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Strong solutions of a stochastic differential equation with irregular random drift[☆]

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Abstract

We present a well-posedness result for strong solutions of one-dimensional stochastic differential equations (SDEs) of the form

$$\mathrm{d}X = u(\omega, t, X)\,\mathrm{d}t + \frac{1}{2}\sigma(\omega, t, X)\partial_x \sigma(\omega, t, X)\,\mathrm{d}t + \sigma(\omega, t, X)\,\mathrm{d}W(t),$$

where the drift coefficient u is random and irregular, with a weak derivative satisfying $\partial_x u = q$ for some $q \in L^p_\omega L^\infty_t (L^2_x \cap L^1_x)$, $p \in [1, \infty)$. The random and regular noise coefficient σ may vanish. The main contribution is a pathwise uniqueness result under the assumptions that $\mathbb{E}\,\|q(t) - q(0)\|^2_{L^2(\mathbb{R})} \to 0$ as $t \downarrow 0$, and u satisfies the one-sided gradient bound $q(\omega, t, x) \le K(\omega, t)$, where the process $K(\omega, t) > 0$ exhibits an exponential moment bound of the form $\mathbb{E}\exp\bigl(p \int_t^T K(s)\,\mathrm{d}s\bigr) \lesssim t^{-2p}$ for small times t, for some $p \ge 1$. This study is motivated by ongoing work on the well-posedness of the stochastic Hunter–Saxton equation, a stochastic perturbation of a nonlinear transport equation that arises in the modelling of the director field of a nematic liquid crystal. In this context, the one-sided bound acts as a selection principle for dissipative weak solutions of the stochastic partial differential equation.

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1. Introduction

1.1. Main result

In this paper, we prove strong existence and pathwise uniqueness for a class of one-dimensional SDEs with rough random drift $u = u(\omega, t, x)$ and a noise coefficient $\sigma = \sigma(\omega, t, x)$ that is random and possibly degenerate. We fix a stochastic basis $\mathcal{S} = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ consisting of a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a complete right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Moreover, we fix a standard Brownian motion W on $\mathcal{S}$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

We are interested in strong solutions X, i.e., $\mathbb{P}$-almost surely continuous and $\{\mathcal{F}_t\}_{t\geq 0}$-adapted stochastic processes X satisfying

$$\mathrm{d}X = u(\omega, t, X)\,\mathrm{d}t + \frac{1}{4}\left(\sigma^2\right)'(\omega, t, X)\,\mathrm{d}t + \sigma(\omega, t, X)\,\mathrm{d}W, \quad X(0) = x \in \mathbb{R}, \tag{1.1}$$

where σ' denotes the x-derivative of $\sigma = \sigma(\omega, t, x)$, so that $\frac{1}{4}\left(\sigma^2\right)' = \frac{1}{2}\sigma\partial_x\sigma$. For a deterministic, sufficiently regular $\sigma = \sigma(x)$, the SDE (1.1) can be written as

$$\mathrm{d}X = u(\omega, t, X)\,\mathrm{d}t + \sigma(X)\circ\mathrm{d}W, \tag{1.2}$$

where $\circ$ denotes the Stratonovich differential.

The random drift u is assumed to belong to the class

$$\begin{aligned}\mathcal{H} := \Bigg\{ & u(\omega, t, x) = \xi(\omega, t) + \int_{-\infty}^{x} q(\omega, t, y)\,\mathrm{d}y, \ \ \mathbb{P}\otimes\mathrm{d}t - a.e. \ : \\ & q \in L^p\left(\Omega; L^\infty([0, T]; (L^2\cap L^1)(\mathbb{R}))\right), \ \xi \in L^p(\Omega; L^\infty([0, T])), \ p \in [1, \infty) \Bigg\}.\end{aligned} \tag{1.3}$$

In this way, we have $u \in L^p\left(\Omega; L^\infty([0, T]; \dot{H}^1(\mathbb{R}) \cap \dot{W}^{1,1}(\mathbb{R}))\right)$, the equivalence class of functions that is identical to u in the $\mathbb{P}\otimes\mathrm{d}t\otimes\mathrm{d}x$ – a.e. sense. The semi-normed vector space $\dot{H}^1(\mathbb{R})$ is defined as the subspace of functions in $L^\infty(\mathbb{R})$ having a weak derivative in $L^2(\mathbb{R})$, with semi-norm $|h|_{\dot{H}^1(\mathbb{R})} = \|\partial_x h\|_{L^2(\mathbb{R})}$. The space $\dot{W}^{1,1}(\mathbb{R})$ is similarly defined. For any $u \in \mathcal{H}$, we have that q is the weak spatial derivative of u, $\mathbb{P}\otimes\mathrm{d}t$ – a.e., a fact that is denoted $q = \partial_x u$. Finally, note that for $\mathbb{P}\otimes\mathrm{d}t$ – a.e. (ω, t), the function $x \mapsto u(\omega, t, x)$ is absolutely continuous.

In the deterministic case ($\sigma \equiv 0$), the condition $u \in \dot{H}^1_x$ implies that u is $\frac{1}{2}$-Hölder continuous in x, which does not ensure the uniqueness of solutions (there are simple counterexamples). Given the assumption $\partial_x u \in L^1_\omega L^\infty_t L^2_x$, we clearly have $\mathbb{E}\int_0^T\int_{-\ell}^{\ell}|\partial_x u|\,\mathrm{d}x\,\mathrm{d}t < \infty$, for all $\ell > 0$, and thus $|\partial_x u(\omega, t, x)| < \infty$ a.e. in (ω, t, x). This is not enough to apply the existing theory [11] for SDEs with random coefficients, which requires an explicit bound on $\partial_x u$ (from one side). Roughly speaking, one needs $\partial_x u(\omega, t, x) \leq K_\ell(\omega, t)$, for all $x \in [-\ell, \ell]$, where $\int_0^T K_\ell(\omega, t)\,\mathrm{d}t < \infty$ almost surely.

This paper replaces the condition $K_\ell \in L^1([0, T])$ by a notably weaker condition. Indeed, we assume the following one-sided gradient bound:

$$q(\omega, t, x) = \partial_x u(\omega, t, x) \leq K(\omega, t), \tag{1.4}$$

where $K > 0$ and, for some $p > 1$,

$$\mathbb{E}\exp\left(p\int_{\varepsilon}^{T} K(t)\,\mathrm{d}t\right) \lesssim_{p,T} \varepsilon^{-2p}, \quad \text{for all } \varepsilon \in (0,1). \tag{1.5}$$

Here we use the notation $h_1 \lesssim_\alpha h_2$ if $h_1 \le C(\alpha)h_2$ for some constant C that may depend on α, and non-negative functions h_1, h_2.

The pointwise condition (1.4), (1.5) allows for a logarithmic divergence in the temporal integral of $\partial_x u$, see Remark 3.1. In the deterministic case, (1.5) becomes $\int_\varepsilon^T K(t)\,\mathrm{d}t \lesssim -2\ln(\varepsilon)$, which corresponds to a bound of the form $\partial_x u \le 2/t =: K(t)$ and so $K \notin L^1([0,T])$. The conditions imposed on the random drift u are motivated by [9], where u solves a nonlinear stochastic transport equation, and the one-sided gradient bound (1.4), (1.5) acts as a selection principle for dissipative weak solutions of this SPDE. We will return to the motivation behind the key condition (1.4), (1.5) later. In Section 1.3, we also supply an explicit example of a random drift u for which (1.4), (1.5) hold, but $\partial_x u$ is not integrable on the entire interval $[0,T]$. A further illustration of this type of blow-up can be found in [10, Figs. 2.1, 2.2].

Finally, we require a temporal (right-continuity) condition at $t = 0$:

$$\lim_{t\downarrow 0}\mathbb{E}\,|u(t)-u(0)|^2_{\dot H^1(\mathbb{R})} = 0; \tag{1.6}$$

since $u \in \mathcal{H}$, cf. (1.3), this means that $q(t) \to q(0)$ in $L^2(\mathbb{R})$ as $t \downarrow 0$.

Regarding the noise coefficient σ, let us discuss the case of a deterministic σ first, cf. (1.2). In this case, we assume that $\sigma = \sigma(x)$ satisfies

$$\sigma \in C^2(\mathbb{R}), \quad \sigma', \sigma'', \left(\sigma^2\right)'' \in L^\infty(\mathbb{R}). \tag{1.7}$$

For such a σ, which is necessarily globally Lipschitz continuous and of linear growth, the second derivative $\frac{1}{2}\left(\sigma^2\right)'' = \left(\sigma'\right)^2 + \sigma\sigma''$ is bounded on $\mathbb{R}$. An example ensuring the latter is when σ', σ'' are bounded and σ'' is compactly supported on $\mathbb{R}$; then $\left(\sigma^2\right)'' \lesssim 1$. The noise coefficient σ is allowed to vanish in this work.

The main contribution of this paper is the treatment of the irregular random drift u. However, it turns out that our methods are sufficiently flexible to allow for a wider class of random noise coefficients $\sigma = \sigma(\omega,t,x)$. The conditions defining this class appear somewhat elaborate, but any deterministic $\sigma = \sigma(x)$ satisfying (1.7) belongs to this class. First, we require that $\sigma = \sigma(\omega,t,x)$ is progressively measurable (on $\mathcal{S}$), and that σ, $\left(\sigma^2\right)'$ are globally x-Lipschitz in the sense that

$$|\sigma(\omega,t,x)-\sigma(\omega,t,y)|, \left|\left(\sigma^2\right)'(\omega,t,x)-\left(\sigma^2\right)'(\omega,t,y)\right| \le \Lambda(\omega,t)\,|x-y|, \tag{1.8}$$

where the progressively measurable process $\Lambda(t) = \Lambda(\omega,t)$ exhibits exponential moments,

$$\mathbb{E}\exp\left(p\int_0^T \Lambda^2(t)\,\mathrm{d}t\right) < \infty, \ \forall p > 0. \tag{1.9}$$

The exponential moments (1.9) are used to prove the *existence* of a solution X that belongs (locally) to $L^p(\Omega; C([0,T]))$ for any finite $p \ge 1$. Dropping the requirement of arbitrary p-moments, one can relax (1.9) somewhat.

By Jensen's inequality, the condition (1.9) implies

$$\mathbb{E}\int_0^T \Lambda^2(t)\,\mathrm{d}t < \infty, \tag{1.10}$$

which will be used on certain occasions. Finally, we will also need the following technical conditions:

$$\left\| \left(\sigma^2\right)''(0) \right\|_{L^\infty(\Omega\times\mathbb{R})} < \infty; \tag{1.11a}$$

$$\left(\sigma^2\right)''(t) - \left(\sigma^2\right)''(0) \in L^2(\Omega\times\mathbb{R}), \quad \text{uniformly on } [0,T]; \tag{1.11b}$$

$$\lim_{t\downarrow 0} \mathbb{E} \left\| \left(\sigma^2\right)''(t) - \left(\sigma^2\right)''(0) \right\|^2_{L^2(\mathbb{R})} = 0; \tag{1.11c}$$

$$\mathbb{E}\int_0^T |\sigma(t,0)|^p \, \mathrm{d}t < \infty, \quad \mathbb{E}\int_0^T \left| \left(\sigma^2\right)'(t,0) \right|^p \, \mathrm{d}t < \infty, \quad \forall p \in [1,\infty). \tag{1.11d}$$

Our main result is the following theorem.

Theorem 1.1. *Suppose $u \in \mathcal{H}$, cf. (1.3), satisfies (1.4), (1.5) and (1.6), and σ satisfies (1.8), (1.9), (1.11a), (1.11b), (1.11c), and (1.11d). There exists a unique strong solution of (1.1).*

The central part of Theorem 1.1 is the uniqueness assertion (cf. Theorem 2.2). We prove pathwise uniqueness by a careful estimation of the difference between two solutions, making use of the Tanaka formula, the exponential moment bound (1.5) and a recent stochastic Gronwall inequality [19,22] (see Lemma 2.1). The exponential bound (1.5), along with (1.6), allows us to control the difference between the two solutions for short times $t \le \varepsilon$ ($\varepsilon \ll 1$), which is the main challenge in demonstrating pathwise uniqueness. When $\sigma \equiv 0$, our uniqueness result recovers [22, Prop. A]. The detailed proof reported in Section 2 can be viewed as a (surprisingly non-trivial) stochastic extension of the ODE proof in [22].

In Section 3, we demonstrate existence of strong solutions to the SDE (1.1) (Theorem 3.7). We approximate (1.1) using "one-sided truncation" $\{u_R\}$ of the drift u, and then make use of Krylov's theorem [11] for SDEs with random coefficients to solve (1.1) with $u = u_R$. This produces a family of solutions $\{X_R\}$, indexed by the truncation level R with $R \to \infty$. We show that $\{X_R\}$ constitutes a Cauchy sequence in the space $L^{1/2}(\Omega; C([0,T]))$, with metric $d(X_1, X_2) := \mathbb{E}\sup_{t\in[0,T]} |X_1(t) - X_2(t)|^{1/2}$ (Proposition 3.5) [1, 4.7.62]. The proof of this result proceeds along the lines of the uniqueness argument. The Cauchy property, along with (1.9) and R-independent p-moments of X_R (Lemma 3.3), implies the existence of a limit $X \in L^2(\Omega; C([0,T]))$ such that $X_R \to X$ in $L^2(\Omega; C([0,T]))$. It is straightforward to deduce that X is a solution of (1.1) (Theorem 3.7).

Before discussing the literature on SDEs with irregular drift and the motivation behind our particular class of drift coefficients u, let us supply a relevant example of the process K in (1.4), (1.5).

Remark 1.1. Consider the SDE (1.2) with deterministic $\sigma = \sigma(x)$ satisfying (1.7). According to Section 1.3, it makes sense to impose the condition

$$q(\omega,t,x) = \partial_x u(\omega,t,x) \le C + \frac{e^{-\|\sigma'\|_{L^\infty} W(t)}}{\frac{1}{2}\int_0^t e^{-\|\sigma'\|_{L^\infty} W(s)}\, \mathrm{d}s}, \tag{1.12}$$

where $C \ge 0$ is a constant. Let us verify that q satisfies (1.4), (1.5). With

$$K(\omega,t) := C + \frac{e^{-\|\sigma'\|_{L^\infty} W(t)}}{\frac{1}{2}\int_0^t e^{-\|\sigma'\|_{L^\infty} W(s)}\, \mathrm{d}s} = C + 2\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\frac{1}{2}\int_0^t e^{-\|\sigma'\|_{L^\infty} W(s)}\, \mathrm{d}s\right),$$

we find that $I(\varepsilon) := \mathbb{E}\exp\left(p\int_\varepsilon^T K(\omega,s)\,\mathrm{d}s\right)$ satisfies

$$I(\varepsilon) = e^{pC(T-\varepsilon)}\mathbb{E}\left(\frac{\int_0^T \exp\left(-\left\|\sigma'\right\|_{L^\infty} W(s)\right)\,\mathrm{d}s}{\int_0^\varepsilon \exp\left(-\|\sigma'\|_{L^\infty} W(s)\right)\,\mathrm{d}s}\right)^{2p}.$$

By the Cauchy–Schwarz inequality, $I(\varepsilon) \lesssim_{T,p} (I_-)^{1/2}(I_+)^{1/2}$, where

$$I_+ := \mathbb{E}\left(\int_0^T \exp\bigl(-\left\|\sigma'\right\|_{L^\infty} W(s)\bigr)\,\mathrm{d}s\right)^{4p},$$
$$I_- := \mathbb{E}\left(\int_0^\varepsilon \exp\bigl(-\left\|\sigma'\right\|_{L^\infty} W(s)\bigr)\,\mathrm{d}s\right)^{-4p}.$$

We estimate I_- as follows:

$$\begin{aligned}
I_- &\le \mathbb{E}\left(\varepsilon \min_{s\in[0,\varepsilon]} \exp\left(-\left\|\sigma'\right\|_{L^\infty} W(s)\right)\right)^{-4p}\\
&\le \mathbb{E}\left(\varepsilon \exp\left(-\left\|\sigma'\right\|_{L^\infty} \max_{s\in[0,\varepsilon]} W(s)\right)\right)^{-4p}\\
&= \frac{2}{\sqrt{2\pi\varepsilon}}\int_0^\infty \varepsilon^{-4p}\exp\bigl(4p\left\|\sigma'\right\|_{L^\infty} x - \frac{x^2}{2\varepsilon}\bigr)\,\mathrm{d}x \lesssim_{p,\sigma} \varepsilon^{-4p},
\end{aligned}$$

where we have used that the law on $[0,\infty)$ of $\max_{s\in[0,\varepsilon]} W(s)$ is equivalent to the law of $|W(\varepsilon)|$ [16, Prop. III.3.7], for which we have

$$\mathbb{P}\bigl(|W(t)| \in \mathrm{d}x\bigr) = \frac{2}{\sqrt{2\pi t}} e^{-x^2/(2t)}\,\mathrm{d}x.$$

Similarly, $I_+ \lesssim_{\sigma,T,p} 1$. Hence $I(\varepsilon) \lesssim_{p,T,\sigma} \varepsilon^{-2p}$ (for finite p), i.e., (1.4), (1.5) hold.

1.2. Background

Let us contextualise our result by discussing some previous studies on the well-posedness of SDEs. There is a very rich literature studying the existence and uniqueness of solutions, which begins with Itô's work on SDEs with globally Lipschitz coefficients (see [16, Ch. IX]). Often the Lipschitz condition is too strong. While weak existence is relatively easy to obtain for non-smooth coefficients (via, say, Girsanov's theorem), the construction of strong solutions is a more delicate matter. Strong solutions of SDEs with rough deterministic coefficients have been studied by many authors, beginning with [18,24], and later [5,7,8,12], to mention just a few examples. Most of these works use the Fokker–Planck PDE associated with the SDE, the Krylov estimate, and the Zvonkin transformation, which require the noise coefficient to be non-degenerate (uniformly elliptic). As a consequence, the results hold under very weak conditions on the drift, much weaker than in deterministic ODEs. For recent work on the well-posedness of SDEs with (Sobolev) rough coefficients and degenerate noise, see [2]. A probabilistic approach based on Malliavin calculus (nondegenerate noise) is developed in [13,15]. Most of the cited articles assume additive noise. The works [20,21] consider multiplicative noise under non-degeneracy and Sobolev regularity conditions on the noise coefficient. For a detailed study of one-dimensional SDEs, see the book [3].

The influential paper [6] studied stochastic regularisation in linear transport SPDEs with non-smooth velocity b, for which the *characteristic equation* is

$$\mathrm{d}X = b(t,X)\,\mathrm{d}t + \mathrm{d}W. \tag{1.13}$$

Using the Itô–Tanaka trick and solution regularity of the associated Fokker–Planck equation (a backward parabolic equation), they establish uniqueness of solutions to stochastic transport equations under a regularity condition on b that is weaker than in the DiPerna–Lions–Ambrosio theory of deterministic transport equations. To do so they prove the existence and uniqueness of solutions to the SDE (1.13) with minimal regularity assumptions on b using the short-time smooth flow of the associated backward parabolic equation. In [6, Sec. 6.2] they give negative examples showing that their results do not hold for equations with random drift b, a typical example of which is $b = b(\omega, t, x) = |x - W(t)|^{1/2} \wedge 1$. Whilst this b is locally in $\dot{H}^1(\mathbb{R})$, $\partial_x b$ does not satisfy a one-sided bound of the form (1.4), (1.5). Motivated by [6], there were many additional works studying strong solutions of SDEs like (1.13) with non-smooth drift b, but almost all of them assume that b is deterministic.

Let us turn our attention to SDEs with random coefficients. In [11], Krylov established the existence and uniqueness of strong solutions to

$$\mathrm{d}X = b(\omega, t, X)\,\mathrm{d}t + \sigma(\omega, t, X)\,\mathrm{d}W, \tag{1.14}$$

under some boundedness, monotonicity, and coercivity conditions on the random coefficients b and σ. His proof is based on a detailed convergence analysis of the Euler discretisation scheme. We state Krylov's result as Theorem 3.1, and use it in Section 3 as a part of the existence proof. Because of an indispensable "logarithmic divergence" at $t = 0$, Krylov's theorem does not apply to the SDE (1.1) with u satisfying the one-sided gradient bound (1.4), (1.5).

With a random drift b and $\sigma \equiv 1$ in (1.14), the work [4] partially recovered the results of [6] under an additional condition of Malliavin differentiability of b. The proof employed a Girsanov transformation idea [24], which extends the Itô–Tanaka trick in [6], by considering a backward parabolic SPDE instead of the Fokker–Planck PDE associated with X for a deterministic b. We also refer to [14] for a related result, which allows for the drift $b(\omega, t, x) = b_1(t, x) + b_2(\omega, t, x)$, where the deterministic part b_1 is measurable and of linear growth. In contrast, the random part b_2 is sufficiently smooth in t, x and Malliavin differentiable in ω. These results were extended and sharpened in [23] to the SDE (1.14) with non-degenerate noise and random coefficients b and σ satisfying similar (t, x)-regularity and Malliavin differentiability conditions. An illustrative example of random drift b covered by these recent works is $b(\omega, t, x) = f(t, x, W(t))$ for a function f that is Lipschitz continuous in the last variable. The works [4,14,23] cannot handle the SDE (1.14) with random drift $u \in \mathcal{H}$ satisfying (1.4), (1.5) and (1.6), even if we were to assume that $\sigma(\cdot) > 0$. The proof of our Theorem 1.1 will not use ideas based on the associated backward SPDE, nor will we impose non-degeneracy or Malliavin differentiability conditions on our coefficients.

1.3. Motivation

We conclude this introduction with a brief motivation of the current study, which stems from our ongoing investigation into the uniqueness and dissipation properties of solutions to the stochastic Hunter–Saxton equation [9]

$$\mathrm{d}q + \partial_x (uq)\,\mathrm{d}t - \frac{1}{2}q^2\,\mathrm{d}t + \partial_x (\sigma q) \circ \mathrm{d}W = 0, \qquad \partial_x u = q. \tag{1.15}$$

Existence results, along with a specific distribution for wave-breaking (finite-time blowup and continuation), were derived for the nonlinear transport-type SPDE (1.15) in [9]. These results were derived under the condition that σ is linear. Solutions to (1.15) were constructed from its characteristic equation, namely the SDE (1.1).

Using the Itô–Wentzell theorem and the characteristic equation (1.2), the following Lagrangian formulation of (1.15) can be postulated:

$$\mathrm{d}\mathfrak{Q} = -\frac{1}{2}\mathfrak{Q}^2\,\mathrm{d}t - \sigma'\mathfrak{Q}\circ \mathrm{d}W, \qquad \mathfrak{Q}(0) = q(0,x). \tag{1.16}$$

This SDE can be solved exactly as a *stochastic Verhulst equation*. The solution is

$$\mathfrak{Q}(t,x) = \frac{e^{-\sigma' W(t)}}{\frac{1}{q(0,x)} + \frac{1}{2}\int_0^t e^{-\sigma' W(s)}\,\mathrm{d}s}. \tag{1.17}$$

In [9], we constructed the drift u directly in such a way that it was obvious that (1.1) was well-posed, and $\mathfrak{Q}(t,x) = \partial_x u(t, X(t,x))$ solved (1.16), providing us with a way to construct solutions to the stochastic Hunter–Saxton equation (1.15) along characteristics. The solution to the SDE (1.16) identifies the dissipative solution of the SPDE (1.15) with an Oleĭnik-type (one-sided gradient) bound. This motivates our study of (1.1) with random drift u satisfying (1.12), and thus (1.4), (1.5).

In an ongoing work, we study the uniqueness question for the stochastic Hunter–Saxton equation (1.15). In that work, starting from a solution to the SPDE (1.15), we must derive properties of the solution to the characteristic equation (1.2). The well-posedness theorem in the present paper, which we believe is of independent interest, is needed as a part of that endeavour.

Remark 1.2. Finally, we present an example of a random drift u motivated by (1.15), cf. [9]. Fixing a number $c \in \mathbb{R}$, let $Z_1(t)$ be the unique solution to

$$Z_1(t) = \frac{c^2}{2}\int_0^t Z_1(s)\,\mathrm{d}s + \int_0^t c\,Z_1(s)\,\mathrm{d}W.$$

Fixing a number $v_0 > 0$, we introduce

$$Z_2(t) = Z_1(t) + \exp\left(cW(t) + \int_0^t \frac{\exp\big(-cW(s)\big)}{-v_0 + \frac{1}{2}\int_0^s \exp\big(-cW(r)\big)\,\mathrm{d}r}\,\mathrm{d}s\right).$$

Finally, we set

$$Z_3(t) = \big(Z_2(t) - Z_1(t)\big)\frac{\exp\big(-cW(t)\big)}{-v_0 + \frac{1}{2}\int_0^t \exp\big(-cW(s)\big)\,\mathrm{d}s}.$$

Denote by $T^\star = T^\star(\omega)$ the (blow-up) time for which

$$\lim_{t\uparrow T^\star}\int_0^t \exp\big(-cW(s)\big)\,\mathrm{d}s = 2v_0.$$

Now we define the adapted and continuous drift coefficient u by

$$u(\omega,t,x) = \frac{x - Z_1(t)}{Z_2(t) - Z_1(t)} Z_3(t)\mathbb{1}_{[0,T^\star)\times[Z_1(t),Z_2(t))} + Z_3(t)\mathbb{1}_{[0,T^\star)\times[Z_2(t),\infty)}.$$

Clearly, the gradient

$$\partial_x u(t) = \frac{\exp\big(-cW(t)\big)}{-v_0 + \frac{1}{2}\int_0^t \exp\big(-cW(s)\big)\,\mathrm{d}s}\mathbb{1}_{[0,T^\star)\times[Z_1(t),Z_2(t))}$$

blows up ($\partial_x u \to -\infty$ while $|u|$ remains bounded) as $t \uparrow T^\star$ but evidently (1.12) — and hence (1.4), (1.5) — holds. Besides, $u \in \mathcal{H}$, and one can check that u obeys (1.6). Note that $u(t) \equiv 0$ for all $t > T^\star$, which corresponds to a dissipative solution of the stochastic Hunter–Saxton equation (1.15).

2. Pathwise uniqueness

In this section, we prove the uniqueness part of Theorem 1.1. We make essential use of the stochastic Gronwall inequality established recently by Scheutzow [17]. The proof in [17] relies on a martingale inequality of Burkholder that holds for continuous martingales. Below we recall a mild refinement due to Xie and Zhang [19, Lemma 3.8] which holds for general discontinuous martingales. The stochastic Gronwall lemma provides an upper bound for the pth moment of a process ξ that does not depend on the martingale part M of the inequality. It is this convenient "martingale uniformity" that forces $p \in (0, 1)$.

Lemma 2.1 (*[19]*). *Fix a stochastic basis $\mathcal{S}$. Let $\xi(t)$ and $\eta(t)$ be non-negative adapted processes, $A(t)$ be a non-decreasing adapted process starting at $A(0) = 0$, and M be a local martingale with $M(0) = 0$. Suppose ξ is càdlàg in time and satisfies the following pathwise differential inequality:*

$$\mathrm{d}\xi \le \eta \, \mathrm{d}t + \xi \, \mathrm{d}A + \mathrm{d}M \quad \text{on } [0, T].$$

For any $0 < p < r < 1$ and $t \in [0, T]$,

$$\left(\mathbb{E} \sup_{s \in [0,t]} \xi^p(s) \right)^{1/p} \le C_{p,r} \left(\mathbb{E} \exp\left(\frac{r}{1-r} A(t) \right) \right)^{(1-r)/r} \mathbb{E} \left(\xi(0) + \int_0^t \eta(s) \, \mathrm{d}s \right),$$

where $C_{p,r} = \left(\frac{r}{r-p} \right)^{1/p}$.

We are now in a position to prove the following result.

Theorem 2.2 (*Pathwise uniqueness*). *Suppose that $u \in \mathcal{H}$, cf. (1.3), satisfies (1.4), (1.5) and (1.6), and σ satisfies (1.8), (1.10), (1.11a), (1.11b)* and *(1.11c). Let X_1, X_2 be two (strong) solutions of the SDE (1.1) on $[0, T]$, with $T > 0$ finite. Then*

$$\mathbb{E} \sup_{t \in [0,T]} |X_2(t) - X_1(t)|^{1/2} = 0. \tag{2.1}$$

Consequently, $\mathbb{P}\Big(\big\{ \omega \in \Omega : X_1(\omega, t) = X_2(\omega, t) \; \forall t \in [0, T] \big\} \Big) = 1$, i.e., X_1 and X_2 are indistinguishable.

Proof. Let X_1, X_2, and T be as in the statement of the theorem. Without loss of generality, we assume throughout the proof that

$$|X_i(t)| \le N, \quad t \in [0, T], \quad i = 1, 2, \tag{2.2}$$

for some $N > 0$. Indeed, introducing the stopping time

$$\tau_N := \inf \{ t \in [0, T] : |X_1(t)| > N \;\text{ or }\; |X_2(t)| > N \},$$

we may replace X_i by $\tilde{X}_i(t) := X_i\big(t \wedge \tau_N\big)$, which satisfies $\left|\tilde{X}_i(t)\right| \le N$ for all $t \in (0, \tau_N]$. The SDE for $\tilde{X}_i$ becomes

$$\begin{aligned}\tilde{X}_i(t) &= x + \int_0^{t\wedge\tau_N} u(s, X_i(s))\,\mathrm{d}s + \frac{1}{4}\int_0^{t\wedge\tau_N} \left(\sigma^2\right)'(s, X_i(s))\,\mathrm{d}s \\ &\quad + \int_0^{t\wedge\tau_N} \sigma(s, X_i(s))\,\mathrm{d}W(s) \\ &= x + \int_0^t u\left(s, \tilde{X}_i(s)\right)\,\mathrm{d}s + \frac{1}{4}\int_0^t \left(\sigma^2\right)'\left(s, \tilde{X}_i(s)\right)\,\mathrm{d}s \\ &\quad + \int_0^t \sigma\left(s, \tilde{X}_i(s)\right)\,\mathrm{d}W(s), \qquad t \in [0, \tau_N].\end{aligned}$$

We can therefore apply the upcoming argument to $\tilde{X}_2 - \tilde{X}_1$ on $[0, \tau_N]$ instead of to $X_2 - X_1$ on $[0, T]$, to deduce that

$$\mathbb{E}\sup_{t\in[0,T]} \left|X_2\left(t\wedge\tau_N\right) - X_1\left(t\wedge\tau_N\right)\right|^{1/2} = 0,$$

for any finite N. By the continuity of X_1 and X_2, we have that $\tau_N \to T$ a.s. as $N \to \infty$. Sending $N \to \infty$, Fatou's lemma yields

$$\begin{aligned}&\mathbb{E}\liminf_{N\to\infty}\sup_{t\in[0,T]} \left|X_2\left(t\wedge\tau_N\right) - X_1\left(t\wedge\tau_N\right)\right|^{1/2} \\ &\quad \le \lim_{N\to\infty}\mathbb{E}\sup_{t\in[0,T]} \left|X_2\left(t\wedge\tau_N\right) - X_1\left(t\wedge\tau_N\right)\right|^{1/2} = 0.\end{aligned}$$

Clearly, we also have

$$\begin{aligned}0 &\le \sup_{t\in[0,T]}\liminf_{N\to\infty} \left|X_2\left(t\wedge\tau_N\right) - X_1\left(t\wedge\tau_N\right)\right|^{1/2} \\ &\le \liminf_{N\to\infty}\sup_{t\in[0,T]} \left|X_2\left(t\wedge\tau_N\right) - X_1\left(t\wedge\tau_N\right)\right|^{1/2}.\end{aligned}$$

Since $\tau_N \to T$ a.s., the limit inferior is in fact a limit and it equals $|X_2(t) - X_1(t)|^{1/2}$, and upon taking expectation, this yields (2.1).

In what follows, we consider $X_2 - X_1$ and assume (2.2). We have by linearity

$$\begin{aligned}\mathrm{d}(X_2 - X_1) &= \left(u(t, X_2) - u(t, X_1)\right)\,\mathrm{d}t + \frac{1}{4}\left(\left(\sigma^2\right)'(t, X_2) - \left(\sigma^2\right)'(t, X_1)\right)\,\mathrm{d}t \\ &\quad + \left(\sigma(t, X_2) - \sigma(t, X_1)\right)\,\mathrm{d}W.\end{aligned}$$

Set $Y := |X_2 - X_1|$. By the Tanaka formula,

$$\mathrm{d}Y = \operatorname{sgn}\left(X_2 - X_1\right)\,\mathrm{d}\left(X_2 - X_1\right) + \frac{1}{2}\left(\sigma(t, X_2) - \sigma(t, X_1)\right)^2\,\mathrm{d}L_Y^0(t).$$

Since the local time L_Y^0 at 0 of Y is supported on the zero set of $X_2 - X_1$, which is a subset of the zero set of $\sigma(t, X_2) - \sigma(t, X_1)$, the local time correction term is zero. Set $\phi_\sigma(t) := \left(\sigma(t, X_2) - \sigma(t, X_1)\right)/Y(t)$, which is a process uniformly bounded in absolute value by $\Lambda(t)$ of (1.10). Integrating in time yields

$$\begin{aligned}Y(t) &= \int_0^t \operatorname{sgn}\left(X_2(s) - X_1(s)\right)\int_{X_1(s)}^{X_2(s)}\left(q(s, y) + \frac{1}{4}\left(\sigma^2\right)''(s, y)\right)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_0^t \phi_\sigma(s)Y(s)\,\mathrm{d}W(s),\end{aligned} \tag{2.3}$$

where, in view of (1.10) and (2.2), the last term is a square-integrable martingale starting from zero. In (2.3), we have used the $\mathbb{P}\otimes \mathrm{d}t$-a.e. representation

$$u(\omega, t, X_1(\omega, t)) - u(\omega, t, X_2(\omega, t)) = \int_{X_1(\omega,t)}^{X_2(\omega,t)} q(\omega, t, y)\,\mathrm{d}y.$$

This is justified by the assumption (1.3) that u is absolutely continuous in x ($\mathbb{P}\otimes \mathrm{d}t$-a.e.), so that the compositions $u(t, X_i)$ make $\mathbb{P}\otimes \mathrm{d}t$-a.e. sense.

Making use of (1.11b) in (2.3) and taking the expectation, we obtain

$$\begin{aligned}
\mathbb{E}Y(t) &= \mathbb{E}\int_0^t \operatorname{sgn}\left(X_2(s) - X_1(s)\right) \int_{X_1(s)}^{X_2(s)} q(s, y)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \frac{1}{4}\mathbb{E}\int_0^t \operatorname{sgn}(X_2(s) - X_1(s)) \int_{X_1(s)}^{X_2(s)} \left(\sigma^2\right)''(s, y)\,\mathrm{d}y\,\mathrm{d}s \\
&\le \mathbb{E}\int_0^t Y^{1/2}(s)\,\|q(s)\|_{L^2(\Delta_s)}\,\mathrm{d}s + \frac{1}{4}\left\|\left(\sigma^2\right)''(0)\right\|_{L^\infty(\Omega\times\mathbb{R})}\int_0^t \mathbb{E}Y(s)\,\mathrm{d}s \\
&\quad + \frac{1}{4}\mathbb{E}\int_0^t Y^{1/2}(s)\left\|\left(\sigma^2\right)''(s) - \left(\sigma^2\right)''(0)\right\|_{L^2(\Delta_s)}\mathrm{d}s \\
&\le \int_0^t \left(\mathbb{E}Y(s)\right)^{1/2}\Bigg[\left(\mathbb{E}\,\|q(s)\|^2_{L^2(\Delta_s)}\right)^{1/2} \\
&\qquad\qquad + \frac{1}{4}\left(\mathbb{E}\left\|\left(\sigma^2\right)''(s) - \left(\sigma^2\right)''(0)\right\|^2_{L^2(\Delta_s)}\right)^{1/2}\Bigg]\mathrm{d}s \\
&\quad + \frac{1}{4}\left\|\left(\sigma^2\right)''(0)\right\|_{L^\infty(\Omega\times\mathbb{R})}\int_0^t \mathbb{E}Y(s)\,\mathrm{d}s,
\end{aligned}$$

by the Cauchy–Schwarz inequality. Here, Δ_s denotes the (random) interval

$$\Delta_s = \Big[X_1(s)\wedge X_2(s),\ X_1(s)\vee X_2(s)\Big].$$

Taking the supremum over $t\in[0,\varepsilon]$ on both sides gives

$$\begin{aligned}
&\sup_{t\in[0,\varepsilon]} \mathbb{E}Y(t) \\
&\quad \le \frac{\varepsilon}{4}\left\|\left(\sigma^2\right)''(0)\right\|_{L^\infty(\Omega\times\mathbb{R})}\sup_{t\in[0,\varepsilon]}\mathbb{E}Y(t) + \varepsilon\sup_{t\in[0,\varepsilon]}\left(\mathbb{E}Y(t)\right)^{1/2} \\
&\qquad \times \sup_{t\in[0,\varepsilon]}\left[\left(\mathbb{E}\,\|q(t)\|^2_{L^2(\Delta_t)}\right)^{1/2} + \frac{1}{4}\left(\mathbb{E}\left\|\left(\sigma^2\right)''(t) - \left(\sigma^2\right)''(0)\right\|^2_{L^2(\Delta_t)}\right)^{1/2}\right].
\end{aligned}$$

Fix ε so small that $\frac{\varepsilon}{4}\left\|\left(\sigma^2\right)''(0)\right\|_{L^\infty(\Omega\times\mathbb{R})} \le \frac{1}{2}$. The first term on the right-hand side can be absorbed by the term on the left-hand side. We then divide through by $\sup_{t\in[0,\varepsilon]}\left(\mathbb{E}Y(t)\right)^{1/2}$ and square both sides, eventually arriving at

$$\sup_{t\in[0,\varepsilon]}\mathbb{E}Y(t) \le 8\varepsilon^2 \sup_{t\in[0,\varepsilon]}\mathbb{E}\left[\|q(t)\|^2_{L^2(\mathbb{R})} + \left\|\left(\sigma^2\right)''(t) - \left(\sigma^2\right)''(0)\right\|^2_{L^2(\mathbb{R})}\right] \lesssim \varepsilon^2. \tag{2.4}$$

The estimate (2.4) allows us to control $\mathbb{E}Y(t)$ near $t = 0$. Using the one-sided bound (1.12), which deteriorates near $t = 0$ for every $\omega\in\Omega$, in combination with the quadratic short-time estimate (2.4), we will next deduce a global estimate on the entire time interval $[0, T]$.

Given the short-time estimate (2.4), we begin afresh from (2.3). Again let ε be so small that $\varepsilon \left\| \left(\sigma^2\right)''(0) \right\|_{L^\infty(\Omega\times\mathbb{R})} \leq 2$, and $t > \varepsilon$. We can then write the inequality

$$\begin{aligned} Y(t) &= \int_0^\varepsilon \operatorname{sgn}\left(X_2(s) - X_1(s)\right) \int_{X_1(s)}^{X_2(s)} q(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_\varepsilon^t \operatorname{sgn}\left(X_2(s) - X_1(s)\right) \int_{X_1(s)}^{X_2(s)} q(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \frac{1}{4}\int_0^t \operatorname{sgn}\left(X_2(s) - X_1(s)\right) \int_{X_1(s)}^{X_2(s)} \left(\sigma^2\right)''(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_0^t \phi_\sigma(s) Y(s)\,\mathrm{d}W(s) \\ &\leq \int_0^t \eta_\varepsilon(s)\,\mathrm{d}s + \int_0^t Y(s)\,\mathrm{d}A_\varepsilon(s) + M(t), \end{aligned}$$

where, for $t \in [0,T]$,

$$\begin{aligned} M(t) &:= \int_0^t \phi_\sigma(s) Y(s)\,\mathrm{d}W(s), \\ \eta_\varepsilon(t) &:= \mathbb{1}_{\{t\leq\varepsilon\}} \operatorname{sgn}\left(X_2(t) - X_1(t)\right) \int_{X_1(t)}^{X_2(t)} |q(t,y)|\,\mathrm{d}y, \end{aligned}$$

and,

$$A_\varepsilon(t) := \int_0^t \left[\mathbb{1}_{\{s\geq\varepsilon\}} K(s) + \Lambda(s)\right] \mathrm{d}s, \tag{2.5}$$

for $K(t) = K(\omega,t)$ defined in (1.4), (1.5), and because, from (1.8),

$$\frac{1}{4}\operatorname{sgn}\left(X_2(s) - X_1(s)\right) \int_{X_1(s)}^{X_2(s)} \left(\sigma^2\right)''(s,y)\,\mathrm{d}y \leq \frac{1}{4} Y(s)\Lambda(s).$$

The adapted process η_ε is non-negative. Furthermore, using first the Cauchy–Schwarz inequality and then the short-time estimate (2.4), we have

$$\mathbb{E}\int_0^t \eta_\varepsilon(s)\,ds \leq \int_0^\varepsilon \left(\mathbb{E}Y(s)\right)^{1/2} \left(\mathbb{E}\left\|q(s)\right\|_{L^2(\Delta_s)}^2\right)^{1/2} \mathrm{d}s \lesssim \varepsilon^2 \rho(\varepsilon),$$

where

$$\rho(\varepsilon) := \left(\sup_{s\in[0,\varepsilon]} \mathbb{E}\left\|q(s)\right\|_{L^2(\Delta_s)}^2\right)^{1/2}.$$

We will show that $\rho(\varepsilon) = o(1)$ as $\varepsilon \to 0$. Furthermore, A_ε is a non-decreasing adapted process with $A(0) = 0$. From (1.5) and (1.9),

$$\begin{aligned} \mathbb{E}\exp\left(\mu A_\varepsilon(t)\right) &= \mathbb{E}\exp\left(\mu\int_\varepsilon^t K(s)\,ds + \mu\int_0^t \Lambda(s)\,\mathrm{d}s\right) \\ &\leq \left(\mathbb{E}\exp\left(2\mu\int_\varepsilon^t K(s)\,ds\right)\right)^{1/2} \left(\mathbb{E}\exp\left(2\mu\int_0^t \Lambda(s)\,\mathrm{d}s\right)\right)^{1/2} \\ &\leq C_\mu \varepsilon^{-2\mu}, \end{aligned}$$

for a number μ such that $2\mu = p$, cf. (1.5).

Finally, by (1.10) and (2.2), M is a square-integrable martingale with $M(0) = 0$. Hence, in view of Lemma 2.1, the stochastic Gronwall inequality with $p = \frac{1}{2}$ and a suitable $r \in (1/2, 1)$, we arrive at

$$\left(\mathbb{E} \sup_{s\in[0,t]} Y^{1/2}(s)\right)^2 \leq \left(\frac{2r}{2r-1}\right)^2 \left(\mathbb{E} \exp\Big(\frac{r}{1-r} A_\varepsilon(t)\Big)\right)^{(1-r)/r} \mathbb{E}\left(\int_0^t \eta_\varepsilon(s)\,\mathrm{d}t\right) \stackrel{(1.5)}{\leq} C_r e^{C_\sigma(t-\varepsilon)} \left(\varepsilon^{-2r/(1-r)}\right)^{(1-r)/r} \varepsilon^2 \rho(\varepsilon) \lesssim \rho(\varepsilon), \tag{2.6}$$

where C_r is a constant depending only on r and C_σ is coming from (1.11a).

Next we will show that the right-continuity condition (1.6) ensures that

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,\varepsilon]} \mathbb{E}\, \|q(t)\|^2_{L^2(\Delta_t)} = 0. \tag{2.7}$$

Clearly,

$$\sup_{t\in[0,\varepsilon]} \mathbb{E}\,\|q(t)\|^2_{L^2(\Delta_t)} \leq 2 \sup_{t\in[0,\varepsilon]} \mathbb{E}\,\|q(t)-q(0)\|^2_{L^2(\Delta_t)} + 2 \sup_{t\in[0,\varepsilon]} \mathbb{E}\,\|q(0)\|^2_{L^2(\Delta_t)}.$$

The first term on the right-hand side is bounded by $2\sup_{t\in[0,\varepsilon]} \mathbb{E}\,|u(t)-u_0|^2_{\dot{H}^1(\mathbb{R})}$, which tends to zero by (1.6). Since $|\Delta_t| = Y(t)$, the result (2.4) implies $\mathbb{1}_{\Delta_t} \to 0$, $\mathbb{P}$-a.s., as $t \to 0$. We have $\mathbb{E}\,\|q(0)\|^2_{L^2(\Delta_t)} = \mathbb{E}\left(\mathbb{1}_{\Delta_t} \|q(0)\|^2_{L^2(\mathbb{R})}\right) \to 0$ as $t \to 0$ by the dominated convergence theorem, since $q(0) \in L^2(\Omega \times \mathbb{R})$. This proves (2.7).

Given (2.7) and (1.11c), it follows that $\rho(\varepsilon) = o(1)$ as $\varepsilon \to 0$. As a result, we can send $\varepsilon \to 0$ in (2.6) to reach the conclusion that $\mathbb{E}\sup_{s\in[0,t]} Y^{1/2}(s) = 0$, for any $t \in [0, T]$, which implies the desired result (2.1). □

Remark 2.1. We point out that whilst the result above holds for $q(0) \in L^2(\mathbb{R})$, that is, $q^2(0) \in L^1(\mathbb{R})$, it fails for general $q(t)$ for which the right-continuity limit $\lim_{t\downarrow 0} q^2(t)$ exists only in the sense of measures—but not in L^1 as required by (1.6). An example comes from the deterministic Hunter–Saxton equation with an initial condition of the form $q^2(0) = \delta_0$. Although it is possible to define characteristics for this case, the characteristics emanating from $x = 0$ are not unique. The temporal continuity condition (1.6) is essential.

3. Existence of solution

In this section, we establish the existence of strong solutions for the SDE (1.1) by approximating (1.1) using a truncated coefficient in a way that allows us to apply a well-posedness theorem of Krylov, reproduced below. We then show that the solutions to the approximating SDEs form a Cauchy sequence in an appropriate space, from which we recover a solution to our SDE.

We begin by recalling Krylov's theorem for the well-posedness of SDEs with random coefficients [11, Thm. 1.2].

Theorem 3.1 ([11]). *Let $\mathcal{S}$ be a stochastic basis. Assume that for any $\omega \in \Omega$, $t \geq 0$, and $x \in \mathbb{R}^d$, we have $V(\omega,t,x) \in \mathbb{R}^{d\times d}$ and $b(\omega,t,x) \in \mathbb{R}^d$, and that V and b are continuous in x for any (ω,t), and measurable in (ω,t). Moreover, assume*

(i) *boundedness: for any $T, \ell \in [0,\infty)$, $\omega \in \Omega$, and any matrix norm $\|V\|$,*

$$\int_0^T \sup_{|x|<\ell} \left(|b(t,x)| + \|V(t,x)\|^2\right) \mathrm{d}t < \infty;$$

(ii) *monotonicity: for all* $t, \ell \in [0, \infty)$, $x, y \in B_\ell(0)$, *the ball with radius* ℓ *and centred at the origin, and* $\omega \in \Omega$,

$$2(x-y)\cdot\big(b(t,x)-b(t,y)\big)+\|V(t,x)-V(t,y)\|^2 \le \tilde{K}(t,\ell)\,|x-y|^2\,;$$

(iii) *coercivity: for all* $t, \ell \in [0, \infty)$, $x \in B_\ell(0)$, *and* $\omega \in \Omega$,

$$2x\cdot b(t,x)+\|V(t,x)\|^2 \le \tilde{K}(t,1)\left(1+|x|^2\right);$$

where $\tilde{K}(t, \ell)$ *is an adapted non-negative processes satisfying*

$$\int_0^T \tilde{K}(t,\ell)\,\mathrm{d}t < \infty, \quad \text{for all } \omega \in \Omega,\ T, \ell \in [0,\infty). \tag{3.1}$$

Let X_0 *be an* $\mathcal{F}_0$*-measurable* $\mathbb{R}^d$*-valued random variable. Then the SDE*

$$\mathrm{d}X(t) = b(t, X(t))\,\mathrm{d}t + V(t, X(t))\,\mathrm{d}W(t), \quad X(0) = X_0$$

has a solution which is unique up to indistinguishability. We also have,

$$\mathbb{E}\left(e^{-\alpha(t)}X^2(t)\right) \le x^2+1, \qquad \alpha(t) := \int_0^t \tilde{K}(s,1)\,\mathrm{d}s. \tag{3.2}$$

Remark 3.1 (*Logarithmic divergence*)**.** The monotonicity condition in Theorem 3.1 can be viewed as a one-sided Lipschitz condition. In our motivating example, cf. Remark 1.1 and the one-sided gradient bound (1.12), we have

$$(x-y)\,(u(t,x)-u(t,y)) \le |x-y|^2\left(C+\frac{e^{-\|\sigma'\|_{L^\infty}W(t)}}{\frac{1}{2}\int_0^t e^{-\|\sigma'\|_{L^\infty}W(s)}\,\mathrm{d}s}\right).$$

Unfortunately, the factor multiplying $|x-y|^2$ is not sufficiently well controlled at $t = 0$ to ensure (3.1). There is the possibility of a logarithmic divergence in the temporal integral. As a result, Theorem 3.1 does not apply to our problem.

Next we introduce an approximate SDE by truncating the gradient $q = \partial_x u$. The reason for doing so is explained in Remark 3.1. The strong well-posedness of these approximate SDEs then follows from Theorem 3.1.

Lemma 3.2. *Suppose* $u \in \mathcal{H}$, *cf.* (1.3), *satisfies conditions* (1.4), (1.5) *and* (1.6), *and* σ *satisfies* (1.8), (1.10), (1.11a), (1.11b), *and* (1.11c). *Let* u_R *be the process obtained from* q *by one-sided truncation at level* $R > 0$*:*

$$u_R(t,x) := \int_{-\infty}^x \vartheta_R(q(t,y))\,\mathrm{d}y, \quad \vartheta_R(q) := \begin{cases} q, & \text{if } q \le R,\\ R, & \text{if } q > R.\end{cases} \tag{3.3}$$

The SDE

$$\mathrm{d}X_R = u_R(t,X_R)\,\mathrm{d}t + \frac{1}{4}\left(\sigma^2\right)'(t,X_R)\,\mathrm{d}t + \sigma(t,X_R)\,\mathrm{d}W(t), \quad X(0) = x \in \mathbb{R}, \tag{3.4}$$

has a unique strong solution.

Proof. We take $b = u_R + \frac{1}{4}\left(\sigma^2\right)' = u_R + \frac{1}{2}\sigma'\sigma$ and $V = \sigma$, on $\mathbb{R}^d$ with $d = 1$. The lemma follows from Theorem 3.1 once we have verified conditions (i), (ii), and (iii).

By assumption,

$$\mathbb{E}\,\|u\|^p_{L^\infty([0,T]\times\mathbb{R})} \lesssim \mathbb{E}\,\|q\|^p_{L^\infty([0,T];L^1(\mathbb{R}))} \lesssim_p 1$$

for $p \in [1,\infty)$. Of course, the same bound holds for u_R:

$$\mathbb{E}\,\|u_R\|^p_{L^\infty([0,T]\times\mathbb{R})} \lesssim_p 1. \tag{3.5}$$

From this bound (with $p = 1$),

$$\operatorname*{ess\,sup}_{\substack{t\in[0,T]\\ |x|<\ell}} |u_R(\omega,t,x)| < \infty, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

The Lipschitz condition (1.8) and (1.10), (1.11d) imply

$$\int_0^T \sup_{|x|<\ell} \left(\sigma^2\right)'(t,x)\,\mathrm{d}t \le \int_0^T \left(\left|\left(\sigma^2\right)'(t,0)\right| + \Lambda(t)\ell\right)\,\mathrm{d}t < \infty.$$

Similarly, we have

$$\int_0^T \sup_{|x|<\ell} \sigma^2(t,x)\,\mathrm{d}t \le \int_0^T \left(|\sigma(t,0)| + \Lambda(t)\ell\right)^2 \mathrm{d}t < \infty.$$

Hence

$$\int_0^T \sup_{|x|<\ell} \left(\left|u_R(t,x) + \frac{1}{4}\left(\sigma^2\right)'(t,x)\right| + |\sigma(t,x)|^2\right)\,\mathrm{d}t < \infty,$$

which is (i).

For condition (ii), we have by (1.8) that

$$\begin{aligned}
&2(x-y)\left(u_R(t,x) - u_R(t,y) + \frac{1}{4}\left(\sigma^2\right)'(t,x) - \frac{1}{4}\left(\sigma^2\right)'(t,y)\right) \\
&\qquad\qquad + |\sigma(t,x)-\sigma(t,y)|^2 \\
&\quad \le 2\,|x-y|\left(\left|\int_x^y \vartheta_R(q(t,z))\,\mathrm{d}z\right| + \frac{1}{4}\Lambda(t)\,|x-y|\right) + \Lambda^2(t)\,|x-y|^2 \\
&\quad \le \left(2R + \Lambda(t) + \Lambda^2(t)\right)|x-y|^2 =: \tilde{K}_1(t)\,|x-y|^2,
\end{aligned}$$

and $\tilde{K}_1(t)$ is readily seen to satisfy (3.1) by (1.10).

Finally, condition (iii) is a result of

$$\begin{aligned}
&2x\,u_R(t,x) + \frac{1}{2}x\left(\sigma^2\right)'(t,x) + \sigma^2(t,x) \\
&\quad \le 2\,|x|\,\|u_R(t)\|_{L^\infty(\mathbb{R})} + \frac{1}{2}\,|x|\left(\left|\left(\sigma^2\right)'(t,0)\right| + \Lambda(t)\,|x|\right) + \left(|\sigma(t,0)| + \Lambda(t)\,|x|\right)^2 \\
&\quad \le \left(\|u_R(t)\|_{L^\infty(\mathbb{R})} + \left|\left(\sigma^2\right)'(t,0)\right| + \frac{1}{2}\Lambda(t) + 2\sigma^2(t,0) + \Lambda^2(t)\right)\left(1+x^2\right) \\
&\quad =: \tilde{K}_2(t)\left(1+x^2\right),
\end{aligned}$$

where we have used (1.8), (1.10), and (1.11d). By (3.5), (1.10), and (1.11d), it follows that $\tilde{K}_2$ satisfies (3.1).

If we take $\tilde{K}(t,\ell) = \tilde{K}(t) := \tilde{K}_1(t) + \tilde{K}_2(t)$ (so $\tilde{K}$ is independent of ℓ, but dependent on R), then all three conditions are verified. □

The next lemma supplies R-independent estimates for X_R in $L^p(\Omega; C([0,T]))$ for any finite p. Observe that the L^2-estimate on $X_R(t)$ coming from Theorem 3.1, cf. (3.2), is useless because our K depends on R.

Lemma 3.3. *Let X_R be the solution constructed in Lemma 3.2. Assume in addition that (1.9) and (1.11d) hold. We have the uniform-in-R bound*

$$\mathbb{E} \sup_{t\in[0,T]} |X_R|^p \lesssim_{T,p} |x|^{4p} \lesssim_{x,T,p} 1, \qquad p \in [1,\infty). \tag{3.6}$$

Proof. We make frequent use of the following elementary inequalities, which hold for all $r \geq 2$ and $a, b, \epsilon > 0$:

$$a^{r-1}b \leq \frac{\epsilon(r-1)}{r} a^r + \frac{1}{\epsilon^{r-1} r} b^r, \quad a^{r-2}b^2 \leq \frac{\epsilon(r-2)}{r} a^r + \frac{2}{\epsilon^{(r-1)/2} r} b^r.$$

By Itô's formula, $|X_R(t)|^{2p} = |x|^{2p} + I_1(t) + I_2(t) + I_3(t) + M(t)$, where

$$I_1(t) := 2p \int_0^t \operatorname{sgn}(X_R)\, |X_R|^{2p-1}\, u_R(s, X_R)\, \mathrm{d}s,$$

$$I_2(t) := \frac{p}{2} \int_0^t \operatorname{sgn}(X_R)\, |X_R|^{2p-1} \left(\sigma^2\right)'(s, X_R)\, \mathrm{d}s,$$

$$I_3(t) := p(2p-1) \int_0^t |X_R|^{2p-2}\, \sigma^2(s, X_R)\, \mathrm{d}s,$$

$$M(t) := 2p \int_0^t \operatorname{sgn}(X_R)\, |X_R|^{2p-1}\, \sigma(s, X_R)\, \mathrm{d}W(s).$$

Given (1.8), we readily derive the bounds

$$I_1(t) \leq t\, \|u_R\|^{2p}_{L^\infty([0,T]\times\mathbb{R})} + \tilde{C}_p \int_0^t |X_R|^{2p}\, \mathrm{d}s,$$

$$I_2(t) \leq \tilde{C}_p \int_0^t \left((1 + \Lambda(s))\, |X_R|^{2p} + \left| \left(\sigma^2\right)'(s, 0) \right|^{2p} \right) \mathrm{d}s,$$

$$I_3(t) \leq \tilde{C}_p \int_0^t \left(1 + \Lambda^2(s)\right) |X_R|^{2p}\, \mathrm{d}s + \tilde{C}_p \int_0^t |\sigma(s,0)|^{2p}\, \mathrm{d}s,$$

for a constant $\tilde{C}_p$ depending only p. From this we obtain the inequality

$$|X_R(t)|^{2p} \leq |x|^{2p} + t\, \|u_R\|^{2p}_{L^\infty([0,T]\times\mathbb{R})} + C_p \int_0^t \left| \left(\sigma^2\right)'(s, 0) \right|^{2p} \mathrm{d}s$$
$$+ C_p \int_0^t |\sigma(s,0)|^{2p}\, \mathrm{d}s + C_p \int_0^t \left(1 + \Lambda^2(s)\right) |X_R(s)|^{2p}\, \mathrm{d}s + M(t),$$

for another constant C_p depending only p.

For any $N > 0$, introduce the stopping time

$$\tau_N := \inf\{t \in [0,T] : |X_R(t)| > N\}.$$

By the continuity of X_R we have that $\tau_N \to T$, $\mathbb{P}$-almost surely, as $N \to \infty$. Clearly, for $t \in [0, T]$,

$$\begin{aligned}|X_R(t\wedge\tau_N)|^{2p} \le |x|^{2p} &+ (t\wedge\tau_N)\,\|u_R\|^{2p}_{L^\infty([0,T]\times\mathbb{R})}\\ &+ C_p\int_0^{t\wedge\tau_N}\left|\left(\sigma^2\right)'(s,0)\right|^{2p}\,\mathrm{d}s + C_p\int_0^{t\wedge\tau_N}|\sigma(s,0)|^{2p}\,\mathrm{d}s\\ &+ C_p\int_0^{t\wedge\tau_N}\left(1+\Lambda^2(s)\right)|X_R(s\wedge\tau_N)|^{2p}\,\mathrm{d}s + M(t\wedge\tau_N),\end{aligned}$$

where $t \mapsto M(t\wedge\tau_N)$ is a (square-integrable) martingale starting from zero.

Using the stochastic Gronwall inequality (Lemma 2.1 with exponents $\frac{1}{2}$ and $\frac{2}{3}$),

$$\begin{aligned}&\left(\mathbb{E}\sup_{t\in[0,T]}|X_R(t\wedge\tau_N)|^p\right)^{1/2}\\ &\quad\le\left(\mathbb{E}\exp\left(C_p\int_0^T\left(1+\Lambda^2(t)\right)\mathrm{d}t\right)\right)^{1/2}\\ &\qquad\times\mathbb{E}\left(|x|^{2p}+T\,\|u_R\|^{2p}_{L^\infty([0,T]\times\mathbb{R})}+\int_0^T\left|\left(\sigma^2\right)'(s,0)\right|^{2p}\,\mathrm{d}s+\int_0^T|\sigma(t,0)|^{2p}\,\mathrm{d}t\right).\end{aligned}$$

Given (1.11d) and (3.5), we conclude that

$$\mathbb{E}\sup_{t\in[0,T]}|X_R(t\wedge\tau_N)|^p \lesssim_{x,T,p} 1.$$

Finally, sending $N \to \infty$, we arrive at (3.6). $\square$

To show that $\{X_R\}$ is a Cauchy sequence, we will require some compactness properties of u_R as $R \to \infty$. Since u_R is constructed from u in an explicit manner, this is not difficult to establish:

Lemma 3.4. *Suppose $u \in \mathcal{H}$, cf. (1.3). Let u_R be defined by the construction (3.3). For any finite $p \ge 1$,*

$$\mathbb{E}\,\|u_R - u\|^p_{L^\infty([0,T]\times\mathbb{R})} \xrightarrow{R\uparrow\infty} 0. \tag{3.7}$$

Proof. We have

$$\begin{aligned}\tilde{I}_R(t) := |u_R(t) - u(t)| &= \left|\int_{-\infty}^x \left(\vartheta_R(q(t,y)) - q(t,y)\right)\mathrm{d}y\right|\\ &\le \int_{-\infty}^x |R - q(t,y)|\,\mathbb{1}_{\{q(t,y)>R\}}\,\mathrm{d}y \le \frac{1}{R}\,\|q(t)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

By assumption, for all $p \ge 1$ we have $q \in L^p(\Omega; L^\infty([0,T]; L^2(\mathbb{R})))$ and therefore we can take sup over $[0, T]$ on both sides, and get $\mathbb{E}\left(\sup_{t\in[0,T]}\tilde{I}_R(t)\right)^p \xrightarrow{R\uparrow\infty} 0$. This proves the claim (3.7). $\square$

The next result, which is the main contribution of this section, reveals that $\{X_R\}$ is a Cauchy sequence in $L^{1/2}(\Omega; C([0,T]))$.

Proposition 3.5. *Under the assumptions of Lemma 3.2, suppose in addition that (1.9) is true and also that (1.11d) holds with $p = 2$. The solutions X_R to (3.4), which satisfy the*

R-independent bound $\mathbb{E}\sup_{t\in[0,T]}|X_R(t)| \lesssim_{T,x} 1$ (cf. Lemma 3.3), form a sequence $\{X_R\}$ that is Cauchy in $L^{1/2}(\Omega; C([0,T]))$.

Proof. For $N, R, R' > 0$, define

$$\tau_N^{R,R'} := \inf\{t \in [0,T] : |X_R(t)| > N \text{ or } |X_{R'}(t)| > N\}.$$

Replace X_R by $\tilde{X}_R(t) := X_R\left(t \wedge \tau_N^{R,R'}\right)$, which satisfies $\left|\tilde{X}_R(t)\right| \le N$ for all $t \in \left[0, \tau_N^{R,R'}\right]$. The SDE for $\tilde{X}_R$ becomes

$$\begin{aligned}\tilde{X}_R(t) = x &+ \int_0^t u_R\left(s, \tilde{X}_R(s)\right) \mathrm{d}s + \frac{1}{4}\int_0^t \left(\sigma^2\right)'\left(s, \tilde{X}_R(s)\right) \mathrm{d}s \\ &+ \int_0^t \sigma\left(s, \tilde{X}_R(s)\right) \mathrm{d}W(s), \qquad t \in \left[0, \tau_N^{R,R'}\right].\end{aligned}$$

Applying the upcoming argument to $\tilde{X}_R - \tilde{X}_{R'}$ on the time interval $\left[0, \tau_N^{R,R'}\right]$, where $\tilde{X}_{R'}(\,\cdot\,) := X_{R'}\left(\,\cdot\, \wedge \tau_N^{R,R'}\right)$, we deduce that for any $\delta > 0$ there exists $R_0 = R_0(\delta)$ such that, for all $t \in [0,T]$,

$$\mathbb{E}\sup_{s\in[0,t]}\left|X_R\left(s \wedge \tau_N^{R,R'}\right) - X_{R'}\left(s \wedge \tau_N^{R,R'}\right)\right|^{1/2} < \delta, \quad \text{for all } R, R' \ge R_0,$$

see (3.14). To conclude one notices that $\tau_N^{R,R'} \to T$ as $N \to \infty$, uniformly in R, R'. Indeed, the R-independent bound $\mathbb{E}\sup_{t\in[0,T]}|X_R| \lesssim 1$ (Lemma 3.3) implies

$$\begin{aligned}\mathbb{P}\left(\tau_N^{R,R'} < T\right) &\le \mathbb{P}\left(\sup_{t\in\left[0,\tau_N^{R,R'}\right]}|X_R(t)| \ge N,\ \tau_N^{R,R'} < T\right) \\ &\le \frac{1}{N}\mathbb{E}\sup_{t\in[0,T]}|X_R(t)| \to 0,\end{aligned}$$

as $N \to \infty$, uniformly in R. Hence, $\tau_N^{R,R'} \to T$ as $N \to \infty$, uniformly in R, R'.

Given the preceding discussion, in what follows, there is no loss of generality in assuming that

$$|X_R(t)|, |X_{R'}(t)| \le N, \quad \text{for all } t \in [0,T], \tag{3.8}$$

for some given $N > 0$, when seeking to establish that

$$Y(t) = Y_{R,R'}(t) := |X_R(t) - X_{R'}(t)|, \qquad R, R' \in [0,\infty),$$

satisfies the Cauchy property (3.14). The Tanaka formula gives

$$\begin{aligned}Y(t) = &\int_0^t \operatorname{sgn}(X_R - X_{R'})\,(u_R(s,X_R) - u_R(s,X_{R'}))\,\mathrm{d}s \\ &+ \int_0^t \operatorname{sgn}(X_R - X_{R'})\,(u_R(s,X_{R'}) - u_{R'}(s,X_{R'}))\,\mathrm{d}s \\ &+ \frac{1}{4}\int_0^t \operatorname{sgn}(X_R - X_{R'})\left(\left(\sigma^2\right)'(s,X_R) - \left(\sigma^2\right)'(s,X_{R'})\right)\mathrm{d}s \\ &+ \int_0^t \operatorname{sgn}(X_R - X_{R'})\,(\sigma(s,X_R) - \sigma(s,X_{R'}))\,\mathrm{d}W(s).\end{aligned} \tag{3.9}$$

This is very similar to (2.3), except for the difference $u_R(s, X_{R'}) - u_{R'}(s, X_{R'})$.

First we seek to estimate $Y(t)$ over a short time period $t \in [0, \varepsilon]$. In (3.9), as in the previous section, we write

$$\begin{aligned}
&\int_0^t \operatorname{sgn}(X_R - X_{R'}) \left(u_R(s, X_R) - u_R(s, X_{R'})\right) \mathrm{d}s \\
&\qquad + \frac{1}{4}\int_0^t \operatorname{sgn}(X_R - X_{R'}) \left(\left(\sigma^2\right)'(s, X_R) - \left(\sigma^2\right)'(s, X_{R'})\right) \mathrm{d}s \\
&= \int_0^t \operatorname{sgn}(X_R - X_{R'}) \int_{X_{R'}(s)}^{X_R(s)} \vartheta_R(q(s, y))\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \frac{1}{4}\int_0^t \operatorname{sgn}(X_R - X_{R'}) \int_{X_{R'}(s)}^{X_R(s)} \left(\left(\sigma^2\right)''(s, y) - \left(\sigma^2\right)''(0, y)\right) \mathrm{d}y\,\mathrm{d}s \\
&\quad + \frac{1}{4}\int_0^t \operatorname{sgn}(X_R - X_{R'}) \int_{X_{R'}(s)}^{X_R(s)} \left(\sigma^2\right)''(0, y)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

Estimating by the Cauchy–Schwarz inequality,

$$\begin{aligned}
Y(t) \le &\int_0^t \operatorname{sgn}(X_R - X_{R'}) \left(u_R(s, X_{R'}) - u_{R'}(s, X_{R'})\right) \mathrm{d}s \\
&+ \int_0^t Y^{1/2}(s)\,\|q(s)\|_{L^2(\Delta_s)}\,ds \\
&+ \frac{1}{4}\int_0^t Y^{1/2}(s) \left\|\left(\sigma^2\right)''(s) - \left(\sigma^2\right)''(0)\right\|_{L^2(\Delta_s)} \mathrm{d}s \\
&+ \frac{1}{4}\int_0^t \left\|\left(\sigma^2\right)''(0)\right\|_{L^\infty(\Omega\times\mathbb{R})} Y(s)\,\mathrm{d}s + \int_0^t \phi_\sigma(s) Y(s)\,\mathrm{d}W(s),
\end{aligned} \tag{3.10}$$

where $\phi_\sigma(t) := \left(\sigma(t, X_R) - \sigma(t, X_{R'})\right) / Y(t)$ is a process bounded in absolute value by $\Lambda(t)$ of (1.10). Here, Δ_s denotes the (random) interval

$$\Delta_s = \left[X_R(s) \wedge X_{R'}(s),\ X_R(s) \vee X_{R'}(s)\right].$$

Given (3.8), the last term in (3.10) is a square-integrable martingale starting from zero. Taking the expectation, and estimating as in the proof of Theorem 2.2,

$$\begin{aligned}
\mathbb{E}Y(t) \le &\, t\mathbb{E}\,\|u_R - u_{R'}\|_{L^\infty([0,t]\times\mathbb{R})} \\
&+ \int_0^t (\mathbb{E}Y(s))^{1/2} \left(\mathbb{E}\,\|q(s)\|^2_{L^2(\Delta_s)}\right)^{1/2} ds \\
&+ \frac{1}{4}\int_0^t (\mathbb{E}Y(s))^{1/2} \left(\mathbb{E}\left\|\left(\sigma^2\right)''(s) - \left(\sigma^2\right)''(0)\right\|^2_{L^2(\Delta_s)}\right)^{1/2} \mathrm{d}s \\
&+ \frac{1}{4}\int_0^t \left\|\left(\sigma^2\right)''(0)\right\|_{L^\infty(\Omega\times\mathbb{R})} \mathbb{E}Y(s)\,\mathrm{d}s.
\end{aligned}$$

Taking the supremum over $t \in [0, \varepsilon]$, and applying Young's inequality (in the form $ab = \left(\frac{1}{\sqrt{2\varepsilon}}a\right)\left(\sqrt{2\varepsilon}b\right) \le \frac{1}{4\varepsilon}a^2 + \varepsilon b^2$), we find

$$\begin{aligned}
\sup_{t\in[0,\varepsilon]} \mathbb{E}Y(t) \le &\, \varepsilon\mathbb{E}\,\|u_R - u_{R'}\|_{L^\infty([0,\varepsilon]\times\mathbb{R})} \\
&+ \frac{1}{4}\sup_{t\in[0,\varepsilon]} \mathbb{E}Y(t) + \varepsilon^2 \sup_{t\in[0,\varepsilon]} \mathbb{E}\,\|q(s)\|^2_{L^2(\Delta_s)}
\end{aligned}$$

$$
+\frac{1}{16}\sup_{t\in[0,\varepsilon]}\mathbb{E}Y(s)+\frac{\varepsilon^2}{4}\sup_{t\in[0,\varepsilon]}\mathbb{E}\left\|\left(\sigma^2\right)''(s)-\left(\sigma^2\right)''(0)\right\|_{L^2(\Delta_s)}^2
+\frac{\varepsilon}{4}\left\|\left(\sigma^2\right)''(0)\right\|_{L^\infty(\Omega\times\mathbb{R})}\sup_{t\in[0,\varepsilon]}\mathbb{E}Y(t).
$$

In what follows, we fix ε so small that $\frac{1}{4}+\frac{1}{16}+\frac{\varepsilon}{4}\left\|\left(\sigma^2\right)''(0)\right\|_{L^\infty(\Omega\times\mathbb{R})}\le\frac{1}{2}$. Since ε and R, R' are independent parameters, given (3.7) of Lemma 3.4, we can take $R_0=R_0(\varepsilon)$ so large that

$$
\begin{aligned}
\mathbb{E}\left\|u_R-u_{R'}\right\|_{L^\infty([0,\varepsilon]\times\mathbb{R})}&\le\mathbb{E}\left\|u_R-u_{R'}\right\|_{L^\infty([0,T]\times\mathbb{R})}\\
&\le\varepsilon^{5/2}=\varepsilon^2 o(1),\quad\text{as }\varepsilon\to 0,
\end{aligned}
\tag{3.11}
$$

for all $R, R'\ge R_0(\varepsilon)$. This gives us

$$
\sup_{t\in[0,\varepsilon]}\mathbb{E}Y(t)\le 2\varepsilon^2\Bigg(\varepsilon^{3/2}+\sup_{t\in[0,\varepsilon]}\mathbb{E}\left\|q(s)\right\|_{L^2(\mathbb{R})}^2
+\sup_{t\in[0,\varepsilon]}\mathbb{E}\left\|\left(\sigma^2\right)''(s)-\left(\sigma^2\right)''(0)\right\|_{L^2(\mathbb{R})}^2\Bigg).
$$

Importantly, from (1.11c) and (2.7) we conclude that

$$
\sup_{t\in[0,\varepsilon]}\mathbb{E}Y(t)=\varepsilon^2 o(1),\quad\text{as }\varepsilon\to 0.\tag{3.12}
$$

As in the proof of Theorem 2.2, we estimate Y again (this time on the entire time interval $[0,T]$). From (3.10), we arrive at the integral inequality

$$
Y(t)\le\int_0^t\eta(s)\,\mathrm{d}s+\int_0^t Y(s)\,\mathrm{d}A(s)+M(t),
$$

where, for $t\in[0,T]$,

$$
\begin{aligned}
M(t)&:=\int_0^t\phi_\sigma(s)Y(s)\,\mathrm{d}W,\\
\eta(t)&:=\mathbb{1}_{\{t\le\varepsilon\}}Y^{1/2}(s)\left\|q(s)\right\|_{L^2(\Delta_s)}+T\left\|u_R-u_{R'}\right\|_{L^\infty([0,T]\times\mathbb{R})},
\end{aligned}
$$

and, as in (2.5),

$$
A(t):=\int_0^t\left(\mathbb{1}_{\{s\ge\varepsilon\}}K(s)+\Lambda(s)\right)\,\mathrm{d}s.
$$

Since we have not assumed an exponential moment bound for the difference $\left\|u_R-u_{R'}\right\|_{L^\infty([0,T]\times\mathbb{R})}$, it becomes imperative to include this term as a part of η and not A. The process η is non-negative and, by (1.11c), (3.12) and (3.11), is controlled thus:

$$
\mathbb{E}\int_0^t\eta(s)\,\mathrm{d}s=\varepsilon^2 o(1),\quad\text{as }\varepsilon\to 0.\tag{3.13}
$$

Now we apply Lemma 2.1, the stochastic Gronwall inequality with $p=\frac{1}{2}$ and a suitable $r\in\left(\frac{1}{2},1\right)$. In view of (1.4), (1.5) and (3.13),

$$
\left(\mathbb{E}\sup_{s\in[0,t]}Y^{1/2}(t)\right)^2\le C_r e^{C_\sigma(t-\varepsilon)}\varepsilon^{-2}\varepsilon^2 o(1)=o(1),\quad\text{as }\varepsilon\to 0.
$$

Therefore, given any $\delta > 0$, we can find $\varepsilon = \varepsilon(\delta)$ and $R_0 = R_0(\delta) := R_{\varepsilon(\delta)} \vee R'_{\varepsilon(\delta)}$ such that, for all $t \in [0, T]$,

$$\mathbb{E} \sup_{s\in[0,t]} Y_{R,R'}^{1/2}(s) < \delta, \quad \forall R, R' \geq R_0. \tag{3.14}$$

This concludes the proof of the proposition. □

Proposition 3.5 implies convergence in probability.

Lemma 3.6. *Under the assumptions of Lemma 3.2, suppose in addition that (1.9) is true and also that (1.11d) holds with $p = 2$. Then there exist a $\mathbb{P}$-almost surely continuous and $\{\mathcal{F}_t\}_{t\geq 0}$-adapted stochastic processes $X : \Omega \times [0, \infty) \to \mathbb{R}$ such that*

$$\lim_{R\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} |X_R(t) - X(t)| > \varepsilon \right) = 0, \tag{3.15}$$

for all $\varepsilon > 0$, for all finite $T > 0$.

Proof. By Chebyshev's inequality and Proposition 3.5, we obtain

$$\mathbb{P}\left(\sup_{t\in[0,T]} |X_R(t) - X_{R'}(t)| > \varepsilon \right) \leq \frac{1}{\sqrt{\varepsilon}} \mathbb{E} \sup_{t\in[0,T]} |X_R(t) - X_{R'}(t)|^{1/2} \overset{R,R'\to\infty}{\longrightarrow} 0,$$

so that $\{X_R\}$ is a Cauchy sequence in the space of continuous processes with respect to locally (in t) uniform convergence in probability. Since this space is complete, the lemma follows. □

It remains to identify the limit X as a solution to the original SDE (1.1).

Theorem 3.7 (*Existence of solution*)**.** *Under the assumptions of Theorem 1.1, there exists a strong solution X to the SDE (1.1).*

Proof. Fix a finite number $T > 0$. By Lemma 3.2, there exists a unique strong solution X_R to the SDE (3.4), such that

$$X_R(t) = x + \int_0^t u_R(s, X_R)\, \mathrm{d}s + \frac{1}{4}\int_0^t \left(\sigma^2\right)'(s, X_R)\, \mathrm{d}s + \int_0^t \sigma(s, X_R)\, \mathrm{d}W(s).$$

Let X be the limit process constructed in Lemma 3.6. Then

$$\begin{aligned} I(t) &:= X_R - x - \int_0^t u(s, X)\, \mathrm{d}s - \frac{1}{4}\int_0^t \left(\sigma^2\right)'(s, X)\, \mathrm{d}s - \int_0^t \sigma(s, X)\, \mathrm{d}W(s) \\ &= I_R^{(1)}(t) + I_R^{(2)}(t) + I_R^{(3)}(t) + M_R(t), \end{aligned}$$

where

$$\begin{aligned} &I_R^{(1)}(t) := X(t) - X_R(t), \quad I_R^{(2)}(t) := \int_0^t \left(u_R(s, X_R) - u(s, X)\right) \mathrm{d}s, \\ &I_R^{(3)}(t) := \frac{1}{4}\int_0^t \left(\left(\sigma^2\right)'(s, X_R) - \left(\sigma^2\right)'(s, X) \right) \mathrm{d}s, \\ &M_R(t) := \int_0^t \left(\sigma(s, X_R) - \sigma(s, X)\right) \mathrm{d}W(s). \end{aligned}$$

Because of the path continuity of X, it is enough to prove that $I(t) = 0$ $\mathbb{P}$-almost surely, for any fixed $t \in [0, T]$. To this end, we will verify that

$$I_R^{(1)}(t),\ I_R^{(2)}(t),\ I_R^{(3)}(t),\ M_R(t) \xrightarrow{R\uparrow\infty} 0, \quad \mathbb{P}\text{-a.s.},$$

at least for some subsequence $R_n \to 0$ as $n \to \infty$.

Since convergence in probability, cf. (3.15), implies almost sure convergence along a subsequence, we have

$$\sup_{t\in[0,T]} \left|X_{R_n}(t) - X(t)\right| \xrightarrow{n\uparrow\infty} 0, \quad \mathbb{P}\text{-a.s.}, \tag{3.16}$$

which implies that $I_{R_n}^{(1)}(t) \to 0$, $\mathbb{P}$-almost surely, as $n \to \infty$.

Given (3.7), we have that

$$\left\|u_{R_n} - u\right\|_{L^\infty([0,T]\times\mathbb{R})} \xrightarrow{n\uparrow\infty} 0, \quad \mathbb{P}\text{-a.s.}$$

Using this and the a.s. bound $\|q\|^2_{L^\infty([0,T];L^2(\mathbb{R}))} < \infty$, we obtain

$$\begin{aligned}
\left|I_{R_n}^{(2)}(t)\right| &\le \left|\int_0^t \int_{X_{R_n}}^X \vartheta_{R_n}(q(s,y))\,\mathrm{d}y\,\mathrm{d}s\right| + \left|\int_0^t \left(u_{R_n}(s,X) - u(s,X)\right)\,\mathrm{d}s\right| \\
&\le T\left(\sup_{s\in[0,T]} \left|X_{R_n}(s) - X(s)\right|\right)^{1/2} \left(\|q\|^2_{L^\infty([0,T];L^2(\mathbb{R}))}\right)^{1/2} \\
&\quad + T\left\|u_{R_n} - u\right\|_{L^\infty([0,T]\times\mathbb{R})} \xrightarrow{n\uparrow\infty} 0, \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

By (1.8), (1.10), and (3.16),

$$\begin{aligned}
\left|I_{R_n}^{(3)}(t)\right| &\le \int_0^T \Lambda(s)\left|X_{R_n}(s) - X(s)\right|\,\mathrm{d}s \\
&\le \left(\int_0^T \Lambda(s)\,\mathrm{d}s\right) \sup_{s\in[0,T]} \left|X_{R_n}(s) - X(s)\right| \xrightarrow{n\uparrow\infty} 0, \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

Recall the bound $\mathbb{E}\sup_{t\in[0,T]} |X_R(t)| \lesssim_{T,x} 1$, which holds uniformly in R (here we need (1.9) and (1.11d) with $p = 2$). Set $S(t) := \sup_{n\in\mathbb{N}} \sup_{s\in[0,t]} \left|X_{R_n}(s)\right|$, which is bounded, $\mathbb{P}$-almost surely. For $N \in [0, \infty)$, introduce the stopping time

$$\tau_N := \inf\{t \in [0, T] : S(t) > N\}.$$

Clearly, $\mathbb{P}(\tau_N < t) \to 0$ as $N \to \infty$. By (1.8), (1.10), (3.16) and the dominated convergence theorem,

$$\begin{aligned}
&\mathbb{E}\left|\int_0^{t\wedge\tau_N} \left(\sigma\left(s, X_{R_n}(s)\right) - \sigma\left(s, X(s)\right)\right)\mathrm{d}W(s)\right|^2 \\
&\quad = \mathbb{E}\int_0^t \mathbb{1}_{[0,\tau_N]}(s)\left|\sigma\left(s, X_{R_n}(s)\right) - \sigma\left(s, X(s)\right)\right|^2\,\mathrm{d}s \xrightarrow{n\uparrow\infty} 0.
\end{aligned}$$

As a result, for any $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{P}\left(\left|M_{R_n}(t)\right| > \varepsilon\right) &\le \mathbb{P}\left(\left|M_{R_n}(t)\right| > \varepsilon, \tau_N \ge t\right) + \mathbb{P}\left(\tau_N < t\right) \\
&\le \frac{1}{\varepsilon^2}\mathbb{E}\left|\int_0^t \left(\sigma\left(s, X_{R_n}(s)\right) - \sigma\left(s, X(s)\right)\right)\mathrm{d}W(s)\right|^2 + \mathbb{P}\left(\tau_N < t\right).
\end{aligned}$$

Sending first $n \to \infty$ and then $N \to \infty$, we conclude that $M_{R_n}(t) \xrightarrow{n\to\infty} 0$ in probability, and therefore, along a further subsequence (not relabelled),

$$M_{R_n}(t) \xrightarrow{n\uparrow\infty} 0, \quad \mathbb{P}\text{-a.s.}$$

This completes the proof that X is a solution of the SDE (1.1). □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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