# ALGEBRAIC COBORDISM AND ÉTALE COHOMOLOGY

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ABSTRACT. Thomason's étale descent theorem for Bott periodic algebraic $K$-theory [78] is generalized to any MGL module over a regular Noetherian scheme of finite dimension. Over arbitrary Noetherian schemes of finite dimension, this generalizes the analog of Thomason's theorem for Weibel's homotopy $K$-theory. This is achieved by amplifying the effects from the case of motivic cohomology, using the slice spectral sequence in the case of the universal example of algebraic cobordism. We also obtain integral versions of these statements: Bousfield localization at étale motivic cohomology is the universal way to impose étale descent for these theories. As applications, we describe the étale local objects in modules over these spectra and show that they satisfy the full six functor formalism, construct an étale descent spectral sequence converging to Bott-inverted motivic Landweber exact theories, and prove cellularity and effectivity of the étale versions of these motivic spectra.

## CONTENTS

## 1. Introduction

In constructing the motivic homotopy category $H(S)$ over a base scheme $S$, Morel and Voevodsky [56] use the Nisnevich topology. One inspiration for this choice is the fact that many interesting invariants of schemes such as algebraic $K$-theory and Chow groups satisfy Nisnevich (hyper)descent but not étale (hyper)descent (we refer the reader to §9 for a self-contained discussion on hyperdescent). This failure for $K$-theory has both arithmetic and geometric sources. For example, if $K$-theory did satisfy étale hyperdescent then $K_0$ of many fields will admit contributions from its (generally non-zero) Brauer group which is simply not true since this group is always just $\mathbf{Z}$; see [52, Section 4] for a discussion. Even assuming that the field is algebraically closed, étale hyperdescent combined with the comparison theorem in étale cohomology would prove that profinitely completed algebraic $K$-theory of a complex variety would simply be the profinitely completed topological $K$-theory of its complex points which is again not true; see [22] for the precise relationship.

As a result, the venture of motivic homotopy theory has focused on algebro-geometric invariants which are Nisnevich local in nature. Besides the fact that the aforementioned invariants are representable in the motivic homotopy category, the Nisnevich topology enjoys some "finiteness" properties. If $S$ is noetherian, then the Nisnevich cohomological dimension of $S$ is bounded above by its Krull dimension [38]; this is important for dualizability and compactness questions, and also for the convergence of spectral sequences derived from motivic homotopy theory. Recent work of Clausen-Mathew has extended greatly the scope of these "finiteness" properties of the Nisnevich site [16].

However, there are good reasons to study versions of $H(S)$ and its stable counterpart $SH(S)$ using the étale topology. Indeed, $SH(S)$ suffers from the fact that its "linearization", i.e., Voevodsky's big category of motives DM [63] and subcategories thereof, do not admit the elusive motivic $t$-structure [88, Proposition 4.8]. In recent years, a theory of étale motives has been developed by Ayoub [5] and by Cisinski-Deglise [15], including a full-fledged six functors formalism; for an overview, see [4]. The étale theory has many good properties — from the construction of an integral étale cycle class map [15, Section 7.1] to providing an environment for the conservativity conjecture [4, Section 5.1].

This article is a contribution to our understanding of the *homotopical* analogue of the above *homological* picture. We are interested in a specific question which is, in some sense, quite classical:

**Question 1.1.** What is the difference between $SH(S)$ and $SH_{\text{ét}}(S)$, the stable motivic homotopy categories constructed in the Nisnevich and étale topologies?

These categories agree upon rationalization, at least if $-1$ is a sum of squares (see [13, §12.3, Corollary 16.2.14], and the discussion in Appendix §13), so our question is basically one of torsion. The main difference concerns the descent condition; whereas the Mayer-Vietoris property with respect to distinguished Nisnevich squares is sufficient for descent in $SH(S)$, the more involved étale hyperdescent condition is required in $SH_{\text{ét}}(S)$. An étale hypercover $U_\bullet \to X$ is essentially a generalized Cech nerve construction, where at each stage one allows a further étale covering, and the descent condition amounts to a weak equivalence between $\mathcal{F}(X)$ and the homotopy limit of the diagram $\mathcal{F}(U_\bullet)$ [19, Definition 4.3].

1.0.1. *Thomason's work.* A version of Question 1.1 was answered by Thomason in his seminal paper [78]. Suppose $X$ is a separated, regular, noetherian scheme of finite Krull dimension. For $\ell$ a prime, we say that $X$ is a $T_\ell$-*scheme* if:

(1) $\ell$ invertible in $X$,
(2) the residue fields of $X$ have a uniformly bounded mod-$\ell$ étale cohomological dimension, and admit Tate-Tsen filtrations,
(3) if $\ell = 2$, $\sqrt{-1} \in \mathrm{H}^0(X, \mathscr{O}_X)$.

With these technical hypotheses at hand, Thomason [78, Theorem 2.45] proved the following beautiful result for Bott periodic mod-$\ell^\nu$ algebraic $K$-theory. Let $\text{Ét}_X$ denote the small étale site on $X$ and $\mathsf{Sp}$ is the category of spectra.

**Theorem 1.2.** *Suppose $X$ is a $T_\ell$-scheme and let $\beta \in \epsilon_*(K(-)/\ell^\nu)$ be a Bott element [78, A.7]. Then the presheaf of spectra:*

$$K(-)/\ell^\nu[\beta^{-1}] : \text{Ét}_X^{\text{op}} \to \mathsf{Sp}; U \mapsto K(U)/\ell^\nu[\beta^{-1}],$$

*satisfies étale hyperdescent.*

One concrete manifestation of étale hyperdescent is the strongly convergent descent spectral sequence for $X$ a $T_\ell$-scheme:

$$H_{\text{ét}}^*(X, \underline{\pi}_*^{\text{ét}} K/\ell^\nu[\beta^{-1}]) \Rightarrow \pi_* K/\ell^\nu(X)[\beta^{-1}],$$

which by Gabber rigidity allows for concrete computations of the target groups, see e.g., [78, Examples 4.4, 4.8, 4.18].

In the language of motivic homotopy theory, we may reformulate Theorem 1.2 for the motivic spectrum $\mathsf{KGL}/\ell_X^\nu$ representing mod-$\ell^\nu$ $K$-theory over $X$[1]: the unit of the adjunction between stable motivic homotopy categories:

(1) $$\epsilon^* : \mathrm{SH}(X) \rightleftarrows \mathrm{SH}_{\text{ét}}(X) : \epsilon_*,$$

furnished by étale localization induces an isomorphism in $\mathrm{SH}(X)$:

$$\mathsf{KGL}/\ell_X^\nu[\beta^{-1}] \xrightarrow{\cong} \epsilon_* \epsilon^* \mathsf{KGL}/\ell_X^\nu.$$

Phrased this way, motivic homotopy theory also provides a universal property of $\beta$-inverted $K$-theory as an étale hyperdescent localization.

1.0.2. *Bott-inverted motivic cohomology.* The story in Section 1.0.1 has a parallel in motivic cohomology, considered by the third author in [40]. We write $H_{\mathsf{m}}^{p,q}(X, \mathbf{Z}/n)$ for mod $n$ motivic cohomology of a smooth $k$ scheme $X$.

Suppose $\ell$ is prime to the exponential characteristic of the field $k$ and that $k$ contains a primitive $\ell$th root of unity $\zeta_\ell$. Via the isomorphism $H_{\mathsf{m}}^{0,1}(\mathrm{Spec}\, k; \mathbf{Z}/\ell) \cong \mu_\ell(k)$ we have the element $\tau_\ell^{\mathsf{MZ}} \in H_{\mathsf{m}}^{0,1}(\mathrm{Spec}\, k; \mathbf{Z}/\ell)$ corresponding to $\zeta_\ell$. Even without supposing that $\zeta_\ell$ is in $k$, one can construct (see §6.1) a family of "Bockstein-compatible" generators $\tau_{\ell^\nu}^{\mathsf{MZ}} \in H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(\mathrm{Spec}\, k; \mathbf{Z}/\ell^\nu) \simeq \mathbf{Z}/\ell^\nu$, where $\mathrm{e}(\ell^\nu)$ is short for the exponent of the group $(\mathbf{Z}/\ell^\nu)^\times$. This is analogous to Thomason's construction of the Bott elements $\beta_\nu \in \pi_{n(\nu)}(K(-)/\ell^\nu)$ for suitable $n(\nu)$; corresponding to Thomason's theorem on Bott inverted $K$-theory one has the following result for motivic cohomology.

**Theorem 1.3.** *Let $k$ be a field with exponential characteristic prime to $\ell$ and let $X$ be in $\mathsf{Sm}_k$. Suppose that $\mathrm{cd}_\ell(k) < \infty$. For all $\nu \geq 1$, the natural map:*

$$H_{\mathsf{m}}^{p,q}(X; \mathbf{Z}/\ell^\nu) \to H_{\text{ét}}^p(X; \mu_{\ell^\nu}^{\otimes q})$$

*induces an isomorphism*

$$H_{\mathsf{m}}^{p,*}(X; \mathbf{Z}/\ell^\nu)[(\tau_{\ell^\nu}^{\mathsf{MZ}})^{-1}] \cong H_{\text{ét}}^p(X; \mu_{\ell^\nu}^{\otimes *}).$$

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[1] Since $X$ is regular, we note that homotopy $K$-theory agrees with Thomason-Trobaugh $K$-theory.

In case char $k > 0$, or char $k = 0$ and $\ell$ is odd, or char $k = 0$, $\ell = 2$ and $\sqrt{-1} \in k$, this is [40, Theorem 6.2], proved using essentially the same argument as for Thomason's theorem. The case char $k = 0$, $\ell = 2$ and $\sqrt{-1} \notin k$ is discussed in §6.2.2.

*Remark* 1.4. The Milnor conjecture (the case $\ell = 2$) and the Bloch-Kato conjecture (the case $\ell > 2$), proven by Voevodsky, gives a finer statement about the relationship between motivic mod $\ell^\nu$ cohomology and étale cohomology with $\mu_{\ell^\nu}^{\otimes *}$-coefficients:

*Theorem* 1.5. *(Voevodsky [85, 86]) For $p \leq q$ and $X \in \mathrm{Sm}_k$, the étale sheafification functor induces an isomorphism between motivic and étale cohomology groups:*

$$H_\mathsf{m}^{p,q}(X; \mathbf{Z}/\ell^\nu) \xrightarrow{\cong} H_{\text{ét}}^p(X; \mu_{\ell^\nu}^{\otimes q}).$$

This directly implies Theorem 1.3, even without the additional assumption about the finiteness of the étale-$\ell$-cohomological dimension. However, as we will need these extra conditions for other portions of our argument, we will not need the full power of the Bloch-Kato conjecture for our main results, and can rely on the more elementary arguments that go into the proof of Theorem 1.3.

In motivic terms, we may reformulate the isomorphism of Theorem 1.3 in a more structured way by using the motivic spectrum $\mathsf{M}\mathbf{Z}/\ell_X^\nu$ over $X$ representing mod-$\ell$ motivic cohomology. That is, the unit of the adjunction (1) induces an equivalence in $\mathrm{SH}(X)$:

$$(2) \qquad\qquad \mathsf{M}\mathbf{Z}/\ell_X^\nu[(\tau_{\ell^\nu}^{\mathsf{M}\mathbf{Z}})^{-1}] \xrightarrow{\cong} \epsilon_* \epsilon^* \mathsf{M}\mathbf{Z}/\ell_X^\nu.$$

We interpret the above equivalence as an answer to the linearized version of Question 1.1: the categories $\mathrm{DM}(X, \mathbf{Z}/\ell^\nu)$ and $\mathrm{DM}_{\text{ét}}(X, \mathbf{Z}/\ell^\nu)$ differ only by inversion of the element $\tau_{\ell^\nu}^{\mathsf{M}\mathbf{Z}}$. Indeed, this was formalized and verified by Haesemayer-Hornbostel in [26], at least when $X = \mathrm{Spec}\, k$ and for the subcategories generated by motives of $k$-varieties; we refine their result in Theorem 7.3 and generalize it to non-linear situations in Theorem 1.10.

1.0.3. *Main result.* We can state a version of our main result as follows:

**Theorem 1.6.** *Let $\ell$ be a prime and $S$ be a Noetherian $\mathbf{Z}[\frac{1}{\ell}]$-scheme of finite dimension and assume that for all $x \in S$, $\mathrm{cd}_\ell(k(x)) < \infty$, and let $\mathsf{E}_S$ be an $\mathsf{MGL}_S$-module. Then, there exists an element $\tau_{\ell^\nu} \in \mathsf{MGL}/\ell_{0, -e_{\mathsf{MGL}}(\ell^\nu)}^\nu(S)$ such that:*

(1) *The $\mathsf{MGL}_S$-module obtained by inverting the action of $\tau_{\ell^\nu}$ on $\mathsf{E}_S$,*

$$\mathsf{E}_S/\ell^\nu[(\tau_{\ell^\nu})_S^{-1}]$$

*satisfies étale hyperdescent.*

(2) *The spectrum $\mathsf{E}_S/\ell^\nu[(\tau_{\ell^\nu})_S^{-1}]$ identifies with the étale localization of $\mathsf{E}_S$.*

*Remark* 1.7. Theorem 1.6 has a long and interesting history starting with Thomason's work [78]. Two complementary accounts are Mitchell's survey [52] and Jardine's book [36]. The insight that one can deduce Thomason's theorem from its analogue for motivic cohomology comes from [39]; it is also executed in [68] and [69]. A similar argument for algebraic cobordism over algebraically closed fields was the subject of Quick's thesis in [61].

*Remark* 1.8. When $\ell = 2$ and $\nu = 1$, the mod-2 Moore spectrum does not have a unital multiplication so the Bott-inverted spectrum in Theorem 1.6 has to be defined using (a motivic version) of Oka's module action of the mod-4 Moore spectrum on the mod-2 Moore spectrum (63). This point is discussed in §6.3.5.

1.0.4. *Integral statement.* While the above statement is interesting, an integral statement is of course preferable. The reason for doing this is more than just aesthetics. For example, such a statement would give a lift of the above theorem to the level of categories of modules over highly structured motivic ring spectra as localization usually preserves such structures, while passing to some homotopy cofiber does not.

In algebraic $K$-theory, this was achieved by localizing algebraic $K$-theory spectra at the topological $K$-theory spectrum; see [78, Theorem 2.50]. For algebraic cobordism over the

complex numbers, shades of an integral statement appeared first in Heller's work [27] on semitopological cobordism. The right thing to consider turns out to be the completion (in the sense of, say, [48, Section 2]) at the étale motivic cohomology spectrum $M\mathbf{Z}^{\text{ét}} := \epsilon_* \epsilon^* M\mathbf{Z}$. If $J$ is a set of primes, we denote by $\mathsf{E}_{(J)}$ the localization of $\mathsf{E}$ at $J$.

**Theorem 1.9.** *Under the same assumptions as in Theorem 1.6, the completion of $\mathsf{E}$ at the étale motivic cohomology spectrum $M\mathbf{Z}^{\text{ét}}$ coincides with the étale localization of $\mathsf{E}$ up to localizing at the primes which are invertible in $S$.*

Denote by $L_{M\mathbf{Z}^{\text{ét}}}$ the left adjoint to the inclusion of $M\mathbf{Z}^{\text{ét}}$-local objects, i.e., a Bousfield localization functor. Theorem 1.9 helps us deduce our main result for modules over highly structured ring spectra using the language of stable $\infty$-categories:

**Theorem 1.10.** *Under the hypotheses of Theorem 1.9, assume further that $\mathsf{E}$ is a motivic $\mathcal{E}_\infty$ ring spectrum (such as $\mathsf{MGL}$ itself). Then there are equivalences of stable $\infty$-categories:*

$$\mathrm{Mod}_{L_{M\mathbf{Z}^{\text{ét}}}\mathsf{E}_{(J)}} \simeq \mathrm{Mod}_{\epsilon_* \epsilon^* \mathsf{E}_{(J)}} \simeq \mathrm{Mod}_{\mathsf{E}_{(J)}}^{\text{ét}}.$$

Here, by a motivic $\mathcal{E}_\infty$-ring spectrum we mean a commutative algebra object of the symmetric monoidal $\infty$-category of motivic spectra over a Noetherian scheme of finite dimension $S$. Hence Theorem 1.9 gives a description of the étale-local subcategory of $\mathsf{E}_{(J)}$-modules as modules over a certain motivic ring spectrum.

1.0.5. *Functoriality of étale algebraic cobordism.* Another application concerns the functoriality of étale algebraic cobordism. If we let $\mathrm{Sch}^{(J)}$ be the full subcategory of schemes for which the hypotheses of Theorem 1.9 hold then, restricted to this subcategory of schemes, we show that $\mathsf{MGL}^{\text{ét}}_{(J)}$ forms a Cartesian section of SH. The point is that $f^*$ and $\epsilon_*$ have no *a priori* reason to commute, while $f^*$ commutes with colimits and Bott elements (by construction). Therefore, $f^*$ preserves Bott-inverted algebraic cobordism. In conjunction with Theorem 1.10, we obtain a six functors formalism for $\mathrm{Mod}^{\text{ét}}_{\mathsf{MGL}_{(J)}}$.

**Corollary 1.11.** *The functor*

(3) $$\mathrm{Mod}^{\text{ét}}_{\mathsf{MGL}_{(J)}} : (\mathrm{Sch}^{(J)})^{\text{op}} \to \mathrm{Cat}_\infty,$$

*satisfies the full six functors formalism in the sense of [3], [13]. In particular, $\mathrm{Mod}^{\text{ét}}_{\mathsf{MGL}_{(J)}}$ satisfies localization: for any closed immersion $i : Z \hookrightarrow X$ with open complement $j : U \to X$ and $\mathsf{M} \in \mathrm{Mod}^{\text{ét}}_{\mathsf{MGL}_{(J)}}$, then we have a cofiber sequences*

$$j_! j^* \mathsf{M} \to \mathsf{M} \to i_! i^* \mathsf{M},$$

*and*

$$i_* i^! \mathsf{M} \to \mathsf{M} \to j_* j^* \mathsf{M}.$$

1.0.6. *Subsequent work.* While this paper was under revision, the first and last author, in a joint project with Bachmann, have managed to obtain a stronger étale descent result for the motivic sphere spectrum in [7]. While the general ideas are very similar — via reduction to the case of motivic cohomology — the execution is rather different. In this paper, our technical ingredients are 1) a study of the convergence of localized spectral sequences and 2) a thorough study of étale localization in the context of oriented theories. We believe that these techniques and the results they yield are of independent interest to motivic homotopy theory as well.

1.0.7. *Organization.* Using the language of $\infty$-categories we start Section 2 by reviewing motivic homotopy theory with respect to some topology $\tau$. We recall the adjunctions comparing motivic homotopy theories across different topologies along with Voevodsky's category of motives. In Section 3, we introduce the "slice comparison paradigm" which is a factorization of the unit map of the adjunction comparing different topologies. We recall, in

a slightly more general setting, how to associate to $E \in SH(S)$ the filtered motivic spectra $\{f_q E\}_{q \in \mathbf{Z}}$ of its slice covers. The "slice comparison paradigm" then takes the form:

$$\eta : \operatorname{colim} f_q E \simeq E \xrightarrow{\eta_\infty} \operatorname{colim} \epsilon_* \epsilon^* f_q E \xrightarrow{\eta^\infty} \epsilon_* \epsilon^* E.$$

We explain conditions for when $\eta^\infty$ and $\eta_\infty$ can be turned into equivalences. This is addressed in Sections 4 and 5. In more details, Section 4 studies the effect $\epsilon_* \epsilon^*$ has on connectivity of motivic spectra and thus addresses the convergence properties of the spectral sequence arising from the tower of spectra $\{\epsilon_* \epsilon^* f_q E\}$. These connectivity results are of independent interest. In Section 5, we give conditions for $\eta^\infty$ to be an equivalence, which boils down to understanding situations where the $\epsilon_*$ functor commutes past colimits. In Section 6 we apply this paradigm to prove Theorem 1.6. After dealing with the motivic cohomology spectrum M**Z** we turn to algebraic cobordism MGL, and finally arbitrary Landweber exact theories. The main point is that the result for M**Z** proves the theorem on the level of slices and the "slice comparison paradigm" bridges the gap between the slices and the spectra of interest. In Section 7 we give applications of our main theorems and promote Theorem 1.6 to the level of module categories as in Theorem 1.9. We also prove that étale local algebraic cobordism defines a Cartesian section of SH and thus the category of modules over it satisfy the six functors formalism. In the first appendix (Section 8) we review the basics of periodization (after [29]) and localization in suitable $\infty$-categories, the short second appendix (Section 10) records a well-known result about E-locality, while the third appendix (Section 11) constructs transfers for MGL necessary to pick out Bott elements over general fields.

1.0.8. *Conventions.* Categorical terminologies are to be interpreted in the $\infty$-categorical context; "functor" always means a functor of $\infty$-categories (i.e., a morphism of quasicategories). We freely use the notions of algebras and modules in higher algebra [46].

- We use the following vocabulary of higher category theory:
    - Spc is the $\infty$-category of $\infty$-groupoids (Kan complexes give a concrete model).
    - $\operatorname{Maps}(X, Y)$ is the Kan complex of maps between objects $X, Y \in$ C and $\underline{\operatorname{map}}(X, Y) \in$ C is the internal mapping object.
    - $[X, Y] := \pi_0 \operatorname{Maps}(X, Y)$ denotes homotopy classes of maps between $X, Y \in$ C.
    - $\operatorname{Pr}^{L, \otimes}$ —the symmetric monoidal $\infty$-category of presentable $\infty$-categories and colimit preserving functors — has a full subcategory $\operatorname{Pr}^L_{\mathsf{stab}}$ of *stable* presentable $\infty$-categories.
- If $p : X \to S$ is a smooth $S$-scheme, we always mean that $X$ is of *finite type*. An essentially smooth scheme over $k$ is an inverse limit of smooth $S$-schemes with affine and dominant transition maps.
- If $F$ is a presheaf of abelian groups we denote its $\tau$-sheafification by $a_\tau(F)$, $\tau$ any topology.
- Some conventions in motivic homotopy theory:
    - We denote unstable motivic spheres by $\mathbf{S}^{p,q} := (S^1)^{p-q} \wedge \mathbf{G}_m^q$ for $p \geq q$, the motivic sphere spectrum by $\mathbf{1}$, and motivic suspensions by $\Sigma^{p,q}$.
    - For a motivic spectrum $E$ over a base scheme $S$ we write $E \in SH(S)$, or $E_S$ for emphasis. If the base is clear in the context, we will sometimes drop the subscript and simply write "$E$".

## 2. PRELIMINARIES

In this section, we review the basics of motivic homotopy theory — mainly setting up notation and checking certain statements in the generality of arbitrary topologies; the reader familiar with this theory may feel free to skip this section and return as necessary. In Section 2.1 we review the construction of the presentably symmetric monoidal stable ∞-category of motivic spectra in the generality of arbitrary Grothendieck topologies. In Section 2.2 we compare motivic homotopy categories across different topologies, and in Section 2.4 we review Voevodsky motives in our language. For the following, $S$ is a quasicompact quasiseparated (qcqs) base scheme.

**2.1. $\tau$-motivic homotopy theory.** We first construct the presentably symmetric monoidal ∞-category of $\tau$-*motivic spaces*, $H_\tau(S)$. For our purposes, this ∞-category has $\tau$-*hyperdescent* built into it as we will explain later. We start with the (discrete) category $Sm_S$ of smooth $S$-schemes equipped with a Grothendieck topology $\tau$. Let $P(Sm_S)$ be the ∞-category of presheaves over $S$; it is obtained from $Sm_S$ by freely adjoining arbitrary small colimits [45, Theorem 5.1.5.6]. We denote by $j : Sm_S \to P(Sm_S)$ the Yoneda embedding.

**2.1.1.** Recall that $F \in P(Sm_S)$ is called *homotopy invariant* or $\mathbf{A}^1$-*invariant* if the projection induces an equivalence $F(X) \to F(X \times_S \mathbf{A}^1)$. We denote by $P_{\mathbf{A}^1}(Sm_S)$ the ∞-category of homotopy invariant presheaves. The embedding $P_{\mathbf{A}^1}(Sm_S) \hookrightarrow P(Sm_S)$ preserves limits, and thus admits a left adjoint:

$$L_{\mathbf{A}^1} : P(Sm_S) \to P_{\mathbf{A}^1}(Sm_S),$$

which witnesses $P_{\mathbf{A}^1}(Sm_S)$ as an accessible localization [45, Proposition 5.2.7.4]. A map $f : F \to G$ in $P(Sm_S)$ is an $\mathbf{A}^1$-*weak equivalence* if $L_{\mathbf{A}^1} f : L_{\mathbf{A}^1} F \to L_{\mathbf{A}^1} G$ is an equivalence.

**2.1.2.** One difference between doing motivic homotopy theory with an arbitrary topology $\tau$ and the usual Nisnevich localization is the choice of whether or not to hypercomplete; recall that if $S$ is a base scheme which is quasi-compact and has finite Krull dimension then the Nisnevich ∞-topos is automatically hypercomplete (see [16, Theorem 3.17] for a general version of this statement). In the main body of this paper, we work with the *hypercompletion* of the étale topology, mainly because we want to obtain spectral sequences

with convergent properties to calculate étale versions of motivic Landweber exact theories as in §7.3. However, as we will prove in the appendix, we can remove this hypothesis in §9.

We consider the full subcategory $P_{\tau,\mathsf{hyp}}(\mathrm{Sm}_S) \hookrightarrow P(\mathrm{Sm}_S)$ spanned by $\tau$-*hypersheaves* (also called $\tau$-*local*), i.e., those presheaves $F$ such that for any $\tau$-hypercover $U_\bullet \to U$, the map $F(U) \to \lim_\Delta F(U_\bullet)$ is an equivalence. As above, the embedding $P_{\tau,\mathsf{hyp}}(\mathrm{Sm}_S) \hookrightarrow P(\mathrm{Sm}_S)$ preserves limits and admits a left adjoint:

$$L_\tau : P(\mathrm{Sm}_S) \to P_{\tau,\mathsf{hyp}}(\mathrm{Sm}_S),$$

which witnesses $P_{\mathbf{A}^1}(\mathrm{Sm}_S)$ as an accessible localization. A map $f : F \to G$ is a $\tau$-*weak equivalence* if the induced map $L_\tau f : L_\tau F \to L_\tau G$ is an equivalence.

For the purposes of this paper, we work with $\tau$-hypersheaves. However, we could also consider the $\infty$-category $P_\tau(\mathrm{Sm}_S)$ of $\tau$-*sheaves*, see [45, Section 6.2.2], and its localization functor $L'_\tau : P(\mathrm{Sm}_S) \to P_\tau(\mathrm{Sm}_S)$. The $\infty$-category $P_{\tau,\mathsf{hyp}}(\mathrm{Sm}_S)$ is obtained as a further accessible localization by [45, Theorem 6.5.3.13] (usually called *hypercompletion*):

$$\widehat{(-)} : P_\tau(\mathrm{Sm}_S) \to P_{\tau,\mathsf{hyp}}(\mathrm{Sm}_S).$$

Note that this is the underlying $\infty$-category of the Brown-Joyal-Jardine model structure on simplicial presheaves [45, Proposition 6.5.2.14] and $\tau$-hypersheaves in our sense are equivalent to hypercomplete objects in the sense of [45] by [45, Theorem 6.5.3.12].

2.1.3.   The $\infty$-category $H_\tau(S)$ of $\tau$-*motivic spaces* is the full subcategory of $P(\mathrm{Sm}_S)$ spanned by homotopy invariant $\tau$-hypersheaves. Combining the localizations above, it is characterized by a universal property whose proof follows the one for $\tau = \mathrm{Nis}$ verbatim:

**Proposition 2.1.** *([62, Theorem 2.30], [62, Remark 2.31]) For any $\infty$-category $C$ with small colimits, the composite of localizations $L_{\mathbf{A}^1}L_\tau : P(\mathrm{Sm}_S) \to P_{\tau,\mathsf{hyp}}(\mathrm{Sm}_S) \to H_\tau(S)$ induces a fully faithful functor $\mathrm{Fun}^L(H_\tau(S), C) \to \mathrm{Fun}(\mathrm{Sm}_S, C)$, whose essential image is spanned by functors which are homotopy invariant and satisfy $\tau$-hyperdescent. Furthermore, the Cartesian monoidal structure on $P(\mathrm{Sm}_S)$ descends to a Cartesian monoidal structure on $H_\tau(S)$, and the functor:*

$$L_{\mathbf{A}^1} \circ L_{\tau,\mathsf{hyp}} \circ j : \mathrm{Sm}_S^\times \to P(\mathrm{Sm}_S)^\times \to P_{\tau,\mathsf{hyp}}(\mathrm{Sm}_S)^\times \to H_\tau(S)^\times,$$

*is monoidal.*

Let $L_{\tau,\mathsf{m}} : P(\mathrm{Sm}_S) \to H_\tau(S)$ be short for the *motivic localization* functor $L_{\mathbf{A}^1}L_\tau$. If $X \in \mathrm{Sm}_S$ we write $L_{\tau,\mathsf{m}}(X)$ or, if the context is clear, simply $X$ for the motivic localization $L_{\tau,\mathsf{m}}(j(X))$ of $X$ under the Yoneda embedding.

2.1.4.   From here on, we impose the following assumption on $\tau$:

- A $\tau$-hypersheaf takes coproducts to products, i.e., for any two $S$-schemes $X$, $Y$, the natural map $F(X \coprod Y) \to F(X) \times F(Y)$ is an equivalence.

The Zariski (and hence Nisnevich) topology satisfies the above condition (use the distinguished square obtained by the cover $\{X \to X \coprod Y, Y \to X \coprod Y\}$), and thus also all finer topologies. With this assumption, we see that $P_{\tau,\mathsf{hyp}}(\mathrm{Sm}_S) \subset P_\Sigma(\mathrm{Sm}_S)$, where $P_\Sigma(\mathrm{Sm}_S)$ is the full subcategory of presheaves which transforms coproducts into products. In [45, Section 5.5.5] this is called the *nonabelian derived category*, while (in the context of model categories) these are called *simplicial radditive functors* in [84]. The universal property of $P_\Sigma(\mathrm{Sm}_S)$ is that it is freely generated by the Yoneda image of $\mathrm{Sm}_S$ under sifted colimits and furthermore an object in the Yoneda image is compact [45, Proposition 5.5.8.10]. In general, $P_\Sigma(C)$ is *not* the category of sheaves for some Grothendieck topology [84, Example 3.7].

**Proposition 2.2.**     - *The $\infty$-category $H_\tau(S)$ is generated under sifted colimits by smooth $S$-schemes.*
- *If $\tau$ is at least as fine as the Zariski topology, then the smooth $S$-schemes which are affine generate $H_\tau(S)$.*
- *If the inclusion $H_\tau(S) \hookrightarrow P_\Sigma(\mathrm{Sm}_S)$ preserves filtered colimits, then the generators are compact.*

*Proof.* By the above discussion, we see that since $H_\tau(S) \subset P_\Sigma(Sm_S)$, it is generated by $L_{\tau,m}(X)$ under sifted colimits, where $X$ is a smooth $S$-scheme. The second statement follows by the argument in [37, Lemma 4.3.5]. In more detail, we first note that, by definition:

$$Sm_S \to H_\tau(S); X \mapsto L_{\tau,m}X,$$

is a $\tau$-cosheaf. Thus if $\tau$ is at least as fine as the Zariski topology we may assume that $X$ is separated by picking a Zariski cover of $X$ consisting of affine schemes, where the pairwise intersections are all separated. We may further pick a Zariski cover of $X$ where the pairwise intersections are affine. Applying $L_{\tau,m}$ to the nerve of this cover produces a simplicial object in $P_\Sigma(Sm_S)$ with colimit $L_{\tau,m}X$ (its terms are $L_{\tau,m}$ applied to affine schemes). The last statement is immediate. □

*Remark* 2.3. The inclusion $H_\tau(S) \hookrightarrow P_\Sigma(Sm_S)$ need not preserve filtered colimits, and hence $L_{\tau,m}(X)$ may be noncompact. But $H_{Nis}(S) \hookrightarrow P(Sm_S)$ does preserve filtered colimits. Indeed, being a Nis-hypersheaf is the same as being Nisnevich excisive. Thus we need to check that if $F_i$ is a filtered diagram of Nisnevich excisive presheaves, then the colimit colim $F_i$ taken in $P_\Sigma(Sm_S)$ is Nisnevich excisive. If $Q$ is an elementary distinguished square, then taking the pointwise colimit in $P_\Sigma(Sm_S)$ we see that $(\text{colim}_i F)(Q)$ is a Cartesian square since colimits commute pullbacks in Spc; so colim $F_i$ is Nisnevich excisive.

2.1.5. We may consider the $\infty$-category of pointed objects $H_{\tau,\bullet}(S) := H(S)_{\star/}$ for the final object $\star$ of $H_\tau(S)$. It is also presentable since $H_{\tau,\bullet}(S) \simeq H_\tau(S) \otimes Spc_\bullet$, where the $\otimes$-product is taken in $Pr^L$. There is an adjunction:

$$(-)_+ : H_\tau(S) \rightleftarrows H_{\tau,\bullet}(S) : u_E.$$

The symmetric monoidal structure on $H_\tau(S)$ extends to a monoidal structure on $H_{\tau,\bullet}(S)$ so that $(-)_+ : H_\tau(S) \to H_{\tau,\bullet}(S)$ is a monoidal functor.

2.1.6. Let $CAlg(Pr^L)$ be the $\infty$-category of presentably symmetric monoidal $\infty$-categories and symmetric monoidal colimit preserving functors. To $C^\otimes \in CAlg(Pr^L)$ we can associate the $\infty$-category of $C^\otimes$-modules $Mod_C := Mod_C(Pr^L)$ and $C^\otimes$-algebras $CAlg_C := CAlg(Mod_C)$. The latter can be identified with $CAlg(Pr^L)_{C/}$ [46, Corollary 3.4.1.7]. Hence a $C^\otimes$-algebra object in $C^\otimes$-modules is the data of a presentably symmetric monoidal category D admitting a colimit preserving symmetric monoidal functor $F : C^\otimes \to D^\otimes$. If $X \in C^\otimes$, we let $X$ "act" on D via the functor $F$. Let $CAlg_C[X^{-1}]$ denote the full subcategory of $CAlg_C$ spanned by those $C^\otimes$-algebras on which $X$ acts invertibly.

**Theorem 2.4.** *(Robalo, [62, Proposition 2.9]) Let $C^\otimes \in CAlg(Pr^L)$ and $X \in C^\otimes$ be an object.*

- *There exists a functor:*

$$L_X : CAlg_C \to CAlg_C[X^{-1}],$$

 *which witnesses $CAlg_C[X^{-1}]$ as a localization of $CAlg_C$. Thus for any $D^\otimes \in CAlg_C$ there is a canonical map $D^\otimes \to L_X(D^\otimes)$ in $CAlg_C$ sending $X$ to an invertible object.*
- *The functor $L_X$ fits into the commutative diagram:*

(4)
$$
\begin{array}{ccc}
CAlg_C & \xrightarrow{\;L_X\;} & CAlg_C[X^{-1}] \\
u \downarrow & & \downarrow u \\
Mod_C & \xrightarrow{-\otimes_{C^\otimes} C^\otimes[X^{-1}]} & Mod_{C^\otimes[X^{-1}]},
\end{array}
$$

 *where the horizontal arrows are forgetful functors. Thus if $D^\otimes \in CAlg_C$ the underlying $C^\otimes$-module of $L_X(D^\otimes)$ is equivalent to $D^\otimes \otimes_{C^\otimes} C^\otimes[X^{-1}]$.*

In light of the second point above, we write $L_X(D^\otimes) := D[X^{-1}]$.

2.1.7. Recall that an object $X \in C^{\otimes}$ is *n-symmetric* if there is an invertible two cell that witnesses an equivalence between the cyclic permutation on $X \otimes X \otimes \cdots \otimes X$ and the identity (see [62, Definition 2.16]). For $M \in \mathrm{Mod}_C$ we set:

$$\text{(5)} \qquad \mathrm{Stab}_X(M) := \mathrm{colim}\, M \xrightarrow{-\otimes X} M \xrightarrow{-\otimes X} M \xrightarrow{-\otimes X} \cdots.$$

**Corollary 2.5.** *(Robalo, [62, Corollary 2.22]]) Let $C^{\otimes} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ and let $X$ be an n-symmetric object in $C$. Then for any $M \in \mathrm{Mod}_C$ there is a natural equivalence:*

$$M[X^{-1}] \to \mathrm{Stab}_X(M).$$

2.1.8. $H_\tau(S)^\times_\bullet \in \mathrm{CAlg}(\mathrm{Pr}^L)$ and we invert the 3-symmetric object $(\mathbf{P}^1, \infty) \in H_\tau(S)_\bullet$ [80, Lemma 4.4] to form:

$$\mathrm{SH}_\tau(S) := H_\tau(S)[(\mathbf{P}^1, \infty)^{-1}].$$

The sum of our discussion is the following universal property of $\mathrm{SH}_\tau(S)$.

**Theorem 2.6.** *(Robalo, [62, Corollary 2.39]) Let S be a qcqs base scheme and let:*

$$\theta : \mathrm{Sm}^\times_S \to \mathrm{SH}_\tau(S)^{\otimes},$$

*be the symmetric monoidal functor obtained from $L_{(\mathbf{P}^1, \infty)} \circ (-)_+ \circ L_{\tau,\mathbf{m}} \circ j$. Then for $D \in \mathrm{CAlg}(\mathrm{Pr}^L)$, $\theta$ induces a fully faithful functor:*

$$\mathrm{Fun}^{\otimes, L}(\mathrm{SH}_\tau(S), D) \to \mathrm{Fun}^{\otimes}(\mathrm{Sm}_S, D),$$

*whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{Sm}_S \to D$ which satisfies $\tau$-hyperdescent, homotopy invariance, and such that the cofiber of $F(S) \to F(\mathbf{P}^1)$ induced by the $\infty$-section acts invertibly.*

Unwinding the above theorem we get the standard adjunction:

$$\text{(6)} \qquad \Sigma^\infty_{T,+} : H_\tau(S) \leftrightarrows \mathrm{SH}_\tau(S) : \Omega^\infty_T.$$

2.1.9. The stable analogue of Proposition 2.2 is:

**Proposition 2.7.**
- *The presentably symmetric monoidal stable $\infty$-category $\mathrm{SH}_\tau(S)$ is generated under sifted colimits by $\{\Sigma^{-2n,-n}\Sigma^\infty_T X_+\}_{n \in \mathbf{Z}}$, where $X$ is a smooth $S$-scheme.*
- *If $\tau$ is at least as fine as the Zariski topology, then the smooth $S$-schemes which are affine generate $\mathrm{SH}_\tau(S)$ under sifted colimits.*
- *If the inclusion $H_\tau(S) \hookrightarrow P_\Sigma(\mathrm{Sm}_S)$ preserves filtered colimits, then the generators are compact.*

*Proof.* According to [62, Proposition 2.19], the symmetric monoidal stable $\infty$-category $\mathrm{SH}_\tau(S)$ is calculated as the colimit in $\mathrm{Pr}^{L,\otimes}$ of the diagram:

$$H_{\tau,\bullet}(S) \xrightarrow{-\wedge(\mathbf{P}^1,\infty)\simeq\Sigma^{2,1}} H_{\tau,\bullet}(S) \xrightarrow{-\wedge(\mathbf{P}^1,\infty)\simeq\Sigma^{2,1}(-)} \cdots,$$

and is thus generated under filtered colimits by $\Sigma^{-2n,-n}\Sigma^\infty X$ where $X \in H_\bullet(S)$ from the formula for filtered colimits of presentable categories [45, Lemma 6.3.3.6]. The adjunction $H_\tau(S) \rightleftarrows H(S)_{\tau,\bullet}$ is monadic and in particular any object in $H(S)_{\tau,\bullet}$ can be written as a sifted colimit of objects in $H_\tau(S)$. This implies the first statement since the $\infty$-category $H_\tau(S)$ is generated by smooth $S$-schemes. The next two statements follows immediately from their analogues in Proposition 2.2. $\qquad\square$

For a general topology $\tau$, suspension spectra of schemes need not be compact in the category $\mathrm{SH}_\tau(S)$. In fact its unit can already be noncompact, as the following example illustrates.

**Example 2.8.** We claim the sphere spectrum is noncompact in $\mathrm{SH}_{\text{ét}}(\mathbf{R})$. Suppose for contradiction $\mathbf{1}_\mathbf{R}$ is compact and let $M_i$ be a countable collection of objects in $\mathrm{DM}_{\text{ét}}(\mathbf{R}; \mathbf{Z}/2)$ viewed as objects of $\mathrm{SH}(\mathbf{R})$ via $u_{\mathrm{tr}} : \mathrm{DM}_{\text{ét}}(\mathbf{R}; \mathbf{Z}/2) \to \mathrm{SH}_{\text{ét}}(\mathbf{R})$. Then $\mathrm{Maps}(\mathbf{1}_\mathbf{R}, \oplus u_{\mathrm{tr}}(M_i)) \simeq \oplus \mathrm{Maps}(\mathbf{1}_\mathbf{R}, u_{\mathrm{tr}}(M_i))$, which cannot be the case as explained in [15, Remark 5.4.10] since $\mathrm{cd}_2(\mathbf{R}) = \infty$ (the remark uses $\mathrm{DM}_h(\mathbf{R}; \mathbf{Z}/2)$ which agrees with $\mathrm{DM}_{\text{ét}}(\mathbf{R}; \mathbf{Z}/2)$ after [15, Corollary 5.5.5]).

2.1.10. If $E \in SH_\tau(S)$ and $p, q \in \mathbf{Z}$, the *homotopy presheaf* $\pi_{p,q}^{\mathrm{pre}}(E)$ on $Sm_S$ is defined by:

$$U \mapsto [\Sigma^{p,q}\Sigma_T^\infty U_+, E]_{SH_\tau(S)}.$$

The *$\tau$-homotopy sheaf* of $E$ is the $\tau$-sheafification $\pi_{p,q}^\tau(E) := a_\tau(\pi_{p,q}^{\mathrm{pre}}(E))$. When $\tau = $ Nis we refer to $\pi_{p,q}^\tau(E)$ as the homotopy sheaf.

## 2.2. **Comparing motivic categories.**

2.2.1. Let $S$ be a qcqs scheme and consider two topologies $\sigma, \tau$ on $Sm_S$ such that $\tau$ is finer than $\sigma$ (we write $\sigma \leq \tau$ to signify this); the examples that concern us are $\sigma = $ Nis, $\tau = $ ét. The identity functor $id : Sm_S \to Sm_S$ induces a geometric morphism of $\infty$-topoi $\epsilon_* : P_{\tau,\mathrm{hyp}}(Sm_S) \to P_{\sigma,\mathrm{hyp}}(Sm_S)$; that is, $\epsilon_*$ admits a left adjoint, denoted by $\epsilon^*$, that preserves finite limits [45, Definition 6.3.1.1]. The functor $\epsilon_*$ is obtained by precomposition $F \mapsto F \circ id$ while $\epsilon^*$ is obtained as localization at $\tau$-hypercovers. This functor can be computed via the left Kan extension:

$$\begin{array}{ccc} Sm_S & \xrightarrow{\quad y_\tau \quad} & P_{\tau,\mathrm{hyp}}(Sm_S) \\ {\scriptstyle y_\sigma}\downarrow & \nearrow & \\ P_{\sigma,\mathrm{hyp}}(Sm_S), & {\scriptstyle \epsilon^*} & \end{array}$$

where $y_\sigma$ and $y_\tau$ are the composite of the Yoneda embedding with $\sigma$ and $\tau$-sheafifications respectively. Hence, if $\sigma, \tau$ are subcanonical, $\epsilon^*$ preserves representable sheaves. Since $\epsilon^*$ preserves finite products, and the categories are equipped with the Cartesian monoidal structures, it is symmetric monoidal.

2.2.2. We note that the adjunction above descends to the motivic categories.

**Lemma 2.9.** *In the adjunction:*

$$\epsilon^* : P_{\sigma,\mathrm{hyp}}(Sm_S) \rightleftarrows P_{\tau,\mathrm{hyp}}(Sm_S) : \epsilon_*,$$

$\epsilon^*$ *preserves* $\mathbf{A}^1$*-weak equivalences and* $\epsilon_*$ *preserves homotopy invariant objects.*

*Proof.* If $F \in P_{\tau,\mathrm{hyp}}(Sm_S)$ is homotopy invariant, $\epsilon_*(F)(X \times \mathbf{A}^1) \simeq F(X \times \mathbf{A}^1) \simeq F(X)$ for $X \in Sm_S$, so $\epsilon_*$ preserves homotopy invariant objects. To show $\epsilon^*$ preserves $\mathbf{A}^1$-weak equivalences, we note that by general considerations ([45, Proposition 5.5.4.15]), the class of $\mathbf{A}^1$-weak equivalences is the strong saturation of the maps $\{X \times \mathbf{A}^1 \to X\}$. By [8, Lemma 2.10] and the fact that $\epsilon^*$ preserves colimits, it suffices to prove that $\epsilon^*(X \times \mathbf{A}^1 \to X)$ is an $\mathbf{A}^1$-weak equivalence, which follows from the fact that $\epsilon^*$ preserves products. $\square$

**Proposition 2.10.** *The adjunction of Lemma 2.9 descends to an adjunction:*

$$\epsilon^* : H_\sigma(S) \rightleftarrows H_\tau(S) : \epsilon_*,$$

*such that the following diagram of adjunctions commutes:*

(7)
$$\begin{array}{ccc} & \xrightarrow{\quad \epsilon^* \quad} & \\ H_\sigma(S) & \xleftarrow{\quad \epsilon_* \quad} & H_\tau(S) \\ {\scriptstyle L_{\tau,m}}\Big\Updownarrow{\scriptstyle i} & & {\scriptstyle i}\Big\Updownarrow{\scriptstyle L_{\tau,m}} \\ P_{\sigma,\mathrm{hyp}}(Sm_S) & \xleftarrow[\epsilon^*]{\epsilon_*} & P_{\tau,\mathrm{hyp}}(Sm_S). \end{array}$$

*Proof.* Since $\epsilon_*$ preserves homotopy invariant presheaves by Lemma 2.9, the diagram of right adjoints commute. The corresponding left adjoints commutes by uniqueness of adjoints. $\square$

Upon taking stabilization we obtain the following comparison result.

**Proposition 2.11.** *The adjunction of Proposition 2.10 induces an adjunction:*

$$\epsilon^* : \mathrm{SH}_\sigma(S) \rightleftarrows \mathrm{SH}_\tau(S) : \epsilon_*,$$

*which fits into the following commutative diagram of adjunctions:*

(8)

$$
\begin{array}{ccc}
\mathrm{SH}_\sigma(S) & \xleftarrow{\ \epsilon^* \ }{\overset{\epsilon_*}{\longleftarrow}} & \mathrm{SH}_\tau(S) \\
\Sigma^\infty_{T,+} \Big\uparrow \Big\downarrow \Omega^\infty_T & & \Omega^\infty_T \Big\downarrow \Big\uparrow \Sigma^\infty_{T,+} \\
\mathrm{H}_\sigma(S) & \xleftarrow[\epsilon^*]{\epsilon_*} & \mathrm{H}_\tau(S).
\end{array}
$$

*Proof.* The adjunction of (7) gives rise to the adjunction of (8) via the universal property of stabilization given in Theorem 2.4, which also leads to the diagram of left adjoints commuting. By uniqueness of adjoints, the right adjoints also commute. □

*Remark* 2.12. From the commutation of the left adjoints in Proposition 2.11, we get an equivalence of functors $\Sigma_T \epsilon^* \simeq \epsilon^* \Sigma_T$, and likewise for the right adjoints $\epsilon_* \Sigma_T^{-1} \simeq \Sigma_T^{-1} \epsilon_*$. The latter furnishes natural equivalences:

$$\epsilon_* \simeq \epsilon_* \Sigma_T^{-1} \Sigma_T \simeq \Sigma_T^{-1} \epsilon_* \Sigma_T,$$

and thus:

$$\Sigma_T \epsilon_* \simeq \epsilon_* \Sigma_T.$$

By the same reasoning:

$$\epsilon^* \simeq \epsilon^* \Sigma_T \Sigma_T^{-1} \simeq \Sigma^T \epsilon^* \Sigma_T^{-1},$$

and:

$$\Sigma_T^{-1} \epsilon^* \simeq \epsilon^* \Sigma_T^{-1}.$$

Since $\epsilon^*$ and $\epsilon_*$ are exact and thus commute with $\Sigma^{1,0}$ and $\Sigma^{-1,0}$ we deduce the natural equivalence:

$$\Sigma^{p,q} \epsilon_* \epsilon^* \simeq \epsilon_* \epsilon^* \Sigma^{p,q},$$

for all $p, q \in \mathbf{Z}$ in the $\mathbf{P}^1$-stable motivic homotopy categories.

2.2.3.   We will call the functor:

$$\epsilon^* : \mathrm{SH}_\sigma(S) \to \mathrm{SH}_\tau(S),$$

the *τ-localization* functor. This terminology is justified by the next proposition. Let us define:

$$W_\tau := \{ \mathrm{L}_{\mathbf{m},\sigma}(U \to X) : X \in \mathrm{Sm}_S, U \to X \text{ is a } \tau\text{-hypercover} \},$$

and:

$$\Sigma^\infty_{T,+} W_\tau := \{ \Sigma^{p,q} \Sigma^\infty_{T,+} \mathrm{L}_{\mathbf{m},\sigma}(U \to X) : X \in \mathrm{Sm}_S, U \to X \text{ is a } \tau\text{-hypercover of } X, p, q \in \mathbf{Z} \}.$$

**Proposition 2.13.** *The functors:*

$$\epsilon^* : \mathrm{H}_\sigma(S) \to \mathrm{H}_\tau(S) \text{ and } \epsilon^* : \mathrm{SH}_\sigma(S) \to \mathrm{SH}_\tau(S),$$

*are localizations at $W_\tau$ and $\Sigma^\infty_{T,+} W_\tau$, respectively. In particular, their right adjoints are fully faithful.*

*Proof.* For the unstable case, $\epsilon^*$ and localization at $W_\tau$ satisfy the same universal properties via Proposition 2.1. In slightly more detail, the composite of functors:

$$\mathrm{Sm}_S \to \mathrm{H}_\sigma(S) \to \mathrm{H}_\sigma(S)[W_\tau^{-1}],$$

satisfies the universal property for $\mathrm{H}_\tau(S)$ as stated in Proposition 2.1. This follows similarly for the stabilized version via Theorem 2.6. □

As a consequence of Proposition 2.13 we adopt the following terminology: If $X \in H_\sigma(S)$ is in the full subcategory $H_\tau(S)$ (equivalently, it satisfies $X \simeq \epsilon_* \epsilon^* X$), then $X$ is called a *$\tau$-local object* in $H_\sigma(S)$. Similarly, $E \in SH_\sigma(S)$ is a *$\tau$-local object* in $SH_\sigma(S)$ if it is in the full subcategory $SH_\tau(S)$.

2.2.4. We give a characterization of $\tau$-local objects in terms of $S^1$-spectra, i.e., presheaves of spectra; we will discuss this $\infty$-category in more detail in §4.0.2. Recall $SH_\sigma(S)$ is enriched over the $\infty$-category of spectra. We denote by $\text{maps}(E, F)$ the *mapping spectrum* corresponding to this enrichment. If $E \in SH_\sigma(S)$, define $\Omega^\infty_{\mathbf{G}_m, \sigma} E$ as the presheaf of spectra:

$$X \mapsto \text{maps}(\Sigma^\infty_T X_+, \Sigma^{p,q} E).$$

This presheaf of spectra satisfies $\sigma$-hyperdescent and is $\mathbf{A}^1$-invariant.

**Lemma 2.14.** *An object $E \in SH_\sigma(S)$ is $\tau$-local if and only if $\Omega^\infty_{\mathbf{G}_m, \sigma} \Sigma^{p,q} E$ satisfies $\tau$-hyperdescent for all integers $p, q \in \mathbf{Z}$.*

*Proof.* This follows since for $X \in Sm_S$ the homotopy groups of $\text{maps}(\Sigma^\infty_+ X, \Sigma^{p,q} E)$ are computed as:

$$\epsilon_* \text{maps}(\Sigma^\infty_T X_+, \Sigma^{p,q} E) \simeq [\Sigma^{*,0} \Sigma^\infty_T X_+, \Sigma^{p,q} E]_{SH_\sigma(S)}.$$

$\square$

2.2.5. The reason why $\epsilon^*$ *does not deserve* to be called the $\tau$-sheafification functor is because the diagram:

(9)
$$
\begin{array}{ccc}
SH_\sigma(S) & \xrightarrow{\epsilon^*} & SH_\tau(S) \\
\Big\downarrow{\scriptstyle \Omega^\infty_{\mathbf{G}_m, \sigma}} & & \Big\downarrow{\scriptstyle \Omega^\infty_{\mathbf{G}_m, \tau}} \\
SH^{S^1}_\sigma(S) & \xrightarrow{\epsilon^*} & SH^{S^1}_\tau(S),
\end{array}
$$

*does not* commute. Here $SH^{S^1}_\sigma(S)$ is the $\infty$-category of homotopy invariant presheaves of spectra satisfying $\sigma$-hyperdescent. We offer an example to clarify the situation.

**Example 2.15.** The spectrum $\Sigma^{0,q} M\mathbf{Z}$ represents motivic cohomology in $SH_{Nis}(S)$ with Tate twist $q \in \mathbf{Z}$. By the cancellation theorem [83] there is an equivalence:

$$\Omega^\infty_{\mathbf{G}_m, Nis} \Sigma^{0,q} M\mathbf{Z} \simeq \mathbf{Z}^{tr}(q).$$

Here, the chain complex of presheaves with transfer $\mathbf{Z}^{tr}(q)$ is viewed as an object of $SH^{S^1}_{Nis}(S)$ via the Dold-Kan corespondence. If $\tau = $ ét, then $\epsilon^* M\mathbf{Z}$ computes étale motivic cohomology, see Corollary 2.20. Hence $\Omega^\infty_{\mathbf{G}_m, \text{ét}} \Sigma^{0,q} \epsilon^* M\mathbf{Z} \neq 0$ for $q < 0$ since étale motivic cohomology and étale cohomology coincide for torsion coefficients, cf. Theorem 2.19 . However, $\mathbf{Z}^{tr}(q)$ and thus $\epsilon^* \mathbf{Z}^{tr}(q)$ are 0 for $q < 0$ since negative weight motivic cohomology vanishes.

2.3. **Continuity and localization.** Let $Sch'_S$ be subcategory of the category of Noetherian schemes of finite dimension over $S$ which is, furthermore, *adequate* in the sense of [13, 2.0] and contains Henselizations of schemes. In this situation, we have access to the full six functors formalism as proved by Ayoub in [3] and extended in [13], which we now rapidly review. We refer to [13, Theorem 2.4.50] for an exact form of this formalism that we need; see also [15, Appendix A] for a succint summary.

We have a functor

(10)
$$M : (Sch'_S)^{op} \to Cat_\infty$$

which satisfies the conditions to be *premotivic* as defined in [15, Definition A.1.1] with $\mathcal{P} = Sm$, i.e., the pullback functor $p^*$ admits a left adjoint $p_\#$ whenever $p$ is smooth. Although [15] phrases their axioms in terms of triangulated categories, an extensive discussion of

these axioms in the setting of $\infty$-categories, and the proof that SH is such an example, can be found in Khan's thesis [37, Chapter 2, Section 3].

2.3.1. Two properties of the functor (10) that we will later use are *continuity* and *localization*. Let us review the first property. Let $c = (c_i)_{i \in I}$ be a collection of Cartesian sections of M. We denote by $M_c(X) \subset M(X)$ the smallest thick subcategory [2] of $M(X)$ which contains $f_\# f^* c_{i,X}$ for any $f : Y \to X$ smooth. Following [14, Definition 2.3], we call objects in $M_c(X)$ *c-constructible*. We say that M is *c-generated* if for all $X \in \mathrm{Sch}'_S$, the $\infty$-category $M(X)$ is generated by $M_c(X)$ under all small colimits.

**Definition 2.16.** Suppose that $\mathscr{F} \subset (\mathrm{Sch}'_S)^{\Delta^1}$ is a collection of morphisms in $\mathrm{Sch}'$. We say that M is *continuous with respect to* $\mathscr{F}$ if for any diagram $X : I \to \mathrm{Sch}'_S$ where $I$ is a small cofiltered category and the transition maps are in $\mathscr{F}$ and the limit $X := \lim X_i$ exists in $\mathrm{Sch}'_S$ the canonical map

$$M_c(X') \to \lim_I M_c(X_i).$$

is an equivalence. We say that $M$ satisfies *localization* if for any closed immersion $i : Z \hookrightarrow X$ and its open complement $j : U \to X$ we have a cofiber sequence of $\infty$-categories:

$$M(U) \xrightarrow{j_\#} M(X) \xrightarrow{i^*} M(Z).$$

2.3.2. We will make good use of the following lemma to reduce to the case of fields.

**Lemma 2.17.** *Suppose that we have a functor* $M : (\mathrm{Sch}'_S)^{\mathrm{op}} \to \mathrm{Cat}_\infty$ *of the form* (10) *which satisfies continuity and localization. Then for any* $X \in \mathrm{Sch}'_S$, *the functor*

$$\prod i_x^* : M(X) \to \prod_{i_x : \mathrm{Spec}\, k \to X} M(k),$$

*is conservative.*

*Proof.* This is paraphrase of [6, Corollary 14], which is a consequence of the six functors formalism of [3] and [13]. □

2.4. **Motives.** By a theory of "motives" we mean the stable $\infty$-category of modules over the motivic cohomology spectrum constructed by the third author. Here is a summary of its basic properties"

**Theorem 2.18** (Spitzweck [75]). *Let* $\mathrm{Sch}'$ *denote the category of Noetherian separated schemes of finite dimension. For each* $S \in \mathrm{SH}(S)$ *denote by* $M\mathbf{Z}_S \in \mathrm{SH}(S)$ *the motivic cohomology spectrum constructed in* [75]. *Then the following properties hold.*

(1) $M\mathbf{Z}_S$ *defines a Cartesian section of* $\mathrm{CAlg}(\mathrm{SH}) \to \mathrm{Sch}'$, *i.e., each* $M\mathbf{Z}_S$ *is a canonically an* $\mathcal{E}_\infty$-*algebra and for every map* $p : T \to S$ *the canonical map* $p^* M\mathbf{Z}_S \to M\mathbf{Z}_T$ *is an equivalence of* $\mathcal{E}_\infty$-*algebras.*

(2) *If* $S$ *is smooth over a field* $k$, $M\mathbf{Z}_S$ *is canonically equivalent as* $\mathcal{E}_\infty$-*algebras to the motivic cohomology spectrum in the sense of Voevodsky (see, for example,* [13, Definition 11.2.17]).

(3) *If* $S$ *is a Dedekind domain and* $X \in \mathrm{Sm}_S$, *then there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma_T^\infty X_+, M\mathbf{Z}(q)[p]) \cong H^{p,q}(X; \mathbf{Z})$$

*where* $H^{p,q}_\mathrm{m}(-; \mathbf{Z})$ *denotes the motivic cohomology groups of* [41].

(4) *Suppose that* $S = \mathrm{Spec}\, D$ *is the spectrum Dedekind of a domain of mixed characteristics, $i : \mathrm{Spec}\, k \hookrightarrow S$ an inclusion of a closed point, $j : U := S \setminus \{x\} \hookrightarrow S$ its open complement and $f : \mathrm{Spec}\, K \to S$ the canonical map from the spectrum of the fraction field of $D$. Then there are canonical equivalences*
   (a) $i^! M\mathbf{Z}_S \simeq M\mathbf{Z}_k(-1)[-2]$ *in* $\mathrm{SH}(k)$, *and*

---

[2] Recall that if C is a stable $\infty$-category, a full stable subcategory D $\subset$ C is called *thick* if it is a stable subcategory and contains all retracts of objects.

(b) $f^* M\mathbf{Z}_S \simeq M\mathbf{Z}_K$ in $\mathrm{SH}(K)$.

(5) *In the notation of (3), there is a canonical equivalence in* $\mathrm{SH}^{S^1}(S)$: $\Omega_T^\infty M\mathbf{Z}_S(1) \simeq \Sigma^{-1} \mathscr{O}_S^\times$.

We denote by $\mathrm{Mod}_{M\mathbf{Z}_S}$ the stable $\infty$-category of modules over the spectrum $M\mathbf{Z}_S$. These assemble into a premotivic category:

$$\mathrm{Mod}_{M\mathbf{Z}} : \mathrm{Sch}'^{\mathrm{op}} \to \mathrm{Cat}_\infty,$$

which satisfies continuity and localization as discussed in §2.3; see [75, §10] for details.

2.4.1. Let $\tau$ be a topology finer than Nis and consider the adjunction (19). By the $\tau$-*motivic cohomology spectrum* we mean the spectrum $\epsilon_* \epsilon^* M\mathbf{Z}_S \in \mathrm{SH}(S)$. We will be most interested in $\tau = \text{ét}$, which we now make more concrete in some situations. Let $\mathrm{D}(S_{\text{ét}}, R)$ be the unbounded derived $\infty$-category of étale sheaves on the small étale site of $S$ with coefficients in $R$; its homotopy category is the unbounded derived triangulated category of étale sheaves of $R$-modules on the small étale site. We let $\mathrm{DM}_\tau^{\mathrm{eff}}(S, R)$ (resp. $\mathrm{DM}(S, R)$) the stable $\infty$-category of Voevodsky motives; for a construction in the language used in this paper see [8, §14]. We have the following "relative rigidity theorem," first proved by Suslin over a field and later generalized by Ayoub and Cisinski-Deglise.

**Theorem 2.19** (Suslin, Ayoub, Cisinski-Deglise). *Let $S$ be a noetherian scheme and $R$ a coefficient ring of characteristic $\ell$. If $\ell$ is invertible in the structure sheaf $\mathscr{O}_S$, then the following hold.*

(1) *The infinite suspension functor* $\Sigma_{\mathrm{tr}}^\infty : \mathrm{DM}_{\text{ét}}^{\mathrm{eff}}(S, R) \to \mathrm{DM}_{\text{ét}}(S, R)$ *is an equivalence.*

(2) *There is a symmetric monoidal functor* $\rho_! : \mathrm{D}(S_{\text{ét}}, R) \to \mathrm{DM}_{\text{ét}}(S, R)$ *which is an equivalence of categories with inverse* $\rho^* : \mathrm{DM}_{\text{ét}}(S, R) \to \mathrm{D}(S_{\text{ét}}, R)$ *induced by restriction to the small étale site.*

*Proof.* Part (1) is [15, Corollary 4.1.2] and (2) is [5, Theorem 4.1], [15, Theorem 4.5.2]. □

2.4.2. We obtain the following corollary about étale motivic cohomology. We shall set $\tau = \text{ét}$ and consider the adjunction (1).

**Corollary 2.20.** *Let $k$ be a field and suppose that $m$ is an integer prime to the characteristic of $k$. Then for any essentially smooth scheme over a field $S$, the spectrum $\epsilon_* \epsilon^* M\mathbf{Z}/m_S$ represents étale motivic cohomology, i.e. there is a canonical isomorphism*

$$H_{\text{ét}}^p(S, \mu_m^{\otimes q}) \cong [\mathbf{1}_S, \Sigma^{p,q} \epsilon_* \epsilon^* M\mathbf{Z}/m]_{\mathrm{SH}(S)}.$$

*Proof.* Recall that for any topology $\tau$ finer than Nis, and any coefficient ring $R$, we have an adjunction $R_{\mathrm{tr}} : \mathrm{SH}_\tau(S) \rightleftarrows \mathrm{DM}_\tau(S; R) : u_{\mathrm{tr}}$ (see for example the discussion in [30, (4.1)] and the formula for motivic cohomology as in [21]). By Theorem 2.18.2, $M\mathbf{Z}_S$ represents Voevodsky's motivic cohomology spectrum, i.e., an equivalence $MR \simeq u_{\mathrm{tr}} R_{\mathrm{tr}}(\mathbf{1})$. The claim then follows immediately from Theorem 2.19.2 above for $R = \mathbf{Z}/m$, which implies that $\epsilon_* \epsilon^* u_{\mathrm{tr}} R_{\mathrm{tr}}(\mathbf{1})$ exactly represents the étale cohomology. □

## 3. THE SLICE COMPARISON PARADIGM

Let $\mathsf{E} \in \mathrm{SH}(S)$ be a motivic spectrum. The counit map of the adjunction:

(11) $$\epsilon^* : \mathrm{SH}(S) \rightleftarrows \mathrm{SH}_{\text{ét}}(S) : \epsilon_*,$$

is always an equivalence since $\epsilon_*$ is fully faithful.

We are asking how far away the unit map:

(12) $$\eta : \mathsf{E} \to \epsilon_* \epsilon^* \mathsf{E},$$

is from being an equivalence; *a priori* we only know this when $E = 0$.

A target theorem involves the inversion of some element $\alpha$. The map $\eta$ factors through:

$$\mathsf{E}[\alpha^{-1}] \to \epsilon_* \epsilon^* \mathsf{E},$$

which under favorable conditions will turn out to be an equivalence. Our approach to this question employs the slice spectral sequence and its étale localization. The theory of slices

in motivic homotopy theory has been developed in recent years since its introduction in Voevodsky's "open problems" article [81]. We refer to [65, §2], [66, §3], and [67, §3] for basic properties of slices, some of which we review in a slightly more general setting. For a geometric incarnation in terms of the coniveau tower, see [42], and [43].

For the $q$-th effective cover $f_q E$ of E we can consider the unit map $\eta_q : f_q E \to \epsilon_* \epsilon^* f_q E$. Moreover, applying $\epsilon_* \epsilon^*$ to the natural map $f_q E \to E$ produces $\eta^q : \epsilon_* \epsilon^* f_q E \to \epsilon_* \epsilon^* E$. By passing to the colimits of the diagrams afforded by $\eta_q$ and $\eta^q$ we obtain a factorization of (12):

$$E \simeq \operatorname{colim} f_q E \xrightarrow{\eta_\infty} \operatorname{colim} \epsilon_* \epsilon^* f_q E \xrightarrow{\eta^\infty} \epsilon_* \epsilon^* E.$$

Hence we can break down the question about $\eta$ into two parts:

$\eta_\infty$ The upshot is that the unit maps $\{\eta_q\}_{q \in \mathbf{Z}}$ induce a map from the slice filtration $\{f_q E\}_{q \in \mathbf{Z}}$ of E to the "étale slice filtration" $\{\epsilon_* \epsilon^* f_q E\}_{q \in \mathbf{Z}}$ of $\operatorname{colim} \epsilon_* \epsilon^* f_q E$. We analyze the induced map between the associated spectral sequences. It will never be an isomorphism unless one inverts elements in the slice spectral sequence for E. These are daring operations — on the one hand, we localize a spectral sequence which will generally destroy convergence properties. On the other hand, we have "motivically sheafified" (in the sense of §3.0.7) the slice spectral sequence, and so we need to produce new convergence statements.

$\eta^\infty$ If $\epsilon_*$ commutes past sequential colimits then $\eta^\infty$ is an equivalence. This will typically not happen because $\epsilon_*$ is a right adjoint, but it will be the case under the assumption of finite cohomological dimension.

We address the issues with $\eta_\infty$ in §4 and the issue with $\eta^\infty$ in §5.

3.0.1. Suppose that $E \in \operatorname{CAlg}(\operatorname{SH}(S))$. We begin with a simple generalization of slices to E-module slices. Let $\operatorname{SH}^{\mathrm{eff}}(S)$ be the stable $\infty$-category generated by all suspension spectra $\Sigma_T^\infty X_+$ under retracts and colimits, i.e., the localizing subcategory generated by these objects. For $q \in \mathbf{Z}$, let $\Sigma_T^q \operatorname{SH}^{\mathrm{eff}}(S)$ denote the full localizing subcategory of $\operatorname{SH}(S)$ generated by $\Sigma_T^q E$, where $E \in \operatorname{SH}^{\mathrm{eff}}(S)$. These categories form an exhaustive filtration:

$$\cdots \subset \Sigma_T^q \operatorname{SH}^{\mathrm{eff}}(S) \subset \Sigma_T^{q-1} \operatorname{SH}^{\mathrm{eff}}(S) \subset \cdots \subset \operatorname{SH}^{\mathrm{eff}}(S) \subset \Sigma_T^{-1} \operatorname{SH}^{\mathrm{eff}}(S) \subset \cdots \subset \operatorname{SH}(S).$$

The fully faithful embedding:

$$i_q : \Sigma_T^q \operatorname{SH}^{\mathrm{eff}}(S) \hookrightarrow \operatorname{SH}^{\mathrm{eff}}(S),$$

admits a right adjoint $r_q$; see Proposition 3.2 below for a slightly more general result. Let $f_q$ denote the composite $i_q r_q$.

If $E \in \operatorname{CAlg}(\operatorname{SH}(S))$ there is a presentably symmetric monoidal stable $\infty$-category of E-modules $(\operatorname{Mod}_E, \wedge_E, 1)$.

**Definition 3.1.** Suppose that $E \in \operatorname{CAlg}(\operatorname{SH}(S))$ and is furthermore effective[3], i.e., $E \in \operatorname{SH}(S)^{\mathrm{eff}}$. The stable $\infty$-category of *effective* E-*modules* $\operatorname{Mod}_E^{\mathrm{eff}}$ is the full stable subcategory of $\operatorname{Mod}_E$ generated by $E \wedge \Sigma_T^\infty X_+$ under retracts and colimits, with $X \in \operatorname{Sm}_S$, i.e., the full *localizing* subcategory of $\operatorname{Mod}_E$ generated by the said objects. In particular it is a stable $\infty$-category. For $q \in \mathbf{Z}$, let

$$\Sigma_T^q \operatorname{Mod}_E^{\mathrm{eff}} := \{\Sigma_T^q F : F \in \operatorname{Mod}_E^{\mathrm{eff}}\}.$$

**Proposition 3.2.** *Let E be an effective motivic spectrum. The fully faithful embedding* $i_q^{\mathsf E} : \Sigma_T^q \operatorname{Mod}_E^{\mathrm{eff}} \hookrightarrow \operatorname{Mod}_E$ *admits a right adjoint* $r_q^{\mathsf E} : \operatorname{Mod}_E \to \Sigma_T^q \operatorname{Mod}_E^{\mathrm{eff}}$, *which gives rise to the*

---

[3]With this assumption, the forgetful functor to $\operatorname{SH}(S)$ detects effectivity. Without this assumption, the forgetful functor will not if $E$ is, for example, the spectrum KGL.

*following commutative diagram of adjunctions:*

(13)

$$
\begin{array}{ccc}
\Sigma_T^q \mathrm{Mod}_{\mathsf{E}}^{\mathrm{eff}} & \xleftarrow[r_q^{\mathsf{E}}]{i_q^{\mathsf{E}}} & \mathrm{Mod}_{\mathsf{E}} \\
\mathsf{E}\wedge- \Big\uparrow\Big\downarrow u_{\mathsf{E}} & & u_{\mathsf{E}} \Big\downarrow\Big\uparrow \mathsf{E}\wedge- \\
\Sigma_T^q \mathrm{SH}^{\mathrm{eff}}(S) & \xleftarrow[i_q]{r_q} & \mathrm{SH}(S).
\end{array}
$$

*Proof.* $\mathrm{Mod}_{\mathsf{E}}$ and $\Sigma_T^q\mathrm{Mod}_{\mathsf{E}}$ are presentable stable $\infty$-categories and $i_q^{\mathsf{E}}$ preserves small coproducts and hence all small colimits by [46, Proposition 1.4.4.1.2]. By the $\infty$-categorical adjoint functor theorem [45, Corollary 5.5.2.9] $i_q^{\mathsf{E}}$ admits a right adjoint. It suffices to prove that the diagram of left adjoints commutes. But this follows from the fact that $E$ is effective, the definition of $\Sigma_T^q\mathrm{Mod}_{\mathsf{E}}^{\mathrm{eff}}$ as the full subcategory generated by suspensions of free E-modules together with the fact that left adjoints commute with colimits. $\qquad\square$

Because of the constant appeal to Proposition 3.2, we will assume $E$ to be effective for the rest of this section, unless stated otherwise.

3.0.2. The categories $\Sigma_T^q\mathrm{Mod}_{\mathsf{E}}^{\mathrm{eff}}$ assemble into a filtration:

$$
\cdots \subset \Sigma_T^q\mathrm{Mod}_{\mathsf{E}}^{\mathrm{eff}} \subset \Sigma_T^{q-1}\mathrm{Mod}_{\mathsf{E}}^{\mathrm{eff}} \subset \cdots \subset \mathrm{Mod}_{\mathsf{E}}^{\mathrm{eff}} \subset \Sigma_T^{-1}\mathrm{Mod}_{\mathsf{E}}^{\mathrm{eff}} \subset \cdots \subset \mathrm{Mod}_{\mathsf{E}}.
$$

Setting $f_q^{\mathsf{E}} := i_q^{\mathsf{E}} r_q^{\mathsf{E}}$ we obtain for every E-module M the E-*module slice tower*:

$$
\cdots \to f_{q+1}^{\mathsf{E}}\mathsf{M} \to f_q^{\mathsf{E}}\mathsf{M} \to \cdots \to f_0^{\mathsf{E}}\mathsf{M} \to f_{-1}^{\mathsf{E}}\mathsf{M} \to \cdots \to \mathsf{M}.
$$

We refer to $f_q^{\mathsf{E}}\mathsf{M}$ as the *q-th* E-*effective cover of* M, and the cofiber $s_q^{\mathsf{E}}\mathsf{M}$ of $f_q^{\mathsf{E}}\mathsf{M} \to f_{q+1}^{\mathsf{E}}\mathsf{M}$ as the *q-th* E-*slice of* M.

By construction, we have:

**Proposition 3.3.** *The counit* $i_q^{\mathsf{E}} r_q^{\mathsf{E}}\mathsf{M} = f_q^{\mathsf{E}}\mathsf{M} \to \mathsf{M}$ *witnesses* $f_q^{\mathsf{E}}\mathsf{M}$ *as the universal map from* $\Sigma_T^q\mathrm{Mod}_{\mathsf{E}}^{\mathrm{eff}}$.

3.0.3. The following result compares E-module slices with standard slices.

**Proposition 3.4.** *For* $q \in \mathbf{Z}$*, there are natural equivalences:*

$$
u_{\mathsf{E}} \circ f_q^{\mathsf{E}} \xrightarrow{\simeq} f_q \circ u_{\mathsf{E}} \text{ and } u_{\mathsf{E}} \circ s_q^{\mathsf{E}} \simeq s_q u_{\mathsf{E}}.
$$

*Proof.* To prove the first equivalence note that for $\mathsf{M} \in \mathrm{Mod}_{\mathsf{E}}$, $f_q u_{\mathsf{E}}(\mathsf{M}) \simeq i_q r_q u_{\mathsf{E}}\mathsf{M} \simeq i_q u_{\mathsf{E}} r_q^E \mathsf{M}$ by Proposition 3.2. We claim that $u_{\mathsf{E}} i_q \simeq i_q u_{\mathsf{E}}$. It holds by definition for $\mathsf{E} \wedge \Sigma_T^q \Sigma_T^\infty X_+$. So we are done if $i_q$ and $u_{\mathsf{E}}$ preserve colimits. This holds for $i_q$ since it is localizing, and for $u_{\mathsf{E}}$ since the adjunction $\mathsf{E} \wedge - : \mathrm{SH}(S) \rightleftarrows \mathrm{Mod}_{\mathsf{E}} : u_{\mathsf{E}}$ is monadic. The second equivalence follows readily from the first. $\qquad\square$

3.0.4. We review some basic properties of effective covers and slices for E-modules.

**Proposition 3.5.** *For all* $q \in \mathbf{Z}$*, the functors* $f_q^{\mathsf{E}}$ *and* $s_q^{\mathsf{E}}$ *preserve all small colimits.*

*Proof.* For E the sphere spectrum this is in [73, Corollary 4.6] and the argument is similar in our setting. These are exact functors of stable $\infty$-categories by construction. For $f_q^E$ we need only check that it preserves all small sums. Since $f_q^{\mathsf{E}}$ is the composite of a left adjoint $i_q^{\mathsf{E}}$ and a right adjoint $r_n^{\mathsf{E}}$, it suffices to consider $r_q^{\mathsf{E}}$. We conclude by Lemma 5.4 because $i_q^{\mathsf{E}}$ preserves compact objects. It follows that $s_q^{\mathsf{E}}$ preserves small colimits since it is the cofiber of $f_{q+1}^{\mathsf{E}} \to f_q^{\mathsf{E}}$. $\qquad\square$

**Proposition 3.6.** *For any* $\mathsf{M} \in \mathrm{Mod}_\mathsf{E}$ *and* $q \in \mathbf{Z}$, *the* $q$-th *slice* $s_q^\mathsf{E}\mathsf{M}$ *belongs to* $\Sigma_T^q \mathrm{Mod}_\mathsf{E}^{\mathrm{eff}}$ *and is furthermore right orthogonal to* $\Sigma_T^{q+1} \mathrm{Mod}_\mathsf{E}^{\mathrm{eff}}$.

*Proof.* For the first claim, we note that both $f_{q+1}^\mathsf{E}\mathsf{M}$ and $f_q^\mathsf{E}M$ belong to the full subcategory $\Sigma_T^q \mathrm{Mod}_\mathsf{E}^{\mathrm{eff}}$ of $\mathrm{Mod}_\mathsf{E}$. Since $\Sigma_T^q \mathrm{Mod}_\mathsf{E}^{\mathrm{eff}}$ is closed under cofibers, it follows that $s_q^\mathsf{E}\mathsf{M}$ is in $\Sigma_T^q \mathrm{Mod}_\mathsf{E}^{\mathrm{eff}}$.

For the second claim, it suffices to prove that:

$$\mathrm{Maps}_{\mathrm{Mod}_\mathsf{E}}(\Sigma_T^{q+1}\mathsf{E} \wedge \Sigma_T^\infty X_+, s_q^\mathsf{E}\mathsf{M}) \simeq 0.$$

This follows from the equivalences:

$$
\begin{aligned}
\mathrm{Maps}_{\mathrm{Mod}_\mathsf{E}}(\Sigma_T^{q+1}\mathsf{E} \wedge \Sigma_T^\infty X_+, f_q^\mathsf{E}\mathsf{M}) \quad &\simeq \quad \mathrm{Maps}_{\mathrm{Mod}_\mathsf{E}}(i_{q+1}^\mathsf{E} r_{q+1}^\mathsf{E} \Sigma_T^{q+1}\mathsf{E} \wedge \Sigma_T^\infty X_+, f_q^\mathsf{E}\mathsf{M}) \\
&\simeq \quad \mathrm{Maps}_{\Sigma_T^{q+1}\mathrm{Mod}_\mathsf{E}}(r_{q+1}^\mathsf{E}\Sigma_T^{q+1}\mathsf{E} \wedge \Sigma_T^\infty X_+, r_{q+1}^\mathsf{E} f_q^\mathsf{E}\mathsf{M}) \\
&= \quad \mathrm{Maps}_{\Sigma_T^{q+1}\mathrm{Mod}_\mathsf{E}}(r_{q+1}^\mathsf{E}\Sigma_T^{q+1}\mathsf{E} \wedge \Sigma_T^\infty X_+, r_{q+1}^\mathsf{E} i_q^\mathsf{E} r_q^\mathsf{E}\mathsf{M}) \\
&\simeq \quad \mathrm{Maps}_{\mathrm{Mod}_\mathsf{E}}(i_{q+1}^\mathsf{E} r_{q+1}^\mathsf{E}\Sigma_T^{q+1}\mathsf{E} \wedge \Sigma_T^\infty X_+, i_{q+1}^\mathsf{E} r_{q+1}^\mathsf{E} i_q^\mathsf{E} r_q^\mathsf{E}\mathsf{M}) \\
&= \quad \mathrm{Maps}_{\mathrm{Mod}_\mathsf{E}}(\Sigma_T^{q+1}\mathsf{E} \wedge \Sigma_T^\infty X_+, f_{q+1}^\mathsf{E} f_q^\mathsf{E}\mathsf{M}) \\
&\simeq \quad \mathrm{Maps}_{\mathrm{Mod}_\mathsf{E}}(\Sigma_T^{q+1}\mathsf{E} \wedge \Sigma_T^\infty X_+, f_{q+1}^\mathsf{E}\mathsf{M}),
\end{aligned}
$$

where the first equivalence follows because, by definition, $\Sigma_T^{q+1}\mathsf{E} \wedge \Sigma_T^\infty X_+ \simeq i_{q+1}^\mathsf{E} r_{q+1}^\mathsf{E}\Sigma_T^{q+1}\mathsf{E} \wedge \Sigma_T^\infty X_+$. The second equivalence is by the adjunction between $i_{q+1}^\mathsf{E}$ and $r_{q+1}^\mathsf{E}$, the third is by fully faithfulness of $i_{q+1}$, and the fourth and final is by Lemma 3.7. $\qquad\square$

**Lemma 3.7.** *The natural transformation* $f_{q'}^\mathsf{E} f_q^\mathsf{E} \to f_{q'}^\mathsf{E}$ *is invertible for* $q < q'$, *and* $f_{q'}^\mathsf{E} f_q^\mathsf{E} \to f_q^\mathsf{E}$ *is invertible for* $q' \leq q$. *Hence for* $q \leq q'$ *there is a natural equivalence* $s_{q'}^\mathsf{E} f_q^\mathsf{E} \xrightarrow{\simeq} s_{q'}^\mathsf{E}$.

*Proof.* These follow immediately from the universal property of $f_q^\mathsf{E}$ given in Proposition 3.3, the defining cofiber sequences for the slices, and commutativity of the effective covers. $\quad\square$

**Lemma 3.8.** *The natural map* $\mathrm{colim}_{q \to -\infty} f_q^\mathsf{E} \to id$ *is an equivalence of endofunctors of* $\mathrm{Mod}_\mathsf{E}$.

*Proof.* First note that the slice filtration on categories is exhaustive in the sense that:

$$\underset{q}{\mathrm{colim}}\, \Sigma_T^q \mathrm{Mod}_\mathsf{E}^{\mathrm{eff}} \simeq \mathrm{Mod}_\mathsf{E}.$$

Since the $\infty$-category $\mathrm{Mod}_\mathsf{E}$ is generated by $\mathsf{E} \wedge \Sigma^{2n,n}\Sigma_T^\infty X_+, n \in \mathbf{Z}$, where $X \in \mathrm{Sm}_S$, it suffices to prove that for any $\mathsf{E}$-module $\mathsf{M}$ and any $X \in \mathrm{Sm}_S$ we have:

$$\mathrm{Maps}_{\mathrm{Mod}_\mathsf{E}}(\mathsf{E} \wedge \Sigma^{2n,n}\Sigma_T^\infty X_+, \underset{q \to -\infty}{\mathrm{colim}} f_q^\mathsf{E} M) \simeq \mathrm{Maps}_{\mathrm{Mod}_\mathsf{E}}(\mathsf{E} \wedge \Sigma^{2n,n}\Sigma_T^\infty X_+, \mathsf{M}).$$

By adjunction we are checking that:

$$\mathrm{Maps}_{\mathrm{SH}(S)}(\Sigma^{2n,n}\Sigma_T^\infty X_+, u_\mathsf{E} \underset{q \to -\infty}{\mathrm{colim}} f_q^\mathsf{E} M) \simeq \mathrm{Maps}_{\mathrm{SH}(S)}(\Sigma^{2n,n}\Sigma_T^\infty X_+, u_\mathsf{E} M).$$

Since $u_\mathsf{E}$ commutes with colimits and $u_\mathsf{E} \circ f_q^\mathsf{E} \simeq f_q \circ u_\mathsf{E}$ according to Proposition 3.4, this equivalence follows from the analogous statement for $f_q$ shown in [67, Lemma 3.1]. $\quad\square$

3.0.5.   The cofiber of $f_n^\mathsf{E} \to id$ is called the $(n-1)$-th $\mathsf{E}$-*effective cocover* of $\mathsf{E}$ and is denoted by $f_\mathsf{E}^{n-1}$ (see [66, 3.1] for a discussion in $\mathrm{SH}(S)$). This yields, for any $\mathsf{E}$-module $\mathsf{M}$, a diagram where the horizontal rows are cofiber sequences (see also [67, 3.9]):

(14)
$$
\begin{array}{ccccc}
f^{\mathsf{E}}_{q+1}\mathsf{M} & \longrightarrow & \mathsf{M} & \longrightarrow & f^q_{\mathsf{E}}\mathsf{M} \\
\downarrow & & \downarrow & & \downarrow \\
f^{\mathsf{E}}_q\mathsf{M} & \longrightarrow & \mathsf{M} & \longrightarrow & f^{q-1}_{\mathsf{E}}\mathsf{M} \\
\downarrow & & \downarrow & & \downarrow \\
s^{\mathsf{E}}_q\mathsf{M} & \longrightarrow & 0 & \longrightarrow & s^{\mathsf{E}}_q\mathsf{M}[1],
\end{array}
$$

and the cofiber sequence (see also [67, 3.11]):

$$
\lim_{q\to+\infty} f^{\mathsf{E}}_q\mathsf{M} \to \mathsf{M} \to \lim_{q\to+\infty} f^q_{\mathsf{E}}\mathsf{M}.
$$

The E-*slice completion of* M is defined by setting $\mathrm{sc}_{\mathsf{E}}(\mathsf{M}) := \lim f^q_{\mathsf{E}}\mathsf{M}$. The E-module M is *slice complete* if $\mathsf{M} \to \mathrm{sc}_{\mathsf{E}}(\mathsf{M})$ is an equivalence, or equivalently $\lim_{q\to+\infty} f^q_{\mathsf{E}}\mathsf{M} \simeq 0$.

3.0.6. *Slice spectral sequence.* If M is an E-module, we write

$$
\pi_{p,q}(\mathsf{M})(X) := [\Sigma^{p,q}\mathsf{E} \wedge \Sigma^\infty_T X_+, \mathsf{M}]_{\mathrm{Mod}_{\mathsf{E}}}.
$$

If $X$ is the base scheme we just write $\pi_{p,q}(\mathsf{M}) := [\Sigma^{p,q}\mathsf{E}, \mathsf{M}]_{\mathrm{Mod}_{\mathsf{E}}}$. So if E is the unit sphere **1**, this is just the group $[\Sigma^{p,q}\mathbf{1}, \mathsf{M}]_{\mathrm{SH}(S)}$. When $S = \mathrm{Spec}\, k$ is a field, we recover the value of the homotopy sheaf on $\mathrm{Spec}\, k$: $\underline{\pi}^{\mathrm{Nis}}_{p,q}(\mathsf{M})(k) \simeq \pi_{p,q}(\mathsf{M})$. The E-module slice tower yields the trigraded E-slice spectral sequence:

$$
E^1_{p,q,w}(\mathsf{M})(X) = \pi_{p,w}(s^{\mathsf{E}}_q\mathsf{M})(X) \Rightarrow \pi_{p,w}(\mathsf{M})(X).
$$

The exact couple that gives rise to this spectral sequence (with our preference for its indexing) is written in, for example, [65, (11)]. The differentials go:

$$
d_r(p,q,w) : \pi_{p,w}(s^{\mathsf{E}}_q\mathsf{M})(X) \to \pi_{p-1,w}(s^{\mathsf{E}}_{q+r}\mathsf{M})(X).
$$

We view the trigraded slice spectral sequence as a family of bigraded spectral sequences indexed by the weight $w$. This spectral sequence is always conditionally convergent to the homotopy groups of the slice completion $\pi_{*,*}\mathrm{sc}_{\mathsf{E}}(\mathsf{M})(X)$. If a conditionally convergent spectral sequence collapses at a finite stage, then it is strongly convergent [9, Theorem 7.1]. Convergence of the slice spectral sequence is discussed at length in [44], [66], and [81].

3.0.7. *ét-localization of slices.* Suppose $\tau$ is finer than the Nisnevich topology and there is an adjunction $\epsilon^* : \mathrm{SH}(S) \rightleftarrows \mathrm{SH}_\tau(S) : \epsilon_*$. We define the $q$-th $\tau$-*effective cover* of a motivic spectrum E to be $\epsilon_*\epsilon^* f_q\mathsf{E}$ and refer to the corresponding filtration as the $\tau$-*slice filtration*. Using the cofiber sequence $f_{q+1}\mathsf{E} \to f_q\mathsf{E} \to s_q\mathsf{E}$ we obtain the commutative diagram:

(15)
$$
\begin{array}{ccccc}
f_{q+1}\mathsf{E} & \longrightarrow & f_q\mathsf{E} & \longrightarrow & s_q\mathsf{E} \\
\downarrow & & \downarrow & & \downarrow \\
\epsilon_*\epsilon^* f_{q+1}\mathsf{E} & \longrightarrow & \epsilon_*\epsilon^* f_q\mathsf{E} & \longrightarrow & \epsilon_*\epsilon^* s_q\mathsf{E}.
\end{array}
$$

We call $\epsilon_*\epsilon^* s_q\mathsf{E}$ the $q$-th $\tau$-*slice* because the bottom row is also a cofiber sequence since $\epsilon_*\epsilon^*$ is an exact functor; the object $\epsilon_*\epsilon^* f^q\mathsf{E}$ is the $q$-th $\tau$-*coeffective cover*. The $\tau$-*slice tower* is given by:

(16)
$$
\cdots \to \epsilon_*\epsilon^* f_q\mathsf{E} \to \epsilon_*\epsilon^* f_{q-1}\mathsf{E} \to \cdots.
$$

We refer to the corresponding spectral sequence:

(17)
$$
E^{1,\tau}_{p,q,w}(\mathsf{E}) := \pi_{p,n}(\epsilon_*\epsilon^* s_q\mathsf{E}) \Rightarrow \pi_{p,w}(\mathrm{colim}\, \epsilon_*\epsilon^* f_q\mathsf{E}),
$$

as the $\tau$-*slice spectral sequence*. As above, this give a bigraded spectral sequence for every fixed weight $w$. Some of the issues at stake in this paper are to determine the convergence properties of (17) and identify its filtered target groups with $\pi_{p,w}(\epsilon_*\epsilon^*\mathsf{E})$.

The above has an E-module analogue. If $\mathsf{E} \in \mathrm{CAlg}(\mathrm{SH}(S))$ then $\epsilon^*\mathsf{E} \in \mathrm{CAlg}(\mathrm{SH}_\tau(S))$ since $\epsilon^*$ is a monoidal functor. Thus it makes sense to form the $q$-*th* $\mathsf{E}^\tau$-*effective cover* $f_q^{\mathsf{E}^\tau} M := \epsilon_*\epsilon^* f_q^{\mathsf{E}} M$ and the slices $s_q^{\mathsf{E}^\tau} M := \epsilon_*\epsilon^* s_q^{\mathsf{E}} M$.

3.0.8. There is another possibility for what one might call "étale slices." Let $\mathrm{SH}_{\text{ét}}^{\mathrm{eff}}(S)$ be the stable $\infty$-category generated by the suspension spectra $\Sigma_T^\infty X_+$ under retracts and colimits in $\mathrm{SH}_{\text{ét}}(S)$. We may define $\Sigma_T^q \mathrm{SH}_{\text{ét}}^{\mathrm{eff}}(S)$ as above and build a slice tower in the étale setting. Denote the effective covers by $f_q^{\text{ét}}$ and the slices by $s_q^{\text{ét}}$. The following proposition essentially follows from [81, Remark 2.1], but we elaborate it here for the reader's convenience.

**Proposition 3.9.** *Suppose $k$ is a field containing a primitive $\ell$-th root of unity where $\ell$ is prime to the exponential characteristic of $k$. Then $s_q^{\text{ét}}\epsilon^*\mathsf{MZ}/\ell \simeq 0$ for all $q \in \mathbf{Z}$.*

*Proof.* Since $H_{\text{ét}}^0(k; \mu_\ell) \simeq [\mathbf{1}, \Sigma^{0,1}\epsilon^*\mathsf{MZ}_\ell]_{\mathrm{SH}_{\text{ét}}}$ there is an invertible map $\tau_\ell^{\mathsf{MZ}} : \epsilon^*\mathsf{MZ}/\ell \to \Sigma^{0,1}\epsilon^*\mathsf{MZ}/\ell$ classifying a primitive $\ell$-th root of unity in $k$. By the proof of [65, Lemma 2.1], which makes no use of the Nisnevich topology, there is an equivalence $f_{q+1}^{\text{ét}}\Sigma^{0,1}\epsilon^*\mathsf{MZ}/\ell \xrightarrow{\simeq} \Sigma^{0,1} f_q^{\text{ét}}\epsilon^*\mathsf{MZ}/\ell$ and a commutative diagram:

$$(18) \qquad \begin{array}{ccc} f_{q+1}^{\text{ét}}\epsilon^*\mathsf{MZ}/\ell & \longrightarrow & f_q^{\text{ét}}\epsilon^*\mathsf{MZ}/\ell \\ {\scriptstyle f_{q+1}^{\text{ét}}(\tau_\ell^{\mathsf{MZ}})} \downarrow & & \downarrow {\scriptstyle \tau_\ell^{\mathsf{MZ}}} \\ f_{q+1}^{\text{ét}}\Sigma^{0,1}\epsilon^*\mathsf{MZ}/\ell & \xrightarrow{\ \simeq\ } & \Sigma^{0,1} f_q^{\text{ét}}\epsilon^*\mathsf{MZ}/\ell. \end{array}$$

The vertical arrows are invertible because multiplication by the class of $\tau_\ell^{\mathsf{MZ}}$ is invertible. Thus the top arrow is invertible and its cofiber $s_q^{\text{ét}}\epsilon^*\mathsf{MZ}/\ell \simeq 0$. $\qquad\square$

In fact, one can run the argument in [81, Remark 2.1] to show that $s_0^{\text{ét}}\epsilon^*\mathsf{MGL}/\ell \simeq s_0^{\text{ét}}\epsilon^*\mathsf{MZ}/\ell \simeq 0$. Hence this approach should be abandoned.

## 4. SOME ÉTALE CONNECTIVITY RESULTS

We come to the technical heart of the paper which concerns the convergence of the ét-slice spectral sequence. The main result that we will need is Corollary 4.19. For the rest of the paper, we will only be considering the adjunction:

$$(19) \qquad\qquad \epsilon^* : \mathrm{SH}(S) \rightleftarrows \mathrm{SH}_{\text{ét}}(S) : \epsilon_*.$$

We also employ the following conventions:

- $\mathsf{E}^{\text{ét}} := \epsilon_*\epsilon^*\mathsf{E}$ for the étale localization of $\mathsf{E} \in \mathrm{SH}(S)$,
- $\mathrm{H}(S)$, $\mathrm{SH}^{S^1}(S)$, $\mathrm{SH}(S)$ for $\mathrm{H}_{\mathrm{Nis}}(S)$, $\mathrm{SH}_{\mathrm{Nis}}^{S^1}(S)$, $\mathrm{SH}_{\mathrm{Nis}}(S)$, respectively. Objects in $\mathrm{SH}(S)$ will be called motivic spectra.

We will still have occasion to use $\epsilon^*\mathsf{E}$ when we make arguments internal to $\mathrm{SH}_{\text{ét}}(S)$. We are interested in the spectral sequence (17) for the étale topology. For $X \in \mathrm{Sm}_S$ this takes the form:

$$(20) \qquad E_{p,q,w}^{1,\text{ét}}(\mathsf{E})(X) := \pi_{p,w}(s_q \mathsf{E}^{\text{ét}})(X) \Rightarrow \pi_{p,w}(\mathrm{colim}\, f_q \mathsf{E}^{\text{ét}})(X).$$

We will refer to (20) as the *étale slice spectral sequence*, but be warned that it might not be what the reader expects, as discussed in §3.0.8 and further in Example 4.1. The goal of this section is to prove convergence results about this spectral sequence.

In order to access the convergence of the étale slice spectral sequence, we will need a bound on the connectivity of $\mathsf{E}^{\text{ét}}$ and its étale slices. This is in general a delicate issue, mainly due to the problems pointed out in §2.2.5 and the lack of a connectivity theorem

for the $\mathrm{SH}_{\text{ét}}(S)$ in the sense of [54]. When E is a Landweber exact motivic spectrum, we can produce a bound on the connectivity, measured differently. Producing this bound is achieved in Theorem 4.17. Before we proceed, let us examine the example in §2.2.5 in greater detail.

**Example 4.1.** Suppose $k$ is a field with exponential characteristic coprime to $\ell$. We view the mod-$\ell$ motivic cohomology spectrum $M\mathbf{Z}/\ell$ as a $\mathbf{G}_m$-spectrum in $\mathrm{SH}^{S^1}(S)$, i.e., a sequence of $\mathbf{A}^1$-invariant and Nisnevich-local presheaves of spectra $(\mathsf{E}_i)_{i\in\mathbb{N}}$ equipped with bonding maps $\mathsf{E}_i \to \Omega_{\mathbf{G}_m}\mathsf{E}_{i+1}$, each of which is an equivalence. From this point of view the constituent $S^1$-spectra (again thinking of chain complexes as spectra) of $\Sigma^{0,-1}M\mathbf{Z}/\ell$ are:

$$(0, \mathbf{Z}/\ell[1], \mathbf{Z}/\ell(1)[2], \mathbf{Z}/\ell(2)[3], ...).$$

For each $t \in \mathbb{N}$ we have a bonding map:

$$(21) \qquad \mathbf{Z}/\ell(t-1)[t] \to \Omega_{\mathbf{G}_m}\mathbf{Z}/\ell(t)[t+1],$$

which is an equivalence due to Voevodsky's cancellation theorem [83]. Taking the étale sheafification of each of these individual spectra gives us a sequence of $S^1$-spectra:

$$(22) \qquad (0, \mathrm{L}_{\text{ét}}\mathbf{Z}/\ell[1], \mathrm{L}_{\text{ét}}\mathbf{Z}/\ell(1)[2], \mathrm{L}_{\text{ét}\mathbf{Z}/\ell}(2)[3], ...),$$

each of which is still $\mathbf{A}^1$-invariant and in fact equivalent to $\mu_\ell^{\otimes t}[t+1]$ on account of the identification $\mathrm{L}_{\text{ét}}\mathbf{Z}/\ell(1) \simeq \mu_\ell$. Furthermore, and this is the crux point, we have two cases:

(1) each bonding map (21) induces an equivalence for $t > 1$:

$$\mathrm{L}_{\text{ét}}\mathbf{Z}/\ell(t-1)[t] \to \Omega_{\mathbf{G}_m}\mathrm{L}_{\text{ét}}\mathbf{Z}/\ell(t)[t+1].$$

(2) For $t = 1$ we have:

$$(23) \qquad \Omega_{\mathbf{G}_m}\mathrm{L}_{\text{ét}}\mathbf{Z}/\ell[1] \simeq \mu_\ell^{\otimes-1} \not\simeq 0.$$

Indeed, non-commutativity of the diagram in §2.2.5 boils down to this phenomenon — even if the levelwise étale sheafifcation preserves $\mathbf{A}^1$-invariance, the individual spectrum may no longer be a $\mathbf{G}_m$-spectrum but is, instead, a $\mathbf{G}_m$-*prespectrum* as we saw above — a notion we will revisit in the $\infty$-categorical setting in §4.1.1. Phrased another way, the étale localization functor *does not* commute with taking $\mathbf{G}_m$-loops unstably and only does so after taking spectrification again in the sense elaborated in Remark 2.12.

4.0.1. As Example 4.1 shows one cannot simply apply descent spectral sequence arguments to get connectivity bounds on $\mathsf{E}^{\text{ét}}$. To spell this out let us denote by:

$$\Omega^\infty_{\mathbf{G}_m,\text{ét}} : \mathrm{SH}_{\text{ét}}(S) \to \mathrm{SH}^{S^1}_{\text{ét}}(S),$$

the infinite loop functor in the etale topology. Each of the constituent $S^1$-spectra of $\mathsf{E}^{\text{ét}}$ is given by:

$$(\Omega^\infty_{\mathbf{G}_m,\text{ét}}\mathsf{E}^{\text{ét}}, \Omega^\infty_{\mathbf{G}_m,\text{ét}}\Sigma^{1,1}\mathsf{E}^{\text{ét}}, \Omega^\infty_{\mathbf{G}_m,\text{ét}}\Sigma^{2,2}\mathsf{E}^{\text{ét}}, \cdots).$$

From this we have a descent spectral sequence:

$$H^p_{\text{ét}}(X, \underline{\pi}^{\text{ét}}_{-q}\Omega^\infty_{\mathbf{G}_m,\text{ét}}\Sigma^{i,i}\mathsf{E}^{\text{ét}}) \Rightarrow [X, \mathbf{G}_m^{\wedge i} \wedge \mathsf{E}^{\text{ét}}[p+q]].$$

*A priori* we do not know any relationship between $\underline{\pi}^{\text{ét}}_{-q}(\Omega^\infty_{\mathbf{G}_m,\text{ét}}\Sigma^{i,i}\mathsf{E}^{\text{ét}})$ and its Nisnevich version $\underline{\pi}^{\mathrm{Nis}}_{-q}(\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\mathsf{E})$ since $\Omega^\infty_{\mathbf{G}_m,\text{ét}}\Sigma^{i,i}\mathsf{E}^{\text{ét}}$ is not, in general, the étale sheafification of the Nisnevich-local presheaf of spectra $\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\mathsf{E}$.

4.0.2.   We begin by examining more closely the category of $S^1$-spectra, $\mathrm{SH}_\tau^{S^1}(S)$, i.e., the $\infty$-category of $\tau$-local and $\mathbf{A}^1$-invariant presheaves of spectra. Let us denote by:

$$\mathrm{P}_{\mathrm{Spt}}(S) := \mathrm{Fun}(\mathrm{Sm}_S^{\mathrm{op}}, \mathrm{Spt}),$$

the $\infty$-category of presheaves of spectra on $\mathrm{Sm}_S$. The $\infty$-category $\mathrm{SH}_{\mathbf{A}^1}^{S^1}(S)$ (resp. $\mathrm{SH}_\tau^{S^1}(S)$) is the full subcategory of $\mathrm{P}_{\mathrm{Spt}}(S) := \mathrm{Fun}(\mathrm{Sm}_S^{\mathrm{op}}, \mathrm{Spt})$ spanned by those presheaves of spectra which are $\mathbf{A}^1$-invariant as in §2.1.1 (resp. $\tau$-local as in §2.1.2), with spectra instead of spaces.

We denote by:

$$\mathrm{L}_{\mathbf{A}^1} : \mathrm{P}_{\mathrm{Spt}}(S) \to \mathrm{P}_{\mathrm{Spt},\mathbf{A}^1}(S), \mathrm{L}_\tau : \mathrm{P}_{\mathrm{Spt}}(S) \to \mathrm{P}_{\mathrm{Spt},\tau}(S),$$

and:

$$\mathrm{L}_{\tau,\mathrm{m}} := \mathrm{L}_{\mathbf{A}^1}\mathrm{L}_\tau : \mathrm{P}_{\mathrm{Spt}}(S) \to \mathrm{SH}_\tau^{S^1}(S),$$

the $\mathbf{A}^1$-localization, $\tau$-localization, and motivic localization functors for presheaves of spectra, respectively. The notation is justified since these functors are just the $S^1$-stabilizations in the sense of [46, Section 1.4] of their unstable versions described in §2.1.1, §2.1.2, and §2.1.3.

In order to avoid confusion, we write

$$\mathrm{L}_{\text{ét},\mathrm{m}} = \mathrm{L}_{\mathbf{A}^1}\mathrm{L}_{\text{ét}} : \mathrm{SH}^{S^1}(S) \to \mathrm{SH}_{\text{ét}}^{S^1}(S),$$

for the $\mathbf{A}^1$-localization and ét-sheafification of presheaves of Nis-local spectra, while we write

$$\epsilon^* : \mathrm{SH}(S) \to \mathrm{SH}_{\text{ét}}(S)$$

for étale localization in the sense of Proposition 2.13.

4.0.3.   Before we proceed with motivic homotopy theory, we require some ingredients from the theory of presheaves of spectra. The first is a convergence statement for ét-local presheaves of spectra, see also [34, Lemma 3.4], [47, Corollary 1.3.3.11].

**Proposition 4.2.** *Let $k$ be a field such that $\mathrm{cd}_\ell(k) < \infty$, and let*

$$\cdots \to \mathsf{E}_{i+1} \to \mathsf{E}_i \to \mathsf{E}_{i-1} \to \cdots \to \mathsf{E}_0$$

*be a tower in $\mathrm{P}_{\mathrm{Spt}}(k)$. Let $\mathrm{fib}_i(\mathsf{E})$ denote the fiber of $\mathsf{E}_i \to \mathsf{E}_{i-1}$, and suppose that the following condition holds:*

- *Let $\mathrm{conn}(i)$ denote the connectivity of $\mathrm{fib}_i(\mathsf{E})$ in $\mathrm{P}_{\mathrm{Spt}}(\mathrm{Sm}_k)$. Then for any $n \geq 0$, there exists an $i \gg 0$ such that for any $i' \geq i$, we have $\mathrm{conn}(i') \geq n$.*

*Then for any $\nu \geq 1$ there is a natural equivalence:*

$$\mathrm{L}_{\text{ét}}(\lim_i \mathsf{E}_i / \ell^\nu) \xrightarrow{\simeq} \lim_i \mathrm{L}_{\text{ét}}(\mathsf{E}_i / \ell^\nu).$$

*Proof.* Denote by $\mathsf{E}/\ell^\nu$ the limit of the tower:

$$\cdots \to \mathsf{E}_{i+1}/\ell^\nu \to \mathsf{E}_i/\ell^\nu \to \mathsf{E}_{i-1}/\ell^\nu \to \cdots \to \mathsf{E}_0/\ell^\nu,$$

in $\mathrm{P}_{\mathrm{Spt}}(\mathrm{Sm}_k)$. We have canonical morphisms:

$$\mathsf{E}/\ell^\nu \to \mathsf{E}_i/\ell^\nu \to \mathrm{L}_{\text{ét}}\mathsf{E}_i/\ell^\nu.$$

The inclusion $\mathrm{P}_{\mathrm{Spt},\text{ét}}(k) \hookrightarrow \mathrm{P}_{\mathrm{Spt}}(k)$ preserves limits since it is a right adjoint, and thus $\lim \mathrm{L}_{\text{ét}}\mathsf{E}_i/\ell^\nu$ is ét-local. Our goal is then to prove that the map:

$$(24) \qquad\qquad \mathsf{E}/\ell^\nu \to \lim \mathrm{L}_{\text{ét}}\mathsf{E}_i/\ell^\nu,$$

witnesses an ét-localization. We first claim that for any $n \geq 0$ the map on homotopy sheaves:

$$\underline{\pi}_n^{\text{ét}}(\mathsf{E}/\ell^\nu) \to \underline{\pi}_n^{\text{ét}}(\lim \mathrm{L}_{\text{ét}}\mathsf{E}_i/\ell^\nu),$$

is an isomorphism. On homotopy presheaves, choose $i \gg 0$ such that:

$$\cdots \cong \underline{\pi}_n(\mathsf{E}_{i+2}) \cong \underline{\pi}_n(\mathsf{E}_{i+1}) \cong \underline{\pi}_n(\mathsf{E}_i),$$

which we can by the assumption on the connectivity of $\mathrm{fib}_i(\mathsf{E})$. Therefore, by the Milnor lim-lim$^1$-sequence we have isomorphisms:

(25) $$\underline{\pi}_n(\lim \mathsf{E}_i) \cong \lim \underline{\pi}_n(\mathsf{E}_i) \cong \underline{\pi}_n(\mathsf{E}_i),$$

for all $i \gg 0$. Having this, we obtain the isomorphisms:

$$\underline{\pi}_n^{\text{ét}}(\mathsf{E}/\ell^\nu) = \underline{\pi}_n^{\text{ét}}(\lim_i \mathsf{E}_i/\ell^\nu) \cong a_{\text{ét}}(\underline{\pi}_n \mathsf{E}_i/\ell^\nu) \cong \underline{\pi}_n^{\text{ét}}(\mathsf{E}_i/\ell^\nu) \cong \underline{\pi}_n^{\text{ét}}(\mathsf{L}_{\text{ét}}\mathsf{E}_i/\ell^\nu).$$

Here, the first two isomorphisms are due to (25), and the third isomorphism is due to the fact that étale sheafification does not change stalks [4]. Now the descent spectral sequence:

$$H^p(X, \underline{\pi}_{-q}^{\text{ét}}(\mathsf{E}/\ell^\nu)) \Rightarrow [X, \Sigma^{p+q}\mathsf{L}_{\text{ét}}\mathsf{E}/\ell^\nu]_{\mathrm{SH}_{\text{ét}}(k)},$$

which is strongly convergent due to the cohomological dimension assumption, shows that the map $\mathsf{E}/\ell^\nu \to \lim \mathsf{L}_{\text{ét}}\mathsf{E}_i/\ell^\nu$ is an ét-localization, as desired. $\qquad\square$

4.0.4. Next, recall that we have the descent spectral sequence for ét-local motivic spectra, which boils down to a Bousfield-Kan spectral sequence as in [78, Proposition 1.36].

**Proposition 4.3.** *For* $\mathsf{E} \in \mathrm{SH}(S)$, *there is a spectral sequence:*

(26) $$E_{p,q}^2 = H_{\text{ét}}^p(S, \underline{\pi}_{-q}^{\text{ét}}(\Omega_{\mathbf{G}_m}^\infty \Sigma^{t,t}\mathsf{E}^{\text{ét}})) \Rightarrow [\mathbf{1}_S, \Sigma^{p+q+t,t}\mathsf{E}^{\text{ét}}]_{\mathrm{SH}(S)}.$$

*This spectral sequence is strongly convergent if the homotopy sheaves* $\underline{\pi}_{-q}^{\text{ét}}(\Omega_{\mathbf{G}_m}^\infty \Sigma^{t,t}\mathsf{E}^{\text{ét}})$ *are* $\ell$-torsion *and the residue characteristics of* $S$ *have finite* $\ell$-cohomological dimension or if the homotopy sheaves $\underline{\pi}_{-q}^{\text{ét}}(\Omega_{\mathbf{G}_m}^\infty \Sigma^{t,t}\mathsf{E}^{\text{ét}})$ vanish for $-q \ll 0$.

*Proof.* For any $t \in \mathbf{Z}$, consider the presheaf of $S^1$-spectra $\Omega_{\mathbf{G}_m}^\infty \Sigma^{t,t}\mathsf{E}^{\text{ét}}$. This presheaf satisfies ét-hyperdescent by Proposition 2.13. The general format of the descent spectral sequence (see for example [36, Section 6.1] or [78, Proposition 1.36]) is:

$$H_{\text{ét}}^p(S, \underline{\pi}_{-q}^{\text{ét}}\mathsf{F}) \Rightarrow \pi_0 \mathrm{Maps}_{\mathrm{SH}_{\text{ét}}^{S^1}(S)}(\mathbf{1}_S, \Sigma^{p+q}\mathsf{F}),$$

for $\mathsf{F}$ a presheaf of spectra satisfying ét-hyperdescent. Plugging in $\Omega_{\mathbf{G}_m}^\infty \Sigma^{t,t}\mathsf{E}^{\text{ét}}$ for $\mathsf{F}$ gets us the desired descent spectral sequence. The convergence statements follow from standard convergence criteria for the descent spectral sequence discussed in [78, 5.44-5.48]. $\qquad\square$

**4.1. $\mathbf{G}_m$-prespectra.** We recall that the adjunction (6) factors as the $S^1$-stabilization of $\mathrm{H}_\tau(S)$:

(27) $$\Sigma_{S^1,+}^\infty : \mathrm{H}_\tau(S) \rightleftarrows \mathrm{SH}_\tau^{S^1}(S) : \Omega_{S^1}^\infty,$$

and the process of $(\mathbf{G}_m, 1)$-inversion described in the sense discussed in Corollary 2.5:

(28) $$\Sigma_{\mathbf{G}_m,+}^\infty : \mathrm{SH}_\tau^{S^1}(S) \rightleftarrows \mathrm{SH}_\tau(S) = \mathrm{SH}_\tau^{S^1}(S)[(\mathbf{G}_m, 1)^{-1}] : \Omega_{\mathbf{G}_m}^\infty.$$

By the formula given in (5), the $\infty$-category $\mathrm{SH}_\tau(S)$ is obtained from $\mathrm{SH}_\tau^{S^1}(S)$ by the colimit:

(29) $$\mathrm{SH}_\tau^{S^1}(S) \overset{(\mathbf{G}_m,1)\wedge-}{\to} \mathrm{SH}_\tau^{S^1}(S) \overset{(\mathbf{G}_m,1)\wedge-}{\to} \mathrm{SH}_\tau^{S^1}(S) \to \cdots,$$

computed $\mathrm{Mod}_{\mathrm{H}_\tau(S)}(\mathrm{Pr}^L)$. Since colimits in $\mathrm{Mod}_{\mathrm{H}_\tau(S)}(\mathrm{Pr}^L)$ can also be computed as a limit in $\mathrm{Mod}_{\mathrm{H}_\tau(S)}(\mathrm{Pr}^R)$, where the transition maps are the right adjoints [45, 5.5.3], and the forgetful functor $\mathrm{Mod}_{\mathrm{H}_\tau(S)}(\mathrm{Pr}^R) \to \mathrm{Cat}_\infty$ preserves limits, the underlying $\infty$-category of $\mathrm{SH}_\tau(S)$ is computed as the limit of the diagram:

(30) $$\mathrm{SH}_\tau^{S^1}(S) \overset{\Omega_{\mathbf{G}_m}}{\leftarrow} \mathrm{SH}_\tau^{S^1}(S) \overset{\Omega_{\mathbf{G}_m}}{\leftarrow} \mathrm{SH}_\tau^{S^1}(S) \overset{\Omega_{\mathbf{G}_m}}{\leftarrow} \cdots,$$

---

[4]To see this: note that, by the finiteness assumption on cohomological dimension, we have that Postnikov and hypercompletion coincide [16, Proposition 2.10]. As explained in [16, Construction 2.25], we can use the Godement resolution in order to compute Postnikov sheafification. Therefore we consider the map $\mathsf{E}/\ell^\nu \to L_{\text{ét}}\mathsf{E}/\ell^\nu$ and restrict it to a strictly henselian local ring. Since any strictly henselian local ring does not have any nontrivial étale cover, the Godement resolution is trivial and we the map $\mathsf{E}/\ell^\nu \to L_{\text{ét}}\mathsf{E}/\ell^\nu$ induces an isomorphism on homotopy groups since the Godement resolution commutes with homotopy groups [78, (1.26)].

in the $\infty$-category of $\infty$-categories, $\mathrm{Cat}_\infty$. This formulation lets us speak of $\mathbf{G}_m$-prespectra as limits in $\mathrm{Cat}_\infty$ that can be described "pointwise." The following discussion can be seen as a formulation of the usual notion of "$\mathbf{G}_m$-$S^1$" bispectra following [33], [35], and [87]

4.1.1.  To elaborate on this point, let $L \subset \mathbb{N}$ be the 1-skeleton of the nerve of the poset defined by the natural numbers. The diagram (30) determines a functor:

$$p : L^{\mathrm{op}} \to \mathrm{Cat}_\infty.$$

The $\infty$-categorical Grothendieck construction [45, §3.2] applied to the functor $p$ furnishes a Cartesian fibration:

$$q : \int \mathrm{SH}^{S^1}_\tau(S) \to L,$$

such that the $\infty$-category of Cartesian sections of $q$ is equivalent to the limit of (30), i.e., the $\infty$-category $\mathrm{SH}_\tau(S)$ [45, Corollary 3.3.3.2]. Concretely, this means that to specify an object of $\mathrm{SH}_\tau(S)$ is equivalent to:

- specifying objects $\{\mathsf{E}_i \in \mathrm{SH}^{S^1}_\tau(S)\}_{i \in \mathbb{N}}$,
- and bonding maps $\epsilon_i : \mathsf{E}_i \to \Omega_{\mathbf{G}_m} \mathsf{E}_{i+1}$, which are equivalences.

We shall call such a section a $\mathbf{G}_m$-*spectrum in* $\mathrm{SH}^{S^1}_\tau(S)$. The last condition is implied by the demand that the section be Cartesian. However, we may also speak of the $\infty$-category of $\mathbf{G}_m$-*prespectra in* $\mathrm{SH}^{S^1}_\tau(S)$, which is defined as sections of $q$ which are not necessarily Cartesian. Concretely, specifying a $\mathbf{G}_m$-prespectrum is equivalent to:

- specifying objects $\{\mathsf{E}_i \in \mathrm{SH}^{S^1}_\tau(S)\}_{i \in \mathbb{N}}$,
- and bonding maps $\epsilon_i : \mathsf{E}_i \to \Omega_{\mathbf{G}_m} \mathsf{E}_{i+1}$, which need not be equivalences.

4.1.2.  We denote by $\mathrm{SH}^{\mathrm{pre}}_\tau(S) := \mathrm{Fun}_L(L, \int \mathrm{SH}^{S^1}_\tau(S))$ the $\infty$-category of $\mathbf{G}_m$-prespectra, which is defined as the space of sections of the Cartesian fibration $q$; [45, Corollary 3.3.3.2] implies that there is a fully faithful embedding $u : \mathrm{SH}_\tau(S) \hookrightarrow \mathrm{SH}^{\mathrm{pre}}_\tau(S)$ as the Cartesian sections. According to [29, Page 7] this defines a localization, i.e., $u$ has a left adjoint:

(31) $$Q : \mathrm{SH}^{\mathrm{pre}}_\tau(S) \rightleftarrows \mathrm{SH}_\tau(S) : u,$$

which we call *spectrification*. It is computed by the formula (98) which we discuss in §12.

For the rest of the paper we will specify $\mathbf{G}_m$-prespectra in $\mathrm{SH}^{S^1}_\tau(S)$ by writing a sequence of objects in $\mathrm{SH}^{S^1}_\tau(S)$:

$$(\mathsf{E}_0, \mathsf{E}_1, \cdots), \mathsf{E}_i \in \mathrm{SH}^{S^1}_\tau(S).$$

The bonding maps will always be clear from the context.

4.1.3.  Let $\mathsf{E} \in \mathrm{SH}(S)$ be a motivic spectrum. By the discussion of §4.1.1, we obtain a $\mathbf{G}_m$-spectrum in $\mathrm{SH}^{S^1}(S)$, specified by the sequence of objects in $\mathrm{SH}^{S^1}(S)$:

$$(\Omega^\infty_{\mathbf{G}_m} \mathsf{E}, \Omega^\infty_{\mathbf{G}_m} \Sigma^{1,1} \mathsf{E}, \Omega^\infty_{\mathbf{G}_m} \Sigma^{2,2} \mathsf{E}, \cdots),$$

such that the bonding maps:

$$\Omega^\infty_{\mathbf{G}_m} \Sigma^{i,i} \mathsf{E} \xrightarrow{\cong} \Omega_{\mathbf{G}_m} \Omega^\infty_{\mathbf{G}_m} \Sigma^{i+1,i+1} \mathsf{E},$$

are equivalences for all $i \in \mathbb{N}$. Now, we have an inclusion $\pi_{*,\mathrm{pre}} : \mathrm{SH}^{\mathrm{pre}}_{\mathrm{ét}}(S) \hookrightarrow \mathrm{SH}^{\mathrm{pre}}(S)$ over $L$, where $\mathrm{SH}^{\mathrm{pre}}_{\mathrm{ét}}(S)$ is identified with the full subcategory of $\mathbf{G}_m$-prespectra in $\mathrm{SH}^{S^1}(S)$ spanned by those prespectra such that $(\Omega^\infty_{\mathbf{G}_m} \mathsf{E})$ is ét-local, i.e., sections of $\int \mathrm{SH}^{S^1}(S) \to L$ which lands in $\mathrm{SH}^{S^1}_{\mathrm{ét}}(S) \hookrightarrow \mathrm{SH}(S)$. Since the $\infty$-category of $\mathbf{G}_m$-prespectra in $\mathrm{SH}^{S^1}(S)$ is described as a functor category (being a category of sections) and colimits are computed pointwise in such categories, $\pi_{*,\mathrm{pre}}$ preserves limits and thus admits a left adjoint:

$$\pi^{*,\mathrm{pre}} : \mathrm{SH}^{\mathrm{pre}}(S) \to \mathrm{SH}^{\mathrm{pre}}_{\mathrm{ét}}(S).$$

4.1.4. The functor $\pi^{*,\mathrm{pre}}$ is computed by applying:

$$L_{\text{ét,m}} : \mathrm{SH}^{S^1}(S) \to \mathrm{SH}^{S^1}_{\text{ét}}(S),$$

to each individual presheaves of spectra to obtain a $\mathbf{G}_m$-prespectrum in $\mathrm{SH}^{S^1}_{\text{ét}}(S)$ in the sense of §4.1.1 just discussed:

(32) $$\pi^{*,\mathrm{pre}}\mathsf{E} := (L_{\text{ét,m}}\Omega^\infty_{\mathbf{G}_m}\mathsf{E}, L_{\text{ét,m}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{1,1}\mathsf{E}, L_{\text{ét,m}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{2,2}\mathsf{E}, \cdots).$$

Here, for each $i \in \mathbb{N}$, the map:

$$\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\mathsf{E} \simeq \Omega^\infty_{\mathbf{G}_m}\Omega_{\mathbf{G}_m}\Sigma^{i+1,i+1}\mathsf{E} \xrightarrow{\simeq} \Omega_{\mathbf{G}_m}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i+1,i+1}\mathsf{E} \to \Omega_{\mathbf{G}_m}L_{\text{ét,m}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i+1,i+1}\mathsf{E},$$

induces the bonding map:

$$L_{\text{ét,m}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\mathsf{E} \to \Omega_{\mathbf{G}_m}L_{\text{ét,m}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i+1,i+1}\mathsf{E},$$

since the target is $\mathbf{A}^1$-invariant and ét-local. Indeed, a morphism in $\mathrm{SH}^{\mathrm{pre}}(S)$ is just a natural transformation $f : \mathsf{E} \to \mathsf{F}$ over $L$ between functors $\mathsf{E}, \mathsf{F} : L \to \int \mathrm{SH}^{S^1}(S)$. Hence if each term of $\mathsf{F}$ has étale descent then $f$ must factor through $\pi^{*,\mathrm{pre}}\mathsf{E}$. Note that the bonding maps in $\pi^{*,\mathrm{pre}}\mathsf{E}$ are not necessarily equivalences and so $\pi^{*,\mathrm{pre}}\mathsf{E}$ *does not* necessarily descend to an object of $\mathrm{SH}_{\text{ét}}(S)$.

4.1.5. The following lemma identifies $Q(\epsilon^*\mathsf{E}^{\text{ét}})$ as the ét-localization of E:

**Lemma 4.4.** *The functor* $\epsilon^* : \mathrm{SH}(S) \to \mathrm{SH}_{\text{ét}}(S)$ *is calculated as the composite functor*

$$\mathrm{SH}(S) \xrightarrow{\pi^{*,\mathrm{pre}}} \mathrm{SH}^{\mathrm{pre}}_{\text{ét}}(S) \xrightarrow{Q} \mathrm{SH}_{\text{ét}}(S)^5.$$

*Proof.* Using [45, Proposition 5.2.7.4] we need to check that the essential image of $Q \circ \pi^{*,\mathrm{pre}}$ is the essential image of $\epsilon^*$, i.e., ét-local spectra in the sense of Proposition 2.13 and that $Q \circ \pi^{*,\mathrm{pre}}$ is left adjoint to $Q \circ \pi^{*,\mathrm{pre}}(\mathrm{SH}(S)) \subset \mathrm{SH}(S)$. The first statement follows from the characterization of objects in $\mathrm{SH}_{\text{ét}}(S)$ in Lemma 2.14, the second statement follows since the analogous statements are also true for the localization functors $Q$ and $\pi^{*,\mathrm{pre}}$. $\square$

4.1.6. We will now define a condition on a motivic spectrum $\mathsf{E} \in \mathrm{SH}(S)$ such that its ét-localization can be controlled. To motivate the next definition, let us examine Example 4.1 in the language we just set up.

*Remark* 4.5. In the notation of this section, the $\mathbf{G}_m$-prespectrum in $\mathrm{SH}^{S^1}_{\text{ét}}(S)$ specified in (22) is exactly $\pi^{*,\mathrm{pre}}\Sigma^{0,-1}M\mathbf{Z}/\ell$. Note that (23) tells us that $\pi^{*,\mathrm{pre}}\Sigma^{0,-1}M\mathbf{Z}/\ell$ is *not* equivalent to $Q(\pi^{*,\mathrm{pre}}\Sigma^{0,-1}M\mathbf{Z}/\ell) \simeq \Sigma^{0,-1}\epsilon^*M\mathbf{Z}/\ell$, i.e., it is not a $\mathbf{G}_m$-spectrum in $\mathrm{SH}^{S^1}_{\text{ét}}(S)$. However, we note that each bonding map $\mathbf{Z}/\ell_{\text{ét}}(t-1)[t] \to \Omega_{\mathbf{G}_m}\mathbf{Z}/\ell_{\text{ét}}(t)[t+1]$ is an equivalence for $t \geq 1$. This motivates the next definition.

**Definition 4.6.** Let $\mathsf{E} \in \mathrm{SH}(S)$ and consider the $\mathbf{G}_m$-prespectrum $\pi^{*,\mathrm{pre}}\mathsf{E}$ which is specified by $\{L_{\text{ét,m}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\mathsf{E}\}_{i\in\mathbb{N}}$. We say that $\mathsf{E}$ *is étale-$\mathbf{A}^1$ naive above degree $r$* if for all $i \geq r$:

- the natural map

$$L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\mathsf{E} \to L_{\mathbf{A}^1}L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\mathsf{E} = L_{\text{ét,m}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\mathsf{E}$$

  is an equivalence in $\mathrm{P}_{\text{ét,Spt}}(k)$, i.e., the ét-local presheaf of spectra $L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\mathsf{E}$ is $\mathbf{A}^1$-invariant and,
- the induced map

$$L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\mathsf{E} \to \Omega_{\mathbf{G}_m}L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i+1,i+1}\mathsf{E}$$

  is an equivalence in $\mathrm{P}_{\text{ét,Spt}}(k)$.

With this definition, Example 4.1 states that $\Sigma^{0,-1}M\mathbf{Z}/\ell$ is étale $\mathbf{A}^1$-naive above degree 1. The following lemma justifies the above definition

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[5] In fact, this claim is true for any topology $\tau$.

**Lemma 4.7.** *Suppose that* $E \in SH(S)$ *is étale-$\mathbf{A}^1$-naive above degree $r$. Then for any $i \geq r$, we have an equivalence in* $P_{\text{ét,Spt}}(S)$*:*

$$L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}E \simeq \Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\epsilon^*E.$$

*Proof.* Both presheaves of spectra are ét-local and $\mathbf{A}^1$-invariant and so define objects in $SH^{S^1}_{\text{ét}}(S)$; for the left hand side this is because of the first condition in Definition 4.6, which also tells us that for $n \geq 0$ there is an equivalence in $P_{\text{ét,Spt}}(S)$:

$$L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i+n,i+n}E \simeq L_{\mathbf{A}^1}L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i+n,i+n}E.$$

By Lemma 4.4 we have an equivalence $Q(\pi^{*,\text{pre}}E) \simeq \epsilon^*E$. In other words, for any $i \in \mathbb{N}$, the presheaf of spectra $\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\epsilon^*E$ is the $i$-th term of the $\mathbf{G}_m$-spectrum in $SH^{S^1}_{\text{ét}}(S)$, $Q(\pi^{*,\text{pre}}E)$. According to (98), for each $i \in \mathbb{N}$, $Q(\pi^{*,\text{pre}}E)_i$ is computed by a transfinite colimit whose transition maps are the bonding maps $L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}E \to \Omega_{\mathbf{G}_m}L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i+1,i+1}E$ for $i \geq r$. Since $E$ is $\mathbf{A}^1$-naive above degree $r$ the hypothesis that $i \geq r$ informs us that the transition maps in the colimit are all equivalences. Therefore, we obtain the desired equivalence:

$$\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\epsilon^*E \simeq L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}E.$$

□

4.1.7.   The point of Definition 4.6 is that we can estimate the connectivity of $E^{\text{ét}}$. In general, if $E \in SH(S)$, then we have a descent spectral sequence by the discussion in §4.0.4:

$$H^p_{\text{ét}}(X, \underline{\pi}^{\text{ét}}_{-q}\Omega^\infty_{\mathbf{G}_m}\epsilon^*\Sigma^{i,i}E) \Rightarrow [\Sigma^\infty_T X_+, \epsilon^*\Sigma^{i,i}E[p+q]]_{SH_{\text{ét}}(S)} \simeq [\Sigma^\infty_T X_+, \Sigma^{i,i}E^{\text{ét}}[p+q]]_{SH(S)}$$

Suppose that $E$ is étale-$\mathbf{A}^1$ naive above degree $r$, then for all $X \in Sm_S$, we have the following equivalences of mapping spaces whenever $i \geq r$:

$$\begin{aligned}
\text{Maps}_{SH(S)}(\Sigma^{q,0}\Sigma^\infty_T X_+, \Sigma^{i,i}E^{\text{ét}}) &\simeq \text{Maps}_{SH^{S^1}_{\text{ét}}(S)}(\Sigma^{q,0}\Sigma^\infty_{S^1}X_+, \Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\epsilon^*E) \\
&\simeq \text{Maps}_{SH^{S^1}_{\text{ét}}(S)}(\Sigma^{q,0}\Sigma^\infty_{S^1}X_+, L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}E).
\end{aligned}$$

Here, the last equivalence is exactly Lemma 4.7. Therefore, in this range, the étale homotopy sheaves are computed as:

$$(33) \qquad \underline{\pi}^{\text{ét}}_{q,0}\Sigma^{i,i}E \cong a_{\text{ét}}(\underline{\pi}^{\text{pre}}_q\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}\epsilon^*E) \cong a_{\text{ét}}(\underline{\pi}^{\text{pre}}_q L_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}E) \cong a_{\text{ét}}(\underline{\pi}^{\text{pre}}_q\Omega^\infty_{\mathbf{G}_m}\Sigma^{i,i}E),$$

where the second isomorphism is due to the condition of being étale-$\mathbf{A}^1$ naive, and the last isomorphism follows because $L_{\text{ét}}$ does not change stalks. We record this as a lemma

**Lemma 4.8.** *Suppose that* $E \in SH(S)$ *is étale-$\mathbf{A}^1$ naive above degree $r$. Then for any $i \geq r$, we have a spectral sequence:*

$$(34) \qquad H^p_{\text{ét}}(X, \underline{\pi}^{\text{ét}}_{-q}\Omega^\infty_{\mathbf{G}_m}E_i) \Rightarrow [\Sigma^\infty_T X_+, \Sigma^{i,i}E^{\text{ét}}[p+q]]_{SH(S)}.$$

4.2. **Étale-$\mathbf{A}^1$ naive properties of Landweber exact motivic spectra.** Now we apply the above machinery to investigate the étale-$\mathbf{A}^1$ naive properties of certain MGL modules whose slices are computed in [74], namely the Landweber exact motivic spectra and its slice covers. For the rest of this section, we work over a field $k$ and we also assume that all our Landweber exact spectra are effective. We prove a sequence of lemmas towards establishing some étale-$\mathbf{A}^1$ naive properties of Landweber exact motivic spectra

4.2.1.   We give a rapid review of motivic Landweber exact spectra, the basic reference being [57]. If $N_*$ is a Landweber exact $MU_*[\frac{1}{c}]$-module as in [74, Section 5], then the functor from $SH(S)$ to the category of Adams-graded graded Abelian groups — see [57, Section 3] — given by

$$X \mapsto MGL_{*,*}(X) \otimes_{MU_*[\frac{1}{c}]} N_*$$

is representable by a (motivic) Landweber exact spectrum $E_N$. By construction $E_N$ is a Cartesian section of $SH(-)$ [57, Proposition 8.5]. In fact, [57] proves more: every Landweber

exact spectrum is an object of the $\infty$-category $\mathrm{Mod}_{\mathrm{MGL}}$ [57, Proposition 7.9] and is further-more cellular [57, Proposition 8.4]. The slices of Landweber exact spectra are computed in [74, Theorem 6.1]. For $\mathsf{E}_M$ there is an equivalence $s_q(\mathsf{E}_N) \simeq \Sigma^{2q,q} \mathsf{M}(N_{2q})$ which is compatible with the map $N_* \to E_{N,*,*}$; see [30, Theorem 8.5].

4.2.2.   The form of the slices of Landweber exact spectra results in the following

**Lemma 4.9.** *Let $k$ be a field of exponential characteristic coprime to $\ell$ and assume $\mathrm{cd}_\ell(k) < \infty$. Let $\mathsf{E}_N \in \mathrm{SH}(k)$ be a Landweber exact motivic spectrum associated to a Landweber exact $\mathrm{MU}_*$-module $N_*$. Then for any pair of integers $(r, q)$ we have an equivalence in $\mathrm{P}_{\mathrm{Spt},\text{ét}}(\mathrm{Sm}_k)$:*

$$\text{(35)} \qquad \mathrm{L}_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f_q \mathsf{E}_N/\ell^\nu \xrightarrow{\simeq} \lim_n \mathrm{L}_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f^{n+1} f_q \mathsf{E}_N/\ell^\nu.$$

*Proof.* For a fixed integer $q$ we have the following tower in $\mathrm{SH}(S)$:
$$\text{(36)}$$
$$\cdots \to f^n f_q \mathsf{E}/\ell^\nu \to \cdots \to f^{q+3} f_q \mathsf{E}_N/\ell^\nu \to f^{q+2} f_q \mathsf{E}_N/\ell^\nu \to f^{q+1} f_q \mathsf{E}_N/\ell^\nu \simeq s_q \mathsf{E}_N/\ell^\nu,$$

and thus for any $r \in \mathbf{Z}$, we obtain a tower in $\mathrm{SH}^{S^1}(S)$ after applying $\Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r}$:
$$\text{(37)}$$
$$\cdots \to \Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f^{q+3} f_q \mathsf{E}_N/\ell^\nu \to \Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f^{q+2} f_q \mathsf{E}_N/\ell^\nu \to \Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f^{q+1} f_q \mathsf{E}_N/\ell^\nu \simeq \Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} s_q(\mathsf{E}_N/\ell^\nu).$$

Now, for any $n > q$, we have a cofiber sequence in $\mathrm{SH}(S)$ (see (14)):

$$f^{n+1} f_q \mathsf{E}_N/\ell^\nu \simeq f^{n+1} \mathsf{E}_N/\ell^\nu \to f^n f_q \mathsf{E}_N/\ell^\nu \simeq f^n \mathsf{E}_N/\ell^\nu \to \Sigma^{1,0} s_{n+1} \mathsf{E}_N/\ell^\nu \simeq \Sigma^{1,0}\Sigma^{2n+2,n+1} \mathsf{M} N_*/\ell^\nu,$$

where the last equivalence is due to the computation of the slices of Landweber exact spectra as in [74]. Therefore, the fiber of:

$$\Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f^{n+1} f_q \mathsf{E}_N/\ell^\nu \to \Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f^n f_q \mathsf{E}_N/\ell^\nu,$$

is equivalent to:

$$\Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} s_{n+1}\mathsf{E}_N/\ell^\nu \quad \simeq \quad \Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r}\Sigma^{2n+2,n+1}\mathsf{M} N_*/\ell^\nu$$
$$\simeq \quad \mathbf{Z}/\ell^\nu(2n+2+r)[n+1+r] \otimes N_{2n+2}.$$

Since the connectivity of $\mathbf{Z}/\ell^\nu(2n+2+r)[n+1+r] \otimes N_{2n+2} \in \mathrm{SH}^{S^1}(S)$ tends to $\infty$ as $n \to \infty$, Proposition 4.2 gives us an equivalence in $\mathrm{P}_{\mathrm{Spt},\text{ét}}(\mathrm{Sm}_k)$:

$$\text{(38)} \qquad \mathrm{L}_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f_q \mathsf{E}_N/\ell^\nu \xrightarrow{\simeq} \lim_n \mathrm{L}_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f^{n+1} f_q \mathsf{E}_N/\ell^\nu.$$

$\square$

**Lemma 4.10.** *Let $k$ be a field of exponential characteristic coprime to $\ell$ and assume $\mathrm{cd}_\ell(k) < \infty$. Let $\mathsf{E}_N \in \mathrm{SH}(k)$ be a Landweber exact motivic spectrum. For integers $(r, q)$ the ét-local presheaf of spectra $\mathrm{L}_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f_q \mathsf{E}_N/\ell^\nu$ is $\mathbf{A}^1$-invariant. Hence there is a canonical equivalence in $\mathrm{P}_{\text{ét},\mathrm{Spt}}(k)$:*

$$\mathrm{L}_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f_q \mathsf{E}_N/\ell^\nu \xrightarrow{\simeq} \mathrm{L}_{\mathbf{A}^1}\mathrm{L}_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f_q \mathsf{E}_N/\ell^\nu,$$

*and $\mathrm{L}_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f_q \mathsf{E}_N/\ell^\nu \in \mathrm{SH}^{S^1}_{\text{ét}}(k)$.*

*Proof.* Note that $\mathbf{A}^1$-invariant presheaves in $\mathrm{P}_{\text{ét},\mathrm{Spt}}(\mathrm{Sm}_k)$ are closed under limits. Indeed, if $F : I \to \mathrm{P}_{\text{ét},\mathrm{Spt}}(\mathrm{Sm}_k)$ is a diagram of $\mathbf{A}^1$-invariant ét-local presheaf of spectra, then for any $X \in \mathrm{Sm}_k$ we have natural equivalences:

$$\mathrm{Maps}(\Sigma^\infty_{S^1} X_+, \lim F_i) \quad \simeq \quad \lim_i \mathrm{Maps}(\Sigma^\infty_{S^1} X_+, F_i)$$
$$\simeq \quad \lim_i \mathrm{Maps}(\Sigma^\infty_{S^1}(X \times \mathbf{A}^1)_+, F_i)$$
$$\simeq \quad \mathrm{Maps}(\Sigma^\infty_{S^1}(X \times \mathbf{A}^1)_+, \lim_i F_i).$$

Hence, after Lemma 4.9, it suffices to prove that for any $n \geq q+1$, each of the ét-local presheaf of spectra $\mathrm{L}_{\text{ét}}\Omega^\infty_{\mathbf{G}_m}\Sigma^{r,r} f^{n+1} f_q \mathsf{E}_N/\ell^\nu$ is $\mathbf{A}^1$-invariant. For $n = q+1$, we note that

$L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} s_{n+1} E_N / \ell^\nu \simeq L_{ét}\mathbf{Z}/\ell^\nu(2n+2+r)[n+1+r] \otimes N_{2n+2}$ is $\mathbf{A}^1$-invariant. Indeed, by [49, Theorem 2.3], $L_{ét}\mathbf{Z}/\ell^\nu(q)[p] \simeq \mu^{\otimes q}[p]$ for $q \geq 0$, which are indeed $\mathbf{A}^1$-invariant ét-local presheaves [1, XV, Corollary 2.2].

Suppose the claim holds for all $n' < n$. The property of being $\mathbf{A}^1$-invariant is clearly closed under extensions, and so we conclude using the cofiber sequence in $P_{ét,Spt}(k)$:

$$L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} s_n E_N / \ell^\nu \to L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f^n f_q E_N / \ell^\nu \to L_{ét}\Omega_{\mathbf{G}_m}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f^{n-1} f_q E_N / \ell^\nu.$$

$\square$

**Lemma 4.11.** *Let* $E_N \in SH(k)$ *be a Landweber exact motivic spectrum. Then for all integers* $(r,q)$ *such that* $r > -q$ *there is a canonical equivalence:*

$$L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r-1,r-1} f_q E / \ell^\nu \xrightarrow{\simeq} \Omega_{\mathbf{G}_m} L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f_q E_N / \ell^\nu.$$

*Proof.* First we note that for $q \geq 1$, we have an equivalence for all $p \in \mathbf{Z}$:

$$(39) \qquad \Omega_{\mathbf{G}_m} L_{ét}\mathbf{Z}/\ell^\nu(q)[p] \simeq L_{ét}\Omega_{\mathbf{G}_m}\mathbf{Z}/\ell^\nu(q)[p].$$

Indeed, both sides are equivalent to $\mu_{\ell^\nu}^{\otimes q-1}[p-1]$. With this, we claim that the natural map:

$$(40) \qquad L_{ét}\Omega_{\mathbf{G}_m}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f^n f_q E_N / \ell^\nu \to \Omega_{\mathbf{G}_m} L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f^n f_q E_N / \ell^\nu,$$

is an equivalence; in other words, we claim that $\Omega_{\mathbf{G}_m}$ and $L_{ét}$ commute. This is proved by induction on $n$. For the base case $n = q+1$ we have equivalences in $DM_{ét}^{eff}(k, \mathbf{Z}/\ell)$:

$$\begin{aligned}
\Omega_{\mathbf{G}_m} L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f^{q+1} f_q E_N / \ell^\nu &= \Omega_{\mathbf{G}_m} L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} s_q(E_N / \ell^\nu) \\
&\simeq \Omega_{\mathbf{G}_m} L_{ét}\mathbf{Z}/\ell^\nu[2q+r](q+r) \otimes N_{2n+2} \\
&\simeq L_{ét}\Omega_{\mathbf{G}_m}\mathbf{Z}/\ell^\nu[2q+r](q+r) \otimes N_{2n+2}.
\end{aligned}$$

The first equivalence follows from the computation of the slices of Landweber exact spectra [74], and the second follows from (40) using the hypothesis that $q + r > 0$.

Now assume that (40) has been verified for $n-1$, and consider the commutative diagram:
(41)

$$\begin{array}{ccccc}
L_{ét}\Omega_{\mathbf{G}_m}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} s_n E_N / \ell^\nu & \longrightarrow & L_{ét}\Omega_{\mathbf{G}_m}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f^n f_q E_N / \ell^\nu & \longrightarrow & L_{ét}\Omega_{\mathbf{G}_m}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f^{n-1} f_q E_N / \ell^\nu \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_{\mathbf{G}_m} L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} s_n E_N / \ell^\nu & \longrightarrow & \Omega_{\mathbf{G}_m} L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f^n f_q E_N / \ell^\nu & \longrightarrow & \Omega_{\mathbf{G}_m} L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f^{n-1} f_q E_N / \ell^\nu
\end{array}$$

Since $L_{ét}$ and $\Omega_{\mathbf{G}_m}$ are exact functors, the horizontal rows are cofiber sequences. The leftmost vertical map is an equivalence by the same argument as in the base case since $\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} s_n E_N / \ell^\nu \simeq \mathbf{Z}/\ell^\nu[2n+r](n+r) \otimes N_{2n+2}$, while the rightmost vertical map is an equivalence by the inductive hypothesis. Thus the middle vertical map is an equivalence.

To conclude, we have the following string of equivalences, for $r > -q$:

$$\begin{aligned}
\Omega_{\mathbf{G}_m} L_{ét}\Omega^\infty \Sigma^{r,r} f_q E_N / \ell^\nu &\simeq \Omega_{\mathbf{G}_m} \lim_n L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f^{n+1} f_q E_N / \ell^\nu \\
&\simeq \lim_n \Omega_{\mathbf{G}_m} L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f^{n+1} f_q E_N / \ell^\nu \\
&\simeq \lim_n L_{ét}\Omega_{\mathbf{G}_m}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r,r} f^{n+1} f_q E_N / \ell^\nu \\
&\simeq \lim_n L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r-1,r-1} f^{n+1} f_q E_N / \ell^\nu \\
&\simeq L_{ét} \lim_n \Omega_{\mathbf{G}_m}^\infty \Sigma^{r-1,r-1} f^{n+1} f_q E_N / \ell^\nu \\
&\simeq L_{ét}\Omega_{\mathbf{G}_m}^\infty \Sigma^{r-1,r-1} f_q E_N / \ell^\nu.
\end{aligned}$$

The first equivalence is due to (38), the second is because $\Omega_{\mathbf{G}_m}$ is a right adjoint and thus preserves limits, the third is due to (40), the fourth is because of the canonical equivalence $\Omega_{\mathbf{G}_m}\Omega_{\mathbf{G}_m}^\infty \simeq \Omega_{\mathbf{G}_m}^\infty \Omega_{\mathbf{G}_m}$, the fifth is another application of (38), while the last equivalence is again because $\Omega_{\mathbf{G}_m}$ preserves limits.          $\square$

As a result we have proved:

**Proposition 4.12.** *Let* $E_N \in \mathrm{SH}(k)$ *be a Landweber exact motivic spectrum. Then* $f_q E_N / \ell^\nu$ *is étale-*$\mathbf{A}^1$*-naive above degree* $-q$.

*Proof.* The first point of Definition 4.6 is verified by Lemma 4.10, while the second point at the desired range is verified by Proposition 4.11. □

4.3. **Notions of connectivity in motivic homotopy.** We now use Proposition 4.12 to prove convergence results about the étale slice spectral sequence. We say that a motivic spectrum $E \in \mathrm{SH}(S)$ is:

- *t-connected* [66, Definition 3.16] if for any triple $(p, q, d)$ of integers for which $p - q + d < t$ and every $d$-dimensional $X \in \mathrm{Sm}_S$ the group $[\Sigma^{p,q} \Sigma_T^\infty X_+, E]$ is zero,
- *affine t-connected* if for any triple $(p, q, d)$ of integers for which $p - q + d < t$ and every $d$-dimensional $X \in \mathrm{Sm}_S$ which is affine, the group $[\Sigma^{p,q} \Sigma_T^\infty X_+, E]$ is zero,
- *t-connective* [30, §2.1] if it is contained in the localizing subcategory generated by $\{\Sigma^{p,q} \Sigma_T^\infty X_+\}_{p-q \geq t, X \in \mathrm{Sm}_S}$,
- *locally t-connective* if the homotopy sheaves $\underline{\pi}_{p,q}^{\mathrm{Nis}} E = 0$ for any $p, q \in \mathbf{Z}$ such that $p - q < t$.

If $S$ is a field, then we can say more about the relationships of the above notions.

- Being $t$-connected is stronger than being affine $t$-connected. These notions are, in general, different because the étale cohomological dimension of a scheme $X$ over an algebraically closed field is bounded above by $2\dim(X)$ in general and by $\dim(X)$ if $X$ is affine [50, Corollary 1.4, Remark 1.5(a)]. The latter notion is meant to accommodate certain phenomena in the étale topology, which we will consider later, starting from §4.4.
- $E$ is $t$-connective if and only if it is locally $t$-connective by [30, Theorem 2.3].
- The locally 0-connective motivic spectra form the nonnegative part of the *homotopy t-structure* [53, Section 5.2]. Thus 0-connective motivic spectra and 0-locally connective motivic spectra are nonnegative parts of isomorphic $t$-structures on $\mathrm{SH}(S)$.
- According to [66, Lemma 3.17], $E$ being $t$-connective implies that $E$ is $t$-connected.

4.3.1. Next we turn to convergence properties of the slice spectral sequence.

**Definition 4.13.** A tower of motivic spectra:

$$\{E_q\}_{q \in \mathbf{Z}} = \cdots \to E_{q+1} \to E_q \to \cdots,$$

is called *left bounded with respect to a pair* $(X, w)$ *where* $X \in \mathrm{Sm}_S$ *and* $w \in \mathbf{Z}$ if for every $s \in \mathbf{Z}$, the group $[\Sigma_T^{s,w} \Sigma^\infty X_+, E_q] = 0$ for $q \gg 0$, and is it called *left bounded* if it is left bounded with respect to all $(X, w)$.

For the effective covers in the slice filtration $\{f_q E\}$ the notion of left boundedness is stronger than that of slice completeness; see the discussion in [30, §8.5]. In this case we say that the motivic spectrum $E$ *is left bounded*. Consequently, for $X \in \mathrm{Sm}_S$ and $w \in \mathbf{Z}$ fixed, the slice spectral sequence §3.0.6:

$$E^1_{p,q,w}(E)(X) \Rightarrow [\Sigma^{p,w} \Sigma_T^\infty X_+, E],$$

converges conditionally whenever we have left boundedness with respect to $(X, w)$ due to the slice completeness of $E$. The following illustrates how connectedness implies left boundedness:

**Lemma 4.14.** *Let* $E \in \mathrm{SH}(S)$ *and suppose that for* $q \gg 0$ *there exists an integer* $n \in \mathbf{Z}$ *such that the effective cover* $f_q E$ *is* $(q + n)$*-connected. Then the tower* $\{f_q E\}$ *is left bounded, i.e., the motivic spectrum* $E$ *is left bounded.*

*Proof.* Let us fix $(X, w)$, $s \in \mathbf{Z}$. By assumption the group $[\Sigma^{s,w} \Sigma_T^\infty X_+, f_q E]$ is trivial when $s - w + \dim(X) - n < q$. We conclude by letting $q \to \infty$. □

4.3.2.    For convergence of the étale slice spectral sequence, it turns out that the usual notion of connectedness will not be useful because of the following example.

**Example 4.15.** One might guess that applying the étale localization decreases the connectivity of $\mathsf{E} \in \mathrm{SH}(S)$ by the cohomological dimension of the base scheme. However, we show that $\mathsf{M}\mathbf{Z}^{\text{ét}}/\ell$ is *not* 0-connected when $k$ is an algebraically closed field. Assuming that $\mathsf{M}\mathbf{Z}^{\text{ét}}/\ell$ is 0-connected then for $p - q < 0$ the group:

$$[\Sigma^{p,q}\mathbf{1}, \mathsf{M}\mathbf{Z}^{\text{ét}}/\ell] \cong H_{\text{ét}}^{-p}(k, \mu_\ell^{\otimes -q}) = 0.$$

But for $p = 0$ and $q > 0$ the above group $H_{\text{ét}}^0(k, \mu_\ell^{\otimes -q}) \cong H_{\text{ét}}^0(k, \mu_\ell) \cong \mu_\ell(k) \neq 0$.

However, $\mathsf{M}\mathbf{Z}^{\text{ét}}/\ell$ will be *weightlessly affine 0-connected* (see Definition 4.16). This notion tests vanishing only with respect to the "$S^1$-variable" and affine schemes. It captures the following fact: let $(n, d)$ be a pair of integers such that $n + d < 0$. Then for any affine scheme $X$ of dimension $d$ over an algebraically closed field, the group $[\Sigma^{n,0}\Sigma_T^\infty X_+, \mathsf{M}\mathbf{Z}^{\text{ét}}/\ell] \cong H_{\text{ét}}^{-n}(X; \mu_\ell) = 0$ [1, XIV, Théorème 3.1].

4.3.3.    In the étale-local setting the most relevant notion of connectivity for our purposes is the following one.

**Definition 4.16.** A motivic spectrum $\mathsf{E} \in \mathrm{SH}(S)$ is *weightlessly affine t-connected* if for any pair of integers $(n, d)$ such that $n + d < t$ and any $X \in \mathrm{Sm}_S$ which is an affine scheme of dimension $d$, the group $[\Sigma^{n,0}\Sigma_T^\infty X_+, \mathsf{E}] = 0$.

The above notion is so defined because we are only testing connectivity against generators of $\mathrm{SH}(S)^{\text{eff}}$, namely the collection $\{\Sigma^{n,0}\Sigma_T^\infty X_+\}$ where $n \in \mathbf{Z}$ and $X \in \mathrm{Sm}_S$ for any $X \in \mathrm{Sm}_S$ which is an affine scheme. In spite of the appearance of "effective" in Definition 4.16, $\mathsf{E}$ does not have to be an effective motivic spectrum. Indeed, for our purposes, we will be calculating the effective connectivity of motivic spectra which are of the form $\mathsf{E}^{\text{ét}}$ and there is no reason why $\mathsf{E}^{\text{ét}}$ should be effective since $\epsilon_*$ need not preserve effective objects.

4.4. **Main connectivity results.** The following is our main connectivity result.

**Theorem 4.17.** *Let $k$ be a field of exponential characteristic coprime to $\ell$ and assume $\mathrm{cd}_\ell(k) < \infty$. Let $\mathsf{E}_N \in \mathrm{SH}(k)$ be a Landweber exact motivic spectrum. Then for all integers $(q, w)$ such that $q \geq w$, the spectrum $\Sigma^{-w,-w}(f_q\mathsf{E}_N/\ell^\nu)^{\text{ét}}$ is weightlessly affine $q - \mathrm{cd}_\ell(k)$-connected.*

*Proof.* Proposition 4.12 implies $f_q\mathsf{E}_N/\ell^\nu$ is étale-$\mathbf{A}^1$-naive above degree $-q$. Therefore since $-w \geq -q$, for any $X \in \mathrm{Sm}_k$, the descent spectral sequence (34) takes the form:

$$H_{\text{ét}}^s(X, \underline{\pi}_{-t}^{\text{ét}}(\mathsf{L}_{\text{ét}}\Omega_{\mathbf{G}_m}^\infty\Sigma^{-w,-w}f_q\mathsf{E}_N/\ell^\nu)) \Rightarrow [\Sigma_T^\infty X_+, \Sigma^{-w,-w}(f_q\mathsf{E}_N)^{\text{ét}}/\ell^\nu[s+t]]_{\mathrm{SH}(k)}.$$

Suppose $n + d < q - \mathrm{cd}_\ell(k)$ and $X$ is affine of dimension $d$. We claim the vanishing:

$$[\Sigma^{n,0}\Sigma_T^\infty X_+, \Sigma^{-w,-w}f_q\mathsf{E}_N^{\text{ét}}/\ell^\nu]_{\mathrm{SH}(k)} \cong [\Sigma^{n,0}\epsilon^*\Sigma_T^\infty X_+, \Sigma^{-w,-w}\epsilon^*f_q\mathsf{E}_N/\ell^\nu]_{\mathrm{SH}_{\text{ét}}(k)} = 0.$$

Here, we applied the isomorphism given by the adjunciton $(\epsilon^*, \epsilon_*)$ and §2.12 to commute $\epsilon_*$ past the suspension. Examining the descent spectral sequence, we see that the only contributions come from the terms:

$$H_{\text{ét}}^s(X, \underline{\pi}_{-t}^{\text{ét}}(\mathsf{L}_{\text{ét}}\Omega_{\mathbf{G}_m}^\infty\Sigma^{-w,-w}f_q\mathsf{E}_N/\ell^\nu)),$$

for $s + t = -n$. Whenever $s > d + \mathrm{cd}_\ell(k)$, the latter is trivial for cohomological dimension reasons by [1, XIV, Théorème 3.1] since the étale homotopy sheaves are $\ell$-torsion sheaves.

If $s \leq d + \mathrm{cd}_\ell(k)$, then since we assume $n + d < q - \mathrm{cd}_\ell(k)$, we have that $-t < q$. Thus, the homotopy sheaves $\underline{\pi}_{-t}^{\text{ét}}(\mathsf{L}_{\text{ét}}\Omega_{\mathbf{G}_m}^\infty\Sigma^{-w,-w}f_q\mathsf{E}_N/\ell^\nu) \cong a_{\text{ét}}\underline{\pi}_{-t}^{\text{Nis}}(\Omega_{\mathbf{G}_m}^\infty\Sigma^{-w,-w}f_q\mathsf{E}_N/\ell^\nu)$ vanishes by the connectivity of Landweber exact motivic spectra [30, Lemma 8.11].    $\square$

**Corollary 4.18.** *Under the assumptions in Theorem 4.17 the étale slice tower $\{(f_q\mathsf{MGL})^{\text{ét}}/\ell^\nu\}_{q\in\mathbf{Z}}$ is left bounded.*

*Proof.* Fix $s, w \in \mathbf{Z}$, and an affine $X \in \mathrm{Sm}_k$ of dimension $d$. For $q \gg 0$ we want to show that

$$[\Sigma^{s,w}\Sigma_T^\infty X_+, (f_q\mathsf{MGL}^{\text{ét}})/\ell^\nu] \cong [\Sigma^{s-w,0}\Sigma_T^\infty X_+, \Sigma^{-w,-w}(f_q\mathsf{MGL})^{\text{ét}}/\ell^\nu],$$

vanishes.

When $q \geq w$, $\Sigma^{-w,-w}(f_q\mathsf{MGL}/\ell^\nu)^{\text{ét}}$ is weightlessly affine $q - \mathrm{cd}_\ell(k)$-connected by Theorem 4.17. Hence the desired vanishing holds for all $q > \max\{d + s - w + \mathrm{cd}_\ell(k), w\}$. $\quad\square$

**Corollary 4.19.** *Under the assumptions in Theorem 4.17 we have* $\lim_q(f_q\mathsf{MGL})^{\text{ét}}/\ell^\nu \simeq 0$. *Thus the ét-slice spectral sequence* (17) *for* $\mathsf{MGL}/\ell^\nu$ *is conditionally convergent.*

*Proof.* By Proposition 2.7 it suffices to prove that for all $s, w \in \mathbf{Z}$, and affine $X \in \mathrm{Sm}_k$,

$$[\Sigma^{s,w}X_+, \lim_q(f_q\mathsf{MGL})^{\text{ét}}/\ell^\nu] = 0.$$

To that end we employ the Milnor lim-lim[1] exact sequence:
(42)
$$0 \to \lim_q{}^1[\Sigma^{s+1,w}X_+, (f_q\mathsf{MGL})^{\text{ét}}/\ell^\nu] \to [\Sigma^{s,w}X_+, \lim_q(f_q\mathsf{MGL})^{\text{ét}}/\ell^\nu] \to \lim_q[\Sigma^{s,w}X_+, (f_q\mathsf{MGL})^{\text{ét}}/\ell^\nu] \to 0.$$

By left boundedness of $\{(f_q\mathsf{MGL})^{\text{ét}}/\ell^\nu\}_{q\in\mathbf{Z}}$ in Corollary 4.18 the outer terms in (42) are trivial for $q \gg 0$, and we are done. $\quad\square$

## 5. THE FORGETFUL FUNCTOR $\epsilon_*$ AND COLIMITS

Next we address the other half of the comparison paradigm for the change of topology adjunction of exact functors:

(43) $$\epsilon^* : \mathrm{SH}(S) \rightleftarrows \mathrm{SH}_{\text{ét}}(S) : \epsilon_*.$$

5.0.1. Suppose that $S$ is a scheme and let $\ell$ be a prime. We say that $S$ has *uniformly bounded $\ell$-cohomological dimension* if for all residue fields $k(s)$ of $S$, $\mathrm{cd}_\ell(k(s)) < C_\ell$ for some constant $C_\ell$. The goal of this section is to prove the following theorem

**Theorem 5.1.** *Suppose that $S$ is a Noetherian base scheme. Let $\ell$ be a prime such that $S$ has uniformly bounded $\ell$-cohomological dimension. Then the functor:*

$$\epsilon_* : \mathrm{SH}_{\text{ét}}(S)_{(\ell)} \to \mathrm{SH}(S)_{(\ell)},$$

*preserves colimits.*

The statement that we will need to proceed is the following corollary.

**Corollary 5.2.** *Suppose that $S$ is a Noetherian base scheme. Let $P$ be a set of positive integers which contains primes $p$ for which $S$ does not have uniformly bounded $p$-cohomological dimension. Let $L$ be a product of all primes in $P$ then the functor*

$$\epsilon_* : \mathrm{SH}_{\text{ét}}(k)[\tfrac{1}{L}] \to \mathrm{SH}(k)[\tfrac{1}{L}],$$

*preserves colimits.*

*Proof.* Let $X : I \to \mathrm{SH}_{\text{ét}}(k)[\tfrac{1}{L}], i \mapsto X(i)$ be a small diagram. We need to verify that the natural map $\mathrm{colim}\,\epsilon_*(X(i)) \to \epsilon_*\mathrm{colim}(X(i))$ is an equivalence in $\mathrm{SH}(S)[\tfrac{1}{L}]$. Using the conservative family of localizations:

$$\{\mathrm{SH}(S)[\tfrac{1}{L}] \to \mathrm{SH}(k)_{(\ell)}\}_{\ell\nmid L}^{\ell\text{ prime}},$$

the claim follows from Theorem 5.1. $\quad\square$

5.0.2. To begin proving Theorem 5.1 we note that all compact objects are essentially finite colimits and retracts of $T$-desuspensions of suspension spectra of smooth affine $S$-schemes.

**Lemma 5.3.** *Let $S$ be a base scheme, then $\mathrm{SH}(S)^\omega$ is the smallest stable subcategory of $\mathrm{SH}(S)$ closed under finite colimits and retracts of $\Sigma^{-2n,-n}\Sigma_T^\infty X_+$ where $X$ is a smooth affine $S$-scheme and $n \in \mathbf{Z}$.*

*Proof.* Let C be the smallest subcategory of $\mathrm{SH}(S)$ closed under finite colimits and retracts of $\Sigma^{-2n,-n}\Sigma_T^\infty X_+$ where $X$ is a smooth affine $S$-scheme and $n \in \mathbf{Z}$. Using Proposition 2.7.3, we get that all the $\Sigma^{-2n,-n}\Sigma_T^\infty X_+$ are compact, and thus $\mathrm{C} \subset \mathrm{SH}(S)^\omega$. Now compact objects are stable under finite colimits and retracts [45, Lemma 5.1.6.4.]. Since Proposition 2.7.2 also tells us that $\mathrm{SH}(S)$ is generated by objects in C, we are done. □

5.0.3. Now, since the functors in (43) are exact it suffices to know when $\epsilon_*$ preserves infinite coproducts [46, Proposition 1.4.4.1.3]. We recall the following easy but important lemma:

**Lemma 5.4.** *Suppose $F : \mathrm{C} \rightleftarrows \mathrm{D} : G$ is an adjunction of stable $\infty$-categories where $F, G$ are exact, C and D admit small coproducts, and C is compactly generated. Then $G$ preserves all small coproducts if and only $F$ preserves compact objects.*

*Proof.* If $G$ preserves small coproducts, then [46, Proposition 1.4.4.1.2] implies $G$ preserves all small colimits, and therefore [45, Proposition 5.5.7.2.1] tells us that $F$ preserves compact objects. Conversely, since C is compactly generated it is accessible, and so [45, Proposition 5.5.7.2.2] applies to tell us that $G$ preserves all small colimits. □

5.0.4. We can now proceed to the

*Proof of Theorem 5.1.* After Lemma 5.4 it suffices to prove that $\epsilon^*$ preserves compact objects. Since the étale topology is subcanonical the discussion of §2.2.1 tells us that $\epsilon^*\Sigma_T^\infty X_+ \simeq \Sigma_T^\infty X_+$ for any $X \in \mathrm{Sm}_S$. After Lemma 5.3 we thus need only verify that $\Sigma_T^\infty X_+$ is compact in $\mathrm{SH}_{\text{ét}}(S)_{(\ell)}$ for any $X \in \mathrm{Sm}_S$ which is affine. To do so, we pick a small collection of objects $(\mathsf{E}_j)_{j \in J}$ in $\mathrm{SH}_{\text{ét}}(S)_{(\ell)}$. We compare the descent spectral sequence:

$$(44) \qquad H_{\text{ét}}^p(X, \underline{\pi}_{t-q,t}^{\text{ét}}(\oplus_j \mathsf{E}_j)) \Rightarrow \mathrm{Hom}_{\mathrm{SH}_{\text{ét}}(k)}(\mathbf{G}_m^{\wedge t} \wedge \Sigma_T^\infty X_+, \oplus_j \mathsf{E}[p+q]),$$

with the sum of descent spectral sequences:

$$(45) \qquad \oplus_j H_{\text{ét}}^p(X, \underline{\pi}_{t-q,t}^{\text{ét}}(\mathsf{E}_j)) \Rightarrow \oplus_j \mathrm{Hom}_{\mathrm{SH}_{\text{ét}}(k)}(\mathbf{G}_m^{\wedge t} \wedge \Sigma_T^\infty X_+, \mathsf{E}_j[p+q]).$$

Using the cohomological dimension assumptions on the residue fields, the natural map of spectral sequences (44) → (45) is an isomorphism (see, for example, [15, Lemma 1.1.7]). From the same cohomological dimension assumptions on the residue fields we also deduce strong convergence of the spectral sequences (see the discussion in [78, 5.44-5.48]), so that the isomorphism on $E^2$-pages implies an isomorphism on the abutments.

□

# 6. PROOF OF MAIN THEOREMS

We now prove the main theorems of this paper in the following order:

(1) Construction of Bott elements in motivic cohomology (§6.1).
(2) Proof of Étale Descent for Bott-inverted motivic cohomology for essentially smooth schemes over a field (Theorem 6.7).
(3) Construction of Bott elements in algebraic cobordism (§6.3).
(4) Proof of Étale Descent for Bott-inverted algebraic cobordism for essentially smooth schemes over a field (Theorem 6.26).
(5) Proof of Étale Descent for Bott-inverted algebraic cobordism for Noetherian schemes (Theorem 6.28).
(6) Proof of Étale Descent for Bott-inverted MGL-modules for Noetherian schemes (Theorem 6.29).
(7) Proof of an integral statement (Theorem 6.32).

6.1. **Bott elements in motivic cohomology.** Our goal in this section is to produce *Bott elements* in motivic cohomology over $\mathbf{Z}[\frac{1}{\ell}]$ and, more generally, schemes on which $\ell$ is invertible. To describe this element, we will use the following notation: For a prime number $\ell$ and $\nu \geq 1$ we define:

$$
\mathrm{e}(\ell^\nu) = \begin{cases}
(\ell - 1)\ell^{\nu-1} & \ell \text{ odd} \\
2^{\nu-2} & \ell = 2, \nu \geq 3 \\
2 & \ell = 2, \nu = 2 \\
1 & \ell = 2, \nu = 1.
\end{cases}
$$

Here $\mathrm{e}(\ell^\nu)$ is the exponent of the multiplicative group of units of the cyclic group $\mathbf{Z}/\ell^\nu$. The properties demanded of these elements are summarized in the following proposition:

**Proposition 6.1.** *There exists a collection of elements:*

(46)
$$
\{\tau_{\ell^\nu}^{\mathsf{MZ}} \in H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(\mathbf{Z}[\tfrac{1}{\ell}]; \mathbf{Z}/\ell^\nu)\}_{\nu \geq 1},
$$

*such that:*

(1) *for any $\ell$ odd and $\nu > 1$ and $\ell = 2$ and $\nu \geq 2$, under the reduction map:*

$$
H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(\mathbf{Z}[\tfrac{1}{\ell}]; \mathbf{Z}/\ell^\nu) \xrightarrow{/\ell} H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(\mathbf{Z}[\tfrac{1}{\ell}]; \mathbf{Z}/\ell^{\nu-1}),
$$

*the element $\tau_{\ell^\nu}^{\mathsf{MZ}}$ maps to $(\tau_{\ell^{\nu-1}}^{\mathsf{MZ}})^\ell$.*

(2) *For any field $k$ with $\frac{1}{\ell} \in k$, let $q : \mathrm{Spec}\, k \to \mathrm{Spec}\, \mathbf{Z}[\frac{1}{\ell}]$ be the canonical map. Then the element:*

$$
q^* \tau_{\ell^\nu}^{\mathsf{MZ}} \in H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(k; \mathbf{Z}/\ell^\nu) \cong H_{\text{ét}}^0(k, \mu_{\ell^\nu}^{\mathrm{e}(\ell^\nu)}),
$$

*is a periodicity operator in étale cohomology.*

(3) *Let $\{\tau_{\ell^\nu}^{\mathsf{MZ}'}\}_{\nu \geq 1}$ be another choice of a collection as in (46) which satisfies (1) and (2) above. Then the spectra $\mathsf{MZ}/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MZ}'})^{-1}]$ and $\mathsf{MZ}/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MZ}})^{-1}]$ are equivalent.*

We call any collection as in (46) a *system of $\mathsf{MZ}$ $\ell$-adic Bott elements* while for a fixed $\nu \geq 1$ we call $\tau_{\ell^\nu}^{\mathsf{MZ}}$ an $\mathsf{MZ}$ *mod-$\ell^\nu$ Bott element.* Pulling back along $f : S \to \mathrm{Spec}\, \mathbf{Z}[\frac{1}{\ell}]$ yields a collection

(47)
$$
\{(\tau_{\ell^\nu}^{\mathsf{MZ}})_S := f^* \tau_{\ell^\nu}^{\mathsf{MZ}} \in H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(S; \mathbf{Z}/\ell^\nu)\}_{\nu \geq 1}.
$$

6.1.1. We proceed with the construction of the classes in (6.1). Let $D$ be a Dedekind domain. Then, according to part (5) of Theorem 2.18, we have an isomorphism:

$$
H_{\mathsf{m}}^{1,1}(\mathrm{Spec}\, D; \mathbf{Z}) \cong D^\times.
$$

For any integer $n$, by the universal coefficients theorem, we have an isomorphism:

(48)
$$
H_{\mathsf{m}}^{0,1}(\mathrm{Spec}\, D; \mathbf{Z}/n) \cong \mu_n(D).
$$

Therefore, for any prime $\ell$ and $\nu \geq 1$, we have an $\mathsf{MZ}$ mod-$\ell$ Bott element over the ring $\mathbf{Z}[\frac{1}{\ell}, \zeta_\ell]$ where $\zeta_\ell$ is a primitive $\ell$-th root of unity. Namely, it is the class of:

(49)
$$
\zeta_\ell \in \mu_\ell(D) \cong H_{\mathsf{m}}^{0,1}(\mathrm{Spec}\, \mathbf{Z}[\tfrac{1}{\ell}, \zeta_\ell]; \mathbf{Z}/\ell),
$$

which we denote by $\widetilde{\tau}_\ell^{\mathsf{MZ}}$. The extension of rings $\mathbf{Z}[\frac{1}{\ell}] \to \mathbf{Z}[\frac{1}{\ell}, \zeta_\ell]$ induces a Galois extension of Dedekind schemes $\pi_\ell : \mathrm{Spec}\, \mathbf{Z}[\frac{1}{\ell}, \zeta_\ell] \to \mathrm{Spec}\, \mathbf{Z}[\frac{1}{\ell}]$. The automorphism group of this extension is isomorphic to the group of units $(\mathbf{Z}/\ell)^\times$; we denote the former group by $G_\ell$. Crucially, this group is of order prime to $\ell$. The group $G_\ell$ acts via the cyclotomic character on the subgroup of units in $\mathbf{Z}[\frac{1}{\ell}, \zeta_\ell]$ generated by $\zeta_\ell$:

$$
\chi_\ell : G_\ell \to \mathrm{Aut}(\langle \zeta_\ell \rangle).
$$

**Lemma 6.2.** *Let $\ell$ be an odd prime and suppose that $\nu \geq 1$ then*

(1) *the element $(\widetilde{\tau}_\ell^{\mathsf{MZ}})^{\ell-1} \in H_{\mathsf{m}}^{0,\ell-1}(\mathrm{Spec}\, \mathbf{Z}[\frac{1}{\ell}, \zeta_\ell]; \mathbf{Z}/\ell)$ is invariant under the action of $G_\ell$.*

(2) *the element $(\widetilde{\tau}_\ell^{MZ})^{e(\ell^\nu)} \in H_m^{0,e(\ell^\nu)}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},\zeta_\ell]; \mathbf{Z}/\ell)$ is the reduction mod $\ell$ of an element*

(50)
$$\widetilde{\tau}_{\ell^\nu}^{MZ} \in H_m^{0,e(\ell^\nu)}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},\zeta_\ell]; \mathbf{Z}/\ell^\nu),$$

*which is also $G_\ell$-invariant.*

*Proof.* By the discussion in the previous paragraph, $G_\ell$ acts on the subgroup of $H_m^{0,1}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},\zeta_\ell]; \mathbf{Z}/\ell)$ generated by $\widetilde{\tau}_{\ell^\nu}^{MZ}$ via $\chi_{\ell^\nu}$. Hence, the $\ell-1$-st cup power of $\widetilde{\tau}_{\ell^\nu}^{MZ}$ is $G_\ell$-invariant. Now the $\nu$-th Bockstein map, $\beta_\nu$, fits into the following exact sequence:

$$H_m^{0,*}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},\zeta_\ell], \mathbf{Z}/\ell^\nu) \xrightarrow{/\ell} H_m^{0,*}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},\zeta_\ell], \mathbf{Z}/\ell) \xrightarrow{\beta_\nu} H_m^{1,*}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},\zeta_\ell], \mathbf{Z}/\ell^{\nu-1}),$$

and is a derivation. Therefore, we have that:

$$\beta_\nu((\tau_\ell^{MZ})^{\ell^\nu}) = \ell^\nu \beta_\nu(\tau_\ell^{MZ}) = 0.$$

Hence there is an element $\widetilde{\tau}_{\ell^\nu}^{MZ'} \in H_m^{0,(\ell-1)(\ell^\nu)}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},\zeta_\ell], \mathbf{Z}/\ell^\nu)$ whose reduction mod $\ell$ is $(\widetilde{\tau}_\ell^{MZ})^{\ell-1}$. By naturality of the Bockstein maps and part (1), we obtain the $G_\ell$-invariance statement. $\qquad\square$

The same argument gives the even case with slightly different numerics.

**Lemma 6.3.** *Suppose that $\nu \geq 3$ then*

(1) *the element $(\widetilde{\tau}_\ell^{MZ})^2 \in H_m^{0,2}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},i]; \mathbf{Z}/\ell)$ is invariant under the action of $G_\ell$.*

(2) *the element $(\widetilde{\tau}_\ell^{MZ})^{e(2^\nu)} \in H_m^{0,e(2^\nu)}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},i]; \mathbf{Z}/\ell)$ is the reduction mod $\ell$ of an element:*

(51)
$$\widetilde{\tau}_{2^\nu}^{MZ} \in H_m^{0,e(2^\nu)}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},i]; \mathbf{Z}/\ell^\nu),$$

*which is also $G_\ell$-invariant.*

6.1.2.   We use transfers in motivic cohomology (which we discuss in the generality of MGL-modules in Appendix §11) to descend the Bott element down to $H_m^{*,*}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell}]; \mathbf{Z}/\ell^\nu)$.

**Lemma 6.4.** *Let $\ell$ be a prime and $\nu \geq 1$. The natural map:*

$$\epsilon_\ell^* : H_m^{*,*}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell}]; \mathbf{Z}/\ell^\nu) \to H_m^{*,*}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},\zeta_\ell]; \mathbf{Z}/\ell^\nu),$$

*factors through an isomorphism:*

(52)
$$\epsilon_\ell^* : H_m^{*,*}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell}]; \mathbf{Z}/\ell^\nu) \xrightarrow{\cong} H_m^{*,*}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},\zeta_\ell]; \mathbf{Z}/\ell^\nu)^{G_\ell}$$

*Proof.* Since $G_\ell$ acts trivially on the group $H_m^{*,*}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell}]; \mathbf{Z}/\ell^\nu)$, the map $\epsilon_\ell^*$ factors through the $G_\ell$-invariants of $H_m^{*,*}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},\zeta_\ell]; \mathbf{Z}/\ell^\nu)$. The claim now follows from a standard transfer argument and the fact that $\ell-1$ is coprime to $\ell^\nu$. Indeed, since $M\mathbf{Z}/\ell^\nu$ is an MGL-module, we have a transfer map by Proposition 11.1 :

$$\pi_{\ell*} : H_m^{*,*}(\mathbf{Z}[\frac{1}{\ell},\zeta_\ell]; \mathbf{Z}/\ell^\nu) \to H_m^{*,*}(\mathbf{Z}[\frac{1}{\ell}]; \mathbf{Z}/\ell^\nu).$$

such that

(1) $\pi_{\ell*}\epsilon_\ell^* = (|G_\ell|)\cdot = (\ell-1)\cdot$,

(2) $\epsilon_\ell^*\pi_{\ell*} = \underset{g \in G_\ell}{\Sigma g_*}$, where $g_*$ is the action of $g \in G_\ell$ on the cohomology group.

The first property implies that we have an injection:

$$\epsilon_\ell^* : H_m^{*,*}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell}]; \mathbf{Z}/\ell^\nu) \hookrightarrow H_m^{*,*}(\operatorname{Spec} \mathbf{Z}[\frac{1}{\ell},\zeta_\ell]; \mathbf{Z}/\ell^\nu)^{G_\ell}.$$

The second property tells us if $x$ is an $G_\ell$-invariant element, then:

$$\epsilon_\ell^*\pi_{\ell*}x = \underset{g \in G_\ell}{\Sigma g_*}x = |G_\ell|x.$$

Whence the map:

$$\epsilon_* / |G_\ell| : H_{\mathsf{m}}^{*,*}(\mathrm{Spec}\ \mathbf{Z}[\tfrac{1}{\ell}, \zeta_\ell]; \mathbf{Z}/\ell^\nu)^{G_\ell} \hookrightarrow H_{\mathsf{m}}^{*,*}(\mathrm{Spec}\ \mathbf{Z}[\tfrac{1}{\ell}, \zeta_\ell]; \mathbf{Z}/\ell^\nu) \to H_{\mathsf{m}}^{*,*}(\mathbf{Z}[\tfrac{1}{\ell}]; \mathbf{Z}/\ell^\nu),$$

is a canonical inverse to $i^*$. □

6.1.3. Now, we give a

*Proof of Proposition 6.1.* For $\ell$ odd and $\nu \geq 1$ and $\ell = 2$ and $\nu \geq 3$, Lemmas 6.2 and 6.3, furnishes us with an element:

$$(\widetilde{\tau}_{\ell^\nu}^{\mathsf{MZ}})^{\mathrm{e}(\ell^\nu)} \in H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(\mathrm{Spec}\ \mathbf{Z}[\tfrac{1}{\ell}, \zeta_{\ell^\nu}]; \mathbf{Z}/\ell^\nu)^{G_\ell},$$

depending only on the choice of a primitive $\ell$-th root of unity in $\mathbf{Z}[\tfrac{1}{\ell}, \zeta_\ell]$. The isomorphism in Lemma 6.4, then gives us a *unique* element:

$$\tau_\ell^{\mathsf{MZ}} \in H_{\mathsf{m}}^{0,\ell-1}(\mathrm{Spec}\ \mathbf{Z}[\tfrac{1}{\ell}]; \mathbf{Z}/\ell^\nu).$$

When $\nu > 1$, take $\widetilde{\tau}_{\ell^\nu}^{\mathsf{MZ}} \in H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(\mathrm{Spec}\ \mathbf{Z}[\tfrac{1}{\ell}, \zeta_\ell]; \mathbf{Z}/\ell^\nu)$ as in (51) which is $G_\ell$-invariant by Lemma 6.2.2 and hence gives us a unique element $\tau_{\ell^\nu}^{\mathsf{MZ}} \in H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(\mathrm{Spec}\ \mathbf{Z}[\tfrac{1}{\ell}]; \mathbf{Z}/\ell^\nu)$ by Lemma 6.4. Now, point (1) of Proposition 6.1 follows by the commutativity of the diagram:

(53)
$$\begin{array}{ccc}
H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(\mathrm{Spec}\ \mathbf{Z}[\tfrac{1}{\ell}]; \mathbf{Z}/\ell^\nu) & \xrightarrow{/\ell} & H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(\mathrm{Spec}\ \mathbf{Z}[\tfrac{1}{\ell}]; \mathbf{Z}/\ell^{\nu-1}) \\
\downarrow & & \downarrow \\
H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(\mathrm{Spec}\ \mathbf{Z}[\tfrac{1}{\ell}, \zeta_\ell]; \mathbf{Z}/\ell^\nu) & \xrightarrow{/\ell} & H_{\mathsf{m}}^{0,\mathrm{e}(\ell^\nu)}(\mathrm{Spec}\ \mathbf{Z}[\tfrac{1}{\ell}, \zeta_\ell]; \mathbf{Z}/\ell^{\nu-1}),
\end{array}$$

the naturality of the Bockstein sequences, and the construction of the Bott elements. The fact that they are periodicity operators for étale cohomology follows from the construction using cyclotomic characters and the fact that the $\mathrm{e}(\ell^\nu)$'s are the exponents of the multiplicative group of units of the cyclic group $\mathbf{Z}/\ell^\nu$.

To address the remaining cases, we note that when $\ell = 2$ and $\nu = 1$ there is nothing to show. When $\ell = 2$ and $\nu = 2$, we note that if we take $[i] \in H_{\mathsf{m}}^{0,1}(\mathbf{Z}[\tfrac{1}{2}, i], \mathbf{Z}/4)$, then $[i]^2 = [-1]$ is in $H_{\mathsf{m}}^{0,2}(\mathbf{Z}[\tfrac{1}{2}], \mathbf{Z}/4)$ and we are done as before. The last statement is proved in Lemma 6.5 below. □

6.2. **Bott inverted motivic cohomology.** Let $S$ be a $\mathbf{Z}[\tfrac{1}{\ell}]$-scheme and suppose that we have a collection $\{\tau_{\ell^\nu}^{\mathsf{MZ}}\}$ as in (47), then we may employ the formalism of §8.1 to construct the *Bott-inverted motivic cohomology* spectra $\{\mathsf{MZ}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MZ}})_S^{-1}]\}$. Each $\mathsf{MZ}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MZ}})_S^{-1}]$ is a $\mathcal{E}_\infty$-ring spectrum by Proposition 8.4.

**Lemma 6.5.** *Suppose $\ell$ is coprime to the exponential characteristic of $k$. Then any choice of $\tau_{\ell^\nu}^{\mathsf{MZ}}$ which satisfies the first two statements of Proposition 6.1 gives equivalent Bott inverted motivic cohomology $\mathcal{E}_\infty$-rings.*

*Proof.* Under the cyclotomic character $\kappa \colon G_k \to \mathbf{Z}_\ell^\times$ the Galois module $\mu_{\ell^\nu}^{\otimes i}$ corresponds to the $i$th Tate twist of the $\mathbf{Z}_\ell$-module $\mathbf{Z}/\ell^\nu$ on which $\mathbf{Z}_\ell^\times$ acts by a homomorphism $\mathbf{Z}_\ell^\times \to \mathrm{Aut}(\mathbf{Z}/\ell^\nu)$. Thus different choices of $\mathsf{MZ}$-theoretic mod-$\ell^\nu$ Bott elements coincide up to an automorphism, resulting in an equivalence between the corresponding Bott inverted $\mathcal{E}_\infty$-rings. □

6.2.1.   Using Theorem 2.18.1 and the properties of the Bott element as in Proposition 46 we immediately deduce that for any morphism $f : T \to S$ we have a canonical equivalence of Bott-inverted motivic cohomology:

$$(54) \qquad f^* M\mathbf{Z}_S / \ell^v [(\tau_{\ell^v}^{MZ})^{-1}] \simeq f^* M\mathbf{Z}_T / \ell^v [(\tau_{\ell^v}^{MZ})^{-1}].$$

The equivalence (54) shows that Bott inverted motivic cohomology pulls back, we will also need to know how étale motivic cohomology pulls back, at least along essentially smooth morphisms.

**Lemma 6.6.** *For essentially smooth schemes $S$ and $T$ over a field $k$ and $f : T \to S$ a smooth morphism over* Spec $k$, *there is an equivalence:*

$$(55) \qquad f^* \epsilon_* \epsilon^* M\mathbf{Z}_S / \ell^v \simeq \epsilon_* \epsilon^* M\mathbf{Z}_T / \ell^v.$$

*Proof.* Since $f^*$ commutes with $\epsilon^*$ already on the level of sheaves, we need only prove that $f^*$ commutes with $\epsilon_*$. To see this, note that these functors have left adjoints $f_\#$ and $\epsilon^*$, respectively, and there is an equivalence $f_\# \epsilon^* \simeq \epsilon^* f_\#$ since both sides agree on representables and preserve colimits, being left adjoints. $\qquad\square$

6.2.2.   We now turn to the analog of our main results for motivic cohomology, which is essentially a consequence of Theorem 1.3. We first give the details of the proof of that result. For the reader who would rather skip this proof, one can always rely on Voevodsky's proof of the Bloch-Kato conjecture to yield the weaker Theorem 1.3, as mentioned in Remark 1.4.

*Proof of Theorem 1.3.* As mentioned in the introduction, we need only handle the case char $k = 0$, $\ell = 2$ and $\sqrt{-1} \notin k$. The argument is an addendum to the proof of [40, Theorem 4.5]. Following that proof, we may assume that $k$ is finitely generated over $\mathbf{Q}$. Let $k_0$ be the algebraic closure of $\mathbf{Q}$ in $k$ and let $\bar{k}$ be the algebraic closure of $k$. Following the arguments of *loc. cit.* it suffices to construct a tower of fields

$$k = L_0 \subset L_1 \subset \ldots \subset L_N = \bar{k}$$

such that $L_i$ is Galois over $L_{i-1}$ and each Galois group $\mathrm{Gal}(L_i / L_{i-1})$ has 2-cohomological dimension $\leq 1$. We let $k_1 = k_0(\mu_{2^\infty})$ and $k_2 = \bar{\mathbf{Q}}$, $L_1 = L_0 k_1$, $L_2 = L_0 k_2$. As in *loc. cit.*, one constructs the $L_i$ for $i > 2$ by using a trancendence basis of $L_2$ over $\bar{\mathbf{Q}}$ and $\mathrm{Gal}(L_i / L_{i-1})$ has 2-cohomological dimension $\leq 1$ for $i > 2$, so the only question is for the layers $L_1/L_0$ and $L_2/L_1$.

As a quotient of $\mathrm{Gal}(\bar{k}/k)$, $\mathrm{Gal}(L_1/L_0)$ has finite 2-cohomological dimension. Since $\mathrm{Gal}(L_1/L_0)$ is a open subgroup of $\mathrm{Gal}(\mathbf{Q}(\mu_{2^\infty})/\mathbf{Q}) = \mathbf{Z}_2 \times \mathbf{Z}/2$ and as an open subgroup of $\mathbf{Z}_2 \times \mathbf{Z}/2$ with finite 2-cohomological dimension is isomorphic to $\mathbf{Z}_2$, we have $\mathrm{Gal}(L_1/L_0) \cong \mathbf{Z}_2$ and thus $\mathrm{Gal}(L_1/L_0)$ has 2-cohomological dimension 1. It follows from [70, II, Proposition 9] that $\mathrm{Gal}(k_2/k_1)$ has 2-cohomological dimension 1 and as an open subgroup of $\mathrm{Gal}(k_2/k_1)$, $\mathrm{Gal}(L_2/L_1)$ has 2-cohomological dimension 1 as well. $\qquad\square$

**Theorem 6.7.** *Let $k$ be a field with exponential characteristic prime to $\ell$ and let $f : S \to$ Spec $k$ be an essentially smooth scheme over $k$. Suppose that $\mathrm{cd}_\ell(k) < \infty$. For all $v \geq 1$, the natural map:*

$$M\mathbf{Z}_S / \ell^v \to M\mathbf{Z}_S^{\text{ét}} / \ell^v,$$

*induces an equivalence in* SH$(S)$:

$$M\mathbf{Z}_S / \ell^v [(\tau_{\ell^v}^{MZ})_S^{-1}] \xrightarrow{\simeq} M\mathbf{Z}_S^{\text{ét}} / \ell^v.$$

*Consequently, we have an equivalence on $\ell$-completions:*

$$M\mathbf{Z}_S [(\tau^{MZ})_\ell^\wedge] \simeq M\mathbf{Z}_S^{\text{ét}\wedge}{}_\ell.$$

*Proof.* After Corollary 2.20 and essentially smooth base change (to reduce the essentially smooth case to the smooth case) [30, Appendix A], it suffices to prove that for all smooth $k$-schemes $S$, the canonical map:

$$H_{\mathfrak{m}}^{*,*}(S, \mathbf{Z}/\ell^v)[(\tau_{\ell^v}^{MZ})_S^{-1}] \to H_{\text{ét}}^*(S; \mu_\ell^{\otimes *}),$$

is a isomorphism. This is Theorem 1.3.                                            □

6.3. **Bott elements in algebraic cobordism.** We now proceed to prove étale descent results for Bott-inverted algebraic cobordism. We first construct Bott elements in MGL over a Dedekind domain. To do so we will need further input from [76], where the Hopkins-Morel isomorphism is suitably generalized to Dedekind domains of mixed characteristics. Recall that if $S$ is a base scheme, we have the *Hopkins-Morel map* [75, §11.1]:

$$(56) \qquad \Phi_S : \mathsf{MGL}_S/(x_1, \cdots x_n) \to \mathsf{M}\mathbf{Z}_S.$$

Let $R$ be a ring, and let $\mathbf{1}_{(R)}$ be the motivic Moore spectrum corresponding to $R$. If $R$ is a localization of $\mathbf{Z}$, then smashing with $\mathbf{1}_{(R)}$ computes the $R$-localization of $\mathrm{SH}(S)$. The state-of-the-art of the Hopkins-Morel-Hoyois isomorphism is given by [75, Theorem 11.3].

**Theorem 6.8.** *Let $S$ be a base scheme and let $R$ be a ring such that for any positive residue characteristic of $p$ of $S$, then the prime $p$ is invertible in $R$. Then, the Hopkins-Morel map* (56)*:*

$$(57) \qquad \Phi_S \wedge \mathbf{1}_{(R)} : \mathsf{MGL}_S/(x_1, \cdots x_n) \wedge \mathbf{1}_{(R)} \to \mathsf{M}R_S,$$

*is an equivalence.*

6.3.1. Furthermore, we have the following results by the third author on the slices of MGL.

**Theorem 6.9.** *Let $S$ be an essentially smooth scheme over Dedekind domain $D$. Let $R$ be a localization of $\mathbf{Z}$ such that $\Phi_S \wedge \mathbf{1}_{(R)}$ is an equivalence. Then*

(1) $\mathsf{MGL}_S \wedge \mathbf{1}_{(R)}$ *is slice complete in the sense of* §3.0.5.
(2) *For all $k \in \mathbf{Z}$, the $k$-th effective cover $f_k \mathsf{MGL}_D \wedge \mathbf{1}_{(R)}$ is $k$-connective in the homotopy $t$-structure, i.e., it is in $\mathrm{SH}(D)_{\geq k}$.*
(3) *We have an equivalence of graded $\mathcal{E}_\infty$-algebra objects in $\mathrm{SH}(S)$:*

$$(58) \qquad s_* \mathsf{MGL}_S \wedge \mathbf{1}_{(R)} \simeq \mathsf{M}R_S[x_1, ..., x_q, ...],$$

   *where each $x_q$ is assigned degree $q \in \mathbb{N}$ and corresponds to a copy of $\mathsf{M}R$ suspended $(2q, q)$-times.*

*Proof.* The first statement is [76, Corollary 5.9], the second is [76, Proposition 7.1], the third statement follows from [76, Theorem 3.1], and the highly coherent multiplicativity of the slice filtration [25], while the second statement comes from Theorem 6.8.                              □

   In other words, $(s_q \mathsf{MGL}_S) \wedge \mathbf{1}_{(R)}$ is a sum of $\Sigma^{2q,q} \mathsf{M}R$'s indexed by monomials $x_i$ of total degree $q$. For low dimensional examples:

- $s_0 \mathsf{MGL}_S \wedge \mathbf{1}_{(R)} \simeq \mathsf{M}R_S$
- $s_1 \mathsf{MGL}_S \wedge \mathbf{1}_{(R)} \simeq \Sigma^{2,1} \mathsf{M}R_S\{x_1\}$
- $s_2 \mathsf{MGL}_S \wedge \mathbf{1}_{(R)} \simeq \Sigma^{4,2} \mathsf{M}R_S\{x_1^2\} \vee \Sigma^{4,2} \mathsf{M}R_S\{x_2\}$
- $s_3 \mathsf{MGL}_S \wedge \mathbf{1}_{(R)} \simeq \Sigma^{6,3} \mathsf{M}R_S\{x_1^3\} \vee \Sigma^{6,3} \mathsf{M}R_S\{x_1 x_2\} \vee \Sigma^{6,3} \mathsf{M}R_S\{x_3\}.$

6.3.2.   We note that for any base scheme $S$ $s_*(\mathsf{MGL}_S)$ is a graded $\mathcal{E}_\infty$ algebra over the $\mathcal{E}_\infty$-ring spectrum $s_0 \mathsf{MGL}_S$; see [25, §6 (iv),(v)]. Hence, if the Hopkins-Morel map (56) is an equivalence $s_*(\mathsf{MGL}_S) \wedge \mathbf{1}_{(R)}$ is a graded $\mathcal{E}_\infty$-algebra over $\mathsf{M}R_S$ and the equivalence in (58) indicates the $\mathsf{M}R_S$-algebra structure of $s_*(\mathsf{MGL}_S) \wedge \mathbf{1}_{(R)}$. Now since $\epsilon_* \epsilon^*$ is a lax monoidal functor, applying it to this $\mathcal{E}_\infty$-graded algebra object gives the $\mathcal{E}_\infty$-graded algebra $\epsilon_* \epsilon^* s_*(\mathsf{MGL}_S)$.

**Lemma 6.10.** *Let $S$ be any base scheme. There is an equivalence of graded algebras over $\epsilon_* \epsilon^* s_0 \mathsf{MGL} \simeq \mathsf{M}\mathbf{Z}^{\text{ét}}$:*

$$\epsilon_* \epsilon^* s_*(\mathsf{MGL}_S) \wedge \mathbf{1}_{(R)} \simeq \mathsf{M}R_S^{\text{ét}}[x_1, ..., x_q....].$$

*Proof.* The terms of the graded spectrum $\epsilon_*\epsilon^* s_*(\mathrm{MGL}_S)$ are just shifts of $\epsilon_*\epsilon^* \mathrm{M}\mathbf{Z}_S \simeq \epsilon_* \mathrm{M}\mathbf{Z}_S^{\text{ét}}$, and so we have an equivalence of graded objects. The equivalence of algebras follows from the fact that $\epsilon_*\epsilon^*$ is a lax monoidal functor. In more detail, we first identify $s_0\mathrm{MGL}_S \wedge \mathbf{1}_{(R)}$ with $\mathrm{M}R_S$. Then, a graded $\mathrm{M}R$-algebra is just a commutative algebra object in the $\infty$-category $\mathrm{Fun}(\mathbb{N}^\delta, \mathrm{Mod}_{\mathrm{M}R})$ with respect to the Day convolution monoidal structure. Here $\mathbb{N}^\delta$ is the constant simplicial set on the the set of natural numbers with the monoidal structure obtained from addition. The adjunction:

$$\epsilon^* : \mathrm{Mod}_{\mathrm{M}R_S} \rightleftarrows \mathrm{Mod}_{\mathrm{M}R_S^{\text{ét}}} : \epsilon_*,$$

can be promoted to an adjunction where $\epsilon^*$ is monoidal and $\epsilon_*$ is lax monoidal:

$$\epsilon^* : \mathrm{Fun}(\mathbb{N}^\delta, \mathrm{Mod}_{\mathrm{M}R_S}) \rightleftarrows \mathrm{Fun}(\mathbb{N}^\delta, \mathrm{Mod}_{\mathrm{M}R_S^{\text{ét}}}) : \epsilon_*.$$

The composite $\epsilon_*\epsilon^*$ is lax monoidal, and thus it preserves algebras. $\qquad\square$

The unit map induces a map of graded $\mathcal{E}_\infty$-$\mathrm{M}\mathbf{Z}$-algebras:

$$(59) \quad s_*(\mathrm{MGL}_S) \wedge \mathbf{1}_{(R)} \simeq \mathrm{M}R_S[x_1, ..., x_q, ...] \to \epsilon_*\epsilon^* s_*(\mathrm{MGL}_S) \wedge \mathbf{1}_{(R)} \simeq \mathrm{M}R_S^{\text{ét}}[x_1, ..., x_q....].$$

Now, suppose that $\ell$ is invertible in $S$ and $\nu \geq 1$, we shall consider the map of graded $\mathcal{E}_\infty$-$\mathrm{M}\mathbf{Z}/\ell$-algebras:

$$(60) \quad s_*(\mathrm{MGL}_S/\ell^\nu) \simeq \mathrm{M}\mathbf{Z}_S/\ell^\nu[x_1, ..., x_q, ...] \to \epsilon_*\epsilon^* s_*(\mathrm{MGL}_S/\ell^\nu) \simeq (\mathrm{M}\mathbf{Z}_S^{\text{ét}}/\ell^\nu)[x_1, ..., x_q....].$$

6.3.3. *Bott elements in* $\mathrm{MGL}$. We now choose Bott elements for $\mathrm{MGL}$. Our method extracts Bott elements for $\mathrm{MGL}$ from its slice spectral sequence, i.e., the Bott elements originate from the Bott elements in motivic cohomology in the sense of Proposition 6.1. While there could be other methods to produce the Bott elements for $\mathrm{MGL}$, we go through the trouble of tracing their origins in the slice spectral sequence precisely because the source of étale descent for Bott-inverted $\mathrm{MGL}$ comes from étale descent for Bott inverted motivic cohomology. In other words, we amplify étale descent on the $E_2$-page to the target, after overcoming some difficult convergence issues. Let us begin by recording a vanishing range for motivic cohomology over Dedekind domains.

**Lemma 6.11.** *Suppose that $\mathscr{O}$ is the ring of integers in a number field $F$ such that $\ell$ is invertible in $\mathscr{O}$, then $H_{\mathrm{m}}^{p,q}(\mathrm{Spec}\ \mathscr{O}; \mathbf{Z}/\ell)$ vanishes whenever $p > q$ except when $(p, q) = (2, 1)$.*

*Proof.* By [75, Corollary 7.12], we have a cofiber sequence:

$$\oplus_{\kappa(\mathfrak{p})} i_{\kappa(\mathfrak{p})*} \mathrm{M}\mathbf{Z}/\ell_{\kappa(\mathfrak{p})}(-1)[-2] \to \mathrm{M}\mathbf{Z}/\ell_{\mathscr{O}} \to j_* \mathrm{M}\mathbf{Z}/\ell_F.$$

Therefore, for all $p, q \in \mathbf{Z}$, we have an exact sequence:

$$\oplus_{\kappa(\mathfrak{p})} H_{\mathrm{m}}^{p-2,q-1}(\mathrm{Spec}\ \kappa(\mathfrak{p}); \mathbf{Z}/\ell) \to H_{\mathrm{m}}^{p,q}(\mathrm{Spec}\ \mathscr{O}; \mathbf{Z}/\ell) \to H_{\mathrm{m}}^{p,q}(\mathrm{Spec}\ F; \mathbf{Z}/\ell).$$

By vanishing of motivic cohomology of fields (see, for example, [30, Corollary 4.26]), the only range we need to check is when $p = q + 1$. In this case the right most term is zero. Now we have an exact sequence:

$$H_{\mathrm{m}}^{q,q}(\mathrm{Spec}\ F; \mathbf{Z}/\ell) \to \oplus_{\kappa(\mathfrak{p})} H_{\mathrm{m}}^{q-1,q-1}(\mathrm{Spec}\ \kappa(\mathfrak{p}); \mathbf{Z}/\ell) \to H_{\mathrm{m}}^{q+1,q}(\mathrm{Spec}\ \mathscr{O}; \mathbf{Z}/\ell),$$

Whenever $q > 1$, the first map identifies with the map boundary map in Milnor $K$-theory:

$$K_q^M(F) \xrightarrow{\partial} \oplus_{\kappa(\mathfrak{p})} K_{q-1}^M(\kappa(\mathfrak{p})).$$

which is always a surjection by [51, Theorem 2.3] under the identification of motivic cohomology with Milnor $K$-theory of Nesterenko-Suslin and Totaro; see for example [49, Lecture 5]. $\qquad\square$

6.3.4. The mod-$\ell$ MGL Bott elements are produced from the following.

**Lemma 6.12.** *Any* M$\mathbf{Z}$ *mod-$\ell$ Bott element* $\tau_\ell^{\mathsf{MZ}} \in h^{0,\ell-1}(\mathbf{Z}[\frac{1}{\ell}])$ *defines a permanent cycle in the slice spectral sequence for* $\mathsf{MGL}/\ell_{\mathbf{Z}[\frac{1}{\ell}]}$. *Consequently, there exists an* MGL *mod-$\ell$ Bott element* $\tau_\ell^{\mathsf{MGL}} \in \mathsf{MGL}/\ell_{0,1-\ell}(\mathbf{Z}[\frac{1}{\ell}])$ *that is detected under the edge map by the element* $\tau_\ell^{\mathsf{MZ}}$ *of order $\ell$.*

*Proof.* Consider the slice spectral sequence for $\mathsf{MGL}/\ell_{\mathbf{Z}[\frac{1}{\ell},\zeta_\ell]}$ in weight $w = -1$:
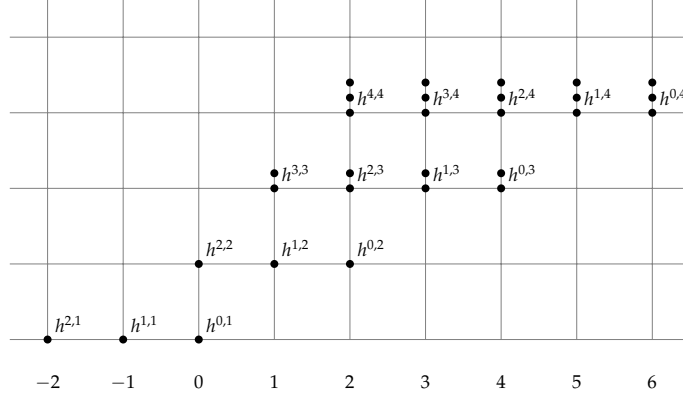


Here, we have written:

$$h^{p,q} := H_{\mathsf{m}}^{p,q}(\mathbf{Z}[\frac{1}{\ell},\zeta_\ell];\mathbf{Z}/\ell),$$

and we stick with this notation for the remainder of the proof. The vanishing range in the display above is furnished by Lemma 6.11.

Now, since no differentials enter or exit the 0-th stem, we obtain the short exact sequence:

$$(61) \qquad 0 \to h^{2,2}(\mathbf{Z}[\frac{1}{\ell},\zeta_\ell]) \to \mathsf{MGL}/\ell_{0,-1}(\mathbf{Z}[\frac{1}{\ell},\zeta_\ell]) \to h^{0,1}(\mathbf{Z}[\frac{1}{\ell},\zeta_\ell]) \to 0.$$

Therefore any choice of $\widetilde{\tau}_\ell^{\mathsf{MZ}}$ corresponding to a generator of $h^{0,1}(\mathbf{Z}[\frac{1}{\ell},\zeta_\ell]) \cong \mu_\ell(\mathbf{Z}[\frac{1}{\ell},\zeta_\ell])$ (as in (49)) determines an element $\tau_\ell^{\mathsf{MGL}} \in \mathsf{MGL}/\ell_{0,-1}(\mathbf{Z}[\frac{1}{\ell},,\zeta_\ell])$. This disposes of the case $\ell = 2$.

Suppose first that $\ell \geq 5$. Consider the Galois extension $\mathbf{Z}[\frac{1}{\ell}] \subset \mathbf{Z}[\frac{1}{\ell},\zeta_\ell]$ with Galois group $G_\ell$. Let $\tau \in \mathsf{MGL}/\ell_{0,-1}(\mathbf{Z}[\frac{1}{\ell},\mu_\ell])$ be picked as in the procedure of the preceding paragraph. Since $\mathsf{MGL}/\ell$ is a homotopy commutative and associative ring spectrum for $\ell \geq 5$, the element $\tau^{\ell-1} \in \mathsf{MGL}/\ell_{0,1-\ell}(\mathbf{Z}[\frac{1}{\ell},\zeta_\ell])$ is Galois invariant. By the transfer constructed in Proposition 11.1, the isomorphism (52) gives a unique element in $\tau_\ell^{\mathsf{MGL}} \in \mathsf{MGL}/\ell_{0,1-\ell}(\mathbf{Z}[\frac{1}{\ell}])$ such that $\tau_\ell^{\mathsf{MGL}}$ maps to $\tau^{\ell-1}$ under the natural map:

$$\mathsf{MGL}/\ell_{0,1-\ell}(\mathbf{Z}[\frac{1}{\ell}]) \to \mathsf{MGL}/\ell_{0,1-\ell}(\mathbf{Z}[\frac{1}{\ell},\zeta_\ell]).$$

Since the element $\tau_\ell^{\mathsf{MZ}}$ is constructed in the same way in motivic cohomology, by first producing it over $\mathbf{Z}[\frac{1}{\ell},\zeta_\ell])$ and then utilizing the transfer (see Proposition 6.1), we may arrange $\tau_\ell^{\mathsf{MGL}}$ to be detected by a M$\mathbf{Z}$-theoretic mod-$\ell$ Bott element $\tau_\ell^{\mathsf{MZ}} \in h^{0,\ell-1} \in E_{0,0,1-\ell}^1(\mathsf{MGL}/\ell)$.

Now suppose that $\ell = 3$. In this case, $\mathsf{MGL}/3$ *does not* have an associative or commutative multiplication (although it does have a unital multiplication) since this is the case for the mod 3 Moore spectrum [59]. We examine the slice spectral sequence of weight $1 - \ell = -2$:
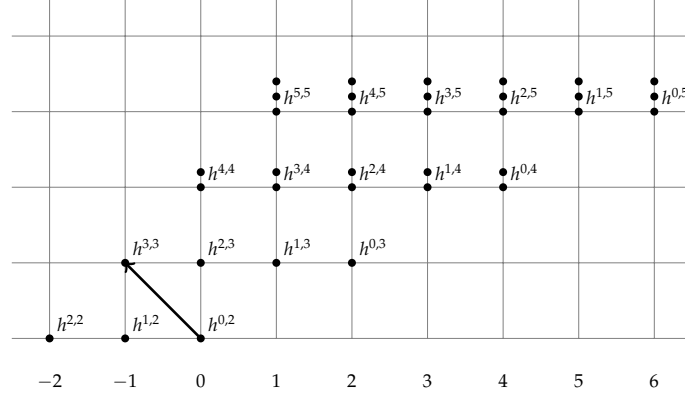
To show $\tau_3^{\mathsf{MZ}}$ is an permanent cycle we need to examine the single differential displayed above. We recall that (over any base) for *all primes* $\ell$, $s_q \mathsf{MGL}/\ell$ is an $\mathsf{MZ}/\ell$-module [25, §6 (iv),(v)] since $s_0 \mathsf{MGL}/\ell \simeq s_0 \mathbf{1}/\ell \simeq \mathsf{MZ}/\ell$ by [43], [81]. The first differential in the slice spectral sequence for $\mathsf{MGL}/\ell$ is induced by the composite map:

$$s_q \mathsf{MGL}/\ell \to \Sigma^{1,0} f_{q+1} \mathsf{MGL}/\ell \to \Sigma^{1,0} s_{q+1} \mathsf{MGL}/\ell.$$

Hence the differential of interest is an operation in the Steenrod algebra $\mathcal{A}_\ell^{*,*} \cong \mathsf{MZ}/\ell^{*,*} \mathsf{MZ}/\ell$ of [31], [82]. As an algebra $\mathcal{A}_\ell^{*,*}$ is generated by the reduced power operations:

$$\mathcal{P}^i \in \mathcal{A}_\ell^{2i(\ell-1), i(\ell-1)},$$

for $i \geq 1$, the Bockstein operation:

$$\beta \in \mathcal{A}_\ell^{1,0},$$

and the operations given by multiplication by mod-$\ell$ motivic cohomology classes in $h^{*,*}$. As noted in [82, §8] the Bockstein operation satisfies:

(62) $$\beta^2 = 0, \beta(uv) = \beta(u)v + (-1)^r u\beta(v); u \in h^{r,*}, v \in h^{*,*}.$$

Now for all odd primes $\ell$, by inspection of degrees we see the reduced power operation $\mathcal{P}^i$ acts trivially on $E_{p,q,w}^1(\mathsf{MGL}/\ell)$ for all $i \geq 1$ (and all weights $w$). Thus all the possibly nontrivial differentials are obtained from powers of $\beta$ and mod-$\ell$ cohomology classes.

To check triviality of the $d_1$-differential exiting bidegree $(0,0)$ we note that all $(3,1)$-operations $h^{0,\ell-1} \to h^{3,\ell}$ arise from powers of $\beta$ and a class $u$ in either $h^{0,1}$ or $h^{1,1}$. If $u \in h^{0,1}$ then $\beta^2 = 0$ ensures triviality. If $u \in h^{1,1}$ then we claim that $\beta(u) = 0$. But now since $\ell = 3$, we note that $H^{2,1}(\text{Spec } \mathbf{Z}[\frac{1}{\ell}, \zeta_\ell], \mathbf{Z}/\ell) = 0$ since the Picard group of Spec $\mathbf{Z}[\frac{1}{\ell}, \zeta_\ell]$ is zero[6]. Indeed, $\mathbf{Z}[\frac{1}{\ell}, \zeta_\ell]$ is unique factorization domain, since it is the localization of the ring of integers of $\mathbb{Q}(\sqrt{-3})$ which has class number one [58, Page 37]. Hence, we conclude from (62).

For $\ell = 3$, there are no more differentials to check as we see from the weight $-2$ spectral sequence displayed above and, therefore, we can conclude that $\tau_3^{\mathsf{MZ}}$ is indeed a permanent cycle. □

6.3.5. Up to §6.4.1, we work over the base scheme Spec $\mathbf{Z}[\frac{1}{\ell}]$. We now seek MGL-theoretic Bott elements for prime powers. This is easily done in the presence of a unital multiplication on $\mathsf{MGL}/\ell^v$ using Bockstein arguments. However, due to the notorious lack of a multiplicative structure on $\mathsf{MGL}/2$[59] one cannot readily make sense of inverting $\tau_2^{\mathsf{MGL}}$ in general. Instead, we will form the Bott inversion of $\mathsf{MGL}/2$ via the action of a mod-4 Bott element $\tau_4^{\mathsf{MGL}}$, and similarly for $\mathsf{MGL}/2^v$ and mod-$2^v$ Bott elements $\tau_{2^v}^{\mathsf{MGL}}$ for all $v \geq 2$. As in [78], at odd primes we argue via a Bockstein spectral sequence. Again, the Bott element $\tau_{\ell^v}^{\mathsf{MGL}}$ is

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[6]This argument does not work for a general prime $\ell$ — particularly those such that $\mathbb{Q}(\sqrt{\ell})$ has class number larger than one.

detected in the slice spectral sequence by the Bott element $\tau_{\ell^\nu}^{\mathsf{MZ}}$ for $\mathsf{MZ}$. For the rest of this section, we write:

$$h^{p,q} := H^{p,q}(\mathrm{Spec}\ \mathbf{Z}[\tfrac{1}{\ell}]; \mathbf{Z}/\ell^\nu),$$

for some prime $\ell$, some $\nu \geq 1$. The context will always make it clear.

6.3.6.  We begin with $\ell = 2$ and the motivic version:

(63)                              $\mathbf{1}/4 \wedge \mathbf{1}/2 \to \mathbf{1}/2,$

of Oka's module action of the mod-4 by the mod-2 Moore spectrum [59, §6]. If $\mathsf{E}$ is a motivic ring spectrum, i.e., a monoid in $\mathrm{Ho}(\mathsf{SH}(S))$, then (63) induces a pairing:

(64)                              $\mathsf{E}/4 \wedge \mathsf{E}/2 \to \mathsf{E}/2.$

According to [66, Proposition 2.24], (64) induces a pairing of slice spectral sequences:

(65)            $E^r_{p,q,w}(\mathsf{E}/4) \otimes E^r_{p',q',w'}(\mathsf{E}/2) \to E^r_{p+p',q+q',w+w'}(\mathsf{E}/2),$

satisfying the Leibniz rule:

(66)      $d^r(a \cdot b) = d^r(a) \cdot b + (-1)^p a \cdot d^r(b)$ for $a \in E^r_{p,q,w}(\mathsf{E}/4), b \in E^r_{p',q',w'}(\mathsf{E}/2).$

When $\nu \geq 2$, we note that $\mathsf{E}/2^\nu$ admits a unital multiplication and its slice spectral sequence is one of (unital, but not necessarily associative) algebras satisfying the same type of Leibniz rule for the differentials as (66).

**Lemma 6.13.** *For $\nu \geq 2$ any $\mathsf{MZ}$-theoretic mod-$2^\nu$ Bott element $\tau_{2^\nu}^{\mathsf{MZ}} \in h^{0,\mathrm{e}(2^\nu)}$ defines a permanent cycle of order $2^\nu$ in the slice spectral sequence for $\mathsf{MGL}/2^\nu$:*

$$(\tau_{2^\nu}^{\mathsf{MZ}})^{2^\nu} \in E^1_{0,0,-2^\nu \mathrm{e}(2^\nu)}(\mathsf{MGL}/2^\nu) = h^{0,2^\nu \mathrm{e}(2^\nu)}.$$

*Hence, there exists an $\mathsf{MGL}$-theoretic mod-$2^\nu$ Bott element $\tau_{2^\nu}^{\mathsf{MGL}} \in \mathsf{MGL}/2^\nu_{0,-2^\nu \mathrm{e}(2^\nu)}(\mathbf{Z}[\tfrac{1}{\ell}])$ that is detected by $(\tau_{2^\nu}^{\mathsf{MZ}})^{2^\nu}$ under the edge map in the slice spectral sequence for $\mathsf{MGL}/2^\nu$.*

*Proof.* From the degree zero part of $s_* \mathsf{MGL}$ (58) it follows that $\tau_{2^\nu}^{\mathsf{MZ}} \in E^1_{0,0,-\mathrm{e}(2^\nu)}(\mathsf{MGL}/2^\nu)$. Since $\nu \geq 2$ the unital multiplicative structure on $\mathsf{MGL}/2^\nu$ yields a pairing:

(67)        $E^r_{p,q,w}(\mathsf{MGL}/2^\nu) \otimes E^r_{p',q',w'}(\mathsf{MGL}/2^\nu) \to E^r_{p+p',q+q',w+w'}(\mathsf{MGL}/2^\nu).$

By the Leibniz rule associated to (67) we deduce the desired vanishing for $r \geq 1$:

(68)                    $d_r((\tau_{2^\nu}^{\mathsf{MZ}})^{2^\nu}) = 2^\nu (\tau_{2^\nu}^{\mathsf{MZ}})^{2^\nu - 1} d_r(\tau_{2^\nu}^{\mathsf{MZ}}) = 0.$

The existence of $\tau_{2^\nu}^{\mathsf{MGL}}$ follows by strong convergence.                                     $\square$

*Remark* 6.14.  We note that $\tau_2^{\mathsf{MZ}}$ need not be a permanent cycle in the slice spectral sequence for $\mathsf{MGL}/2$. The issue is basically that $d_1 : h^{0,2} \to h^{3,3}$ maps $(\tau_2^{\mathsf{MZ}})^2$ to $\rho^3$ via the third Steenrod operation $\mathrm{Sq}^3$, where $\rho \in h^{1,1}$ is the square class of $-1$.

6.3.7.  For odd primes $\ell$, the analogous lemma holds.

**Lemma 6.15.** *Let $\ell$ be an odd prime number. Any choice of an $\mathsf{MGL}$-theoretic mod-$\ell$ Bott element $\tau_\ell^{\mathsf{MGL}}$ yields an $\mathsf{MGL}$-theoretic mod-$\ell^\nu$ Bott element $\tau_{\ell^\nu}^{\mathsf{MGL}} \in \mathsf{MGL}/\ell^\nu_{0,-\mathrm{e}(\ell^\nu)}(\mathbf{Z}[\tfrac{1}{\ell}])$ that is detected by an $\mathsf{MZ}$-theoretic mod-$\ell^\nu$ Bott element in the slice spectral sequence for $\mathsf{MGL}/\ell^\nu$. Furthermore, the mod-$\ell$ reduction of $\tau_{\ell^\nu}^{\mathsf{MGL}}$ equals $(\tau_\ell^{\mathsf{MGL}})^{\ell^{\nu-1}}$.*

*Proof.* For $\nu \geq 2$ there is the Bockstein exact sequence:

(69)        $\mathsf{MGL}/\ell^\nu_{0,-\mathrm{e}(\ell^\nu)}(\mathbf{Z}[\tfrac{1}{\ell}]) \to \mathsf{MGL}/\ell_{0,-\mathrm{e}(\ell^\nu)}(\mathbf{Z}[\tfrac{1}{\ell}]) \xrightarrow{\beta} \mathsf{MGL}/\ell^{\nu-1}_{-1,-\mathrm{e}(\ell^\nu)}(\mathbf{Z}[\tfrac{1}{\ell}]).$

Now choose $\tau_\ell^{MGL} \in MGL/\ell_{0,1-\ell}(\mathbf{Z}[\frac{1}{\ell}])$ as in Lemma 6.12. Then $(\tau_\ell^{MGL})^{\ell^{\nu-1}} \in MGL/\ell_{0,-e(\ell^\nu)}(\mathbf{Z}[\frac{1}{\ell}])$ is detected by $(\tau_\ell^{M\mathbf{Z}})^{\ell^{\nu-1}}$ in the slice spectral sequence for $MGL/\ell_{(\mathbf{Z}[\frac{1}{\ell}])}$. Since $\beta$ is a derivation, we obtain:

$$\beta((\tau_\ell^{MGL})^{\ell^{\nu-1}}) = \ell^{\nu-1}(\tau_\ell^{MGL})^{\ell^{\nu-1}-1}\beta(\tau_\ell^{MGL}) = 0 \in MGL/\ell_{-1,-e(\ell^\nu)}^{\nu-1}(\mathbf{Z}[\frac{1}{\ell}]).$$

By exactness of (69) there exists an element $\tau_{\ell^\nu}^{MGL} \in MGL/\ell_{0,-e(\ell^\nu)}^\nu(\mathbf{Z}[\frac{1}{\ell}])$ mapping to $(\tau_\ell^{MGL})^{\ell^{\nu-1}}(\mathbf{Z}[\frac{1}{\ell}])$. Using the analogous properties for $\tau_{\ell^\nu}^{M\mathbf{Z}} \in M\mathbf{Z}/\ell_{0,-e(\ell^\nu)}^\nu(\mathbf{Z}[\frac{1}{\ell}])$ and naturality of the Bockstein sequences, we may arrange so that $\tau_{\ell^\nu}^{MGL}$ is detected by $\tau_{\ell^\nu}^{M\mathbf{Z}}$.          □

With these choices of Bott elements in mind and the notation introduced in §6, we define:

$$e_{MGL}(\ell^\nu) = \begin{cases} e(\ell^\nu) & \ell \text{ odd} \\ 2^\nu e(2^\nu) & \ell = 2. \end{cases}$$

**Definition 6.16.** Let $\ell$ be a prime, $\nu \geq 1$ and suppose that $S$ is a $\mathbf{Z}[\frac{1}{\ell}]$-scheme with structure map $f : S \to \text{Spec } \mathbf{Z}[\frac{1}{\ell}]$. Then an MGL *mod-$\ell^\nu$ Bott element* on $S$ is the the class:

$$(\tau_{\ell^\nu}^{MGL})_S := f^*\tau_{\ell^\nu}^{MGL} \in MGL/\ell_{0,-e_{MGL}(\ell^\nu)}^\nu(S).$$

where $\tau_{\ell^\nu}^{MGL} \in MGL/\ell_{0,-e_{MGL}(\ell^\nu)}^\nu(\mathbf{Z}[\frac{1}{\ell}])$ obtained in Lemmas 6.15 and 6.13. We call the collection of element $\{(\tau_{\ell^\nu}^{MGL})_S\}_{\nu \geq 1}$ a *system of $\ell$-adic* MGL *Bott elements on $S$.*

6.4. **Bott inverted algebraic cobordism.** We made choices when defining Bott elements for MGL. As for the case of M$\mathbf{Z}$ in Lemma 6.5 we note that the corresponding Bott inverted algebraic cobordism spectrum is independent of the choice of a Bott element. We invert the Bott elements again using the discussion in §8.1.1, i.e., in the homotopy category instead of the ∞-category. This is in contrast with the situation we previously described with motivic cohomology and stems from the lack of coherent multiplication in Moore spectra.

**Lemma 6.17.** *Suppose that* $\tau_{\ell^\nu}^{MGL}, \tau_{\ell^\nu}'^{MGL}$ *are two choices of* MGL*-theoretic mod-$\ell^\nu$ Bott elements as in Lemma 6.13 for $\ell = 2$, and Lemma 6.15 for $\ell$ odd. Then the resulting Bott inverted spectra* $MGL/\ell^\nu[(\tau_{\ell^\nu}^{MGL})^{-1}]$ *and* $MGL/\ell^\nu[(\tau_{\ell^\nu}'^{MGL})^{-1}]$ *are equivalent.*

*Proof.* First, suppose that $\ell$ is an odd prime. When $\nu = 1$, the Bott elements are constructed from the corresponding Bott elements for M$\mathbf{Z}/\ell$; see the proof of Lemma 6.12. Since $MGL/\ell$ has a unital multiplication, we may form the Bott inverted spectrum $MGL/\ell[(\tau_\ell^{MGL})^{-1}]$ via the formula in (90). In this case Lemma 6.5 and the convergence of the Bott inverted slice spectral sequence (Proposition 6.23 below) gives us the desired result.

For $\nu \geq 2$, the elements are obtained via Bockstein spectral sequences. Therefore, any two choices of $\tau_{\ell^\nu}^{MGL}$ differ by an $\ell$-divisible element, and so the difference acts nilpotently on the spectrum $MGL/\ell^\nu$. Hence, the results of inverting both elements, using the formula in (90), are again equivalent (compare with the case of algebraic $K$-theory discussed in [22]).

Now let $\ell = 2$. If $\nu = 1$, then the Bott inversion is constructed using the pairing (64). Indeed, by Lemma 6.13, we may pick a mod-4 Bott element $\tau_4^{MGL} \in \pi_{0,-8}MGL/4$. Under the paring (64), MGL/2 is a left MGL/4-module, and $MGL/\ell^\nu[(\tau_2^{MGL})^{-1}]$ is then obtained by inverting the map:

$$\tau_4^{MGL} \cdot : MGL/2 \to \Sigma^{0,-8}MGL/2,$$

in the sense discussed in §8.1.1, again using (90). If $\nu \geq 2$, then the spectrum $MGL/2^\nu$ has an unital multiplication, and so we can form the Bott inverted spectrum $MGL/2^\nu[(\tau_{2^\nu}^{MGL})^{-1}]$ as in the case of odd primes.

The independence of choices for both cases follows by the same arguments as in the odd case using the strong convergence of the Bott inverted slice spectral sequence, and the fact that any two choices of the Bott elements involved again differ by a 2-divisible element.          □

After Lemma 6.17 we define:

**Definition 6.18.** For each prime $\ell$ and any $\nu \geq 1$, the *Bott inverted algebraic cobordism spectrum* $\mathrm{MGL}/\ell^\nu[(\tau_{\ell^\nu}^{\mathrm{MGL}})^{-1}]$ is the spectrum obtained from inverting the Bott elements as described in Lemma 6.17.

6.4.1. In order to access the homotopy groups of Bott inverted algebraic cobordism, we first study the Bott inverted slice spectral sequence. For an odd prime $\ell$ and $\nu \geq 1$, we invert the action of $\tau_{\ell^\nu}^{\mathrm{MZ}} \in \pi_{0,-\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}(s_0\mathrm{MGL}/\ell^\nu)$ on the graded $\mathrm{MZ}/\ell$-module $s_*\mathrm{MGL}/\ell^\nu$. Explicitly, by the procedure described in §8.1.1, we start with the map in $\mathrm{SH}(k)$:

$$(70) \qquad \tau_{\ell^\nu}^{\mathrm{MZ}} \cdot : \Sigma^{0,-\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}\mathbf{1} \wedge s_*\mathrm{MGL}/\ell^\nu \overset{\tau_{\ell^\nu}^{\mathrm{MZ}}\wedge id}{\to} s_0\mathrm{MGL}/\ell^\nu \wedge s_*\mathrm{MGL}/\ell^\nu \to s_*\mathrm{MGL}/\ell^\nu,$$

and apply $\Sigma^{0,\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}$ to obtain the map:

$$(71) \qquad \Sigma^{0,\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}\tau_{\ell^\nu}^{\mathrm{MZ}} \cdot : \ s_*\mathrm{MGL}/\ell^\nu \to \Sigma^{0,\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}s_*\mathrm{MGL}/\ell^\nu.$$

The graded motivic spectrum $s_*\mathrm{MGL}/\ell^\nu[(\tau_{\ell^\nu}^{\mathrm{MZ}})^{-1}]$ is then calculated by the colimit of graded $s_0\mathrm{MGL}/\ell^\nu$-modules:

$$(72)$$
$$s_*\mathrm{MGL}/\ell^\nu \overset{\Sigma^{0,\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}\tau_{\ell^\nu}^{\mathrm{MZ}}\cdot}{\to} \Sigma^{0,\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}s_*\mathrm{MGL}/\ell^\nu \overset{\Sigma^{0,2\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}\tau_{\ell^\nu}^{\mathrm{MZ}}\cdot}{\to} \Sigma^{0,2\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}s_*\mathrm{MGL}/\ell^\nu \cdots .$$

Since the element $\tau_{\ell^\nu}^{\mathrm{MGL}}$ is detected by $\tau_{\ell^\nu}^{\mathrm{MZ}}$ by Lemma 6.15, we obtain a spectral sequence:

$$(73) \qquad E_{p,q,w}^*(\mathrm{MGL}/\ell^\nu)[(\tau_{\ell^\nu}^{\mathrm{MZ}})^{-1}] \Rightarrow \mathrm{MGL}/\ell^\nu[(\tau_{\ell^\nu}^{\mathrm{MGL}})^{-1}].$$

We will give more details on the construction of this spectral sequence in §6.5.1 below.

Let us now deal with the prime 2 and $\nu = 1$. Lemma 6.13 picks out an element $(\tau_4^{\mathrm{MZ}})^4 \in E_{0,0,-8}^1(\mathrm{MGL}/4)$. We then define:

$$(74) \qquad \tau_4^{\mathrm{MZ}} \cdot : \Sigma^{0,-8}\mathbf{1} \wedge s_*\mathrm{MGL}/2 \overset{(\tau_4^{\mathrm{MZ}})^4\wedge id}{\to} s_0\mathrm{MGL}/4 \wedge s_*\mathrm{MGL}/2 \to s_*\mathrm{MGL}/2,$$

where the last map is defined using the pairing (64). As in (72), we define the graded motivic spectrum $s_*\mathrm{MGL}/2[(\tau_2^{\mathrm{MZ}})^{-1}]$ as the colimit of graded spectra:

$$(75) \qquad s_*\mathrm{MGL}/2 \overset{\Sigma^{0,8}\tau_4^{\mathrm{MZ}}\cdot}{\to} \Sigma^{0,8}s_*\mathrm{MGL}/2 \overset{\Sigma^{0,16}\tau_4^{\mathrm{MZ}}\cdot}{\to} \Sigma^{0,16}s_*\mathrm{MGL}/2 \cdots .$$

For $\nu \geq 2$, we use the unital pairing on $\mathrm{MGL}/2^\nu$ and the element $(\tau_{2^\nu}^{\mathrm{MZ}})^{2^\nu} \in E_{0,0,-\mathrm{e}_{\mathrm{MGL}}(2^\nu)}^1(\mathrm{MGL}/2^\nu)$ to get a map:

$$(76) \ (\tau_{2^\nu}^{\mathrm{MZ}})^{2^\nu} \cdot : \Sigma^{0,-\mathrm{e}_{\mathrm{MGL}}(2^\nu)}\mathbf{1} \wedge s_*\mathrm{MGL}/2^\nu \overset{(\tau_{2^\nu}^{\mathrm{MZ}})^{2^\nu}\wedge id}{\to} s_0\mathrm{MGL}/2^\nu \wedge s_*\mathrm{MGL}/2^\nu \to s_*\mathrm{MGL}/2^\nu,$$

which we invert using the same formula as in (72). In any event, this gives us a spectral sequence of the form (73) in all cases. The key point to address in these spectral sequences is their *convergence* which we will provide a proof of.

6.5. **Proof of Theorem 1.6 for algebraic cobordism.** We now have all the ingredients to prove Theorem 1.6 for MGL. We begin by factoring the unit map.

**Lemma 6.19.** *Let $S$ be a $\mathbf{Z}[\frac{1}{\ell}]$-scheme, $\ell$ a prime and $\nu \geq 1$ and consider the map:*

$$(\tau_{\ell^\nu}^{\mathrm{MGL}})_S : \Sigma^{0,-\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}\mathbf{1}_S \to \mathrm{MGL}_S/\ell^\nu,$$

*classifying the Bott element as in Definition 6.16. Then, the map in $\mathrm{SH}_{\text{ét}}(S)$*

$$\epsilon^*(\tau_{\ell^\nu}^{\mathrm{MGL}})_S : \Sigma^{0,-\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}\epsilon^*\mathbf{1}_S \to \epsilon^*\mathrm{MGL}_S/\ell^\nu$$

*is invertible and thus $(\tau_{\ell^\nu}^{\mathrm{MGL}})_S$ acts invertibly on $\mathrm{MGL}_{*,*}^{\text{ét}}$.*

*Proof.* Over any base $S$, the unit map factors as:

$$\mathrm{MGL}_S/\ell^\nu \to \mathrm{colim}(f_q\mathrm{MGL}_S)^{\text{ét}}/\ell^\nu \to \mathrm{MGL}_S^{\text{ét}}/\ell^\nu,$$

where the first map induces a commutative diagram:

(77)

$$
\begin{array}{ccc}
\mathrm{MGL}/\ell^\nu{}_{0,-\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}(S) & \longrightarrow & (\mathrm{colim}(f_q\mathrm{MGL})^{\text{ét}}/\ell)_{0,-\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}(S) \\
\downarrow & & \downarrow \\
\pi_{0,-\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}s_0\mathrm{MGL}/\ell(S) \cong H^{0,\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}(S,\mathbf{Z}/\ell^\nu) & \longrightarrow & \pi_{0,-\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}(s_0\mathrm{MGL})^{\text{ét}}/\ell^\nu(S) \cong H^0_{\text{ét}}(S,\mu_{\ell^\nu}^{\otimes \mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}).
\end{array}
$$

Consider $(\tau_{\ell^\nu}^{\mathrm{MGL}})_S \in \mathrm{MGL}/\ell^\nu{}_{0,-\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}$; it suffices to prove that it maps to an invertible element via the top horizontal map of diagram (77). To verify the claim, it suffices to check the case that $S = \mathrm{Spec}\ k$, i.e., the spectrum of a field $k$ (using, for example, the fact that $\mathrm{SH}_{\text{ét}}$ has the full six functor formalism by [3, Corollaire 4.5.47]).

By construction $\tau_{\ell^\nu}^{\mathrm{MGL}} \in \mathrm{MGL}/\ell^\nu{}_{0,-\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}(k)$ maps to $\tau_{\ell^\nu}^{\mathrm{MZ}} \in H^{0,\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}(k,\mathbf{Z}/\ell^\nu)$ via the left vertical arrow. Since we are over a field, $\tau_{\ell^\nu}^{\mathrm{MZ}}$ maps to a periodicity operator in $H^0_{\text{ét}}(k,\mu_{\ell^\nu}^{\otimes \mathrm{e}_{\mathrm{MGL}}(\ell^\nu)})$ via the bottom horizontal map. This element acts isomorphically on the étale slice spectral sequence. Identifying the étale slice spectral sequence as the inverted slice spectral sequence by Proposition 6.20, such a periodicity operator survives to an element in $\mathrm{MGL}^{\text{ét}}/\ell^\nu{}_{0,-\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}(k)$ since one can check that the computations in Lemmas 6.12, 6.13, and 6.15 are not affected by this inversion procedure. Thus we conclude that this element survives to the value of $\tau_{\ell^\nu}^{\mathrm{MGL}}$ via the top vertical map, as desired.

$\square$

6.5.1. *Field case.* We first prove the case when the base is a field. To do this, we first give more details on the construction of the spectral sequence (73) since we will need some of the notation in our discussion of convergence. Suppose $\{{}^iE\}_{i\in\mathbb{N}}$ is a collection of spectral sequences with filtered graded groups $\{{}^iG\}_{i\in\mathbb{N}}$ together with maps ${}^i\phi : {}^iE \to {}^{i+1}E$ which are compatible with the maps ${}^i\psi : {}^iG \to {}^{i+1}G$ in the evident sense.

We say ${}^i\phi$ is *of degree $a$* if it takes an element $x$ of degree $|x|$ to an element of degree $|x| + a$. By passing to the colimit of the system $(\{{}^iE\}, {}^i\phi)_{i\in\mathbb{N}}$ we obtain a spectral sequence with respect to the evident filtration on the target group $G$:

(78)                         $E := \mathrm{colim}\ {}^iE \Rightarrow G := \mathrm{colim}\ {}^iG.$

In our main example of interest, we apply $\pi_{*,*}$ to the diagrams (72) and (75), whence we get ${}^iE := E^*_{p,q,w}(\mathrm{MGL}/\ell^\nu)$ for all $i$, together with degree $(0, -\mathrm{e}_{\mathrm{MGL}}(\ell^\nu))$ maps:

$$\tau_{\ell^\nu}^{\mathrm{MZ}} : {}^iE \to {}^{i+1}E,$$

which are compatible with the multiplication by $\tau_{\ell^\nu}^{\mathrm{MGL}}$-map:

$$\tau_{\ell^\nu}^{\mathrm{MGL}} \cdot : \pi_{p,w}\mathrm{MGL}/\ell^\nu \to \pi_{p,-\mathrm{e}_{\mathrm{MGL}}(\ell^\nu)}\mathrm{MGL}/\ell^\nu,$$

since $\tau_{\ell^\nu}^{\mathrm{MZ}}$ detects $\tau_{\ell^\nu}^{\mathrm{MGL}}$ by the definition of the latter element as in Lemmas 6.13 and 6.15.

6.5.2. By the discussion in §6.4.1 we can identify the first page of the Bott inverted slice spectral sequence and the étale slice spectral sequence.

**Proposition 6.20.** *Let $k$ be a field with exponential characteristic prime to $\ell$ and let $f : S \to \mathrm{Spec}\ k$ be an essentially smooth scheme over $k$. The unit map $\mathrm{MGL}_S \to \epsilon_*\epsilon^*\mathrm{MGL}_S$ induces a natural isomorphism of spectral sequences:*

$$E^*_{p,q,w}(\mathrm{MGL}_S/\ell^\nu)[(\tau_{\ell^\nu}^{\mathrm{MZ}})_S^{-1}] \simeq E^{*,\text{ét}}_{p,q,w}(\mathrm{MGL}_S/\ell^\nu).$$

*Proof.* By Theorem 6.7 we obtain equivalences:

$$s_* \mathsf{MGL}_S / \ell^\nu [(\tau_{\ell^\nu}^{\mathsf{MZ}})^{-1}] \quad \simeq \quad \mathsf{MZ}_S / \ell^\nu [x_1, ..., x_q, ...][(\tau_{\ell^\nu}^{\mathsf{MZ}})_S^{-1}]$$

$$\simeq \quad \mathsf{MZ}_S^{\text{ét}} / \ell^\nu [x_1, ..., x_q, ...],$$

under the natural maps from $s_* \mathsf{MGL} / \ell_S^\nu$. Indeed for $\ell$ an odd prime this is straightforward. For the prime 2 and $\nu \geq 1$, the first term in the above equivalence is inverted using the $2^\nu$-th power of the $\tau_{2^\nu}^{\mathsf{MZ}}$ as in (76) and so Theorem 6.7 still applies. When $\ell = 2$ and $\nu = 1$, we use the inversion procedure described in (74), which is slightly different from the case described in Theorem 6.7. However, the element $\tau_4^{\mathsf{MZ}}$ acts invertibly in mod-2 étale cohomology, which is all one needs to obtain the second equivalence.

Note that, as proved in Lemma 6.17, the above equivalences are independent of the various choices involved in choosing the Bott elements. □

6.5.3. For Proposition 6.20 to be useful we need to address convergence results. In general, inverting an element in a strongly convergent spectral sequence may destroy strong convergence (see [90, Section 2] for toy examples.) In the notation of §6.5.1, assume that $^i E$ converges strongly to $^i G$, where the $q$-th filtration of $^i G$ is denoted by $^i F_q$. Denote by $^i E_q$ the $q$-th filtration degree of $^i E$. We require $^i E_q = 0$ for $q < 0$. The convergence result we need follows closely the proof of [90, Proposition 3] but tailor-made for our needs.

**Lemma 6.21.** *Suppose that $\{^i E\}$ is a directed system of strongly convergent spectral sequences of bigraded groups $^i E_q^{p,w}$ where $q$ is the filtration degree such that:*

(1) *The maps $^i\phi : {}^i E \to {}^{i+1} E$ preserve filtration,*
(2) *The maps are of degree $(0, a)$ for some $a \in \mathbf{Z}$,*
(3) *For fixed $p$, there is an $M(p) > 0$ such that for all $i > M(p)$ the group $^i E_q^{p,w} = 0$ for $q < N(p)$.*

*In this case, (78) is strongly convergent in the sense of [9, Definition 5.2].*

*Proof.* First we check weak convergence. Let $E := \text{colim} \, {}^i E$ be the colimit spectral sequence. As usual $E_\infty^q = \lim_q Z_\infty^q / B_\infty^q$ where $Z_\infty^q = \lim_r Z_r^q$ is the group of infinite cycles and $^i B_\infty^q = \text{colim}_r B_r^q$. We want to show there is a natural isomorphism:

$$\operatorname*{colim}_i {}^i F_q / {}^i F_{q+1} \xrightarrow{\cong} E_\infty^q.$$

Since $^i E$ is strongly convergent, in particular weakly convergent we have:

(79) $${}^i F_q / {}^i F_{q+1} \cong {}^i Z_\infty^q / {}^i B_\infty^q,$$

where, similarly, $^i Z_\infty^q = \lim_r {}^i Z_r^q$ and $^i B_\infty^q = \text{colim}_r {}^i B_r^q$. Moreover, we have:

$$\operatorname*{colim}_i {}^i F_q / {}^i F_{q+1} \cong \operatorname*{colim}_i \lim_r Z_r^q / {}^i B_r^q \cong \lim_r \operatorname*{colim}_i {}^i Z_r^q / {}^i B_r^q = E_\infty^q.$$

Here the first isomorphism comes from (79), the second comes from assumption (3) (so that the limit term is finite), and the last equality is by definition.

Next we check completeness. We denote by $F_q = \text{colim}_i {}^i F_q$, i.e., the induced filtration on the colimit group. We claim that there is an isomorphism:

$$\lim_q G / F_q \cong G.$$

Using the definition $G^b / F_q^b := \text{colim}_i ({}^i G^{b+ia} / {}^i F_q^{b+ia})$ we obtain:

$$\lim_q G^b / F_q^b \cong \lim_q \operatorname*{colim}_i ({}^i G^{b+ia} / {}^i F_q^{b+ia}).$$

On the other hand, by completeness of each $^i E$ there is an isomorphism:

$$G^b \cong \operatorname*{colim}_i ({}^i G^{i+ba}) \cong \operatorname*{colim}_i \lim_q ({}^i G^{i+ba}) / ({}^i F_q^{i+ba}).$$

Since filtered colimits commute with finite limits of abelian groups it suffices to prove that $\lim_q ({}^iG^{i+ba})/({}^iF_q^{i+ba})$ is a finite limit for $i \gg 0$. For fixed $(p,w)$, assumption (3) implies that ${}^iF_q^{i+ba}/{}^iF_{q+1}^{i+ba} = 0$ whenever $i > M(p)$ and $q \geq N(p)$. Since the filtration ${}^iF$ of ${}^iG$ is Hausdorff, ${}^iF_q^{i+ba} = 0$. Thus the limit $\lim_q ({}^iG^{i+ba})/({}^iF_q^{i+ba})$ is attained at a finite stage.

Next we check that the filtration $F_q$ of $G$ is *Hausdorff*, that is, $\cap_q F_q = 0$. We claim that for every nonzero element $x \in G^b$, there exists some ${}^ix \in {}^iG^{b+ia}$ such that:

- ${}^ix$ maps to $x$ under the canonical map to ${}^iG \to G$,
- ${}^ix$ is detected by an element ${}^iy \in {}^iE$ which is nonzero for all iterated compositions of ${}^if$.

Choose an element ${}^jx'$ that maps to $x \neq 0$. Then ${}^{k+j}\psi(x') \neq 0$ are all nonzero for all $k \geq 0$. Let us write ${}^ky \in {}^{k+j}E^{b+k,a}$ for an element that detects ${}^{k+j}\psi(x')$. We may assume $k > M(p)$, so that the filtration degree of ${}^ky$ is less than $N(p)$. As $k$ increases, the filtration degree of ${}^ky$ can increase but it must eventually be constant, because otherwise ${}^ky$ would be zero and therefore it cannot detect ${}^{k+j}\psi(x') \neq 0$. Where the filtration degree becomes constant is exactly where we find ${}^ix$; in other words we let $i = k + j$ where $k \gg 0$.

Note that ${}^ky$ survives to the $E_\infty$ page of ${}^iE$ since ${}^{k+j}g(x') \neq 0$ for all $k \geq 0$. Suppose that $q$ is its filtration degree. Since the spectral sequences ${}^iE$ are strongly convergent for all $i$, we have that ${}^{k+j}\psi(x')$ is indeed in filtration $q$ and not in any higher filtration. Since this always happens as $i$ tends to $\infty$, we are done. $\qquad\square$

6.5.4.   In order to apply Lemma 6.21 to algebraic cobordism we check for vanishing lines.

**Lemma 6.22.** *Let $k$ be a field with exponential characteristic prime to $\ell$. The group $E^1_{p,q,w}(\mathsf{MGL}_k/\ell^v) = 0$ if (1) $p > 2q$, (2) $q + w > p$, or (3) $2q > p + \mathrm{cd}_\ell(k)$.*

*Proof.* Let us write $L_q$ for the set of degree $q$ monomials in the polynomial generators $x_i$ in the slice calculation (58). The $E^1$-term $E^1_{p,q,w}(\mathsf{MGL}_k/\ell^v)$ of the slice spectral sequence for $\mathsf{MGL}_k/\ell^v$ is given as:

$$\pi_{p,w}(s_q\mathsf{MGL}_k/\ell^v) \cong \oplus_{L_q} \pi_{p,w}(\Sigma^{2q,q}\mathsf{MZ}_k/\ell^v) \cong \oplus_{L_q} H^{2q-p,q-w}(k; \mathbf{Z}/\ell^v).$$

Hence (1) and (2) follow from standard vanishing results in motivic cohomology. By the Bloch-Kato conjecture, away from the region specifed in (2), there is an isomorphism

$$H^{2q-p,q-w}(k; \mathbf{Z}/\ell^v) \cong H^{2q-p}_{\text{ét}}(k, \mu_{\ell^v}^{\otimes q-w}).$$

Thus (3) follows by the definition of $\mathrm{cd}_\ell(k)$. $\qquad\square$

**Proposition 6.23.** *The spectral sequence $E^1_{p,q,w}(\mathsf{MGL}_k/\ell^v)[(\tau_{\ell^v}^{\mathsf{MZ}})_k^{-1}]$ is strongly convergent.*

*Proof.* The convergence of the original spectral sequence is established in [30, Theorem 8.12]. In the notation of Lemma 6.21 we let ${}^iE_q^{p,w} := E^1_{p,q,w}$. Part (1) of Lemma 6.21 is satisfied since $\tau_{\ell^v}^{\mathsf{MZ}}$ is in slice filtration zero. Part (2) of Lemma 6.21 is satisfied since multiplication by $\tau_{\ell^v}^{\mathsf{MZ}}$ is of degree $(0, e_{\mathsf{MGL}}(\ell^v))$. Part (3) of Lemma 6.22 implies part (3) of Lemma 6.21 is satisfied for $N(p) = \frac{p + \mathrm{cd}_\ell(k)}{2}$. $\qquad\square$

We can now proceed to prove our main theorems for essentially smooth schemes over a field.

**Lemma 6.24.** *Suppose $f : T \to S$ is an essentially smooth morphism of base schemes. Then for $\mathsf{E} \in \mathrm{SH}(S)$ there is an equivalence $f^*(\mathsf{E}_{\geq d}) \simeq (f^*\mathsf{E})_{\geq d}$.*

*Proof.* This is exactly [30, Lemma 2.2]. $\qquad\square$

**Lemma 6.25.** *For essentially smooth schemes $S$ and $T$ over a field $k$ and $f : T \to S$ a smooth morphism over $\mathrm{Spec}\, k$, there are equivalences:*

(1) $f^*(\mathsf{MGL}_S/\ell^v[(\tau_{\ell^v}^{\mathsf{MGL}})_S^{-1}]) \simeq f^*(\mathsf{MGL}_S/\ell^v)[(\tau_{\ell^v}^{\mathsf{MGL}})_T^{-1}] \simeq \mathsf{MGL}_T/\ell^v[(\tau_{\ell^v}^{\mathsf{MGL}})_T^{-1}]$, *and*

(2) $f^* \mathsf{MGL}^{\mathrm{\acute{e}t}}_S / \ell^\nu \simeq \mathsf{MGL}^{\mathrm{\acute{e}t}}_T / \ell^\nu$.

*Proof.* Statement (1) follows from the construction of Bott elements. For (2) we need to prove that $f^*$ commutes with $\epsilon_*$. These functors have left adjoints $f_\#$ and $\epsilon^*$, respectively, and there is an equivalence $f_\# \epsilon^* \simeq \epsilon^* f_\#$ since both sides agree on representables and preserve colimits. $\qquad\square$

**Theorem 6.26.** *Suppose $k$ is a field, $\ell$ is prime to the exponential characteristic of $k$ such that $\mathrm{cd}_\ell(k) < \infty$, and let $S$ be an essentially smooth $k$-scheme. Then, for all $\nu \geq 1$ the unit of the adjunction* (19) *induces a natural equivalence in* $\mathrm{SH}(S)$:

$$\mathsf{MGL}/\ell^\nu [(\tau^{\mathsf{MGL}}_{\ell^\nu})^{-1}_S] \xrightarrow{\simeq} \mathsf{MGL}^{\mathrm{\acute{e}t}}/\ell^\nu.$$

*Proof.* First assume that the field is perfect. Lemmas 6.24 and 6.25 reduce to the case of $S = \mathrm{Spec}\, k$. Over perfect fields, the motivic homotopy sheaves are strictly $\mathbf{A}^1$-invariant [53, Remark 5.1.13], and isomorphisms are detected on generic points of smooth $k$-schemes [55, Example 2.3, Proposition 2.8]. For every finitely generated extension $L$ of $k$ and $p, q \in \mathbf{Z}$ we claim there is a naturally induced isomorphism:

$$\pi_{p,q} \mathsf{MGL}/\ell^\nu [(\tau^{\mathsf{MGL}}_{\ell^\nu})^{-1}](L) \xrightarrow{\cong} \pi_{p,q} \mathsf{MGL}^{\mathrm{\acute{e}t}}/\ell^\nu(L).$$

To wit, recall the unit map $\mathsf{MGL}/\ell^\nu \to \mathsf{MGL}^{\mathrm{\acute{e}t}}/\ell^\nu)$ factors though:

$$\eta_\infty : \mathsf{MGL}/\ell^\nu \simeq \mathrm{colim}\, f_q \mathsf{MGL}/\ell^\nu \to \mathrm{colim}(f_q \mathsf{MGL})^{\mathrm{\acute{e}t}}/\ell^\nu,$$

and:

$$\eta^\infty : \mathrm{colim}(f_q \mathsf{MGL})^{\mathrm{\acute{e}t}}/\ell^\nu \to \epsilon_* \epsilon^* \mathsf{MGL}/\ell^\nu.$$

Under our assumptions, $\eta^\infty$ is an equivalence since $\epsilon_*$ preserves colimits by Corollary 5.2. Lemma 6.19 shows $\eta_\infty$ factors through $\mathsf{MGL}/\ell^\nu [(\tau^{\mathsf{MGL}}_{\ell^\nu})^{-1}] \to \mathsf{MGL}^{\mathrm{\acute{e}t}}/\ell$. By Proposition 6.20 the maps between the corresponding spectral sequences are isomorphisms. Corollary 4.19 (which applies since $\mathrm{cd}_\ell(L) < \infty$ by, for example, [71, Theorem 28]) and Proposition 6.23 inform us that these spectral sequences are strongly convergent. Therefore we have an isomorphism on homotopy groups $\pi_{p,q}$.

To promote the statement to non-perfect fields we perform a standard continuity argument. So let $k$ be an arbitrary field. It suffices to prove that the comparison map $\mathsf{MGL}_k/\ell^\nu [(\tau^{\mathsf{MGL}}_{\ell^\nu})^{-1}] \to \mathsf{MGL}^{\mathrm{\acute{e}t}}_k/\ell^\nu$ is an equivalence. Choose an essentially smooth morphism $f : \mathrm{Spec}\, k \to \mathrm{Spec}\, L$ where $L$ is perfect. According to Lemma 6.25 the comparison map $\mathsf{MGL}_k/\ell^\nu [(\tau^{\mathsf{MGL}}_{\ell^\nu})^{-1}] \to \mathsf{MGL}^{\mathrm{\acute{e}t}}_k/\ell^\nu$ is $f^*$ of the comparison map over $\mathrm{Spec}\, L$. Write $f$ as a cofiltered limit of maps $f_\alpha : S_\alpha \to \mathrm{Spec}\, k$ where each $S_\alpha \to \mathrm{Spec}\, k$ is smooth. Furthermore any $X \in \mathrm{Sm}_k$ can be written as a limit $X \simeq \lim X_\alpha$ where each $X_\alpha$ is a smooth $S_\alpha$-scheme. Now for any $p, q \in \mathbf{Z}$ and any $X \in \mathrm{Sm}_k$, continuity [30, Lemma A.7] gives us equivalences

(80)
$$\mathrm{Maps}(\Sigma^{p,q}_T X_+, \mathsf{MGL}_k/\ell^\nu [(\tau^{\mathsf{MGL}}_{\ell^\nu})^{-1}]) \simeq \operatorname*{colim}_\alpha \mathrm{Maps}(\Sigma^{p,q}_T X_{\alpha+}, f^*_\alpha \mathsf{MGL}_L/\ell^\nu [(\tau^{\mathsf{MGL}}_{\ell^\nu})^{-1}])$$

and

(81)
$$\mathrm{Maps}(\Sigma^{p,q}_T X_+, \mathsf{MGL}^{\mathrm{\acute{e}t}}_k/\ell^\nu) \simeq \operatorname*{colim}_\alpha \mathrm{Maps}(\Sigma^{p,q}_T X_\alpha +, f^*_{\alpha+} \mathsf{MGL}^{\mathrm{\acute{e}t}}_L/\ell^\nu)$$

respecting the comparison map which goes (80) $\to$ (81). Since each term of the colimit is an equivalence from the case of smooth schemes over perfect fields, we are done. $\qquad\square$

6.5.5. *General case.* We now proceed to prove the general case. We would like to thank Tom Bachmann for help in improving the next theorem to its current level of generality.

**Theorem 6.27.** *Let $\ell$ be a prime, $S$ be a Noetherian $\mathbf{Z}[\frac{1}{\ell}]$-scheme of finite dimension, and assume that for all $x \in S$, $\mathrm{cd}_\ell(k(x)) < \infty$. Then for all $\nu \geq 1$, $\mathsf{MGL}_S/\ell^\nu [(\tau^{\mathsf{MGL}}_{\ell^\nu})^{-1}_S]$ is étale local.*

*Proof.* Let $U \in \mathrm{Sm}_S$ and let $U_\bullet \to U$ be a hypercover. After Proposition 2.13, we need to prove that the map:

$$\mathrm{Maps}(\Sigma^{p,q}\Sigma_T^\infty U_+, \mathsf{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MGL}})_S^{-1}]) \to \lim \mathrm{Maps}(\Sigma^{p,q}\Sigma_T^\infty U_{\bullet,+}, \mathsf{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MGL}})_S^{-1}])$$

is an equivalence for all $p, q \in \mathbf{Z}$. Fix $p, q \in \mathbf{Z}$ and take the cofiber, $C(p,q)$, of the map $\mathrm{colim}\,\Sigma^{p,q}\Sigma_T^\infty U_{\bullet,+} \to \Sigma^{p,q}\Sigma_T^\infty U_+$ in $\mathrm{SH}(S)$. Our goal now is to prove that:

$$\mathrm{Maps}(C(p,q), \mathsf{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MGL}})_S^{-1}]) \simeq 0.$$

First let us assume that $\mathsf{MGL}_S/\ell^\nu$ is a unital ring spectrum so that $\mathsf{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MGL}})_S^{-1}]$ is as well (since the functor of $(\tau_{\ell^\nu}^{\mathsf{MGL}})_S$-inversion is lax monoidal). In this situation we have an isomorphism:

$$[C(p,q), \mathsf{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MGL}})_S^{-1}]] \cong [\mathsf{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MGL}})_S^{-1}] \wedge C(p,q), \mathsf{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MGL}})_S^{-1}]].$$

We claim that $\mathsf{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MGL}})_S^{-1}] \wedge C(p,q) \simeq 0$. Since $\tau^{\mathsf{MGL}}$-inverted algebraic cobordism is stable under base change by Lemma 6.25, we may check this on fields by an application of Lemma 2.17, where we apply Theorem 6.26 which, in particular, tells us that $\tau^{\mathsf{MGL}}$-inverted algebraic cobordism is ét-local.

The only case that is not covered by the preceding argument is when $\ell = 2$ and $\nu = 1$, i.e., $\mathsf{MGL}_S/2$. We argue as follows: since $\mathsf{MGL}_S/4$ is a unital ring spectrum, we have a retract diagram:

$$\mathsf{MGL}_S/2 \wedge \mathbf{1} \overset{\mathrm{id}\wedge\eta}{\to} \mathsf{MGL}_S/2 \wedge \mathsf{MGL}_S/4 \to \mathsf{MGL}_S/2,$$

where the second map is induced by Oka's module action (63). This induces a retract diagram of $\mathsf{MGL}_S/4$-modules

$$\mathsf{MGL}_S/2[(\tau_2^{\mathsf{MGL}})_S^{-1}] \wedge \mathbf{1} \overset{\mathrm{id}\wedge\eta}{\to} \mathsf{MGL}_S/2 \wedge \mathsf{MGL}_S/4[(\tau_2^{\mathsf{MGL}})_S^{-1}] \to \mathsf{MGL}_S/2[(\tau_2^{\mathsf{MGL}})_S^{-1}].$$

Now, since $\mathsf{MGL}_S/4[(\tau_2^{\mathsf{MGL}})_S^{-1}]$ is étale local by the previous paragraph and $\mathsf{MGL}_S/2[(\tau_2^{\mathsf{MGL}})_S^{-1}]$ is, by definition, obtained by inverting the Bott element from $\mathsf{MGL}_S/4$ (see Lemma 6.13), we conclude. $\square$

**Theorem 6.28.** *Let $\ell$ be a prime, $S$ be a Noetherian $\mathbf{Z}[\frac{1}{\ell}]$-scheme of finite dimension, and assume that for all $x \in S$, $\mathrm{cd}_\ell(k(x)) < \infty$. Then for all $\nu \geq 1$ the unit of the adjunction (19):*

$$\mathsf{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MGL}})_S^{-1}] \overset{\simeq}{\to} \mathsf{MGL}_S^{\text{ét}}/\ell^\nu.$$

*is an equivalence in $\mathrm{SH}(S)$.*

*Proof.* Theorem 6.27 tells us that $\mathsf{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MGL}})_S^{-1}]$ is étale local, while Lemma 6.19 tells us that $(\tau_{\ell^\nu}^{\mathsf{MGL}})_S$ is invertible after applying the étale localization endofunctor $\epsilon_*\epsilon^*$. The theorem then follows immediately. $\square$

**Theorem 6.29.** *Let $\ell$ be a prime, $S$ be a Noetherian $\mathbf{Z}[\frac{1}{\ell}]$-scheme of finite dimension, and assume that for all $x \in S$, $\mathrm{cd}_\ell(k(x)) < \infty$. Let $\nu \geq 1$ and suppose that $\mathsf{MGL}/\ell^\nu$ is a unital ring spectrum [7]. Then, for any $\mathsf{E} \in \mathrm{Mod}_{\mathsf{MGL}}$ the unit of the adjunction (19):*

$$\mathsf{E}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MGL}})_S^{-1}] \overset{\simeq}{\to} \mathsf{E}_S^{\text{ét}}/\ell^\nu.$$

*is an equivalence in $\mathrm{SH}(S)$.*

*Proof.* Just as in the proof of Theorem 6.28, and following the notation of the proof of Theorem 6.27, it suffices to check that:

$$[C(p,q), \mathsf{E}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathsf{MGL}})_S^{-1}]] \simeq 0.$$

---

[7]The only case missing is $\ell = 2$, $\nu = 1$.

Since $\mathrm{MGL}/\ell^\nu$ is a unital ring spectrum, $\mathrm{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathrm{MGL}})_S^{-1}]$ is as well, and thus we have an equivalence:

$$[C(p,q), \mathsf{E}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathrm{MGL}})_S^{-1}]] \simeq [C(p,q) \wedge \mathrm{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathrm{MGL}})_S^{-1}], \mathsf{E}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathrm{MGL}})_S^{-1}]].$$

But then $C(p,q) \wedge \mathrm{MGL}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathrm{MGL}})_S^{-1}] \simeq 0$ by Theorem 6.28.

$\square$

Since motivic cohomology is an MGL-algebra, we also note the following improvement to Theorem 6.7.

**Corollary 6.30.** *Let $\ell$ be a prime, $S$ be a Noetherian $\mathbf{Z}[\frac{1}{\ell}]$-scheme of finite dimension, and assume that for all $x \in S$, $\mathrm{cd}_\ell(k(x)) < \infty$. Let $\nu \geq 1$. Then the unit of the adjunction (19):*

$$\mathrm{M}\mathbf{Z}_S/\ell^\nu \to \mathrm{M}\mathbf{Z}_S^{\text{ét}}/\ell^\nu,$$

*induces an equivalence in* $\mathrm{SH}(S)$:

$$\mathrm{M}\mathbf{Z}_S/\ell^\nu[(\tau_{\ell^\nu}^{\mathrm{M}\mathbf{Z}})_S^{-1}] \xrightarrow{\simeq} \mathrm{M}\mathbf{Z}_S^{\text{ét}}/\ell^\nu.$$

*Proof.* We remark that the case not covered by Theorem 6.29, i.e., $\ell = 2, \nu = 1$ can be proved by the same argument as in Theorem 6.28 noting that $\mathrm{M}\mathbf{Z}/2$ does have an $\mathcal{E}_\infty$-ring structure by construction [75, §4.1.1].

$\square$

6.6. **Integral statements.** In the following we promote our $\ell$-local results to an integral statement — at least after inverting some primes. First, let us investigate the rational results.

**Lemma 6.31.** *If $S$ is a Noetherian scheme of finite dimension, there is a canonical equivalence in* $\mathrm{SH}(S)_\mathbf{Q}$:

$$L_{\mathrm{M}\mathbf{Q}}\mathrm{MGL}_S \xrightarrow{\simeq} L_{\mathrm{M}\mathbf{Z}^{\text{ét}}}\mathrm{MGL}_{S,\mathbf{Q}}.$$

*Proof.* We show $\mathrm{MGL}_S \to L_{\mathrm{M}\mathbf{Z}^{\text{ét}}}\mathrm{MGL}_{S,\mathbf{Q}}$ is an $\mathrm{M}\mathbf{Q}$-equivalence and $L_{\mathrm{M}\mathbf{Z}^{\text{ét}}}\mathrm{MGL}_{S,\mathbf{Q}}$ is $\mathrm{M}\mathbf{Q}$-local. For the $\mathrm{M}\mathbf{Q}$-equivalence note that $\mathrm{MGL}_S \to L_{\mathrm{M}\mathbf{Z}^{\text{ét}}}\mathrm{MGL}_{S,\mathbf{Q}}$ factors as:

$$\mathrm{MGL}_S \to L_{\mathrm{M}\mathbf{Z}^{\text{ét}}}\mathrm{MGL}_S \to L_{\mathrm{M}\mathbf{Z}^{\text{ét}}}\mathrm{MGL}_{S,\mathbf{Q}}.$$

Upon smashing with $\mathrm{M}\mathbf{Z}_S^{\text{ét}}$ the first map becomes an equivalence, and upon smashing with $\mathrm{M}\mathbf{Q}$ the second map becomes an equivalence. For $\mathrm{M}\mathbf{Q}$-locality, by Morel's rational decomposition theorem, $\mathrm{SH}(S)_\mathbf{Q} \simeq \mathrm{DM}(S, \mathbf{Q}) \times \mathrm{SH}(S)_\mathbf{Q}^-$ [13, Theorem 16.2.13]. It remains to note that $L_{\mathrm{M}\mathbf{Z}^{\text{ét}}}\mathrm{MGL}_{S,\mathbf{Q}} \in \mathrm{DM}(S, \mathbf{Q})$. This is because it receives a map of ring spectra from $\mathrm{MGL}_S$ and hence is an oriented theory [13, Theorem 14.2.16] which means that $\eta$ acts by zero, in particular it cannot lie in $\mathrm{SH}(S)_\mathbf{Q}^-$ as the first Hopf map $\eta$ acts invertibly on this subcategory of $\mathrm{SH}(S)_\mathbf{Q}$.

$\square$

6.6.1. We can now glue our results to prove the following result up to inverting some primes.

**Theorem 6.32.** *Let $S$ be a Noetherian scheme of finite dimension. Let $J$ be a collection of primes which are all invertible in $S$ such that for $\ell \in J$ and any $x \in S$, $\mathrm{cd}_\ell(k(x)) < \infty$. Then, the canonical map*

$$\mathrm{MGL}_S \to \mathrm{MGL}_S^{\text{ét}},$$

*induces an equivalence in* $\mathrm{Mod}_{\mathrm{MGL}_{S,(J)}}$:

$$L_{\mathrm{M}\mathbf{Z}^{\text{ét}}}\mathrm{MGL}_{S,(J)} \xrightarrow{\simeq} \mathrm{MGL}_{S,(J)}^{\text{ét}}.$$

*Proof.* To begin with, consider the canonical map $\mathrm{MGL}_{S,(J)} \to \mathrm{MGL}_{S,(J)}^{\text{ét}}$. We claim that it factors through $\mathrm{MGL}_{S,(J)} \to L_{\mathrm{M}\mathbf{Z}^{\text{ét}}}\mathrm{MGL}_{S,(J)}$. To do so, we first need to prove that $\mathrm{MGL}_{S,(J)}^{\text{ét}}$ is $\mathrm{M}\mathbf{Z}^{\text{ét}}$-local, i.e., the canonical map (which exists since the target is étale-local) $\mathrm{MGL}_S^{\text{ét}} \to L_{\mathrm{M}\mathbf{Z}^{\text{ét}}}\mathrm{MGL}_S^{\text{ét}}$ is an equivalence in $\mathrm{SH}(S)$.

By the arithmetic fracture square in Theorem 8.15 we reduce to checking the equivalence on rationalization and $\ell$-adic completions. Lemma 6.31 tells us that $L_{\mathsf{MQ}}\mathsf{MGL}_S \simeq L_{\mathsf{MZ}^{\text{ét}}}\mathsf{MGL}_{S,\mathbb{Q}}$, while motivic Landweber exactness tells us that $\mathsf{MGL}_{S,\mathbb{Q}}$ is exactly computed as $L_{\mathsf{MQ}}\mathsf{MGL}_S \simeq \mathsf{MGL}_S \wedge \mathsf{MQ}_S$, as explained in [57, Corollary 10.6].

For a prime $\ell$, Lemma 6.34 reduces to proving that $\mathsf{MGL}_{S\ \ell}^{\text{ét}\wedge}$ is $\mathsf{MZ}_S^{\text{ét}}/\ell$-local. Since the $\infty$-category of $\mathsf{MZ}_S^{\text{ét}}/\ell$-local objects is colocalizing, i.e., stable and closed under small limits, it suffices to check that $\mathsf{MGL}_S^{\text{ét}}/\ell^\nu$ is $\mathsf{MZ}_S^{\text{ét}}/\ell$-local for $\nu \geq 1$. Since local objects are closed under cofiber sequences such as $\mathsf{MGL}_S^{\text{ét}}/\ell^\nu \to \mathsf{MGL}_S^{\text{ét}}/\ell^{\nu-1} \to \mathsf{MGL}_S^{\text{ét}}/\ell$ we may assume $\nu = 1$. It remains to prove that $\mathsf{MGL}_S^{\text{ét}}/\ell$ is $\mathsf{MZ}_S^{\text{ét}}/\ell$-local. By conditional convergence, see Corollary 4.19, there is an equivalence:

$$\lim (f^q \mathsf{MGL}_S)^{\text{ét}}/\ell \simeq \mathsf{MGL}_S^{\text{ét}}/\ell.$$

Since $\mathsf{MGL}_S$ is effective, $(s_0\mathsf{MGL}_S)^{\text{ét}}/\ell \simeq (f_{-1}\mathsf{MGL}_S)^{\text{ét}}/\ell$ and $(f_{-1}\mathsf{MGL}_S)^{\text{ét}}/\ell$ is $(s_0\mathsf{MGL}_S)^{\text{ét}}/\ell \simeq \mathsf{MZ}_S^{\text{ét}}/\ell$-local, which is the first induction step. Assuming $(f^{q'}\mathsf{MGL}_S)^{\text{ét}}/\ell$ is $\mathsf{MZ}_S^{\text{ét}}/\ell$-local for all $q' < q$, then $(f^q\mathsf{MGL}_S)^{\text{ét}}/\ell$ is $\mathsf{MZ}_S^{\text{ét}}/\ell$-local by the cofiber sequence:

$$(s_q\mathsf{MGL}_S)^{\text{ét}}/\ell \to (f^{q'}\mathsf{MGL}_S)^{\text{ét}}/\ell \to (f^{q-1}\mathsf{MGL}_S)^{\text{ét}}/\ell.$$

Indeed the term $(f^{q-1}\mathsf{MGL}_S)^{\text{ét}}/\ell$ is $\mathsf{MZ}^{\text{ét}}/\ell$-local by hypothesis. The slices $s_q\mathsf{MGL}_S/\ell$ are $s_0\mathsf{MGL}_S/\ell$-modules by [25, Section 6 (v)] and the functor $\epsilon_*\epsilon^*$ is lax monoidal, and thus preserves $\mathcal{E}_\infty$-algebras and modules. Hence $(s_q\mathsf{MGL}_S)^{\text{ét}}/\ell$ is a module over $\mathsf{MZ}_S^{\text{ét}}/\ell$. We conclude that the limit is $\mathsf{MZ}_S^{\text{ét}}/\ell$-local, since the $\infty$-category of $\mathsf{MZ}_S^{\text{ét}}/\ell$-local objects is colocalizing and thus closed under small limits.

Therefore, we have a comparison map $L_{\mathsf{MZ}^{\text{ét}}}\mathsf{MGL}_{S,(J)} \to \mathsf{MGL}_{S,(J)}^{\text{ét}}$ which we want to prove is an equivalence. Using the arithmetic fracture square in Theorem 8.15 again it suffices to check the equivalence rationally and on $\ell$-adic completions. Rationally there are equivalences:

$$L_{\mathsf{MZ}^{\text{ét}}}\mathsf{MGL}_{S,\mathbb{Q}} \simeq L_{\mathsf{MQ}}\mathsf{MGL}_S \simeq \mathsf{MGL}_{S,\mathbb{Q}} \simeq \mathsf{MGL}_{S,\mathbb{Q}}^{\text{ét}},$$

where the first is due to Lemma 6.31, the second to [57, Theorem 10.5], and the third holds since rational oriented theories are étale local [13, Corollary 14.2.16, Theorem 14.3.4].

Now to check at the primes. If $\ell \notin J$, then $\ell$ is invertible in $\mathsf{MGL}_{S,(J)}$ so that $\mathsf{MGL}_{S,(J)}/\ell^\nu$ is zero for all $\nu \geq 1$ and thus the claim follows trivially. If $\ell \in J$, using Lemma 6.34 below, it suffices to prove that $\mathsf{MGL}_{S,(J)\ell}^{\wedge} \to \mathsf{MGL}_{S,(J)\ell}^{\text{ét}\ \wedge}$ is an $\mathsf{MZ}_S^{\text{ét}}/\ell$-equivalence. This claim follows from Lemma 6.33, which uses our main Theorem 6.28.

$\square$

**Lemma 6.33.** *With the notation of Theorem 6.32, let $\ell \in J$ such that $\mathsf{MGL}_S/\ell^\nu$ is a unital ring spectrum. Then for all $\nu \geq 1$, the canonical map:*

$$\mathsf{MGL}_S \to \mathsf{MGL}_S^{\text{ét}},$$

*induces a canonical equivalence:*

(82)          $$\mathsf{MZ}_S^{\text{ét}} \wedge \mathsf{MGL}_S/\ell^\nu \xrightarrow{\simeq} \mathsf{MZ}_S^{\text{ét}} \wedge \mathsf{MGL}_S^{\text{ét}}/\ell^\nu.$$

*Proof.* The map $\mathsf{MGL}_S \to \mathsf{MGL}_S^{\text{ét}}$ induces a map $\mathsf{MZ}_S^{\text{ét}} \wedge \mathsf{MGL}_S/\ell^\nu \to \mathsf{MZ}_S^{\text{ét}} \wedge \mathsf{MGL}_S^{\text{ét}}/\ell^\nu$, after modding out by $\ell^\nu$ and applying $\mathsf{MZ}_S^{\text{ét}}\wedge$. To prove the result, we proceed via the following string of equivalences:

$$
\begin{aligned}
\mathsf{M}\mathbf{Z}^{\text{ét}}_S \wedge \mathsf{MGL}_S/\ell^\nu \;\;
&\simeq \;\; (\mathsf{MGL}_S/(x_1,\cdots x_n,\cdots))^{\text{ét}} \wedge \mathsf{MGL}_S/\ell^\nu \\
&\simeq \;\; (\operatorname{colim}\mathsf{MGL}_S)^{\text{ét}} \wedge \mathsf{MGL}_S/\ell^\nu \\
&\simeq \;\; \operatorname{colim}\mathsf{MGL}^{\text{ét}}_S \wedge \mathsf{MGL}_S/\ell^\nu \\
&\simeq \;\; \operatorname{colim}(\mathsf{MGL}_S/\ell^\nu)^{\text{ét}} \wedge \mathsf{MGL}_S \\
&\simeq \;\; \operatorname{colim}(\mathsf{MGL}_S/\ell^\nu)[(\tau^{\mathsf{MGL}_S}_{\ell^\nu})^{-1}] \wedge \mathsf{MGL}_S \\
&\simeq \;\; \operatorname{colim}((\mathsf{MGL}_S/\ell^\nu) \wedge \mathsf{MGL}_S)[(\tau^{\mathsf{MGL}_S}_{\ell^\nu})^{-1}] \\
&\simeq \;\; (\mathsf{M}\mathbf{Z}_S \wedge \mathsf{MGL}_S)^{\text{ét}} \\
&\simeq \;\; \mathsf{M}\mathbf{Z}^{\text{ét}}_S \wedge \mathsf{MGL}^{\text{ét}}_S.
\end{aligned}
$$

Here, the first equivalence is due to Theorem 6.8 which holds under the stated hypotheses, the second is merely rewriting the $\mathsf{MGL}_S/(x_1,\cdots x_n,\cdots)$ term as the colimit it is, the third is because of Theorem 5.1 which holds under the stated hypotheses, the fourth holds simply because étale localization is exact, the fifth is our Theorem 6.28, the sixth holds because $\tau_{\ell^\nu}$-inversion can take place in either SH or in $\mathrm{Mod}_{\mathsf{MGL}/\ell^\nu}$, the seventh is a consequence of Theorem 6.29, the eighth is because étale localization is closed under $\wedge$. $\qquad\square$

**Lemma 6.34.** *For* $\mathsf{M} \in \mathrm{SH}(S)$ *and all primes* $\ell$ *there is a canonical equivalence of endofunctors:*

$$
(L_\mathsf{M})^\wedge_\ell \simeq L_{\mathsf{M}/\ell} : \mathrm{SH}(S) \to \mathrm{SH}(S).
$$

*Proof.* We need to prove that for any $\mathsf{E} \in \mathrm{SH}(S)$, $(L_\mathsf{M}\mathsf{E})^\wedge_\ell$ is $\mathsf{M}/\ell$-local and the map $\mathsf{E} \to (L_\mathsf{M}\mathsf{E})^\wedge_\ell$ is an $\mathsf{M}/\ell$-equivalence in $\mathrm{SH}(S)$. To prove that $(L_\mathsf{M}\mathsf{E})^\wedge_\ell$ is $\mathsf{M}/\ell$-local, we let $\mathsf{F}$ be an $\mathsf{M}/\ell$-acyclic spectrum and first consider $\mathrm{Maps}(\mathsf{F}, L_\mathsf{M}\mathsf{E}/\ell)$. This is equivalent to $\mathrm{Maps}(\mathsf{F}, L_\mathsf{M}\mathsf{E} \wedge \mathbf{1}/\ell) \simeq \mathrm{Maps}(\mathsf{F} \wedge (\mathbf{1}/\ell)^\vee, L_\mathsf{M}\mathsf{E})$. Since $L_\mathsf{M}\mathsf{E}$ is a left adjoint and compatible with the monoidal structure, this is furthermore equivalent to $\mathrm{Maps}(L_\mathsf{M}(F) \wedge L_\mathsf{M}(1/\ell)^\vee, L_\mathsf{M}\mathsf{E})$ which is contractible by the assumption on $\mathsf{F}$. By induction we get the same result for $\mathbf{1}/\ell^\nu$ and thus the result upon completion. $\qquad\square$

6.6.2. Applying the same argument as in Theorem 6.32 for any $\mathsf{E} \in \mathrm{Mod}_{\mathsf{MGL}}$ and using Theorem 6.29 we get

**Theorem 6.35.** *Let $S$ be a Noetherian scheme of finite dimension. Suppose that $J$ is a collection of primes which are all invertible in $S$ and are such that for any $\ell \in J$ and any $x \in S$, $\mathrm{cd}_\ell(k(x)) < \infty$. Then, for any $\mathsf{E} \in \mathrm{Mod}_{\mathsf{MGL}}$ there is a naturally induced equivalence in $\mathrm{Mod}_{\mathsf{MGL}_{S,(J)}}$:*

$$
L_{\mathsf{M}\mathbf{Z}^{\text{ét}}}\mathsf{E}_{S,(J)} \xrightarrow{\cong} \mathsf{E}^{\text{ét}}_{S,(J)}.
$$

6.6.3. We explain how to recover Thomason's theorem in our setting. The motivic spectrum KGL representing algebraic $K$-theory is Landweber exact since the multiplicative formal group defines a Landweber exact $\mathrm{MU}_*$-algebra [57]. Recall further that for Noetherian scheme $S$ we have for all $p, q \in \mathbf{Z}$ a Voevodsky's isomorphism [80, Theorem 6.9], [12, Théorème 2.20]:

$$
\mathsf{KGL}_{-p,-q}(S) \cong \mathsf{KGL}^{p,q}(S) \cong \mathrm{KH}_{2q-p}(S),
$$

where KH denote Weibel's homotopy $K$-theory [89]. When $S$ is furthermore regular KH is equivalent to Thomason-Trobaugh algebraic $K$-theory.

Under the Todd genus map $\mathsf{MGL} \to \mathsf{KGL}$, the Bott element $\tau^{\mathsf{MGL}}_{\ell^\nu} \in \mathsf{MGL}/\ell^\nu_{0,-e_{\mathsf{MGL}}(\ell^\nu)}$ maps to $\tau^{\mathsf{KGL}}_{\ell^\nu} \in \mathsf{KGL}_{0,-e_{\mathsf{MGL}}(\ell^\nu)}$. Theorem 6.29 shows there is an equivalence:

$$
\text{(83)} \qquad\qquad \mathsf{KGL}/\ell^\nu[(\tau^{\mathsf{KGL}}_{\ell^\nu})^{-1}] \xrightarrow{\cong} \mathsf{KGL}^{\text{ét}}/\ell^\nu.
$$

Applying $\Omega^\infty_T$ to (83) tells us that $\mathsf{KGL}/\ell^\nu[(\tau^{\mathsf{KGL}}_{\ell^\nu})^{-1}]$ satisfies étale hyperdescent since $\Omega^\infty_T\mathsf{KGL}^{\text{ét}}/\ell^\nu$ does by the characterization of étale local objects in $\mathrm{SH}(S)$ in Lemma 2.13. This

recovers Thomason's main theorem from [78] when the base scheme $S$ is regular Noetherian and proves the analogous result for homotopy $K$-theory in general.

We sketch how our Bott element specializes to one picked by Thomason for $\ell \neq 2$. By [22, Bott Elements] the $K$-theoretic Bott elements are also chosen using Bockstein arguments, in parallel to our maneuvers in §6.3.5 and so we may assume $\nu = 1$. In Thomason's case, this Bott element is also chosen using Galois descent (again, see [22, Bott Elements]) along the extension $k(\zeta_\ell)/k$. This again parallels what we do in §6.3.

Working over $S = \mathbf{Z}[\frac{1}{\ell}, \zeta_\ell]$, we pick an MGL-theoretic Bott element in $\mathsf{MGL}/\ell_{0,-1}(S)$ which maps to a KGL-theoretic Bott element in $\mathsf{KGL}/\ell_{0,-1}(S) \simeq K_2/\ell(k)$. By its origin from the roots of unity, the KGL-theoretic Bott element is clearly an element that maps to $\zeta_\ell \in K_1(S)$ under the connecting homomorphism $\beta : K_2/\ell(S) \to K_1(S)$, just as Thomason picks his Bott element in this case [78, §A.7].

## 7. APPLICATIONS AND CONSEQUENCES

In this section, we gather some applications and consequences of our theorems above.

7.1. **Descriptions of étale-local module categories.** First, we generalize the result in [26] which gives an equivalence between modules over Bott inverted motivic cohomology and étale motives over a field. In particular we will describe the stable $\infty$-category of MGL-modules with étale descent. To begin, suppose that $\mathsf{E} \in \mathrm{CAlg}(\mathrm{SH}(S))$ and denote by $\mathrm{Mod}_\mathsf{E}^{\text{ét}}$ the full subcategory spanned by $\mathsf{E}$-modules which are ét-local, i.e., its image under the forgetful functor $u_E : \mathrm{Mod}_\mathsf{E} \to \mathrm{SH}(S)$ lands in $\mathrm{SH}_{\text{ét}}(S)$. By definition, we have the following diagram of adjoints

(84)
$$
\begin{array}{ccc}
\mathrm{Mod}_\mathsf{E} & \underset{\epsilon_*}{\overset{\epsilon^*}{\longleftrightarrow}} & \mathrm{Mod}_\mathsf{E}^{\text{ét}} \\
{\scriptstyle -\wedge\mathsf{E}}\Big\updownarrow{\scriptstyle u_\mathsf{E}} & & {\scriptstyle u_\mathsf{E}^{\text{ét}}}\Big\updownarrow{\scriptstyle (-\wedge\mathsf{E})^{\text{ét}}} \\
\mathrm{SH}(S) & \underset{\epsilon^*}{\overset{\epsilon_*}{\longleftrightarrow}} & \mathrm{SH}_{\text{ét}}(S).
\end{array}
$$

**Proposition 7.1.** *Let $S$ be a scheme, $\mathsf{E} \in \mathrm{CAlg}(\mathrm{SH}(S))$. Then we have a canonical adjunction of* $\mathrm{SH}(S)$*-modules:*

(85)
$$F_\mathsf{E} : \mathrm{Mod}_{\mathsf{E}^{\text{ét}}} \rightleftarrows \mathrm{Mod}_\mathsf{E}^{\text{ét}} : G_\mathsf{E},$$

*which is an equivalence whenever the composite of the right adjoints in* (84)*:*

$$\mathrm{Mod}_\mathsf{E}^{\text{ét}} \to \mathrm{SH}(S),$$

*preserves small colimits.*

*Proof.* For this proof, we denote the composite right adjoint of (84) by $R : \mathrm{Mod}_\mathsf{E}^{\text{ét}} \to \mathrm{SH}(S)$ and the left adjoint by $L : \mathrm{SH}(S) \to \mathrm{Mod}_\mathsf{E}^{\text{ét}}$. We remark that it factors through $\mathrm{Mod}_{\mathsf{E}^{\text{ét}}}$ since it takes the unit object in $\mathrm{Mod}_\mathsf{E}^{\text{ét}}$ to $\mathsf{E}^{\text{ét}}$ and is lax monoidal, being the right adjoint of a strong monoidal functor. The resulting functor $G_\mathsf{E} : \mathrm{Mod}_\mathsf{E}^{\text{ét}} \to \mathrm{Mod}_{\mathsf{E}^{\text{ét}}}$ is the right adjoint of the desired adjunction (84), from which the left adjoints exist by the adjoint functor theorem.

Now, suppose that the functor $R : \mathrm{Mod}_\mathsf{E}^{\text{ét}} \to \mathrm{SH}(S)$ preserve colimits (in other words, it preserves filtered colimits since it is an exact functor of stable $\infty$-categories). By [23, Theorem 3.6], it suffices to verify the projection formula [23, Definition 3.5]: for any $\mathsf{M} \in \mathrm{SH}(S)$ the canonical map:

$$RL(\mathbf{1}) \wedge \mathsf{M} \simeq \mathsf{E}^{\text{ét}} \wedge \mathsf{M} \to RL(\mathsf{M}),$$

is an equivalence. To prove this, since both functors preserves colimits in the M-variable, it suffices to prove the claim for $M = \Sigma^{p,q}\Sigma_+^\infty X$ for any $p, q \in \mathbf{Z}$. By Remark 2.12, we may assume that $p, q = 0$ we are then left to prove that the map:

$$RL(\mathbf{1}) \wedge \Sigma_+^\infty X \simeq \mathsf{E}^{\text{ét}} \wedge M \to RL(\Sigma_+^\infty X),$$

is an equivalence for $X$ a smooth $S$-scheme. In this case $X$ is a smooth $S$-scheme with structure map $f$, then we have equivalences in $\mathrm{SH}(S)$:

$$
\begin{aligned}
\mathsf{E}^{\text{ét}} \wedge \Sigma_T^\infty X_+ &= \epsilon_* \epsilon^*(\mathsf{E}) \otimes f_\# f^* \mathbf{1} \\
&\simeq f_\#(f^* \epsilon_* \epsilon^* \mathsf{E} \otimes \mathbf{1}) \\
&\simeq f_\#(\epsilon_* \epsilon^* f^* \mathsf{E}) \\
&\simeq \epsilon_* \epsilon^*(f_\# f^* \mathsf{E}) \\
&= RL(\Sigma_T^\infty X_+),
\end{aligned}
$$

Here, the first equivalence is the smooth projection formula (see, for example, [13, 1.1.26]), the second equivalence follows from the argument of Lemma 6.25 for SH using that $f$ is smooth, and the last equivalence follows from the equivalence of the transformation $f_\# \epsilon_* \to \epsilon_* f_\#$ on representables which can be seen to hold unstably from the definitions of these functors since the étale topology is subcanonical. $\square$

7.1.1. We freely use the machinery of periodization and localization reviewed in §8. In our context we are interested in $\mathsf{E} \in \mathrm{CAlg}(\mathrm{SH}(S))$ and a map $\alpha : x \to \mathsf{E}$ in $\mathrm{Mod}_\mathsf{E}$ corresponding to a homotopy element in $\mathsf{E}$. We consider the full subcategory $P_\alpha(\mathrm{Mod}_\mathsf{E}) \subset \mathrm{Mod}_\mathsf{E}$ spanned by $\alpha$-periodic objects. By §8.1.3, $\mathsf{E}[\alpha^{-1}] := P_\alpha \mathsf{E} \in \mathrm{CAlg}(\mathrm{Mod}_\mathsf{E})$, i.e., the periodization remains an $\mathcal{E}_\infty$-algebra. Therefore, we may consider the stable $\infty$-category of modules over $P_\alpha \mathsf{E}$ in $\mathsf{E}$-modules:

$$\mathrm{Mod}_{\mathsf{E}[\alpha^{-1}]} := \mathrm{Mod}_{P_\alpha E}(\mathrm{Mod}_\mathsf{E}).$$

First we identify $\alpha$-periodic modules with modules over the $\mathcal{E}_\infty$-ring of the $\alpha$-periodization of $\mathsf{E}$.

**Proposition 7.2.** *Let $\alpha : x \to \mathsf{E}$ be a morphism in $\mathrm{Mod}_\mathsf{E}$ where $x$ is an invertible object. Then there is an equivalence of $\infty$-categories:*

$$P_\alpha \mathrm{Mod}_\mathsf{E} \simeq \mathrm{Mod}_{\mathsf{E}[\alpha^{-1}]}.$$

*Proof.* According to Proposition 8.4 for $M \in \mathrm{Mod}_\mathsf{E}$, we have an equivalence between $P_\alpha M$ and the colimit $Q_\alpha M$ of the diagram:

$$M \to \underline{\mathrm{map}}(x, M) \to \underline{\mathrm{map}}(x^{\otimes 2}, M) \to \cdots.$$

Moreover, there are equivalences:

$$
\begin{aligned}
Q_\alpha M &\simeq \mathrm{colim}(M \to M \otimes x^\vee \to M \otimes (x^\vee)^{\otimes 2} \to \cdots) \\
&\simeq \mathrm{colim}\, M \otimes (1 \to x^\vee \to (x^\vee)^{\otimes 2} \to \cdots) \\
&\simeq M \otimes \mathsf{E}[\alpha^{-1}],
\end{aligned}
$$

where the first follows from dualizability (since it is, in fact, invertible) of $x$ and the second holds because $\mathrm{Mod}_\mathsf{E}$ is presentably symmetric monoidal, i.e., the tensor product commutes with colimits. Thus every $\alpha$-periodic object is canonically an $\mathsf{E}[\alpha^{-1}]$-module. In other words, $P_\alpha$ factors through $P_\alpha : \mathrm{Mod}_\mathsf{E} \to \mathrm{Mod}_{\mathsf{E}[\alpha^{-1}]}$.

Conversely, since the full subcategory of $\alpha$-local objects is the essential image of $Q_\alpha$, every $\alpha$-local object is a module over $\mathsf{E}[\alpha^{-1}]$. Thus $P_\alpha : \mathrm{Mod}_\mathsf{E} \rightleftarrows P_\alpha \mathrm{Mod}_\mathsf{E} : u$ induces the adjunction:

$$- \otimes E[\alpha^{-1}] : \mathrm{Mod}_{\mathsf{E}[\alpha^{-1}]} \rightleftarrows P_\alpha \mathrm{Mod}_\mathsf{E} : u.$$

Since every object in $P_\alpha \mathrm{Mod}_\mathsf{E}$ is of the form $Q_\alpha M$, the functor is essentially surjective. Fully faithfulness follows since $Q_\alpha$ is computed by tensoring with $- \otimes E[\alpha^{-1}]$ as shown above. $\square$

In other words, $\alpha$-periodization is a smashing localization.

7.1.2.    In the case of motivic cohomology we generalize the main result of [26] in a highly coherent setting.

**Theorem 7.3.** *Let $\ell$ be a prime, $S$ be a Noetherian $\mathbf{Z}[\frac{1}{\ell}]$-scheme of finite dimension. Further assume that for all $x \in S$, $\mathrm{cd}_\ell(k(x)) < \infty$. There is an equivalence of stable $\infty$-categories:*

$$\epsilon^* : P_{\tau_{\ell^\nu}^{\mathsf{M}\mathbf{Z}}}\mathrm{Mod}_{\mathsf{M}\mathbf{Z}_S/\ell^\nu} \rightleftarrows \mathrm{Mod}_{\mathsf{M}\mathbf{Z}_S^{\text{ét}}/\ell^\nu} : \epsilon_*.$$

*Suppose further that $S$ is a regular Noetherian scheme of finite dimension, over a field $k$ whose exponential charateristic is coprime to $\ell$, then there is an equivalence:*

$$\epsilon^* : P_{\tau_{\ell^\nu}^{\mathsf{M}\mathbf{Z}}} \mathrm{DM}(S, \mathbf{Z}/\ell^\nu) \rightleftarrows \mathrm{DM}_{\text{ét}}(S; \mathbf{Z}/\ell^\nu) : \epsilon_*.$$

*Proof.* We remark that, by construction [75, §4.1.1], $\mathsf{M}\mathbf{Z}_S^{\text{ét}}/\ell^\nu$ is an $\mathcal{E}_\infty$-ring spectrum. The claim then follows immediately from Theorem 6.30 and Proposition 7.2. The last statement follows from the identification of DM with modules over motivic cohomology. This latter result is obtained by a combination of the main theorem of [63] over characteristic zero, Theorem 5.8 of [31] over positive characteristics and its generalization in [14, Theorem 3.1]. □

7.1.3.    The following is a version for MGL and, more generally, $\mathcal{E}_\infty$-algebras in MGL modules. This is where the strength of our integral statement in Theorem 6.32 comes into play since the motivic spectrum $L_{\mathsf{M}\mathbf{Z}^{\text{ét}}}\mathsf{E}_{\mathsf{M},(J)}$ is an $\mathcal{E}_\infty$-ring.

**Theorem 7.4.** *Let $S$ be a Noetherian scheme, and $J$ be a collection of primes which are all invertible in $S$ such that for any $\ell \in J$ and any $x \in S$, $\mathrm{cd}_\ell(k(x)) < \infty$. Then, the adjunction:*

$$\epsilon_* : \mathrm{Mod}_{\mathsf{MGL}_S} \rightleftarrows \mathrm{Mod}_{\mathsf{MGL}_S}^{\text{ét}} : \epsilon_*,$$

*induces an equivalence of presentably symmetric monoidal stable $\infty$-categories:*

$$\epsilon^* : \mathrm{Mod}_{L_{\mathsf{M}\mathbf{Z}^{\text{ét}}}\mathsf{MGL}_{S,(J)}} \rightleftarrows \mathrm{Mod}_{\mathsf{MGL}_{S,(J)}}^{\text{ét}} : \pi_*.$$

*More generally, if $\mathsf{E}_N$ is an $\mathcal{E}_\infty$-ring in $\mathrm{Mod}_{\mathsf{MGL}_S}$, then there is an equivalence of stable $\infty$-categories:*

$$\epsilon^* : \mathrm{Mod}_{L_{\mathsf{M}\mathbf{Z}^{\text{ét}}}\mathsf{E}_{N,(J)}} \rightleftarrows \mathrm{Mod}_{\mathsf{E}_{N,(J)}}^{\text{ét}} :: \pi_*.$$

*Proof.* First, note that for every $\mathsf{E} \in \mathrm{SH}(S)$ the localization $L_\mathsf{E}$ is compatible with the monoidal structure in the sense of [46, Definition 2.2.1.6, Remark 2.2.1.7]: for any E-equivalence $f : X \to Y$ and $Z \in \mathrm{SH}(S)$, $f \wedge Z : X \wedge Z \to Y \wedge Z$ is an E-equivalence. Therefore, by [46, Proposition 2.2.1.9], $L_{\mathsf{M}\mathbf{Z}^{\text{ét}}\mathsf{MGL}_{(J)}}$ is an $\mathcal{E}_\infty$-algebra and Theorem 6.32 gives an equivalence of $\mathcal{E}_\infty$-algebras in $\mathrm{SH}(S)$: $L_{\mathsf{M}\mathbf{Z}^{\text{ét}}}\mathsf{MGL}_{(J)} \simeq \mathsf{MGL}_{(J)}^{\text{ét}}$. The claimed result then follows from the second part of Proposition 7.1, whose hypothesis is verified by Theorem 5.1. □

7.2.  **Base change and six functors.**  Since $f^*$ and $\epsilon_*$ has no reason to commute, it is not clear that étale localization of a motivic spectrum commutes with base-change. Our description of étale local spectra as Bott-inverted spectra allows us to prove some base change results.

**Theorem 7.5.** *Let $\ell$ be a prime and $T, S$ be a Noetherian $\mathbf{Z}[\frac{1}{\ell}]$-scheme of finite dimension and assume that for all $x \in S$, $\mathrm{cd}_\ell(k(x)) < \infty$. Let $f : T \to S$ be a morphism, then the canonical map:*

$$f^*\mathsf{E}_S^{\text{ét}}/\ell^\nu \to \mathsf{E}_T^{\text{ét}}/\ell^\nu,$$

*is an equivalence*

*Proof.* The claim follows immediately from Theorem 6.28, the fact that $f^*$ commutes with colimits, and our choice of Bott elements from Definition 6.16. □

Now, let $\mathrm{Sch}'_{\mathbf{Z}[\frac{1}{\ell}]}$ be the full subcategory of $\mathrm{Sch}_{\mathbf{Z}[\frac{1}{\ell}]}$ spanned by Noetherian schemes whose field points have finite $\ell$-cohomological dimension. It follows that:

**Corollary 7.6.** *For for any $n \geq 1$:*

$$X \mapsto \mathsf{MGL}_X^{\text{ét}}/\ell^v, \mathsf{MZ}_X^{\text{ét}}/\ell^v,$$

*defines Cartesian sections of* $\mathrm{SH} \to \mathrm{Sch}'_{\mathbf{Z}[\frac{1}{\ell}]}$.

We also obtain an integral statement. Let $J$ is a collection of primes and let $\mathrm{Sch}^{(J)} \subset \mathrm{Sch}$ be the full subcategory of schemes spanned by those Noetherian schemes $X$ such that all primes in $J$ are invertible in $X$, and any $\ell \in J$ and any $x \in X$, $\mathrm{cd}_\ell(k(x)) < \infty$.

**Corollary 7.7.** *The functor:*

$$X \mapsto \mathsf{MGL}_{X,(J)}^{\text{ét}},$$

*defines a Cartesian section of* $\mathrm{CAlg}(\mathrm{SH}) \to \mathrm{Sch}^{(J)}$.

*Proof.* By Theorem 6.32, it suffices to prove that for any morphism $f : X \to Y$ in $\mathrm{Sch}^{(J)}$, the canonical map:

$$f^* L_{\mathsf{MZ}^{\text{ét}}} \mathsf{MGL}_{(J),Y} \to L_{\mathsf{MZ}^{\text{ét}}} \mathsf{MGL}_{(J),X},$$

is an equivalence. We note that for any scheme $S$, $L_{\mathsf{MZ}_{(J)}^{\text{ét}}} \mathsf{MGL}_{(J),S} \simeq L_{\mathsf{MZ}^{\text{ét}}} \mathsf{MGL}_{(J),S}$. Hence, by Lemma 8.16, it then suffices to prove that $\mathsf{MZ}_{(J)}^{\text{ét}}$ defines a Cartesian section of $\mathrm{SH} \to \mathrm{Sch}^{(J)}$. This follows from Corollary 7.6. $\square$

Thanks to Corollary (7.6), it is standard (see, for example [75, §10]) that we can form a stable homotopy 2-functor:

$$(86) \qquad \mathrm{ho}(\mathrm{Mod}_{\mathsf{MGL}/\ell^v}) : \mathrm{Sch}_{\mathbf{Z}[\frac{1}{\ell}]}^{'\text{op}} \to \mathrm{TriCat},$$

in the sense of [3, Definition 1.4.1] which thus satisfies the full six functors formalism by [3, Scholie 1.4.2]. Corollary (7.6) also feeds into the formalism of premotivic categories which we have already encountered. By [13, Theorem 2.4.50], we also obtain the six functor formalism for $\mathrm{Mod}_{\mathsf{MGL}/\ell^v}$. The same remark applies to:

$$(87) \qquad \mathrm{Mod}_{\mathsf{MGL}_{(J)}^{\text{ét}}} : (\mathrm{Sch}^{(J)})^{\mathrm{op}} \to \mathrm{Cat}_\infty,$$

using Corollary 7.7. Combining this with Theorem 7.4, we obtain

**Corollary 7.8.** *The functor*

$$(88) \qquad \mathrm{Mod}_{\mathsf{MGL}_{(J)}}^{\text{ét}} : (\mathrm{Sch}^{(J)})^{\mathrm{op}} \to \mathrm{Cat}_\infty,$$

*satisfies the full six functors formalism.*

7.3. **The étale hyperdescent spectral sequence.** After the discussion in §4.0.4, we obtain an immediate reward in the form of descent spectral sequences. When $N = \mathsf{KU}_*$, so that $\mathsf{E}_N = \mathsf{KGL}$ is the algebraic $K$-theory spectrum, we get back Thomason's result [78, Theorem 4.1].

**Theorem 7.9.** *Let* $\mathsf{E} \in \mathrm{SH}(S)$ *be a Landweber exact spectrum. There exist strongly convergent spectral sequences of the form:*

$$H_{\text{ét}}^p(S, \underline{\pi}_{-q,-t}^{\text{ét}}(\mathsf{E}_N^{\text{ét}}/\ell^v)) \Rightarrow \mathrm{Hom}_{\mathrm{SH}(S)}(\mathbf{1}_S, \Sigma^{0,t}\mathsf{E}_N^{\text{ét}}/\ell^v[(\tau_{\ell^v}^{\mathsf{MGL}})_S^{-1}][p+q]),$$

$$H_{\text{ét}}^p(S, \underline{\pi}_{-q,-t}^{\text{ét}}((\mathsf{E}_N^{\text{ét}})_\ell^\wedge)) \Rightarrow \mathrm{Hom}_{\mathrm{SH}(S)}(\mathbf{1}_S, \Sigma^{0,t}(\mathsf{E}_N^{\text{ét}})_\ell^\wedge[(\tau_{\ell^v}^{\mathsf{MGL}})_S^{-1}][p+q]),$$

*and:*

$$H_{\text{ét}}^p(S, \underline{\pi}_{-q,-t}^{\text{ét}}(L_{\mathsf{MZ}^{\text{ét}}}\mathsf{E}_{N,(J)})) \Rightarrow \mathrm{Hom}_{\mathrm{SH}(S)}(\mathbf{1}_S, \Sigma^{0,t}L_{\mathsf{MZ}^{\text{ét}}}\mathsf{E}_{N,(J)}[p+q]).$$

7.3.1.   In order to access the spectral sequence above, we need to compute the étale homo-topy sheaves:

$$\underline{\pi}_{p,q}^{\text{ét}}(E_N^{\text{ét}}/\ell^\nu) : \text{Sm}_S^{\text{op}} \to \text{Ab}, \underline{\pi}_{p,q}^{\text{ét}}((E_N^{\text{ét}})_\ell^\wedge) : \text{Sm}_S^{\text{op}} \to \text{Ab}.$$

These étale sheaves also assemble into étale sheaves of graded abelian groups:

$$\oplus_{p,q}\underline{\pi}_{p,q}^{\text{ét}}(E_N^{\text{ét}}/\ell^\nu) : \text{Sm}_S^{\text{op}} \to \text{grAb}, \oplus_{p,q}\underline{\pi}_{p,q}^{\text{ét}}((E_N^{\text{ét}})_\ell^\wedge) : \text{Sm}_S^{\text{op}} \to \text{grAb}.$$

We claim these sheaves are locally constant (see, for example, [49, Definition 6.7]). There is an adjunction between the small and big étale (discrete) topoi over $S$:

$$\rho^* : \text{Shv}_{\text{ét}}(\text{Ét}_S) \rightleftarrows \text{Shv}_{\text{ét}}(\text{Sm}_S) : \rho_*.$$

This gives rise to an adjunction between sheaves of abelian groups:

$$\rho^* : \text{AbShv}_{\text{ét}}(\text{Ét}_S) \rightleftarrows \text{AbShv}_{\text{ét}}(\text{Sm}_S) : \rho_*,$$

and its graded variant:

$$\rho^* : \text{grAbShv}_{\text{ét}}(\text{Ét}_S) \rightleftarrows \text{grAbShv}_{\text{ét}}(\text{Sm}_S) : \rho_*.$$

**Proposition 7.10.** *For any $p, q \in \mathbf{Z}$, the counit map yields an isomorphism:*

$$\rho^*\rho_*\underline{\pi}_{p,q}^{\text{ét}}(E_N^{\text{ét}}/\ell^\nu) \xrightarrow{\cong} \underline{\pi}_{p,q}^{\text{ét}}(E_N^{\text{ét}}/\ell^\nu).$$

*Consequently the counit map on $\ell$-completions:*

$$\rho^*\rho_*\underline{\pi}_{p,q}^{\text{ét}}((E_N^{\text{ét}})_\ell^\wedge) \to \underline{\pi}_{p,q}^{\text{ét}}((E_N^{\text{ét}})_\ell^\wedge),$$

*and the graded variants:*

$$\rho^*\rho_* \oplus_{p,q} \underline{\pi}_{p,q}^{\text{ét}}(E_N^{\text{ét}}/\ell^\nu) \to \oplus_{p,q}\underline{\pi}_{p,q}^{\text{ét}}(E_N^{\text{ét}}/\ell^\nu),$$

$$\rho^*\rho_* \oplus_{p,q} \underline{\pi}_{p,q}^{\text{ét}}((E_N^{\text{ét}})_\ell^\wedge) \to \oplus_{p,q}\underline{\pi}_{p,q}^{\text{ét}}((E_N^{\text{ét}})_\ell^\wedge),$$

*are isomorphisms.*

*Proof.* The counit map

$$\rho^*\rho_*\underline{\pi}_{p,q}^{\text{ét}}(E_N^{\text{ét}}/\ell^\nu) \to \underline{\pi}_{p,q}^{\text{ét}}(E_N^{\text{ét}}/\ell^\nu)$$

is a map of étale sheaves on $\text{Sm}_S$. To check the isomorphism, we consider stalks. Since the residue field of the strict Henselization of the local ring of a smooth $S$-scheme is separably closed, we may assume that $S$ is the spectrum of a separably closed field. In this case, $\underline{\pi}_{p,q}^{\text{ét}}(E_N^{\text{ét}}/\ell^\nu)$ is the locally constant sheaf for the abelian group $\underline{\pi}_{p,q}^{\text{ét}}(E_N^{\text{ét}}/\ell^\nu)(\text{Spec } k)$ by [28, Theorem 0.3]. The other cases follow readily.          □

7.3.2.   As the proof of Proposition 7.10 indicates, it will be useful to know the value of the homotopy sheaves over a separably closed field. First we have the following computation:

**Proposition 7.11.** *If $k$ is an algebraically closed field, we have isomorphisms of graded abelian groups:*

$$\oplus_{p,q}\text{MGL}^{\text{ét}}/\ell_{p,q}^\nu(k) \cong \oplus_p\text{MU}/\ell_p^\nu[(\tau_{\ell^\nu}^{\text{MGL}})_S^{-1}],$$

*and:*

$$\oplus_{p,q}((\text{MGL}^{\text{ét}})_\ell^\wedge)_{p,q}(k) \cong \oplus_p(\text{MU}_\ell^\wedge)_p[(\tau_{\ell^\nu}^{\text{MGL}})_S^{-1}].$$

*Proof.* From the computation of $s_*\text{MGL}$ reviewed in §6.3.1, and the collapse of the Bott inverted spectral sequence over an algebraically closed field we have an isomorphism:

$$(89) \qquad \oplus_{p,q} \text{MGL}/\ell_{p,q}^\nu(k)[(\tau_{\ell^\nu}^{\text{MGL}})_S^{-1}] \cong \oplus_p\text{MU}/\ell_p^\nu[(\tau_{\ell^\nu}^{\text{MGL}})_S^{-1}].$$

The desired isomorphisms follow as an instance of Theorem 6.26.          □

More generally, we have:

**Proposition 7.12.** *If $k$ is an algebraically closed field, we have isomorphisms of graded abelian groups:*

$$\oplus_{p,q}\mathsf{E}_N^{\text{ét}}/\ell_{p,q}^v(k) \cong \oplus_p N_p/\ell^v[(\tau_{\ell^v}^{\mathsf{MGL}})_S^{-1}],$$

*and:*

$$\oplus_{p,q}((\mathsf{E}_N^{\text{ét}})_\ell^\wedge)_{p,q}(k) \cong \oplus_p (N_\ell^\wedge)_p[(\tau_{\ell^v}^{\mathsf{MGL}})_S^{-1}].$$

*Proof.* There is an isomorphism of graded abelian groups $\mathsf{E}_{N*,*} \cong \mathsf{MGL}_{*,*} \otimes_{\mathsf{MU}_*} N_*$ [57, §7]. Therefore we have isomorphisms:

$$
\begin{aligned}
\oplus_{p,q}\mathsf{E}_N/\ell_{p,q}^v(k)[(\tau_{\ell^v}^{\mathsf{MGL}})_S^{-1}] &\cong (\oplus_{p,q}\mathsf{MGL}/\ell_{p,q}^v(k)[(\tau_{\ell^v}^{\mathsf{MGL}})_S^{-1}]) \otimes_{\mathsf{MU}_*} N \\
&\cong (\oplus_p\mathsf{MU}/\ell_p^v[(\tau_{\ell^v}^{\mathsf{MGL}})_S^{-1}]) \otimes_{\mathsf{MU}_*} N_* \\
&\cong \oplus_p N_p/\ell^v[(\tau_{\ell^v}^{\mathsf{MGL}})_S^{-1}].
\end{aligned}
$$

The desired isomorphisms follow as an instance of Theorem 6.26. $\qquad\square$

Propositions 7.10 and 7.12 show the coefficients of the descent spectral sequences displayed in Theorem 7.9 are locally constant sheaves on a periodized version of $N_*$.

### 7.4. Cellularity.

Recall the $\infty$-category $\mathsf{SH}(S)^{\mathsf{cell}}$ of *cellular spectra* is the full localizing subcategory of $\mathsf{SH}(S)$ generated by $\Sigma^{p,q}\mathbf{1}_S$ for $p, q \in \mathbf{Z}$ [20]. While the left adjoint functor $\epsilon^*$ preserves cellularity, this isn't quite clear for its right adjoint $\epsilon_*$.

**Theorem 7.13.** *Let $\ell$ be a prime and $S$ be a Noetherian $\mathbf{Z}[\frac{1}{\ell}]$-scheme of finite dimension and assume that for all $x \in S$, $\mathrm{cd}_\ell(k(x)) < \infty$. Suppose that $\mathsf{E} \in \mathsf{SH}(S)$ is an $\mathsf{MGL}$-module which is cellular, then $\mathsf{E}^{\text{ét}}/\ell^v$ is cellular.*

*Proof.* Follows from Theorem 6.28 because cellular objects are closed under colimits and thus $\mathsf{E}/\ell^v[(\tau_{\ell^v}^{\mathsf{MGL}})_S^{-1}]$ is cellular. $\qquad\square$

*Remark 7.14.* It is unclear whether the $\ell$-completion $\mathsf{E}_\ell^\wedge$ is actually cellular. This is a delicate issue as cellular objects are not closed under infinite limits, see [79, §5].

## 8. APPENDIX A: PERIODIZATION AND LOCALIZATIONS

### 8.1. Periodization.

We begin by reviewing the process of periodization/inversion of objects in a presentably symmetric monoidal $\infty$-category. Our main reference is [29, Section 3]. The set-up is as follows: $\mathsf{C}$ is a presentably symmetric monoidal $\infty$-category and $S$ a collection of objects of $\mathsf{C}_{/\mathbf{1}}$ i.e., objects $x$ equipped with a map $\alpha : x \to \mathbf{1}$. Let $\mathsf{M}$ be a $\mathsf{C}^\otimes$-module.

8.1.1. Let us first recall the periodization procedure which does not take into account coherence. We have already used this procedure in §6.4 and in other places throughout the paper. Here, we consider the category $\mathrm{Ho}(\mathsf{C})$ as a symmetric monoidal category.

Suppose $m \in \mathrm{Ho}(\mathsf{C})$ admits a unital multiplication (which is not necessarily associative) and $x \in \mathrm{Ho}(\mathsf{C})$ acquires a unital pairing $m \otimes x \to x$. If $m$ is an associative monoid in $\mathrm{Ho}(\mathsf{C})$ and $x$ is an $m$-module, we may take the module action as this pairing. Let $\alpha : y \to m$ be a map from an $\otimes$-invertible object $y$. For example if $\mathsf{C} = \mathsf{SH}(S)$, then $y$ could be $\Sigma^{p,q}\mathbf{1}$ and $\alpha : \Sigma^{p,q}\mathbf{1} \to m$ is a homotopy element. We define $\alpha \cdot : x \to y^{-1} \otimes x$ by applying $y^{\otimes -1}$ to the "multiplication by $\alpha$ map":

$$y \otimes x \xrightarrow{\alpha} m \otimes x \to x.$$

The $\alpha$-inversion $x[\alpha^{-1}]$ of $x$ is defined as the colimit of the diagram:

$$(90) \qquad\qquad x \xrightarrow{\alpha\cdot} y^{-1} \otimes x \xrightarrow{\mathrm{id}\,\otimes\alpha\cdot} y^{-1} \otimes y^{-1} \otimes x \cdots.$$

8.1.2.   Let us now tackle the homotopy coherent situation.

**Definition 8.1.** The $\infty$-category $P_S\mathsf{M}$ of *S-periodic objects in* $\mathsf{M}$ is the full subcategory of $\mathsf{M}$ spanned by objects $m \in M$ such that for all $\alpha : x \to \mathbf{1}$ in $S$, the map $\alpha^* : \mathrm{Hom}(\mathbf{1}, m) \simeq m \to \mathrm{Hom}(x, m)$ is an equivalence.

**Example 8.2.** Let $n : \mathbf{1} \to \mathbf{1}$ be the multiplication by $n$ map in SH, the stable $\infty$-category of spectra. A spectrum is $\{n : \mathbf{1} \to \mathbf{1}\}$-periodic if and only if $n$ acts invertibly.

**Proposition 8.3.** *Suppose the domains of elements in $S$ are $\kappa$-compact. Then the fully faithful embedding of $P_S\mathsf{M}$ into $\mathsf{M}$ admits a left adjoint $P_S : \mathsf{M} \to P_S\mathsf{M}$ which witnesses the latter as an $\kappa$-accessible localization at $\{\mathrm{id}_m \otimes \alpha\}$, where $m \in M$, and $\alpha : x \to \mathbf{1} \in S$.*

*Proof.* It is clear that limits of periodic objects are also periodic. To see that $P_S\mathsf{M} \hookrightarrow \mathsf{M}$ preserves $\kappa$-filtered colimits, note that for all $\alpha : x \to \mathbf{1} \in S$ we have, by compactness of $x$,:

$$\mathrm{Maps}(x, \mathrm{colim}_\alpha E_\alpha) \simeq \mathrm{colim}_\alpha \mathrm{Maps}(x, E_\alpha) \simeq \mathrm{colim}_\alpha E_\alpha,$$

and thus the left adjoint exists. By the adjunction $\mathrm{Hom}(m \otimes x, n) \simeq \mathrm{Hom}(m, \mathrm{Hom}(x, n))$, we see that we are exactly inverting the maps $\mathrm{id}_m \otimes x : m \otimes x \to m \otimes \mathbf{1} \simeq m$. $\qquad\square$

8.1.3.   The localization turns out to be compatible with the monoidal structure in the sense of [46, Proposition 2.2.1.9]. Thus $A \in \mathrm{CAlg}(\mathsf{C})$ implies $P_S A \in \mathrm{CAlg}(\mathsf{C})$, and the map $A \to P_S A$ is one of algebras.

8.1.4.   Let $S_0$ denote the set of domains of the morphisms in $S$. By the machinery explained in Theorem 2.4 we can consider the monoidal inversion $\mathsf{M}[S_0^{-1}]$ for which there is an equivalence of symmetric monoidal $\infty$-categories [29, Proposition 3.2]:

$$P_S\mathsf{M}[S_0^{-1}] \rightleftarrows P_S\mathsf{M}.$$

Thus every $S$-periodic object in $\mathrm{CAlg}(\mathsf{C})$ uniquely defines an $S$-periodic object in $\mathrm{CAlg}(\mathsf{C}[S_0^{-1}])$.

8.1.5.   An explicit formula for periodization is given in [29, Section 3]. We are interested in inverting a single map $\alpha : x \to \mathbf{1}$, so the formula explained prior to Theorem 3.8 of [29] simplifies to:

$$Q_\alpha m := \mathrm{colim}\, m \xrightarrow{\alpha^*} \underline{\mathrm{map}}(x, m) \xrightarrow{\alpha^*} \underline{\mathrm{map}}(x, \underline{\mathrm{map}}(x, m)) \simeq \underline{\mathrm{map}}(x^{\otimes 2}, m) \cdots.$$

According to [29, Theorem 3.8], this formula computes exactly the $\alpha$-periodiziation in certain cases. For our purposes it suffices to prove a simpler result.

Suppose $\Gamma$ is a symmetric monoidal 1-groupoid, and $\mathsf{D}$ is a 1-category. The 1-category $\mathsf{D}^\Gamma$ of $\Gamma$-*graded objects in* $\mathsf{D}$ is the functor category $\mathrm{Fun}(\Gamma, \mathsf{D})$. For $F : \Gamma \to \mathsf{D}$ we set $F_\gamma := F(\gamma)$ and for $\gamma, \gamma' \in \Gamma$ we set $F_{\gamma+\gamma'} := F(\gamma \otimes \gamma')$. If $\Gamma$ is the groupoid of integers with a unique invertible morphism and monoidal structure given by addition, then $\mathsf{D}^\Gamma$ is the category of graded objects in $\mathsf{D}$. If $\Gamma := \mathrm{Ho}(\mathrm{Pic}(\mathrm{SH}(S)))$, where $\mathrm{Pic}(\mathrm{SH}(S))$ denotes $\otimes$-invertible objects in $\mathrm{SH}(S)$, then $\mathrm{Ab}^\Gamma$ corresponds to homotopy groups graded by $\otimes$-invertible objects. The point of the above definition is that we often have a functor $\pi : \mathsf{C} \to \mathsf{D}$ such that the collection of functors $\{\pi(\underline{\mathrm{map}}(\gamma, -))\}_{\gamma\in\Gamma}$ is conservative for $\Gamma \subset \mathsf{C}$ a 1-subgroupoid. This reduces the checking of certain higher coherences to simpler 1-categorical coherences.

**Proposition 8.4.** *Let $\mathsf{C}$ be a presentably symmetric monoidal $\infty$-category and let $\alpha : x \to \mathbf{1}$ be a morphism. Suppose we are given the following data:*

(1) *$\Gamma \subset \mathsf{C}$ a 1-subgroupoid of $\mathsf{C}$ containing the morphism $\alpha$ and the permutation isomorphisms $\sigma_n : x^{\otimes n} \to x^{\otimes n}$ for all $n \geq 1$.*
(2) *A filtered colimit preserving functor:*

$$\pi : \mathsf{C} \to \mathsf{D},$$

*so that the functors $\{\pi(\underline{\mathrm{map}}(\gamma, -)) =: \pi_\gamma(-)\}_{\gamma\in\Gamma}$ form a conservative family.*

(3) *For any $m \in C$, consider the diagram $\pi(\underline{\mathrm{map}}(-,m)) \in D^\Gamma$. Then there exists $n \gg 0$ such that for any $m \geq n$ there is a commutative diagram in $D$:*

(91)
$$
\begin{array}{ccc}
\pi(\underline{\mathrm{map}}(-,m))_{\Sigma_m x} & \xrightarrow{\mathrm{id}} & \pi(\underline{\mathrm{map}}(-,m))_{\Sigma_m x} \\
{\scriptstyle \sigma_{n,*}} \downarrow & \nearrow & \\
\pi(\underline{\mathrm{map}}(-,m))_{\Sigma_m x}. & \mathrm{id} &
\end{array}
$$

*Then for $x$ any $n$-symmetric object and $n \geq 2$, we have $Q_\alpha \simeq P_\alpha$.*

*Proof.* For $m \in C$ we claim that $Q_\alpha m$ is $\alpha$-periodic, i.e., $\alpha^* : Q_\alpha m \to \underline{\mathrm{map}}(x, Q_\alpha m)$ is an equivalence in C. In effect, consider the (solid) commutative diagram:

$$
\begin{array}{ccccccccc}
m & \longrightarrow & \underline{\mathrm{map}}(x,m) & \longrightarrow & \underline{\mathrm{map}}(x^{\otimes 2},m) & \longrightarrow & \cdots & \longrightarrow & Q_\alpha m \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & & & \downarrow {\scriptstyle \alpha^*} \\
\underline{\mathrm{map}}(x,m) & \longrightarrow & \underline{\mathrm{map}}(x,\underline{\mathrm{map}}(x,m)) \simeq \underline{\mathrm{map}}(x^{\otimes 2},m) & \longrightarrow & \underline{\mathrm{map}}(x,\underline{\mathrm{map}}(x^{\otimes 2},m)) & \longrightarrow & \cdots & \longrightarrow & \underline{\mathrm{map}}(x, Q_\alpha m).
\end{array}
$$

We construct a quasi-inverse to $\alpha^*$ by constructing the indicated dotted arrows rendering the diagrams commutative. Letting the dotted arrows be identities, we see that while the top triangles commute, the bottom triangles *do not necessarily commute*. The problem is that the horizontal arrow is given by $\alpha^*$, while the vertical arrow is given by $\underline{\mathrm{map}}(\mathrm{id}, \alpha^*)$. Hence, after adjunction, the two maps differ by a cyclic permutation. However, for $\gamma \in \Gamma$, applying $\pi_\gamma$ to the above solid diagram yields a diagram in the 1-category D:

$$
\begin{array}{ccccccccc}
\pi_\gamma m & \longrightarrow & \pi_{\gamma+x} m & \longrightarrow & \pi_{\gamma+x+x} m & \longrightarrow & \cdots & \longrightarrow & \mathrm{colim}_n \, \pi_{\gamma+nx} m \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & \downarrow {\scriptstyle \pi_\gamma \alpha^*} \\
\pi_{\gamma+x} m & \longrightarrow & \pi_{\gamma+x+x} m & \longrightarrow & \pi_{\gamma+x+x+x} m & \longrightarrow & \cdots & \longrightarrow & \mathrm{colim}_n \, \pi_{\gamma+nx} \underline{\mathrm{map}}(x, Q_\alpha m).
\end{array}
$$

Here we have identified the last terms with $\pi_\gamma(Q_\alpha m)$ using the fact that $\pi$ preserves filtered colimits. By the conservativity of $\pi_\gamma$, we need only prove that $\pi_\gamma(\alpha^*)$ is an isomorphism. To see this, we examine the diagram:

$$
\begin{array}{ccc}
 & & \pi_{\gamma+kx} m \\
 & \nearrow & \downarrow \\
\pi_{\gamma+kx} m & \longrightarrow & \pi_{\gamma+(k+1)x} m.
\end{array}
$$
with id on the upper arrow.

If this diagram commutes for $k \gg 0$, then we can produce the desired inverse. The commutativity of the above diagram is thus given by the third hypothesis.

It remains to apply [29, Lemma 3.3] which states that if $Q_\alpha m$ is $\alpha$-periodic then it must coincide with $P_\alpha m$. $\qquad\square$

Note that the third hypothesis is a weaker form of the condition that the map $\alpha : x \to 1$ is $n$-symmetric since we need only check a commutative diagram of 1-categories. According to the coherence results of Dugger in [18, Proposition 4.20-21] for invertible objects, the conclusion of Proposition 8.4 holds for $C = SH(S)$ and $\Gamma = \mathrm{Ho}(\mathrm{Pic}(SH(S))$.

8.1.6. We shall apply Proposition 8.4 to $C = SH(S)$ (resp. $\mathrm{Mod}_E$ for $E \in \mathrm{CAlg}(SH(S)$ or a presheaf of $\mathcal{E}_\infty$-ring spectra), and to $\Gamma$ the 1-subgroupoid of $SH(S)$ (resp. $\mathrm{Mod}_E$) spanned by $\Sigma^{2n,n} \Sigma_T^\infty X_+$, $n \in \mathbf{Z}$ (resp. $E \wedge \Sigma^{2n,n} \Sigma_T^\infty X_+$, $n \in \mathbf{Z}$) and all invertible 1-morphisms. We let $\pi := [S^0, -]$ so that $\{\pi(\underline{\mathrm{map}}(\Sigma^{2n,n} \Sigma_T^\infty X_+, -))\}_\Gamma$ (resp. $\{\pi(\underline{\mathrm{map}}(E \wedge \Sigma^{2n,n} \Sigma_T^\infty X_+, -))\}_\Gamma$) forms a conservative family.

8.1.7. *Examples.* We review some examples relevant for the main body of the paper.

**Example 8.5.** (*K*-theory) We work with $C = \mathrm{Mod}_K$. As explained in [29, Example 3.4] there is a map:

$$\gamma : \Sigma^\infty((\mathbf{P}^1 \setminus 0) \coprod_{\mathbf{G}_m} \mathbf{A}^1) \to \Sigma^\infty \Sigma(\mathbf{G}_m, 1) \to K,$$

in presheaves of spectra on $\mathrm{Sm}_S$, which defines a map $\gamma : K \wedge \Sigma^\infty(\mathbf{P}^1 \setminus 0 \coprod_{\mathbf{G}_m})\mathbf{A}^1 \to K$ in $\mathrm{Mod}_K$. The nonconnective Bass-Thomason-Trobaugh *K*-theory is the presheaf of $\mathcal{E}_\infty$-ring spectra defined by the periodization $K^B := Q_\gamma K$ in $\mathrm{Mod}_K$. Taking $L_m$ of $K$ (actually $L_{\mathbf{A}^1}$ suffices) $\gamma$-periodization coincides with $\beta$-periodization, where $\beta$ is the second map defining $\gamma$. Using Proposition 8.4, we obtain the motivic $\mathcal{E}_\infty$-ring spectrum representing algebraic *K*-theory KGL such that $\Omega^\infty\mathrm{KGL} \simeq L_m K^B$. This is the main content of [12]. See also [24] and [77] for another perspective of how Bott inversion on the infinite projective space gives KGL. In the language of this paper, this is also discussed in [29, Section 5], in the equivariant setting.

**Example 8.6.** (Étale Cohomology) We work with $C = \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(S; \mathbf{Z}/\ell)$. If $\ell$ is prime to all the residue characteristics of $S$, the presheaf $\mu_\ell$ is a homotopy invariant étale sheaf with transfers; thus $\mu_\ell \in C$. The choice of an $\ell$-th root of unity is a map $\mathbf{Z}/\ell^{\text{ét}} \to \mu_\ell$; since $\mu_\ell$ is invertible in C, this amounts to a map $\tau : \mu_\ell^{\otimes -1} \to \mathbf{Z}/\ell^{\text{ét}}$. Then $\mathrm{H}\mu_\ell := Q_\tau \mathbf{Z}/\ell^{\text{ét}}$ represents étale cohomology with $\mu_\ell$-coefficients. By Proposition 8.4, $\mathrm{H}\mu_\ell$ is an object of $\mathrm{DM}_{\text{ét}}(S; \mathbf{Z}/\ell)$. Recall that $\mathrm{DM}_{\text{ét}}(S; \mathbf{Z}/\ell) \simeq C$ and $\mathbf{Z}/\ell^{\text{ét}}(1) \simeq \mu_\ell$. Since $\tau : \mathbf{Z}/\ell^{\text{ét}} \to \mathrm{Hom}(\mu_\ell, \mathbf{Z}/\ell^{\text{ét}})$ is an equivalence, the unit object in C is $\tau$-periodic and $\mathrm{H}\mu_\ell \simeq \mathbf{Z}/\ell^{\text{ét}}$ in $\mathrm{DM}_{\text{ét}}(S; \mathbf{Z}/\ell)$. The spectrum $\mathrm{H}_{\text{ét}}\mu_\ell := u_{\mathrm{tr}}\mathrm{H}\mu_\ell$ in $\mathrm{SH}_{\text{ét}}(S)$ represents étale cohomology with $\mu_\ell$-coefficients in the sense that $[\Sigma^{p,q}\Sigma^\infty_T X_+, \mathrm{H}_{\text{ét}}\mu_\ell] \cong H^{-p}_{\text{ét}}(X, \mu_\ell^{\otimes -q})$.

**Example 8.7.** (Inverting elements in $\mathrm{SH}(S)$) Suppose that $E \in \mathrm{CAlg}(\mathrm{SH}(S))$ so that $\mathrm{Mod}_E$ is a symmetric monoidal $\infty$-category (inverting elements for a noncommutative algebra is trickier). Given $F \to E$ in $\mathrm{SH}(S)$, we get $\alpha : E \wedge F \to E$ in $\mathrm{Mod}_E$ and set $E[\alpha^{-1}] := Q_\alpha E$. More generally, given $\alpha : F \to E$, we may $\alpha$-periodize any E-module M by setting $M[\alpha^{-1}] := P_\alpha M$.

8.2. **Localization.** A theory for Bousfield localization of motivic spectra, in the language of stable model categories was carried out in [64]. We quickly adopt their results to $\infty$-categories; the results stated here are surely well-known to experts. Throughout $(C, \otimes, \mathbf{1})$ is a presentably symmetric monoidal stable $\infty$-category.

**Definition 8.8.** For $M, E, F \in C$ we define:
   (1) $f : E \to F$ in C to be an M-*equivalence* if $f \otimes id_M$ is an equivalence,
   (2) E to be M-*acyclic* if $E \otimes M \simeq 0$,
   (3) F to be M-*local* if for any M-acyclic object E, the mapping space $\mathrm{Maps}(E, F)$ is contractible.

In the case that $C = \mathrm{SH}$, the stable $\infty$-category of spectra, this is just a homotopy-theoretic refinement of the definition of [10]. In the case of $C = \mathrm{SH}(S)$, which is our primary case of interest, this is just an $\infty$-categorical reformulation of the definitions of [64, Appendix A].

From now on we assume in addition that C is presentably symmetric monoidal. In many cases M will at least be an $\mathcal{E}_1$-algebra object in C.

**Proposition 8.9.** *Any* M-*module is* M-*local.*

*Proof.* Suppose that F is an M-acyclic object. Then, for any M-module E, we have:

$$\begin{aligned}
\mathrm{Maps}_C(F, E) &\simeq \mathrm{Maps}_{\mathrm{Mod}_M}(F \wedge M, E) \\
&\simeq \mathrm{Maps}_{\mathrm{Mod}_M}(0, E) \\
&\simeq 0.
\end{aligned}$$

$\square$

8.2.1. Let $C_M \subset C$ denote the full subcategory of C spanned by the M-local objects. We would like to construct a *Bousfield localization functor*:

$$L_M : C \to C_M,$$

as a left adjoint to the inclusion $C_M \subset C$. In this situation, we also call $L_M$ an M-*completion* functor. Such a functor is produced in [45, Proposition 5.5.4.15]. Indeed, the collection of all morphisms $f$ such that $f \otimes \mathrm{id}_M$ is an equivalence forms a strongly saturated class by Proposition [45, Proposition 5.5.4.16], and thus the inclusion $C_M$ has a left adjoint. The following lemma is a slight refinements of this existence result.

**Lemma 8.10.** *Let* C *be a presentable stable ∞-category. Suppose* $\ell : C_0^\perp \subset C$ *is a full subcategory, and let* $r : C_0 \subset C$ *be the full subcategory spanned by objects* $X$ *in* C *for which* $\mathrm{Maps}(Y, X) \simeq 0$ *for all* $Y \in C_0^\perp$. *Then the following are equivalent:*

(1) *the inclusion* $r : C_0 \subset C$ *admits a left adjoint* $L : C \to C_0$ *such that* $r \circ L : C \to C$ *is exact and* $\kappa$-*accessible;*

(2) *the full subcategory* $r : C_0 \subset C$ *is presentable, stable, and closed under* $\kappa$-*filtered colimits under sufficiently large* $\kappa$.

*Furthermore, the following equivalent statements:*

(3) *the inclusion* $\ell : C_0^\perp \subset C$ *admits a right adjoint* $R : C \to C_0^\perp$ *such that* $\ell \circ R : C \to C$ *is exact and accessible;*

(4) *the full subcategory* $\ell : C_0^\perp \subset C$ *is presentable and stable,*

*imply the above two statements.*

*Proof.* Assuming (1) we get $C_0 \simeq LC$. By [46, Lemma 1.1.3.3], we need only show that $C_0$ is stable under cofibers and translations. Stability under cofibers follows since $L$ is a left adjoint and hence preserves cofibers in C. To see stability under translations, it suffices to prove that $X[-1] \in C_0$ for any $X \in C_0$. We have a cofiber sequence in C, $X[-1] \to 0 \to X$. Then $rL(X[-1]) \to rL(0) \simeq 0 \to rL(X) \simeq X$ is still a cofiber sequence in C by exactness of $r \circ L$, and is thus a fiber sequence by the stability of C. Now since the inclusion $r$ is fully faithful and a right adjoint, $r$ creates limits and thus $L(X[-1]) \to 0 \to X$ is a fiber sequence in $C_0$, and therefore the shift $X[-1]$ is indeed in $C_0$. The presentability of $C_0$ follows from [45, Proposition 5.5.4.15], by letting $S$ be all morphisms of the form $rL(f)$. In particular $C_0$ is $\kappa$-accessible for some large enough cardinal $\kappa$, and thus admits $\kappa$-filtered colimits.

Assume that (2) holds. Since C and $C_0$ are presentable, $r$ preserves small limits and $\kappa$-filtered colimits for some cardinal $\kappa$. We deduce from the adjoint functor theorem [45, Corollary 5.5.2.9] that $r$ admits a left adjoint $L : C \to C_0$. By [45, Proposition 5.4.7.7], we see that both $L$ and $r$ are accessible. By [46, Proposition 1.1.4.1] both $r$ and $L$ are exact, and thus their composite is accessible and exact.

The equivalence between (3) and (4) follows analogously by replacing left with right adjoints and using the adjoint functor theorem [45, Corollary 5.5.2.9] with right adjoints.

Next, assuming (3) we show (1). The cofiber of the counit transformation $\ell R \to \mathrm{id}$ yields an endofunctor $L' : C \to C$. Since $\ell R$ and id are exact and accessible, $L'$ is exact and accessible. We claim that its essential image lies in $C_0$ and that the resulting functor $L : C \to C_0$ is the desired left adjoint. For any $X \in C$, we show that $L'(X) \in C_0$. If $Y \in C_0^\perp$ there is a cofiber sequence of spectra:

$$\mathrm{Maps}(Y, \ell R(X)) \to \mathrm{Maps}(Y, X) \to \mathrm{Maps}(Y, L'(X)).$$

By the assumption on $Y$, $\mathrm{Maps}(Y, \ell R(X)) \simeq \mathrm{Maps}(Y, X)$ and therefore the final term is contractible and indeed the functor lies in the essential image. To check that it is left adjoint, it suffices to show that for any $X' \in C_0$, the map $L'(X) \to X$ induces an equivalence $\mathrm{Maps}(X, X') \to \mathrm{Maps}(L'(X), X')$. To this end we consider the defining cofiber sequence $\ell R(X) \to X \to L'(X)$. It induces a cofiber sequence of spectra:

$$\mathrm{Maps}(L'(X), X') \to \mathrm{Maps}(X, X') \to \mathrm{Maps}(\ell R(X), X'),$$

where the final term is contractible since $X' \in C_0$. $\square$

*Remark* 8.11. The following standard warning applies: even though $L$ preserves colimits, the composite $r \circ L$ may not. An example is the process of étale sheafification: if $r \circ L$ preserves filtered colimits then representables would remain compact but, as discussed earlier, this is not the case without finiteness of cohomological dimension. In the case of localization at an $\mathcal{E}_1$-ring spectrum E, $r \circ L$ preserves colimits if and only if it is computed at a spectrum F by smashing F with $r \circ L(\mathbf{1})$.

8.2.2. *Example: $\ell$-completion.* Let $\mathsf{M} = \mathbf{1}/\ell$ for $1 < \ell \in \mathbb{N}$. In this case M is *not* an $\mathcal{E}_1$-object in $\mathrm{SH}(S)$. The Bousfield localization functor:

$$L_{\mathbf{1}/\ell} : \mathrm{SH}(S) \to \mathrm{SH}(S)_{\mathbf{1}/\ell},$$

admits an explicit formula. Define the *$\ell$-adic completion* functor as:

$$(-)_\ell^\wedge : \mathrm{SH}(S) \to \mathrm{SH}(S); \ \ \mathsf{E}_\ell^\wedge := \lim(\mathsf{E} \leftarrow \mathsf{E}/\ell \leftarrow \mathsf{E}/\ell^2 \leftarrow \cdots),$$

where the transition maps $\mathsf{E}/\ell^{k+1} \to \mathsf{E}/\ell^k$ are induced by the natural equivalence at the bottom row of the following diagram of cofiber sequences:

(92)

$$
\begin{array}{ccccccc}
E & \xrightarrow{\times \ell} & E & \longrightarrow & \mathsf{E}/\ell & \longrightarrow & \Sigma^{1,0}E \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \times \ell^k}\downarrow & & \downarrow & & \downarrow {\scriptstyle \Sigma^{1,0}\mathrm{id}} \\
E & \xrightarrow{\times \ell^{k+1}} & E & \longrightarrow & \mathsf{E}/\ell^{k+1} & \longrightarrow & \Sigma^{1,0}E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathsf{E}/\ell^k & \xrightarrow{\simeq} & C & \longrightarrow & 0.
\end{array}
$$

**Proposition 8.12.** *The Bousfield localization functor* $L_{\mathbf{1}/\ell} : \mathrm{SH}(S) \to \mathrm{SH}(S)_{\mathbf{1}/\ell}$ *coincides with the $\ell$-adic completion functor* $(-)_\ell^\wedge : \mathrm{SH}(S) \to \mathrm{SH}(S)$.

*Proof.* The proof in [64, Example 3.5] applies in the $\infty$-categorical setting. $\qquad\square$

Thus the $\infty$-category $\mathrm{SH}(S)_{\mathbf{1}/\ell}$ is the *$\ell$-adic completion* of $\mathrm{SH}(S)$; we write $\mathrm{SH}(S)_\ell^\wedge := \mathrm{SH}(S)_{\mathbf{1}/\ell}$.

8.2.3. *Example: localization.* Applying Lemma 8.10 to $\mathsf{E} \in \mathrm{SH}(S)$ we can construct an adjunction:

$$L_\mathsf{E} : \mathrm{SH}(S) \rightleftarrows \mathrm{SH}(S)_\mathsf{E} : r_\mathsf{E},$$

where $\mathrm{SH}(S)_\mathsf{E}$ is the full subcategory of E-local objects.

**Proposition 8.13.** *If* $\mathsf{M} \in \mathrm{SH}(S)$, *the inclusion* $i : \mathrm{SH}(S)_\mathsf{M} \subset \mathrm{SH}(S)$ *admits a left adjoint* $L_\mathsf{M} : \mathrm{SH}(S) \to \mathrm{SH}(S)_\mathsf{M}$.

*Proof.* Let $\mathrm{SH}(S)_\mathsf{M}^\perp$ denote the M-acyclic objects. To verify condition (4) of Lemma 8.10, the nontrivial thing to check is that $\mathrm{SH}(S)_\mathsf{M}^\perp$ is $\kappa$-accessible for some large cardinal $\kappa$. In other words, it is of the form $\mathrm{Ind}_\kappa(\mathsf{C}_0)$ for a small $\infty$-category $\mathsf{C}_0$. For $\kappa$ we choose the minimal cardinal larger than $\bigcup_{p,q,X \in \mathrm{Sm}_S} |\mathsf{M}_{p,q}(X)|$, where $|-|$ denotes the cardinality of a set. Let $\mathsf{C}_0$ be the full subcategory of $\mathrm{SH}(S)_\mathsf{M}$ spanned by objects $A$ such that $|\mathrm{Hom}(\Sigma^{p,q}X_+, A)| < \kappa$. The objects of $\mathsf{C}_0$ form a set and is thus a small $\infty$-category. The verification that this choice of $\kappa$ works is in [64, Appendix A]. $\qquad\square$

8.2.4. *Fracture squares.* We record Bousfield localization fracture squares in the context of motivic homotopy theory [60, Theorem A.1].

**Theorem 8.14.** *Let* $\mathsf{E}, \mathsf{F}, \mathsf{G} \in \mathrm{SH}(S)$, *and suppose that* $\mathsf{E} \wedge L_\mathsf{F}\mathsf{G} \simeq \mathsf{E} \wedge L_\mathsf{F}L_\mathsf{E}\mathsf{G} \simeq 0$. *Then there is a Cartesian square in* $\mathrm{SH}(S)$:

$$
\begin{array}{ccc}
L_{\mathsf{E}\vee\mathsf{F}}\mathsf{G} & \xrightarrow{\eta_\mathsf{E}} & L_\mathsf{E}\mathsf{G} \\
{\scriptstyle \eta_\mathsf{F}}\downarrow & & \downarrow \\
L_\mathsf{F}\mathsf{G} & \xrightarrow{L_\mathsf{F}\eta_\mathsf{E}} & L_\mathsf{F}L_\mathsf{E}\mathsf{G}.
\end{array}
$$

A special case we will use repeatedly is the so-called arithmetic fracture square.

**Theorem 8.15.** *For any* $E \in SH(S)$ *there is a Cartesian square in* $SH(S)$:

$$
\begin{CD}
E @>{\eta_E}>> \prod_\ell E_\ell^\wedge \\
@VVV @VVV \\
E_Q @>{L_F \eta_E}>> (\prod_\ell E_\ell^\wedge)_Q.
\end{CD}
$$

8.2.5.    Finally, we record a lemma about localization and the six functors formalism.

**Lemma 8.16.** *Suppose* M *is a Cartesian section of* $SH(-)$ *and* $f : S \to T$ *is a map of schemes. Then for* $E \in SH(T)$ *there is a canonical equivalence:*

$$
f^* L_M E \overset{\simeq}{\to} L_M f^* E.
$$

*Proof.* First we show that $f_*$ preserves M-local objects. If $M \in SH(S)$ is M-local and $F \in SH(T)$ is M-acyclic, then $f^* F$ is M-acyclic since $M \wedge f^* F \simeq f^*(M \wedge F) \simeq 0$ (here we use that M is Cartesian and $f^*$ is monoidal). As a result, $[F, f_* E] \simeq [f^* F, E] \simeq 0$ and thus the following diagram of right adjoints commute:

$$
\begin{CD}
SH(S)_M @>>> SH(S) \\
@V{f_*}VV @VV{f_*}V \\
SH(T)_M @>>> SH(T).
\end{CD}
$$

Thus the diagram with the corresponding left adjoints commutes too, i.e., $f^* L_M \simeq L_M f^*$.    $\square$

## 9. APPENDIX B: HYPERDESCENT

In this paper, we have meant for $SH_{\text{ét}}(S)$ to be the version of stable étale motivic homotopy theory built from hypercovers. There is a version, which we denote by $\widetilde{SH}_{\text{ét}}(S)$ built from Čech covers: this is obtained by inverting

$$
\{\Sigma^{p,q} \Sigma_{T,+}^\infty L_{m,\text{ét}}(U \hookrightarrow X) : X \in Sm_S, U \hookrightarrow X \text{ is an ét-sieve of } X, p, q \in \mathbf{Z}\}.
$$

We also have an adjunction:

$$
\widetilde{\pi}^* : SH(S) \rightleftarrows \widetilde{SH}_{\text{ét}}(S) : \widetilde{\pi}_*,
$$

and we denote $\widetilde{\pi}^* \widetilde{\pi}_* E := \widetilde{E}^{\text{ét}}$. We have a comparison map $\widetilde{E}^{\text{ét}} \to E^{\text{ét}}$.

**Lemma 9.1.** *Let* $\ell$ *be a prime and* $S$ *be a Noetherian* $\mathbf{Z}[\frac{1}{\ell}]$-*scheme of finite dimension and assume that for all* $x \in S$, $cd_\ell(k(x)) < \infty$. *Then for all* $v \geq 1$, *the map* $\widetilde{MGL_S}^{\text{ét}}/\ell^v \to MGL_S^{\text{ét}}/\ell^v$ *is an equivalence.*

*Proof.* The lemma asserts that $\widetilde{MGL_S}^{\text{ét}}/\ell^v$ is, in fact, hypercomplete. By the criterion of [16, Theorem 4.37], it then suffices to prove the claim for fields. Unpacking the proof for the field case §6.5.1, our arguments assert that Bott inverted $MGL_S/\ell^v$ is filtered by the étale version of the slice tower and the associated graded are spectra representing étale cohomology each of which has hyperdescent. Since hypercomplete sheaves are stable under limits (it is a Bousfield localization of the category of sheaves), an induction along the Postnikov tower tells us that Bott inverted $MGL_S/\ell^v$ is, in fact, hypercomplete and is equivalent to both $\widetilde{MGL_S}^{\text{ét}}/\ell^v$ and $MGL_S^{\text{ét}}/\ell^v$.    $\square$

Using Lemma 9.1 we can replace our results expressing various Bott-inverted motivic spectra as the localization at étale hypercovers by the analogous statement for étale covers instead.

## 10. APPENDIX C: E-SLICE COMPLETENESS AND E-LOCALITY

Here we record the following result related to [32, Lemma 4.1] which we will not need directly but inspired the proof of Theorem 6.32, and is generally useful.

**Proposition 10.1.** *Let* $\mathsf{E}$ *be a motivic* $\mathcal{E}_\infty$*-ring spectrum. Suppose* $\mathsf{M} \in \mathrm{Mod}_\mathsf{E}$ *is* $\mathsf{E}$*-slice complete and eventually effective, i.e., there exists* $q \in \mathbb{N}$ *such that* $\Sigma_T^d \mathsf{E} \in \mathrm{Mod}_\mathsf{E}^{\mathrm{eff}}(S)$*. Then* $\mathsf{M}$ *is* $s_0^\mathsf{E} \mathsf{E}$*-local.*

*Proof.* By the discussions in §3.0.5 $\mathsf{M}$ is $\mathsf{E}$-slice complete if it is equivalent to $\lim f_\mathsf{E}^q \mathsf{M}$. Since the $\infty$-category of $s_0^\mathsf{E} \mathsf{E}$-local objects is colocalizing, thus closed under limits, it suffices to prove $f_\mathsf{E}^q \mathsf{M}$ is $s_0^\mathsf{E} \mathsf{E}$-local for all $q \in \mathbf{Z}$. Without loss of generality we may assume that $\mathsf{M}$ is $\mathsf{E}$-effective and so we may assume $q \geq -1$. In this case, $f_0^\mathsf{E} \mathsf{M} \simeq \mathsf{M}$ and thus the cofiber sequence $f_1^\mathsf{E} \mathsf{M} \to f_0^\mathsf{E} \mathsf{M} \simeq \mathsf{E} \to f_\mathsf{E}^{-1} \mathsf{M}$ implies that $f_\mathsf{E}^{-1} \mathsf{M} \simeq s_0^\mathsf{E} \mathsf{M}$. Since $s_0^\mathsf{E} \mathsf{M}$ is a module over the $\mathcal{E}_\infty$-ring spectrum $s_0^\mathsf{E} \mathsf{E}$ [25, Section 6 (v)], we have that $f_\mathsf{E}^{-1} \mathsf{M}$ is $s_0^\mathsf{E} \mathsf{E}$-local. Assume now that $f_\mathsf{E}^{q'} \mathsf{M}$ is $s_0^\mathsf{E} \mathsf{E}$-local for all $q' < q$. We claim $f_\mathsf{E}^q \mathsf{M}$ is $s_0^\mathsf{E} \mathsf{E}$-local. To see this, we use the cofiber sequence $s_q^\mathsf{E} \mathsf{M} \to f_\mathsf{E}^q \mathsf{M} \to f_\mathsf{E}^{q-1} \mathsf{M}$. Since $s_q^\mathsf{E} \mathsf{M}$ is an $s_0^\mathsf{E} \mathsf{M}$-module, the claim follows from the inductive hypothesis. □

*Remark* 10.2. Suppose $S = \mathrm{Spec}\, k$ is a perfect field of finite cohomological dimension and exponential characteristic $c$. Then according to [44, Theorem 4] objects in $\mathrm{SH}(k)^{\mathrm{proj}}[\frac{1}{c}]$ are slice complete and thus $s_0(\mathbf{1}[\frac{1}{c}]) \simeq \mathsf{M}\mathbf{Z}[\frac{1}{c}]$-local.

## 11. APPENDIX D: TRANSFERS FOR MGL

In this appendix we prove the following statement about transfers for MGL-modules:

**Proposition 11.1.** *Suppose that* $A \subset B$ *is a finite Galois extension of commutative rings of degree* $d$ *with Galois group* $G := \mathrm{Aut}_A(B)$*, i.e., the map* $A \to B$ *is finite étale and* $A \cong B^G$*. Denote by* $p : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ *the map on prime spectra. Then there exists a transfer map:*

$$p_* : \mathsf{E}^{*,*}(\mathrm{Spec}\, B) \to \mathsf{E}^{*,*}(\mathrm{Spec}\, A)$$

*such that*

(1) *The map* $p_*$ *is* $G$*-equivariant.*
(2) *The composite* $p_* p^*$ *is multiplication by* $|G| = d$*.*
(3) *The composite* $p^* p_* = \sum_{g \in G} g_*$*.*

*Proof.* According to [17, Definition 3.3.2] we have the *Gysin morphism* $f_* : \mathsf{E}^{*,*}(X) \to \mathsf{E}^{*,*}(Y)$ for any proper morphism $f : X \to Y$. In the situation of finite étale extensions, this map preserves degrees and this is our transfer map of interest. This map is constructed via cupping with the fundamental class, see [17, Section 2] for a construction which is functorial along gci morphisms, whence the map $f_*$ is $G$-equivariant; in fact in the situation of étale extensions the construction of this fundamental class is easy and clearly independent of any factorization of morphisms [17, Example 2.1.2]. This proves property (1).

Property (2) is a consequence of Proposition [17, Proposition 3.3.9]; indeed since $A \to B$ is finite étale we can let $L_0/S$ in *loc. cit.* to be trivial and hence $p^* p_* = |G| = d$. To prove property (3), we have the following Cartesian diagram

(93)
$$
\begin{array}{ccc}
\coprod_{g \in G} \mathrm{Spec}\, B & \xrightarrow{\pi_1} & \mathrm{Spec}\, B \\
\pi_2 \downarrow & & \downarrow p \\
\mathrm{Spec}\, B & \xrightarrow{p} & \mathrm{Spec}\, A
\end{array}
$$

of schemes. By base change we have an equality $p^* p_* = \pi_{2*} \pi_1^*$, whence we compute the composite

$$\mathsf{E}^{*,*}(\mathrm{Spec}\, B) \xrightarrow{\pi_1^*} \mathsf{E}^{*,*}(\coprod_{g \in G} \mathrm{Spec}\, B) \cong \bigoplus_{g \in G} \mathsf{E}^{*,*}(\mathrm{Spec}\, B) \xrightarrow{\pi_{2*}} \mathsf{E}^{*,*}(\mathrm{Spec}\, B).$$

Now, we note that $\pi_{2*}$ identifies with the fold map, while $\pi_1^*(x) = (x, (g_1)_* x, \cdots, (g_{i*})x, \cdots)$ for $g_i \in G$, whence we have the desired formula. $\qquad\square$

## 12. APPENDIX E: TRANSFINITE CONSTRUCTIONS

Let $\tau$ be a topology on $\mathrm{Sm}_S$ where $S$ is a qcqs scheme. We review two transfinite constructions used in this paper.

12.0.1. *Motivic localization.* Recall that if $\tau = \mathrm{Nis}$, then the motivic localization functor $L_{\mathrm{Nis},\mathrm{m}} : P(\mathrm{Sm}_S) \to H(S)$ is obtained by a filtered colimit:

$$(94) \qquad L_{\mathrm{Nis},\mathrm{m}} \simeq \operatorname*{colim}_{n \to \infty} (L_{\mathrm{Nis}} \circ \mathrm{Sing}^{\mathbf{A}^1})^{\circ n} \simeq (L_{\mathrm{Nis}} \mathrm{Sing}^{\mathbf{A}^1})^{\circ \mathbb{N}}.$$

Here $\mathrm{Sing}^{\mathbf{A}^1}$ is the construction introduced in [56] for an arbitrary site with an interval; this formula in our situation is reviewed in [2, Definition 4.23]. The following is simply an $\infty$-categorical formulation of [56, Lemma 3.21].

**Proposition 12.1.** *Let $\tau$ be a topology on $\mathrm{Sm}_S$ and $\kappa$ be regular cardinal such that the endofunctor:*

$$P(\mathrm{Sm}_S) \xrightarrow{L_\tau} P_\tau(\mathrm{Sm}_S) \hookrightarrow P(\mathrm{Sm}_S),$$

*preserves $\kappa$-filtered colimits. Then we have an equivalence of endofunctors on $P(\mathrm{Sm}_S)$:*

$$(95) \qquad L_{\tau,\mathrm{m}} \simeq (L_\tau \mathrm{Sing}^{\mathbf{A}^1})^{\circ \kappa}.$$

*Proof.* We sketch a modification of the proof in [2, Theorem 4.27]. Let $\Phi := L_\tau \mathrm{Sing}^{\mathbf{A}^1}$. We need to check that (1) $\Phi^{\circ \kappa} X$ is $\tau$-local for any $X \in P(\mathrm{Sm}_S)$, (2) $\Phi^{\circ \kappa} X$ is $\mathbf{A}^1$-invariant, and (3) $\Phi^{\circ \kappa} X$ preserves $L_\tau$ and $L_{\mathbf{A}^1}$-equivalences. The proofs for (2) and (3) are the same as for the Nisnevich topology in [2, Theorem 4.27] with $\mathbb{N}$ replaced by the cardinal $\kappa$. The only difference is (1): by definition the presheaf $L_\tau \mathrm{Sing}^{\mathbf{A}^1} X$ is $\tau$-local. The presheaf $\Phi^{\circ \kappa} X$ is a $\kappa$-filtered colimit of $\tau$-local presheaves taken in $P(\mathrm{Sm}_S)$. Hence we are done if the inclusion $P_\tau(\mathrm{Sm}_S) \hookrightarrow P(\mathrm{Sm}_S)$ preserves $\kappa$-filtered colimits. This is true by hypothesis. $\qquad\square$

**Corollary 12.2.** *For $X \in \mathrm{Sm}_S$ the objects $X \in H_\tau(S), \Sigma_{S^1}^\infty X_+ \in \mathrm{SH}_\tau^{S^1}(S)$, and $\Sigma_T^\infty X_+ \in \mathrm{SH}_\tau(S)$ are $\kappa$-compact.*

*Proof.* The case of $X$ follows from (95) since $L_{\tau,\mathrm{m}}$ is a transfinite composite of endofunctors in $P(\mathrm{Sm}_S)$ that preserve $\kappa$-compact objects ($\mathrm{Sing}^{\mathbf{A}^1}$ preserves compact objects since the category of $\mathbf{A}^1$-invariant presheaves are closed under filtered colimits). To check that $\Sigma_{S^1}^\infty X_+$ (resp. $\Sigma_T^\infty X_+$) is $\kappa$-compact, it suffices to check that $\Omega_{S^1}^\infty$ (resp. $\Omega_T^\infty$) is $\kappa$-compact. This follows since $\mathrm{Maps}_{H_\tau(S)_\bullet}(S^1, -)$ (resp. $\mathrm{Maps}_{H_\tau(S)_\bullet}(T, -)$) is $\kappa$-compact because $S^1$ (resp. $T$) is the pushout of $\kappa$-compact objects, and hence $\kappa$-compact. $\qquad\square$

12.0.2. *Spectrification.* In the notation of §4.1.1, there is an adjunction:

$$Q : \mathrm{SH}_\tau^{\mathrm{pre}}(S) \rightleftarrows \mathrm{SH}_\tau(S) : u.$$

Intuitively, the functor $Q$ should be computed as:

$$(96) \qquad Q(\mathsf{E})_i \simeq \operatorname*{colim}_n E_i \xrightarrow{\epsilon} \Omega_{\mathbf{G}_m} \mathsf{E}_{i+1} \xrightarrow{\Omega \epsilon} \Omega_{\mathbf{G}_m}^2 \mathsf{E}_{i+2} \cdots = \operatorname*{colim}_n \Omega_{\mathbf{G}_m}^n \mathsf{E}_{i+n}.$$

This is only true whenever the endofunctor $\Omega_{\mathbf{G}_m} : \mathrm{SH}_\tau^{S^1}(S) \to \mathrm{SH}_\tau^{S^1}(S)$ preserves filtered colimits. In general, we have to perform a transfinite construction. We define an endofunctor on $\mathrm{SH}_\tau^{\mathrm{pre}}(S)$ as:

$$(97) \qquad s : \mathrm{SH}_\tau^{\mathrm{pre}}(S) \to \mathrm{SH}_\tau^{\mathrm{pre}}(S), \quad s(\mathsf{E})_i = \mathrm{Maps}_{\mathrm{SH}_\tau^{S^1}(S)}(\mathbf{G}_m, \mathsf{E}_{i+1}).$$

**Proposition 12.3.** *Let $\tau$ be a topology on $\mathrm{Sm}_S$ and $\kappa$ be a regular cardinal such that the functor $L_\tau : \mathrm{P}(\mathrm{Sm}_S) \to \mathrm{P}_\tau(\mathrm{Sm}_S)$ preserves $\kappa$-filtered colimits. Then $Q : \mathrm{SH}^{\mathrm{pre}}_\tau(S) \rightleftarrows \mathrm{SH}_\tau(S)$ is computed on $\mathsf{E} \in \mathrm{SH}^{\mathrm{pre}}_\tau(S)$ as:*

$$(98) \qquad\qquad Q(\mathsf{E})_i \simeq s^{\circ\kappa}(\mathsf{E})_i.$$

*Proof.* The formula in (98) is given in [29, Page 7] using the fact that $\mathbf{G}_m$ is $\kappa$-compact in $\mathrm{SH}^{S^1}_\tau(S)$; it is the cofiber of the map of $\kappa$-compact objects, $S^0 \to \Sigma^\infty_{S^1} \mathbf{A}^1 \setminus 0_+$. $\qquad\square$

*Remark* 12.4. If a site has $< \tau$ arrows, then the $L_\tau$ preserves $\tau$-filtered colimits as explained in [11, Proposition 3.4.16]. So, for example, when $\tau = $ ét, setting $\kappa = \aleph_1$ suffices.

### 13.   APPENDIX F: COMPLEMENTS ON THE RATIONAL ÉTALE MOTIVIC SPHERE SPECTRUM

In this section, we piece together known facts about the rational motivic sphere spectrum and describe $\mathbf{1}^{\text{ét}}_{S,\mathbb{Q}}$, its rational counterpart.

13.0.1.   Recall that $\mathrm{D}_{\mathbf{A}^1}(S;\mathbb{Q})$ denote the $\mathbf{A}^1$-*derived category* (see [55, Section 5.2] for a reference). As an $\infty$-category $\mathrm{D}_{\mathbf{A}^1}(S;\mathbb{Q})$ is constructed in parallel as in $\mathrm{SH}(S)_\mathbb{Q}$. First we consider $\mathrm{D}^{S^1}_{\mathbf{A}^1}(S;\mathbb{Q})$, which is the full subcategory of presheaves of complexes of rational vector spaces on $\mathrm{Sm}_S$, $\mathrm{P}_{\mathrm{D}(\mathbb{Q})}(\mathrm{Sm}_S)$, spanned by objects which are $\mathbf{A}^1$-invariant and satisfy Nisnevich descent. The $\infty$-category $\mathrm{D}^{S^1}_{\mathbf{A}^1}(S;\mathbb{Q})$ is presentably symmetric monoidal and stable. Recall that there is an equivalence

$$\mathrm{Spt}_\mathbb{Q} \simeq \mathrm{D}(\mathbb{Q})$$

between the $\infty$-category of rational spectra and the $\infty$-category of category of (unbounded) complexes of $\mathbb{Q}$-vector spaces; this is a result of the basic fact that the rationalized sphere spectrum, $S^0_\mathbb{Q}$, is equivalent to the rationalized Eilenberg-MacLane spectrum $H\mathbb{Q}$ and $\mathrm{D}(\mathbb{Q})$ is the $\infty$-category of $H\mathbb{Q}$-modules; see [72] for details. Therefore we have an equivalence of $\infty$-categories:

$$(99) \qquad\qquad \mathrm{SH}^{S^1}(S)_\mathbb{Q} \simeq \mathrm{D}^{S^1}_{\mathbf{A}^1}(S)_\mathbb{Q}.$$

In the same way as discussed in §2.1, we can invert the object $\mathbb{Q}(\mathbf{P}^1, \infty)$ of $\mathrm{D}^{S^1}_{\mathbf{A}^1}(S)_\mathbb{Q}$ to obtain the $\infty$-category $\mathrm{D}_{\mathbf{A}^1}(S)_\mathbb{Q}$ or the *rational $\mathbf{A}^1$-derived category*. The equivalence (99) persists after $\otimes$-inverting $\Sigma^\infty_{S^1}(\mathbf{P}^1, \infty)$ on the left and $\mathbb{Q}(\mathbf{P}^1, \infty)$ on the right; see the discussion preceding [13, (5.3.35.2)]:

**Proposition 13.1.** *For any qcqs base scheme $S$ there is a canonical equivalence:*

$$(100) \qquad\qquad \mathrm{SH}(S)_\mathbb{Q} \simeq \mathrm{D}_{\mathbf{A}^1}(S, \mathbb{Q}).$$

*Hence there is an equivalence after étale localizations:*

$$(101) \qquad\qquad \mathrm{SH}_{\text{ét}}(S)_\mathbb{Q} \simeq \mathrm{D}_{\text{ét},\mathbf{A}^1}(S, \mathbb{Q}).$$

13.0.2.   Any story on rational motivic homotopy theory starts with the decomposition of $\mathrm{SH}(S)_\mathbb{Q}$. According to Morel [53, Section 6.1], there is an involution on the sphere spectrum, i.e. an endomorphism:

$$\mathrm{sw} : \mathbf{1}_S \to \mathbf{1}_S$$

such that $\mathrm{sw}^2 = \mathrm{id}$. This gives a decomposition of the motivic sphere spectrum with 2-inverted into $+$ and $-1$-eigenspectra:

$$\mathbf{1}_S[\tfrac{1}{2}] \simeq \mathbf{1}^+_S[\tfrac{1}{2}] \oplus \mathbf{1}^-_S[\tfrac{1}{2}],$$

where $\tau$ acts on $\mathbf{1}^+_S[\tfrac{1}{2}]$ (resp. $\mathbf{1}^-_S[\tfrac{1}{2}]$) by $+1$ (resp. $-1$). This decomposition of the motivic sphere spectrum then induces a decomposition:

$$\mathrm{SH}(S)[\tfrac{1}{2}] \simeq \mathrm{SH}(S)[\tfrac{1}{2}]^- \times \mathrm{SH}(S)[\tfrac{1}{2}]^+,$$

and this a decomposition:

$$\mathrm{SH}(S)_{\mathbf{Q}} \simeq \mathrm{SH}(S)_{\mathbf{Q}}^{-} \times \mathrm{SH}(S)_{\mathbf{Q}}^{+}.$$

Hence we also obtain a decomposition upon étale localization:

$$\mathrm{SH}_{\text{ét}}(S)_{\mathbf{Q}} \simeq \mathrm{SH}_{\text{ét}}(S)_{\mathbf{Q}}^{-} \times \mathrm{SH}_{\text{ét}}(S)_{\mathbf{Q}}^{+}.$$

13.0.3.   The main result of rational étale motives is:

**Theorem 13.2** (Cisinski-Déglise). *For any Noetherian, geometrically unibranch scheme $S$ of finite dimension there is an equivalence of $\infty$-categories:*

$$\mathrm{DM}(S)_{\mathbf{Q}} \simeq \mathrm{SH}_{\text{ét}}(S)_{\mathbf{Q}} \simeq \mathrm{SH}(S)_{\mathbf{Q}}^{+}.$$

*In particular, $\mathrm{SH}_{\text{ét}}(S)_{\mathbf{Q}}^{-}$ vanishes.*

*Proof.* This is a matter of parsing definitions. In [13, Theorem 16.2.18] it is proved that the rational étale $\mathbf{A}^1$-derived category is equivalent to "Beilinson motives" for any Noetherian scheme of finite dimension. Under étale localization on both categories, the equivalence of Proposition 13.1 tells us that $\mathrm{SH}_{\text{ét}}(\mathbf{Q})$ is naturally equivalent to the rational étale $\mathbf{A}^1$-derived category. On the other hand in [13, Theorem 16.1.4], it is proved that the category "Beilinson motives" is equivalent to $\mathrm{DM}(S)_{\mathbf{Q}}$ supposing further that $S$ is geometrically unibranch. The last statement follows from [13, Lemma 16.2.19] where it is proved that $\mathbf{1}_{\mathbf{Q}}^{-}$ vanishes in $\mathrm{SH}_{\text{ét}}(\mathbf{Q})$ and thus the negative part of $\mathrm{SH}_{\text{ét}}(S)_{\mathbf{Q}}$ vanishes.

$\square$

## References

[1] *Théorie des topos et cohomologie étale des schémas. Tome 3* (1973), vi+640. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat. ↑4.2.2, 4.15, 4.4

[2] B. Antieau and E. Elmanto, *A primer for unstable motivic homotopy theory*, Surveys on recent developments in algebraic geometry, 2017, pp. 305–370. ↑12.0.1, 12.0.1

[3] J. Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I*, Astérisque **314** (2007), x+466 pp. (2008). ↑1.11, 2.3, 2.3.2, 6.5, 7.2

[4] ———, *A guide to (étale) motivic sheaves*, Seoul ICM Proceedings 2014, available at `http://user.math.uzh.ch/ayoub/PDF-Files/ICM2014.pdf`. ↑1

[5] ———, *La réalisation étale et les opérations de Grothendieck*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 1, 1–145, DOI 10.24033/asens.2210. ↑1, 2.4.1

[6] T. Bachmann, *Motivic and real étale stable homotopy theory*, Compos. Math. **154** (2018), no. 5, 883–917, DOI 10.1112/S0010437X17007710. ↑2.3.2

[7] T. Bachmann, E. Elmanto, and Østvær Paul Arne, *On étale motivic spectra and Voevodsky's convergence conjecture* (2020), available at `arXiv:2003.04006`. ↑1.0.6

[8] T. Bachmann and M. Hoyois, *Norms in motivic homotopy theory*, Astérisque **425** (2021), 207, DOI 10.24033/ast. ↑2.2.2, 2.4.1

[9] J. M. Boardman, *Conditionally convergent spectral sequences*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), 1999, pp. 49–84, DOI 10.1090/conm/239/03597. ↑3.0.6, 6.21

[10] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281, DOI 10.1016/0040-9383(79)90018-1. ↑8.2

[11] F. Borceux, *Handbook of categorical algebra. 3* **52** (1994), xviii+522. Categories of sheaves. ↑12.4

[12] D.-C. Cisinski, *Descente par éclatements en K-théorie invariante par homotopie*, Ann. of Math. (2) **177** (2013), no. 2, 425–448, DOI 10.4007/annals.2013.177.2.2. ↑6.6.3, 8.5

[13] D.-C. Cisinski and F. Déglise, *Triangulated categories of mixed motives*, Springer Monographs in Mathematics, Springer, Cham, [2019] ©2019. ↑1, 1.11, 2.3, 2.3.2, 2, 6.6, 6.6.1, 7.1, 7.2, 13.0.1, 13.0.3

[14] ———, *Integral mixed motives in equal characteristic*, Doc. Math. **Extra vol.: Alexander S. Merkurjev's sixtieth birthday** (2015), 145–194. ↑2.3.1, 7.1.2

[15] ———, *Étale motives*, Compos. Math. **152** (2016), no. 3, 556–666, DOI 10.1112/S0010437X15007459. ↑1, 2.8, 2.3, 2.3, 2.4.1, 5.0.4

[16] D. Clausen and A. Mathew, *Hyperdescent and etale K-theory* (2019), available at `arXiv:1905.06611`. To appear in Invent. Math. ↑1, 2.1.2, 4, 9

[17] F. Déglise, *Bivariant theories in motivic stable homotopy*, Doc. Math. **23** (2018), 997–1076. ↑11

[18] D. Dugger, *Coherence for invertible objects and multigraded homotopy rings*, Algebr. Geom. Topol. **14** (2014), no. 2, 1055–1106, DOI 10.2140/agt.2014.14.1055. ↑8.1.5

[19] D. Dugger, S. Hollander, and D. C. Isaksen, *Hypercovers and simplicial presheaves*, Math. Proc. Cambridge Philos. Soc. **136** (2004), no. 1, 9–51, DOI 10.1017/S0305004103007175. MR2034012 ↑1

[20] D. Dugger and D. C. Isaksen, *Motivic cell structures*, Algebr. Geom. Topol. **5** (2005), 615–652, DOI 10.2140/agt.2005.5.615. ↑7.4

[21] B. I. Dundas, O. Röndigs, and P. A. Østvær, *Motivic functors*, Doc. Math. **8** (2003), 489–525. MR2029171 ↑2.4.2

[22] W. Dwyer, E. M. Friedlander, V. Snaith, and R. W. Thomason, *Algebraic K-theory eventually surjects onto topological K-theory*, Invent. Math. **66** (1982), no. 3, 481–491, DOI 10.1007/BF01389225. ↑1, 6.4, 6.6.3

[23] E. Elmanto and H. K. Kolderup, *On modules over motivic ring spectra*, Ann. K-Theory **5** (2020), no. 2, 327–355, DOI 10.2140/akt.2020.5.327. ↑7.1

[24] D. Gepner and V. Snaith, *On the motivic spectra representing algebraic cobordism and algebraic K-theory*, Doc. Math. **14** (2009), 359–396. ↑8.5

[25] J. J. Gutiérrez, O. Röndigs, M. Spitzweck, and P. A. Østvær, *Motivic slices and coloured operads*, J. Topol. **5** (2012), no. 3, 727–755, DOI 10.1112/jtopol/jts015. ↑6.3.1, 6.3.2, 6.3.4, 6.6.1, 10

[26] C. Haesemayer and J. Hornbostel, *Motives and etale motives with finite coefficients*, K-Theory **34** (2005), no. 3, 195–207, DOI 10.1007/s10977-005-1563-6. ↑1.0.2, 7.1, 7.1.2

[27] J. Heller, *Motivic strict ring spectra representing semi-topological cohomology theories*, Homology Homotopy Appl. **17** (2015), no. 2, 107–135, DOI 10.4310/HHA.2015.v17.n2.a7. ↑1.0.4

[28] J. Hornbostel and S. Yagunov, *Rigidity for Henselian local rings and $\mathbb{A}^1$-representable theories*, Math. Z. **255** (2007), no. 2, 437–449, DOI 10.1007/s00209-006-0049-4. ↑7.3.1

[29] M. Hoyois, *Cdh descent in equivariant homotopy K-theory*, Doc. Math. **25** (2020), 457–482, DOI 10.1007/s00031-019-09528-5. ↑1.0.7, 4.1.2, 8.1, 8.1.4, 8.1.5, 8.1.5, 8.5, 12.0.2

[30] _____, *From algebraic cobordism to motivic cohomology*, J. Reine Angew. Math. **702** (2015), 173–226, DOI 10.1515/crelle-2013-0038. ↑2.4.2, 4.2.1, 4.3, 4.3.1, 4.4, 6.2.2, 6.3.3, 6.5.4, 6.5.4, 6.5.4

[31] M. Hoyois, S. Kelly, and P. A. Østvær, *The motivic Steenrod algebra in positive characteristic*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 12, 3813–3849, DOI 10.4171/JEMS/754. ↑6.3.4, 7.1.2

[32] M. Hoyois, A. Krishna, and P. A. Østvær, $\mathbb{A}^1$-*contractibility of Koras-Russell threefolds*, Algebr. Geom. **3** (2016), no. 4, 407–423, DOI 10.14231/AG-2016-019. ↑10

[33] M. Hovey, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra **165** (2001), no. 1, 63–127. ↑4.1

[34] J. F. Jardine, *Simplicial presheaves*, J. Pure Appl. Algebra **47** (1987), no. 1, 35–87. ↑4.0.3

[35] _____, *Motivic symmetric spectra*, Doc. Math. **5** (2000), 445–552. ↑4.1

[36] _____, *Generalized etale cohomology theories*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 2010. Reprint of the 1997 edition [MR1437604]. ↑1.7, 4.0.4

[37] A. Khan, *Motivic homotopy theory in derived algebraic geometry* (2016), available at https://www.preschema.com/papers/thesis.pdf. ↑2.1.4, 2.3

[38] K. Kato and S. Saito, *Global class field theory of arithmetic schemes*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), 1986, pp. 255–331, DOI 10.1090/conm/055.1/862639. ↑1

[39] M. Levine, *K-Theory and motivic cohomology of schemes*, available at http://www.math.uiuc.edu/K-theory/336/{\ell}ot.pdf. ↑1.7

[40] _____, *Inverting the motivic Bott element*, K-Theory **19** (2000), no. 1, 1–28, DOI 10.1023/A:1007874218371. ↑1.0.2, 1.0.2, 6.2.2

[41] _____, *Techniques of localization in the theory of algebraic cycles*, J. Algebraic Geom. **10** (2001), no. 2, 299–363. ↑3

[42] _____, *Chow's moving lemma and the homotopy coniveau tower*, K-Theory **37** (2006), no. 1-2, 129–209, DOI 10.1007/s10977-006-0004-5. ↑3

[43] _____, *The homotopy coniveau tower*, J. Topol. **1** (2008), no. 1, 217–267, DOI 10.1112/jtopol/jtm004. ↑3, 6.3.4

[44] _____, *Convergence of Voevodsky's slice tower*, Doc. Math. **18** (2013), 907–941. ↑3.0.6, 10.2

[45] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR2522659 (2010j:18001) ↑2.1, 2.1.1, 2.1.2, 2.1.4, 2.1.9, 2.2.1, 2.2.2, 3.0.1, 4.1, 4.1.1, 4.1.2, 4.1.5, 5.0.2, 5.0.3, 8.2.1, 8.2.1

[46] _____, *Higher algebra*, available at http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf. ↑1.0.8, 2.1.6, 3.0.1, 4.0.2, 5.0.3, 5.0.3, 7.1.3, 8.1.3, 8.2.1

[47] _____, *Spectral Algebraic Geometry*, available at http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf. ↑4.0.3

[48] A. Mathew, N. Naumann, and J. Noel, *Nilpotence and descent in equivariant stable homotopy theory*, Adv. Math., posted on 2017, DOI 10.1016/j.aim.2016.09.027. ↑1.0.4

[49] C. Mazza, V. Voevodsky, and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006. ↑4.2.2, 6.3.3, 7.3.1

[50] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980. ↑4.3

[51] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1969/1970), 318–344, DOI 10.1007/BF01425486. ↑6.3.3

[52] S. A. Mitchell, *Hypercohomology spectra and Thomason's descent theorem*, Algebraic K-theory (Toronto, ON, 1996), 1997, pp. 221–277. ↑1, 1.7

[53] F. Morel, *An introduction to $\mathbb{A}^1$-homotopy theory*, Contemporary developments in algebraic *K*-theory, 2004, pp. 357–441. ↑4.3, 6.5.4, 13.0.2

[54] ———, *The stable $\mathbb{A}^1$-connectivity theorems*, *K*-Theory **35** (2005), no. 1-2, 1–68, DOI 10.1007/s10977-005-1562-7. ↑4

[55] ———, **A**$^1$-*algebraic topology over a field*, Lecture Notes in Mathematics, Vol. 2052, Springer Verlag, Berlin, 2012. ↑6.5.4, 13.0.1

[56] F. Morel and V. Voevodsky, **A**$^1$-*homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. **90** (1999), 45–143. ↑1, 12.0.1

[57] N. Naumann, M. Spitzweck, and P. A. Østvær, *Motivic Landweber exactness*, Doc. Math. **14** (2009), 551–593. ↑4.2.1, 6.6.1, 6.6.3, 7.3.2

[58] J. Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. ↑6.3.4

[59] S. Oka, *Multiplications on the Moore spectrum*, Mem. Fac. Sci. Kyushu Univ. Ser. A **38** (1984), no. 2, 257–276, DOI 10.2206/kyushumfs.38.257. ↑6.3.4, 6.3.5, 6.3.6

[60] K. M. Ormsby and P. A. Østvær, *Stable motivic $\pi_1$ of low-dimensional fields*, Adv. Math. **265** (2014), 97–131, DOI 10.1016/j.aim.2014.07.024. MR3255457 ↑8.2.4

[61] G. Quick, *Stable étale realization and étale cobordism*, Adv. Math. **214** (2007), no. 2, 730–760, DOI 10.1016/j.aim.2007.03.005. ↑1.7

[62] M. Robalo, *K-theory and the bridge from motives to noncommutative motives*, Adv. Math. **269** (2015), 399–550, DOI 10.1016/j.aim.2014.10.011. ↑2.1, 2.4, 2.1.7, 2.5, 2.6, 2.1.9

[63] O. Röndigs and P. A. Østvær, *Modules over motivic cohomology*, Adv. Math. **219** (2008), no. 2, 689–727, DOI 10.1016/j.aim.2008.05.013. ↑1, 7.1.2

[64] ———, *Rigidity in motivic homotopy theory*, Math. Ann. **341** (2008), no. 3, 651–675, DOI 10.1007/s00208-008-0208-5. ↑8.2, 8.2, 8.2.2, 8.2.3

[65] ———, *Slices of hermitian K-theory and Milnor's conjecture on quadratic forms*, Geom. Topol. **20** (2016), no. 2, 1157–1212, DOI 10.2140/gt.2016.20.1157. ↑3, 3.0.6, 3.0.8

[66] O. Röndigs, M. Spitzweck, and P. A. Østvær, *The first stable homotopy groups of motivic spheres*, Ann. of Math. (2) **189** (2019), no. 1, 1–74, DOI 10.4007/annals.2019.189.1.1. ↑3, 3.0.5, 3.0.6, 4.3, 6.3.6

[67] ———, *The motivic Hopf map solves the homotopy limit problem for K-theory*, Doc. Math. **23** (2018), 1405–1424. ↑3, 3.0.4, 3.0.5, 3.0.5

[68] A. Rosenschon and P. A. Østvær, *The homotopy limit problem for two-primary algebraic K-theory*, Topology **44** (2005), no. 6, 1159–1179, DOI 10.1016/j.top.2005.04.004. ↑1.7

[69] ———, *Descent for K-theories*, J. Pure Appl. Algebra **206** (2006), no. 1-2, 141–152, DOI 10.1016/j.jpaa.2005.06.019. ↑1.7

[70] J. P. Serre, *Cohomologie Galoisienne*, Springer Verlag; Berlin-Heidelberg-New York, 1962. Lecture Notes in Mathematics, No. 5. ↑6.2.2

[71] S. S. Shatz, *Profinite groups, arithmetic, and geometry*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1972. Annals of Mathematics Studies, No. 67. ↑6.5.4

[72] B. Shipley, *H$\mathbb{Z}$-algebra spectra are differential graded algebras*, Amer. J. Math. **129** (2007), no. 2, 351–379, DOI 10.1353/ajm.2007.0014. ↑13.0.1

[73] M. Spitzweck, *Relations between slices and quotients of the algebraic cobordism spectrum*, Homology Homotopy Appl. **12** (2010), no. 2, 335–351. ↑3.0.4

[74] ———, *Slices of motivic Landweber spectra*, J. K-Theory **9** (2012), no. 1, 103–117, DOI 10.1017/is010008019jkt128. ↑4.2, 4.2.1, 4.2.2, 4.2.2

[75] ———, *A commutative $\mathbb{P}^1$-spectrum representing motivic cohomology over Dedekind domains*, Mém. Soc. Math. Fr. (N.S.) **157** (2018), 110. ↑2.18, 2.4, 6.3, 6.3, 6.3.3, 6.5.5, 7.1.2, 7.2

[76] ———, *Algebraic cobordism in mixed characteristic*, Homology Homotopy Appl. **22** (2020), no. 2, 91–103, DOI 10.4310/hha.2020.v22.n2.a5. ↑6.3, 6.3.1

[77] M. Spitzweck and P. A. Østvær, *The Bott inverted infinite projective space is homotopy algebraic K-theory*, Bull. Lond. Math. Soc. **41** (2009), no. 2, 281–292, DOI 10.1112/blms/bdn124. ↑8.5

[78] R. W. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 3, 437–552. ↑(document), 1.0.1, 1.0.1, 1.2, 1.0.1, 1.7, 1.0.4, 4.0.4, 4.0.4, 4, 5.0.4, 6.3.5, 6.6.3, 7.3

[79] B. Totaro, *Adjoint functors on the derived category of motives*, J. Inst. Math. Jussieu **17** (2018), no. 3, 489–507, DOI 10.1017/S1474748016000104. ↑7.14

[80] V. Voevodsky, *$\mathbb{A}^1$-homotopy theory*, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998), 1998, pp. 579–604. ↑2.1.8, 6.6.3

[81] ———, *Open problems in the motivic stable homotopy theory. I*, Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), 2002, pp. 3–34. ↑3, 3.0.6, 3.0.8, 3.0.8, 6.3.4

[82] ———, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 1–57, DOI 10.1007/s10240-003-0009-z. ↑6.3.4

[83] ———, *Cancellation theorem*, Doc. Math. **Extra vol.: Andrei A. Suslin sixtieth birthday** (2010), 671–685. ↑2.15, 4.1

[84] ———, *Simplicial radditive functors*, J. K-Theory **5** (2010), no. 2, 201–244, DOI 10.1017/is010003026jkt097. ↑2.1.4

[85] _____ , *Motivic cohomology with* **Z**/2*-coefficients*, Publ. Math. IHES **98** (2003), 59–104. ↑1.5
[86] _____ , *On motivic cohomology with* **Z**/l*-coefficients*, Ann. of Math. (2) **174** (2011), no. 1, 401–438, DOI 10.4007/annals.2011.174.1.11. ↑1.5
[87] V. Voevodsky, O. Röndigs, and P. A. Østvær, *Voevodsky's Nordfjordeid lectures: motivic homotopy theory*, Motivic homotopy theory, 2007, pp. 147–221. ↑4.1
[88] V. Voevodsky, A. Suslin, and E. M. Friedlander, *Cycles, transfers, and motivic homology theories* **143** (2000), vi+254. ↑1
[89] C. A. Weibel, *Homotopy algebraic K-theory*, Algebraic *K*-theory and algebraic number theory (Honolulu, HI, 1987), 1989, pp. 461–488, DOI 10.1090/conm/083/991991. ↑6.6.3
[90] G. M. Wilson, *The eta-inverted sphere over the rationals*, Algebr. Geom. Topol. **18** (2018), no. 3, 1857–1881, DOI 10.2140/agt.2018.18.1857. ↑6.5.3

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