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Free Probabilistic and Poisson–Lie Geometric Methods for Quantum Groups

On Strong 1-Boundedness and Coboundary Lie Bialgebras

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"I have approximate knowledge of many things."
Adventure Time S01E18.

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* at the University of Oslo. The research presented here was conducted at the University of Oslo under the supervision of Førsteamanuensis Makoto Yamashita and Professor Sergey Neshveyev. This work was partially supported by the Norwegian Research Council through project 30087 "Quantum Symmetry", and by the Trond Mohn Foundation via the project "Pure Mathematics in Norway".

The thesis is a collection of three papers, presented in order of when the main results were obtained. This differs slightly from the order of publication (on arXiv), which would put Paper III before Paper II. The papers are preceded by an introductory chapter that relates them to each other and provides background information and motivation for the work.

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Floris Eelke Elzinga
Oslo, May 2022

List of Papers

Paper I

Elzinga, F. “Strong 1-Boundedness of Unimodular Orthogonal Free Quantum Groups”. In: *Infinite Dimensional Analysis, Quantum Probability and Related Topics*. Vol. 24, no. 2 (2021), Paper No. 2150012, 23. DOI: 10.1142/S0219025721500120.

Paper II

Elzinga, F. and Yamashita, M. “Poisson–Lie Group Structures on Semidirect Products”. Submitted to *Journal of Noncommutative Geometry.*

Paper III

Brannan, M., Elzinga, F., Harris, S. and Yamashita, M. “Crossed Product Equivalence of Quantum Automorphism Groups”. Submitted to *International Mathematics Research Notices*.

Contents

Introduction

This thesis is concerned with two very different flavours of quantum groups. On the one hand there are the universal compact quantum groups of Kac type, which form one of the most well-behaved classes of quantum groups. We will primarily study their associated reduced operator algebras, the von Neumann algebraic variant of which will always be a type II_1-factor. As such, one can use the powerful techniques of free probability to derive structural results about these operator algebras. On the other hand we want to consider examples of locally compact quantum groups that arise as quantizations of real simple Lie groups. Here the obstacles are much more immediate and fundamental, and we will have to content ourselves with studying some shadows of quantum phenomena in geometry and Lie theory.

The introduction begins by recalling the notion of compact quantum groups and some of their history, before discussing some important examples in more detail. An excellent reference for this material is the book by Neshveyev and Tuset [NT13]. We then proceed to give an overview of the relevant parts of free probability theory, a thorough account of which can be found in [MS17]. At this point we are ready to summarize Papers I and III.

We then move on to briefly discuss locally compact quantum groups and some of the most prominent examples. Finally, we review some notions from the theory of Poisson–Lie groups and their infinitesimal models (see [CP95] for more) and we summarize Paper II.

1 The Road to Compact Quantum Groups

The modern theory of operator algebraic quantum groups is by now over 30 years old, and in this time it has grown into a mature field with connections to many other areas of mathematics. Rather than diving into the long history of the term ‘quantum group’, let us instead motivate the modern operatic algebraic definition best suited to our purposes.

Let G be a topological group, which we will take to be compact for the moment for simplicity. Then by the celebrated Gelfand Duality we can recover the structure of G as a topological space from the commutative C*-algebra $C(G)$ of continuous $\mathbb{C}$-valued functions on G.

It is a natural question whether it is possible to add data to $C(G)$ that also allows for recovery of the group structure. Looking at the group axioms, we need to encode the multiplication, inverses, and the unit element at the level of functions. On the group level, we have continuous maps

$$\nabla\colon G \times G \to G,\ \gamma\colon G \to G,\ e\colon \{*\} \to G.$$

The map ∇ encoding the multiplication has to be associative, meaning that

$$\nabla \circ (\nabla \times \iota) = \nabla \circ (\iota \times \nabla).$$

Here, ι is a symbol used for generic identity maps. The inversion map γ is an involution, and an 'antihomomorphism' with respect to ∇ in the sense that

$$\gamma \circ \nabla = \nabla \circ \Sigma \circ (\gamma \times \gamma),$$

where $\Sigma\colon G \times G \to G \times G$ is the flip map. One can similarly write down the other group axioms in terms of these functions.

We now 'dualize' these functions and their relations to $C(G)$. This results in maps

$$\Delta\colon C(G) \to C(G \times G),\ S\colon C(G) \to C(G),\ \varepsilon\colon C(G) \to \mathbb{C},$$

which are defined by

$$(\Delta f)(g, h) = f(gh),\ (Sf)(g) = f(g^{-1}),\ \varepsilon(f) = f(e),$$

respectively. The map Δ is a unital $*$-homomorphism satisfying $(\Delta\otimes\iota)\Delta = (\iota\otimes\Delta)\Delta$, which is dual to the associativity condition for ∇. The assignment ε is also a $*$-homomorphism and interacts with Δ according to the rule $(\varepsilon\otimes\iota)\Delta = \iota = (\iota\otimes\varepsilon)\Delta$. Finally, S is an involutive linear map, and its compatibility condition reads $m(S \otimes \iota) = \varepsilon(\cdot)1 = m(\iota \otimes S)$, where m is the map $m\colon C(G \times G) \to C(G)$ given by sending a function to its restriction to the diagonal in $G \times G$.

It is now tempting to follow the philosophy of noncommutative geometry and pass to a more general class of objects than compact groups by allowing the commutativity of the function algebra to be violated. This is analogous to how one passes from compact topological spaces to compact quantum spaces described by a function algebra which is allowed to be any unital C*-algebra. Such a more general category of group-like objects then deserves to be called the category of *compact quantum groups*. However, in this case things are not quite so straightforward and a naive approach quickly runs into problems. Primarily, it turns out that the maps S and ε will generally become unbounded when one passes to noncommutative C*-algebras.

The way forward was found by Woronowicz in the 80s. First, he restricted to compact subgroups of $\mathrm{GL}(n, \mathbb{C})$ and wrote down the definition of a *compact matrix quantum group* in [Wor87a]. One starts with a unital C*-algebra A which is of a special form, namely, it is generated by the entries $(u_{ij})_{i,j=1}^n$ of an operator matrix u such that both u and $\overline{u}$ (entry-wise adjoint) are invertible. One then assumes that the map $\Delta\colon A \to A \otimes A$ defined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$$

is a unital $*$-homomorphism. This is a very concrete definition, and most of the quantum groups that we will encounter in the papers below are of this form.

Nevertheless, this definition is not general enough to cover all of the examples that deserve to be called a compact quantum group[1]. In [Wor98] Woronowicz introduced the following more abstract definition, which subsumes the previous one.

Definition 1.1. A *compact quantum group* $\mathbb{G}$ consists of a pair (A, Δ), where A is a unital C*-algebra and $\Delta\colon A \to A \otimes A$ is a unital $*$-homomorphism such that

(i) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ (*coassociativity*);

(ii) the sets $\{(a \otimes 1)\Delta(b) | a, b \in A\}$ and $\{(1 \otimes a)\Delta(b) | a, b \in A\}$ are total in $A \otimes A$, meaning for each set that the span of its elements is dense (*cancellation property*).

We call Δ the *comultiplication* or *coproduct*. Following the example of Gelfand Duality, we talk about $\mathbb{G}$ as a 'virtual' object, and hence often write $A = C(\mathbb{G})$.

For the moment it seems like the maps S and ε have disappeared, but we will see them again later on. In fact, it is a nice feature of the theory of compact quantum groups that the two properties above suffice to capture all of the group-like structure. Let us indicate how this works in case the function algebra $C(\mathbb{G})$ is commutative. Gelfand Duality immediately gives us a compact Hausdorff space G and Δ becomes a continuous map from $C(G)$ to $C(G \times G)$. Coassociativity then implies that G is a compact semigroup, and the cancellation property moreover implies that G has the structure of a compact semigroup with cancellation. However, any such object is necessarily a compact group by standard arguments. Let us remark for completeness that in the other direction the cancellation property is an easy consequence of the Stone–Weierstraß Theorem.

There is another piece of compact group technology that makes sense in the quantum setting and is easily obtainable from the definition above. This is the Haar state h on $C(\mathbb{G})$, which satisfies

$$(\iota \otimes h)\Delta(a) = h(a)1 = (h \otimes \iota)\Delta(a)$$

for all $a \in C(\mathbb{G})$. An important ingredient in its construction is the realization that Δ can be used to define a convolution product for bounded linear functionals on $C(\mathbb{G})$. Consequently we can define the Hilbert space of square integrable functions $L^2(\mathbb{G})$ on $\mathbb{G}$ through the Gelfand–Naimark–Segal (GNS) Construction. Accordingly, we also obtain two reduced operator algebras associated to $\mathbb{G}$, namely the C*-algebra $C^r(\mathbb{G})$ as the image of $C(\mathbb{G})$ under the h-GNS representation, and its von Neumann algebraic closure $L^\infty(\mathbb{G})$. These operator algebras will be the primary objects of interest in two of the articles below.

[1] This is clear already at the level of genuine compact groups, as any compact group with a faithful finite dimensional representation is automatically a Lie group. As a concrete example of a compact group without such a representation once can take $\prod_n \mathrm{U}(n)$.

Thus a compact quantum group $\mathbb{G}$ comes with three operator algebraic models, namely $C(\mathbb{G})$, $C^r(\mathbb{G})$, and $L^\infty(\mathbb{G})$. There is also an algebraic model that is based instead on $\mathcal{O}(G)$, the Hopf $*$-algebra of regular functions on the compact group G. This algebra is spanned by coefficients of the finite dimensional unitary representations of G, and so to make sense of this in our framework, we first need to briefly discuss the representation theory of compact quantum groups.

There is a straightforward way to adapt the definition of a representation of a compact group into the one for a compact quantum group. Indeed, a *unitary representation* of a compact quantum group $\mathbb{G}$ is given by a finite dimensional Hilbert space $\mathcal{H}$ and a unitary element $U \in B(\mathcal{H}) \otimes C(\mathbb{G})$ such that $(\iota \otimes \Delta)U = U_{12}U_{13}$. Here we are using leg-numbering notation, where we have as many subscripts as the tensor has factors and the numbers indicate to which 'leg' in a larger tensor product the factors are mapped, with units filling any gaps. This is illustrated by the example $(a \otimes b)_{31} = (b \otimes 1 \otimes a)$. In case we have a compact group G, this is asking for a continuous function U from G into the unitary operators on $\mathcal{H}$ such that $U(gh) = U(g)U(h)$ as functions on $G \times G$.

From this point on, the (abstract) representation theory of compact quantum groups is almost identical to the that of compact groups. In particular Peter–Weyl Theory remains valid, Schur's Lemma holds, and we can take tensor products and duals of representations. This leads to the structure of a rigid C*-tensor category, denoted $\mathrm{Rep}(\mathbb{G})$.

Here however, some subtle and interesting differences with the classical theory emerge. The categorical dimension on $\mathrm{Rep}(\mathbb{G})$, often called the *quantum dimension*, need not assign integer values to representations. Moreover, in the classical case, the map that flips the factors in a tensor product of two representations is always an intertwiner, but this fails to be true in the quantum setting. Finally, the contragredient to a representation can fail to be a representation and in general it is necessary to conjugate the contragredient by some matrix to obtain the dual representation. It should be noted that these phenomena are strongly related to each other.

We can now define the $*$-algebra $\mathcal{O}(\mathbb{G})$ of coefficients of representations of $\mathbb{G}$. This $*$-algebra sits densely inside $C(\mathbb{G})$, and it turns out that the coproduct Δ sends $\mathcal{O}(\mathbb{G})$ into the algebraic tensor product $\mathcal{O}(\mathbb{G}) \otimes \mathcal{O}(\mathbb{G})$. In fact, we can turn $\mathcal{O}(\mathbb{G})$ into a Hopf $*$-algebra by introducing the counit ε and antipode S as those maps characterized by

$$(\iota \otimes \varepsilon)(U) = 1, \ (\iota \otimes S)(U) = U^*,$$

for any finite dimensional unitary representation U. This is reminiscent of the naive approach we took at the start of this section, but in general we cannot pass to a C*-closure, as the map S will typically not be bounded. This should be intuitively clear to any reader familiar with modular theory, and indeed the modular theory of the Haar state is very interesting, but we will not need it.

As an aside, note that Hopf $*$-algebras are precisely those $*$-algebras with a 'well-behaved' unitary representation theory. It is a pleasing fact that any Hopf

$*$-algebra that can be generated by the coefficients of its unitary representations must come from a compact quantum group [DK94]. In essence this works because one can construct a Haar functional by projecting onto the span of the trivial representation.

Another upside of the Hopf $*$-algebraic picture is that we can talk about the (algebraic) dual algebra, which is again a Hopf $*$-algebra in a natural way by taking the transpose of the product as the coproduct and so on. Let $\mathbb{G}$ be a compact quantum group and $\mathcal{O}(\mathbb{G})$ its algebra of regular functions, then dual Hopf $*$-algebra gives rise to a so-called *discrete quantum group*
$hatbG$, which it is said to be the *dual* quantum group to $\mathbb{G}$.

Before discussing the most important examples of 'genuine' compact quantum groups for us, we will end this section with some elaborations on the duality above in the classical case. Aside from $\mathcal{O}(G)$, another important class of examples of Hopf $*$-algebras are the group algebras associated to discrete groups. Let Γ be a discrete group, then for any $s \in \Gamma$ the coproduct is given by $\Delta s = s \otimes s$, the counit is the map $s \mapsto 1$, and the antipode sends s to s^{-1}. In fact, any discrete group Γ defines a compact quantum group $\hat{\Gamma}$ in this way, considering instead $C(\hat{\Gamma}) = C^*_r(\Gamma)$ with the same formula for the coproduct. The algebraic model $\mathcal{O}(\hat{\Gamma})$ then becomes the group algebra $\mathbb{C}[\Gamma]$. In terms of duality we hence identify $\mathcal{O}(\mathbb{G}) = \mathbb{C}[\hat{\mathbb{G}}]$. This is the point of view on the free orthogonal quantum groups that we take in Paper I.

Notice that the coproduct on $\hat{\Gamma}$ is *cocommutative* in the sense that applying the flip map after Δ gives Δ again. The Hopf algebra $\mathcal{O}(G)$ is of course commutative, but not cocommutative, and $\mathbb{C}[\Gamma]$ is not commutative unless Γ itself is. It is easy to see that the dual of a commutative Hopf algebra is cocommutative, and vice versa. In particular the dual to a Hopf algebra that is both commutative and cocommutative is again of this type.

This is of course related to the famous result that the Pontryagin dual of a compact Abelian group is a discrete Abelian group. However, the dual of a possibly non-Abelian compact group is in general only a discrete quantum group. A major motivation for the development of (general) quantum groups was to find a category of group-like objects that is closed under such a duality [Tak69]. The formalism of compact and discrete quantum groups accomplishes this.

2 Examples of Compact Quantum Groups

2.1 The Quantum SU(2) Group

Shortly before Woronowicz laid out his general theory of compact matrix quantum groups, he established significant parts of the theory for a particular example, namely his famous q-deformation of SU(2) [Wor87b]. Woronowicz was motivated by the idea of applying the process of quantization from physics to important symmetry groups within that field. The group SU(2) is such a group because it encodes the symmetries of the *spin* degree of freedom of elementary particles. Classically, one would usually not expect to be able to find a 'continuous deformation' of such Lie groups, since for example connected simply connected complex simple Lie groups are classified by discrete structures (their Dynkin diagrams).

Woronowicz's approach was the following. One can define SU(2) as the group of 2×2 complex unitary matrices that have determinant equal to 1. The idea is to try and find a way to deform the determinant condition. For this, start with the following definition of the determinant. Consider $\mathbb{C}^2$, then $\wedge^2\mathbb{C}^2$ is one-dimensional, and for any non-zero vector ξ in it, the determinant of a matrix u in M_2 is the number such that

$$(u \otimes u)\xi = \det(u)\xi.$$

In particular, u has determinant 1 if and only if

$$(u \otimes u)\xi = \xi.$$

Now we can try to allow general (unit) vectors ζ from $\mathbb{C}^2 \otimes \mathbb{C}^2$ in this relation. Woronowicz first showed that is necessary to allow the unitary matrix u to have entries in a C*-algebra and to have the vector ζ be of the form

$$\zeta \propto e_1 \otimes e_2 - qe_2 \otimes e_1$$

for some real number q. From this one can derive the relations in the definition below, which Woronowicz also showed to be sufficient.

Definition 2.1 (Woronowicz' $\mathrm{SU}_q(2)$ [Wor87b])**.** For any $-1 \leq q \leq 1$ we define $C(\mathrm{SU}_q(2))$ to be the universal C*-algebra generated by the operators α and γ such that γ is normal satisfying and the following relations,

$$\alpha^*\alpha + \gamma^*\gamma = 1,\ \alpha\alpha^* + q^2\gamma^*\gamma,\ \alpha\gamma = q\gamma\alpha,\ \alpha\gamma^* = q\gamma^*\alpha.$$

We define the coproduct on the generators as

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma,\ \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

We can demystify these relations somewhat if we present $\mathrm{SU}_q(2)$ concretely as a compact matrix quantum group. Note that $C(\mathrm{SU}_q(2))$ is generated by the entries of the matrix

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix},$$

and that the relations above are precisely those that make u unitary and ensure that

$$\det_q(u) = \alpha\alpha^* - q\gamma(-q\gamma^*) = 1.$$

The formulas for the coproduct then become the standard ones for compact matrix quantum groups. Notice that we recover $\mathrm{SU}(2) \cong \mathrm{SU}_1(2)$.

The example of $\mathrm{SU}_q(2)$ fits into a whole family of examples that come from Lie groups. Namely, if G is a simply connected semisimple compact Lie group, there is always a one-parameter family $(G_q)_{q>0}$ of compact quantum groups such that $G \cong G_1$. These are called the *Drinfeld–Jimbo q-deformations* of G, see [Ros90] and [CP95, Section 10.1.E]. We will have more to say about other examples of operator algebraic quantum groups coming from Lie groups in a later section.

2.2 The Free Quantum Groups

Now, we turn to three families of examples that do not come from some sort of deformation quantization. Instead, these examples are 'universal' in some sense within the category of compact quantum groups (or a subcategory). The first two families come with a matrix parameter, and were first defined by Wang [Wan95] in case this matrix is the identity. The more general definition is due to van Daele and Wang [VW96].

Definition 2.2 ([VW96; Wan95])**.** Let $F, Q \in \mathrm{GL}(n, \mathbb{C})$ for some $n \geq 2$. Denote by $C(U_Q^+)$ the universal C*-algebra generated by the entries of a matrix $(v_{ij})_{i,j=1}^n$ such that both v and $Q\overline{v}Q^{-1}$ are unitary, where $\overline{v} = (v_{ij}^*)_{i,j=1}^n$. Denote by $C(O_F^+)$ the universal C*-algebra generated by the entries of a matrix $(u_{ij})_{i,j=1}^n$ such that u is unitary and $F\overline{u}F^{-1} = u$. This defines two compact matrix quantum groups, called the *free unitary quantum group* U_Q^+ and the *free orthogonal quantum group* O_F^+ respectively. The entire family is sometimes referred to as the family of *free quantum groups*.

The free unitary quantum groups are universal in the sense that any compact matrix quantum group can be realised as a quantum subgroup of one of them. More precisely, this means that for every compact matrix quantum group $\mathbb{G}$ there is a surjective $*$-homomorphism from some $C(U_Q^+)$ to $C(\mathbb{G})$ that intertwines the coproducts. This is a quantum analogue of the classical theorem that any compact Lie group admits a faithful unitary representation, and this was in fact a motivation to consider these quantum groups.

The conjugation by the matrices Q and F appear in the definition precisely because of the fact mentioned above that the contragredient to a representation is not automatically the dual. The free orthogonal quantum group has the additional requirement that the defining representation is self-dual, and is universal among this class of compact matrix quantum groups. This class includes $\mathrm{SU}_q(2)$, as it can in fact be realised as a free orthogonal quantum group for a suitable 2×2 matrix.

While the free quantum groups are (in general) not q-deformations, they still bear resemblance to the classical unitary and orthogonal groups. This is clearest when one makes the choice $Q = I_n = F$, where it is simple to see that the Abelianization of $C(U_n^+)$ is isomorphic to $C(\mathrm{U}(n))$ and that of $C(O_n^+)$ is isomorphic to $C(\mathrm{O}(n))$. Here we have used the customary notation $U_{I_n}^+ = U_n^+$, and similarly for the free orthogonal quantum group. For this reason the free quantum groups are sometimes called *liberations* of these classical groups [BS09].

Let us briefly say some words about the discrete quantum group dual to the free orthogonal quantum group. This discrete quantum group is often denoted as $\mathbb{F}O_F$, and in this dual picture the function algebra $C(O_F^+)$ is instead interpreted as the full group C*-algebra $C^*(\mathbb{F}O_F)$. If one takes the quotient of $C^*(\mathbb{F}O_n)$ by the ideal generated by the off-diagonal elements of u, one obtains the full group C*-algebra of the free product group $*_{i=1}^n \mathbb{Z}_2$. In Paper I, we will take this point of view on the free orthogonal quantum group, since we will need to talk about its 'quantum Cayley graph', which is more natural from a discrete standpoint.

From this point on we will only discuss free unitary quantum groups with $Q = I_n$. For the free orthogonal quantum group we shall be slightly more inclusive and allow both $F = I_n$ and $F = J_{2m}$, where J_{2m} is the standard symplectic matrix in $2m$ dimensions. For the latter, we will use the notation $O_{J_{2m}}^+ = O_{2m}^{+J}$, and $\mathbb{F}O_{J_{2m}} = \mathbb{F}O_{2m}^J$ on the dual side. The reason for this restriction is that (up to isomorphism) these are the only matrices for which the associated free quantum groups have a tracial Haar state.

A compact quantum group $\mathbb{G}$ whose Haar state is a trace is said to be of *Kac type*, and its discrete dual is called *unimodular*. An immediate consequence is that the quantum group von Neumann algebra $L^\infty(\mathbb{G}) = \mathcal{L}(\hat{\mathbb{G}})$ is then a finite von Neumann algebra.

Let us suggestively call the quantum group von Neumann algebras of the free quantum groups of Kac type the *free quantum group factors*. These have been extensively studied and many of their properties are known [Ban96; BC07; Bra12; Bra14; Cas21; CFY14; FV15; Fre13; Iso15; VV07]. For instance, they are indeed factors, and hence type II_1-factors. Banica established a deep link between these II_1-factors and the free probability theory of Voiculescu, which we shall describe in a subsequent section.

2.3 The Quantum Automorphism Groups

So far we have been discussing compact quantum groups for their own sake, rather than as a 'collection' of symmetries of some other object. Motivated by a question of Connes, Wang [Wan98] investigated what it should mean for a finite (noncommutative) space to posses 'quantum symmetry'.

In the paradigm of noncommutative geometry, a finite noncommutative space is described by a finite dimensional C*-algebra, with the classical n-point space corresponding to the commutative C*-algebra $\mathbb{C}^n$. It turns out that we want to talk about finite noncommutative *measured* spaces instead, that is to say a pair (B, ψ) where B is a finite dimensional C*-algebra and ψ is a state on B, which we will assume to be tracial. Let us now define what it means for a compact quantum group to act on (B, ψ), which one can easily obtain by dualizing the classical definition.

Definition 2.3. A *left action* of a compact quantum group $\mathbb{G}$ on a finite noncommutative measured space is a unital $*$-homomorphism $\delta\colon B \to \mathcal{O}(\mathbb{G}) \otimes B$ such that $(\iota \otimes \delta)\delta = (\Delta \otimes \iota)\delta$, $(\varepsilon \otimes \iota)\delta = \iota$, and $(\iota \otimes \psi)\delta = \psi(\cdot)1$.

Definition 2.4 ([Wan98])**.** Let (B, ψ) be a finite noncommutative measured space. We define the *quantum automorphism group* of (B, ψ) to be the universal compact quantum group admitting an action on (B, ψ). It is denoted by $\mathrm{Aut}^+(B, \psi)$ and it is of Kac type.

While this definition is very nicely packaged, it is not particularly illuminating and of course one needs to show that such a quantum group exists (although uniqueness is clear by abstract nonsense). Banica instead provided a presentation of $\mathrm{Aut}^+(B, \psi)$ as a compact matrix quantum group [Ban99]. He also showed that there is a link between the free quantum groups and the quantum automorphism groups of the full matrix algebras M_n. Classically, the group of automorphisms of M_n is isomorphic to the projective orthogonal group $\mathbb{P}\mathrm{O}(n)$ through acting by conjugation. Banica defined the projective free orthogonal quantum group $\mathbb{P}O_n^+$ and showed that it is isomorphic to $\mathrm{Aut}^+(M_n, \mathrm{tr})$.

The quantum automorphism groups for the classical n-point space with the normalized counting measure admit a particularly nice presentation, and will feature prominently in Paper III. It is commonly referred to as the *quantum permutation group* S_n^+, and it can be defined as the compact matrix quantum group whose defining representation is given by a *magic unitary* matrix. This means that all of its entries are projections, and that the sum of these projections along any row or column is the identity operator. Unsurprisingly, its C*-algebra $C(S_n^+)$ is a liberation of the C*-algebra of continuous functions on S_n. What is a little more surprising is that S_n^+ is isomorphic to S_n for $n = 1, 2, 3$, and that as soon as $n \geq 4$, S_n^+ is a genuine, not finite, compact quantum group.

Keeping in mind the universality properties of the free quantum groups, one might hope that the quantum permutation groups are universal for finite quantum groups, but this is too much to expect [BBN12]. However, it is trivial that the entire collection of quantum automorphism groups is universal for finite quantum groups, since every finite quantum group $\mathbb{G}$ acts on the finite noncommutative measured space $(C(\mathbb{G}), h)$ and is therefore a subgroup of $\mathrm{Aut}^+(C(\mathbb{G}), h)$.

3 Free Probability and Strong 1-Boundedness

3.1 A Brief Introduction to Free Probability

The various operator algebras constructed out of (discrete) groups form some of the central families of examples in the entire field. In particular the free groups $\mathbb{F}_n$ give rise to very interesting but also very difficult questions. Perhaps the most basic such question is which isomorphisms exist among the reduced group C*-algebras $C_r^*(\mathbb{F}_n)$ or the group von Neumann algebras $\mathcal{L}(\mathbb{F}_n)$ respectively, or whether any exist at all. These questions are as old as the field of operator algebras, and only the C*-version has received a definitive answer, when in 1982 in [PV82], Pimsner and Voiculescu constructed an exact sequence in K-theory with which they could show that all the $C_r^*(\mathbb{F}_n)$ are distinct.

At that time, essentially nothing was known about the free group factors beyond that they were not isomorphic to the hyperfinite II_1-factor $\mathcal{R}$, established by Murray and von Neumann through property Γ [MN43]. However, this changed when Voiculescu decided to study free products from a probabilistic point of view [Voi85], leading to his celebrated theory of *free probability*. Using the ideas of free probability, it could be shown that the free group factors admit no Cartan subalgebra [Voi96], hence they cannot be realized through a group-measure space construction, and that the free group factors are either all isomorphic or all distinct [Dyk94; Rǎd94].

In the theory, the algebras of random variables are modelled by tracial von Neumann algebras and the concept of independence is modelled by the free product. Classically, one would take $L^\infty(X,\mu)$ as the algebra of (bounded) random variables on a probability space (X,μ), and one would model two independent random variables by taking their tensor product. Nevertheless, strong analogies between free and classical probability theory exist and are a source of inspiration. Important examples are Voiculescu's Central Limit Theorem [Voi85] and his asympototic freeness for random matrices [Voi91; Voi98a].

We succinctly review the definitions and results from free probability theory that will be relevant for the rest of this thesis, starting with the notion of free independence itself.

Definition 3.1. Let $X = (X_1, \ldots, X_n)$ be an n-tuple of elements in a tracial von Neumann algebra $(\mathcal{M}, \tau)$. Denote by $\mathcal{X}_i$ the $*$-algebra generated by X_i. We say that X is *freely independent* or a *free family* if whenever we have $1 \leq i_1, \ldots, i_m \leq n$, $i_k \neq i_{k+1}$, and $y_k \in \mathcal{X}_{i_k}$ such that $\tau(y_k) = 0$, then we also have that $\tau(y_1 \cdots y_m) = 0$.

Note that while we define free independence using $*$-algebras, this is enough to ensure that the same conclusion holds if we are allowed to pick the y_k to lie in $\mathcal{X}_{i_k}''$ instead.

Definition 3.2. If X is an n-tuple as above, its *joint moments* are the numbers $\{\tau(X_{i_1}^{\varepsilon_1} \cdots X_{i_m}^{\varepsilon_m})\}$ with $\varepsilon_i \in \{1, *\}$, $1 \leq i_j \leq n$, and $m \in \mathbb{Z}_{\geq 0}$.

For a single self-adjoint element X_0, its moments are just the numbers $\{\tau((X_0)^k)\}$, which can be computed through spectral theory as

$$\int_{\mathbb{R}} t^k \, \mathrm{d}\mu_{X_0}(t),$$

where μ_{X_0} is the spectral measure of X_0 with respect to τ. From a probabilistic point of view, the moments and this measure are the important objects. Accordingly, call a self-adjoint element S in a tracial von Neumann algebra $(\mathcal{M}, \tau)$ *semicircular* if its spectral measure with respect to τ is

$$\mathrm{d}\mu_S(t) = \frac{1}{2\pi}\sqrt{4 - x^2}\chi_{[-2,2]}(x) \, \mathrm{d}\lambda(x),$$

where λ is the Lebesgue measure. Semicircular elements are to free probability what Gaußian distributions are to classical probability. Their odd moments vanish, and their even moments are given by the Catalan numbers. There are also *circular elements*, which are those of the form $(S_1+iS_2)/\sqrt{2}$ for S_1 and S_2 free semicircular elements.

If the tuple X happens to generate the ambient von Neumann algebra, it turns out that their joint moments characterize the von Neumann algebra in the following sense.

Theorem 3.3. *Let $(\mathcal{M},\tau)$ and $(\mathcal{N},\sigma)$ be tracial von Neumann algebras. Assume that X and Y are n-tuples generating $\mathcal{M}$ and $\mathcal{N}$ respectively. If the joint moments of X with respect to τ agree with the joint moments of Y with respect to σ, then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic. Moreover, one can take the extension of $X_i \mapsto Y_i$ to be the isomorphism.*

Corollary 3.4. *Let $(\mathcal{M},\tau)$ be a tracial von Neumann algebra. Assume that it can be generated by an n-tuple X of free normal elements, each having a diffuse spectral measure with respect to τ. Then $\mathcal{M}$ is isomorphic to $\mathcal{L}(\mathbb{F}_n)$.*

The corollary follows by using measurable functional calculus to deform the generators X_i in a suitable way. For example, the group-like generators of $\mathcal{L}(\mathbb{F}_n)$ are what are known as *Haar unitaries*. That is to say, they are unitaries and their spectral measure is the Haar measure on the unit circle. Moreover, these generators form a free family. It is straightforward to write down an explicit function that deforms a Haar unitary into a semicircular element, and so we can also identify $\mathcal{L}(\mathbb{F}_n)$ with the von Neumann algebra generated by n free semicircular elements.

We now move on to Voiculescu's relative microstates free entropy and the derived notion of microstates free entropy dimension. These will play a central role in Papers I and III.

Assume once again that we have an n-tuple X of self-adjoint elements in a tracial von Neumann algebra $(\mathcal{M},\tau)$. Since entropy is a powerful tool in classical probability and information theory, we want to introduce a free analogue. It turns out that there are several approaches to defining such a thing, and it is unclear whether the different definitions give rise to equivalent objects[2].

The notion that we shall make the most use of is the so-called *microstates* approach [Voi94]. Its definition is inspired both by statistical physics and the philosophy that (random) matrices are a suitable finite-dimensional approximate model of free probability. Here, we mean approximation in the sense of moments, which is natural given the theorem above.

Definition 3.5 ([Voi94])**.** For $\ell, k \in \mathbb{Z}_{\geq 1}$ and $\varepsilon > 0$, an (ℓ,k,ε)-microstate is an n-tuple of $k \times k$ self-adjoint matrices $(A_1,\ldots,A_n)$ such that

$$|\tau(X_{i_1}\cdots X_{i_m}) - \operatorname{tr}(A_{i_1}\cdots A_{i_m})| < \varepsilon$$

[2] The proposed proof of MIP*=RE [Ji+20] and its implied refutation of the Connes Embedding Problem would have as a consequence that some notions are different.

for all $m \leq \ell$. In words, the tuple of matrices approximate the moments of degree at most ℓ with a tolerance of ε. The set of all (ℓ, k, ε)-microstates is denoted by $\Gamma(X; \ell, k, \varepsilon)$.

Note that the space of n-tuples of self-adjoint $k \times k$ matrices is a real Euclidean space, and hence admits a Lebesgue measure, which we will still denote by λ. There is also the notion of a *relative microstate* which we will need. For this, assume that Y is an m-tuple of self-adjoint elements in $\mathcal{M}$. The set of microstates of X relative to Y is then

$$\Gamma(X : Y; \ell, k, \varepsilon) = p_n \Gamma(X \cup Y; \ell, k, \varepsilon),$$

where p_n is the projection onto the first n factors. We recover the usual microstates of X by taking Y to be the empty tuple.

Definition 3.6 ([Voi96])**.** The *microstates free entropy* of X *relative* to Y is

$$\chi(X : Y) = \lim_{\ell \to \infty} \lim_{\varepsilon \to 0} \limsup_{k \to \infty} \left(\frac{1}{k^2} \ln \lambda(\Gamma(X : Y; \ell, k, \varepsilon)) + \frac{n}{2} \ln k \right).$$

It can be shown that this is well-defined if we allow for the value $-\infty$.

The normalizations in the definition are necessary because we need to look at the volume of microstates as a fraction of the volume of a certain ball (for details see [MS17, Section 7.5]). This quantity χ, setting Y to be the empty tuple for the moment, has many nice properties. For instance, it is subadditive

$$\chi(X_1, \ldots, X_n) \leq \chi(X_1, \ldots, X_k) + \chi(X_{k+1}, X_n) \leq \chi(X_1) + \cdots + \chi(X_n),$$

and if X is a free family then it is even additive

$$\chi(X_1, \ldots, X_n) = \chi(X_1) + \cdots + \chi(X_n).$$

There is an integral formula available for the free entropy of a single random variable [Voi94]. As a consequence, a sufficient condition for $\chi(Y_0)$ to be finite is that Y_0 admits a bounded density with respect to the Lebesgue measure on $\mathbb{R}$. Moreover, in the case that X generates $\mathcal{M}$, one can show that several von Neumann algebraic properties imply that $\chi(X)$ cannot be finite. For instance, this holds for property Γ, having a Cartan subalgebra, or not being prime [Ge98; Voi96]. However, the free entropy is finite for a free familiy of semicircular elements, which thus has dramatic consequences for the structure of the free group factors.

Since the particular value of the free entropy of X is not so illuminating, it is often more useful to consider the following derived quantity.

Definition 3.7 ([Voi96])**.** Let S be an n-tuple of free semicircular elements, that is also free from X. The *(modified) microstates free entropy dimension* of X is then the number

$$\delta_0(X) = n + \limsup_{\varepsilon \downarrow 0} \frac{\chi(X + \varepsilon S : S)}{|\ln \varepsilon|}.$$

It can be shown that $\delta_0(X) \leq n$.

One should interpret the combination $X_i + \varepsilon S_i$ as regularizing the X_i by a free analogue of 'mollifying with a Gaußian'. The free entropy dimension of a free n-tuple of semicircular elements is precisely n. It is unknown whether or not δ_0 is a W*-invariant (it is not a total invariant [Bro05]). There is another commonly used version of the free entropy dimension, δ^*, which comes from a 'non-microstates' definition of a free entropy [Voi98b], but it will not play a significant role in the sequel. The only fact we will need about it is the deep result of Biane–Capitaine–Guionnet [BCG03] that microstates free entropy is always dominated by non-microstates free entropy.

3.2 Free Group Factors and Free Quantum Group Factors

Let us now link the theories of the free compact quantum groups and free probability. When Banica worked out the representation theories of O_F^+ and U_Q^+ in the late 90s, he discovered that, up to rescaling, the characters of their defining representations are semicircular and circular respectively with respect to the Haar state [Ban96; Ban97]. In the case of O_F^+ for instance, this is equivalent to saying that the dimensions of the spaces of intertwiners from u to $u^{\otimes 2n}$ is the n-th Catalan number, and that there are no intertwiners between u and an odd tensor power. Moreover, the combinatorics of these (and other compact quantum group) representation categories are similar to those that appear in free probability, most notably the combinatorics of non-crossing partitions [BS09].

From here on we stick to the free quantum groups of Kac type, which are U_n^+, O_n^+, and O_{2m}^{+J}. We often tacitly assume that $n \geq 3$ and $m \geq 2$. Closer investigation of the moments of the generators of the free quantum groups uncovered even more relations to free probability [BCZ09; BC07; Bra14]. It turns out that asymptotically, these generators behave as a free (semi)circular family in a strong sense. This suggests that there should be structural similarities between the free group factors and the free quantum group factors. Indeed, Banica had already observed in [Ban97] that $L^\infty(U_2^+)$ and $\mathcal{L}(\mathbb{F}_2)$ are isomorphic.

In fact, there is a large amount of results in the literature that establish such structural similarities, of which we mention some here. On the von Neumann algebra level, the free quantum group factors are indeed II_1-factors, see [VV07] and [CFY14, Appendix]. Moreover, both they and the free group factors are strongly solid [FV15; Iso15; OP10] and thus without Cartan subalgebras (see also [Oza04; Voi96]), both are full and hence prime [CFY14; Con74; VV07] (but see also [Ge98; Ște98]), and they are all Connes embeddable [BCV17; Con76]. In the setting of II_1-factors, fullness is equivalent to not having property Γ.

Furthermore, the free quantum groups share many group-like properties with the free group factors as well. None of them are amenable [Ban97], they all have the Haagerup property [Bra12; Haa79], and they are all weakly amenable with Cowling–Haagerup constant equal to 1 [CH85; CH89; Fre13] (see also [CFY14]).

Thus one is lead to the question when there are isomorphisms between the free group factors and the free quantum group factors. For the free unitary quantum groups, we know that such an isomorphism is possible when $n = 2$, but very little is known beyond this result and we will not have much to say on the topic. The case of the free orthogonal quantum groups is one of the main themes of this thesis, but before we can say more we need to introduce a bit of L^2-cohomology.

In [Kye08] Kyed introduced ℓ^2-Betti numbers for unimodular discrete quantum groups, following work of Lück [Lüc98] and Connes–Shlyakhtenko [CS05]. These generalize the ℓ^2-Betti numbers for discrete groups. Now, the first ℓ^2-Betti number of a free group remembers the number of generators, since $\beta_1^{(2)}(\mathbb{F}_n) = n - 1$ for $n \geq 2$. However, it was shown by Vergnioux [Ver12] and Bichon [Bic13] that $\beta_1^{(2)}(\hat{O}_n^+) = 0 = \beta_1^{(2)}(\hat{O}_{2m}^{+J})$, while $\beta_1^{(2)}(\hat{U}_n^+) = 1$ [KR17]. Thus one does not expect any isomorphisms beyond the one we have already discussed.

An important step in the direction of turning this into a proof of the absence of isomorphisms is to combine an estimate of Connes–Shlyakhtenko [CS05] connecting free entropy dimension to ℓ^2-Betti numbers with [BCG03] to give

$$\delta_0(w) \leq 1 - \beta_0^{(2)}(\hat{\mathbb{G}}) + \beta_1^{(2)}(\hat{\mathbb{G}}).$$

Here, $\mathbb{G}$ can be any of O_n^+, U_n^+, and O_{2m}^{+J}, and w is then the corresponding defining representation. The eagle-eyed reader will object that w is not a tuple of self-adjoint elements, except when $\mathbb{G} = O_n^+$, but this is not important since δ_0 turns out to only depend on the $*$-algebra generated by the tuple [Voi98a].

If one plugs in the known values of the ℓ^2-Betti numbers, we find that $\delta_0(u) \leq 1$ for O_n^+ (and O_{2m}^{+J}), and that $\delta_0(v) \leq 2$ for U_n^+. This is a good start, but recall that it is unknown whether δ_0 of a tuple only depends on the generated von Neumann algebra. In the next section, we will introduce a strengthening of the inequality $\delta_0(X) \leq 1$ due to Jung which will turn out to be a W*-invariant.

3.3 Strong 1-Boundedness

Let us say that a tracial von Neumann algebra for which one has that $\delta_0(X) \leq 1$ for any generating tuple has *weak property J*. The first examples of von Neumann algebras with weak property J appeared already in Voiculescu's paper introducing δ_0 itself [Voi96]. There he shows that any tracial von Neumann algebra containing a regular diffuse hyperfinite von Neumann subalgebra (usually called a Cartan subalgebra for II_1-factors) satisfies this property, and concludes from this that the free group factors cannot have such subalgebras. He shows that the same holds if the von Neumann algebra has property Γ. Building on these ideas, Ge and Shen could show that any von Neumann algebra that is not prime has weak property J, and the same for group von Neumann algebras of certain property (T) groups [GS02].

Subsequently, major progress was made in this direction by Jung [Jun03]. Jung computed the value of δ_0 for any tuple that generates a hyperfinite von Neumann algebra, and could conclude that it is a W*-invariant in this case. Additionally, he proved that for a Connes embeddable diffuse tracial von Neumann algebra, any generating tuple X satisfies $1 \leq \delta_0(X)$. Consequently, free entropy dimension is an invariant for the class of Connes embeddable von Neumann algebras that admit a regular diffuse hyperfinite subalgebra, have property Γ, are non-prime, or are diffuse hyperfinite. Seeking to single out a more fundamental property that implied weak property J, he made the following definitions.

Definition 3.8 ([Jun07])**.** Let $(\mathcal{M}, \tau)$ be a tracial von Neumann algebra, and X an n-tuple of self-adjoint elements in $\mathcal{M}$, and let $r > 0$. The tuple X is called *r-bounded* if for ε small enough, we have the estimate

$$\chi(X + \varepsilon S : S) \leq (r - n)|\ln \varepsilon| + K,$$

for some constant $K \geq 0$ not depending on ε. If Y is another such tuple that is 1-bounded and in addition contains some Y_i with finite free entropy, then we say that Y is *strongly* 1*-bounded*.

Comparing the definitions of δ_0 and r-boundedness for a tuple X, one sees that this is indeed a strengthening of the bound $\delta_0(X) \leq r$. Jung then proved the truly remarkable result that any tracial von Neumann algebra that can be generated by a strongly 1-bounded tuple has weak property J. This class of von Neumann algebras contains all of the examples mentioned above.

In another article, Jung provided sufficient conditions for r-boundedness [Jun16], which we were reproved and generalized by Shlyakhtenko [Shl21]. The gist of their result is that whenever an n-tuple X satisfies a system of sufficiently regular algebraic relations, it is r-bounded for some r that can be computed explicitly from the relations (it is morally n minus the rank of the associated Jacobian matrix). When the von Neumann algebra is a group von Neumann algebra for a finitely generated and finitely presented group sofic Γ, Shlyakhtenko could determine that $r = 1 - \beta_0^{(2)}(\Gamma) + \beta_1^{(2)}(\Gamma)$, a combination that we have already seen above. This, among other things, recovers the property (T) examples since the first ℓ^2-Betti number of such groups always vanishes [Sha00].

It was observed by Brannan and Vergnioux that Shlyakhtenko's arguments go through for the quantum group von Neumann algebras of the Kac type free quantum groups [BV18]. Their strategy was to take the defining relations of O_n^+ and establish their regularity by connecting them to the so-called quantum Cayley tree for the discrete dual $\hat{O}_n^+$, a notion due to Vergnioux [Ver05]. Since the relevant ℓ^2-Betti numbers were known to vanish, they could conclude in this way that the standard generators of $L^\infty(O_n^+)$ form a 1-bounded set. To complete the argument and obtain strong 1-boundedness for $L^\infty(O_n^+)$, they relied on explicit computations of the spectral measures of the standard generators with respect to the Haar state due to Banica, Collins, and Zinn-Justin [BCZ09].

Thus, Brannan and Vergnioux could show that $L^\infty(O_n^+)$ is never a free group factor. However, natural follow-up questions immediately present themselves.

- Is the same true for the other Kac type free orthogonal quantum groups O_{2m}^{J+}?

- What about other universal compact quantum groups with vanishing first ℓ^2-Betti number? This includes for example the quantum permutation groups, and more generally the quantum automorphism groups for finite dimensional C*-algebras B equipped with their Plancherel trace, i.e., the unique tracial state on B that is also a δ-form [Kye+17].

- Finally, we expect that $L^\infty(U_n^+)$ is never strongly 1-bounded based on the ℓ^2-Betti number estimate, but can we prove this?

In Papers I and III we address all of these questions, and more. We provide complete answers to the first and last questions, and make significant progress on the second. The next two sections provide a summary of these two papers.

4 Summary of Paper I

In the first paper we investigate the strong 1-boundedness of the other family of Kac type free orthogonal quantum groups, which is O_{2m}^{J+}. The strategy is fundamentally the same as for O_n^+, but a few additional obstacles need to be overcome. First, the canonical generators of $L^\infty(O_{2m}^{J+})$ are not self-adjoint and so one needs a suitable choice of self-adjoint generators that keep the relations manageable and preserve the link to the quantum Cayley tree. This is achieved by introducing a decomposition of the fundamental representation in terms of Pauli matrices.

Second, detailed information about the spectral measures of the generators is not available in the case of O_{2m}^{J+}. Brannan and Vergnioux relied on [BCZ09], where the main steps are to pass to $\mathrm{SU}_q(2)$ for a suitable q and then do concrete calculations in an explicit representation of this compact quantum group. However, this approach does not carry over into the twisted setting of O_{2m}^{J+}. While the Weingarten calculus developed in [BC07] can still be applied to the twisted case to find the moments of the generators, it is not clear how to proceed as in [BCZ09] and find the corresponding measure.

Our solution is to avoid the problem entirely and instead prove a technical lemma that allows, under mild regularity conditions, to enlarge a tuple without spoiling r-boundedness. This is then used to append the character of the defining representation to the tuple of canonical generators. As we mentioned above, Banica showed that this is a semicircular element, so in particular it has finite free entropy, and we obtain a strongly 1-bounded generating set. Note that this also removes the dependence of the original proof for O_n^+ on the highly non-trivial calculations performed in [BCZ09].

5 Summary of Paper III

Paper III is in collaboration with Michael Brannan, Samuel J. Harris, and Makoto Yamashita (the author's supervisor). In this paper we develop tools to transfer various properties between compact quantum groups and their associated operator algebras and apply them to the quantum automorphism groups. The starting point consists of the following two facts. First, it is known that $\mathrm{Aut}^+(M_n)$ occurs as the 'even' part of O_n^+ [Ban99]. Recall that two compact quantum groups are said to be *monoidally equivalent* if there is a unitary monoidal equivalence between their representation categories [BRV06]. Such a monoidal equivalence can be implemented by an object called the *linking algebra*. Then the second fact is that the monoidal equivalence of the quantum automorphism groups is completely understood.

In particular, $\dim B$ is a complete invariant for the equivalence class of $\mathrm{Aut}^+(B,\psi)$ where ψ is the Plancherel trace [RV10]. Hence, if $\dim B = n$, then we obtain that $\mathrm{Aut}^+(B,\psi)$ is monoidally equivalent to the quantum permutation group S_n^+. The linking algebra implementing this for $B = M_n$ can have a finite dimensional representation, which was first noticed by Brannan–Ganesan–Harris in the context of non-local games [BGH21]. This is a rather rare property that we will leverage extensively in the paper.

In fact, we prove that if two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ are monoidally equivalent with the linking algebra admitting a finite dimensional representation, then one can transfer several finite dimensional approximation properties through this monoidal equivalence, namely residual finite-dimensionality of $\mathcal{O}(\mathbb{G}_i)$ and Connes embeddability of $L^\infty(\mathbb{G}_i)$. Since these properties are known to hold for S_n^+ [BCF20], this establishes them for all quantum automorphism groups.

Inspired by all this, we also construct 2-cocycles on S_n^+ which are induced from a finite subgroup Γ and we show that the induced cocycle twist can be used to realize all quantum automorphism groups $\mathrm{Aut}^+(B,\psi)$ with $\dim B = n$ and ψ the Plancherel trace. Moreover, this allows us to prove crossed product equivalences between such $\mathcal{O}(\mathrm{Aut}^+(B,\psi))$ and $\mathcal{O}(S_n^+)$, namely that

$$\mathcal{O}(S_n^+) \rtimes \Gamma^2 \cong \mathcal{O}(\mathrm{Aut}^+(B,\psi)) \rtimes \Gamma^2$$

as tracial $*$-algebras.

We then use these twists to also pass inner unitarity of S_n^+ [BCF20] to the other quantum automorphism groups, using a result from [BB10]. Inner unitarity is a strong form of residual finite-dimensionality, where it is asked that the Hopf $*$-algebra admits a $*$-homomorphism into some full matrix algebra M_k such that its kernel contains no non-zero Hopf $*$-ideal. Furthermore, by Takesak–Takai Duality the crossed product equivalence gives rise to finite index embeddings

$$L^\infty(\mathrm{Aut}^+(B,\psi)) \hookrightarrow M_{|\Gamma|^2} \otimes L^\infty(S_n^+).$$

In order to apply this to strong 1-boundedness, we establish the permanence of strong 1-boundedness under finite index subfactors. Therefore, the strong 1-boundedness of $L^\infty(O_n^+)$ implies strong 1-boundedness of $L^\infty(\mathrm{Aut}^+(M_n,\mathrm{tr}))$, which then implies strong 1-boundedness of any $L^\infty(\mathrm{Aut}^+(B,\psi))$ for which $\dim B$ is a square. Moreover, we realize $L^\infty(U_n^+)$ as a finite index subfactor of a von

Neumann algebra that is not strongly 1-bounded. Hence it cannot be strongly 1-bounded itself, and is hence never isomorphic to an orthogonal free quantum group factor.

We conclude the paper with another construction of the crossed product equivalence and the finite index embeddings, this time using tools from quantum information theory and non-local games. This construction has the benefit of producing explicit maps and actions, and allows us to use the group Γ itself instead of Γ^2.

6 Locally Compact Quantum Groups

So far, we have been almost exclusively discussing compact quantum groups (even of Kac type). A more general notion of a locally compact quantum group is of course desirable, since there are many such interesting classical groups, and also here there is the opportunity to find a category of group-like objects closed under Pontryagin duality. However, it turned out to be far more difficult to find the 'correct' axioms for locally compact quantum groups, and even the currently accepted ones could be said to be not entirely satisfactory.

While in the compact case the existence of the Haar state could easily be derived from the axioms, it is not known how to achieve this in the locally compact setting. Indeed, the existence of left and right invariant Haar weights has to be assumed. We now present the von Neumann algebraic definition of a locally compact quantum group due to Kustermans and Vaes.

Definition 6.1 ([KV03]). A *locally compact quantum group* $\mathbb{G}$ consists of a von neumann algebra $\mathcal{M}$, a normal unital $*$-homomorphism $\Delta\colon \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$, and two normal semifinite faithful weights φ and ψ on $\mathcal{M}$, satisfying

- *coassociativty*: as maps between $\mathcal{M}$ and $\mathcal{M}^{\otimes 3}$ we have that

$$(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta;$$

- *left invariance*: for all $x \in \mathcal{M}^+$ with $\varphi(x) < \infty$, and any normal state ω on $\mathcal{M}$, it holds that $\varphi((\omega \otimes \iota)\Delta(x)) = \varphi(x)\omega(1)$;
- *right invariance*: for all $y \in \mathcal{M}^+$ with $\psi(y) < \infty$, and any normal state σ on $\mathcal{M}$, it holds that $\psi((\iota \otimes \sigma)\Delta(y)) = \psi(y)\sigma(1)$;

For a locally compact quantum group $\mathbb{G}$, we will write $L^\infty(\mathbb{G})$ for the associated von Neumann algebra $\mathcal{M}$.

An early example of a locally compact quantum group was constructed by Woronowicz [Wor91a]. He started with the group of matrices

$$\mathrm{E}(2)^\sim = \left\{ \begin{pmatrix} v & n \\ 0 & v^* \end{pmatrix} \middle| \, v, n \in \mathbb{C},\ |v| = 1 \right\},$$

which is a double cover of the group $\mathrm{E}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ of rigid motions of the plane. Now choose $q \in (0,1)$ and consider the following two operators acting on $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$,

$$v(e_i \otimes e_j) = e_{i-1} \otimes e_j, \;\; n(e_i \otimes e_j) = q^i e_i \otimes e_{j+1}.$$

Note that the operator n is normal but unbounded. We take the von Neumann algebra generated by v and the bounded spectral projections of n as $L^\infty(\mathrm{E}_q(2)^\sim)$. Then $L^\infty(\mathrm{E}_q(2)^\sim)$ is isomorphic to $B(\ell^2(\mathbb{Z}))\bar{\otimes}\mathcal{L}(\mathbb{Z})$. The quantum group structure now comes from the coproduct

$$\Delta(v) = v \otimes v, \;\; \Delta(n) = n \otimes v \dot{+} v^* \otimes n,$$

where $\dot{+}$ is the sum of unbounded operators. This formula is dictated by the matrix presentation we started with, in the same way as for compact matrix quantum groups, but we see that we cannot interpret it in a purely algebraic way. This is a sign of many more technical difficulties to come.

An important example where many such technical obstacles arise is $\mathrm{SU}(1,1)$. We can start again with a matrix presentation, say

$$\mathrm{SU}(1,1) = \left\{ \left(\begin{array}{cc} z & \overline{w} \\ w & \overline{z} \end{array} \right) \middle| \, z, w \in \mathbb{C}, \;\; |z|^2 - |w|^2 = 1 \right\},$$

and then try to quantize as we did for $\mathrm{SU}_q(2)$, since there is 'only a minus sign difference' as

$$\mathrm{SU}(2) = \left\{ \left(\begin{array}{cc} z & -\overline{w} \\ w & \overline{z} \end{array} \right) \middle| \, z, w \in \mathbb{C}, \;\; |z|^2 + |w|^2 = 1 \right\}.$$

Unfortunately, Woronowicz discovered that this does not give a locally compact quantum group, as the tensor product of representations cannot be consistently defined [Wor91b]. It was suggested by Korogodsky [Kor94] on the basis of Poisson–Lie geometric arguments that one should consider instead the normalizer $\mathrm{SU}(1,1)^\sim$ of $\mathrm{SU}(1,1)$ inside $\mathrm{SL}(2,\mathbb{C})$. This gives a larger Hopf $*$-algebra with a canonical projection onto $\mathcal{O}(\mathrm{SU}(1,1))$, while the rest comes from the other connected component of the normalizer.

From here, it is still a difficult and delicate task to actually construct $\mathrm{SU}_q(1,1)^\sim$. It was carried out by Kustermans and Koelink in [KK03]. Korogodsky [Kor94] had found all of the relevant representations of the q-deformation of the Hopf $*$-algebraic model of $\mathrm{SU}(1,1)^\sim$, and Kustermans–Koelink 'glued' together a subset of these representations to generate an operator algebra. There are severe technical obstacles to overcome when doing this, since the representations are in terms of unbounded operators with non-unique extensions. Moreover, it turns out that one needs to include an additional operator that is 'invisible' in the Hopf $*$-algebraic picture. Its function is to connect the two parts of the decomposition coming from the canonical projection. This then gives $L^\infty(\mathrm{SU}_q(1,1)^\sim)$, but defining the rest of the locally compact quantum group structure requires extensive use of the theory of second order q-difference operators and q-hypergeometric function theory.

There are preciously few examples beyond these two. Some of these other examples arise as deformations of complex simple Lie groups, and there is a general scheme to quantize such groups [PW00]. However, no such general scheme is known for real simple Lie groups, and the extended quantum $\mathrm{SU}(1,1)$ group we have just described is in fact the only example in this class. Moreover, its construction was carried out 'by hand' and gives no clues as to how to generalize the procedure.

Hence it is all the more remarkable that there is a strong relation between $\mathrm{E}_q(2)^{\sim}$, $\mathrm{SU}_q(1,1)^{\sim}$ and $\mathrm{SU}_q(2)$. It was discovered by de Commer [Com11; Com12a; Com12b] when he investigated actions of locally compact quantum groups on type I-factors, which he calls *quantum torsors*. A natural example of such actions comes from the quantization of the action of $\mathrm{SU}(2)$ on its homogeneous space $\mathrm{SU}(2)/\mathrm{U}(1)$, which is a 2-sphere. This yields an action of $\mathrm{SU}_q(2)$ on the *quantum spheres* S^2_{qc} of Podleś for $c = 0, \infty$ [Pod87].

As the von Neumann algebric models of the coordinate algebras of the Podleś spheres are type I-factors, the action of $\mathrm{SU}_q(2)$ on them can be implemented by a unitary. To be more precise, one has a faithful normal unital $*$-homomorphism $\alpha\colon L^\infty(S^2_{qc}) \to L^\infty(\mathrm{SU}_q(2)) \otimes L^\infty(S^2_{qc})$ such that $(\iota\otimes\alpha)\alpha = (\Delta\otimes\iota)\alpha$, and then one can construct a unitary $\mathcal{G}$ such that $\alpha(x) = \mathcal{G}^*(1\otimes x)\mathcal{G}$. The quantum group structure of any locally compact quantum group can be encoded in a similar way by defining the *multiplicative unitary* W, which is such that $\Delta(y) = W^*(1\otimes y)W$ [BS93]. Combining W with $\mathcal{G}$ one can define a new locally compact quantum group which keeps the same underlying von Neumann algebra but has a new coproduct. Applying this to $\mathrm{SU}_q(2)$ acting on the *standard quantum sphere* S^2_{q0} produces $\mathrm{E}_q(2)^{\sim}$, and acting on the *equatorial quantum sphere* $S^2_{q\infty}$ yields $\mathrm{SU}_q(1,1)^{\sim}$.

After this result was established, there was hope that such techniques could be used to construct more examples of locally compact quantum groups, but this has unfortunately not materialized. Additionally, there was no classical geometric picture that could explain this quantum phenomenon and why these three particular groups show up. The motivation behind Paper II was precisely to try and establish such a picture, and for this we turn to Poisson structures on Lie groups.

7 Poisson–Lie Geometry and Deformation Quantization

Let M be a manifold, then a *Poisson structure* on M is a $\mathbb{R}$-bilinear map $\{\cdot,\cdot\}\colon C^\infty(M)\times C^\infty(M)\to C^\infty(M)$, with $C^\infty(M)$ the smooth $\mathbb{R}$-valued functions on M, such that the Poisson bracket is skew-symmetric and satisfies the Leibniz and Jacobi identities. Alternatively we can encode the Poisson structure into a bivector $\Pi \in \wedge^2 TM$ satisfying $\{f,g\} = \langle \mathrm{d}f\otimes \mathrm{d}g, \Pi\rangle$. Let F be a smooth map between Poisson manifolds N and M, then it is a *Poisson map* if it intertwines the

Poisson brackets, i.e.,

$$\{f_1 \circ F, f_2 \circ F\}_N = \{f_1, f_2\}_M \circ F$$

for all $f_1, f_2 \in C^\infty(M)$. The product $M \times N$ is also a Poisson manifold when endowed with the bracket

$$\{g_1, g_2\}_{M\times N}(p, q) = \{g_1(\cdot, q), g_2(\cdot, q)\}_M(p) + \{g_1(p, \cdot), g_2(p, \cdot)\}_N(q),$$

where $p \in M$, $q \in N$, and $g_1, g_2 \in C^\infty(M \times N)$.

The following informal example will motivate why we are considering this geometric structure. Consider a particle moving in 1 dimension in some potential V. Then classically its motion is governed by Newton's Second Law $\ddot{x} = \dot{p} = -V'(x)$, which is a second order ODE and hence requires position and momentum at an initial time to be integrated. So the phase space is $\mathbb{R}^2$ and has coordinates x and p. These two coordinate functions generate $C^\infty(\mathbb{R}^2)$, and we have the natural Poisson structure $\{x, p\} = 1$. Then Newton's second law can be packaged in terms of this Poisson structure by introducing the *Hamiltonian* $H = p^2/2 + V(x)$, where $p^2/2$ is the kinetic energy term, and saying that any function $f \in C^\infty(\mathbb{R}^2)$ on the state space evolves in time according to the differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\}.$$

Indeed, this recovers

$$\begin{aligned}\dot{x} &= \{x, p^2/2\} = p\{x, p\} = p,\\ \dot{p} &= \{p, V(x)\} = V'(x)\{p, x\} = -V'(x).\end{aligned}$$

In quantum mechanics, this system would instead be described by the Schrödinger Equation for the wave function of the particle, which lives in $L^2(\mathbb{R})$. One obtains predictions about the position and momentum of the particle by evaluating non-commuting operators X and P on the vector state given by the wave function. The failure of X and P to commute is measured by a constant denoted $\hbar$. More precisely it holds that $[X, P] = i\hbar$. The Correspondence Principle now asserts that if one sends $\hbar$ to 0, one should recover the classical description of the system. Concretely, we see that

$$\lim_{\hbar\to 0} \frac{1}{i\hbar}[X, P] = \{x, p\},$$

and the claim is that this should be true much more generally.

Formally, the idea is that if one has a family of quantum spaces $\mathbb{X}_h$ depending on some parameter h such that $h = 0$ corresponds to a classical space X, then one should interpret $C(\mathbb{X}_h)$ for $h > 0$ as being modelled on $C(X)$ with a deformed multiplication $\cdot_h$. One should of course have that $f \cdot_h g \to fg$ as h goes to zero, but according to the Correspondence Principle we should also have that

$$\lim_{h\to 0} \frac{1}{ih}(f \cdot_h g - g \cdot_h f) = \{f, g\}$$

for a Poisson bracket on X.

This limit where h goes to 0 is called the *semiclassical limit*, and the upshot of the discussion above is that the semiclassical limit of a parametrized family of quantum spaces deforming some classical space should correspond to a Poisson structure on this classical space. Since the spaces we are interested in deforming also carry a group structure, we need a refinement of the notion of a Poisson structure for groups.

Definition 7.1 ([Dri83])**.** A Lie group G is called a *Poisson–Lie group* if it is equipped with a Poisson structure such that the multiplication map $G \times G \to G$ is a Poisson map between $G \times G$ (with the product Poisson structure) and G.

Any connected compact semisimple Lie group G possesses a non-trivial (meaning the bracket is not the zero map) Poisson–Lie group structure coming from the structure theory of the complexification of its Lie algebra [LW90]. This simultaneously gives a non-trivial Poisson–Lie group structure on the split real form of the complexification. In the case of SU(2), this Poisson–Lie structure is precisely the one that is recovered in the semiclassical limit $q \to 1$ of $\mathrm{SU}_q(2)$ [LW90]. Moreover, the complexification of SU(2) is $\mathrm{SL}(2, \mathbb{C})$, whose split real form $\mathrm{SL}(2, \mathbb{R})$ is isomorphic to $\mathrm{SU}(1, 1)$. However, it should be noted that it is unknown whether $\mathrm{SU}_q(1, 1)^{\sim}$ can be viewed as a deformation quantization.

A *Poisson–Lie subgroup* of a Poisson–Lie group G is a Lie subgroup H that is simultaneously a *Poisson submanifold*, i.e., the Poisson bracket restricts to one on $C^\infty(H)$. There is a unique structure of a Poisson manifold on the quotient G/H, which is called a *Poisson homogeneous space* of G. Moreover, the translation action of G on G/H is such that the action map $G \times G/H \to G/H$ is a Poisson map, which is called a *Poisson action* of G.

We now return to SU(2) and its homogeneous space S^2. The Poisson structure Π_1 on S^2 that it inherits as a homogeneous space is SU(2)-covariant, and it turns out that all SU(2)-covariant Poisson structures on S^2 are of the form $\Pi_1 + c\Pi_0$, where $c \in \mathbb{R}$ and Π_0 is the standard symplectic (Poisson) structure on S^2 that comes from viewing $TS^2 \hookrightarrow \mathbb{R}^3$, restricting the cross product to obtain a 2-form, and dualizing to a bivector [LW90].

There is a correspondence between these Poisson structures with $c \in [0, 1]$ and the quantum spheres of Podleś. For $c = 1$, the bivector vanishes at a single point, and this partitions the sphere into a point and a complementary disk. When $0 \leq c < 1$, the bivector vanishes instead on a circle, which is of maximal diameter when $c = 0$, and this partitions the sphere into the points on this circle, and two complementary disks. These are precisely the primitive ideal spaces of the quantum spheres, with $c = 1$ corresponding to the standard quantum sphere, and $c = 0$ corresponding to the equatorial quantum sphere. The correct Poisson structures on S^2 are also obtained in the semiclassical limit, and there is a consistent deformation quantization scheme [She91].

These results already strongly suggest that Poisson–Lie geometry is an appropriate place to look for a classical shadow of de Commer's twisting result.

Moreover, Stachura recently made progress towards writing the so-called quantum κ-Poincaré group as a deformation quantization of a Poisson–Lie group [Sta17; Sta19], expanding upon work of Zakrzewski [Zak94; Zak97]. Stachura realizes the appropriate Poisson–Lie group structure as dual to the total space of a certain Lie algebroid, which has a canonical Poisson manifold structure [Cou90]. The Lie groupoid giving rise to the Lie algebroid is constructed out of a matched pair of subgroups $B, C < G$. In Paper II we investigate this recipe in greater generality, paying special attention to the case where the matched pair of subgroups comes from the Iwasawa decomposition of a real simple Lie group. Before we give a summary of Paper II, we need to introduce the infinitesimal picture for Poisson–Lie groups.

Let G be a Poisson–Lie group with Poisson bivector Π, then one can define a map $\eta\colon G \to \mathfrak{g} \otimes \mathfrak{g}$ by right-translating Π_g into $\mathfrak{g} \otimes \mathfrak{g}$ for every $g \in G$. Here $\mathfrak{g} = T_e G$ is the associated Lie algebra. The compatibility of the Poisson and Lie group structures turn the map η into a 1-cocycle on G with values in $\mathfrak{g} \otimes \mathfrak{g}$. This means that η satisfies the relation

$$\eta(gh) = \eta(g) + (\mathrm{Ad}_g \otimes \mathrm{Ad}_g)\,\eta(h).$$

Taking the derivative of this map yields the following definition.

Definition 7.2 ([Dri83])**.** A *Lie bialgebra* consists of a Lie algebra $\mathfrak{g}$ and a linear map $\delta\colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that

- δ is skew-symmetric;
- $\delta^*\colon \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie bracket on $\mathfrak{g}^*$;
- δ is a 1-cocycle on $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$, meaning

$$\delta([X, Y]) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)\delta(Y) - (\mathrm{ad}_Y \otimes 1 + 1 \otimes \mathrm{ad}_Y)\delta(X).$$

The map δ is then called the *cocommutator*.

A particularly nice class of Lie bialgebras $\mathfrak{g}$ are those for which the cocommutator δ is a 1-coboundary, that is, $\delta(X) = \delta_r(X) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)(r)$ for some $r \in \mathfrak{g} \otimes \mathfrak{g}$. Such Lie bialgebras are called *coboundary Lie bialgebras*, and the element r is usually called the *r-matrix*. There are of course some restrictions on the choice of r if δ_r is to define a 1-cocycle. These are that $r_{12} + r_{21}$ be $\mathfrak{g}$-invariant and that $[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$ also be $\mathfrak{g}$-invariant (in $\mathfrak{g}^{\otimes 3}$). The stronger condition $[[r, r]] = 0$ is the famous (classical) Yang-Baxter Equation. It is particularly easy to integrate δ_r to a 1-cocycle on G, namely one sets $\eta_r(g) = (\mathrm{Ad}_g \otimes \mathrm{Ad}_g)(r) - r$. The Poisson brackets associated to these 1-cocycles on G are called *Sklyanin brackets* [Skl82].

Another way to package a Lie bialgebra structure is the notion of a Manin triple [Dri83]. This is a triple of Lie algebras $(\mathfrak{h}, \mathfrak{h}_+, \mathfrak{h}_-)$ with a non-degenerate symmetric bilinear form on $\mathfrak{h}$ such that $\mathfrak{h}_\pm$ are Lie subalgebras of $\mathfrak{h}$ and isotropic with respect to the form, and $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$ as a vector space. In this picture, Lie bialgebra structures on $\mathfrak{g}$ are precisely the Manin triples such that $\mathfrak{h}_+ = \mathfrak{g}$. This form of the definition reveals that we can flip the roles of $\mathfrak{g}$ and $\mathfrak{g}^*$ to also obtain a Lie bialgebra structure on $\mathfrak{g}^*$. This gives rise to the notion of a *Poisson dual*. A simple observation that we will elaborate on in Paper II is that the Poisson duals of SU(2) and SU(1, 1), with their canonical Poisson–Lie structures, are isomorphic, and that there also is a Poisson–Lie structure on E(2) such that its dual is isomorphic to that of the others.

8 Summary of Paper II

Paper II is in collaboration with Makoto Yamashita, who is the supervisor of the author.

Let $B, C < G$ be a matched pair of subgroups, that is, B and C are closed subgroups of G with trivial intersection such that BC is open in G. Then one has a Lie groupoid $\mathcal{G}_B = BC \cap CB$ over B. The total space E of the dual of the associated Lie algebroid can be identified with the Lie group $B \ltimes \mathfrak{b}^0$. Here, $\mathfrak{b}^0 \subset \mathfrak{g}^*$ is the annihilator of the Lie algebra of B viewed as an additive group, on which B acts by the coadjoint action. There is a canonical Poisson structure on E due to Courant [Cou90], and we show that it in fact always defines a Poisson–Lie group structure on E. We accomplish this by showing that the map η (as above) induced by the Poisson bivector is a 1-cocycle, and we discuss an alternative proof using methods of Zakrzewski [Zak90] in the case of a double Lie group, i.e., when the matched pair is such that $BC = G$. Additionally, we explain how the groupoid C*-algebra of $\mathcal{G}_B$ can be interpreted as a deformation quantization, and when this coincides with the bicrossed product construction [VV03].

Then we investigate in detail the case when the matched pair comes from the compact and solvable parts of the Iwasawa decomposition of a real simple Lie group with finite center. Taking the compact part as the base of the groupoid, we prove that the induced Poisson–Lie groups have coboundary Lie bialgebras and we give a simple formula for their r-matrices. This is illustrated in detail with the example of SU(p, 1) for $p \geq 2$.

Another example we work out is the matched pair of U(1) and the $(ax + b)$-group inside SU(1, 1). Then the group E is isomorphic to $\mathrm{E}(2)^\sim$, but the induced Poisson–Lie structure is different from the one underlying Woronowicz' $\mathrm{E}_q(2)^\sim$. However, we present another deformation scheme on the level of Lie algebras that does simultaneously produce the correct Lie bialgebra structures on $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$, and $\mathfrak{e}(2)$. We also explain how these Lie bialgebras are related by 2-cocycles, providing a formal analogue on the level of Lie bialgebras of de Commer's result [Com11].

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Papers

Paper I

Strong 1-Boundedness of Unimodular Orthogonal Free Quantum Groups

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Abstract

Recently, Brannan and Vergnioux showed that the orthogonal free quantum group factors $\mathcal{L}\mathbb{F}O_M$ have Jung's strong 1-boundedness property, and hence are not isomorphic to free group factors. We prove an analogous result for the other unimodular case, where the parameter matrix is the standard symplectic matrix in $2N$ dimensions J_{2N}. We compute free derivatives of the defining relations by introducing self-adjoint generators through a decomposition of the fundamental representation in terms of Pauli matrices, resulting in 1-boundedness of these generators. Moreover, we prove that under certain conditions, one can add elements to a 1-bounded set without losing 1-boundedness. In particular this allows us to include the character of the fundamental representation, proving strong 1-boundedness.

Contents

I.1 Introduction

The C*-algebras and von Neumann algebras associated to discrete groups form a rich and important class of examples. The theory of discrete quantum groups, dual to Woronowicz's compact quantum groups [Wor87; Wor98], has in recent years proven itself to be another fruitful source of interesting C*-algebras and von Neumann algebras. The discrete duals of the free orthogonal and free unitary quantum groups of Van Daele and Wang [VW96; Wan95], depending on an invertible complex $N \times N$ matrix parameter Q, have been particularly well studied.

Write $\mathbb{F}O(Q)$ for the orthogonal free quantum group associated to a general Q and let J_{2N} be the standard symplectic matrix in $2N$ dimensions. We will use the notations $\mathbb{F}O_N = \mathbb{F}O(I_N)$ and $\mathbb{F}O_{2N}^J = \mathbb{F}O(J_{2N})$ for the unimodular orthogonal free quantum groups. These two cases are of particular interest, as their associated quantum group von Neumann algebras $\mathcal{L}\mathbb{F}O_N$ and $\mathcal{L}\mathbb{F}O_{2N}^J$ share many properties with the free group factors [Ban96; BC07; Bra12; Bra14; Cas21; CFY14; FV15; Fre13; Iso15; VV07]. Whether or not they could be isomorphic to a free group factor $\mathcal{L}\mathbb{F}_M$ remained open for over 20 years, until it was recently settled for $Q = I_N$ by Brannan and Vergnioux [BV18]. They distinguish $\mathcal{L}\mathbb{F}O_N$ from the free group factors by proving that it satisfies strong 1-boundedness, a free probabilistic property due to Jung [Jun07]. The main result of the present paper is that this property also holds when $Q = J_{2N}$.

Theorem (See Theorem I.5.1)**.** *The orthogonal free quantum group von Neumann algebras $\mathcal{L}\mathbb{F}O_{2N}^J$ are strongly 1-bounded for $N \geq 2$.*

Combined with the work of Brannan and Vergnioux, this yields the following corollary.

Corollary. *Let $Q \in \mathrm{GL}_N(\mathbb{C})$, $N \geq 3$, be such that $Q\overline{Q} \in \mathbb{C}I_N$ and such that $\mathbb{F}O(Q)$ is unimodular. Then $\mathcal{L}\mathbb{F}O(Q)$ is not isomorphic to any finite von Neumann algebra admitting a tuple of self-adjoint generators whose (modified) free entropy dimension exceeds 1. In particular this excludes being isomorphic to any (interpolated) free group factor.*

Evidence pointing towards this outcome had already appeared in the literature. Vergnioux [Ver12] and Bichon [Bic13] proved that the first L^2-Betti number vanishes for both $\mathbb{F}O_N$ and $\mathbb{F}O_{2N}^J$. Using this, it can be shown that Voiculescu's modified microstates free entropy dimension δ_0 and non-microstates free entropy dimension δ^* [Voi96; Voi98] give different results for the canonical set of generators in $\mathcal{L}\mathbb{F}O_N$ or $\mathcal{L}\mathbb{F}O_{2N}^J$, and $\mathcal{L}\mathbb{F}_M$ respectively [BCV17].

It is unknown whether or not free entropy dimension is a von Neumann algebra invariant in general, but this is the case for strongly 1-bounded von Neumann algebras [Jun07]. In a finite von Neumann algebra $\mathcal{M}$ with faithful normal tracial state τ, a finite tuple $X_1, \dots, X_n \in \mathcal{M}$ of self-adjoint elements is called 1-*bounded* (without the 'strong') if it satisfies a condition that is slightly stronger than $\delta_0(X_1, \dots, X_n) \leq 1$ (see Section I.2.4). If $\mathcal{M}$ admits self-adjoint generators

$X_1, \dots, X_n$ that form a 1-bounded tuple, and at least one of the X_i has finite free entropy, $\mathcal{M}$ is said to be *strongly* 1*-bounded*. Jung introduced these definitions and showed that for a strongly 1-bounded von Neumann algebra $\mathcal{N}$, any finite set of self-adjoint generators $Y_1, \dots, Y_m \in \mathcal{N}$ must satisfy $\delta_0(Y_1, \dots, Y_m) \leq 1$. This forbids $\mathcal{N}$ being isomorphic to any interpolated free group factor $\mathcal{L}\mathbb{F}_r$ for $1 < r \leq \infty$ [Jun07, Section 3].

Checking directly that the canonical generators of $\mathcal{L}\mathbb{F}O_N$ and $\mathcal{L}\mathbb{F}O_{2N}^J$ form a 1-bounded set turns out to be difficult. Instead, the strategy of [BV18] for $\mathbb{F}O_N$ relies on results of Jung [Jun16] and Shlyakhtenko [Shl21]. The quantum group von Neumann algebra $\mathcal{L}\mathbb{F}O_N$ has N^2 self-adjoint operators $u = (u_{ij})_{i,j=1}^N$ as its canonical set of generators. These generators satisfy some polynomial relations F, i.e. $F(u) = 0$ in $\mathcal{L}\mathbb{F}O_N$. One then considers the free derivatives $\partial F(u)$ of the relations F with respect to the generators u_{ij}. The results of Jung and Shlyakhtenko now say that in order to conclude 1-boundedness of u, it is sufficient to prove that the operator $D = \partial F(u)^* \partial F(u)$ is of determinant class and has rank $N^2 - 1$ (see Section I.2.4 for details).

Brannan and Vergnioux achieve this by computing the operator D and relating it to something called the *edge-reversing operator* on the *quantum Cayley tree* due to Vergnioux [Ver05; Ver12]. Regularity results for this edge-reversing operator are proved in [BV18] for many $\mathbb{F}O(Q)$, including the cases $Q = I_N, J_{2N}$. The computation of the rank of D proceeds by expressing the rank in terms of L^2-Betti numbers, which are known for all orthogonal free quantum groups. To complete the proof, there are calculations by Banica, Collins, and Zinn-Justin [BCZ09] which imply that every u_{ij} individually has finite free entropy.

There are two obstacles to generalising this proof to the case of $\mathbb{F}O_{2N}^J$. The first is that the canonical generators are no longer self-adjoint, complicating the determination of ∂F. We will remedy this by choosing a convenient set of self-adjoint generators using a decomposition of the fundamental representation in terms of Pauli matrices, which have simple algebraic properties and relations. Fortunately, the connection to the edge-reversing operator remains intact, allowing us to conclude that our new set of generators is 1-bounded.

The second obstacle is that calculations like [BCZ09] are not available for $\mathbb{F}O_{2N}^J$. We sidestep this by proving a technical result of independent interest, inspired by a relative free entropy estimate due to Voiculescu [Voi96]. This lemma states that under certain regularity conditions, one is allowed to add redundant elements to a generating set without spoiling 1-boundedness. This works in particular if the redundant element is a noncommutative polynomial in the generators. It is a result of Banica that the character of the fundamental representation of $\mathbb{F}O_{2N}^J$ is a semicircular element [Ban96], and hence possesses finite free entropy. As the fundamental character is a linear combination of generators, we have completed the proof. Note that this method also applies to $\mathbb{F}O_N$, removing the dependence on the non-trivial results of [BCZ09].

The remainder of this paper is structured as follows. In Section I.2, we recall the necessary facts and definitions about orthogonal free quantum groups, their corepresentation theory, quantum Cayley graphs, and free probability. In Section I.3, we introduce generators for $\mathcal{L}\mathbb{F}O_{2N}^{J}$, compute their free derivatives, and show how this results in 1-boundedness. In Section I.4, we prove a technical lemma stating conditions under which one is allowed to enlarge a 1-bounded set without destroying 1-boundedness. Finally, in Section I.5 we prove our main result and discuss some consequences.

I.2 Preliminaries

We will keep our notations and conventions close to [BV18]. Generally, the letters H, K, and L represent (separable) Hilbert spaces, and $\mathcal{K}(H)$ or $\mathcal{U}(H)$ denotes the compact or unitary operators on the Hilbert space H respectively. All von Neumann algebras are assumed to have a separable predual. We write $H \otimes K$ for the tensor product of Hilbert spaces, and the same symbol is also used for the minimal tensor product of C*-algebras. Put Σ for the map $H \otimes K \to K \otimes H$ that flips the tensor legs. The Greek letter ι will be used as a generic symbol for any identity map. We will also make use of leg numbering notation, which we will explain by example. If x, y are elements of a unital algebra $\mathcal{A}$, then $\mathcal{A}^{\otimes 3} \ni (x \otimes y)_{31} = y \otimes 1 \otimes x$, while $\mathcal{A}^{\otimes 4} \ni (x \otimes y)_{13} = x \otimes 1 \otimes y \otimes 1$, and so on. It will always be clear from the context in which space the tensors lie. For an operator V on $H \otimes H$, we have for instance that $V_{32} = \iota \otimes (\Sigma V \Sigma)$ on $H \otimes H \otimes H$. We write I_N for the $N \times N$ identity matrix and J_{2N} denotes the standard $2N \times 2N$ symplectic matrix

$$J_{2N} = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}.$$

I.2.1 Orthogonal Free Quantum Groups

For brevity, we will discuss discrete quantum groups within the context of $\mathbb{F}O(Q)$.

Definition I.2.1. Let $N \geq 2$ and $Q \in \mathrm{GL}_N(\mathbb{C})$ such that $Q\overline{Q} \in \mathbb{C}I_N$, where the bar denotes taking the adjoint (i.e. complex conjugate) entry-wise. Then the *orthogonal free* quantum group $\mathbb{F}O(Q)$ is given by the unital Woronowicz C*-algebra

$$C^*\mathbb{F}O(Q) = \left\langle u_{ij} \;\middle|\; 1 \leq i, j \leq N,\ u \text{ unitary},\ Q\overline{u}Q^{-1} = u \right\rangle, \tag{I.1}$$

where u denotes the matrix $(u_{ij})_{ij} \in M_N(\mathbb{C}) \otimes C^*\mathbb{F}O(Q)$. The matrix u is the *fundamental representation* of $\mathbb{F}O(Q)$, and the *coproduct* $\Delta\colon C^*\mathbb{F}O(Q) \to C^*\mathbb{F}O(Q) \otimes C^*\mathbb{F}O(Q)$ takes the form

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}$$

on its entries. The coproduct Δ is a co-associative unital $*$-homomorphism satisfying the *cancellation property* that the subspaces

$$\operatorname{span}\{(x \otimes 1)\Delta(y) \mid x, y \in C^*\mathbb{F}O(Q)\} \subset C^*\mathbb{F}O(Q) \otimes C^*\mathbb{F}O(Q),$$

$$\operatorname{span}\{(1\otimes x)\Delta(y) \mid x, y \in C^*\mathbb{F}O(Q)\} \subset C^*\mathbb{F}O(Q)\otimes C^*\mathbb{F}O(Q),$$

are dense.

These algebras come with a unique invariant state h, called the *Haar* state, where invariance means that $(h\otimes \iota)\Delta(x) = h(x)1 = (\iota\otimes h)\Delta(x)$ for all $x \in C^*\mathbb{F}O(Q)$. If h is a trace, then $\mathbb{F}O(Q)$ is said to be *unimodular*. It is known (see [Bra17, Section 9.1]) that $\mathbb{F}O(Q)$ is unimodular when either $Q = I_N$ or $Q = J_{2N}$ (up to isomorphism). Hence we introduce the special notations $\mathbb{F}O_N = \mathbb{F}O(I_N)$ and $\mathbb{F}O^J_{2N} = \mathbb{F}O(J_{2N})$.

One also has an involutive $*$-anti-automorphism R of $C^*\mathbb{F}O(Q)$ such that $\Delta R = (R\otimes R)\Sigma\Delta$, called the *unitary antipode*. The ordinary *antipode* S is an anti-automorphism of the $*$-algebra generated by the u_{ij} with the property that $(\iota\otimes S)(u) = u^*$. In the unimodular case, the maps R and S are the same.

Applying the GNS construction to the Haar state h gives a Hilbert space $\ell^2\mathbb{F}O(Q) = H_Q$ with canonical cyclic unit vector ξ_0 implementing h as a vector state. This representation gives rise to the *reduced* quantum group C*-algebra $C^*_r\mathbb{F}O(Q)$ and the *quantum group von Neumann algebra* $\mathcal{L}\mathbb{F}O(Q)$ in the usual ways.

On $C^*_r\mathbb{F}O(Q)$, the comultiplication Δ is implemented by an operator $V \in \mathcal{U}(H_Q\otimes H_Q)$ as $\Delta(y) = V(y\otimes 1)V^*$. This *multiplicative unitary* V is defined explicitly by $V(x\xi_0\otimes y\xi_0) = \Delta(x)(1\otimes y)(\xi_0\otimes\xi_0)$ for $x, y \in C^*\mathbb{F}O(Q)$, and witnesses the *pentagon equation* $V_{12}V_{13}V_{23} = V_{23}V_{12}$. The unitary antipode R descends to give an involutive unitary U on H_Q by $U(x\xi_0) = R(x)\xi_0$ for $x \in C^*\mathbb{F}O(Q)$.

We recall some facts about the orthogonal free quantum groups and the parallels to the free group factors on the von Neumann algebraic level. If one takes an identity matrix I_N in the Definition (I.1) above, the orthogonal free quantum groups $\mathbb{F}O_N$ are obtained. This family is both a liberation of $C(O_N)$ and its diagonal quotient (setting all off-diagonal elements to zero) are related to the full group C*-algebra of the N-fold free product group $\mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ [Wan95]. This explains the $\mathbb{F}$ and the O appearing in $\mathbb{F}O_N$.

As we are taking the point of view of discrete quantum groups, we use the notation $C^*\mathbb{F}O_N$ to underline the analogy with the full group C*-algebra mentioned above. If one takes the point of view of compact quantum groups instead, the notation $C^*\mathbb{F}O_N = C^u(O^+_N)$ is more natural in light of the relation to the orthogonal group O_N. The original notation $A_o(N)$ (and more generally $A_o(Q)$) of van Daele and Wang is also common. For general Q, we have a family of deformations of this Woronowicz C*-algebra that still satisfy many of the same properties.

The analogy with free groups becomes stronger when one considers approximation properties. It is a result of Banica [Ban96] that $\mathbb{F}O(Q)$ is 'generically' non-amenable, that is if and only if $N \geq 3$. De Commer, Freslon, and Yamashita [CFY14] proved that $\mathbb{F}O(Q)$ has the Haagerup property and is weakly amenable with Cowling–Haagerup constant 1 (also referred to as the CCAP or CMAP), generalising results by Brannan [Bra12] and Freslon [Fre13].

This trend continues on the von Neumann algebraic level. By [Cas21; FV15; Iso15] it holds that $\mathcal{L}\mathbb{F}O(Q)$ is strongly solid and has no Cartan subalgebra. With some restrictions on Q, Vaes and Vergnioux [VV07] showed that $\mathcal{L}\mathbb{F}O(Q)$ is a full factor and hence prime. In particular, if $QQ^* = I_N$ and $N \geq 3$, then $\mathcal{L}\mathbb{F}O(Q)$ is a factor of type II_1. Recall that $\mathbb{F}O(Q)$ is unimodular for $Q = I_N, J_{2N}$. Thus the analogy between the orthogonal free quantum group von Neumann algebras $\mathcal{L}\mathbb{F}O_N$ and $\mathcal{L}\mathbb{F}O^J_{2N}$ on one hand and the free group factors $\mathcal{L}\mathbb{F}_M$ on the other is especially striking. It was even shown that the series $\{\mathcal{L}\mathbb{F}O_N\}$ has free group factor-like asymptotics in a strong sense [BC07; Bra14].

I.2.2 Corepresentations

All constructions in this section are general, but we state them for $\mathbb{F}O(Q)$. We refer to [NT13] for the general theory of the representation categories of discrete and compact quantum groups.

A *unitary corepresentation* of $\mathbb{F}O(Q)$ on a Hilbert space H is defined as a unitary operator v which lies in the multiplier algebra $M(\mathcal{K}(H) \otimes C^*\mathbb{F}O(Q))$ and which interacts with the comultiplication as

$$(\iota \otimes \Delta)v = v_{12}v_{13} \in M(\mathcal{K}(H) \otimes C^*\mathbb{F}O(Q) \otimes C^*\mathbb{F}O(Q)).$$

The fundamental representation u and the multiplicative unitary V are important examples.

Taking all finite dimensional unitary corepresentations of $\mathbb{F}O(Q)$ as objects and their intertwiners as morphisms yields a rigid C*-tensor category when equipped with the obvious direct sum and the tensor product $v \otimes w = v_{13}w_{23}$. Write v_{triv} for the trivial corepresentation on $\mathbb{C}$ represented by $1 \in C^*\mathbb{F}O(Q)$, and choose a set of representatives $\mathrm{Irr}(Q)$ of the irreducible corepresentations such that u and v_{triv} are among them. If $v \in \mathrm{Irr}(Q)$, write H_v for its Hilbert space.

The algebraic direct sum $\bigoplus_{v\in\mathrm{Irr}(Q)} B(H_v)$ is dense in H_Q. Restricting the multiplicative unitary V to this subspace gives the decomposition $V = \sum_{v\in\mathrm{Irr}(Q)} v$ acting by left multiplication. Using the c_0 direct sum instead, one forms the *dual algebra*

$$c_0(\mathbb{F}O(Q)) = c_0(Q) = \bigoplus_{v\in\mathrm{Irr}(Q)}^{c_0} B(H_v),$$

again acting by left multiplication on the subspace defined above. It turns out that $V \in M(c_0(Q) \otimes C^*_r\mathbb{F}O(Q))$. There are two minimal central projections $p_0, p_1 \in Z(M(c_0(Q)))$ such that $p_0H_Q = B(H_{v_{\mathrm{triv}}}) \cong \mathbb{C}\xi_0$ and $p_1H_Q = B(H_u) \cong M_N(\mathbb{C})$. Note that $p_0p_1 = 0$ and $Up_1 = p_1U$.

I.2.3 Quantum Cayley Trees

To the pair $\mathbb{F}O(Q)$ and p_1, one can associate a *quantum Cayley tree* [Ver05]. This consists of the following four pieces of data. We have the Hilbert spaces H_Q and

$K_Q = H_Q \otimes p_1 H_Q$, to be thought of as the vertex and edge spaces respectively. There is a bounded linear operator E from K_Q to $H_Q \otimes H_Q$, called the *boundary operator*, given by restricting the multiplicative unitary V to K_Q. Finally, we have the important *edge-reversing operator* $\Theta = \Sigma(1 \otimes U)V(U \otimes U)\Sigma \in B(K_Q)$ (this uses $Up_1 = p_1 U$). Note that Θ need not be involutive, but it is unitary.

Let us explain how this generalises the classical Cayley graph. Let G be a discrete group, and consider its group C*-algebra C^*G with the coproduct $\Delta(g) = g \otimes g$. It is easy to see that Δ is *cocommutative*, that is $\Sigma\Delta = \Delta$. A standard fact in this context is that the unitary antipode R is given by $R(g) = g^{-1}$. Passing to the reduced group C*-algebra $C^*_r G$, we write $\{\delta_g\}$ for the orthonormal basis of $\ell^2 G$ given by the point-indicator sequences, and $\lambda\colon C^*G \to B(\ell^2 G)$ for the left regular representation. The definition of the multiplicative unitary V becomes

$$V(\lambda_g\delta_e \otimes \lambda_h\delta_e) = (\lambda_g \otimes \lambda_g)(1 \otimes \lambda_h)(\delta_e \otimes \delta_e) = (\lambda_g\delta_e \otimes \lambda_{gh}\delta_e).$$

The vertex Hilbert space is now just $\ell^2 G$. The right analogue of p_1 in this context turns out to be the indicator sequence of a set $H \subset G$, not containing the neutral element e and closed under inverses. As the boundary operator E is just a restriction of V, we see that the 'boundary' of an edge $(\delta_g \otimes \delta_h)$ is $(\delta_g \otimes \delta_{gh})$. Thus we should view $(\delta_g \otimes \delta_h)$ as an edge in the classical Cayley graph that starts at g, and whose endpoint is given by right translating by h, i.e. gh. Accordingly, the edge-reversing operator acts as

$$\begin{aligned}\Theta(\delta_g \otimes \delta_h) &= \Sigma(1 \otimes U)V(\delta_{h^{-1}} \otimes \delta_{g^{-1}})\\ &= \Sigma(1 \otimes U)(\delta_{h^{-1}} \otimes \delta_{h^{-1}g^{-1}})\\ &= \delta_{gh} \otimes \delta_{h^{-1}}.\end{aligned}$$

I.2.4 Free Probability and Determinant Class Operators

Throughout this section $(\mathcal{M}, \tau)$ is a finite von Neumann algebra with faithful normal tracial state τ. Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ be self-adjoint elements in $\mathcal{M}$. In [Voi94], Voiculescu introduced the *microstates free entropy* $\chi(X_1, \dots, X_n)$. This relies on the notion of microstates $\Gamma(X_1, \dots, X_n; \ell, k, \varepsilon)$ of $X_1, \dots, X_n$, which are n-tuples of $k \times k$ self-adjoint complex matrices that approximate the moments of the X_i up to degree ℓ within precision ε. The microstates free entropy χ is then a normalised limit over the logarithm of the volume of sets of microstates.

For later use, we state a finiteness result for the microstates free entropy of a single self-adjoint element $X \in \mathcal{M}$.

Lemma I.2.2. *Let $X = X^* \in \mathcal{M}$ and write μ_X for its spectral distribution with respect to τ. If μ_X admits an essentially bounded density with respect to the Lebesgue measure on $\mathbb{R}$, then $\chi(X)$ is finite.*

This is a direct consequence of the formula

$$\chi(X) = \iint \log|s - t| \mathrm{d}\mu_X(s)\mathrm{d}\mu_X(t) + \frac{3}{4} + 2^{-1}\log(2\pi),$$

which can be found in Proposition 4.5 of [Voi94].

We next recall the *relative* microstates free entropy

$$\chi(X_1, \dots, X_n : Y_1, \dots, Y_m)$$

from [Voi96]. This is defined in the same way, except one considers *relative* microstates $\Gamma(X_1, \dots, X_n : Y_1, \dots, Y_m; \ell, k, \varepsilon)$. These are the projections onto the first n factors of the microstates $\Gamma(X_1, \dots, X_n, Y_1, \dots, Y_m; \ell, k, \varepsilon)$. We record some of its properties that will be used later.

Proposition I.2.3. *The relative microstates free entropy satisfies*

- *Domination by the microstates free entropy and global upper bound*

$$\begin{aligned}\chi(X_1, \dots, X_n : Y_1, \dots, Y_n) &\leq \chi(X_1, \dots, X_n) \\ &\leq \frac{n}{2} \log \left[\frac{2\pi e}{n} \tau \left(X_1^2 + \cdots + X_n^2 \right) \right].\end{aligned}$$

- χ *is 'subadditive'*

$$\begin{aligned}\chi(X_1, \dots, X_n : Y_1, \dots, Y_m) \leq& \chi(X_1, \dots, X_p : X_{p+1}, \dots, X_n, Y_1, \dots, Y_m) \\ &+ \chi(X_{p+1}, \dots, X_n : X_1, \dots, X_p, Y_1, \dots, Y_m).\end{aligned}$$

- *Let $Z_1, \dots, Z_q \in \mathcal{M}$ be self-adjoint and lying in the von Neumann algebra generated by $Y_1, \dots, Y_m$, then*

$$\chi(X_1, \dots, X_n : Y_1, \dots, Y_m) \leq \chi(X_1, \dots, X_n : Z_1, \dots, Z_q).$$

- *If $Y_p, \dots, Y_m$ lie in the von Neumann algebra generated by $X_1, \dots, X_n$ and $Y_1, \dots, Y_{p-1}$, we have*

$$\chi(X_1, \dots, X_n : Y_1, \dots, Y_m) = \chi(X_1, \dots, X_n : Y_1, \dots, Y_{p-1}).$$

This leads us to the definition of the *modified free entropy dimension* δ_0 [Voi96]. Without loss of generality (replacing $\mathcal{M}$ by a free product if necessary) we can assume that there is a free family of standard semicircular elements $S_1, \dots, S_n$ that are also free from the X_i. Now define

$$\delta_0(X_1, \dots, X_n) = n + \limsup_{\varepsilon \to 0} \frac{\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n : S_1, \dots, S_n)}{|\log \varepsilon|}. \tag{I.2}$$

It turns out that $\delta_0(X_1, \dots, X_n) \leq n$, and this inequality is saturated when the X_i form a free standard semicircular family. Thus the free group factor $\mathcal{L}\mathbb{F}_M$ admits an M-tuple of generators such that their modified free entropy dimension is precisely M.

An important goal of free probability theory is to decide whether δ_0 is a von Neumann algebraic invariant. That is, is it true that when $X_1, \dots, X_n$ and

$Y_1, \ldots, Y_m$ generate isomorphic von Neumann algebras, then $\delta_0(X_1, \ldots, X_n) = \delta_0(Y_1, \ldots, Y_m)$? An affirmative answer to this would solve the long-standing free group factor isomorphism problem.

Jung made progress in this direction when he introduced the notion of strong 1-boundedness and showed that every generating set of a strongly 1-bounded von Neumann algebra has modified free entropy dimension less than 1 [Jun07]. Hence any such von Neumann algebra is not isomorphic to a free group factor $\mathcal{L}\mathbb{F}_M$ with $M \geq 2$. The most convenient definition in our case is not the original one, but rather the equivalent final bullet point of Corollary 1.4 in [Jun07].

Definition I.2.4. Let $\alpha > 0$, then $X_1, \ldots, X_n$ is *α-bounded* if

$$\limsup_{\varepsilon \to 0} \left[\chi(X_1 + \varepsilon S_1, \ldots, X_n + \varepsilon S_n \colon S_1, \ldots, S_n) + (n - \alpha)|\log \varepsilon|\right] < \infty. \quad \text{(I.3)}$$

If in addition to being 1-bounded, at least one of the X_i satisfies $\chi(X_i) > -\infty$, we say that $X_1, \ldots, X_n$ are *strongly 1-bounded*.

Comparing (I.3) with the definition (I.2) of δ_0, one sees that α-boundedness is a strengthening of the estimate $\delta_0(X_1, \ldots, X_n) \leq \alpha$. An alternate way to state the definition of α-boundedness is to say that for small ε there is a constant $K \geq 0$, depending only on the X_i, such that

$$\chi(X_1 + \varepsilon S_1, \ldots, X_n + \varepsilon S_n \colon S_1, \ldots, S_n) \leq (\alpha - n)|\log \varepsilon| + K.$$

Recalling Lemma I.2.2, upgrading 1-boundedness to strong 1-boundedness can be achieved by showing that one of X_i has a sufficiently regular spectral measure μ_{X_i}.

Remark I.2.5. There is another approach to defining a free notion of entropy, called χ^*, also due to Voiculescu [Voi98]. Instead of going through microstates, χ^* is defined through the notions of conjugate variables and free Fisher information. This leads to a non-microstates free entropy dimension δ^*, and an analogous definition of α-boundedness for δ^*. It is a deep result of Biane, Capitaine, and Guionnet [BCG03] that $\chi^*(X_1, \ldots, X_n) \geq \chi(X_1, \ldots, X_n)$ (and so also larger than the relative microstates free entropy). Consequently, α-boundedness for δ^* implies α-boundedness for δ_0.

In the remainder of this section, let us introduce some terminology necessary to state a result of Jung [Jun16] reproved by Shlyakhtenko [Shl21].

Let $T_1, \ldots, T_n$ be formal noncommuting indeterminates, and write $\mathbb{C}\langle T_1, \ldots, T_n\rangle$ for their unital algebra of noncommutative polynomials. For each $1 \leq i \leq n$, define a map

$$\partial_i \colon \mathbb{C}\langle T_1, \ldots, T_n\rangle \to \mathbb{C}\langle T_1, \ldots, T_n\rangle \otimes \mathbb{C}\langle T_1, \ldots, T_n\rangle,$$

by the relations

$$\partial_i T_j = \delta_{ij}(1 \otimes 1), \qquad \partial_i(P_1 P_2) = (\partial_i P_1)(1 \otimes P_2) + (P_1 \otimes 1)(\partial_i P_2),$$

where $P_1, P_2 \in \mathbb{C}\langle T_1, \dots, T_n\rangle$. When we equip $\mathbb{C}\langle T_1, \dots, T_n\rangle^{\otimes 2}$ with the $\mathbb{C}\langle T_1, \dots, T_n\rangle$-bimodule structure $P_1 \cdot (P_2 \otimes P_3) \cdot P_4 = (P_1 P_2 \otimes P_3 P_4)$, the ∂_i become derivations.

For a vector of such polynomials $P = (P_1, \dots, P_m) \in \mathbb{C}\langle T_1, \dots, T_n\rangle^m$, we define

$$\partial P = \sum_{i=1}^{n} \sum_{j=1}^{m} (\partial_i P_j) \otimes e_j \otimes e_i^* \in \mathbb{C}\langle T_1, \dots, T_n\rangle^{\otimes 2} \otimes M_{m\times n}(\mathbb{C}).$$

We now want to evaluate such expressions in self-adjoint $X_1, \dots, X_n \in \mathcal{M}$, where $\mathcal{M}$ is still a finite von Neumann algebra with faithful normal tracial state τ. This results in $\partial P(X_1, \dots, X_n)$, which we view as an element in $\mathcal{M} \otimes \mathcal{M}^{\mathrm{op}} \otimes M_{m\times n}(\mathbb{C})$. Equip $L^2\mathcal{M} \otimes L^2\mathcal{M}^{\mathrm{op}}$ with the right $\mathcal{M} \otimes \mathcal{M}^{\mathrm{op}}$-module structure $(\xi \otimes \eta)\cdot(x \otimes y^{\mathrm{op}}) = (\xi x \otimes y^{\mathrm{op}}\eta)$. Then $\partial P(X_1, \dots, X_n)$ is a bounded right $\mathcal{M} \otimes \mathcal{M}^{\mathrm{op}}$-module map from $L^2\mathcal{M} \otimes L^2\mathcal{M}^{\mathrm{op}} \otimes \mathbb{C}^n$ to $L^2\mathcal{M} \otimes L^2\mathcal{M}^{\mathrm{op}} \otimes \mathbb{C}^m$. Consequently, we can define the *rank* of $\partial P(X_1, \dots, X_n)$, denoted $\mathrm{rank}(\partial F(X_1, \dots, X_n))$, as the Murray–von Neumann dimension of the closure of its image.

Finally, recall that when $A \in M_n(\mathbb{C})$ is strictly positive we have the identity

$$\det(A) = \exp(\mathrm{Tr}(\log(A))).$$

This motivates the definition of the *Fuglede–Kadison–Lück determinant* $\det_{FKL}$ on $(\mathcal{M}, \tau)$. Let $x \in \mathcal{M}$, and write $\mu_{|x|}$ for the spectral distribution of $|x|$ with respect to τ. Then

$$\det_{FKL}(x) = \exp\left(\int_{0^+}^{\infty} \log(s) \mathrm{d}\mu_{|x|}(s)\right),$$

when the integral is finite, and zero else. We say that x is of *determinant class* (with respect to τ) if $\det_{FKL}(x) \neq 0$.

Theorem I.2.6 ([Jun16, Theorem 6.9] and [Shl21, Theorem 2.5])**.** *Let $\mathcal{M}$ be a finite von Neumann algebra with faithful normal tracial state τ, and $X_1, \dots, X_n \in \mathcal{M}$ self-adjoint. Assume that there is a vector $F \in \mathbb{C}\langle T_1, \dots, T_n\rangle^m$ such that*

$$F(X_1, \dots, X_n) = 0 \text{ and } \det_{FKL}\left[\partial F(X_1, \dots, X_n)^* \partial F(X_1, \dots, X_n)\right] \neq 0.$$

Then it holds that $X_1, \dots, X_n$ are α-bounded (for both δ_0 and δ^*) with

$$\alpha = n - \mathrm{rank}\left(\partial F(X_1, \dots, X_n)\right).$$

I.3 Generators, Relations, and 1-Boundedness

I.3.1 Generators

We now fix $Q = J_{2N}$ and consider $\mathbb{F}O_{2N}^J = \mathbb{F}O(J_{2N})$. Recall the $2N \times 2N$ matrix of canonical generators u. Let us split u up into four $N \times N$ pieces as

$$u = \begin{pmatrix} u_{(1)} & u_{(2)} \\ u_{(3)} & u_{(4)} \end{pmatrix}.$$

Writing out the last relation in the definition (I.1) of $C^*\mathbb{F}O_{2N}^J$, one obtains

$$\begin{pmatrix} u_{(1)} & u_{(2)} \\ u_{(3)} & u_{(4)} \end{pmatrix} = \begin{pmatrix} \overline{u_{(4)}} & -\overline{u_{(3)}} \\ -\overline{u_{(2)}} & \overline{u_{(1)}} \end{pmatrix}.$$

Therefore, u must be of the form

$$u = \begin{pmatrix} A^u + iC^u & B^u + iD^u \\ -B^u + iD^u & A^u - iC^u \end{pmatrix}, \tag{I.4}$$

where $A^u, \dots, D^u$ are $N \times N$ matrices of self-adjoint operators (consisting of real and imaginary parts of the canonical generators) from $C^*\mathbb{F}O_{2N}^J$. Thus $\overline{A^u} = A^u$, and so on, and we write $(A^u)_{ij} = a_{ij}^u$ $(1 \le i, j \le N)$, and so on. The reasons for this slightly clunky notation will become clear in the next section. We use the convention that the alphabetical indices $i, j, k, \cdots$ run from 1 to N, and Greek indices from the beginning of the alphabet (e.g., $\alpha, \beta, \gamma, \dots$) run over $\{a, b, c, d\}$. Motivated by the above, we will usually interpret $M_{2N}(\mathbb{C}) \cong M_2(\mathbb{C}) \otimes M_N(\mathbb{C})$.

The above form (I.4) for u can be nicely expressed in terms of the matrices

$$\tau_a = I_2, \qquad \tau_b = i\sigma_y, \qquad \tau_c = i\sigma_z, \qquad \tau_d = i\sigma_x,$$

where $\sigma_{x,y,z}$ are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Namely,

$$\begin{aligned} u &= \tau_a A^u + \tau_b B^u + \tau_c C^u + \tau_d D^u \\ &= \sum_{ij,\alpha} \left(\tau_\alpha \otimes E_{ij} \otimes \alpha_{ij}^u \right) \\ &= \sum_{ij,\alpha} \left(E_{ij}^\alpha \otimes \alpha_{ij}^u \right). \end{aligned} \tag{I.5}$$

Here, we have suppressed the tensor products in the first equality (an abuse of notation we will keep committing), used the standard matrix units $E_{ij} \in M_N(\mathbb{C})$ in the second, and defined $E_{ij}^\alpha = \tau_\alpha \otimes E_{ij}$ in the last. Thus we are using the E_{ij}^α as our basis for $M_{2N}(\mathbb{C})$. Notice that in this form

$$\begin{aligned} u^* &= \tau_a (A^u)^t - \tau_b (B^u)^t - \tau_c (C^u)^t - \tau_d (D^u)^t \\ &= \sum_{ij} \left(E_{ij}^a \otimes a_{ji}^u - E_{ij}^b \otimes b_{ji}^u - E_{ij}^c \otimes c_{ji}^u - E_{ij}^d \otimes d_{ji}^u \right). \end{aligned} \tag{I.6}$$

Remark I.3.1. As an aside, it already follows from the proof of Theorem 5.1 in [BCV17] that δ_0 and δ^* of this set of generators is 1 (but 1-boundedness is of course slightly stronger than this). To see this, note that the inequality (13) above the aforementioned theorem collapses due to the vanishing of the L^2-Betti numbers of $\mathbb{F}O^J_{2N}$ [Bic13]. To obtain Connes embeddability of $\mathcal{L}\mathbb{F}O^J_{2N}$, notice that it lies inside the graded twist $\mathcal{L}\mathbb{F}O_{2N} \rtimes \mathbb{Z}_2$ (where $\mathbb{Z}_2$ acts on u by conjugating with J_{2N}), which is in $\mathcal{L}\mathbb{F}O_{2N} \otimes M_2(\mathbb{C})$ obtained by the crossed product by the dual action. Connes embeddability of $\mathcal{L}\mathbb{F}O_{2N} \otimes M_2(\mathbb{C})$ follows from the Connes embeddability of $\mathcal{L}\mathbb{F}O_{2N}$.

I.3.2 Relations

In this section we compute the free derivatives of the defining relations with respect to the generators fixed in the previous section. Let $F = (F^{(1)}, F^{(2)})$ be the vector containing the defining relations (I.1), in the form $F(u) = 0$. So $F^{(1)}(u) = u^*u - I_{2N}$ and $F^{(2)}(u) = uu^* - I_{2N}$. Here, $F(u)$ is shorthand for $F(u_{11}, \dots, u_{2N,2N})$, and similar notation will be used throughout the remainder of the paper.

Let $a_{ij}, \dots, d_{ij}$, $1 \leq i, j \leq N$, be $(2N)^2$ self-adjoint noncommuting formal indeterminates, and set $\mathcal{C} = \mathbb{C}\langle a_{11}, \dots, d_{NN}\rangle$. When we evaluate in the actual operators, $c_{k\ell}$ will for instance correspond to $c^u_{k\ell}$. Accordingly, collect the formal indeterminates into matrices $A = \sum_{ij} a_{ij} \otimes E_{ij}$ and so on. Thus we view $F \in \mathcal{C} \otimes (M_{2N}(\mathbb{C}) \oplus M_{2N}(\mathbb{C}))$, where we consider $M_{2N}(\mathbb{C})$ to just be a linear space. Keeping in mind Equations (I.5) and (I.6), we get the explicit polyonomials

$$F^{(1)} = \left(A^t\tau_a - B^t\tau_b - C^t\tau_c - D^t\tau_d\right)\left(A\tau_a + B\tau_b + C\tau_c + D\tau_d\right) - I_{2N},$$
$$F^{(2)} = \left(A\tau_a + B\tau_b + C\tau_c + D\tau_d\right)\left(A^t\tau_a - B^t\tau_b - C^t\tau_c - D^t\tau_d\right) - I_{2N}.$$

When evaluating, we will take the generators $a^u_{ij}, \dots, d^u_{ij}$ in their 'reduced' form acting on H. This is due to the fact that we want to investigate properties of the von Neumann algebra $\mathcal{L}\mathbb{F}O^J_{2N}$, which is represented on H, the GNS space of $C^*\mathbb{F}O^J_{2N}$ coming from the Haar state.

Our goal in this section is to determine

$$\partial F(A^u, B^u, C^u, D^u) \in B(H) \otimes B(H) \otimes B(M_{2N}(\mathbb{C}); M_{2N}(\mathbb{C}) \oplus M_{2N}(\mathbb{C})),$$

and express it in terms of the quantum group theoretic data coming from $\mathbb{F}O^J_{2N}$. The result is stated in the lemma below, whose proof constitutes one of the main technical components of this article and should be viewed as analogous to [BV18, Lemma 4.2]. Recall from Section I.2.2 that there is a copy $M_{2N}(\mathbb{C}) \cong p_1H$. This identification will be important for the next lemma.

Lemma I.3.2. *On $H \otimes H \otimes p_1H$ it holds that*

$$\partial F^{(1)}(A^u, B^u, C^u, D^u)^*\partial F^{(1)}(A^u, B^u, C^u, D^u) = 2 + 2\mathfrak{Re}[W],$$

where $W = V_{31}(1 \otimes U \otimes U)V_{32}(1 \otimes U \otimes 1)$. The same relation is true for $F^{(2)}$.

Proof. Since we are going to take free derivatives of $F^{(1)}$ and $F^{(2)}$, we can ignore the I_{2N} terms. Let us first focus on $F^{(2)}$, which can be written out using the algebraic relations of the τ's to read

$$F^{(2)} = F_a^{(2)}\tau_a - F_b^{(2)}\tau_b - F_c^{(2)}\tau_c - F_d^{(2)}\tau_d,$$

with

$$\begin{aligned} F_a^{(2)} &= AA^t + BB^t + CC^t + DD^t, & F_b^{(2)} &= AB^t + DC^t - BA^t - CD^t, \\ F_c^{(2)} &= AC^t + BD^t - CA^t - DB^t, & F_d^{(2)} &= AD^t + CB^t - DA^t - BC^t. \end{aligned}$$

Now, by definition $\partial F^{(2)}$ is the map such that

$$\partial F^{(2)}(E_{ij}^{\alpha}) = \sum_{k\ell,\beta} \partial_{ij}^{\alpha}\left(F^{(2)}\right)_{k\ell}^{\beta}.$$

Here, ∂_{ij}^{a} for instance refers to taking the free partial derivative with respect to a_{ij}. By linearity of ∂, we can compute the free derivatives of the four pieces $F_{a,b,c,d}^{(2)}$ separately.

We perform the computation for $F_a^{(2)}$ in detail, the others are similar. By definition

$$\begin{aligned} \partial_{ij}^{\alpha}\left(F_a^{(2)}\tau_a\right)_{k\ell}^{\beta} &= \delta_{a\beta}\partial_{ij}^{\alpha}\left[\left(\sum_{m,\gamma}\gamma_{km}\gamma_{\ell m}\right)\otimes E_{k\ell}^{a}\right] \\ &= \delta_{a\beta}\left(\sum_{m}\left[\delta_{ik}\delta_{jm}(1\otimes\alpha_{\ell m}) + \delta_{i\ell}\delta_{jm}(\alpha_{km}\otimes 1)\right]\right)\otimes E_{k\ell}^{a} \\ &= \delta_{a\beta}\left[\delta_{ik}(1\otimes\alpha_{\ell j}) + \delta_{i\ell}(\alpha_{kj}\otimes 1)\right]\otimes E_{k\ell}^{a}. \end{aligned}$$

So that

$$\left[\partial\left(F_a^{(2)}\tau_a\right)\right](E_{ij}^{\alpha}) = \sum_{\ell}(1\otimes\alpha_{\ell j}\otimes E_{i\ell}^{a}) + \sum_{k}(\alpha_{kj}\otimes 1\otimes E_{ki}^{a}).$$

Now notice that

$$E_{i\ell}^{a} = (T\lambda_{\ell j}T\otimes\vartheta_{a,\alpha})E_{ij}^{\alpha}, \qquad E_{ki}^{a} = (\lambda_{kj}T\otimes\vartheta_{a,\alpha})\,E_{ij}^{\alpha},$$

where T and λ_{ij} are the transpose map and left multiplication by E_{ij} respectively, acting on $M_N(\mathbb{C})$, and $\vartheta_{\alpha,\beta}$ is the rank one operator on $M_2(\mathbb{C})$ that sends τ_β to τ_α. Thus

$$\partial\left(F_a^{(2)}\tau_a\right) = \sum_{ij,\alpha}(1\otimes\alpha_{\ell j}\otimes T\lambda_{ij}T\otimes\vartheta_{a,\alpha}) + \sum_{k\ell,\beta}(\beta_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{a,\beta})$$

Analogously one finds that

$$\begin{aligned}\partial\left(F_b^{(2)}\tau_b\right) = &+\sum_{ij}(1\otimes b_{ij}\otimes T\lambda_{ij}T\otimes\vartheta_{b,a}) - \sum_{k\ell}(b_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{b,a})\\ &-\sum_{ij}(1\otimes a_{ij}\otimes T\lambda_{ij}T\otimes\vartheta_{b,b}) + \sum_{k\ell}(a_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{b,b})\\ &-\sum_{ij}(1\otimes d_{ij}\otimes T\lambda_{ij}T\otimes\vartheta_{b,c}) + \sum_{k\ell}(d_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{b,c})\\ &+\sum_{ij}(1\otimes c_{ij}\otimes T\lambda_{ij}T\otimes\vartheta_{b,d}) - \sum_{k\ell}(c_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{b,d})\,,\end{aligned}$$

$$\begin{aligned}\partial\left(F_c^{(2)}\tau_c\right) = &+\sum_{ij}(1\otimes c_{ij}\otimes T\lambda_{ij}T\otimes\vartheta_{c,a}) - \sum_{k\ell}(c_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{c,a})\\ &+\sum_{ij}(1\otimes d_{ij}\otimes T\lambda_{ij}T\otimes\vartheta_{c,b}) - \sum_{k\ell}(d_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{c,b})\\ &-\sum_{ij}(1\otimes a_{ij}\otimes T\lambda_{ij}T\otimes\vartheta_{c,c}) + \sum_{k\ell}(a_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{c,c})\\ &-\sum_{ij}(1\otimes b_{ij}\otimes T\lambda_{ij}T\otimes\vartheta_{c,d}) + \sum_{k\ell}(b_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{c,d})\,,\end{aligned}$$

$$\begin{aligned}\partial\left(F_d^{(2)}\tau_d\right) = &+\sum_{ij}(1\otimes d_{ij}\otimes T\lambda_{ij}T\otimes\vartheta_{d,a}) - \sum_{k\ell}(d_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{d,a})\\ &-\sum_{ij}(1\otimes c_{ij}\otimes T\lambda_{ij}T\otimes\vartheta_{d,b}) + \sum_{k\ell}(c_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{d,b})\\ &+\sum_{ij}(1\otimes b_{ij}\otimes T\lambda_{ij}T\otimes\vartheta_{d,c}) - \sum_{k\ell}(b_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{d,c})\\ &-\sum_{ij}(1\otimes a_{ij}\otimes T\lambda_{ij}T\otimes\vartheta_{d,d}) + \sum_{k\ell}(a_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\vartheta_{d,d})\,.\end{aligned}$$

The next step is to rewrite the rank one operators $\vartheta_{\alpha,\beta}$ in the right way. Let us investigate what the action of the antipode S looks like in terms of the self-adjoint generators from Section I.3.1. A quick computation yields

$$S(a_{ij}^u) = a_{ji}^u, \qquad S(b_{ij}^u) = -b_{ji}^u, \qquad S(c_{ij}^u) = -c_{ji}^u, \qquad S(d_{ij}^u) = -d_{ji}^u.$$

Compare this with

$$(E_{ij}^a)^* = E_{ji}^a, \qquad (E_{ij}^b)^* = -E_{ji}^b, \qquad (E_{ij}^c)^* = -E_{ji}^c, \qquad (E_{ij}^d)^* = -E_{ji}^d.$$

Thus write Γ for the linear extension of the map $\Gamma\tau_a = \tau_a$, $\Gamma\tau_{b,c,d} = -\tau_{b,c,d}$ on $M_2(\mathbb{C})$. Recall the operator U from Section I.2.1, which was induced by the unitary antipode R. As we are in the unimodular case, R is the same as S. Hence we can decompose $U = (T \otimes \Gamma)$ on $p_1 H \cong M_{2N}(\mathbb{C}) \cong M_2(\mathbb{C}) \otimes M_N(\mathbb{C})$ when we evaluate in $a^u_{ij}, \ldots, d^u_{ij}$.

We have already written the $M_N(\mathbb{C})$ leg of $\partial F^{(2)}$ in terms of multiplication operators and transposes, so this suggests that we should find expressions for $\vartheta_{\alpha,\beta}$ in terms of $\lambda_{a,b,c,d}$ (left multiplication by $\tau_{a,b,c,d}$), Γ, and $P_{a,b,c,d}$ which are the projections onto $\tau_{a,b,c,d}$ in $M_2(\mathbb{C})$. For example, $\vartheta_{d,b} = \Gamma\lambda_c\Gamma P_b = -\lambda_c\Gamma P_b$.

With this the above relations become

$$\begin{aligned}\partial\left(F_a^{(2)}\tau_a\right) = &+\sum_{ij}(1\otimes a_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_a\Gamma P_a)+\sum_{k\ell}(a_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_a\Gamma P_a)\\ &+\sum_{ij}(1\otimes b_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_b\Gamma P_b)+\sum_{k\ell}(b_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_b\Gamma P_b)\\ &+\sum_{ij}(1\otimes c_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_c\Gamma P_c)+\sum_{k\ell}(c_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_c\Gamma P_c)\\ &+\sum_{ij}(1\otimes d_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_d\Gamma P_d)+\sum_{k\ell}(d_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_d\Gamma P_d)\,,\end{aligned}$$

$$\begin{aligned}\partial\left(F_b^{(2)}\tau_b\right) = &-\sum_{ij}(1\otimes b_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_b\Gamma P_a)-\sum_{k\ell}(b_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_b\Gamma P_a)\\ &-\sum_{ij}(1\otimes a_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_a\Gamma P_b)-\sum_{k\ell}(a_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_a\Gamma P_b)\\ &-\sum_{ij}(1\otimes d_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_c\Gamma P_d)-\sum_{k\ell}(d_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_d\Gamma P_c)\\ &-\sum_{ij}(1\otimes c_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_d\Gamma P_c)-\sum_{k\ell}(c_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_c\Gamma P_d)\,,\end{aligned}$$

$$\begin{aligned}\partial\left(F_c^{(2)}\tau_c\right) = &-\sum_{ij}(1\otimes c_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_c\Gamma P_a)-\sum_{k\ell}(c_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_c\Gamma P_a)\\ &-\sum_{ij}(1\otimes d_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_d\Gamma P_b)-\sum_{k\ell}(d_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_d\Gamma P_b)\\ &-\sum_{ij}(1\otimes a_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_a\Gamma P_c)-\sum_{k\ell}(a_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_a\Gamma P_c)\\ &-\sum_{ij}(1\otimes b_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_b\Gamma P_d)-\sum_{k\ell}(b_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_b\Gamma P_d)\,,\end{aligned}$$

$$\begin{aligned}\partial\left(F_d^{(2)}\tau_d\right) = &-\sum_{ij}(1\otimes d_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_d\Gamma P_a)-\sum_{k\ell}(d_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_d\Gamma P_a)\\ &-\sum_{ij}(1\otimes c_{ij}\otimes T\lambda_{ij}T\otimes\Gamma\lambda_c\Gamma P_b)-\sum_{k\ell}(c_{k\ell}\otimes 1\otimes\lambda_{k\ell}T\otimes\lambda_c\Gamma P_b)\end{aligned}$$

$$
\begin{aligned}
&- \sum_{ij} (1 \otimes b_{ij} \otimes T\lambda_{ij}T \otimes \Gamma\lambda_b\Gamma P_c) - \sum_{k\ell} (b_{k\ell} \otimes 1 \otimes \lambda_{k\ell}T \otimes \lambda_b \Gamma P_c) \\
&- \sum_{ij} (1 \otimes a_{ij} \otimes T\lambda_{ij}T \otimes \Gamma\lambda_a\Gamma P_d) - \sum_{k\ell} (a_{k\ell} \otimes 1 \otimes \lambda_{k\ell}T \otimes \lambda_a \Gamma P_d) .
\end{aligned}
$$

Since

$$
\partial F^{(2)} = \partial\left(F_a^{(2)}\tau_a\right) - \partial\left(F_b^{(2)}\tau_b\right) - \partial\left(F_c^{(2)}\tau_c\right) - \partial\left(F_d^{(2)}\tau_d\right),
$$

we obtain the compact formula

$$
\partial F^{(2)} = \sum_{ij,\alpha} \left(1 \otimes \alpha_{ij} \otimes \left[(T \otimes \Gamma)\, \lambda_{ij}^{\alpha}\, (T \otimes \Gamma)\right]\right) + \sum_{k\ell,\beta} \left(\beta_{k\ell} \otimes 1 \otimes \left[\lambda_{k\ell}^{\beta}\, (T \otimes \Gamma)\right]\right),
$$

where $\lambda_{ij}^{\alpha} = \lambda_{ij} \otimes \lambda_{\alpha}$.

By the same techniques it can be shown that

$$
\begin{aligned}
\partial F^{(1)} = &+ \sum_{ij} \left(1 \otimes a_{ij} \otimes \left[(T \otimes \Gamma)\, \lambda_{ji}^{a}\right]\right) + \sum_{k\ell} \left(a_{k\ell} \otimes 1 \otimes \lambda_{\ell k}^{a}\right) \\
&- \sum_{ij} \left(1 \otimes b_{ij} \otimes \left[(T \otimes \Gamma)\, \lambda_{ji}^{b}\right]\right) - \sum_{k\ell} \left(b_{k\ell} \otimes 1 \otimes \lambda_{\ell k}^{b}\right) \\
&- \sum_{ij} \left(1 \otimes c_{ij} \otimes \left[(T \otimes \Gamma)\, \lambda_{ji}^{c}\right]\right) - \sum_{k\ell} \left(c_{k\ell} \otimes 1 \otimes \lambda_{\ell k}^{c}\right) \\
&- \sum_{ij} \left(1 \otimes d_{ij} \otimes \left[(T \otimes \Gamma)\, \lambda_{ji}^{d}\right]\right) - \sum_{k\ell} \left(d_{k\ell} \otimes 1 \otimes \lambda_{\ell k}^{d}\right) .
\end{aligned}
$$

Now we evaluate the 'formal' expressions above in the 'actual' operators. Let us start with $\partial F^{(1)}$. Note that we are taking $a_{ij}^{u}, \ldots, d_{ij}^{u}$ to act on H, i.e. as elements of $C_r^*\mathbb{F}O_{2N}^J \subset \mathcal{L}\mathbb{F}O_{2N}^J$. Due to the bimodule structure on $\mathcal{C}$, elements in the first tensor leg act from the left, but in the second leg they act from the right. It is simple to check that in the unimodular case, the right multiplication ρ on H of $x \in C_r^*\mathbb{F}O_{2N}^J$ can be written $\rho(x) = US(x)U$.

Keeping in mind the identification of U restricted to $p_1 H$ with $(T \otimes \Gamma)$ discussed above,

$$
\begin{aligned}
\partial F^{(1)}(A^u, \ldots, D^u) = &+ \sum_{ij} (1 \otimes U \otimes U)\left(1 \otimes a_{ji}^{u} \otimes \lambda_{ji}^{a}\right)(1 \otimes U \otimes 1) \\
&+ \sum_{k\ell} \left(a_{k\ell}^{u} \otimes 1 \otimes \lambda_{\ell k}^{a}\right) \\
&+ \sum_{ij} (1 \otimes U \otimes U)\left(1 \otimes b_{ji}^{u} \otimes \lambda_{ji}^{b}\right)(1 \otimes U \otimes 1) \\
&- \sum_{k\ell} \left(b_{k\ell}^{u} \otimes 1 \otimes \lambda_{\ell k}^{b}\right) \\
&+ \sum_{ij} (1 \otimes U \otimes U)\left(1 \otimes c_{ji}^{u} \otimes \lambda_{ji}^{c}\right)(1 \otimes U \otimes 1)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{k\ell} (c^u_{k\ell} \otimes 1 \otimes \lambda^c_{\ell k}) \\
&+ \sum_{ij} (1 \otimes U \otimes U) \left(1 \otimes d^u_{ji} \otimes \lambda^d_{ji}\right) (1 \otimes U \otimes 1) \\
&- \sum_{k\ell} \left(d^u_{k\ell} \otimes 1 \otimes \lambda^d_{\ell k}\right),
\end{aligned}
$$

as an element of $B(H \otimes H \otimes p_1 H)$. This can be written more compactly as

$$
\begin{aligned}
\partial F^{(1)}(A^u, \ldots, D^u) = &+ (1 \otimes U \otimes U) \left[\sum_{ij,\alpha} 1 \otimes \alpha^u_{ij} \otimes \lambda^\alpha_{ij}\right] (1 \otimes U \otimes 1) \\
&+ \sum_{k\ell} \left[a^u_{k\ell} \otimes 1 \otimes \lambda^a_{\ell k} - b^u_{k\ell} \otimes 1 \otimes \lambda^b_{\ell k}\right. \\
&\qquad \left. - c^u_{k\ell} \otimes 1 \otimes \lambda^c_{\ell k} - d^u_{k\ell} \otimes 1 \otimes \lambda^d_{\ell k}\right].
\end{aligned}
$$

Notice that due to Equation (I.5), left multiplication by u on $(p_1 H) \otimes H$ looks like $\sum_{ij,\alpha}(\lambda^\alpha_{ij} \otimes \alpha^u_{ij})$. This is also the restriction of the multiplicative unitary V to $(p_1 H) \otimes H$ by the decomposition discussed in Section I.2.2. Thus, using leg numbering notation and recalling also Equation (I.6) yields

$$
\partial F^{(1)}(A^u, \ldots, D^u) = (1 \otimes U \otimes U)\, V_{32}\, (1 \otimes U \otimes 1) + V^*_{31}.
$$

Similarly

$$
\partial F^{(2)}(A^u, \ldots, D^u) = (1 \otimes U \otimes U)\, V^*_{32}\, (1 \otimes U \otimes U) + V_{31}\, (1 \otimes 1 \otimes U).
$$

Setting $W = V_{31}(1 \otimes U \otimes U)V_{32}(1 \otimes U \otimes 1)$, it is now a simple matter to see that

$$
\partial F^{(1)}(A^u, B^u, C^u, D^u)^* \partial F^{(1)}(A^u, B^u, C^u, D^u) = 2 + 2\mathfrak{Re}\,[W].
$$

For $F^{(2)}(A^u, \ldots, D^u)$ it holds that

$$
\begin{aligned}
&\partial F^{(2)}(A^u, B^u, C^u, D^u)^* \partial F^{(2)}(A^u, B^u, C^u, D^u) \\
&\qquad = 2 + 2\mathfrak{Re}\,[(1 \otimes U \otimes U)V_{32}(1 \otimes U \otimes U)V_{31}(1 \otimes 1 \otimes U)],
\end{aligned}
$$

which reduces to the desired result upon commuting V_{31} with the terms in front of it. This is allowed because the two terms only act simultaneously on the third tensor leg, where the terms lie in $Uc_0(\mathbb{F}O^J_{2N})U$ and $c_0(\mathbb{F}O^J_{2N})$ respectively, which commute. One way to check this is to use the fact that $c_0(\mathbb{F}O^J_{2N})$ can be recovered from V by applying the slice maps $(\iota \otimes \varphi)(V)$, with φ coming from the predual of $B(H)$, and taking the closed linear span. ■

I.3.3 1-Boundedness

In this section we prove 1-boundedness of the generator set $a_{ij}^u, \ldots, d_{ij}^u$. Given the calculation of $\partial F(A^u, \ldots, D^u)$ from the previous section, the rest of the arguments are the same as those for the case $\mathbb{F}O_M$ covered in [BV18], but we reproduce some of them here for convenience and completeness.

It remains to determine the rank of $\partial F(A^u, \ldots, D^u)$ and to show that it is of determinant class.

Lemma I.3.3. rank $\partial F(A^u, \ldots, D^u) = (2N)^2 - 1$

Proof. The proof of Lemma 4.1 of [BV18], where the rank of this operator for $\mathbb{F}O_M$ is computed, goes through unchanged, as the L^2-Betti numbers of $\mathbb{F}O_{2N}^J$ also vanish due to [Bic13, Theorem 6.6] (but see also [Ver12, Section 5]). ■

Theorem I.3.4 (cf. [BV18, Theorem 3.5])**.** *Let $\Theta = U_1 V_{21} U_1 U_2$ be the edge-reversing operator on the quantum Cayley tree of $\mathbb{F}O_{2N}^J$. View $1 + \mathfrak{Re}\,[\Theta]$ as an operator in $U\mathcal{L}\mathbb{F}O_{2N}^J U \otimes B(p_1 H)$. Then it is of determinant class with respect to $h \otimes \mathrm{Tr}$.*

Proof. The proof is the same as the one of Theorem 3.5 in [BV18]. Although it is stated there only for $\mathbb{F}O_M$, it is also valid for $\mathbb{F}O_{2N}^J$. This is due to the fact that the result only depends on the general theory of quantum Cayley graphs [Ver05; Ver12] valid for all $\mathbb{F}O(Q)$ with $Q \in \mathrm{GL}_M(\mathbb{C})$, $M \geq 2$, $Q\overline{Q} \in \mathbb{C}I_M$, and $\mathrm{qdim}(u) > 2$ (see the remark at the start of Section 3 in [BV18]), and on the Haar state being a trace. ■

Proposition I.3.5. $\partial F(A^u, \ldots, D^u)^* \partial F(A^u, \cdots, D^u)$ *is of determinant class with respect to $h \otimes h \otimes \mathrm{Tr}$.*

Proof. Write $\tilde{V} = \Sigma(1 \otimes U)V(1 \otimes U)\Sigma$ and notice that $W = V_{31} U_2 U_3 V_{32} U_2$. We will conjugate W by unitaries Ω as $\Omega^* W \Omega$ to relate it to Θ. First conjugate by $U_2 \Sigma_{23}$ to obtain

$$\Sigma_{23} U_2 V_{31} U_2 U_3 V_{32} U_2 U_2 \Sigma_{23} = U_3 V_{21} U_3 U_2 \Sigma_{23} V_{32} \Sigma_{23} = V_{21} U_2 V_{23}.$$

Next, conjugate by U_1 to find

$$U_1 V_{21} U_2 V_{23} U_1 = U_1 V_{21} U_1 U_2 V_{23} = \Sigma_{12} U_2 V_{12} U_2 \Sigma_{12} U_2 V_{23} = \tilde{V}_{12} U_2 V_{23}.$$

Finally, conjugate by $V_{23}^* V_{13}^*$ to arrive at

$$V_{13} V_{23} \tilde{V}_{12} U_2 V_{23} V_{23}^* V_{13}^* = V_{13} V_{23} \tilde{V}_{12} U_2 V_{13}^*.$$

Now use the formula $V_{13} V_{23} \tilde{V}_{12} = \tilde{V}_{12} V_{13}$ of Baaj and Skandalis, which can be found in Proposition 6.1 of [BS93]. Thus

$$V_{13} V_{23} \tilde{V}_{12} U_2 V_{13}^* = \tilde{V}_{12} V_{13} U_2 V_{13}^* = \tilde{V}_{12} U_2.$$

Comparing with the definition of Θ, we see that $\tilde{V}_{12}U_2 = \Theta \otimes 1$, and we can conclude that W is unitarily conjugate to $\Theta \otimes 1$. On account of Lemma I.3.2, we also have that $\partial F(A^u, \ldots, D^u)^* \partial F(A^u, \cdots, D^u)$ is unitarily conjugate to $4(1 + \mathfrak{Re}[\Theta \otimes 1])$.

We now consider what happens to $h \otimes h \otimes \mathrm{Tr}$ under this conjugation process. The Haar state h is implemented as a vector state by $\xi_0 \in H$, and Tr is implement by some finite sum of vector states by finite dimensionality. Thus, let $\zeta \in p_1 H$ and compute

$$V_{23}^* V_{13}^* U_1 U_2 \Sigma_{23}(\xi_0 \otimes \xi_0 \otimes \zeta) = V_{23}^* V_{13}^*(\xi_0 \otimes \zeta \otimes \xi_0) = V_{23}^*(\xi_0 \otimes \zeta \otimes \xi_0).$$

Hence $h \otimes h \otimes \mathrm{Tr}$ is transformed into $(h \otimes \mathrm{Tr} \otimes h)(V_{23} \cdot V_{23}^*)$. Note that the last two legs of $1 + \mathfrak{Re}[\Theta \otimes 1]$ lie in the finite dimensional algebra $B(p_1 H) \otimes 1$. By finite dimensionality, $(\mathrm{Tr} \otimes h)(V \cdot V^*)$ is dominated by some multiple of the standard trace $(\mathrm{Tr} \otimes h)$ on this algebra. Thus we can use Theorem I.3.4 to conclude that $1 + \mathfrak{Re}[\Theta \otimes 1]$ is of determinant class with respect to $(h \otimes \mathrm{Tr} \otimes h)(V_{23} \cdot V_{23}^*)$. Therefore $\partial F(A^u, \ldots, D^u)^* \partial F(A^u, \cdots, D^u)$ is of determinant class with respect to $h \otimes h \otimes \mathrm{Tr}$, as desired. ∎

Corollary I.3.6. *The set of self-adjoint generators $a_{ij}^u, \ldots, d_{ij}^u$ of $\mathcal{L}\mathbb{F}O_{2N}^J$ is 1-bounded.*

Proof. Combine Lemma I.3.3 and Proposition I.3.5 with Theorem I.2.6. ∎

I.4 Adding Elements to an r-Bounded Set

Let $\mathcal{M}$ be a finite von Neumann algebra with faithful normal tracial state τ, and let $X_1, \ldots, X_n \in \mathcal{M}$ be self-adjoint. In this section we prove a lemma that allows us to add certain redundant elements to the set $X_1, \ldots, X_n$ while preserving r-boundedness. We achieve this using ideas from Proposition 6.9 in [Voi94] and its analogue Proposition 6.12 in [Voi96].

Let $Y_1, \ldots, Y_m$ also be self-adjoint elements in $\mathcal{M}$ such that $Y_1, \ldots, Y_m \in W^*(X_1, \ldots, X_n)$. Before stating the lemma, we introduce a distance function that measures how far away the Y_j lie from the von Neumann algebras generated by semicircular perturbations of the X_i. Let $S_1, \ldots, S_n$ be a free standard semicircular family, free from the X_i, and set

$$d_2(Y_j; X_1, \ldots, X_n)(\varepsilon) = \inf \left\{ \|Y_j - T\|_2 \,|\, T \in W^*(X_1 + \varepsilon S_1, \ldots, X_n + \varepsilon S_n) \right\}.$$

Lemma I.4.1. *Let $\mathcal{M}$ be a finite von Neumann algebra with faithful normal tracial state τ. Suppose that $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are self-adjoint elements such that $Y_1, \ldots, Y_m \in W^*(X_1, \ldots, X_n)$* (redundancy)*. Assume moreover that $\varepsilon^{-1} d_2(Y_j; X_1, \ldots, X_n)(\varepsilon)$ is bounded around $\varepsilon = 0$ for all $1 \leq j \leq m$* (regularity)*. Then if $\{X_1, \ldots, X_n\}$ is an r-bounded set, so is $\{X_1, \ldots, X_n, Y_1, \ldots, Y_m\}$.*

Proof. Note that it suffices to prove the case $m = 1$. Without loss of generality we can extend $S_1, \dots, S_n$ to a free standard semicircular family $S_1, \dots, S_{n+1}$, still free from the X_i. Recalling Definition I.2.4, we need to show that

$$\limsup_{\varepsilon\to 0}[\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n, Y_1 + \varepsilon S_{n+1} : S_1, \dots, S_{n+1}) \\ + (n + 1 - r)|\log \varepsilon|] < \infty.$$

Write T_1 for the conditional expectation of Y_1 onto $W^*(X_1+\varepsilon S_1, \dots, X_n+\varepsilon S_n)$, then by Proposition 1.11 in [Voi96] and the redundancy assumption we have

$$\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n, Y_1 + \varepsilon S_{n+1} : S_1, \dots, S_{n+1}) \\ = \chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n, Y_1 - T_1 + \varepsilon S_{n+1} : S_1, \dots, S_{n+1}).$$

By subadditivity ((ii) of Proposition I.2.3), we can split this in half as

$$\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n, Y_1 + \varepsilon S_{n+1} : S_1, \dots, S_{n+1}) \\ \leq \chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n : Y_1 - T_1 + \varepsilon S_{n+1}, S_1, \dots, S_{n+1}) \\ + \chi(Y_1 - T_1 + \varepsilon S_{n+1} : X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n, S_1, \dots, S_{n+1}).$$

Consider the first term on the right hand side. By (iv) of Proposition I.2.3,

$$\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n : Y_1 - T_1 + \varepsilon S_{n+1}, S_1, \dots, S_{n+1}) \\ = \chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n : S_1, \dots, S_{n+1}),$$

as $Y_1 - T_1 + \varepsilon S_{n+1} \in W^*(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n, S_1, \dots, S_{n+1})$. To get rid of the trailing semicircular S_{n+1}, note that we may apply (iii) of Proposition I.2.3, as $S_1, \dots, S_n \in W^*(S_1, \dots, S_{n+1})$. So

$$\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n, Y_1 + \varepsilon S_{n+1} : S_1, \dots, S_{n+1}) \\ \leq \chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n : S_1, \dots, S_n) \\ + \chi(Y_1 - T_1 + \varepsilon S_{n+1} : X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n, S_1, \dots, S_{n+1}).$$

Let us now focus on the second term on the right hand side. By (i) of Proposition I.2.3, we may replace the relative microstates free entropy by the ordinary microstates free entropy, as we are only after upper bounds. So

$$\chi(Y_1 - T_1 + \varepsilon S_{n+1} : X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n, S_1, \dots, S_{n+1}) \\ \leq \chi(Y_1 - T_1 + \varepsilon S_{n+1}).$$

Apply the linear change of variable formula for χ to it (Proposition 3.6 (b) in [Voi94]), with transformation 'matrix' ε. This yields

$$\chi(Y_1 - T_1 + \varepsilon S_{n+1}) = \log \varepsilon + \chi\left(\varepsilon^{-1}(Y_1 - T_1) + S_{n+1}\right).$$

Using again (i) of Proposition I.2.3, we estimate

$$\chi\big(\varepsilon^{-1}(Y_1 - T_1) + S_{n+1}\big) \leq \frac{1}{2}\log\left\{2\pi e\,\tau\left[\left(\varepsilon^{-1}(Y_1 - T_1) + S_{n+1}\right)^2\right]\right\}.$$

Thus, if we can control $\left\|\varepsilon^{-1}(Y_1 - T_1) + S_{n+1}\right\|_2$ uniformly in ε, we obtain a constant upper bound. For this use the triangle inequality and our regularity assumption to obtain

$$\left\|\varepsilon^{-1}(Y_1 - T_1) + S_{n+1}\right\|_2 \leq \varepsilon^{-1} d_2(Y_1; X_1, \dots, X_n)(\varepsilon) + \|S_{n+1}\|_2 \leq C'.$$

In total we have

$$\begin{aligned}&\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n, Y_1 + \varepsilon S_{n+1} : S_1, \dots, S_{n+1}) \\ &\qquad \leq \chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n : S_1, \dots, S_n) + \log \varepsilon + C.\end{aligned}$$

To complete the proof, combine all of the above to get

$$\begin{aligned}\limsup_{\varepsilon \to 0} \Big[&\chi\big(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n, Y_1 + \varepsilon S_{n+1} : S_1, \dots, S_{n+1}\big) \\ &\qquad + (n+1-r)|\log \varepsilon| \Big] \\ &\leq \limsup_{\varepsilon \to 0} \Big[\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n : S_1, \dots, S_n) + \log \varepsilon + C \\ &\qquad + (n+1-r)|\log \varepsilon| \Big] \\ &= C + \limsup_{\varepsilon \to 0} \Big[\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n : S_1, \dots, S_n) \\ &\qquad + (\log \varepsilon + |\log \varepsilon|) + (n-r)|\log \varepsilon| \Big] \\ &= C + \limsup_{\varepsilon \to 0} \Big[\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n : S_1, \dots, S_n) \\ &\qquad + (n-r)|\log \varepsilon| \Big] \\ &< \infty,\end{aligned}$$

as we assumed that $\{X_1, \dots, X_n\}$ is r-bounded. ■

Remark I.4.2. The ideas used in the proof above can be used show that the result is also true when δ_0 is replaced by δ^*. In fact the proof is simpler.

I.5 Main Result

In this section we present our main results and discuss some corollaries.

Theorem I.5.1. *The orthogonal free quantum group von Neumann algebras $\mathcal{L}\mathbb{F}O_{2N}^J$ are strongly* 1*-bounded when* $N \geq 2$.

Proof. We check that the fundamental character $\chi^u = (\mathrm{Tr} \otimes \iota)(u) = 2(a_{11}^u + \cdots + a_{NN}^u)$ satisfies the requirements of Proposition I.4.1. The redundancy assumption is trivial, and for the regularity assumption simply note that plugging in the obvious candidate gives a bound

$$\begin{aligned} d_2(\chi^u; a_{11}^u, \dots, d_{NN}^u)(\varepsilon) &\leq \|\chi^u - 2\,(a_{11}^u + \varepsilon S_{11}^a + \cdots + a_{NN}^u + \varepsilon S_{NN}^a)\|_2 \\ &= \|2\varepsilon S_{11}^a + \dots 2\varepsilon S_{NN}^a\|_2 \\ &\leq 2N\varepsilon. \end{aligned}$$

Here S_{ij}^α is a free standard semicircular family, free from $a_{11}^u, \dots, d_{NN}^u$. Thus, the set of generators $\{a_{11}^u, \dots, d_{NN}^u, \chi^u\}$ is also 1-bounded by Corollary I.3.6 and Proposition I.4.1.

By [Ban96], χ^u is a semicircular element and hence possesses a continuous density with respect to the Lebesgue measure. Lemma I.2.2 then allows us to conclude that $\chi(\chi^u)$, i.e. the microstates free entropy of the fundamental character, is finite. We conclude that $\mathcal{L}\mathbb{F}O_{2N}^J$ is strongly 1-bounded. ∎

Remark I.5.2. The proof of Theorem I.5.1 also extends to strong 1-boundedness with respect to δ^* when combined with Remark I.4.2 and recalling that the proof of Corollary I.3.6 also goes through for δ^* due to the statement of Theorem I.2.6.

Corollary I.5.3. *Let $Q \in \mathrm{GL}_M(\mathbb{C})$, $M \geq 3$, be such that $Q\overline{Q} \in \mathbb{C}I_M$ and $\mathbb{F}O(Q)$ is unimodular, then $\mathcal{L}\mathbb{F}O(Q)$ is not isomorphic to any finite von Neumann algebra admitting a tuple of self-adjoint generators whose (modified) free entropy dimension exceeds 1. In particular this excludes being isomorphic to any (interpolated) free group factor.*

Proof. By the discussion at the start of section 9.1 in [Bra17], it follows that (up to isomorphism) the only two family of matrices satisfying the assumptions are the identity matrices I_M, and when $M = 2N$ the standard symplectic matrices J_{2N}. These two cases are covered by Corollary 4.4 in [BV18] and Theorem I.5.1 above. ∎

In fact, the class of von Neumann algebras to which $\mathcal{L}\mathbb{F}O(Q)$ cannot be isomorphic contains all countable free products of finitely generated, diffuse, tracial, Connes embeddable von Neumann algebras by Lemma 3.7 of [Jun07]. The free perturbation algebras of Brown [Bro05] are also in this class.

Remark I.5.4. One might hope to extend these results to the (discrete duals of the) quantum permutation groups S_n^+ with $n \geq 4$ [Wan98]. Indeed, from Theorem 5.2 in [Kye+17] we know that their L^2-Betti numbers vanish (this even holds for general quantum automorphisms groups of finite dimensional C*-algebras equipped with their Markov trace). Hence Equation (13) from [BCV17] implies that the standard generating set (see Equations (3.1)–(3.3) in [Wan98]) has free entropy dimension 1. Nevertheless, these generators satisfy non-homogeneous and non-trivial linear relations, and so taking their free derivatives is unfortunately not well-defined.

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