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THE LOCAL-ORBIFOLD CORRESPONDENCE FOR SIMPLE NORMAL CROSSING PAIRS

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Abstract For X a smooth projective variety and $D = D_1 + \cdots + D_n$ a simple normal crossing divisor, we establish a precise cycle-level correspondence between the genus 0 local Gromov–Witten theory of the bundle $\oplus_{i=1}^{n} \mathcal{O}_X(-D_i)$ and the maximal contact Gromov–Witten theory of the multiroot stack $X_{D,\vec{r}}$. The proof is an implementation of the rank-reduction strategy. We use this point of view to clarify the relationship between logarithmic and orbifold invariants.

Key words and phrases: Gromov–Witten invariants, orbifolds, local Gromov–Witten invariants, product formula, rank reduction

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Contents

1. Introduction

Let X be a smooth projective variety and $D = D_1 + \cdots + D_n$ a simple normal crossing divisor with nef components D_i. We study the relationship between the genus 0 local Gromov–Witten theory of $\oplus_{i=1}^{n} \mathcal{O}_X(-D_i)$ and the genus 0 orbifold Gromov–Witten theory of the multiroot stack $X_{D,\vec{r}}$. Our main result is a positive answer to [27, Conjecture 1.8]:

Theorem A ((Theorem 2.1).). *Let β be a curve class on X with $d_i := D_i \cdot \beta > 0$ for $i \in \{1,\ldots,n\}$. For r_i pairwise coprime and sufficiently large, the following identity holds on the moduli space $\mathsf{K}_{0,m}(X,\beta)$ of stable maps to X:*

$$\rho_\star \left[\mathsf{K}^{\max}_{0,(I_1,\ldots,I_m)}(X_{D,\vec{r}},\beta)\right]^{\mathrm{virt}} = \left(\Pi_{i=1}^{n}(-1)^{d_i-1}\right)\left(\cup_{j=1}^{m} \mathrm{ev}_j^\star \left(\cup_{i\in I_j} D_i\right)\right) \cap \left[\mathsf{K}_{0,m}\left(\oplus_{i=1}^{n}\mathcal{O}_X(-D_i),\beta\right)\right]^{\mathrm{virt}},$$

where $I_j \subseteq \{1,\ldots,n\}$ records the set of divisors which the marking x_j is tangent to (see §2.1 for details), and ρ is the morphism forgetting the orbifold structures.

This generalizes the smooth divisor local-logarithmic correspondence [29] to the simple normal crossing setting, by interpreting the orbifold theory of the multiroot stack as an alternative to the logarithmic theory [26].

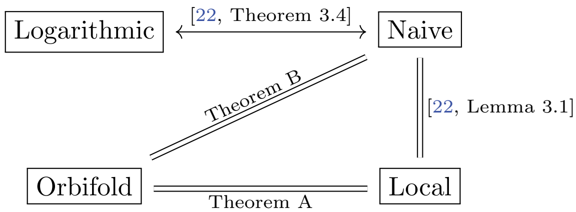
When D is smooth, Theorem A follows from previous results equating both local and orbifold invariants with relative invariants [1, 29, 28]. For general D, the key observation is that both the local and orbifold theories satisfy a product formula over the space of stable maps to X. Theorem A follows immediately, by bootstrapping from the smooth divisor case. This is another manifestation of the 'rank reduction' technique in Gromov–Witten theory [2, 22].

1.1. Logarithmic Gromov–Witten theory

Unlike the local and orbifold theories, the logarithmic theory does not satisfy a naive product formula over the space of stable maps to X. This observation was used in [22] to produce counterexamples to the local-logarithmic conjecture. The same reasoning shows that the orbifold invariants also differ from the logarithmic invariants (and it is easy to find counterexamples beyond the maximal contact setting). In fact, Corollary 3.4 equates the orbifold invariants with the so-called *naive invariants*, introduced in [21, §3] and studied in [22]:

Theorem B ((Corollary 3.4).). *The orbifold invariants of the multiroot stack coincide with the naive invariants, and hence differ from the logarithmic invariants. This holds for arbitrary choices of contact orders.*

In summary, there are four genus 0 maximal contact theories associated to a simple normal crossings pair: logarithmic, orbifold, naive, and local. They are related as follows:



The orbifold, local, and naive theories all coincide up to combinatorial factors. The logarithmic theory differs in a more essential way, though there is an in-principle procedure which relates it to the other three.

Despite the failure of the cycle-level local-logarithmic correspondence, there are many choices of targets and insertions for which the correspondence does hold on the numerical level. This occurs when the insertions kill the correction terms described in [22, Theorem 3.4]. In [7, 8, 9], numerous instances of the numerical local-logarithmic correspondence are established: for toric varieties, log Calabi–Yau surfaces and orbifold log Calabi–Yau surfaces; in [22, §5], the numerical correspondence is established for product geometries. As a corollary of Theorem A, all of these logarithmic invariants coincide with the corresponding orbifold invariants.

These different theories are approached using very different techniques. Torus localization, in various guises, has been applied to compute both orbifold and local invariants [18, 11, 12]; logarithmic invariants, on the other hand, are typically calculated using tropical correspondence theorems and scattering diagrams [23, 15]. Depending on the context, one technique may be more effective than another. These correspondences provide bridges between different techniques, thus increasing the roster of tools available for computations in Gromov–Witten theory as a whole.

1.2. Relation to previous work

The smooth divisor case of Theorem A follows by combining the orbifold-logarithmic correspondence [1, 28] with the strong form [13, 27] of the local-logarithmic correspondence [29]. Some cases of Theorem A for normal crossing divisors were numerically verified in [27, §5.2], by computing the J-functions of both sides.

1.3. User's guide

We provide two approaches to rank reduction. The first (§2) uses the iterative construction of root stacks and the projection formula, and relies on a local-orbifold correspondence

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for certain smooth orbifold pairs (Theorem 2.2). The second (§3) uses a product formula for orbifold invariants over the space of stable maps to the coarse moduli space (Theorem 3.1). This holds for arbitrary tangency orders but requires a positivity assumption. The identification of orbifold and naive invariants (Corollary 3.4) is an immediate consequence.

2. Rank reduction I: Projection formula

2.1. Geometric setup

Fix a smooth projective variety X and a simple normal crossing divisor $D = D_1 + \cdots + D_n \subseteq X$. For a tuple of pairwise coprime and sufficiently large integers $\vec{r} = (r_1, \ldots, r_n)$, we form the associated multiroot stack

$$\mathcal{X} = X_{D,\vec{r}}.$$

Consider m marked points $x_1, \ldots, x_m$ and fix an ordered partition of the index set $\{1, \ldots, n\}$ into disjoint subsets $I_1, \ldots, I_m$ such that $\cap_{i \in I_j} D_i$ is nonempty for each $j \in \{1, \ldots, m\}$. Fix a curve class $\beta \in H_2^+(X)$ such that $d_i := D_i \cdot \beta > 0$ for all i.

We consider a moduli problem of genus 0 stable maps relative to (X, D), such that the marking x_j has maximal contact order d_i to each divisor D_i with $i \in I_j$. Notice that some of the I_j may be empty, corresponding to markings with no tangency conditions. Some markings may have positive contact orders along several divisors simultaneously, which implies in particular that they should be mapped to the intersection.

This moduli problem determines associated discrete data for a moduli problem of orbifold stable maps to the multiroot stack $\mathcal{X}$, by taking each marking x_j to have twisting index

$$s_j = \prod_{i \in I_j} r_i.$$

The twisted sector insertion in

$$\mu_{s_j} = \prod_{i \in I_j} \mu_{r_i}$$

coincides with the tuple of tangency orders, since the twisting indices on source and target are the same [10, §2.1]. We denote the associated moduli space by

$$\mathsf{K}^{\max}_{0,(I_1,\ldots,I_m)}(\mathcal{X}, \beta)$$

and let ρ denote the morphism which forgets the orbifold structures:

$$\rho \colon \mathsf{K}^{\max}_{0,(I_1,\ldots,I_m)}(\mathcal{X}, \beta) \to \mathsf{K}_{0,m}(X, \beta).$$

2.2. Local-orbifold correspondence

Our main result is a cycle-level correspondence between the multiroot orbifold theory and the local theory of the associated split vector bundle, proving [27, Conjecture 1.8]:

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Theorem 2.1. *For r_i sufficiently large, we have*

$$\rho_\star\left[\mathsf{K}^{\max}_{0,(I_1,\ldots,I_m)}(\mathcal{X},\beta)\right]^{\text{virt}} = \left(\Pi_{i=1}^{n}(-1)^{d_i-1}\right)\left(\cup_{j=1}^{m}\operatorname{ev}_j^\star\left(\cup_{i\in I_j}D_i\right)\right) \cap\left[\mathsf{K}_{0,m}\left(\oplus_{i=1}^{n}\mathcal{O}_X(-D_i),\beta\right)\right]^{\text{virt}}.$$

The case $n=1$ follows by combining the smooth divisor local-logarithmic correspondence [29] in its strong form (see [13, Introduction] or [27, equation (2)]), together with the smooth divisor logarithmic-orbifold correspondence [1, 28].

Proof. We proceed by induction on n. The base case $n=1$ was already discussed. For the induction step, consider the root stack

$$\mathcal{Z}=X_{(D_1,\ldots,D_{n-1}),(r_1,\ldots,r_{n-1})}.$$

Letting $p\colon\mathcal{Z}\to X$ be the morphism to the coarse moduli space and $\mathcal{D}_n=p^{-1}D_n$, we have

$$\mathcal{X}=\mathcal{Z}_{\mathcal{D}_n,r_n}.$$

The ordered partition $(I_1,\ldots,I_m)$ of $\{1,\ldots,n\}$ induces a partition $(J_1,\ldots,J_m)$ of $\{1,\ldots,n-1\}$ by setting $J_j=I_j\setminus\{n\}$. Consider the tower of moduli spaces

$$\mathsf{K}^{\max}_{0,(I_1,\ldots,I_m)}(\mathcal{X},\beta)\xrightarrow{\ \psi\ }\mathsf{K}^{\max}_{0,(J_1,\ldots,J_m)}(\mathcal{Z},\beta)\xrightarrow{\ \varphi\ }\mathsf{K}_{0,m}(X,\beta),\qquad \rho=\varphi\circ\psi.$$

The induction hypothesis gives

$$\varphi_\star\left[\mathsf{K}^{\max}_{0,(J_1,\ldots,J_m)}(\mathcal{Z},\beta)\right]^{\text{virt}} = \left(\Pi_{i=1}^{n-1}(-1)^{d_i-1}\right)\left(\cup_{j=1}^{m}\operatorname{ev}_j^\star\left(\cup_{i\in J_j}D_i\right)\right) \cap\left[\mathsf{K}_{0,m}\left(\oplus_{i=1}^{n-1}\mathcal{O}_X(-D_i),\beta\right)\right]^{\text{virt}}, \tag{1}$$

and Theorem 2.2 establishes a local-orbifold correspondence for the smooth orbifold pair $(\mathcal{Z},\mathcal{D}_n)$, giving

$$\begin{aligned}\psi_\star\left[\mathsf{K}^{\max}_{0,(I_1,\ldots,I_m)}(\mathcal{X},\beta)\right]^{\text{virt}} &= (-1)^{d_n-1}\operatorname{ev}_{j_n}^\star\mathcal{D}_n\cap\left[\mathsf{K}^{\max}_{0,(J_1,\ldots,J_m)}(\mathcal{O}_{\mathcal{Z}}(-\mathcal{D}_n),\beta)\right]^{\text{virt}}\\ &=(-1)^{d_n-1}\operatorname{ev}_{j_n}^\star\mathcal{D}_n\cdot\mathrm{e}\left(\mathrm{R}^1\pi_\star f^\star\mathcal{O}_{\mathcal{Z}}(-\mathcal{D}_n)\right)\cap\left[\mathsf{K}^{\max}_{0,(J_1,\ldots,J_m)}(\mathcal{Z},\beta)\right]^{\text{virt}},\end{aligned} \tag{2}$$

where $j_n\in\{1,\ldots,m\}$ is the unique index such that $n\in I_{j_n}$. Since $\mathcal{D}_n=p^\star D_n$ is pulled back from X, we have

$$\begin{gathered}\operatorname{ev}_{j_n}^\star\mathcal{D}_n=\varphi^\star\operatorname{ev}_{j_n}^\star D_n\\ \mathrm{e}\left(\mathrm{R}^1\pi_\star f^\star\mathcal{O}_{\mathcal{Z}}(-\mathcal{D}_n)\right)=\varphi^\star\mathrm{e}\left(\mathrm{R}^1\pi_\star f^\star\mathcal{O}_X(-D_n)\right).\end{gathered}$$

The latter equation follows from the projection formula and the following fact. The pullback $\varphi^\star C$ of the universal curve on $\mathsf{K}_{0,m}(X,\beta)$ coincides with the coarsening of the universal curve

$$\mathcal{C}\to\mathsf{K}^{\max}_{0,(J_1,\ldots,J_m)}(\mathcal{Z},\beta),$$

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which makes the composite universal map $\mathcal{C} \to \mathcal{Z} \to X$ representable. Therefore the structure sheaves of the universal curves are preserved by push-forward [4, Theorem 3.1]. The result then follows from equations (1) and (2), the projection formula for φ, and the splitting of the obstruction bundle for the local theory of $\oplus_{i=1}^{n}\mathcal{O}_X(-D_i)$. □

2.3. Local-orbifold correspondence for smooth orbifold pairs

It remains to establish the local-orbifold correspondence for the smooth orbifold pair $(\mathcal{Z},\mathcal{D}_n)$, used in the preceding proof.

Theorem 2.2. *With notation as in the proof of Theorem 2.1, we have*

$$\psi_\star \left[\mathsf{K}^{\max}_{0,(I_1,\dots,I_m)}(\mathcal{X},\beta)\right]^{\mathrm{virt}} = (-1)^{d_n-1}\,\mathrm{ev}^\star_{j_n}\mathcal{D}_n \cap \left[\mathsf{K}^{\max}_{0,(J_1,\dots,J_m)}(\mathcal{O}_{\mathcal{Z}}(-\mathcal{D}_n),\beta)\right]^{\mathrm{virt}}.$$

We establish this result only in the setting we require, namely when $\mathcal{Z}$ is a multiroot stack and $\mathcal{D}_n$ is a divisor pulled back from the coarse moduli space. The proof adapts the arguments of [29], but subtleties arise due to the twisted sectors of $\mathcal{Z}$, which encode tangencies with respect to the divisors $D_1,\dots,D_{n-1}$. These complicate a crucial dimension count in the proof (§2.3.3) and also affect the final multiplicity calculation (§2.3.4).

Remark 2.3. It is unclear whether the correspondence holds in greater generality. If the divisor has generic stabilizer, then the dimension count (§2.3.3) can fail, so at best the result must be established via other methods. Moreover, in this case the multiplicity arising from the contribution of the special graph (§2.3.4) is different, hinting that any generalization of the correspondence will in fact require a new formulation.

2.3.1. Setting up the degeneration formula.

Let $\mathfrak{X}$ be the degeneration to the normal cone of $\mathcal{D}_n \subseteq \mathcal{Z}$, and let M be the degeneration to the normal cone of $D_n \subseteq X$.

Lemma 2.4. *$\mathfrak{X}$ is the root stack of M along the strict (equivalently, total) transforms of the divisors $D_i \times \mathbb{A}^1$ for $i \in \{1,\dots,n-1\}$.*

Proof. The divisors $D_i \times \mathbb{A}^1$ intersect the blowup center $D_n \times \{0\}$ transversely, hence the strict and total transforms coincide. Denote these by $T_i \subseteq M$. Each T_i admits a root on $\mathfrak{X}$, namely the pullback of the rooted divisor $\mathcal{D}_i \subseteq \mathcal{Z}$ along the composition $\mathfrak{X} \to \mathcal{Z} \times \mathbb{A}^1 \to \mathcal{Z}$. By the universal property of the root stack, we obtain a morphism

$$\mathfrak{X} \to M_{(T_1,\dots,T_{n-1}),(r_1,\dots,r_{n-1})},$$

and a local calculation shows that this is an isomorphism. □

The general fiber of the family $\mathfrak{X} \to \mathbb{A}^1$ is

$$\mathcal{Z} = X_{(D_1,\dots,D_{n-1}),(r_1,\dots,r_{n-1})}.$$

The central fiber consists of two components $\mathcal{Z}$ and $\mathcal{Y}$ meeting along $\mathcal{D}_n$. Here $\mathcal{Y}$ is obtained by rooting the bundle $Y = \mathbb{P}_{D_n}\left(\mathrm{N}_{D_n|X} \oplus \mathcal{O}_{D_n}\right)$ along the divisors $\pi^{-1}(D_i \cap D_n)$

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for $i \in \{1,\dots,n-1\}$. There is a Cartesian square

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & Y \\ {\scriptstyle\pi}\downarrow & \square & \downarrow{\scriptstyle\pi} \\ \mathcal{D}_n & \longrightarrow & D_n, \end{array}$$

where we note that $\mathcal{D}_n$ is itself a multiroot stack along a simple normal crossing divisor

$$\mathcal{D}_n = (D_n)_{(E_1,\dots,E_{n-1}),(r_1,\dots,r_{n-1})},$$

where $E_i = D_i \cap D_n \subseteq D_n$. We let $\mathcal{E}_i = E_i/r_i \subseteq \mathcal{D}_n$ be the corresponding gerby divisor.

Each connected component of the rigidified inertia stack $\overline{\mathcal{I}}(\mathcal{D}_n)$ is a rigidification of a closed stratum $\bigcap_{i\in I}\mathcal{E}_i$ of the divisor $\mathcal{E}_1+\cdots+\mathcal{E}_{n-1} \subseteq \mathcal{D}_n$ (including the stratum $\mathcal{D}_n$ corresponding to the empty intersection). This rigidification is obtained from $\bigcap_{i\in I} E_i$ by rooting along the intersection with E_j for all $j \notin I$. This description of the twisted sectors is crucial for understanding the structure of the degeneration formula later.

Finally, $\mathcal{D}_0 \subseteq \mathcal{Y}$ will denote the section of the bundle consisting of its intersection with $\mathcal{Z}$, and $\mathcal{D}_\infty \subseteq \mathcal{Y}$ will denote the intersection of the central fiber of $\mathfrak{X}$ with the strict transform $\mathfrak{D}$ of $\mathcal{D}_n \times \mathbb{A}^1$.

Consider $\mathfrak{L} = \operatorname{Tot}\mathcal{O}_{\mathfrak{X}}(-\mathfrak{D})$. This forms a family of (nonproper) targets over $\mathbb{A}^1$. The general fiber is $\operatorname{Tot}\mathcal{O}_{\mathcal{Z}}(-\mathcal{D}_n)$ and the central fiber is a union of $\mathcal{Z}\times\mathbb{A}^1$ and $\operatorname{Tot}\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_\infty)$.

We apply the degeneration formula [3] to $\mathfrak{L}$. The components of the central fiber are indexed by bipartite graphs Γ. The vertices $v \in \Gamma$ are partitioned into $\mathcal{Z}$-vertices and $\mathcal{Y}$-vertices, and the associated moduli spaces K_v are spaces of maps to expansions of the rooted pairs

$$\left(\mathcal{Z}\times\mathbb{A}^1, \mathcal{D}_n\times\mathbb{A}^1\right) \qquad \text{and} \qquad \left(\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_\infty), \mathcal{D}_0\times\mathbb{A}^1\right),$$

respectively, as defined in [3, §3]. Working with expansions is inconsequential, since the push-forward of the virtual class matches that of the space of maps to the corresponding root stack without expansions [1, Theorem 2.2]. We denote the twisting index by r_n. In the original formulation [3, §3.4], r_n is required to be divisible by all contact orders at the gluing nodes, but by [28] this condition can be removed without affecting the invariants. We assume therefore that r_n is large and coprime to each of $r_1,\dots,r_{n-1}$.

The component K_Γ associated to Γ maps to the fiber product

$$\begin{array}{ccccc} \mathsf{K}_\Gamma & \xrightarrow{\Phi} & \mathsf{F}_\Gamma & \longrightarrow & \prod_v \mathsf{K}_v \\ & & \downarrow & \square & \downarrow \\ & & \prod_e \overline{\mathcal{I}}\left(\mathcal{D}_n\times\mathbb{A}^1\right) & \xrightarrow{\Delta} & \prod_e \overline{\mathcal{I}}\left(\mathcal{D}_n\times\mathbb{A}^1\right)^2 \end{array} \tag{3}$$

with respect to the evaluation maps to the rigidified inertia stack of the join divisor. The virtual class of K_Γ pushes forward to a multiple of the virtual class with which F_Γ is endowed by virtue of this diagram; the virtual degree of the morphism Φ is well understood [3, Proposition 5.9.1]. Each space K_v decomposes as a disjoint union of substacks obtained as preimages of connected components of the inertia stack.

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After pushing forward to the space of stable maps to $\mathcal{Z}$, the degeneration formula gives an equality of classes

$$\left[\mathsf{K}^{\max}_{0,(J_1,\ldots,J_m)}(\mathcal{O}_{\mathcal{Z}}(-\mathcal{D}_n),\beta)\right]^{\mathrm{virt}} = \sum_{\Gamma} \frac{1}{|E(\Gamma)|!} \cdot \Psi_\star[\mathsf{K}_\Gamma]^{\mathrm{virt}}, \tag{4}$$

where Ψ is the composition

$$\mathsf{K}_\Gamma \to \mathsf{K}(\mathfrak{L}_0) \to \mathsf{K}(\mathcal{Z}).$$

Let $j = j_n$ be the index of the marking at which we wish to impose tangency to D_n (as in the proof of Theorem 2.1), and cap both sides of equation (4) with $\mathrm{ev}_j^\star \mathfrak{D}$. The left-hand side gives the local invariants of $\mathcal{O}_{\mathcal{Z}}(-\mathcal{D}_n)$ capped with $\mathrm{ev}_j^\star \mathcal{D}_n$. Our aim is to show that all but one of the terms on the right-hand side vanish.

2.3.2. First vanishing: $\mathcal{Z}$-vertices. Suppose first that there is a $\mathcal{Z}$-vertex $v \in \Gamma$ with $k > 1$ adjacent edges. For each adjacent edge e, the corresponding evaluation map factors (locally) through a specific component of the rigidified inertia stack. Such a component is obtained by rigidifying a (possibly empty) intersection of the divisors $\mathcal{E}_i$ in $\mathcal{D}_n$. We denote this by $\mathcal{E}_e$. The product of evaluation maps thus takes the form

$$\mathsf{K}_v \to \prod_e \left(\mathcal{E}_e \times \mathbb{A}^1\right).$$

However, since the source curve is proper, the map to affine space is constant. This ensures that all the point evaluations agree – that is, the map factors through the closed substack

$$\left(\prod_e \mathcal{E}_e\right) \times \mathbb{A}^1 \hookrightarrow \prod_e \left(\mathcal{E}_e \times \mathbb{A}^1\right).$$

We now follow the argument of [29, Lemma 3.1]. There is a Cartesian diagram

$$\begin{array}{ccc}
\mathsf{F}_\Gamma & \longrightarrow & \mathsf{K}_v \times \prod_{v'} \mathsf{K}_{v'} \\
\downarrow & \square & \downarrow \\
\left(\prod_e \mathcal{E}_e\right) \times \mathbb{A}^1 \times \prod_{e'} \overline{\mathcal{I}}\left(\mathcal{D}_n \times \mathbb{A}^1\right) & \xrightarrow{\widetilde{\Delta}} & \left(\left(\prod_e \mathcal{E}_e\right) \times \mathbb{A}^1\right)^2 \times \prod_{e'} \overline{\mathcal{I}}\left(\mathcal{D}_n \times \mathbb{A}^1\right)^2 \\
\downarrow \iota & \square & \downarrow \iota' \\
\prod_e \left(\mathcal{E}_e \times \mathbb{A}^1\right) \times \prod_{e'} \overline{\mathcal{I}}\left(\mathcal{D}_n \times \mathbb{A}^1\right) & \xrightarrow{\Delta} & \prod_e \left(\mathcal{E}_e \times \mathbb{A}^1\right)^2 \times \prod_{e'} \overline{\mathcal{I}}\left(\mathcal{D}_n \times \mathbb{A}^1\right)^2.
\end{array}$$

The excess intersection formula [14, Theorem 6.3] gives

$$\Delta^! = \mathrm{c}_{k-1}(E) \cap \widetilde{\Delta}^!,$$

where E is the excess bundle, which in this case [14, Example 6.3.2] is equal to

$$E = \widetilde{\Delta}^\star \mathrm{N}_{\iota'}/\mathrm{N}_\iota,$$

which is clearly trivial if $k > 1$. It follows that $\Delta^! = 0$ and so the contribution of Γ vanishes.

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2.3.3. Second vanishing: $\mathcal{Y}$-vertices. We conclude that the only graphs which can contribute are those with a single $\mathcal{Y}$-vertex. Let $v \in \Gamma$ be such a vertex. This corresponds to a space of expanded maps to the rooted pair $\left(\mathcal{O}_{\mathcal{Y}}(-\mathcal{D}_\infty), \mathcal{D}_0 \times \mathbb{A}^1\right)$. Recall that $\mathcal{Y}$ is a projective bundle over the divisor $\mathcal{D}_n$. Suppose that in the discrete data for K_v, either

- the curve class is not a multiple of the fiber class or
- there are at least three special points.

This ensures that the corresponding moduli space of stable maps $\mathsf{K}_v(\mathcal{D}_n)$ to the base of the bundle is well defined. There is a projection

$$\mathsf{K}_v \to \mathsf{K}_v(\mathcal{D}_n), \tag{5}$$

and we claim that the virtual class pushes forward to zero along this morphism. Since there is a natural bijection between the twisted sectors in $\mathcal{Y}$ and the twisted sectors in $\mathcal{D}_n$, the age contributions to the virtual dimensions coincide. From this, one deduces

$$\operatorname{vdim} \mathsf{K}_v = \operatorname{vdim} \mathsf{K}_v(\mathcal{D}_n) + 2,$$

and so the claim holds if we show that formula (5) satisfies the virtual push-forward property [20, Definition 3.1] (we note that this dimension count can fail if $\mathcal{D}_n$ is allowed to have generic stabilizer). By [1, Theorem 2.2], it is equivalent to show that

$$\mathsf{K}_v\left(\mathcal{Y}_{\mathcal{D}_0, r_n}\right) \to \mathsf{K}_v(\mathcal{D}_n)$$

satisfies the virtual push-forward property. For this we adapt the arguments of [29, §4]. Let

$$s = \Pi_{i=1}^{n} r_i, \qquad t = \Pi_{i=1}^{n-1} r_i = s/r_n.$$

By representability, the stabilizer groups of the source curve of any stable map to $\mathcal{Y}_{\mathcal{D}_0, r_n}$ (resp., $\mathcal{D}_n$) must have order dividing s (resp., t). We denote the monoids of effective curve classes by

$$A = H_2^+\left(\mathcal{Y}_{\mathcal{D}_0, r_n}\right), \qquad B = H_2^+(\mathcal{D}_n).$$

Now consider the following diagram, involving moduli stacks of prestable twisted curves with homology weights:

$$\begin{array}{ccccc} \mathsf{K}_v\left(\mathcal{Y}_{\mathcal{D}_0, r_n}\right) & \xrightarrow{\upsilon} & \mathsf{G} & \longrightarrow & \mathsf{K}_v(\mathcal{D}_n) \\ & \searrow & \downarrow & \square & \downarrow \\ & & \mathfrak{M}_A^{s-\mathrm{tw}} & \xrightarrow{\nu} & \mathfrak{M}_B^{t-\mathrm{tw}}, \end{array} \tag{6}$$

in which the morphism ν contracts unstable curve components with vertical homology class and coarsens the r_n–twisting (this morphism is the composition of an étale cover followed by a root construction).

From the short exact sequence of relative tangent bundles associated to the smooth projection $\mathcal{Y}_{\mathcal{D}_0, r_n} \to \mathcal{D}_n$, we obtain a compatible triple for the triangle in diagram (6).

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We note that unlike when the target is a variety, we may have

$$H^1\left(\mathcal{C}, f^\star \mathrm{T}_{\mathcal{Y}_{\mathcal{D}_0,r_n}/\mathcal{D}_n}\right) \neq 0$$

if components of $\mathcal{C}$ are mapped into the rooted divisor. Thus the morphism v is not typically smooth, but it is always virtually smooth, which is sufficient. The arguments given in [29, Lemma 5.1 and Proposition 5.3] then apply verbatim, showing that the virtual push-forward property holds and the contribution of Γ vanishes.

2.3.4. Contribution of the special graph. We conclude that the only graphs Γ which contribute are those with a single $\mathcal{Y}$-vertex v_1 with at most two special points and curve class a multiple of the fiber class F. Since v_1 must contain at least one node as well as the marking x_j, we are left with a single graph Γ, consisting of

- a $\mathcal{Z}$-vertex v_0 supporting all the markings except x_j and with curve class $\beta_0 = \beta$; and
- a $\mathcal{Y}$-vertex v_1 supporting the marking x_j and with curve class $\beta_1 = d_n \cdot F$ for $d_n = \mathcal{D}_n \cdot \beta$.

These are connected along a single edge e, and diagram (3) reduces to the following:

$$\begin{array}{ccccc} \mathsf{K}_\Gamma & \longrightarrow & \mathsf{F}_\Gamma & \longrightarrow & \mathsf{K}_{v_0} \times \mathsf{K}_{v_1} \\ & & \downarrow & \square & \downarrow \\ & & \overline{\mathcal{I}}\left(\mathcal{D}_n \times \mathbb{A}^1\right) & \longrightarrow & \overline{\mathcal{I}}\left(\mathcal{D}_n \times \mathbb{A}^1\right)^2. \end{array}$$

Recall that the component K_Γ of the central fiber is virtually finite over the fiber product F_Γ. Let

$$J_j \subseteq \{1, \ldots, n-1\}$$

be the subset recording those divisors among $D_1, \ldots, D_{n-1}$ which the marking x_j is tangent to. This tangency is encoded in twisted sector insertions which are imposed on both the general and central fibers. In K_{v_1} these correspond to age constraints with respect to the bundles

$$\mathcal{O}_\mathcal{Y}\left(\pi^{-1}\mathcal{E}_i\right) = \pi^\star \mathcal{O}_{\mathcal{D}_n}(\mathcal{E}_i).$$

Since the curve class is a multiple of a fiber, these bundles have zero degree when pulled back to the source curve. It follows from parity considerations that K_{v_1} is empty unless the nodal marking q corresponding to the edge e also carries twisted sector insertions, which are opposite to those at x_j. This means we must have

$$\mathrm{age}_q\, \pi^\star \mathcal{O}_{\mathcal{D}_n}(\mathcal{E}_i) = 1 - \mathrm{age}_{x_j}\, \pi^\star \mathcal{O}_{\mathcal{D}_n}(\mathcal{E}_i)$$

for all $i \in J_j$. By the inversion of the band in the evaluation maps, we then have the opposite ages for the nodal marking q on K_{v_0}. It follows that the vertex v_0 contributes the orbifold invariants of the root stack $\mathcal{Z}_{\mathcal{D}_n,r_n} = \mathcal{X}$ with twisted sector insertions imposing maximal tangency of a single marking q with respect to all divisors D_i for $i \in I_j = J_j \cup \{n\}$, as required.

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For the contribution of v_1, notice that ev_q takes values in a component of $\overline{\mathcal{I}}(\mathcal{D}_n)$ which is naturally isomorphic to the rigidification of

$$\bigcap_{i \in J_j} \mathcal{E}_i.$$

We denote this rigidification by $\mathcal{E}_{J_j}$. A direct calculation shows that

$$\mathrm{vdim}\,\mathsf{K}_{v_1} = \dim \mathcal{E}_{J_j} + 1.$$

There is a divisorial insertion $\mathrm{ev}_j^\star \mathcal{D}_\infty$ on K_{v_1}, and the contribution of v_1 can be expressed as the unique $m \in \mathbb{Q}$ such that

$$(\mathrm{ev}_q)_\star \left(\mathrm{ev}_j^\star \mathcal{D}_\infty \cap [\mathsf{K}_{v_1}]^{\mathrm{virt}} \right) = m \cdot \left[\mathcal{E}_{J_j} \right].$$

This can be computed by restricting to the fiber of a general point in $\mathcal{E}_{J_j}$. Working locally around this point, the gerbes $\mathcal{E}_i$ become trivial, so that we obtain a space of maps to

$$\mathbb{P}(r_n, 1) \times \prod_{i \in J_j} \mathcal{B}\mu_{r_i}.$$

The maps to the $\mathcal{B}\mu_{r_i}$ are uniquely determined, and each has an automorphism factor of $1/r_i$. This cancels with the automorphism factor arising from the Chen–Ruan intersection pairing on the inertia stack of the join divisor [3, §5.2.3].

We are left with a computation on $\mathbb{P}(r_n, 1)$. The contribution is a local invariant capped with an insertion of $\mathrm{ev}_j^\star(\infty)$. The latter insertion can be factored out via the divisor axiom; the obstruction bundle of the local theory is pulled back along the map forgetting a marking, since the structure sheaves of the universal curves are preserved by push-forward along stabilization. The remaining local invariant can be computed by localization. The end result [17, (21)] is

$$(d_n) \left(\frac{(-1)^{d_n - 1}}{d_n^2} \right) = \frac{(-1)^{d_n - 1}}{d_n},$$

which combines with the gluing factor d_n appearing in the degeneration formula to complete the proof of Theorem 2.2. □

3. Rank reduction II: Relative product formula

Having established the main Theorem 2.1, we now present an alternative approach, also based on the rank-reduction philosophy. While this approach is less general, requiring a positivity assumption, we have chosen to include it because the ‘relative product formula’ it uses provides valuable insight into the geometry of maps to the multiroot stack, and clarifies the relationship to logarithmic invariants. Moreover, the main result does not require the maximal contact assumption.

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3.1. Convex embeddings

As before, fix a smooth projective variety X and a simple normal crossing divisor $D = D_1 + \cdots + D_n \subseteq X$. To ease notation, we will assume from now on that $n = 2$; the extension to the general case follows by induction.

We will assume throughout this section that there exist a simple normal crossing pair $(P, H = H_1 + H_2)$ with P convex and a closed embedding $X \hookrightarrow P$ such that $D_i = X \cap H_i$ for each i. In this situation we call (X,D) a *convex embedding*. Two important cases encompassed by this definition are

(1) X convex and D_i arbitrary;

(2) X arbitrary and D_i very ample.

All definitions and proofs will be given first in the case, where X itself is convex, and then extended to convex embeddings via virtual pullback.

3.2. Relative product formula for root stacks

As in §2, we fix discrete data for a moduli problem of genus 0 relative stable maps to (X,D): a curve class $\beta \in H_2^+(X)$, a number of marked points m, and specified tangency orders to D_1 and D_2 at the marked points. Note that we do not require the contact orders to be maximal at this point.

Choose large coprime integers r_1 and r_2 and consider the root stacks

$$\mathcal{X}_1 = X_{D_1,r_1}, \qquad \mathcal{X}_2 = X_{D_2,r_2}.$$

These both have X as their coarse moduli space. For each $\mathcal{X}_i$ we can set up data for a moduli space of orbifold stable maps, by taking every marking to have twisting index r_i. The twisted sector insertion in μ_{r_i} coincides with the tangency order, since the twisting indices on source and target are the same [10, §2.1]. Consider now the multiroot stack

$$\mathcal{X} = \mathcal{X}_1 \times_X \mathcal{X}_2.$$

Just as before, we may construct discrete data for a space of orbifold stable maps to $\mathcal{X}$. Markings tangent to both D_1 and D_2 will have twisting index $r_1 r_2$, and the twisted sector insertion is the unique element of $\mu_{r_1 r_2}$ which maps to the correct pair of tangencies under the canonical isomorphism $\mu_{r_1 r_2} = \mu_{r_1} \times \mu_{r_2}$. From now on the discrete data will be suppressed from the notation.

In this section we show that the theory of orbifold stable maps satisfies a relative product formula over the space of maps to the coarse moduli space. To be more precise, we have the following:

Theorem 3.1. *There exists a diagram*

$$\begin{array}{ccccc}
\mathsf{K}(\mathcal{X}) & \xrightarrow{\nu} & \mathcal{P} & \longrightarrow & \mathsf{K}(\mathcal{X}_1) \times \mathsf{K}(\mathcal{X}_2) \\
 & & \downarrow{\scriptstyle \rho} & \square & \downarrow{\scriptstyle \rho_1 \times \rho_2} \\
 & & \mathsf{K}(X) & \xrightarrow{\Delta_{\mathsf{K}(X)}} & \mathsf{K}(X) \times \mathsf{K}(X)
\end{array} \tag{7}$$

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such that, when X is convex, we have

$$\nu_*[\mathsf{K}(\mathcal{X})]^{\mathrm{virt}} = \Delta^!_{\mathsf{K}(X)}\big([\mathsf{K}(\mathcal{X}_1)]^{\mathrm{virt}} \times [\mathsf{K}(\mathcal{X}_2)]^{\mathrm{virt}}\big). \tag{8}$$

Proof. The morphism $\nu\colon \mathsf{K}(\mathcal{X}) \to \mathcal{P}$ is obtained by taking relative coarse moduli spaces (see [5, §9] and [4, Theorem 3.1]). For each i, the partial coarsening $\mathcal{C} \to \mathcal{C}_i$ is initial among maps $\mathcal{C} \to \mathcal{Y}$ through which the map $\mathcal{C} \to \mathcal{X}_i$ factors and is representable.

We call a twisted curve an *r-curve* (for some positive integer r) if the order of every stabilizer group divides r. Since a stable map $\mathcal{C}_i \to \mathcal{X}_i$ must be representable, it follows that $\mathcal{C}_i$ is an r_i-curve.

A point of the fiber product $\mathcal{P}$ consists of the data of two stable maps $\mathcal{C}_1 \to \mathcal{X}_1$ and $\mathcal{C}_2 \to \mathcal{X}_2$ which induce the same underlying map $C \to X$ on coarse moduli.

Lemma 3.2. *Suppose that r_1 and r_2 are coprime, and let $\mathcal{C}_1,\mathcal{C}_2$ be r_1-, r_2-curves with the same coarse curve C. Then the normalisation*

$$\mathcal{C} = (\mathcal{C}_1 \times_C \mathcal{C}_2)^{\sim} \tag{9}$$

is a twisted curve.

Proof. If $p \in C$ is a marking on the coarse curve with local equation z, then the local model for each $\mathcal{C}_i$ is given by

$$\mathcal{C}_i = [(x_i^{r_i} = z)/\mu_{r_i}].$$

The fiber product is therefore $[(x_1^{r_1} = x_2^{r_2})/\mu_{r_1 r_2}]$. Note that this is not a twisted curve (unless $r_1 = 1$ or $r_2 = 1$). On the other hand, since r_1 and r_2 are coprime, the normalization of the fiber product is given by

$$\big[\mathbb{A}^1_y/\mu_{r_1 r_2}\big],$$

where $y^{r_1} = x_2, y^{r_2} = x_1$. The computation around a node is entirely analogous except that the base must also be normalized, around the divisor where the node persists. □

Remark 3.3. This phenomenon is related to the issue of saturation in logarithmic geometry, via the correspondence between twisted curves and extensions of logarithmic structures [24]. Indeed, the monoid $\mathbb{N}e_1 \oplus_{r_1,\mathbb{N},r_2} \mathbb{N}e_2$ is not saturated: in the groupification $\mathbb{Z}^2/(r_1,-r_2) \cong \mathbb{Z}$, the image of the generator e_1 is divisible by r_2:

$$e_1 = (a_1 r_1 + a_2 r_2)e_1 = a_1 r_2 e_2 + a_2 r_2 e_1 = r_2(a_1 e_2 + a_2 e_1),$$

where $a_1, a_2 \in \mathbb{Z}$ are such that $a_1 r_1 + a_2 r_2 = 1$. Similarly the image of e_2 is divisible by r_1.

The twisted curve $\mathcal{C}$ carries a natural map to $\mathcal{X}$ which is clearly representable. We thus have a Cartesian diagram

$$\begin{array}{ccc} \mathsf{K}(\mathcal{X}) & \xrightarrow{\ \nu\ } & \mathcal{P} \\ \downarrow{\scriptstyle\varphi} & \square & \downarrow{\scriptstyle\psi} \\ \mathfrak{M}^{r_1 r_2-\mathrm{tw}} & \longrightarrow & \mathfrak{M}^{r_1-\mathrm{tw}} \times_{\mathfrak{M}} \mathfrak{M}^{r_2-\mathrm{tw}}, \end{array} \tag{10}$$

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where the bottom morphism is the normalization. The morphism φ carries a natural perfect obstruction theory. We will now construct a compatible perfect obstruction theory for ψ. The diagram

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathsf{K}(\mathcal{X}_1)\times_{\mathfrak{M}}\mathsf{K}(\mathcal{X}_2) \\ \downarrow & \square & \downarrow \\ \mathsf{K}(X) & \xrightarrow{\Delta} & \mathsf{K}(X)\times_{\mathfrak{M}}\mathsf{K}(X) \end{array}$$

is Cartesian. Using the convexity assumption, there is a perfect obstruction theory for Δ given by

$$(\pi_{0\star} f_0^\star \mathrm{T}_X)^\vee[1], \tag{11}$$

where π_0 is the universal coarse curve. This pulls back to a perfect obstruction theory for $\mathcal{P} \to \mathsf{K}(\mathcal{X}_1)\times_{\mathfrak{M}}\mathsf{K}(\mathcal{X}_2)$. The latter space carries a perfect obstruction theory over $\mathfrak{M}^{r_1-\mathrm{tw}}\times_{\mathfrak{M}}\mathfrak{M}^{r_2-\mathrm{tw}}$ given by

$$(\pi_{1\star} f_1^\star \mathrm{T}_{\mathcal{X}_1} \oplus \pi_{2\star} f_2^\star \mathrm{T}_{\mathcal{X}_2})^\vee. \tag{12}$$

We thus have a triangle with perfect obstruction theories

$$\mathcal{P} \xrightarrow{(11)} \mathsf{K}(\mathcal{X}_1)\times_{\mathfrak{M}}\mathsf{K}(\mathcal{X}_2) \xrightarrow{(12)} \mathfrak{M}^{r_1-\mathrm{tw}}\times_{\mathfrak{M}}\mathfrak{M}^{r_2-\mathrm{tw}}, \qquad \psi : \mathcal{P} \to \mathfrak{M}^{r_1-\mathrm{tw}}\times_{\mathfrak{M}}\mathfrak{M}^{r_2-\mathrm{tw}}$$

and wish to build an obstruction theory for ψ giving a compatible triple. There are natural morphisms $\mathrm{T}_{\mathcal{X}_i} \to p_i^\star \mathrm{T}_X$ on $\mathcal{X}_i$. We therefore obtain

$$\pi_{1\star} f_1^\star \mathrm{T}_{\mathcal{X}_1} \oplus \pi_{2\star} f_2^\star \mathrm{T}_{\mathcal{X}_2} \to \pi_{0\star} f_0^\star \mathrm{T}_X.$$

(As in the proof of Theorem 2.1, this follows from the projection formula and the fact that the structure sheaves of the various universal curves are preserved by push-forwards along coarsening maps; see [4, Theorem 3.1].) Dualizing, shifting, and taking the cone, we obtain

$$(\pi_{1\star} f_1^\star \mathrm{T}_{\mathcal{X}_1} \oplus \pi_{2\star} f_2^\star \mathrm{T}_{\mathcal{X}_2})^\vee \to \mathbf{E}_\psi \to (\pi_{0\star} f_0^\star \mathrm{T}_X)^\vee[1] \xrightarrow{[1]}.$$

Several applications of the Four Lemma then show that $\mathbf{E}_\psi$ is a relative perfect obstruction theory for ψ.

Finally, we wish to compare the obstruction theories of ψ and φ in diagram (10). For any root stack $\mathcal{Y} = Y_{D,r}$ with gerby divisor $\mathcal{D}$, a local computation gives the following exact sequence:

$$0 \to \mathrm{T}_{\mathcal{Y}} \to p^\star \mathrm{T}_Y \to \mathcal{O}_{(r-1)\mathcal{D}}(r\mathcal{D}) \to 0.$$

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From this we obtain a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{T}_{\mathcal{X}} & \longrightarrow & p^\star \mathrm{T}_X & \longrightarrow & \bigoplus_{i=1}^2 \mathcal{O}_{(r_i-1)\mathcal{D}_i}(r_i\mathcal{D}_i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & p_1^\star \mathrm{T}_{\mathcal{X}_1} \oplus p_2^\star \mathrm{T}_{\mathcal{X}_2} & \longrightarrow & p^\star \mathrm{T}_X \oplus p^\star \mathrm{T}_X & \longrightarrow & \bigoplus_{i=1}^2 \mathcal{O}_{(r_i-1)\mathcal{D}_i}(r_i\mathcal{D}_i) & \longrightarrow & 0, \end{array}$$

and an application of the snake lemma produces the following exact sequence on $\mathcal{X}$:

$$0 \to \mathrm{T}_{\mathcal{X}} \to p_1^\star \mathrm{T}_{\mathcal{X}_1} \oplus p_2^\star \mathrm{T}_{\mathcal{X}_2} \to p^\star \mathrm{T}_X \to 0. \tag{13}$$

Applying $\pi_\star f^\star$, we see that the pullback of the perfect obstruction theory for ψ coincides with the perfect obstruction theory for φ in diagram (10). The theorem then follows by the commutativity of virtual pullback and push-forward [19, Theorem 4.1], since the bottom horizontal arrow in diagram (10) is proper of degree 1. □

3.3. Local-orbifold correspondence

With the relative product formula established, we can now give a straightforward proof of Theorem 2.1 in the convex setting.

Proof of Theorem 2.1 for convex targets. Consider again diagram (7). Theorem 3.1 gives the following relation in $\mathsf{K}(X)$:

$$(\rho\circ\nu)_\star[\mathsf{K}(\mathcal{X})]^{\mathrm{virt}} = (\rho_1)_\star[\mathsf{K}(\mathcal{X}_1)]^{\mathrm{virt}} \cdot (\rho_2)_\star[\mathsf{K}(\mathcal{X}_2)]^{\mathrm{virt}}.$$

Specializing to the maximal contact setting, the result immediately follows from the local-orbifold correspondence for smooth divisors and the splitting of the obstruction bundle for the local theory of $\mathcal{O}_X(-D_1)\oplus\mathcal{O}_X(-D_2)$. □

This result can be generalized to convex embeddings via virtual pullback methods. This is a fairly routine affair: see for instance [6, Appendix A]. Since the arguments in §2 already establish the result in full generality, we omit the details here.

3.4. Comparison with naive invariants

Recall from [21, 22] that for a simple normal crossing pair (X,D) with X convex, the naive virtual class is defined (in genus 0) as the product of logarithmic virtual classes

$$[\mathsf{N}(X \mid D)]^{\mathrm{virt}} := \prod_{i=1}^{n} (\rho_i)_\star[\mathsf{K}(X \mid D_i)]^{\mathrm{virt}}$$

inside $\mathsf{K}(X)$. We also obtain a refined class on the fiber product $\mathsf{N}(X \mid D)$, but we are mostly interested in its push-forward to $\mathsf{K}(X)$. This definition extends to arbitrary convex embeddings via virtual pullback. An immediate consequence of Theorem 3.1 is an identification of orbifold and naive invariants.

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Corollary 3.4. *For* (X,D) *a convex embedding, the relation*

$$\rho_\star[\mathsf{K}(X_{D,\vec{r}})]^{\mathrm{virt}} = [\mathsf{N}(X \mid D)]^{\mathrm{virt}}$$

holds inside $\mathsf{K}(X)$ *(for compatible choices of contact orders).*

Given this, the (counter)examples presented in [22, §1] and [21, §3.4] show that the orbifold invariants and logarithmic invariants differ in general, and that this defect is not restricted to the maximal contact setting.

The naive spaces provide an alternative perspective for probing the geography and invariants of the multiroot spaces. The iterated blowup construction of [22] gives a method for comparing the logarithmic invariants to the naive/orbifold invariants; see also [25, 16] for treatments of related ideas.

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