)$ is obtained as follows. For each generic point z of $G(F/Y,D;r)$, let n_z be the length of $\mathcal{O}_z$ over $\mathcal{O}_{q(z)}$ provided this length is finite and $q(z) \notin Y(\infty)$; otherwise, let n_z be 0. Let $\overline{\{q(z)\}}$ be the closure of $\{q(z)\}$.

$$U(D,r) := \sum_z n_z\overline{\{q(z)\}}.$$

In addition, let $U(D,0)$ denote the fundamental cycle of Y, and set $U(D,-r) := 0$.

Finally, set $u(D,r) := [U(D,r)]$. It's a class on Y.

Proposition 5.2. *Fix $r \geq 1$. Assume that the pair $(F/Y, D)$ is r-generic. Then $U(D, r)$ has pure codimension r, and its support is just the closure of $Y(r\mathbf{A}_1)$.*

Proof. This follows from the second part of Lemma 2.6, with $\mathbf{D} = r\mathbf{A}_1$: Let z be a generic point of $G(F/Y, D; r)$. Then $\operatorname{cod}_z(G(F/Y, D; r), H(r)) \leq 3r$ because $\operatorname{Hilb}^{3r}_{D/Y}$ is the zero scheme of a section of a locally free sheaf of rank $3r$ on $\operatorname{Hilb}^{3r}_{F/Y}$ by [1, Proposition 4, p. 5]. ■

Proposition 5.3. *Fix $r \geq 1$. Then the following formula holds:*

$$u(D, r) = \frac{1}{r}\pi_{\#} \sum_{i \geq 2} (-1)^i \beta_{i\#} u(D_i, r_i), \tag{5.1}$$

where $\beta_i \colon X_i \to F$ denote the inclusions.

Proof. Notice that the sum in (5.1) is finite, as $r_i = r - \binom{i+1}{2} + 2$ by (2.3) and as $U(D, s) = 0$ for $s < 0$ by Definition 5.1.

Let's first explain set-theoretically why the formula should hold. Consider a closed point $y \in Y(r)$. The curve D_y has precisely r nodes, and if we blow up one of the nodes, $x \in X_2$ say, the strict transform $(D_2)_x$ has $r-1$ nodes. Hence, above $y \in Y(r)$, we get r points $x \in X_2(r-1)$. But not all $r-1$-nodal curves of D_2/X_2 arise in this way: if $x \in X_3$ is an (ordinary) triple point of $D_{\pi(x)}$, then $(D_2)_x = (D_3)_x + E_x$, hence it has the 3 nodes $(D_3)_x \cap E_x$. Therefore, set-theoretically, $X_2(r-1)$ consists of two parts, one mapping $r:1$ to $Y(r)$, the other mapping $1:1$ to $Y(\mathbf{D}_4 + (r-4)\mathbf{A}_1)$. The second part is equal to the part of $X_3(r-4)$ not contained in X_4. The part contained in X_4 is equal to $X_4(r-8)$ minus the part contained in X_5, and so on.

The fact that this reasoning is valid on the cycle level is precisely what Lemma 3.3 shows. It therefore only remains to show that there is a natural map

$$G(F_2/X_2, D_2; r-1) \smallsetminus G(F_3/X_3, D_3; r-4) \to G(F/Y, D; r)$$

which is $r:1$. If $\mathcal{Z}' \to T$ is a T-point of $G(F_2/X_2, D_2; r-1) \smallsetminus G(F_3/X_3, D_3; r-4)$ (i.e., a family of $r-1$ double points in $D_2 \subset F_2$ over T, none of which lie on E), we send it to the image $\varphi'(\mathcal{Z}' \cup 2E)$ in F to get a family over T of r double points in D. This map induces an $r:1$ map from the components of the cycle $U(D_2, r-1)$ that are supported on $X_2 \smallsetminus X_3$ to $U(D, r)$. ■

Notice that the case $r = 8$ of (5.1) looks different from (4.6) on p. 230 of [15]. Indeed, in [15], we used $u(D_2, 7)$ to denote what we denote by $\frac{1}{7!}P_7(a_\bullet(D_2)) \cap [X_2]$ here. However, the mathematics is consistent.

Our main result is Theorem 5.4, the first part of which was proved in [16]. We now prove the last part of the theorem, namely the part concerning the expression for $u(D, r) = [U(D, r)]$, where $U(D, r)$ is the cycle introduced in Definition 5.1.

Recall from Section 2 that for a given pair $(F/Y, D)$, the subscheme $X_i \subset F$ denotes the scheme of zeros of the natural section σ_i of $\mathcal{P}^{i-1}_{F/Y}(D)$. After making some simplifications, we prove the theorem when $X_4 = \varnothing$. This proof is easy, and it yields the case $r \leq 7$. We then consider the case $r = 8$, which is more difficult due to the presence of nonreduced fibers in codimension $r_2 = 7$ in the family of curves of the induced pair $(F_2/X_2, D_2)$.

Theorem 5.4 (Main)**.** *Let $\pi \colon F \to Y$ be a smooth projective family of surfaces, and D a relative effective divisor. Assume Y is Cohen–Macaulay and equidimensional. Fix an integer $r \geq 0$, and assume*

(i) if $Y(\infty) \neq \varnothing$, we have $\operatorname{cod} Y(\infty) \geq r+1$,

(ii) the pair $(F/Y, D)$ is r-generic.

Then either $Y(r\mathbf{A}_1)$ is empty, or it has pure codimension r; in either case, its closure $\overline{Y(r\mathbf{A}_1)}$ is the support of a natural nonnegative cycle $U(D,r)$.

Let $b_s(D)$ be the polynomial in v, w_1, w_2 output by Algorithm 2.3 *in* [16], *set $a_s(D) := \pi_{\#}b_s(D)$, and let P_r be the rth Bell polynomial. Assume $r \leq 8$, and if $r = 8$, then $(F/Y, D)$ is strongly 8-generic. Then the rational equivalence class $u(D,r) := [U(D,r)]$ is given by the formula*

$$u(D,r) = \frac{1}{r!}P_r(a_1(D), \ldots, a_r(D)) \cap [Y].$$

Proof. First of all, we may assume $Y(\infty) = \varnothing$. Indeed, $\operatorname{cod} Y(\infty) \geq r+1$ by hypothesis. Hence we may replace Y by $Y \smallsetminus Y(\infty)$, and thus assume that all the fibres of $\pi|_D\colon D \to Y$ are reduced.

Second, $\operatorname{cod} X_i = \binom{i+1}{2}$ for $i = 2, 3, 4$ by Corollary 2.9. Therefore, X_i is a local complete intersection in F, and F is smooth over the Cohen–Macaulay scheme Y, hence is Cohen–Macaulay, and so X_i is too. Since Y is equidimensional, so is F, and hence so is X_i.

By [16, Lemma 2.4, p. 73] there are at most finitely many $\mathbf{D}$ such that $Y(\mathbf{D})$ is nonempty; hence, we may remove all $Y(\mathbf{D})$ with $\operatorname{cod} Y(\mathbf{D}) \geq r+1$. If $x \in X_i$ is a closed point, $i \geq 5$, then $D_{\pi(x)}$ contains a point of multiplicity at least i; hence, the minimal Enriques diagram $\mathbf{D}$ of $D_{\pi(x)}$ satisfies $\operatorname{cod} \mathbf{D} \geq \binom{i+1}{2} - 2 \geq 13$ by the formula for $\operatorname{cod} \mathbf{D}$ in [15, p. 217]. Therefore, $\operatorname{cod} Y(\mathbf{D}) \geq 13 > r$, since $r \leq 8$. Hence $X_i = \varnothing$ for $i \geq 5$. If $x \in X_4$, then $x \in D_{\pi(x)}$ has multiplicity ≥ 4; hence, $\operatorname{cod} \mathbf{D} \geq 8$. If $\operatorname{cod} \mathbf{D} \geq 9$, then $\operatorname{cod} Y(\mathbf{D}) \geq 9$; so $Y(\mathbf{D}) = \varnothing$. Hence we have $\operatorname{cod} \mathbf{D} = 8$. But the only diagram with a root of multiplicity 4 and codimension 8 is the diagram $X_{1,0}$ corresponding to an ordinary quadruple point; see [15, Figures 2–6, p. 218].

The recursive formula of Proposition 5.3 applies. It gives, for $r \leq 8$,

$$ru(D,r) = \pi_{\#}\sum_{i=2}^{4}(-1)^i \beta_{i\#}u(D_i, r_i),$$

where $r_2 = r-1$, $r_3 = r-4$ and $r_4 = r-8$.

Proposition 5.5. *The theorem holds if $X_4 = \varnothing$ and $Y(\infty) = \varnothing$.*

Proof. The proof is by induction on r. For $r = 1$, we have

$$\begin{aligned} u(D,1) &= \pi_{\#}\beta_{2\#}u(D_2, 0) = \pi_{\#}[X_2] = \pi_{\#}x_2 \cap [Y] \\ &= \pi_{\#}b_1(D) \cap [Y] = a_1(D) \cap [Y] = P_1(a_1(D)) \cap [Y]. \end{aligned}$$

Assume next that $r \geq 2$ and that the theorem holds for all families verifying the hypotheses of the theorem with r replaced by $r' < r$. In particular, the statement then holds for the induced pairs $(F_i/X_i, D_i)$, for $i = 2, 3$, defined in Section 2.1. Indeed, $X_i(\infty) = \varnothing$, and $(F_i/X_i, D_i)$ is r_i-generic by Proposition 2.8; that is, (ii) of the theorem holds with r replaced by r_i.

To simplify the notation, let us write

$$P_m(z_\bullet) := P_m(z_1, \ldots, z_m), \tag{5.2}$$

where P_m is the mth Bell polynomial and $z_1, \ldots, z_m$ are variables. Then we get

$$r!u(D,r) = \pi_{\#}\big(\beta_{2\#}P_{r-1}(a_\bullet(D_2)) - (r-1)!/(r-4)!\beta_{3\#}P_{r-4}(a_\bullet(D_3))\big) \cap [Y].$$

By definition, $a_s(D_i) = \pi_{i\#}b_s(D_i)$. By applying (4.4) to the polynomials $b_s(D)$ (cf. [16, Algorithm 2.3]),

$$\beta_{i\#}P_m(a_\bullet(D_i)) = P_m(\pi^* a_\bullet(D) + Q(i, b_\bullet(D))) \cdot x_i(D).$$

By the binomial property of the Bell polynomials [3, equation (4.9), p. 265], we have

$$P_m(\pi^*a_\bullet(D)+Q(i,b_\bullet(D)))=\sum_{k=0}^{m}\binom{m}{k}P_{m-k}(\pi^*a_\bullet(D))P_k(Q(i,b_\bullet(D))).$$

Plugging this in and using the definition of $b_s(D)$ and $a_s(D)$, we get

$$r!u(D,r)=\sum_{k=0}^{r-1}\binom{r-1}{k}P_{r-1-k}(a_\bullet(D))a_{k+1}(D)\cap[Y]=P_r(a_\bullet(D))\cap[Y],$$

where the last equality follows from the recursive property of the Bell polynomials [3, equation (4.2), p. 263]. ■

When $r\le 7$, we can remove all $Y(\mathbf{D})$ with $\operatorname{cod}\mathbf{D}\ge 8$. If $\mathbf{D}$ contains a root of multiplicity ≥ 4, then $\operatorname{cod}\mathbf{D}\ge\binom{4+1}{2}-2=8$, hence we may assume $X_4=\varnothing$. This proves the theorem for $r\le 7$.

Assume $r=8$. By Proposition 5.3, we have

$$8u(D,8)=\pi_\#\big(\beta_{2\#}u(D_2,7)-\beta_{3\#}u(D_3,4)+\beta_{4\#}u(D_4,0)\big).$$

The induced pairs $(F_i/X_i,D_i)$, $i=3,4$ satisfy the conditions of the theorem, with r replaced by r_i, hence, by the case $r\le 7$ of the theorem:

$$u(D_3,4)=\frac{1}{4!}P_4(a_\bullet(D_3))\cap[X_3]\qquad\text{and}\qquad u(D_4,0)=[X_4].$$

Note that, since F is Cohen–Macaulay, $[X_i]=x_i\cap[F]$ with x_i as in Algorithm 2.3 in [16]. The induced pair $(F_2/X_2,D_2)$ does not satisfy the conditions for r replaced by r_2. Indeed, note that $D_2|_{F_4}=(D-2E)|_{F_4}=(D-4E+2E)|_{F_4}=D_4+2E|_{F_4}$ and that D_4 is relative effective on F_4/X_4 by Lemma 2.2; hence, D_2 has nonreduced fibers above X_4. So $X_2(\infty)=X_4$, and hence has codimension $r_2=7$ in X_2.

However, from what we have seen above, if we restrict the family $F_2\to X_2$ to $X_2\smallsetminus X_4$, then $\frac{1}{7!}P_7(a_\bullet(D_2))\cap[X_2\smallsetminus X_4]$ is the class of the 7-nodal curves of that family. So the difference

$$\frac{1}{7!}P_7(a_\bullet(D_2))\cap[X_2]-u(D_2,7)\tag{5.3}$$

is the *correction term* we are looking for. It is the class of a cycle of codimension 7, supported on the codimension 7 subscheme X_4 of X_2. As Theorem 7.5 shows, (5.3) is equal to $C[X_4]$, where the constant C is an integer, which is independent of the given pair $(F/Y,D)$. Hence it suffices to compute C in any particular case; for example, in [16, Example 3.8, p. 80], we worked out the case of 8-nodal quintic plane curves, and found $C=3280$. Thus Theorem 5.4 is proved. ■

Remark 5.6. Assume $(F/Y,D)$ is the direct sum of two pairs $(F'/Y,D')$ and $(F''/Y,D'')$ over the same base. Then the r-nodal curves of $D\to Y$ consists of the unions of the $(r-i)$-nodal curves of $D'\to Y$ and the i-nodal curves of $D''\to Y$ for $i=0,\dots,r$. Hence, the existence of a universal polynomial for r-nodal curves implies that the generating series for $(F/Y,D)$ is equal to the product of the generating series for $(F'/Y,D')$ and $(F''/Y,D'')$. This fact was observed by Göttsche in the case of a trivial family [9, p. 525], and by Laarakker in the general case [19, Section 5.1, p. 4935]. In the case of a trivial family, Göttsche observed that this multiplicativity implies that the universal polynomials are Bell polynomials. However, as observed by Laarakker, this conclusion does not follow in the case of a nontrivial family.

Let $a_j(D)$, $a_j(D')$, $a_j(D'')$ be the classes, introduced in Theorem 5.4, for the three pairs. Clearly, $a_j(D) = a_j(D') + a_j(D'')$. So (for $r \leq 8$), in the notation of (5.2),

$$u(D,r) = \frac{1}{r!}P_r(a_\bullet(D)) \cap [Y] = \frac{1}{r!}P_r(a_\bullet(D') + a_\bullet(D'')) \cap [Y].$$

By the binomial property of the Bell polynomials, the right-hand side is equal to

$$\sum_{i=0}^{r} \frac{1}{(r-i)!}P_{r-i}(a_\bullet(D'))\frac{1}{i!}P_i(a_\bullet(D'')) \cap [Y].$$

Hence the Bell polynomial shape of the universal polynomials is in agreement with the multiplicative property of the generating series of $(F/Y, D)$.

6 An expression for the correction term

We now find an expression for the correction term (5.3). First, in Section 6.1 we define some useful schemes. Then in Lemma 6.2, we give an expression for $u(D_2, 7)$, obtained via repeated use of the recursion formula of Proposition 5.3. Then in Section 6.3, we introduce classes $e(W_i)$ on X_2 of cycles on X_4. Finally, in Proposition 6.4, we express (5.3) as a linear combination of the $e(W_i)$.

6.1. Some important schemes. Let $\big(F_2^{(1)}/X_2^{(0)}, D_2^{(1)}\big) := (F_2/X_2, D_2)$ be the induced pair of $(F/Y, D)$. Define recursively $\big(F_2^{(j+1)}/X_2^{(j)}, D_2^{(j+1)}\big)$ as the induced pair of $\big(F_2^{(j)}/X_2^{(j-1)}, D_2^{(j)}\big)$. Let $\big(F_3^{(j+1)}/X_3^{(j)}, D_3^{(j+1)}\big)$ be the induced pair of $\big(F_2^{(j)}/X_2^{(j-1)}, D_2^{(j)}\big)$. Let $X_2\big(D_3^{(j+1)}\big) \subset F_3^{(j+1)}$ be the zero scheme of the section of $\mathcal{P}^1_{\pi_3^{(j+1)}}\big(D_3^{(j+1)}\big)$ induced by that defining the divisor $D_3^{(j+1)}$.

Let $D^{(j+1)} - E^{(j)}$ denote the restriction of the divisor $D^{(j+1)} - {\varphi^{(j+1)}}^{-1}E^{(j)}$ to $F^{(j+1)}|_{X_2(D_3^{(j)})}$. This divisor is effective; so it induces a section of the restriction of $\mathcal{P}^1_{\pi^{(j+1)}}\big(D^{(j+1)} - {\varphi^{(j+1)}}^{-1}E^{(j)}\big)$. Let $X_2\big(D^{(j+1)} - E^{(j)}\big)$ denote its scheme of zeros. Let $D^{(j+2)} - E^{(j)}$ denote the restriction of $D^{(j+2)} - {\varphi^{(j+2)}}^{-1}{\varphi^{(j+1)}}^{-1}E^{(j)}$ to $F^{(j+2)}|_{X_2(D^{(j+1)}-E^{(j)})}$, and $X_2\big(D^{(j+2)} - E^{(j)}\big)$ the scheme of zeros of the induced section of the restriction of $\mathcal{P}^1_{\pi^{(j+2)}}\big(D^{(j+2)} - {\varphi^{(j+2)}}^{-1}{\varphi^{(j+1)}}^{-1}E^{(j)}\big)$.

Form these five equidimensional schemes of dimension $\dim X_2 - 7$, or $\dim X_4$:

$$\begin{aligned}
\mathfrak{X}_2^{(7)} &:= \overline{X_2^{(7)} \smallsetminus X_2^{(7)}|_{X_4}},\\
\mathfrak{X}_3^{(4)} &:= \overline{X_3^{(4)} \smallsetminus X_3^{(4)}|_{X_4}},\\
\mathfrak{X}_2\big(D_3^{(4)}\big) &:= \overline{X_2\big(D_3^{(4)}\big) \smallsetminus X_2\big(D_3^{(4)}\big)|_{X_4}},\\
\mathfrak{X}_2\big(D^{(4)} - E^{(3)}\big) &:= \overline{X_2\big(D^{(4)} - E^{(3)}\big) \smallsetminus X_2\big(D^{(4)} - E^{(3)}\big)|_{X_4}},\\
\mathfrak{X}_2\big(D^{(4)} - E^{(2)}\big) &:= \overline{X_2\big(D^{(4)} - E^{(2)}\big) \smallsetminus X_2\big(D^{(4)} - E^{(2)}\big)|_{X_4}}.
\end{aligned}$$

For $j = 2, \ldots, 7$, consider the composed map $\pi^{(j)} \circ \cdots \circ \pi^{(1)} \colon F^{(j)} \to F$, and let $\pi_j \colon F^{(j)}|_{X_2} \to X_2$ denote its restriction.

Lemma 6.2. *Then*

$$\begin{aligned}
u(D_2, 7) = \frac{1}{7!}\pi_{7\#}\big[\mathfrak{X}_2^{(7)}\big] - \frac{3!}{7!}\pi_{4\#}\big[\mathfrak{X}_3^{(4)}\big] - \frac{4!}{7!}\pi_{4\#}\big[\mathfrak{X}_2\big(D_3^{(4)}\big)\big] - \frac{5!}{7!2!}\pi_{4\#}\big[\mathfrak{X}_2\big(D^{(4)} - E^{(3)}\big)\big]\\
- \frac{6!}{7!3!}\pi_{4\#}\big[\mathfrak{X}_2\big(D^{(4)} - E^{(2)}\big)\big].
\end{aligned}$$

Proof. As $(F/Y, D)$ is 8-generic, the successive induced pairs are j-generic (for appropriate j) by Proposition 2.8. Thus the lemma follows from repeated use of the recursion formula of Proposition 5.3. ■

6.3. The classes $e(W_i)$. Next we find an expression for each term appearing in the formula for $u(D_2, 7)$ in Lemma 6.2. We just consider $\mathfrak{X}_2^{(7)}$, since the other schemes can be studied in a similar way and their classes have similar expressions.

By definition, $X_2^{(j)}$ is the scheme of zeros of the section $\sigma_2^{(j)}$ of $\mathcal{P}^1_{\pi_2^{(j)}}\big(D_2^{(j)}\big)$ induced by the section $\sigma^{(j)}$ defining $D_2^{(j)}$. For $j = 1, \ldots, 6$, set

$$\mathfrak{X}_2^{(j)} := \overline{X_2^{(j)} \smallsetminus X_2^{(j)}|_{X_4}}, \qquad \mathfrak{F}_2^{(j+1)} := F_2^{(j+1)}|_{\mathfrak{X}_2^{(j)}}, \tag{6.1}$$

$$W_1^{(j)} := \big(X_2^{(j)}|_{\mathfrak{X}_2^{(j-1)}}\big)|_{X_4}. \tag{6.2}$$

Then $X_2^{(j)}|_{\mathfrak{X}_2^{(j-1)}} = \mathfrak{X}_2^{(j)} \cup W_1^{(j)}$. Notice the $\mathfrak{F}_2^{(j)}$ are equidimensional of dimension $\dim X_2 - j + 3$ and that $\operatorname{cod}\big(\mathfrak{X}_2^{(j)}, \mathfrak{F}_2^{(j)}\big) = 3$. Thus $\dim \mathfrak{X}_2^{(j)} = \dim X_2 - j$.

To determine the dimensions of the "excess schemes" $W_1^{(j)}$, consider the fibers of $W_1^{(j)} \to W_1^{(j-1)} \cap \mathfrak{X}_2^{(j-1)}$. Starting with $x \in X_4$, we have $\big(D_2^{(1)}\big)_x = \Gamma_x \cup 2E_x^{(1)}$, where Γ_x is the strict transform of $D_{\pi(x)}$ under the blowup of $F_{\pi(x)}$ at x. Hence, a local calculation yields $\big(W_1^{(1)}\big)_x$, which is $\big(X_2^{(1)}\big)_x$, is equal to $E_x^{(1)}$ with four embedded points at the intersections of Γ_x with $E_x^{(1)}$. Thus $\dim W_1^{(1)} = \dim X_4 + 1$.

Next take $z \in \big(W_1^{(1)}\big)_x$, but $z \notin \Gamma_x$. Then the fiber $\big(W_1^{(2)}\big)_z$ is the strict transform of $E_x^{(1)}$ plus four embedded points. If $z \in \Gamma_x \cap E_x^{(1)}$, then $\big(W_1^{(2)}\big)_z$ has an additional point, namely, the intersection of the strict transform of Γ_x with $E_z^{(2)}$. Hence $\dim W_1^{(2)} = \dim X_4 + 2$. Continuing, we get

$$\dim W_1^{(3)} = \dim X_4 + 3 = \dim X_2 - 4 = \dim X_2^{(3)} - 1.$$

Thus $W_1^{(j)} \subset X_2^{(j)}$ and $\mathfrak{X}_2^{(j)} = X_2^{(j)}$ for $j \leq 3$.

For $j = 4$ we get $\dim W_1^{(4)} = \dim X_4 + 4 = \dim X_2 - 3 = \dim X_2^{(4)} + 1$. Then $\dim W_1^{(4)} \cap \mathfrak{X}_2^{(4)} \leq \dim X_2^{(4)} - 1$. Hence $\dim W_1^{(5)} \leq \dim X_2^{(4)} - 1 + 1 = \dim X_2^{(5)} + 1$. Continuing, we get $\dim W_1^{(j)} \leq \dim X_2^{(j)} + 1$ for $j \geq 4$.

To simplify notation, set $\mathcal{P}^{(j)} := \mathcal{P}^1_{\pi^{(j)}}\big(D^{(j)}\big)$. Consider $\mathcal{P}^{(7)}$ restricted to $\mathfrak{F}_2^{(7)}$. The scheme of zeros of its section $\sigma_2^{(7)}$ is equal to $\mathfrak{X}_2^{(7)} \cup W_1^{(7)}$. Blow up $\mathfrak{F}_2^{(7)}$ along $W_1^{(7)}$ and apply the residual formula for top Chern classes [8, Example 14.1.4, p. 245]. After pushing down to $\mathfrak{F}_2^{(7)}$, we find

$$\big[\mathfrak{X}_2^{(7)}\big] = c_3\big(\mathcal{P}^{(7)}\big) \cap \big[\mathfrak{F}_2^{(7)}\big] + \big\{c\big(\mathcal{P}^{(7)}\big) \cap s\big(W_1^{(7)}, \mathfrak{F}_2^{(7)}\big)\big\}_{\dim X_4}. \tag{6.3}$$

Here is why (6.3) holds.

Let σ' denote the induced section of $\mathcal{P}^{(7)}$ twisted by the ideal sheaf of the exceptional divisor on the blowup of $\mathfrak{F}_2^{(7)}$. Let $\mathbb{Z}(\sigma')$ denote the localized top Chern class of the pullback of $\mathcal{P}^{(7)}$ with respect to σ'. The zero scheme $Z(\sigma')$ is equal to the strict transform of $\mathfrak{X}_2^{(7)}$, hence has codimension 3 in the blowup of $\mathfrak{F}_2^{(7)}$. It follows from [8, Proposition 14.1(b), p. 244] that $\mathbb{Z}(\sigma')$ is the class of a positive cycle with support $Z(\sigma')$. Since the blowup of $\mathfrak{F}_2^{(7)}$ need not be Cohen–Macaulay, we cannot immediately conclude that $\mathbb{Z}(\sigma') = [Z(\sigma')]$. However, since $\mathfrak{F}_2^{(7)} \smallsetminus \mathfrak{F}_2^{(7)}|_{X_4}$

is Cohen–Macaulay, the restrictions of $\mathbb{Z}(\sigma')$ and $[Z(\sigma')]$ agree above $\mathfrak{X}_2^{(7)} \smallsetminus \mathfrak{X}_2^{(7)}|_{X_4}$; hence they are equal.

Since $\big[\mathfrak{F}_2^{(7)}\big] = \big[F_2^{(7)}|_{\mathfrak{X}_2^{(6)}}\big] = \pi^{(7)*}\big[\mathfrak{X}_2^{(6)}\big]$, the pushdown of the first term under $\pi^{(7)}$ gives $\pi_{\#}^{(7)} c_3\big(\mathcal{P}^{(7)}\big) \cap \big[\mathfrak{X}_2^{(6)}\big]$ by the projection formula. We then replace $\big[\mathfrak{X}_2^{(6)}\big]$ by the analogue of equation (6.3) and push down the resulting terms by $\pi^{(6)}$.

Continuing this way, we get a formula for $\pi_{7\#}\big[\mathfrak{X}_2^{(7)}\big]$. To simplify notation, set

$$\begin{aligned} d_j(W_1) &:= \big\{c\big(\mathcal{P}^{(7-j)}\big) \cap s\big(W_1^{(7-j)}, \mathfrak{F}_2^{(7-j)}\big)\big\}_{\dim X_4+j} \qquad \text{for} \quad j = 0, \dots, 3, \\ e(W_1) &:= \pi_{\#}^{(1)} \cdots \pi_{\#}^{(7)} d_0(W_1) + \pi_{\#}^{(1)} \cdots \pi_{\#}^{(6)}\big(\pi_{\#}^{(7)} c_3\big(\mathcal{P}^{(7)}\big) d_1(W_1)\big) \\ &\quad + \pi_{\#}^{(1)} \cdots \pi_{\#}^{(5)}\big(\pi_{\#}^{(6)}\big(\pi_{\#}^{(7)} c_3\big(\mathcal{P}^{(7)}\big) c_3\big(\mathcal{P}^{(6)}\big) d_2(W_1)\big)\big) \\ &\quad + \pi_{\#}^{(1)} \cdots \pi_{\#}^{(4)}\big(\pi_{\#}^{(5)}\big(\pi_{\#}^{(6)}\big(\pi_{\#}^{(7)} c_3\big(\mathcal{P}^{(7)}\big) c_3\big(\mathcal{P}^{(6)}\big) c_3\big(\mathcal{P}^{(5)}\big)\big) d_3(W_1)\big)\big)\big). \end{aligned}$$

Note that the $d_j(W_1)$ are classes in $A^*\big(W_1^{(7-j)}\big)$, and that restricting the map π_{7-j} gives a proper map $W_1^{(7-j)} \to X_4$.

Then the resulting formula for $\pi_{7\#}[\mathfrak{X}_2^{(7)}]$ is the following:

$$\pi_{7\#}\big[\mathfrak{X}_2^{(7)}\big] = \pi_{\#}^{(1)}\big(\pi_{\#}^{(2)}\big(\cdots\big(\pi_{\#}^{(7)} c_3\big(\mathcal{P}^{(7)}\big)\cdots\big) c_3\big(\mathcal{P}^{(2)}\big)\big) c_3\big(\mathcal{P}^{(1)}\big)\big) \cap [X_2] + e(W_1).$$

Similarly, we obtain formulas for the classes $\pi_{4\#}\big[\mathfrak{X}_3^{(4)}\big]$ and $\pi_{4\#}\big[\mathfrak{X}_2\big(D_3^{(4)}\big)\big]$ and $\pi_{4\#}\big[\mathfrak{X}_2\big(D^{(4)} - E^{(3)}\big)\big]$ and $\pi_{4\#}\big[\mathfrak{X}_2\big(D^{(4)} - E^{(2)}\big)\big]$. For $i = 2, \dots, 5$, define the classes $e(W_i)$ on X_2 correspondingly.

Proposition 6.4. *The correction term* (5.3) *is equal to*

$$\frac{1}{7!} P_7(a_\bullet(D_2)) \cap [X_2] - u(D_2, 7) = \frac{1}{7!} e(W_1) - \frac{3!}{7!} e(W_2) - \frac{4!}{7!} e(W_3) - \frac{5!}{7!2!} e(W_4) - \frac{6!}{7!3!} e(W_5).$$

Proof. Recall that the classes $a_s(D_2)$ on X_2 are obtained by pushing down the classes $b_s(D_2)$ on F_2 obtained by applying Algorithm 2.3 of [16, p. 72] to the pair $(F_2/X_2, D_2)$. In the case that $X_4 = \varnothing$, the Algorithm would have produced the formula $u(D_2, 7) = \frac{1}{7!} P_7(a_\bullet(D_2)) \cap [X_2]$. Removing the classes $e(W_i)$, we get

$$\begin{aligned} \frac{1}{7!} P_7(a_\bullet(D_2)) \cap [X_2] = \frac{1}{7!}\big(\pi_{7\#}\big[\mathfrak{X}_2^{(7)}\big] - e(W_1)\big) &- \frac{3!}{7!}\big(\pi_{4\#}\big[\mathfrak{X}_3^{(4)}\big] - e(W_2)\big) \\ &- \frac{4!}{7!}\big(\pi_{4\#}\big[\mathfrak{X}_2\big(D_3^{(4)}\big)\big] - e(W_3)\big) \\ &- \frac{5!}{7!2!}\big(\pi_{4\#}\big[\mathfrak{X}_2\big(D^{(4)} - E^{(3)}\big)\big] - e(W_4)\big) \\ &- \frac{6!}{7!3!}\big(\pi_{4\#}\big[\mathfrak{X}_2\big(D^{(4)} - E^{(2)}\big)\big] - e(W_5)\big). \end{aligned}$$

Lemma 6.2 now yields the asserted formula. ∎

7 Independence of the correction term

In this section, we prove Theorem 7.5, which asserts that the correction term (5.3) is equal to $C[X_4]$, where C is independent of the strongly 8-generic pair $(F/Y, D)$ with $Y(\infty) = \varnothing$. We work locally analytically on F at a general closed point x in X_4. Section 7.1 describes the local setup. Lemma 7.2 asserts that locally we have the properness we need to pushdown classes. Lemma 7.3 asserts that the key classes $e(W_i)$ pull back to their local counterparts $e\big(\widehat{W}_i\big)$.

Lemma 7.4 asserts that the coefficient in $e\big(\widehat{W}_i\big)$ of $\big[\widehat{X}_4\big]$ depends only on the analytic type of the ordinary quadruple point $x \in D_{\pi(x)}$; namely, on the cross ratio of the four tangents at x. Its proof requires $(F/Y, D)$ to be strongly 8-generic. Finally, we prove Theorem 7.5 by exhibiting a pair $(F/Y, D)$, where X_4 is irreducible and where any given value of the cross ratio appears at some $x \in X_4$.

7.1. The local setup. Fix an 8-generic pair $(F/Y, D)$ with $Y(\infty) = \varnothing$, and a general closed point $x \in X_4$. By *general*, we mean that x lies on a single irreducible component Z of X_4 and that x is an ordinary quadruple point of $D_{\pi(x)}$. Let us arrange for every $x \in X_4$ to be general as follows. First, if two components Z' and Z'' of X_4 meet, then $\dim(Z' \cap Z'') < \dim X_4$. So $\operatorname{cod}\pi(Z' \cap Z'') > \operatorname{cod}\pi(X_4)$. But $\operatorname{cod}(\pi(X_4), Y) = 8$ by (2.11) as $(F/Y, D)$ is 8-generic. Hence we may discard $Z' \cap Z''$. Second we may discard the locus of $y \in Y$, where D_y has a singularity x worse than an ordinary quadruple point, again because $(F/Y, D)$ is 8-generic.

Set $\widetilde{F} := \operatorname{Spec}\mathcal{O}_{F,x}$ and $\widetilde{Y} := \operatorname{Spec}\mathcal{O}_{Y,\pi(x)}$ and $\widetilde{D} := \operatorname{Spec}\mathcal{O}_{D,x}$. Denote the induced pair of $\big(\widetilde{F}/\widetilde{Y}, \widetilde{D}\big)$ by $\big(\widetilde{F}_2/\widetilde{X}_2, \widetilde{D}_2\big)$. The bundles of relative principal parts are compatible not only with the base change $\widetilde{Y} \to Y$, but also with the maps $\widetilde{F} \to F$ and $\widetilde{F}^{(j)} \to F^{(j)}$; cf. [11, Proposition 16.4.14, p. 22]. So although the $\widetilde{X}_i$ for $i \geq 2$ are defined in terms of $\big(\widetilde{F}/\widetilde{Y}, \widetilde{D}\big)$, we have $\widetilde{X}_i = \operatorname{Spec}\widetilde{\mathcal{O}}_{X_i,x}$. Similarly, the schemes constructed in Section 6 for $(F_2/X_2, D_2)$ induce the corresponding schemes for $\big(\widetilde{F}_2/\widetilde{X}_2, \widetilde{D}_2\big)$. Denote by $\widetilde{W}_i^{(j)}$ the scheme corresponding to $W_i^{(j)}$; see (6.2).

Next consider the completions of the local rings, giving us a pair $\big(\widehat{F}/\widehat{Y}, \widehat{D}\big)$. Replacing the principal parts bundles by their completions, cf. [6, Example 16.14, p. 416], construct the $\widehat{X}_i$, the induced pair $\big(\widehat{F}_2/\widehat{X}_2, \widehat{D}_2\big)$, and the corresponding schemes of Section 6. Since the complete principal parts bundles are pullbacks, $\widehat{X}_i = \operatorname{Spec}\widehat{\mathcal{O}}_{X_i,x}$, and all the schemes of Section 6 for $(F_2/X_2, D_2)$ pull back to the corresponding schemes for $\big(\widehat{F}_2/\widehat{X}_2, \widehat{D}_2\big)$. Denote by $\widehat{W}_i^{(j)}$ the scheme corresponding to $\widetilde{W}_i^{(j)}$, so to $W_i^{(j)}$.

The classes $e(W_i)$ of Section 6.3 are sums of pushdowns of classes on the $W_i^{(j)}$. By Lemma 7.2 below, the $\widehat{W}_i^{(j)}$ are proper over $\widehat{X}_2$; hence, we may form the corresponding classes $e(\widehat{W}_i)$ for the pair $\big(\widehat{F}_2/\widehat{X}_2, \widehat{D}_2\big)$. Denote them by $e(\widehat{W}_i)$.

Let $\epsilon\colon \widehat{X}_2 \to X_2$ denote the composition of the flat maps $\widehat{X}_2 \to \widetilde{X}_2$ and $\widetilde{X}_2 \to X_2$. Then $\big[\widehat{X}_4\big] = \epsilon^*[X_4]$, and Lemma 7.3 asserts $e(\widehat{W}_i) = \epsilon^* e(W_i)$.

Each $e(W_i)$ is the class of a cycle U_i on X_4 of dimension $\dim X_4$. Say that the component Z of X_4 containing x appears in U_i with coefficient C_i' and in the fundamental cycle $|X_4|$ with coefficient C_i''. Set $C_i := C_i'/C_i''$. Then the cycles U_i and $C_i|X_4|$ become equal after restriction to a neighborhood of Z, so the classes $e(W_i)$ and $C_i[X_4]$ do too. Thus

$$e\big(\widehat{W}_i\big) = C_i\big[\widehat{X}_4\big]; \tag{7.1}$$

furthermore, C_i is independent of the choice of x in Z.

Lemma 7.2. *The schemes $\widehat{W}_i^{(j)}$ are proper over $\widehat{X}_2$.*

Proof. Let us first show that the schemes $\widetilde{W}_i^{(j)}$ and $W_i^{(j)}|_{\widetilde{X}_2}$ have the same support. It suffices to consider only the $W_1^{(j)}$ as the other cases are similar.

Let $E_j \subset F^{(j)}$ be the union of the strict transforms of the exceptional divisors $E^{(1)}, \ldots, E^{(j)}$; see [17, Definition 3.5, p. 423]. It follows from the description in Section 6 that $W_1^{(j)}$ is supported in E_j. The fibers of the exceptional divisor of $\widetilde{F}_2$ are the same as the corresponding fibers of the exceptional divisor of F_2. Hence, above $x \in \widetilde{X}_2$, the fiber of $E^{(j)}$ lies in $\widetilde{F}_2^{(j)}$. Thus $W_1^{(j)}|_{\widetilde{X}_2}$ has support in $\widetilde{F}_2^{(j)}$, and it is the same as the support of $\widetilde{W}_i^{(j)}$; moreover, this support is proper over $\widetilde{X}_2$.

For $w \in W_i^{(j)}|_{\widetilde{X}_2}$, the map $\mathcal{O}_{F_2^{(j)},w} \twoheadrightarrow \mathcal{O}_{W^{(j)},w}$ pulls back to $\mathcal{O}_{\widetilde{F}_2^{(j)},w} \twoheadrightarrow \mathcal{O}_{\widetilde{W}^{(j)},w}$. But clearly $\mathcal{O}_{F_2^{(j)},w} = \mathcal{O}_{\widetilde{F}_2^{(j)},w}$; hence $\mathcal{O}_{W_i^{(j)},w} = \mathcal{O}_{\widetilde{W}^{(j)}1,w}$. Hence $\widetilde{W}_i^{(j)} = W_i^{(j)}|_{\widetilde{X}_2}$. Therefore, $\widetilde{W}_i^{(j)}$ is proper over $\widetilde{X}_2$. Thus the pullback $\widehat{W}_i^{(j)}$ is proper over $\widehat{X}_2$. ■

Lemma 7.3. *We have* $e(\widehat{W}_i) = \epsilon^* e(W_i)$.

Proof. It suffices to check that each summand of $e(W_i)$ pulls back to the corresponding summand of $e(\widehat{W}_i)$. Here we only consider the first summand of $e(W_1)$, since the other cases are similar.

In (6.1), we defined the schemes $\overline{F}_2^{(j)}$. Denote by $\epsilon_2^{(j)} \colon \widehat{\overline{F}}_2^{(j)} \to \overline{F}_2^{(j)}$ the map induced by the map ϵ defined in Section 7.1. Notice that, as $\epsilon_2^{(7)}$ is flat,

$$s\big(\widehat{W}_1^{(7)}, \widehat{\overline{F}}_2^{(7)}\big) = s\big(\epsilon_2^{(7)-1} W_1^{(7)}, \epsilon_2^{(7)-1}\overline{F}_2^{(7)}\big) = \epsilon_2^{(7)*} s\big(W_1^{(7)}, \overline{F}_2^{(7)}\big);$$

cf. [8, Proposition 4.2(b), p. 74]. But $\widehat{\pi}_{\#}^{(j)} \epsilon_2^{(j)*} = \epsilon_2^{(j-1)*} \pi_{\#}^{(j)}$ by (4.2). Thus

$$\begin{aligned} e\big(\widehat{W}_i\big) &= \widehat{\pi}_{\#}^{(1)} \cdots \widehat{\pi}_{\#}^{(7)} \big\{ c\big(\widehat{\mathcal{P}}^{(7)}\big) \cap s\big(\widehat{W}_1^{(7)}, \widehat{\overline{F}}_2^{(7)}\big)\big\}_{\dim X_4} \\ &= \widehat{\pi}_{\#}^{(1)} \cdots \widehat{\pi}_{\#}^{(7)} \big\{ \epsilon_2^{(7)*} c\big(\mathcal{P}^{(7)}\big) \cap \epsilon_2^{(7)*} s\big(W_1^{(7)}, \overline{F}_2^{(7)}\big)\big\}_{\dim X_4} \\ &= \widehat{\pi}_{\#}^{(1)} \cdots \widehat{\pi}_{\#}^{(6)} \epsilon_2^{(6)*} \pi_{\#}^{(7)} \big\{ c\big(\mathcal{P}^{(7)}\big) \cap s\big(W_1^{(7)}, \overline{F}_2^{(7)}\big)\big\}_{\dim X_4} \\ &= \cdots = \epsilon^* \pi_{\#}^{(1)} \cdots \pi_{\#}^{(7)} \big\{ c\big(\mathcal{P}^{(7)}\big) \cap s\big(W_1^{(7)}, \overline{F}_2^{(7)}\big)\big\}_{\dim X_4} \\ &= \epsilon^* e(W_i), \end{aligned}$$

as desired. ■

Lemma 7.4. *Assume* $(F/Y, D)$ *is strongly* 8*-generic. Then* C_i *depends just on the analytic type of* $D_{\pi(x)}$ *at* x*, but is otherwise inpenendent of the choice of* $(F/Y, D)$.

Proof. Recall that x is an ordinary quadruple point of $\widehat{D}_{\widehat{\pi}(x)}$. Let $(\mathbb{V}/\mathbb{B}, \mathbb{D})$ be its versal deformation; see [12, Example 14.0.1, p. 101 and Theorem 14.1, p. 103]. Recall how $(\mathbb{V}/\mathbb{B}, \mathbb{D})$ is constructed. Take variables $t_1, \dots, t_9, u, v$. Identify $\widehat{F}_{\pi(x)}$ with $\operatorname{Spec} k[[u, v]]$. Say $\widehat{D}_{\pi(x)}$ is defined by $f(u, v)$ in $k[[u, v]]$, and choose $g_1, \dots, g_9$ in $k[u, v]$ whose classes in $k[[u, v]]/(f, f_u, f_v)$ form a basis of that vector space. Then

$$\mathbb{B} := \operatorname{Spec} k[t_1, \dots, t_9] \qquad \text{and} \qquad \mathbb{D} := \operatorname{Spec} B[[u, v]]\Big/\Big(f + \sum t_i g_i\Big).$$

Note that $(\mathbb{V}/\mathbb{B}, \mathbb{D})$ depends just on the analytic type of $D_{\pi(x)}$ at x.

Since $x \in \widehat{D}_{\pi(x)}$ is an ordinary quadruple point, $f = f_4 + f_5 + \cdots$, where f_4 is a product of independent linear forms. Choose the g_i so that only $g_9 \in (u, v)^4$. Define $\mathbb{B}_4$ by the vanishing of $t_1, \dots, t_8$. Then $b \in \mathbb{B}$ lies in $\mathbb{B}_4$ iff the fiber $\mathbb{D}_b$ has a quadruple point.

Recall from [12, Theorem 14.1, p. 103] that there exists a map $\delta \colon \widehat{Y} \to \mathbb{B}$ such that $\widehat{D}$ and $\mathbb{D} \times_{\mathbb{B}} \widehat{Y}$ become isomorphic after completion along their fibers over $\widehat{\pi}(x)$. Since $\widehat{D}$ is complete at x, it is already complete along its fiber. Form the completions $\widehat{\mathbb{V}}$, $\widehat{\mathbb{B}}$, $\widehat{\mathbb{D}}$, $\widehat{\mathbb{B}}_4$ at the origin b_0 of $\mathbb{B}$. Then $\delta \colon \widehat{Y} \to \mathbb{B}$ factors through a map $\widehat{\delta} \colon \widehat{Y} \to \widehat{\mathbb{B}}$, and $\widehat{D}$ is isomorphic to the completion of $\widehat{\mathbb{D}} \times_{\widehat{\mathbb{B}}} \widehat{Y}$ along its fiber over $\widehat{\pi}(x)$. Each subscheme $\widehat{X}_i$ of $\widehat{D}$ is the pullback of the corresponding subscheme of $\widehat{\mathbb{D}} \times_{\widehat{\mathbb{B}}} \widehat{Y}$, which, in turn, is the pullback of the corresponding subscheme $\widehat{\mathbb{X}}_i$ of $\widehat{\mathbb{D}}$.

Let us show $\widehat{\delta}\colon \widehat{Y} \to \widehat{\mathbb{B}}$ is flat. Notice $\widehat{\delta}\big(\widehat{\pi}\big(\widehat{X}_4\big)\big) \subset \widehat{\mathbb{B}}_4$. But $\widehat{\delta}\big(\widehat{\pi}\big(\widehat{X}_4\big)\big) \neq \{b_0\}$ because $(F/Y, D)$ is strongly 8-generic. However, $\dim \widehat{\mathbb{B}}_4 = 1$. It follows that $\operatorname{cod}\big(\delta^{-1}(b_0), \widehat{\pi}\big(\widehat{X}_4\big)\big) = 1$. Now, $\operatorname{cod}\big(\widehat{\pi}(\widehat{X}_4), \widehat{Y}\big) = 8$ by (2.11) since $(F/Y, D)$ is 8-generic. So $\operatorname{cod}\big(\widehat{\delta}^{-1}(b_0), \widehat{Y}\big) = 9$. But $\widehat{Y}$ is Cohen–Macaulay, and $\widehat{\mathbb{B}}$ is smooth. Thus $\widehat{\delta}$ is flat by [10, Proposition 15.4.2, p. 230].

Form the class $e\big(\widehat{\mathbb{W}}_i\big)$ for $\big(\widehat{\mathbb{V}}/\widehat{\mathbb{B}}, \widehat{\mathbb{D}}\big)$ analogous to the class $e\big(\widehat{W}_i\big)$ for $\big(\widehat{F}/\widehat{Y}, \widehat{D}\big)$. The map $\widehat{\mathbb{X}}_2 \times_{\widehat{\mathbb{B}}} \widehat{Y} \to \widehat{\mathbb{X}}_2$ is flat, as it is induced by $\widehat{\delta}$. Since the map $\widehat{X}_2 \to \mathbb{X}_2 \times_{\mathbb{B}} \widehat{Y}$ is also flat, we can argue as in the proof of Lemma 7.3 to conclude that $e\big(\widehat{\mathbb{W}}_i\big)$ pulls back to $e\big(\widehat{W}_i\big)$. Owing to the same flatness, the fundamental class $\big[\widehat{\mathbb{X}}_4\big]$ on $\widehat{\mathbb{X}}_2$ pulls back to the fundamental class $\big[\widehat{X}_4\big]$ on $\widehat{X}_2$.

Form the equation $e\big(\widehat{\mathbb{W}}_i\big) = \mathbb{C}_i\big[\widehat{\mathbb{X}}_4\big]$ on $\widehat{\mathbb{X}}_2$ analogous to $e\big(\widehat{W}_i\big) = C_i\big[\widehat{X}_4\big]$ on $\widehat{X}_2$, see (7.1). The former equation pulls back to the latter owing to the preceding paragraph. Thus $C_i = \mathbb{C}_i$. But $\mathbb{C}_i$ depends just on $(\mathbb{V}/\mathbb{B}, \mathbb{D})$, so just on the analytic type of $D_{\pi(x)}$ at x. Thus C_i depends just on the analytic type of $D_{\pi(x)}$ at x. ∎

Theorem 7.5. *Assume $(F/Y, D)$ is strongly 8-generic. Then the correction term (5.3) is equal to $C[X_4]$, where C is independent of the choice of $(F/Y, D)$.*

Proof. By Lemma 7.4, each C_i depends just on the analytic type of $D_{\pi(x)}$ at x; that is, on the cross ratio of the four tangents at x. By the last line in Section 7.1, furthermore, C_i is independent of the choice of x in Z. Below, we exhibit a pair where X_4 is irreducible and where any given value of the cross ratio appears at some $x \in X_4$. It follows that C_i is independent of the choice of $(F/Y, D)$. Finally, Proposition 6.4 now implies that C is independent too, as desired

To build the pair, say k is the base field, take variables $t_1, \dots, t_8, t, u, v$, and set

$$\mathcal{A} := k\left[t_1, \dots, t_8, t, \frac{1}{t}, \frac{1}{t-1}\right], \qquad \mathcal{B} := A[u, v], \qquad \mathcal{C} := \mathcal{B}/(g),$$

where

$$g := t_1 + t_2 u + t_3 v + t_4 u^2 + t_5 uv + t_6 v^2 + t_7 u^2 v + t_8 uv^2 + uv(u - v)(u - tv).$$

Set $Y := \operatorname{Spec} \mathcal{A}$ and $F := \mathbb{P}^2_k \times Y$ and $D := \overline{\operatorname{Spec} \mathcal{C}}$.

Then $X_4 \subset \operatorname{Spec} \mathcal{B}$. Its ideal $\mathcal{I}$ is generated by the partial derivatives with respect to u and v of g up to order three; so $I = (t_1, \dots, t_8, u, v)$. It follows that $X_4 = \operatorname{Spec}(\mathcal{B}/\mathcal{I}) = \operatorname{Spec} k\big[t, \frac{1}{t}, \frac{1}{t-1}\big]$. Thus X_4 is irreducible.

Given $c \in k$, let $x \in X_4$ be the point with $t = c$. Then the fiber $D_{\pi(x)} \subset \mathbb{P}^2_k$ is equal to the four lines $uv(u - v)(u - cv)$ through $(0 : 0 : 1)$ with cross ratio equal to c. Thus all cross ratios appear in this family, as desired. ∎

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