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ABSTRACT

We prove Lieb type convexity and concavity results for trace functionals associated with positive operator monotone (decreasing) functions and certain monotone concave functions, together with strictly positive linear maps of matrices. This gives a partial generalization of Hiai's recent work on trace functionals associated with power functions, by allowing positive operator monotone decreasing functions instead of power maps. Our proof is based on variational formula for trace functionals based on the Legendre transform, and a strengthened convexity of positive operator monotone decreasing functions in a previous work of Kirihata and the second named author. We also provide the generalization to the framework of unital tracial C -algebras based on Petz's work.

1. Introduction

In a breakthrough paper on concavity of quantum entropy [1], Lieb proved concavity and convexity properties of functionals of the form

$$(A, B) \mapsto \mathrm{Tr}(A^p K B^q K^*)$$

defined on the space of positive definite matrices. Since then this has been expanded in many ways, see Refs [2–8] and references therein. One definitive form is given by Hiai [7], who proved (among other configurations) the joint convexity of functionals of the form

$$(A, B) \mapsto \mathrm{Tr}\, h \left(\Phi(A^{-p})^{-1/2} \Psi(B^{-q}) \Phi(A^{-p})^{-1/2} \right) \quad (A, B > 0)$$

for strictly positive linear maps $\Phi \colon M_m(\mathbb{C}) \to M_k(\mathbb{C})$, $\Psi \colon M_n(\mathbb{C}) \to M_k(\mathbb{C})$, $0 < p, q \le 1$, and certain nondecreasing concave function h. His proof is based on an elegant use of the variational formula for trace functionals based on the Legendre transform which can be traced back to Ref. [4], together with intricate relation between operator majorization and matrix norms.

In this short note, we prove a variant of this result, where we allow functional calculus by arbitrary positive operator monotone decreasing functions inside positive maps instead

of power maps but impose a stronger condition on h. A similar result for the geometric mean of positive matrices [9] was recently proved by Kian and Seo [10].

Our proof is a combination of the variational method with concave functions, and one-variate convexity of operator valued maps of the form $h(\Phi(f(A)))$ for operator monotone h and positive operator monotone decreasing f established by Kirihata and the second named author in Ref. [11], which was motivated by certain operator log-convexity of such f due to Ando and Hiai [12].

Besides allowing a bigger class of functions inside the positive linear maps Φ and Ψ, we also give analogous results in the framework of C*-algebras endowed with tracial states. The overall strategy is essentially the same as the case of matrices, but we rely on Petz's work [13, 14] on trace inequalities for tracial von Neumann algebras, and an adaptation of Hiai's variational formula to this setting.

2. Preliminaries

We denote the set of positive invertible matrices by $M_n(\mathbb{C})^{++}$, and the set of selfadjoint matrices by $M_n(\mathbb{C})_{\mathrm{sa}}$. A linear map $\Phi \colon M_n(\mathbb{C}) \to M_k(\mathbb{C})$ is *strictly positive* if it sends $M_n(\mathbb{C})^{++}$ into $M_k(\mathbb{C})^{++}$. A real function $f(x)$ for $x > 0$ is *operator monotone* when the functional calculus $f(A)$ for $A \in M_n(\mathbb{C})^{++}$ with arbitrary n preserves order relation, that is, $A \le B$ in $M_n(\mathbb{C})^{++}$ implies $f(A) \le f(B)$. For details, we refer to standard references such as in Ref. [15].

For positive numbers a and b, we denote their arithmetic and harmonic means by

$$a \triangledown b = \frac{a+b}{2}, \quad a \mathbin{!} b = \frac{2}{a^{-1} + b^{-1}}.$$

These admit obvious generalization to operator transforms for $A, B \in M_n(\mathbb{C})^{++}$.

Let $h(x)$ be a nondecreasing and concave function for $x > 0$, satisfying $\lim_{x \to \infty} h(x) x^{-1} = 0$. Its *Legendre transform* is given by

$$\check{h}(t) = \inf_{x > 0} tx - h(x).$$

Then $\check{h}$ satisfies the same assumptions as h [7, Lemma A.2].

Now, let us list key ingredients of our proof. First is the variational formula for trace functionals associated with concave functions.

Proposition 2.1 ([7, Lemma A.2]): *Let h be as above. For any positive matrix $A \in M_n(\mathbb{C})^{++}$, we have*

$$\operatorname{Tr} h(A) = \inf_{B \in M_n(\mathbb{C})^{++}} \operatorname{Tr}(AB - \check{h}(B)).$$

Next is a stronger form of convexity for positive operator monotone functions, as follows.

Proposition 2.2 ([11, Theorem 3.1] [1]): *Let $g(x)$ be an operator monotone function, and $f(x)$ be a positive operator monotone decreasing function, both for $x > 0$. For any strictly*

positive linear map $\Phi\colon M_n(\mathbb{C}) \to M_k(\mathbb{C})$, *the map*

$$M_n(\mathbb{C})^{++} \to M_k(\mathbb{C})_{\mathrm{sa}}, \quad A \mapsto g(\Phi(f(A)))$$

is convex.

We will use the consequence of the above for $g(x) = -x^{-1}$, in the following form: $\Phi(f(A))^{-1}$ is concave in $A \in M_n(\mathbb{C})^{++}$ when f is positive operator monotone decreasing.

We also use the Jensen inequality and monotonicity for trace functionals, which can be stated as follows.

Proposition 2.3 ([16, Theorem 2.4]): *Let f be a convex function defined on some interval J, and $C_1, \ldots, C_k$ be elements of $M_n(\mathbb{C})$ such that $\sum_i C_i^* C_i = I_n$. Then, for any $A_1, \ldots, A_k \in M_n(\mathbb{C})_{\mathrm{sa}}$ such that $\sigma(A_i) \subset J$, we have*

$$\mathrm{Tr} f\left(\sum_i C_i^* A_i C_i\right) \leq \mathrm{Tr}\left(\sum_i C_i^* f(A_i) C_i\right)$$

We use this in the following form, by taking $J = (0, \infty)$ and $f = -h$: let $h(x)$ be a concave function for $x > 0$. Then $A \mapsto \mathrm{Tr}\, h(A)$ is concave for $A \in M_n(\mathbb{C})^{++}$. Thus, in fact the main result of Ref. [14] is enough for us.

Proposition 2.4: *Let $f(x)$ be a monotone function with domain J. Suppose that we have $A \leq B$ and $\sigma(A), \sigma(B) \subset J$ for $A, B \in M_n(\mathbb{C})_{\mathrm{sa}}$. Then we have*

$$\mathrm{Tr}(f(A)) \leq \mathrm{Tr}(f(B)).$$

Proof: This is well known to experts, but here is a sketch of the proof. From the minimax principle, we see that the ordered eigenvalues of A and B satisfy $\lambda_i(A) \leq \lambda_i(B)$ for $i = 1, \ldots, n$. Collecting the inequalities $f(\lambda_i(A)) \leq f(\lambda_i(B))$, we obtain the claim. ∎

3. Main result

When $h(x)$ is a real function defined for $x > 0$, put

$$\tilde{h}(x) = -h(x^{-1}).$$

3.1. Convexity

When $h(x)$ is an monotone function for $x > 0$ such that $\tilde{h}$ is concave and $\lim_{x \to 0} h(x)x = 0$, we put

$$\check{h}(t) = \inf_{x>0} tx - \tilde{h}(x) = \inf_{x>0} tx + h(x^{-1}).$$

Note that $\check{h}$ is well defined as the Legendre transform $\tilde{\check{h}}$.

Furthermore, we will consider the class of functions $h(x)$ for $x > 0$ satisfying

$$h(x \triangledown y) \geq h(x) \triangledown h(y) \geq h(x \,!\, y). \tag{1}$$

The first inequality is the usual concavity condition. The second can be interpreted as concavity of $\tilde{h}$, hence this class is closed under the transform $h \mapsto \tilde{h}$. One motivating example comes from operator monotone functions, as follows.

Proposition 3.1: *Suppose that $h(x)$ is operator monotone for $x > 0$. Then it satisfies* (1).

Proof: This observation can be traced back to [12], but let us repeat it here for the reader's convenience. First, operator monotonicity of $h(x)$ for $x > 0$ implies concavity $h(x \triangledown y) \geq h(x) \triangledown h(y)$. Next, as $\tilde{h}$ is also operator monotone, it is again concave. As remarked above, this can be expressed as $h(x \,!\, y) \leq h(x) \triangledown h(y)$ up to change of variables. ∎

Remark 3.1: Recall that any operator monotone function $h(x)$ for $x > 0$ can be written as

$$h(x) = c_0 + c_1 x + \int \frac{x\lambda - 1}{x + \lambda} \, d\mu(\lambda)$$

for some $c_1 \geq 0$ and a finite measure μ on $[0, \infty)$. If μ does not have atom on 0, we have $\lim_{x \to 0} x h(x) = 0$.

We are now ready to state and prove our main result.

Theorem 3.1: *Suppose that $h(x)$ is a monotone function for $x > 0$ such that $\tilde{h}$ is concave, $\lim_{x \to 0} h(x)x = 0$, and that $\check{h}$ satisfies* (1). *Let $f(x)$ and $g(x)$ be positive operator monotone decreasing functions for $x > 0$, and let $\Phi \colon M_m(\mathbb{C}) \to M_k(\mathbb{C})$ and $\Psi \colon M_n(\mathbb{C}) \to M_k(\mathbb{C})$ be strictly positive linear maps. Then, the map*

$$M_m(\mathbb{C})^{++} \times M_n(\mathbb{C})^{++} \to \mathbb{R}, \quad (A, B) \to \operatorname{Tr} h\left(\Phi(f(A))^{1/2}\Psi(g(B))\Phi(f(A))^{1/2}\right)$$

is jointly convex.

Proof: Let us write $A' = \Phi(f(A))$ and $B' = \Psi(g(B))$. We have

$$\operatorname{Tr} h\left(\Phi(f(A))^{1/2}\Psi(g(B))\Phi(f(A))^{1/2}\right) = -\operatorname{Tr} \tilde{h}\left(A'^{-1/2}B'^{-1}A'^{-1/2}\right).$$

Thus, it is enough to prove the joint concavity of $\operatorname{Tr} \tilde{h}(A'^{-1/2}B'^{-1}A'^{-1/2})$ in A and B. We closely follow the proof of [7, Theorem 5.2]. We first get

$$\operatorname{Tr} \tilde{h}\left(A'^{-1/2}B'^{-1}A'^{-1/2}\right) = \inf_{Y \in M_k(\mathbb{C})^{++}} \operatorname{Tr}\left(Y A'^{-1/2}B'^{-1}A'^{-1/2} - \check{h}(Y)\right)$$

from Proposition 2.1. Putting $Z = A'^{-1/2}YA'^{-1/2}$, we can rewrite this as

$$\inf_{Z \in M_k(\mathbb{C})^{++}} \operatorname{Tr}\left(Z^{1/2}B'^{-1}Z^{1/2} - \check{h}(Z^{1/2}A'Z^{1/2})\right). \tag{2}$$

Given $A_i \in M_m(\mathbb{C})^{++}$ and $B_i \in M_n(\mathbb{C})^{++}$ for $i = 1, 2$, let us fix $Z_0 \in M_k(\mathbb{C})^{++}$ that almost achieves the infimum (2) for $A = A_1 \triangledown A_2$ and $B = B_1 \triangledown B_2$. By Proposition 2.2

applied to the operator monotone function $-x^{-1}$, the map

$$B \mapsto Z_0^{1/2} B'^{-1} Z_0^{1/2} = \left(Z_0^{-1/2} \Psi(g(B)) Z_0^{-1/2} \right)^{-1}$$

is concave, hence we obtain

$$Z_0^{1/2} B'^{-1} Z_0^{1/2} \geq \left(Z_0^{1/2} B_1'^{-1} Z_0^{1/2} \right) \triangledown \left(Z_0^{1/2} B_2'^{-1} Z_0^{1/2} \right).$$

As for the term involving A', by assumption on h the function $h' = \tilde{h}$ is concave and monotone. Thus $C \mapsto \operatorname{Tr} h'(C)$ is concave and monotone for $C \in M_k(\mathbb{C})^{++}$ by Propositions 2.3 and 2.4. This observation and the concavity of $A \mapsto Z_0^{-1/2} \Phi(f(A))^{-1} Z_0^{-1/2}$ imply that

$$A \mapsto -\operatorname{Tr} \check{h} \left(Z_0^{1/2} A' Z_0^{1/2} \right) = -\operatorname{Tr} \check{h} \left(Z_0^{1/2} \Phi(f(A)) Z_0^{1/2} \right)$$

is concave, hence we obtain

$$-\operatorname{Tr} \check{h} \left(Z_0^{1/2} A' Z_0^{1/2} \right) \geq \left(-\operatorname{Tr} \check{h} \left(Z_0^{1/2} A_1' Z_0^{1/2} \right) \right) \triangledown \left(-\operatorname{Tr} \check{h} \left(Z_0^{1/2} A_2' Z_0^{1/2} \right) \right).$$

Thus we see that (2) is bounded from below by

$$\frac{1}{2} \left(\inf_{Z_1, Z_2} \operatorname{Tr} \left(Z_1^{1/2} B_1'^{-1} Z_1^{1/2} - \check{h} \left(Z_1^{1/2} A_1' Z_1^{1/2} \right) \right) \right.$$
$$\left. + \operatorname{Tr} \left(Z_2^{1/2} B_2'^{-1} Z_2^{1/2} - \check{h} \left(Z_2^{1/2} A_2' Z_2^{1/2} \right) \right) \right),$$

where Z_1 and Z_2 separately run over $M_k(\mathbb{C})^{++}$. We thus obtained

$$\operatorname{Tr} \tilde{h} \left(A'^{-1/2} B'^{-1} A'^{-1/2} \right) \geq \operatorname{Tr} \tilde{h} \left(A_1'^{-1/2} B_1'^{-1} A_1'^{-1/2} \right) \triangledown \operatorname{Tr} \tilde{h} \left(A_2'^{-1/2} B_2'^{-1} A_2'^{-1/2} \right),$$

which is what we wanted. ∎

The above theorem applies for the following cases.

- $h(x) = \log x; \check{h}(x) = 1 + \log x$.
- $h(x) = x^r$ for $0 < r$; $\check{h}(x) = r^{1/(r+1)}(1 + r^{-1}) x^{r/(r+1)}$. For $r = 1$ we recover [11, Theorem 4.2].
- $h(x) = -x^{-r}$ for $0 < r \leq \frac{1}{2}$; $\check{h}(x) = r^{r/(1-r)}(r-1) x^{r/(r-1)}$. Put another way,

$$\operatorname{Tr} \left(\Phi(f(A))^{1/2} \Psi(g(B)) \Phi(f(A))^{1/2} \right)^{-r}$$

is concave in A and B for such r.

Remark 3.2: If $\check{h}$ is operator monotone, we can avoid using Propositions 2.3 and 2.4 as the map

$$M_m(\mathbb{C})^{++} \to M_k(\mathbb{C})^{++}, \quad A \mapsto -\check{h}(Z_0^{1/2} \Phi(f(A)) Z_0^{1/2})$$

would be operator concave. The above examples all satisfy this additional assumption.

Remark 3.3: By [7, Theorem 5.2],

$$\operatorname{Tr} h \left(\Phi(A^{-p})^{-1/2} \Psi(B^{-q}) \Phi(A^{-p})^{-1/2} \right)$$

is convex if $h(x)$ is a nondecreasing function for $x > 0$ such that either of $h(x^{-(1+p)})$ or $h(x^{-(1+q)})$ is convex. The above examples of h fall under this setting. For $h(x) = -x^{-r}$, the bound on r is sharp as seen from the case of $p = q = 1$, $A = B$, and $\Phi(A) = A = \Psi(A)$, see also [5] for a more precise condition on h that depends on p and q.

Remark 3.4: The claim of Theorem 3.1 does not hold if one relaxes the assumption on f and g to be just operator convex (and positive). Indeed, $f(x) = x^2$ is operator convex, but the formula of the theorem can even fail to be separately convex with a choice like $h(x) = \log x$.

3.2. Concavity

The concave analogue, which is easier, goes as follows.

Theorem 3.2: Let $h(x)$ be a concave monotone function for $x > 0$ such that $\lim_{x \to \infty} h(x) x^{-1} = 0$, and that $\check{h}$ satisfies (1). Let $f(x)$ and $g(x)$ be positive operator monotone functions for $x > 0$, and let $\Phi \colon M_m(\mathbb{C}) \to M_k(\mathbb{C})$ and $\Psi \colon M_n(\mathbb{C}) \to M_k(\mathbb{C})$ be strictly positive linear maps. Then, the map

$$M_m(\mathbb{C})^{++} \times M_n(\mathbb{C})^{++} \to \mathbb{R}, \quad (A, B) \to \operatorname{Tr} h \left(\Phi(f(A))^{1/2} \Psi(g(B)) \Phi(f(A))^{1/2} \right)$$

is jointly concave.

We omit the proof as it is completely analogous to that of Theorem 3.1. The above theorem applies for the following cases.

- $h(x) = \log x; \check{h}(x) = 1 + \log x$.
- $h(x) = x^r$ for $0 < r \le \frac{1}{2}; \check{h}(x) = r^{r/(1-r)}(r-1)x^{r/(r-1)}$.
- $h(x) = -x^{-r}$ for $0 < r; \check{h}(x) = r^{1/(r+1)}(1+r^{-1})x^{r/(r+1)}$.

4. C*-Algebraic setting

The above results have straightforward generalization to the setting of unital C*-algebras with tracial states. In this section $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ denote unital C*-algebras, and τ denotes a tracial state on $\mathfrak{C}$. We use notations such as $\mathfrak{A}^{++}$ and $\mathfrak{A}_{\mathrm{sa}}$ analogous to the case of matrix algebras.

First let us establish a generalization of Proposition 2.1 to this setting.

Proposition 4.1: Let $h(x)$ be a concave function for $x > 0$ such that $\lim_{x \to \infty} h(x)x^{-1} = 0$. For any $A \in \mathfrak{C}^{++}$, we have

$$\tau(h(A)) = \inf_{B \in \mathfrak{C}^{++}} \tau(AB - \check{h}(B)).$$

Proof: Let $\mathfrak{M}$ be the von Neumann algebraic closure of $\mathfrak{C}$ in the GNS representation associated with τ. We denote the extension of τ to $\mathfrak{M}$ again by τ. Let $\mathfrak{N}$ be the von Neumann subalgebra of $\mathfrak{M}$ generated by the image of A. Then there is a (unique) τ-preserving conditional expectation $E\colon \mathfrak{M} \to \mathfrak{N}$.

As $-\check{h}(x)$ is convex for $x > 0$, and E is unital positive map, we have

$$-\tau(\check{h}(E(B))) \leq -\tau(E(\check{h}(B))) = -\tau(\check{h}(B))$$

for any $B \in \mathfrak{M}^{++}$ by [14, Corollary]. Combined with $\tau(AB) = \tau(AE(B))$, we obtain

$$\inf_{B \in \mathfrak{M}^{++}} \tau(AB - \check{h}(B)) = \inf_{B \in \mathfrak{N}^{++}} \tau(AB - \check{h}(B)).$$

The right hand side is equal to $\inf_f \tau(Af(A) - \check{h}(f(A)))$, where f runs over bounded nonnegative Borel measurable functions on $\sigma(A)$, hence it is equal to $h(A)$. By a standard approximation argument, the same infimum is achieved when f runs over nonnegative continuous functions on $\sigma(A)$.

Finally, together with the obvious inequality

$$\inf_{B \in \mathfrak{M}^{++}} \tau(AB - \check{h}(B)) \leq \inf_{B \in \mathfrak{C}^{++}} \tau(AB - \check{h}(B)),$$

we obtain the claim. ∎

Proposition 2.2 holds in this setting, as [11, Theorem 3.1] was already proved for in such generality. The rest is quite well known, as follows.

Proposition 4.2 ([17]): *Let f be a convex function defined on some interval J, and $C_1, \ldots, C_k$ be elements of $\mathfrak{A}$ such that $\sum_i C_i^* C_i = 1$. Then, for any $A_1, \ldots, A_k \in \mathfrak{A}_{\mathrm{sa}}$ such that $\sigma(A_i) \subset J$, we have*

$$\tau\left(f\left(\sum_i C_i^* A_i C_i\right)\right) \leq \tau\left(\sum_i C_i^* f(A_i) C_i\right)$$

Again the setting of [14] is enough for us, as we only need to deal with convex functions defined for $x > 0$.

Proposition 4.3 ([13, Theorem 2]): *Let $f(x)$ be a monotone function with domain J. Suppose that we have $A \leq B$ and $\sigma(A), \sigma(B) \subset J$ for $A, B \in \mathfrak{A}_{\mathrm{sa}}$. Then we have*

$$\tau(f(A)) \leq \tau(f(B)).$$

With above results at hand, the proof of our main results carry over to C*-algebraic setting, and we obtain the following.

Theorem 4.1: *Let $f(x)$, $g(x)$, and $h(x)$ be real functions for $x > 0$ as in Theorem 3.1. Let $\Phi\colon \mathfrak{A} \to \mathfrak{C}$ and $\Psi\colon \mathfrak{B} \to \mathfrak{C}$ be strictly positive linear maps. Then the map*

$$\mathfrak{A}^{++} \times \mathfrak{B}^{++} \to \mathbb{R}, \quad (A, B) \to \tau\left(h\left(\Phi(f(A))^{1/2}\Psi(g(B))\Phi(f(A))^{1/2}\right)\right) \qquad (3)$$

is jointly convex.

Theorem 4.2: *Let $f(x)$, $g(x)$, and $h(x)$ be real functions for $x > 0$ as in Theorem* 3.2. *Let* $\Phi : \mathfrak{A} \to \mathfrak{C}$ *and* $\Psi : \mathfrak{B} \to \mathfrak{C}$ *be strictly positive linear maps. Then, the map*

$$\mathfrak{A}^{++} \times \mathfrak{B}^{++} \to \mathbb{R}, \quad (A, B) \to \tau\left(h\left(\Phi(f(A))^{1/2}\Psi(g(B))\Phi(f(A))^{1/2}\right)\right)$$

is jointly concave.

Note

1. We note an unfortunate typo in that paper, B^{++} in [11, Theorem 3.1] should read B_{sa}.

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References

[1] Lieb EH. Convex trace functions and the Wigner-Yanase-Dyson conjecture. Adv Math. 1973;11:267–288. MR0332080.
[2] Epstein H. Remarks on two theorems of E. Lieb. Commun Math Phys. 1973;31:317–325. MR0343073.
[3] Carlen EA, Lieb EH. A Minkowski type trace inequality and strong subadditivity of quantum entropy. Differential operators and spectral theory. Providence (RI): Amer. Math. Soc.; 1999. p. 59–68. MR1730503.
[4] Carlen EA, Lieb EH. A Minkowski type trace inequality and strong subadditivity of quantum entropy. II. Convexity and concavity. Lett Math Phys. 2008;83(2):107–126. MR2379699.
[5] Hiai F. Concavity of certain matrix trace and norm functions. Linear Algebra Appl. 2013;439(5):1568–1589. MR3067823.
[6] Carlen EA, Frank RL, Lieb EH. Some operator and trace function convexity theorems. Linear Algebra Appl. 2016;490:174–185. MR3429039.
[7] Hiai F. Concavity of certain matrix trace and norm functions. II. Linear Algebra Appl. 2016;496:193–220. MR3464069.
[8] Zhang H. From Wigner-Yanase-Dyson conjecture to Carlen-Frank-Lieb conjecture. Adv Math. 2020;365:107053, 18. MR4064777.
[9] Pusz W, Woronowicz SL. Functional calculus for sesquilinear forms and the purification map. Rep Math Phys. 1975;8(2):159–170. MR0420302.
[10] Kian M, Seo Y. Jointly convex mappings related to Lieb's theorem and Minkowski type operator inequalities. Anal Math Phys. 2021;11(2):1–33. MR4227812.
[11] Kirihata M, Yamashita M. Strengthened convexity of positive operator monotone decreasing functions. Math Scand. 2020;126(3):559–567. MR4156435.
[12] Ando T, Hiai F. Operator log-convex functions and operator means. Math Ann. 2011;350(3): 611–630. MR2805638.
[13] Petz D. Spectral scale of selfadjoint operators and trace inequalities. J Math Anal Appl. 1985;109(1):74–82. MR796042.

[14] Petz D. Jensen's inequality for positive contractions on operator algebras. Proc Amer Math Soc. 1987;99(2):273–277. MR870784.

[15] Bhatia R. Matrix analysis. New York (NY): Springer-Verlag; 1997. (Graduate Texts in Mathematics 169). MR1477662.

[16] Hansen F, Pedersen GK. Jensen's operator inequality. Bull Lond Math Soc. 2003;35(4):553–564. MR1979011.

[17] Harada T, Kosaki H. Trace Jensen inequality and related weak majorization in semi-finite von Neumann algebras. J Oper Theory. 2010;63(1):129–150. MR2606886.