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Mean-field backward stochastic differential equations and applications[☆]

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ABSTRACT

In this paper we study the linear mean-field backward stochastic differential equations (mean-field BSDE) of the form

$$\begin{cases} dY(t) &= -[\alpha_1(t)Y(t)+\beta_1(t)Z(t)+\int_{\mathbb{R}_0}\eta_1(t,\zeta)K(t,\zeta)\nu(d\zeta)+\alpha_2(t)\mathbb{E}[Y(t)] \\ &\quad +\beta_2(t)\mathbb{E}[Z(t)]+\int_{\mathbb{R}_0}\eta_2(t,\zeta)\mathbb{E}[K(t,\zeta)]\nu(d\zeta)+\gamma(t)]dt \\ &\quad +Z(t)dB(t)+\int_{\mathbb{R}_0}K(t,\zeta)\tilde{N}(dt,d\zeta),\ t\in[0,T], \\ Y(T) &= \xi. \end{cases} \tag{0.1}$$

where (Y,Z,K) is the unknown solution triplet, B is a Brownian motion, $\tilde{N}$ is a compensated Poisson random measure, independent of B. We prove the existence and uniqueness of the solution triplet (Y,Z,K) of such systems. Then we give an explicit formula for the first component $Y(t)$ by using partial Malliavin derivatives. To illustrate our result we apply them to study a mean-field recursive utility optimization problem in finance.

1. Introduction

In this paper, we are interested in the following linear *mean-field BSDE* in the unknown triplet $(Y,Z,K)\in S^2\times L^2\times H^2_\nu$:

$$\begin{cases} dY(t) &= -[\alpha_1(t)Y(t)+\beta_1(t)Z(t)+\int_{\mathbb{R}_0}\eta_1(t,\zeta)K(t,\zeta)\nu(d\zeta) \\ &\quad +\alpha_2(t)\mathbb{E}[Y(t)] \\ &\quad +\beta_2(t)\mathbb{E}[Z(t)]+\int_{\mathbb{R}_0}\eta_2(t,\zeta)\mathbb{E}[K(t,\zeta)]\nu(d\zeta)+\gamma(t)]dt \\ &\quad +Z(t)dB(t)+\int_{\mathbb{R}_0}K(t,\zeta)\tilde{N}(dt,d\zeta),\ t\in[0,T], \\ Y(T) &= \xi. \end{cases} \tag{1.1}$$

Here $B(t)=B(t,\omega)$ and $\tilde{N}(dt,d\zeta)=N(dt,d\zeta)-\nu(d\zeta)dt$ are Brownian motion and compensated Poisson random measure, respectively, on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F}=\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ and ν is the Lévy measure of N. We assume that $B(t)$ and $N(dt,d\zeta)$ are independent and that $\int_{\mathbb{R}_0}\zeta^2\nu(d\zeta)<\infty$. The spaces S^2,L^2,H^2_ν, the random variable ξ and the coefficients $(\alpha_i,\beta_i,\eta_i)_{i=1,2}$ and γ will be specified later. We obtain the explicit representation for the solution triplet (Y,Z,K). To this end, we introduce the concept of partial Malliavin derivatives with respect to the Brownian motion B and with respect to the Poisson random measure N.

This explicit solution representation is then applied to solve an optimal consumption rate problem for the following mean-field additive recursive utility described by the following mean-field linear BSDE

$$\begin{cases} dY(t) &= -[\alpha_1(t)Y(t)+\beta_1(t)Z(t)+\int_{\mathbb{R}_0}\eta_1(t,\zeta)K(t,\zeta)\nu(d\zeta) \\ &\quad +\alpha_2(t)\mathbb{E}[Y(t)] \\ &\quad +\beta_2(t)\mathbb{E}[Z(t)]+\int_{\mathbb{R}_0}\eta_2(t,\zeta)\mathbb{E}[K(t,\zeta)]\nu(d\zeta)-c(t)]dt \\ &\quad +Z(t)dB(t)+\int_{\mathbb{R}_0}K(t,\zeta)\tilde{N}(dt,d\zeta), \\ Y(T) &= \xi\,. \end{cases}$$

An important idea to solve the optimization problem is to rewrite the above equation as

$$Y(t)=\mathbb{E}^{\mathcal{F}_t}[\xi\Gamma(t,T)+\int_t^T\Gamma(t,s)\{(\alpha_2(s),\beta_2(s),\eta_2(s,\zeta))V(s,\zeta) \\ -c(s)\}ds],\ t\in[0,T],\ \ \mathbb{P}\text{-a.s.},$$

where Γ and V will be specified later.

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Here are some motivation and background for our study. The system (1.1) can be considered as the limit of the following systems of interacting particles

$$\begin{cases} dY^{i,n}(t) &= -[\alpha_1(t)Y^{i,n}(t) + \beta_1(t)Z^{i,n}(t) \\ &\quad +\int_{\mathbb{R}_0}\eta_1(t,\zeta)K^{i,n}(t,\zeta)\nu(d\zeta) \\ &\quad +\alpha_2(t)\frac{1}{n}\sum_{l=1}^{n}Y^{j,n}(t) + \beta_2(t)\frac{1}{n}\sum_{l=1}^{n}Z^{j,n}(t)+ \\ &\quad \int_{\mathbb{R}_0}\eta_2(t,\zeta)\frac{1}{n}\sum_{l=1}^{n}K^{j,n}(t,\zeta)\nu(d\zeta)+\gamma(t)]dt \\ &\quad +Z^{i,n}(t)dB^i(t) + \int_{\mathbb{R}_0}K^{i,n}(t,\zeta)\tilde{N}^i(dt,d\zeta),\ t\in[0,T], \\ &\qquad 1\le i\le n, \\ Y^{i,n}(T) &= \xi, \end{cases}$$

as the number of particles n goes to infinity assuming that all B^i and $\tilde{N}^i$, $i = 1, 2, \ldots$, are independent.

Let us mention that the general well-posedness of mean-field BSDE in the triple $(Y, Z, K) \in S^2 \times L^2 \times H^2_\nu$ of the form

$$Y(t) = \xi + \int_t^T \mathbb{E}'[f(s, Y'(s), Z'(s), K'(s,\cdot), Y(s), Z(s), K(s,\cdot))]ds$$
$$- \int_t^T Z(s)dB(s) - \int_t^T \int_{\mathbb{R}_0} K(s,\zeta)\tilde{N}(dt,d\zeta),$$

where

$$\begin{aligned} &\mathbb{E}'[f(s, Y'(s), Z'(s), K'(s,\cdot), Y(s), Z(s), K(s,\cdot))](\omega) \\ &= \mathbb{E}'[f(s, Y'(s), Z'(s), K'(s,\cdot), Y(s,\omega), Z(s,\omega), K(s,\omega,\cdot))] \\ &= \int_\Omega f(s,\omega',\omega, Y(s,\omega'), Z(s,\omega'), K(s,\omega',\cdot), Y(s,\omega), Z(s,\omega), \\ &\quad K(s,\omega,\cdot))\mathbb{P}(d\omega'), \end{aligned}$$

was studied by Li and Min [1]. We refer also to Buckdahn et al. [2] for mean-field BSDE without jumps.

Mean-field BSDE's also represent interesting models in finance, for example models of risk measures and recursive utilities. For example let us consider a class of recursive utilities by means of mean-field BSDE with jumps, for a concave driver f, as follows

$$\begin{cases} dY(t) = -f(t, Y(t), Z(t), K(t,\cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t,\cdot)], c(t))dt \\ \qquad\qquad +Z(t)dB(t) + \int_{\mathbb{R}_0}K(t,\zeta)\tilde{N}(dt,d\zeta), \\ Y(T) = \xi. \end{cases} \tag{1.2}$$

The process $c(t) \ge 0$ is the consumption rate. Then the corresponding recursive utility $U_f(c)$ of the consumption c is the value $Y(0)$ at $t = 0$

$$\begin{aligned} Y(0) = \xi + \int_0^T &f(s, Y(s), Z(s), K(s,\cdot), \mathbb{E}[Y(s)], \mathbb{E}[Z(s)], \\ &\mathbb{E}[K(s,\cdot)], c(s))ds \\ &-\int_0^T Z(s)dB(s) - \int_0^T\int_{\mathbb{R}_0}K(s,\zeta)\tilde{N}(ds,d\zeta). \end{aligned}$$

It is interesting to find the consumption rate $\hat{c}$ which maximizes the mean-field recursive utility $U_f(c) = Y(0)$, which will be done in this work.

This problem of finding the optimal consumption rate in the above context can be viewed as a generalization to mean-field (and jumps) of the classical recursive utility of Duffie and Epstein [3]. See also Duffie and Zin [4] and Kreps and Parteus [5]. Standard BSDE's (without mean-field terms) were first introduced in their linear form by Bismut [6] in connection with a stochastic version of the Pontryagin maximum principle. Subsequently, this theory was extended by Pardoux and Peng [7] to the nonlinear case. The first work applying BSDE to finance was the paper by El Karoui et al. [8] where they studied several applications to option pricing and recursive utilities. All the above mentioned works are in the Brownian motion framework (continuous case). The discontinuous case is more involved, especially concerning the comparison principle which requires additional assumptions. Tang and Li [9] proved an existence and uniqueness result in the case of a natural filtration associated with a Brownian motion and a Poisson random measure.

Here is the organization of the paper. In Section 2, we introduce the partial Malliavin derivatives with respect to the Brownian motion B and with respect to Poisson random measure N. Section 3 establishes existence and uniqueness of a solution to a mean-field BSDE with jumps. Moreover, we give a closed formula for linear mean-field BSDE with jumps. Finally, in Section 4 we solve a mean-field recursive utility optimization problem.

2. Partial malliavin derivatives

In this section we give a brief account of the partial Malliavin derivatives with respect to the Brownian motion B and with respect to compensated Poisson random measures N. These derivatives will be used when we obtain the explicit solution formula in the linear case.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{0\le t\le T}$. The expectation on this probability space is denoted by $\mathbb{E}$ and the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_t)$ is denoted by $\mathbb{E}^{\mathcal{F}_t}(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_t)$. Let $(B(t), 0 \le t \le T)$ be a Brownian motion. Let $(N([0,t]), \mathcal{B}), 0 \le t \le T, \mathcal{B} \subseteq \mathbb{R}_0 = \mathbb{R} - \{0\} \in \mathcal{B}(\mathbb{R})$ be a Poisson random measure. Denote by $\nu(\mathcal{B})$ its associated Lévy measure so that $\mathbb{E}[N([0,t]), \mathcal{B}] = \nu(\mathcal{B})t$. Let $\tilde{N}(\cdot)$ denote the compensated Poisson measure of N defined by $\tilde{N}(dt,d\zeta) := N(dt,d\zeta) - \nu(d\zeta)dt$. We assume that $\mathcal{F}_t = \sigma(B(s), N([0,s]), \mathcal{B}), 0 \le s \le t, \mathcal{B} \in \mathcal{B}(\mathbb{R}_0)$. Any square integrable functional $F \in L^2(\Omega, \mathcal{F}, \mathbb{P}) = L^2(\mathbb{P})$ can be written as

$$F = \sum_{m,n=0}^{\infty} I_{m,n}(f_{m,n}), \tag{2.1}$$

where $f_{m,n}(s,t,\zeta) = f_{m,n}(s_1,\ldots,s_m; t_1,\zeta_1,\ldots,t_n,\zeta_n)$ is a function of $m+n$ variables which is symmetric in the first m variables $s = (s_1,\ldots,s_n)$ and the last n-variables $(t,\zeta) = ((t_1,\zeta_1),\ldots,(t_n,\zeta_n))$ satisfying

$$\int_{[0,T]^{m+n}\times\mathbb{R}^n} |f_{m,n}(s,t,\zeta)|^2 ds_1\cdots ds_m dt_1\cdots dt_n \nu(d\zeta_1)\cdots\nu(d\zeta_n) < \infty \tag{2.2}$$

and

$$I_{m,n}(f_{m,n}) = \int_{[0,T]^{m+n}\times\mathbb{R}^n} f_{m,n}(s,t,\zeta)dB(s_1) \cdots dB(s_m)\tilde{N}(dt_1,d\zeta_1)\cdots\tilde{N}(dt_n,d\zeta_n) \tag{2.3}$$

is the mixed multiple integral. It is easy to see that

$$\mathbb{E}(F^2) = \sum_{m,n=1}^{\infty} m!n!\int_{[0,T]^{m+n}\times\mathbb{R}^n}|f_{m,n}(s,t,\zeta)|^2 ds_1\cdots ds_m dt_1 \cdots dt_n\nu(d\zeta_1)\cdots\nu(d\zeta_n). \tag{2.4}$$

We define the Malliavin derivative as $D = (D_t, D_{t,\zeta})$ (where D_t denotes the *partial* Malliavin derivative with respect to the Brownian motion and $D_{t,\zeta}$ denotes the *partial* Malliavin derivative with respect to the compensated Poisson process) as follows

Definition 2.1. We say that F is in $\mathbb{D}_{1,2}$ if

$$\sum_{m,n=1}^{\infty}(m+n)m!n!\int_{[0,T]^{m+n}\times\mathbb{R}^n}|f(s,t,\zeta)|^2 ds_1\cdots ds_m dt_1 \cdots dt_n\nu(d\zeta_1)\cdots\nu(d\zeta_n) < \infty. \tag{2.5}$$

We define

$$D_r I_{m,n}(f_{m,n}) = m I_{m-1,n}(f_{m,n}(r,\cdot,\cdot,\cdot))$$
$$= \int_{[0,T]^{m+n-1}\times\mathbb{R}^n} f_{m,n}(s_1,\ldots,s_{m-1},r;t,\zeta)$$
$$dB(s_1)\cdots dB(s_{m-1})\tilde{N}(dt_1,d\zeta_1)\cdots\tilde{N}(dt_n,d\zeta_n); \quad (2.6)$$

and

$$D_{t,\zeta} I_{m,n}(f_{m,n}) = n I_{m,n-1}(f_{m,n}(\cdot,\cdot,(t,\zeta)))$$
$$= n\int_{[0,T]^{m+n-1}\times\mathbb{R}^{n-1}} f_{m,n}(s_1,\ldots,s_m;t_1,\zeta_1,\ldots,t_{n-1},\zeta_{n-1},t,\zeta)$$
$$dB(s_1)\cdots dB(s_m)\tilde{N}(dt_1,d\zeta_1)\cdots\tilde{N}(dt_{n-1},d\zeta_{n-1}). \quad (2.7)$$

For more details we refer to Di Nunno et al. [10], Section 3.1, Section 12.1 and Section 12.5.

The (classical) Malliavin derivative D_t was originally introduced by Malliavin as a stochastic calculus of variation used to prove results about smoothness of densities of solutions of stochastic differential equations in $\mathbb{R}^n$ driven by Brownian motion. The domain of definition of the Malliavin derivative is the subspace $\mathbb{D}_{1,2}$ of $L^2(\mathbb{P})$. We refer to Nualart [11], Di Nunno et al. [10], and to Hu [12] for information about the Malliavin derivative D_t for Brownian motion and, more generally, for Lévy processes.

We give some properties of the Malliavin derivative, referring to Di Nunno et al. [10] Chapters 3, 12, for proofs and more details:

(i) (Chain rule I) If $F \in L^2(\mathcal{F}_T,\mathbb{P})$ and $\varphi \in C^1(\mathbb{R})$, then

$$D_t(\varphi(F)) = \varphi'(F)D_tF, \quad t\in[0,T]. \quad (2.8)$$

(ii) (Chain rule II) If $G \in L^2(\mathcal{F}_T,\mathbb{P})$ and $\varphi \in C^1(\mathbb{R})$, then

$$D_{t,\zeta}(\varphi(G)) = \varphi(G + D_{t,\zeta}G) - \varphi(G), \quad (t,\zeta)\in[0,T]\times\mathbb{R}_0. \quad (2.9)$$

In particular, note the product rule for jumps:

$$D_{t,\zeta}(FG) = FD_{t,\zeta}G + GD_{t,\zeta}F + (D_{t,\zeta}F)(D_{t,\zeta}G). \quad (2.10)$$

(iii) Suppose that $F \in L^2(\mathcal{F}_t,\mathbb{P})$. Then $D_rF = D_{r,\zeta}F = 0$ for all $r < t$, $\zeta \in \mathbb{R}_0$.

(iv) Suppose $\varphi \in L^2(\lambda\times\mathbb{P})$ is adapted and that $\psi \in L^2(\lambda\times\nu\times\mathbb{P})$ is predictable. Then

$$D_r\Big(\int_0^T \varphi(t)dt\Big) = \int_r^T D_r\varphi(t)dt$$
$$D_r\Big(\int_0^T \varphi(t)dB(t)\Big) = \int_r^T D_r\varphi(t)dB(t) + \varphi(r)$$
$$D_{r,z}\Big(\int_0^T\int_{\mathbb{R}_0}\psi(t,\zeta)\nu(d\zeta)dt\Big) = \int_r^T\int_{\mathbb{R}_0} D_{r,z}\psi(t,\zeta)\nu(d\zeta)dt$$
$$D_{r,z}\Big(\int_0^T\int_{\mathbb{R}_0}\psi(t,\zeta)\tilde{N}(dt,d\zeta)\Big) = \int_r^T\int_{\mathbb{R}_0} D_{r,z}\psi(t,\zeta)\tilde{N}(dt,d\zeta) + \psi(r,z).$$

(v) Representation of BSDE solution:
Suppose that $(p(t),q(t),r(t,\zeta))$ solves a BSDE of the form

$$\begin{cases} dp(t) &= -g(t,p(t),q(t),r(t,\cdot))dt + q(t)dB(t) \\ &\quad + \int_{\mathbb{R}_0} r(t,\zeta)\tilde{N}(dt,d\zeta), 0\le t\le T, \\ p(T) &= F \in L^2(\mathcal{F}_T,P). \end{cases}$$

Then

$$q(t) = D_{t^-}p(t) \quad (:= \lim_{\epsilon\to 0} D_{t-\epsilon}p(t))$$

and

$$r(t,\zeta) = D_{t^-,\zeta}p(t).$$

3. Mean-field BSDE's

3.1. Existence and uniqueness of the solution

We define the following spaces for the solution triplet:

- S^2 consists of the $\mathbb{F}$-adapted càdlàg processes $Y : \Omega\times[0,T]\to\mathbb{R}$, equipped with the norm

  $\| Y \|^2_{S^2} := \mathbb{E}[\sup_{t\in[0,T]} |Y(t)|^2] < \infty.$

- L^2 consists of the $\mathbb{F}$-predictable processes $Z : \Omega\times[0,T]\to\mathbb{R}$, with

  $\| Z \|^2_{L^2} := \mathbb{E}[\int_0^T |Z(t)|^2\,dt] < \infty.$

- L^2_ν consists of Borel functions $K : \mathbb{R}_0\to\mathbb{R}$, such that

  $\| K \|^2_{L^2_\nu} := \int_{\mathbb{R}_0} |K(\zeta)|^2\nu(d\zeta) < \infty.$

- H^2_ν consists of $\mathbb{F}$-predictable processes $K : \Omega\times[0,T]\times\mathbb{R}_0\to\mathbb{R}$, such that for any fixed $t\in[0,T]$, $K(t,\zeta)$ is any element in L^2_ν and

  $\| K \|^2_{H^2_\nu} := \mathbb{E}[\int_0^T\int_{\mathbb{R}_0} K(t,\zeta)^2\nu(d\zeta)dt] < \infty.$

- $L^2(\Omega,\mathcal{F}_T)$ is the set of square integrable random variables which are $\mathcal{F}_T$-measurable.

Let d be a natural number and let

$f : \Omega\times[0,T]\times\mathbb{R}^2\times L^2_\nu\times\mathbb{R}^d\to\mathbb{R}$,

be a $\mathcal{F}_t$-progressively measurable function. We consider the following mean-field BSDE

$$\begin{cases} dY(t) &= -f(t,Y(t),Z(t),K(t,\cdot),\mathbb{E}[\varphi(Y(t),Z(t),K(t,\cdot))])dt \\ &\quad +Z(t)dB(t) \\ &\quad\quad + \int_{\mathbb{R}_0} K(t,\zeta)\tilde{N}(dt,d\zeta), \\ Y(T) &= \xi. \end{cases} \quad (3.1)$$

Definition 3.1. A process

$(Y,Z,K)\in S^2\times L^2\times H^2_\nu$

is said to be a *solution triplet* to the mean-field BSDE (3.1) with terminal condition $Y(T)=\xi$ if

$\int_0^T |f(s,Y(s),Z(s),K(s,\cdot),\mathbb{E}[\varphi(Y(s),Z(s),K(s,\cdot))])|\,ds < +\infty$ $\mathbb{P}$-a.s.,

and

$$Y(t) = \xi + \int_t^T f(s,Y(s),Z(s),K(s,\cdot),\mathbb{E}[\varphi(Y(t),Z(t),K(t,\cdot))])ds$$
$$-\int_t^T Z(s)dB(s) - \int_t^T\int_{\mathbb{R}_0} K(s,\zeta)\tilde{N}(ds,d\zeta),\ t\in[0,T]. \quad (3.2)$$

where $\xi\in L^2(\Omega,\mathcal{F}_T)$ is called the terminal condition and f is the generator.

To obtain the existence and uniqueness of a solution we make the following set of assumptions.

Assumption 3.2. For the driver f we assume

(a) f is square integrable with respect to t:

$\mathbb{E}[\int_0^T |f(t,0,0,0,0)|^2 dt] < \infty.$

(b) There exists a constant $C > 0$, such that for all $t \in [0, T]$ and for all $y_1, y_2, z_1, z_2 \in \mathbb{R}$, $k_1, k_2 \in L^2(\nu)$ and $\mu_1, \mu_2 \in \mathbb{R}^d$,

$$|f(t, y_1, z_1, k_1, \mu_1) - f(t, y_2, z_2, k_2, \mu_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \|k_1 - k_2\|_{L^2(\nu)} + |\mu_1 - \mu_2|), \quad \mathbb{P}\text{-a.s.}$$

For the mean functional, we assume

(c) For each $t \in [0, T]$, the (vector valued) function $\varphi : \Omega \times [0, T] \times \mathbb{R}^2 \times L^2_\nu \to \mathbb{R}^d$ is assumed to be continuously differentiable with bounded partial derivatives, such that

$$|\tfrac{\partial \varphi}{\partial y}(y, z, k)| + |\tfrac{\partial \varphi}{\partial z}(y, z, k)| + \|\nabla_k \varphi(y, z, k)\|_{L^2_\nu} \leq C',$$

for a given constant $C' > 0$ and $\nabla_k \varphi(y, z, k)$ is the Fréchet derivative of φ with respect to k.

The following result is slightly different from what is known in the literature:

Theorem 3.3. *Under Assumption 3.2, the mean-field BSDE (3.2) has a unique solution.*

The proof is given in Appendix.

Remark 3.4. In the above theorem if we take $d = 3$, $\varphi_i(x_1, x_2, x_3) = x_i$ for $i = 1, 2, 3$, we see that the following mean-field BSDE has a unique solution

$$\begin{cases} dY(t) &= -f(t, Y(t), Z(t), K(t, \cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t, \cdot)])dt \\ &\quad + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t, \zeta)\tilde{N}(dt, d\zeta),\ t \in [0, T], \\ Y(T) &= \xi, \end{cases}$$

where $f : \Omega \times [0, T] \times \mathbb{R}^2 \times L^2_\nu \times \mathbb{R}^3 \to \mathbb{R}$ satisfies Assumption 3.2.

3.2. Linear mean-field BSDE

In this section, we shall find the closed formula corresponding to the linear mean-field BSDE of the form

$$\begin{cases} dY(t) &= -[\alpha_1(t)Y(t) + \beta_1(t)Z(t) + \int_{\mathbb{R}_0} \eta_1(t, \zeta)K(t, \zeta)\nu(d\zeta) \\ &\quad + \alpha_2(t)\mathbb{E}[Y(t)] \\ &\quad + \beta_2(t)\mathbb{E}[Z(t)] + \int_{\mathbb{R}_0} \eta_2(t, \zeta)\mathbb{E}[K(t, \zeta)]\nu(d\zeta) + \gamma(t)]dt \\ &\quad + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t, \zeta)\tilde{N}(dt, d\zeta),\ t \in [0, T], \\ Y(T) &= \xi, \end{cases} \tag{3.3}$$

where the coefficients $\alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t), \eta_1(t, \cdot) > -1$, $\eta_2(t, \cdot)$ are given deterministic functions; $\gamma(t)$ is a given $\mathbb{F}$-adapted process and $\xi \in L^2(\Omega, \mathcal{F}_T)$ is a given $\mathcal{F}_T$-measurable random variable. By the solution formula for standard linear BSDE, the solution of the above linear mean-field BSDE (3.3) can be written as follows.

$$\begin{aligned} Y(t) = \mathbb{E}^{\mathcal{F}_t}[\xi \Gamma(t, T) + \textstyle\int_t^T \Gamma(t, s)\{\alpha_2(s)\mathbb{E}[Y(s)] + \beta_2(s)\mathbb{E}[Z(s)] \\ + \textstyle\int_{\mathbb{R}_0} \eta_2(s, \zeta)\mathbb{E}[K(s, \zeta)]\nu(d\zeta) + \gamma(s)\}ds], \quad t \in [0, T], \end{aligned} \tag{3.4}$$

where $\Gamma(t, s)$ is the solution of the following linear SDE

$$\begin{cases} d\Gamma(t, s) &= \Gamma(t, s)[\alpha_1(t)dt + \beta_1(t)dB(t) + \int_{\mathbb{R}_0} \eta_1(t, \zeta)\tilde{N}(dt, d\zeta)], \\ &\quad s \in [t, T], \\ \Gamma(t, t) &= 1. \end{cases} \tag{3.5}$$

Since we are in one dimension, Eq. (3.5) can be solved explicitly and the solution is given by

$$\begin{aligned} \Gamma(t, s) = \exp\{&\textstyle\int_t^s \beta_1(r)dB(r) + \int_t^s (\alpha_1(r) - \frac{1}{2}(\beta_1(r))^2)dr \\ &+ \textstyle\int_t^s \int_{\mathbb{R}_0} (\ln(1 + \eta_1(r, \zeta)) - \eta_1(r, \zeta))\nu(d\zeta)dr \\ &+ \textstyle\int_t^s \int_{\mathbb{R}_0} \ln(1 + \eta_1(r, \zeta))\tilde{N}(dr, d\zeta)\}. \end{aligned}$$

Notice that

$$\mathbb{E}\Gamma(t, s) = \exp\{\textstyle\int_t^s \alpha_1(r)dr\}. \tag{3.6}$$

To solve (3.4) we take the expectation on both sides of (3.4). Denoting $\overline{Y}(t) := \mathbb{E}[Y(t)]$, $\overline{Z}(t) := \mathbb{E}[Z(t)]$, and $\overline{K}(t, \zeta) := \mathbb{E}[K(t, \zeta)]$, we obtain

$$\begin{aligned} \overline{Y}(t) = \mathbb{E}[\xi \Gamma(t, T) + \textstyle\int_t^T \Gamma(t, s)\{\alpha_2(s)\overline{Y}(s) + \beta_2(s)\overline{Z}(s) \\ + \textstyle\int_{\mathbb{R}_0} \eta_2(s, \zeta)\overline{K}(s, \zeta)\nu(d\zeta) + \gamma(s)\}ds],\ t \in [0, T]. \end{aligned} \tag{3.7}$$

To find equations for $\overline{Z}(t)$ and $\overline{K}(t, \zeta)$ we write the original Eq. (3.3) as a forward one:

$$\begin{aligned} Y(t) = Y(0) + &\textstyle\int_0^t [\alpha_1(s)Y(s) + \alpha_2(s)\overline{Y}(s) + \beta_1(s)Z(s) + \beta_2(s)\overline{Z}(s) \\ &+ \textstyle\int_{\mathbb{R}_0} (\eta_1(s, \zeta)K(s, \zeta) + \eta_2(s, \zeta)\overline{K}(s, \zeta))\nu(d\zeta) + \gamma(s)]ds \\ &+ \textstyle\int_0^t Z(s)dB(s) + \int_0^t \int_{\mathbb{R}_0} K(s, \zeta)\tilde{N}(ds, d\zeta), \quad t \in [0, T], \end{aligned}$$

for some deterministic initial value $Y(0)$. We compute the Malliavin derivative of $Y(t)$ for all $r < t$ as follows:

$$\begin{aligned} D_r Y(t) = &\textstyle\int_r^t D_r[\alpha_1(s)Y(s) + \alpha_2(s)\overline{Y}(s) + \beta_1(s)Z(s) + \beta_2(s)\overline{Z}(s) \\ &+ \textstyle\int_{\mathbb{R}_0} (\eta_1(s, \zeta)K(s, \zeta) + \eta_2(s, \zeta)\overline{K}(s, \zeta))\nu(d\zeta) + \gamma(s)]ds \\ &+ \textstyle\int_r^t D_r Z(s)dB(s) + Z(r). \end{aligned}$$

Letting $r \to t-$, we get that $Z(t) = D_{t-}Y(t)$, which we will denote by $D_t Y(t)$ for simplicity. Thus, to find $Z(t)$ we only need to compute $D_t Y(t)$. We shall use the expression (3.4) for $Y(t)$ and the identity

$$D_t \mathbb{E}^{\mathcal{F}_t}[F] = \mathbb{E}^{\mathcal{F}_t}[D_t F].$$

We also note by the chain rule that $D_t \Gamma(t, T) = \Gamma(t, T)\beta_1(t)$. Then

$$\begin{aligned} Z(t) = \mathbb{E}^{\mathcal{F}_t}[D_t \xi \Gamma(t, T) + \xi \Gamma(t, T)\beta_1(t) + \textstyle\int_t^T \Gamma(t, s)\beta_1(t)\{\alpha_2(s)\overline{Y}(s) \\ + \beta_2(s)\overline{Z}(s) + \textstyle\int_{\mathbb{R}_0} \eta_2(s, \zeta)\overline{K}(s, \zeta)\nu(d\zeta) + \gamma(s)\}ds]. \end{aligned}$$

Taking the expectation, we have

$$\begin{aligned} \overline{Z}(t) = \mathbb{E}[D_t \xi \Gamma(t, T) + \beta_1(t)\mathbb{E}(\xi \Gamma(t, T)) + \textstyle\int_t^T \mathbb{E}(\Gamma(t, s))\beta_1(t)\{\alpha_2(s)\overline{Y}(s) \\ + \beta_2(s)\overline{Z}(s) + \textstyle\int_{\mathbb{R}_0} \eta_2(s, \zeta)\overline{K}(s, \zeta)\nu(d\zeta) + \gamma(s)\}ds]. \end{aligned} \tag{3.8}$$

Similarly, we have $K(t, \zeta) = D_{t,\zeta}Y(t)$ and (again by the chain rule) $D_{t,\zeta}\Gamma(t, s) = \eta_1(t, s)\Gamma(t, s)$, and this yields (keeping in mind the product rule for jumps (2.10))

$$\begin{aligned} K(t, \zeta) = \mathbb{E}^{\mathcal{F}_t}[&D_{t,\zeta}\xi \Gamma(t, T) + \xi \Gamma(t, T)\eta_1(t, \zeta) \\ &+ D_{t,\zeta}\xi \Gamma(t, T)\eta_1(t, \zeta) \\ &+ \textstyle\int_t^T \Gamma(t, s)\eta_1(t, \zeta)\{\alpha_2(s)\overline{Y}(s) + \beta_2(s)\overline{Z}(s) \\ &+ \textstyle\int_{\mathbb{R}_0} \eta_2(s, \zeta)\overline{K}(s, \zeta)\nu(d\zeta) + \gamma(s)\}ds]. \end{aligned}$$

Taking the expectation yields

$$\begin{aligned} \overline{K}(t, \zeta) = \mathbb{E}[&D_{t,\zeta}\xi \Gamma(t, T) + \xi \Gamma(t, T)\eta_1(t, \zeta) + D_{t,\zeta}\xi \Gamma(t, T)\eta_1(t, \zeta) \\ &+ \textstyle\int_t^T \Gamma(t, s)\eta_1(t, \zeta)\{\alpha_2(s)\overline{Y}(s) \\ &+ \textstyle\int_{\mathbb{R}_0} \eta_2(s, \zeta)\overline{K}(s, \zeta)\nu(d\zeta) + \gamma(s)\}ds]. \end{aligned} \tag{3.9}$$

Eqs. (3.7), (3.8) and (3.9) can be used to obtain $\bar{Y}, \bar{Z}, \bar{K}$. In fact, we let

$$V = \begin{pmatrix} V_1(t) \\ V_2(t) \\ V_3(t,\zeta) \end{pmatrix} = \begin{pmatrix} \bar{Y}(t) \\ \bar{Z}(t) \\ \bar{K}(t,\zeta) \end{pmatrix} \in L^2 \times L^2 \times H_\nu^2,$$

and (denoting $\mathbb{A}(t,s) = \exp\left\{\int_t^s \alpha_1(u)du\right\}$)

$$A^0(t,s,\zeta) = \left(A^0_{ij}(t,s,\zeta)\right)_{1\le i,j\le 3} \tag{3.10}$$
$$= \begin{pmatrix} \mathbb{A}(t,s)\alpha_2(s) & \mathbb{A}(t,s)\beta_2(s) & \mathbb{A}(t,s)\eta_2(s,\zeta) \\ \mathbb{A}(t,s)\beta_1(t)\alpha_2(s) & \mathbb{A}(t,s)\beta_1(t)\beta_2(s) & \mathbb{A}(t,s)\beta_1(t)\eta_2(s,\zeta) \\ \mathbb{A}(t,s)\eta_1(t,\zeta)\alpha_2(s) & \mathbb{A}(t,s)\eta_1(t,\zeta)\beta_2(s) & \mathbb{A}(t,s)\eta_1(t,\zeta)\eta_2(s,\zeta) \end{pmatrix}.$$

Define a mapping A from $V = (V_1, V_2, V_3)^T \in L^2 \times L^2 \times H_\nu^2$ to itself by

$$(AV)_i(t,\zeta) = \sum_{j=1}^2 \int_t^T A^0_{ij}(t,s)V_j(s)ds + \int_t^T \int_{\mathbb{R}_0} A^0_{i3}(t,s,\zeta)V_3(s,\zeta)\nu(d\zeta)\,ds. \tag{3.11}$$

Then (3.7), (3.8) and (3.9) can be written as

$$V = F + AV\,, \tag{3.12}$$

where

$$F = F(t,\zeta) = \begin{pmatrix} \mathbb{E}(\xi\Gamma(t,T)) + \int_t^T \gamma(s)ds \\ \mathbb{E}[D_t\xi\Gamma(t,T) + \beta_1(t)\xi\Gamma(t,T)] + \int_t^T \gamma(s)ds \\ \mathbb{E}[D_{t,\zeta}\xi\Gamma(t,T) + \xi\Gamma(t,T)\eta_1(t,\zeta) + D_{t,\zeta}\xi\Gamma(t,T)\eta_1(t,\zeta)] + \int_t^T \gamma(s)ds \end{pmatrix}. \tag{3.13}$$

Note that the operator norm of A, $\|A\|$, is less than 1 if t is close enough to T. Therefore there exists $\delta > 0$ such that $\|A\| < 1$ if we restrict the operator to the interval $[T-\delta, T]$ for some $\delta > 0$ small enough. In this case the linear equation (3.12) can now be solved easily as follows:

$$(I - A)V = F,$$

or

$$V = (I-A)^{-1}F = \sum_{n=0}^\infty A^n F\,; \quad t \in [T-\delta, T]. \tag{3.14}$$

Next, using $V(T-\delta)$ as the terminal value of the corresponding BSDE in the interval $[T-2\delta, T-\delta]$ and repeating the argument above, we find that there exists a solution V of the BSDE in this interval, given by the equation

$$V(t,\zeta) = V(T-\delta,\zeta) + A^{T-\delta}(t,\cdot,\zeta)V(\cdot); \quad T-2\delta \le t \le T-\delta. \tag{3.15}$$

Proceeding by induction we end up with a solution on the whole interval $[0,T]$. We summarize this as follows:

Theorem 3.5 (*Closed Formula*)**.** *Assume that $\alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t), \eta_1(t,\cdot), \eta_2(t,\cdot)$ are given bounded deterministic functions and that $\gamma(t)$ is $\mathbb{F}$-adapted and $\xi \in L^2(\Omega, \mathcal{F}_T)$.*

- *Then the first component $Y(t)$ of the solution triplet (Y,Z,K) of the linear mean-field BSDE (3.3) can be written on its closed formula as follows*

$$Y(t) = \mathbb{E}^{\mathcal{F}_t}[\xi\Gamma(t,T) + \int_t^T \Gamma(t,s)\{(\alpha_2(s), \beta_2(s), \eta_2(s,\zeta))V(s,\zeta) + \gamma(s)\}ds],\ t \in [0,T]\,,\ \mathbb{P}\text{-a.s.}, \tag{3.16}$$

where

$$\Gamma(t,s) = \exp\{\int_t^s \beta_1(r)dB(r) + \int_t^s (\alpha_1(r) - \tfrac{1}{2}(\beta_1(r))^2)dr + \int_t^s\int_{\mathbb{R}_0} (\ln(1+\eta_1(r,\zeta)) - \eta_1(r,\zeta))\nu(d\zeta)dr + \int_t^s\int_{\mathbb{R}_0} \ln(1+\eta_1(r,\zeta))\tilde{N}(dr,d\zeta)\}.$$

and, inductively,

$$V(t,\zeta) = V(T-k\delta,\zeta) + A^{T-k\delta}(t,\cdot,\zeta)V(\cdot), \quad T-(k+1)\delta \le t \le T-k\delta, \quad k = 0,1,2,\ldots \tag{3.17}$$

Or, equivalently,

$$V(t,\zeta) = \sum_{n=0}^\infty (A^{T-k\delta}(t,\cdot,\zeta))^n V(T-k\delta,\cdot), \quad T-(k+1)\delta \le t \le T-k\delta, \quad k = 0,1,2,\ldots \tag{3.18}$$

where $A^S, S > 0$ is given by (3.10) and $V(T,\zeta) = F(T,\zeta)$.

- *The second component $Z(t) = D_tY(t)$ is given by*

$$Z(t) = \mathbb{E}^{\mathcal{F}_t}[D_t\xi\Gamma(t,T) + \beta_1(t)\xi\Gamma(t.T)\xi\Gamma(t,T) + \beta_1(t)\int_t^T \Gamma(t,s)\{(\alpha_2(s), \beta_2(s), \eta_2(s,\zeta))V(s,\zeta) + \gamma(s)\}ds],\quad t \in [0,T]\,,\ \mathbb{P}\text{-a.s.}, \tag{3.19}$$

- *The third component $K(t,\zeta) = D_{t,\zeta}Y(t)$ is given by*

$$K(t,\zeta) = \mathbb{E}^{\mathcal{F}_t}[\Gamma(t,\zeta)\{\eta_1(t,\zeta)\xi + D_{t,\zeta}\xi + \eta_1(t)D_{t,\zeta}\xi\} + \eta_1(t)\int_t^T \Gamma(t,s)\{(\alpha_2(s), \beta_2(s), \eta_2(s,\zeta))V(s,\zeta) + \gamma(s)\}ds],\quad t \in [0,T]\,,\ \mathbb{P}\text{-a.s.}, \tag{3.20}$$

4. Optimization of mean-field recursive utility

We consider in this section a mean-field recursive utility process $Y(t)$, defined as a part of the solution of the following mean-field BSDE

$$\begin{cases} dY(t) &= -g(t, Y(t), Z(t), K(t,\cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t,\cdot)], \mathcal{C}(t))dt \\ &\quad + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t,\zeta)\tilde{N}(dt,d\zeta),\ t \in [0,T]\,, \\ Y(T) &= \xi. \end{cases}$$

We denote by $\mathcal{U}$, the set of all consumption processes. For each $\mathcal{C}(t) \in \mathcal{U}$, the driver $g : \Omega \times [0,T] \times \mathbb{R}^2 \times L_\nu^2 \times \mathbb{R}^2 \times L_\nu^2 \times \mathcal{U} \to \mathbb{R}$ and the terminal value $\xi \in L^2(\Omega, \mathcal{F}_T)$. Suppose that $(y,z,k,\bar{y},\bar{z},\bar{k},\mathcal{C}) \mapsto g(t,y,z,k,\bar{y},\bar{z},\bar{k},\mathcal{C})$ is concave for each $t \in [0,T]$. The driver

$$g(t, Y(t), Z(t), K(t,\cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t,\cdot)], \mathcal{C}(t))$$

represents the instantaneous utility at time t of the consumption rate $\mathcal{C}(t) \ge 0$, such that

$$\mathbb{E}[\int_0^T |g(t,0,0,0,0,0,0,\mathcal{C}(t))|^2 dt] < \infty, \text{ for all } t \in [0,T]\,.$$

We call a process $\mathcal{C}(t)$ a consumption rate process if $\mathcal{C}(t)$ is predictable and $\mathcal{C}(t) \ge 0$ for each t $\mathbb{P}$-a.s. Then $Y(t) = Y_g(0)$ is called a mean-field recursive utility process of the consumption $\mathcal{C}(\cdot)$, and the number $U(\mathcal{C}) = Y_g(0)$ is called the total mean-field recursive utility of $\mathcal{C}(\cdot)$. This is an extension to mean-field (and jumps) of the classical recursive utility concept of Duffie and Epstein [3]. See also Duffie and Zin [4], Kreps and Parteus [5], El Karoui et al. [8]. Finding the consumption rate $\widehat{\mathcal{C}}$ which maximizes its total mean-field recursive utility is an interesting application to finance of mean-field stochastic control theory.

4.1. Optimization problem

We discuss now the optimization problem related to the recursive utility. The wealth process $X(t) = X^{\mathcal{C}}(t)$ is given by the following linear SDE

$$\begin{cases} dX(t) &= [b_0(t) - \mathcal{C}(t)]X(t)dt + \sigma_0(t)X(t)dB(t) \\ &\quad + \int_{\mathbb{R}_0} \gamma_0(t,\zeta)X(t)\tilde{N}(dt,d\zeta),\ t \in [0,T], \\ X(0) &= x_0, \end{cases} \tag{4.1}$$

where the initial value $x_0 > 0$, and the functions b_0, σ_0, γ_0 are assumed to be deterministic functions, $\mathcal{C}$ is our relative consumption rate at time t, assumed to be a càdlàg $\mathbb{F}$-adapted process. We assume that $\int_0^T \mathcal{C}(t)dt < \infty$ $\mathbb{P}$-a.s. This implies that our wealth process $X(t) > 0$ for all t $\mathbb{P}$-a.s. Define the recursive utility process $Y(t) = Y^{\mathcal{C}}(t)$ by the linear mean-field BSDE in the unknown triplet $(Y, Z, K) = (Y^{\mathcal{C}}, Z^{\mathcal{C}}, K^{\mathcal{C}}) \in S^2 \times L^2 \times H_\nu^2$, by

$$\begin{cases} dY(t) &= -[\alpha_0(t)Y(t) + \alpha_1(t)\mathbb{E}[Y(t)] + \beta_0(t)Z(t) \\ &\quad +\beta_1(t)\mathbb{E}[Z(t)] \\ &\quad +\int_{\mathbb{R}_0}\{\eta_0(t,\zeta)K(t,\zeta) + \eta_1(t,\zeta)\mathbb{E}[K(t,\zeta)]\}\nu(d\zeta) \\ &\quad + \ln(\mathcal{C}(t)X(t))]dt \\ &\quad + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t,\zeta)\tilde{N}(dt,d\zeta),\ t \in [0,T], \\ Y(T) &= \theta X(T), \end{cases} \tag{4.2}$$

where $\theta = \theta(\omega) > 0$ is in $L^2(\Omega, \mathcal{F}_T)$ and $\alpha_0, \alpha_1, \beta_0, \beta_1, \eta_0, \eta_1$ are given deterministic functions with $\eta_0(t,\zeta), \eta_1(t,\zeta) > -1$.

From the closed formula (3.20), the first component $Y(t)$ of the solution triplet of Eq. (4.2) can be written as

$$\begin{aligned} Y(t) &= \mathbb{E}^{\mathcal{F}_t}[\theta X(T)\Gamma(t,T) \\ &\quad + \int_t^T \Gamma(t,s)\{(\alpha_1(s), \beta_1(s), \eta_1(s,\zeta))V(s,\zeta) \\ &\quad + \ln(\mathcal{C}(s)X(s))\}ds],\ t \in [0,T], \end{aligned}$$

where

$$\begin{aligned} \Gamma(t,s) = \exp\{&\textstyle\int_t^s \beta_0(r)dB(r) + \int_t^s (\alpha_0(r) - \frac{1}{2}(\beta_0(r))^2)dr \\ &+ \textstyle\int_t^s\int_{\mathbb{R}_0}(\ln(1+\eta_0(r,\zeta)) - \eta_0(r,\zeta))\nu(d\zeta)dr \\ &+ \textstyle\int_t^s\int_{\mathbb{R}_0}\ln(1+\eta_0(r,\zeta))\widetilde{N}(dr,d\zeta)\}. \end{aligned}$$

and

$$V = \sum_{n=0}^{\infty} A^n F.$$

We want to maximize the recursive utility of the consumption, which is given by

$$\begin{aligned} Y(0) &= \mathbb{E}[Y(0)] \\ &= \mathbb{E}[\theta X(T)\Gamma(0,T) + \int_0^T \Gamma(0,s)\{(\alpha_1(s), \beta_1(s), \eta_1(s,\zeta))V(s,\zeta) \\ &\quad + \ln(\mathcal{C}(s)X(s))\}ds]. \end{aligned} \tag{4.3}$$

Note that we can write

$$F = F(t,\zeta) = F_0 + (\int_t^T \gamma(s)ds)F_1,$$

where

$$F_0 = \begin{pmatrix} \mathbb{E}[\xi\Gamma(t,T)] \\ \mathbb{E}[D_t\xi\Gamma(t,T) + \beta_1(t)\xi\Gamma(t,T)] \\ \mathbb{E}[D_{t,\zeta}\xi\Gamma(t,T) + \xi\Gamma(t,T)\eta_1(t,\zeta) + D_{t,\zeta}\xi\Gamma(t,T)\eta_1(t,\zeta)] \end{pmatrix},$$

and

$$F_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

This gives

$$V = \sum_{n=0}^{\infty} A^n F = V_0 + (\int_t^T \gamma(s)ds)G,$$

where

$$V_0 = \sum_{n=0}^{\infty} A^n F_0 \quad \text{and} \quad G = \sum_{n=0}^{\infty} A^n F_1.$$

Hence

$$(\alpha_1, \beta_1, \eta_1)V = (\alpha_1, \beta_1, \eta_1)V_0 + (\int_t^T \gamma(s)ds)(\alpha_1, \beta_1, \eta_1)G,$$

and

$$\begin{aligned} &\int_0^T \Gamma(0,s)(\alpha_1(s), \beta_1(s), \eta_1(s,\zeta))V(s,\zeta)ds \\ &= \int_0^T \Gamma(0,s)(\alpha_1(s), \beta_1(s), \eta_1(s,\zeta))V_0(s)ds \\ &+ \int_0^T \Gamma(0,s)(\alpha_1(s), \beta_1(s), \eta_1(s,\zeta))(\int_s^T \gamma(r)dr)G(s)ds. \end{aligned}$$

By change of variables, we get

$$\begin{aligned} &\int_0^T \Gamma(0,s)(\alpha_1(s), \beta_1(s), \eta_1(s,\zeta))V(s,\zeta)ds \\ &= \overline{V}_0 + \int_0^T (\int_0^r \Gamma(0,s)(\alpha_1(s), \beta_1(s), \eta_1(s,\zeta))G(s)ds)\gamma(r)dr \\ &= \overline{V}_0 + \int_0^T \overline{\Gamma}(0,t)\gamma(t)dt, \end{aligned}$$

where

$$\overline{V}_0 = \int_0^T \Gamma(0,s)(\alpha_1(s), \beta_1(s), \eta_1(s,\zeta))V_0(s)ds$$

and

$$\overline{\Gamma}(0,t) = \int_0^t \Gamma(0,s)(\alpha_1(s), \beta_1(s), \eta_1(s,\zeta))Gds.$$

does not depend on γ.

We conclude that $Y(0) = \mathbb{E}[Y(0)]$ can be written, with $\gamma(s) = \ln(\mathcal{C}(s)X(s))$,

$$Y(0) = J(\mathcal{C}) + \mathbb{E}[\overline{V}_0],$$

where

$$J(\mathcal{C}) = \mathbb{E}[\theta X(T)\Gamma(0,T) + \int_0^T \overline{\Gamma}(0,s)\ln(\mathcal{C}(s)X(s))ds]. \tag{4.4}$$

Since $\mathbb{E}[\overline{V}_0]$ does not depend on $\mathcal{C}$ on X, it suffices to maximize $J(\mathcal{C})$.

Since this performance functional does not involve Y, but only X and $\mathcal{C}$ and otherwise known coefficients, we see that the problem is reduced to a standard optimal control problem for the system described by (4.1) and the performance functional $J(\mathcal{C})$. Hence, we can approach the problem by applying a standard version of the stochastic maximum principle for optimal control of jump diffusion, as presented in e.g. Framstad et al. [13], as follows:

The Hamiltonian to this optimization problem, $H : [0,T] \times \mathbb{R}^3 \times L^2(\nu) \times \mathbb{R}^3 \times \mathcal{U} \times \mathbb{R}^2 \times L^2(\nu) \times \mathbb{R} \to \mathbb{R}$, is defined by

$$\begin{aligned} &H(t,x,\mathcal{C},p,q,r(\cdot)) \\ &= (b_0(t) - \mathcal{C})xp + \sigma_0(t)xq + \textstyle\int_{\mathbb{R}_0}\gamma_0(t,\zeta)xr(\zeta)\nu(d\zeta) + \overline{\Gamma}(0,t)\ln(\mathcal{C}x), \end{aligned}$$

and the adjoint processes p, q, r, are defined by the BSDE

$$\begin{cases} dp(t) &= -\Big[(b_0(t) - \mathcal{C}(t))p(t) + \sigma_0(t)q(t) \\ &\quad + \int_{\mathbb{R}_0}\gamma_0(t,\zeta)r(t,\zeta)\nu(d\zeta) + \frac{\overline{\Gamma}(0,t)}{X(t)}\Big]dt \\ &\quad + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,\zeta)\tilde{N}(dt,d\zeta),\ t \in [0,T], \\ p(T) &= \theta\Gamma(0,T). \end{cases} \tag{4.5}$$

Differentiating H with respect to $\mathcal{C}$, we obtain

$$\frac{\partial}{\partial \mathcal{C}}H(t) = \frac{\overline{\Gamma}(0,t)}{\mathcal{C}(t)} - X(t)p(t).$$

The first order necessary condition of optimality of H yields:

$$\widehat{C}(t) = \frac{\overline{\Gamma}(0,t)}{\widehat{X}(t)\widehat{p}(t)}, \tag{4.6}$$

where $\widehat{X}(t)$ and $\widehat{p}(t)$ are the solutions of Eqs. (4.1) and (4.5) respectively, corresponding to the optimal control $\widehat{C}(t)$.

Substituting this into the BSDE (4.5), we get

$$\begin{cases} d\widehat{p}(t) &= -\Big[b_0(t)\widehat{p}(t) + \sigma_0(t)\widehat{q}(t) + \int_{\mathbb{R}_0}\gamma_0(t,\zeta)\widehat{r}(t,\zeta)\nu(d\zeta)\Big]dt \\ &\quad + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,\zeta)\tilde{N}(dt,d\zeta),\ t\in[0,T], \\ \widehat{p}(T) &= \theta\Gamma(0,T). \end{cases} \tag{4.7}$$

This equation for p is a standard linear BSDE, with the following solution

$$\widehat{p}(t) = \mathbb{E}^{\mathcal{F}_t}[\theta\Gamma(0,T)\Lambda(t,T)], \quad t\in[0,T], \tag{4.8}$$

where $\Lambda(t,s)$ is the solution of the following linear SDE

$$\begin{cases} d\Lambda(t,s) &= \Lambda(t,s)[b_0(t)dt + \sigma_0(t)dB(t) \\ &\quad + \int_{\mathbb{R}_0}\gamma_0(t,\zeta)\tilde{N}(dt,d\zeta)], \quad s\in[t,T], \\ \Lambda(t,t) &= 1. \end{cases} \tag{4.9}$$

We summarize what we have proved as follows:

Theorem 4.1. *The optimal control $\widehat{C}(t)$ of the optimal recursive consumption problem (4.4) is given in feedback form by*

$$\widehat{C}(t) = \frac{\overline{\Gamma}(0,t)}{\widehat{X}(t)\widehat{p}(t)}, \tag{4.10}$$

where $\widehat{X}(t)$ and $\widehat{p}(t)$ are given by (4.1) and (4.8)–(4.9), *respectively.*

CRediT authorship contribution statement

Nacira Agram: Writing, Reduction and also on Reviewing and Editing. **Yaozhong Hu:** Writing, Reduction and also on Reviewing and Editing. **Bernt Øksendal:** Writing, Reduction and also on Reviewing and Editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Proof of Theorem 3.3

For $t\in[0,T]$ and all $\beta>0$, we introduce the norm

$$\|(Y,Z,K)\|^2_{\mathbb{H}_\beta} := \mathbb{E}[\int_0^T e^{\beta t}\{|Y(t)|^2 + |Z(t)|^2 + \int_{\mathbb{R}_0}|K(t,\zeta)|^2\nu(d\zeta)\}dt].$$

The space $\mathbb{H}_\beta$ equipped with this norm is an Hilbert space. Define the mapping $\Phi : \mathbb{H}_\beta \to \mathbb{H}_\beta$ by $\Phi(y,z,k) = (Y,Z,K)$ where $(Y,Z,K)\in S^2\times L^2\times H^2_\nu$ $(\subset L^2\times L^2\times H^2_\nu)$ is defined by

$$\begin{cases} dY(t) &= -f(t,y(t),z(t),k(t,\cdot),\mathbb{E}[\varphi(y(t),z(t),k(t,\cdot))])dt \\ &\quad + Z(t)dB(t) + \int_{\mathbb{R}_0}K(t,\zeta)\tilde{N}(dt,d\zeta),\ t\in[0,T], \\ Y(T) &= \xi. \end{cases}$$

To prove the theorem it suffices to prove that Φ is contraction mapping in $\mathbb{H}_\beta$ under the norm $\|\cdot\|_\beta$ for sufficiently small β. For two arbitrary triplet $(y^1,z^1,k^1),(y^2,z^2,k^2)$ and (Y^1,Z^1,K^1), (Y^2,Z^2,K^2), we denote their difference by $\widetilde{y} = y^1 - y^2$ and $\widetilde{Y} = Y^1 - Y^2$ and similarly for z,k,Z and K. Applying Itô's formula to $e^{\beta t}|\widetilde{Y}(t)|^2$

$$\begin{aligned} &\mathbb{E}[\int_0^T e^{\beta t}\{\beta|\widetilde{Y}(t)|^2 + |\widetilde{Z}(t)|^2 + \int_{\mathbb{R}_0}|\widetilde{K}(t,\zeta)|^2\nu(d\zeta)\}dt] \\ &= 2\mathbb{E}[\int_t^T e^{\beta t}\widetilde{Y}(t)\{f(t,y^1(t),z^1(t),k^1(t,\cdot),\mathbb{E}[\varphi(y^1(t),z^1(t),k^1(t,\cdot))]) \\ &- f(t,y^2(t),z^2(t),k^2(t,\cdot),\mathbb{E}[\varphi(y^2(t),z^2(t),k^2(t,\cdot))])\}dt]. \end{aligned}$$

By the Lipschitz property of the map f, the mean value theorem, standard majorization and by choosing $\beta = 1 + 12\overline{C}^2$ ($\overline{C}$ depends only on C and C'), it follows that

$$\begin{aligned} &\mathbb{E}[\int_0^T e^{\beta t}\{|\widetilde{Y}(t)|^2 + |\widetilde{Z}(t)|^2 + \int_{\mathbb{R}_0}|\widetilde{K}(t,\zeta)|^2\nu(d\zeta)\}dt] \\ &\le \tfrac{1}{2}\mathbb{E}[\int_0^T e^{\beta t}\{|\widetilde{y}(t)|^2 + |\widetilde{z}(t)|^2 + \int_{\mathbb{R}_0}|\widetilde{k}(t,\zeta)|^2\nu(d\zeta)\}dt], \end{aligned}$$

Consequently, we get

$$\|(\widetilde{Y},\widetilde{Z},\widetilde{K})\|^2_\beta \le \tfrac{1}{2}\|(\widetilde{y},\widetilde{z},\widetilde{k})\|^2_\beta,$$

and Φ is then a contraction mapping. The theorem now follows by a standard fixed point theorem.

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