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Analysis and Numerical Treatment of Nonlinear Hyperbolic Conservation Laws on Graphs

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This work is dedicated to my grandfather Erwin Etschel.

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* at the University of Oslo. The research presented here was conducted at the University of Oslo, under the supervision of associate professor Ulrik Skre Fjordholm and professor Nils Henrik Risebro.

The thesis is a collection of three papers. The papers are preceded by an introductory chapter that relates them to each other and provides background information and motivation for the work and wrapped up by a concluding chapter that gives an outlook on potential future research. The first and third papers are joint work with Ulrik Skre Fjrodholm and Nils Henrik Risebro. I am the sole author of the remaining paper.

Acknowledgements

I want to express my sincere gratitude to:

My supervisors, Prof. Dr. Ulrik Skre Fjordholm and Prof. Dr. Nils Henrik Risebro for their support in writing my thesis and for sharing their immense knowledge with me.

Find people you admire and ask how they got where they are. Seek book recommendations. Isn't that what Socrates would do? **– Ryan Holiday**

My previous mentors, Prof. Dr. Peter Knabner, Prof. Dr. Vadym Aizinger and Prof. Dr. Andreas Rupp, for giving me the chance to work with them.

Mentors have their own strengths and weaknesses. The good ones allow you to develop your own style and then to leave them when the time is right. **– Robert Greene**

My parents, Dieter Musch and Petra Musch, for supporting me since the first day.

Like it or not, children are and will always be their own beings; but they need great love and guidance to come to full humanness. **– Jon Kabat-Zinn**

My sisters, Melanie Karl and Simone Musch, for regularly reminding me of the "I am normal" paradox.

The greatest obstacle to accurately identifying someone else's style is what I call the "I am normal" paradox. That is, our hypothesis that the world should look to others as it looks to us. **– Chris Voss**

All my friends, for making this a fun journey full of unexpected adventures.

[...] If a space is too rich in possibilities, you are often better off moving around at random rather than exploring it in a systematic way. [...] That way you are free from time to time to choose a path that, although it may not seem the best one at first, promises to lead to increasingly better options down the road. **– Cedric Villani**

A great future is lying ahead. We have only just begun!

Markus Musch
Oslo, May 2022

List of Papers

Paper I

Fjordholm, U. S., Musch, M., Risebro, N. H. “Well-Posedness Theory for Nonlinear Scalar Conservation Laws on Networks”. *Published in Networks & Heterogeneous Media, Vol. 17, no. 1 (2022), pp. 101-128.*

Paper II

Musch, M. “Convergence Rates of Numerical Schemes for Nonlinear Conservation Laws on Graphs with Boundary Nodes”. *Submitted for publication.*

Paper III

Fjordholm, U. S., Musch, M., Risebro, N. H. “Well-posedness and convergence of a finite volume method for conservation laws on networks”. *To appear in SIAM Journal on Numerical Analysis.*

Contents

Introduction

Partial differential equations have become an invaluable tool for modeling processes in fields like engineering, science, economics and beyond. For a mathematical model to be useful, we need to know that there exists at least one solution to the equations that constitute the model of our observed phenomenon. Furthermore, it would be even better to know that the equations have exactly one solution, because what would our model be worth if it gave us several contradicting solutions to our problem. Lastly, it would be desirable for the solution to experience changes in the outcome which are in some way proportional to changes of the initial state. If not, small measurement errors in our observation of the process we want to model could lead to vastly different outcomes in our predictions. This property we call stability.

These three properties, existence, uniqueness, and stability of a solution constitute well-posedness after the definition of the 20th century french mathematician Jacques Hadamard. In this thesis we want to address the question of well-posedness of non-linear hyperbolic conservation laws on networks.

The finite volume method plays an essential role in the numerical analysis of partial differential equations especially of the hyperbolic type. Furthermore, in many cases it is not possible to find an analytical solution to a given problem. If we know, though, that a given numerical scheme converges towards a well-posed solution we can compute approximate solutions with a high degree of accuracy, to get a clear idea of what the solution looks like. This is particularly relevant in applied sciences.

In addition to investigating well-posedness for hyperbolic conservation laws on networks, we will also show a numerical scheme and give an error bound on approximations constructed with this method. We present test runs of numerical experiments obtained by an implementation of said method on a computer.

To approximate a solution for a given equation with the finite volume method, one partitions the given spatial domain into a finite number of volumes and takes the average of the unknown quantity as an approximation. In case one is looking at a time-dependent problem, an appropriate discretization of the time dimension has to be applied as well, in the simplest case by a forward Euler scheme. Though higher order schemes exist, a lot of work is necessary to make them well-behaved. The difficulties of constructing higher order methods is in stark contrast to the finite element method, where higher order approximations are oftentimes a simple extension of the first order case. Finite element methods, though, have difficulties dealing with discontinuous solutions which are standard in hyperbolic problems so while often being the method of choice for parabolic problems, for hyperbolic problems a finite volume method will often be chosen over a finite element approximation. A third class of schemes which try to combine the advantages of finite volume and finite element methods are Discontinuous Galerkin methods. Those allow for an easy adaption to higher order approximations while still being able to handle discontinuous solutions very well. The price one pays for this is computational efficiency. Most recently the so called hybridized discontinuous Galerkin method has been developed and investigated. This method keeps many of the advantages of DG methods while at the same time being more computationally efficient.

While it is interesting to know how the method behaves when the space and time discretization are being chosen at ever higher resolution for practical purposes, namely to know how well one can approximate the problem in question on a computer, the limit case of the space and time discretization parameters going to zero is relevant to the analysis of PDE problems. If we can show that a numerical approximation converges as the discretization is being chosen finer and finer, then we gain knowledge of the existence of a solution to our analytical problem. This was one point on our agenda of proving well-posedness. If we already know that in case a solution exists, and that this solution is unique, we can conclude that the solution the numerical scheme is converging to must be the unique solution to our problem. Uniqueness of a solution can be concluded from a stability result. If we know that the difference of two solutions of the same equation with different initial data is bounded by the difference of the initial data, we get uniqueness of our solution by choosing the same initial data. This is the strategy which we want to apply here to show well-posedness of our problems.

While finite volume methods may appear unsophisticated at first, they still play a large role in industrial applications. This is also due to the fact that they are easily parallelizable.

Nonlinear Scalar Hyperbolic Conservation Laws on Networks

In this section we want to give a slightly more technical introduction to the problem we will be investigating in this thesis.

We consider nonlinear hyperbolic conservation laws on networks and focus in particular on the scalar, one-dimensional case

$$u_t + f(u)_x = 0 \tag{0.0.1}$$

on a network. Here, $u = u(x, t)$ is the unknown *conserved variable* and f is a scalar *flux function* defined either on $\mathbb{R}$ or some subinterval. We aim to make sense of the conservation law on a directed graph and obtain existence, uniqueness, stability and approximability results.

Consider a network represented by a connected and directed graph. We tag the edges of this graph with an index k and impose on each edge a scalar conservation law

$$\begin{aligned} u_t^k + f^k(u^k)_x &= 0, \qquad x \in D^k,\ t > 0 \\ u^k(x,0) &= \bar{u}^k(x), \qquad x \in D^k \end{aligned} \tag{0.0.2}$$

for some spatial domain $D^k \subset \mathbb{R}$. The initial data $\bar{u}^k$ is given. (Here and in the remainder, a superscript k will refer to an edge or a vertex.) We may think of edges as pipes or roads and the vertices as intersections, with the convention that the direction of travel is in the positive x-direction, as shown in Figure 1.

The reason why these types of equations have been studied extensively during the last decades is their many applications. The first big group of applications is problems

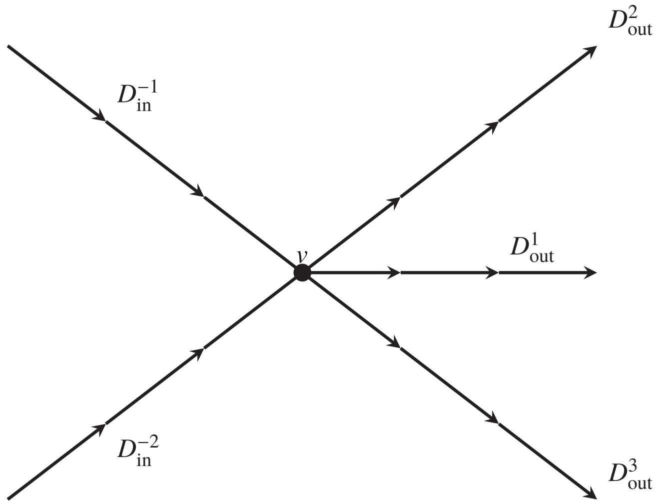

Figure 1: A star shaped network with two ingoing and three outgoing edges.

of fluid flow such as water flowing in irrigation channels, gas flowing in pipelines and blood flow **[BCG13**; **G10]**. Other problems which do not necessarily describe actual fluids can be approximated by equations of fluid dynamics satisfactorily under suitable assumptions. These include traffic flow on a road of networks, air traffic management, supply chains and data and telecommunication networks **[BCG13**; **G10]**.

The first work for hyperbolic conservation laws on networks was done by Holden and Risebro in 1995 **[HR95]**. In this very first work, Holden and Risebro investigate a front tracking algorithm to show existence of a solution to a traffic flow problem. The traffic flow problem is characterized by flux functions which are concave and bell shaped. A good overview over the research that had been done since 1995 up until 2010 can be found in the review article **[CG10]**. In the last decade, the major developments were focused on vanishing viscosity solutions **[ACD17**; **CD19**; **CD20]** and numerical methods **[ACD17**; **T20]**.

It is well-established that nonlinear hyperbolic conservation laws develop shocks in finite time. Therefore, solutions are always understood in the weak sense. Unfortunately, weak solutions to nonlinear hyperbolic conservation laws turn out to be non-unique, and additional conditions, usually referred to as entropy conditions, are imposed to select a unique solution. If the flux function is continuous then the theory of entropy solutions is covered by Kruzkhov's theory **[K70]**. For conservation laws with discontinuous fluxes the choice of entropy conditions is not obvious, and different physical models might lead to different entropy conditions. Although suitable entropy conditions can yield uniqueness, different entropy conditions are known to yield different solutions. The approach we chose here is to construct a numerical approximation of the continuous problem and then let the discretization in space and time become infinitely small. We show that the limit of the numerical approximation exists.

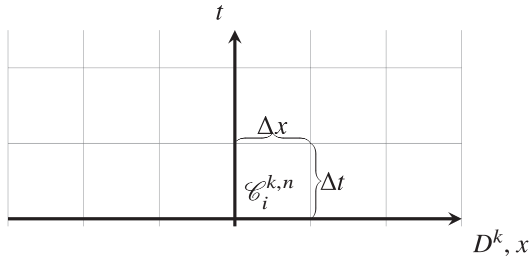

Figure 2: The grid

Finite Volume Methods for Nonlinear Scalar Hyperbolic Conservation Laws on Networks

Now, we want to take a look at how to reasonably construct a finite volume method on a network. A good starting point for investigations into networks is always conservation laws with discontinuous fluxes on the line, since the network case can be seen as a generalization of this case.

With regards to numerical methods for conservation laws with discontinuous fluxes, a lot of research has been done, one of the earliest being an article by Towers in 2000 **[T00]**. Later, in 2002, Towers together with Karlsen and Risebro investigated an upwind method for degenerate parabolic equations with discontinuous coefficient **[KRT02]**. After that, an Enquist–Osher scheme has been investigated as well **[BKT09]**.

Not many works are available on numerical schemes for conservation laws on networks, though we have to mention the work by Andreianov, Coclite and Donadello from 2016 **[ACD16]**, as well as the work by Towers from 2020 **[T20]**. The scheme in the former paper is implicit in the joint node, while the scheme in the later paper is fully explicit.

Now we want to have a look at the exact scheme we are going to investigate in this thesis. First, let us look at a single edge D^k. This is just an interval. We can therefore discretize the physical domain into cells

$$\mathscr{C}_i^k = D^k \cap \left(x_{i-1/2},\ x_{i+1/2}\right),$$

and the space time domain into rectangles

$$\mathscr{C}_i^{k,n} = \mathscr{C}_i^k \times \left[t^n, t^{n+1}\right],$$

with side length Δx, $\Delta t > 0$ as seen in Figure 2. Here, $t^n = n\Delta t$, and $x_{i+1/2} = (i + 1/2)\Delta x$ for integer index values n, i where Δt, Δx are the discretization parameters.

Now we apply the well known explicit finite volume method on the real line,

$$u_i^{k,n} \approx \frac{1}{\Delta x} \int_{\mathscr{C}_i^k} u^k\left(x, t^n\right) dx \qquad \forall i.$$

Afterwards we want to connect the different edges D^k. As is being emphasized in red in Figure 3, this is done by one central cell on the node. We compute the cell average of

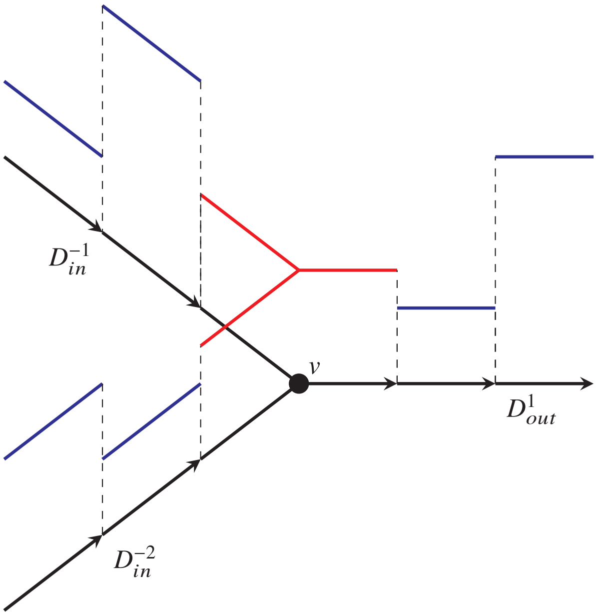

Figure 3: Three-dimensional visualization of the approximation by averages

this cell as

$$\frac{u_0^{n+1} - u_0^n}{\Delta t} + \frac{1}{\Delta x_0}\left(\sum_{k\in\mathscr{I}_{\mathrm{out}}} F_{1/2}^{k,n} - \sum_{k\in\mathscr{I}_{\mathrm{in}}} F_{-1/2}^{k,n}\right) = 0. \tag{0.0.3}$$

which is what distinguishes the finite volume method from a finite difference method on the real line.

Research Questions Considered in this Thesis

We want to give an overview of the research questions which will be considered in this thesis, and explain why they are highly relevant problems.

General Framework and Monotone Fluxes

Due to the non-linearity the complexity of the problem in question is considerable. Therefore it is not uncommon to start out considering only relatively simple flux functions such as monotone fluxes or convex fluxes when approaching a new model. Even though this might seem simple at first it turns out that even this restricted case is quite complicated. More general flux functions can then be considered later on.

While the case of only monotone fluxes is relatively simple, these equations have important applications, nonetheless. For example, nonlinear scalar hyperbolic conservation laws on networks with monotone fluxes can be used to model gas flow in a network of pipelines by means of Burgers' equation.

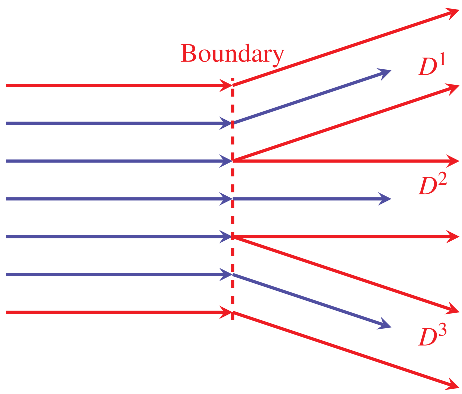

Figure 4: The boundary

Boundary Conditions

While there is no issue to look at edges of infinite length from a mathematical perspective, from an application viewpoint this is not reasonable. To make our model more realistic we want to include nodes with only ingoing or only outgoing edges as shown in Figure 4 which will serve as boundary nodes. On these boundary nodes a value for the flux will be given. This leads us to an initial boundary value problem.

Convergence Rates

If we want to use our numerical scheme in practical applications it is important to know how precisely the numerical solution will approximate the actual solution at a given degree of grid refinement. This is the question of the convergence rate of a numerical method. The very first result in this regard was obtained by Kuznetsov in 1976 **[K76]**. If we have a grid with spatial grid size Δx, a classical result for conservation laws on the line gives us a convergence rate for monotone methods of

$$\|u(\cdot, T) - u^{\Delta t}(\cdot, T)\|_{L^1(\Omega)} \leqslant C\sqrt{\Delta x}.$$

Here, u is a solution at time $T > 0$ and $u^{\Delta t}$ is an approximate solution at time T to the equation (0.0.2). That is to say that the L^1-norm of the difference between the approximation and the actual solution is bounded by the square root of the grid size.

General Flux Functions

After having a firm grasp of the easier cases like all fluxes being monotone, one will want to investigate more complex cases. This comes at the cost of increased complexity, though. More sophisticated means of proof will be necessary to show the desired results.

Outline

The rest of the thesis is a collection of three papers followed by a conclusion section.

In the first paper we construct a very general framework for numerical methods for scalar non-linear hyperbolic conservation laws on networks. This includes a general stability result as well as L^1-contractiveness, an L^∞-bound and Lipschitz-continuity in time for consistent, conservative and monotone finite volume schemes. The general framework, though, depends on the existence of a large enough set of solutions which are constant in time, so called stationary solutions. We show the existence of such sets for the case of monotone as well as bell-shaped flux functions. The fluxes can differ on each edge of the network. We do this by constructing numerical approximations and then let the discretization parameters go to zero. For monotone fluxes we also show convergence towards the entropy solution via a Crandall–Majda type argument.

The results described in this paper were gained in collaboration with Ulrik Skre Fjordholm and Nils Henrik Risebro.

In the second paper we want to present a numerical scheme for which we can show a convergence rate of $\sqrt{\Delta x}$ where Δx is the gridsize of the spatial discretization. We show that a particular numerical approximation has a convergence rate of $\sqrt{\Delta x}$ where Δx is the gridsize of the spatial discretization.

Furthermore, to make the model more realistic we want to address the question of boundary conditions. In particular we want to include nodes into our graph which has only outgoing or only ingoing edges and a given function of the value of the influx or outflux of the preserved quantity.

In the third paper we want to extend the results from Paper 1 to non-monotone flux functions. In this case we consider Lipschitz continuous flux functions $f^k\colon [0,\overline{\alpha}] \to [0,\infty)$ satisfying $f^k(0) = 0$ and $f^k(u) = 0$ for $u \geqslant \alpha^k$, with finitely many extrema $0 = v_0^k < v_1^k < \cdots < v_{m^k}^k = \alpha^k$, where $\alpha^k \in (0,\infty)$, $k \in \mathscr{I}$ are given.

Such flux functions include, but are not restricted to, bell-shaped fluxes which are used to model traffic flow.

We gain an existence and uniqueness result for non-linear scalar hyperbolic conservation laws on networks with non-monotone, non-concave fluxes. We do this by constructing numerical approximations and then let the discretization parameters go to zero.

The results described in this paper were gained in collaboration with Ulrik Skre Fjordholm and Nils Henrik Risebro.

The final section gives a short summary of our findings and an outlook on relevant open questions.

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