

HIGHER ORDER POLAR AND RECIPROCAL POLAR LOCI

RAGNI PIENE

ABSTRACT. In this note we introduce higher order polar loci as natural generalizations of the classical polar loci, replacing the role of tangent spaces by that of higher order osculating spaces. The close connection between polar loci and dual varieties carries over to a connection between higher order polar loci and higher order dual varieties. We generalize the duality between the degrees of polar classes of a variety and those of its dual variety to varieties that are reflexive to a higher order. In particular, the degree of the “top” (highest codimension) polar class of order k is equal to the degree of the k th dual variety. We define higher order Euclidean normal bundles and use them to define higher order reciprocal polar loci and classes. We give examples of how to compute the degrees of the higher order polar and reciprocal polar classes in some special cases: curves, scrolls, and toric varieties.

To Bill Fulton on the occasion of his 80th birthday.

1. INTRODUCTION

The theory of polar varieties, or polar loci, has a long and rich history. The terminology *pole* and *polar* goes back at least to Servois (1811) and Gergonne (1813). Poncelet’s more systematic treatment of *reciprocal polars* was presented in his “Mémoire sur la théorie générale des polaires réciproques” in 1824, though it did not appear in print until 1829. For a discussion of the – at times heated – debate between Poncelet and Gergonne concerning the principles of “reciprocity” versus that of “duality”, see [10]. Apparently, in the end, every geometer adopted both terms and used them interchangeably. Less known, perhaps, is the work of Bobillier, who (in 1828) was the first to replace the conic as “directrice” by a curve of arbitrary degree d , so that the polar curve of a point is a curve of degree $d-1$, and the polar points of a line is the intersection of the polar curves of the points on the line, hence equal to $(d-1)^2$ points, the number of base points of the corresponding pencil. For a summary of this early history, see [10] and [13].

In this note we introduce higher order polar loci as natural generalizations of the classical polar loci. The close connection between polar loci and dual varieties carries over to a connection between higher order polar loci and higher order dual varieties. In particular, the degree of the “top” (highest codimension) polar class of order k is equal to the degree of the k th dual variety.

In a series of papers, Bank, Giusti, Heintz, Mbakop, and Pardo introduced what they called “dual polar varieties” and used them to find real solutions to polynomial equations. These loci were studied further in [19] for real plane curves, and more generally in [26], under the name of “reciprocal polar varieties.” These variants of polar loci are defined with respect to a quadric, in order to get a notion of orthogonality, and sometimes with respect also to a hyperplane at infinity. The

orthogonality enables one to define *Euclidean normal bundles*, as studied in [4], [7], and [26]. For example, the (generic) Euclidean distance degree introduced in [7] is the degree of the “top” reciprocal polar class (see [26]). Note that the definition of reciprocal polar loci given here differs slightly from the one given in [26], but the degrees of the classes are the same.

In the next section we recall the definition of the classical polar loci and their classes, and their relation to the Mather Chern classes. In the third section we define the higher order polar loci and their classes and discuss how the latter can be computed. In the case of a smooth, k -regular variety, the k th order polar classes can be expressed in terms of its Chern classes and the hyperplane class, and there are also other cases when it is possible to compute these classes. In the fourth section, we introduce the higher order Euclidean normal bundles and use them, in the following section, to define higher order reciprocal polar loci and classes. The three last sections give examples of how to compute the degrees of the higher order polar and reciprocal polar classes in some special cases: curves, scrolls, and toric varieties.

Acknowledgments. I am grateful to Patrick Popescu-Pampu for alerting me to the work of Bobillier and to the thesis [10]. I thank the referee for asking a “natural question,” which is answered in Theorem 6.

2. POLAR LOCI AND MATHER CHERN CLASSES

Let V be a vector space of dimension $n + 1$ over an algebraically closed field of characteristic 0. The polar loci of an m -dimensional projective variety $X \subset \mathbb{P}(V)$ are defined as follows: Let $L_i \subset \mathbb{P}(V)$ be a linear subspace of codimension $m - i + 2$. The i th polar locus of X with respect to L_i is

$$M_i(L_i) := \overline{\{P \in X_{\text{sm}} \mid \dim(T_{X,P} \cap L_i) \geq i - 1\}},$$

where $T_{X,P}$ denotes the projective tangent space to X at the (smooth) point P . Note that $M_0(L_0) = X$. The rational equivalence class $[M_i(L_i)]$ is independent of L_i for general L_i , and will be denoted $[M_i]$ and called the i th polar class of X [23, Prop. (1.2), p. 253]. The classes $[M_i]$ are *projective invariants* of X : the i th polar class of a (general) linear projection of X is the projection of the i th polar class of X , and the i th polar class of a (general) linear section is the linear section of the i th polar class (see [23, Thm. (4.1), p. 269; Thm. (4.2), p. 270]).

Let us recall the definition of the Mather Chern class $c^M(X)$ of an m -dimensional variety X . Let $\tilde{X} \subseteq \text{Grass}_m(\Omega_X^1)$ denote the Nash transform of X , i.e., \tilde{X} is the closure of the image of the rational section $X \dashrightarrow \text{Grass}_m(\Omega_X^1)$ given by the locally free rank m sheaf $\Omega_X^1|_{X_{\text{sm}}}$. The Mather Chern class of X is $c^M(X) := \nu_*(c(\Omega^\vee) \cap [\tilde{X}])$, where Ω is the tautological sheaf on $\text{Grass}_m(\Omega_X^1)$ and $\nu: \tilde{X} \rightarrow X$.

In 1978 we showed the following, generalizing the classical Todd–Eger formulas to the case of singular varieties:

Theorem 1. [24, Thm. 3] *The polar classes of X are given by*

$$[M_i] = \sum_{j=0}^i (-1)^j \binom{m-j+1}{m-i+1} h^{i-j} \cap c_j^M(X),$$

where $h := c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ is the class of a hyperplane. Vice versa, the Mather Chern classes of X are given by

$$c_i^M(X) = \sum_{j=0}^i (-1)^j \binom{m-j+1}{m-i+1} h^{i-j} \cap [M_j].$$

3. HIGHER ORDER POLAR LOCI

Let $X \subset \mathbb{P}(V)$ be a projective variety of dimension m , and $P \in X$ a general point. There is a sequence of osculating spaces to X at P :

$$\{P\} \subseteq T_{X,P} = \text{Osc}_{X,P}^1 \subseteq \text{Osc}_{X,P}^2 \subseteq \text{Osc}_{X,P}^3 \subseteq \cdots \subseteq \mathbb{P}(V),$$

defined via the sheaves of principal parts of $\mathcal{O}_X(1)$ as follows. Let $V_X := V \otimes \mathcal{O}_X$ denote the trivial $(n+1)$ -bundle on X , and consider the k -jet map (see e.g. [22, p. 492])

$$j_k: V_X \rightarrow \mathcal{P}_X^k(1).$$

Let $X_{k\text{-cst}} \subseteq X$ denote the open dense where the rank of j_k is constant, say equal to m_k+1 . Then for $P \in X_{k\text{-cst}}$, $\text{Osc}_{X,P}^k = \mathbb{P}((j_k)_P(V)) \subseteq \mathbb{P}(V)$. Note that $m_0 = 0$, $m_1 = m$, and $\dim \text{Osc}_{X,P}^k = m_k$.

Assume $m_k < n$. Let $L_{k,i} \subset \mathbb{P}(V)$ be a linear subspace of codimension $m_k - i + 2$. The i th polar locus of order k of X with respect to $L_{k,i}$ is

$$M_{k,i}(L_{k,i}) := \overline{\{P \in X_{k\text{-cst}} \mid \dim(\text{Osc}_{X,P}^k \cap L_{k,i}) \geq i - 1\}}.$$

Note that $M_{1,i}(L_{1,i}) = M_i(L_{1,i})$ is the classical i th polar locus with respect to $L_{1,i}$, and that $M_{k,0}(L_{k,0}) = X$ for all k .

The (m_k+1) -quotient $V_{X_{k\text{-cst}}} \rightarrow j_k(V_{X_{k\text{-cst}}})$ gives a rational section

$$X \dashrightarrow \text{Grass}_{m_k+1}(V_X) = X \times \text{Grass}_{m_k+1}(V).$$

Its closure $\tilde{X}^k \subseteq X \times \text{Grass}_{m_k+1}(V)$, together with the projection map $\nu^k: \tilde{X}^k \rightarrow X$, is called the k th Nash transform of $X \subset \mathbb{P}(V)$. Note that $\nu^1 = \nu: \tilde{X}^1 = \tilde{X} \rightarrow X$ is the usual Nash transform. Let $V_{\tilde{X}^k} \rightarrow \mathcal{P}^k$ denote the induced (m_k+1) -quotient. We call \mathcal{P}^k the k th order osculating bundle of X . The projection map $\varphi_k: \tilde{X}^k \rightarrow \text{Grass}_{m_k+1}(V)$ is the k th associated map of $X \subset \mathbb{P}(V)$ (see [28] and [21]).

Theorem 2. *The class of $M_{k,i}(L_{k,i})$ is independent of $L_{k,i}$, for general $L_{k,i}$, and is given by*

$$[M_{k,i}] = \nu_*^k(c_i(\mathcal{P}^k) \cap [\tilde{X}^k]).$$

Proof. The proof is similar to that of [23, Prop. (1.2), p. 253]. \square

We call $[M_{k,i}]$ the i th polar class of order k of X .

Proposition 3. *Let $V' \subseteq V$ be a general subspace, with $\dim V' > m_k + 1$, and let $f: X \rightarrow \mathbb{P}(V')$ denote the corresponding linear projection. Then the image of the i th polar class of order k of $X \subset \mathbb{P}(V)$ is the same as the i th polar class of order k of $f(X) \subset \mathbb{P}(V')$.*

Proof. The proof is similar to that of [23, Thm. (4.1), p. 269]. \square

In the case $k = 1$, we proved [23, Thm. (4.2), p. 270] the following result, which does not have an obvious generalization to the case $k \geq 2$.

Proposition 4. *Let $\mathbb{P}(W) \subset \mathbb{P}(V)$ be a (general) linear subspace of codimension s . Set $Y := X \cap \mathbb{P}(W)$. For $0 \leq i \leq m - s$, the i th polar class of $Y \subset \mathbb{P}(W)$ is equal to the intersection of the i th polar class of X with $\mathbb{P}(W)$.*

Recall the definition of higher order dual varieties (introduced in [25]). The points of the dual projective space $\mathbb{P}(V)^\vee := \mathbb{P}(V^\vee)$ are the hyperplanes $H \subset \mathbb{P}(V)$. The k th order dual variety of X is

$$X^{(k)} := \overline{\{H \in \mathbb{P}(V)^\vee \mid H \supseteq \text{Osc}_{X,P}^k, \text{ for some } P \in X_{k\text{-cst}}\}}.$$

In particular, $X^{(1)} = X^\vee$ is the dual variety of X .

Set $\mathcal{K}^k := \text{Ker}(V_{\tilde{X}^k} \rightarrow \mathcal{P}^k)$; it is a $(n - m_k)$ -bundle. Then $X^{(k)} \subset \mathbb{P}(V)^\vee$ is equal to the image of $\mathbb{P}((\mathcal{K}^k)^\vee) \subset \tilde{X}^k \times \mathbb{P}(V)^\vee$ via the projection on the second factor. Let $p : \mathbb{P}((\mathcal{K}^k)^\vee) \rightarrow X$ and $q : \mathbb{P}((\mathcal{K}^k)^\vee) \rightarrow X^{(k)}$ denote the projections.

Proposition 5. *Let $L \subset \mathbb{P}(V)^\vee$ be a linear subspace of codimension $n - m_k + i - 1$ and set $L_{k,i} := L^\vee \subset \mathbb{P}(V)^{\vee\vee} = \mathbb{P}(V)$. Then*

$$p(q^{-1}(X^{(k)} \cap L)) = M_{k,i}(L_{k,i}).$$

Proof. If L is general, then so is $L_{k,i}$. The dimension of $L_{k,i}$ is $n - 1 - \dim L = n - 1 - (m_k - i + 1) = n - m_k + i - 2$, hence the codimension of $L_{k,i}$ is $m_k - i + 2$. If $H \in X^{(k)} \cap L$, then there is a $P \in X_{k\text{-cst}}$ such that $(P, H) \in \mathbb{P}((\mathcal{K}^k)^\vee)$. Hence $\text{Osc}_{X,P}^k \subseteq H$ and $L_{k,i} \subseteq H$, so the intersection $\text{Osc}_{X,P}^k \cap L_{k,i}$ has dimension $\geq i - 1$. Therefore $P \in M_{k,i}(L_{k,i})$. Conversely, if $P \in M_{k,i}(L_{k,i})$, then $\dim \text{Osc}_{X,P}^k \cap L_{k,i} \geq i - 1$, so that $\text{Osc}_{X,P}^k$ and $L_{k,i}$ do not span $\mathbb{P}(V)$. Hence there is a hyperplane H that contains both these spaces, and so $H \in X^{(k)} \cap L$. \square

The “expected dimension” of $X^{(k)}$ is equal to the dimension of $\mathbb{P}((\mathcal{K}^k)^\vee)$, which is $m + n - m_k - 1$. Let $\delta_k := m + n - m_k - 1 - \dim X^{(k)}$ denote the k th dual defect of X . Let $\overline{X} \subset \mathbb{P}(V) \times \mathbb{P}(V)^\vee$ denote the image of $\mathbb{P}((\mathcal{K}^k)^\vee)$, and let $\overline{X^{(k)}} \subset \mathbb{P}(V)^\vee \times \mathbb{P}(V) \cong \mathbb{P}(V) \times \mathbb{P}(V)^\vee$ denote the corresponding variety constructed for $X^{(k)}$. It was shown in [25, Prop. 1, p. 336] that $\overline{X} \subseteq \overline{X^{(k)}}$, so that equality holds iff their dimensions are equal. In this case, we say that X is k -reflexive, and we then have $X = (X^{(k)})^{(k)}$. For example, a non-degenerate curve $X \subset \mathbb{P}(V)$ is $(n - 1)$ -reflexive (see Section 6).

Theorem 6. *Assume X is k -reflexive. Then the degree of the i th polar class of order k of $X^{(k)}$ is given by*

$$\deg[M_{k,i}^\vee] = \deg[M_{k,m-\delta_k-i}].$$

In particular, the degree of $X^{(k)}$ is equal to the degree of the $(m - \delta_k)$ th polar class $[M_{k,m-\delta_k}]$ of order k of X .

Proof. For $k = 1$, this is [13, Thm. (4), p. 189], where it follows immediately from the definition of the ranks (corresponding to the degrees of the polar classes). Here we use Proposition 5: if h and h^\vee denote the hyperplane classes of $\mathbb{P}(V)$ and $\mathbb{P}(V)^\vee$ respectively, then the class $[M_{k,m-\delta_k-i}]$ is the pushdown to X of the class $(h^\vee)^{n-m_k+m-\delta_k-i-1} \cap [\overline{X}]$. By definition, $\delta_k = m + n - 1 - m_k - m^\vee$, where $m^\vee := \dim X^{(k)}$, so this is the same as $(h^\vee)^{m^\vee-i} \cap [\overline{X}]$. Hence its degree is the degree of $h^{n-1-m_k^\vee+i}(h^\vee)^{m^\vee-i} \cap [\overline{X}]$, where m_k^\vee denotes the dimension of a general k th osculating space of $X^{(k)}$. Similarly, the class $[M_{k,i}^\vee]$ is the pushdown to $X^{(k)}$ of the class $h^{n-m_k^\vee+i-1} \cap [\overline{X^{(k)}}] = h^{n-m_k^\vee+i-1} \cap [\overline{X}]$ and has degree equal to the degree of $(h^\vee)^{m^\vee-i} h^{n-1-m_k^\vee+i} \cap [\overline{X}]$.

Note that $i = m - \delta_k$ is the largest $i \leq m$ such that $[M_{k,i}] \neq 0$. \square

Assume $X \subset \mathbb{P}(V)$ is generically k -regular, i.e., the map $j_k: V_X \rightarrow \mathcal{P}_X^k(1)$ is generically surjective. The k th Jacobian ideal \mathcal{J}_k is the $\binom{m+k}{k}$ th Fitting ideal of $\mathcal{P}_X^k(1)$ [21, (2.9)]. Note that $\mathcal{J}_1 = F^{m+1}(\mathcal{P}_X^1(1)) = F^{m+1}(\mathcal{P}_X^1) = F^m(\Omega_X^1)$ is the ordinary Jacobian ideal. Let $\pi_k: X^k \rightarrow X$ denote the blow-up of \mathcal{J}_k . Then, by [29, 5.4.3], setting $\mathcal{A}_k := \text{Ann}_{\pi_k^* \mathcal{P}_X^k(1)}(F^{\binom{m+k}{k}}(\pi_k^* \mathcal{P}_X^k(1)))$, $\pi_k^* \mathcal{P}_X^k(1)/\mathcal{A}_k$ is a $\binom{m+k}{k}$ -bundle. Let \mathcal{I}_k denote the 0th Fitting ideal of the cokernel of the map $V_{X^k} \rightarrow \pi_k^* \mathcal{P}_X^k(1)/\mathcal{A}_k$ and $\bar{\pi}_k: \bar{X}^k \rightarrow X^k$ the blow-up of \mathcal{I}_k . Then [23, Lemma (1.1), p.252] the image $\bar{\mathcal{P}}^k$ of $\bar{V}_{\bar{X}^k}$ in $\bar{\pi}_k \pi_k^* \mathcal{P}_X^k(1)/\mathcal{A}_k$ is a $\binom{m+k}{k}$ -bundle. Hence we get a $\binom{m+k}{k}$ -quotient $V_{\bar{X}^k} \rightarrow \bar{\mathcal{P}}^k$ which agrees with $V_X \rightarrow \mathcal{P}_X^k(1)$ above $X_{k\text{-cst}}$. Note that, as discussed in the case $k = 1$ in [23, p. 255], the map $\pi_k \circ \bar{\pi}_k: \bar{X}^k \rightarrow X$ factors via the k th Nash transform $\nu^k: \tilde{X}^k \rightarrow X$. In particular, we have

$$[M_{k,i}] = \pi_{k*} \bar{\pi}_{k*} (c_i(\bar{\mathcal{P}}^k) \cap [\bar{X}^k]).$$

In some cases, the Chern classes of $\bar{\mathcal{P}}^k$ can be computed in terms of the Chern classes of $\mathcal{P}_X^k(1)$ and the invertible sheaves $\mathcal{J}_k \mathcal{O}_{\bar{X}^k}$ and $\mathcal{I}_k \mathcal{O}_{\bar{X}^k}$. We shall see in Section 6 that this is the case when X is a curve. Another case is the following.

Proposition 7. *Assume $X \subset \mathbb{P}(V)$ is smooth and generically k -regular, and that $m_k = n - 1$. Then*

$$[M_{k,i}] = c_1(\mathcal{P}_X^k(1))^i \cap [X] - \sum_{j=0}^{i-1} \binom{i}{j} c_1(\mathcal{P}_X^k(1))^j \cap s_{m-i+j}(I_k, X),$$

where $s_{m-i+j}(I_k, X) = -\bar{\pi}_{k*} (c_1(\mathcal{I}_k \mathcal{O}_{\bar{X}^k})^{i-j} \cap [\bar{X}^k])$ denote the Segre classes of the subscheme $I_k \subset X$ defined by the ideal $\mathcal{I}^k := F^0(\text{Coker } j^k)$.

Proof. Since $m_k = n - 1$, the (locally free) kernel $\bar{\mathcal{K}}^k := \text{Ker}(V_{\bar{X}^k} \rightarrow \bar{\mathcal{P}}^k)$ has rank 1, hence $c_i(\bar{\mathcal{P}}^k) = c_1(\bar{\mathcal{P}}^k)^i$ holds. We also have $\Lambda^n \bar{\mathcal{P}}^k \cong \Lambda^n \pi_k^* \mathcal{P}_X^k(1) \otimes \mathcal{I}^k \mathcal{O}_{\bar{X}^k}$, hence $c_1(\bar{\mathcal{P}}^k) = c_1(\pi_k^* \mathcal{P}_X^k(1)) + c_1(\mathcal{I}^k \mathcal{O}_{\bar{X}^k})$, so the result follows by applying the projection formula. \square

4. HIGHER ORDER EUCLIDEAN NORMAL BUNDLES

Let V and V' be vector spaces of dimensions $n + 1$ and n , and $V \rightarrow V'$ a surjection. Set $H_\infty := \mathbb{P}(V') \subset \mathbb{P}(V)$, and call it the *hyperplane at infinity*. A non-degenerate quadratic form on V' defines an isomorphism $V' \cong (V')^\vee$ and a non-singular quadric $Q_\infty \subset H_\infty$.

Let $L' := \mathbb{P}(W) \subset \mathbb{P}(V')$ be a linear space, and set

$$K := \text{Ker}(V'^\vee \cong V' \rightarrow W) \subset V'^\vee.$$

The *polar* of L' with respect to Q_∞ is the linear space $L'^\perp := \mathbb{P}(K^\vee) \subset \mathbb{P}(V')$.

Given a linear space $L \subset \mathbb{P}(V)$ and a point $P \in L$, define the *orthogonal space to L at P* by

$$L_P^\perp := \langle P, (L \cap H_\infty)^\perp \rangle.$$

If $L \not\subset H_\infty$, the dimension of $(L \cap H_\infty)^\perp$ is equal to $n - 1 - (\dim L - 1) - 1 = n - \dim L - 1$, and the dimension of L_P^\perp is $n - \dim L$ if $P \notin (L \cap H_\infty)^\perp$ and

$n - \dim L - 1$ if $P \in (L \cap H_\infty)^\perp$. Note that if $P \in (L \cap H_\infty)^\perp$, then $P \in L \cap H_\infty$ and hence $P \in P^\perp$. This implies that $P \in Q_\infty$ and $L \cap H_\infty \subseteq T_{Q_\infty, P}$.

Let $X \subset \mathbb{P}(V)$ be a variety of dimension m . Assume $X \not\subseteq H_\infty$. Let $P \in X$ be a non-singular point. The *Euclidean normal space* to X at P with respect to H_∞ and Q_∞ is the linear space $N_{X, P} := T_{\tilde{X}, P}^\perp$. Let $X_1 \subseteq X$ denote the set of non-singular points P such that $T_{X, P} \not\subseteq H_\infty$, and such that $T_{X, P} \cap H_\infty \not\subseteq T_{Q_\infty, P}$ if $P \in Q_\infty$. Then $\dim N_{X, P} = n - m$ for $P \in X_1$.

By replacing tangent spaces by higher order osculating spaces we may define higher order Euclidian normal spaces. Let $k \geq 1$ be such that the dimension m_k of a general k th order osculating space to X is less than n . For $P \in X_{k\text{-cst}}$, set

$$N_{X, P}^k := (\text{Osc}_{X, P}^k)^\perp.$$

Let $X_k \subseteq X_{k\text{-cst}}$ denote the set of points P such that $\text{Osc}_{X, P}^k \not\subseteq H_\infty$, and such that $\text{Osc}_{X, P}^k \cap H_\infty \not\subseteq T_{Q_\infty, P}$ if $P \in Q_\infty$. Then $\dim N_{X, P}^k = n - m_k$ for $P \in X_k$.

With notations as in Section 3, consider the exact sequence

$$0 \rightarrow \mathcal{K}^k \rightarrow V_{\tilde{X}^k} \rightarrow \mathcal{P}^k \rightarrow 0.$$

Assume the hyperplane $H_\infty = \mathbb{P}(V') \subset \mathbb{P}(V)$ is *general*. Then it follows from [22, Lemma (4.1), p. 483] that the dual $V_{\tilde{X}^k}^{\prime \vee} \rightarrow (\mathcal{K}^k)^\vee$ of the composed map $\mathcal{K}^k \rightarrow V_{\tilde{X}^k} \rightarrow V_{\tilde{X}^k}^{\prime \vee}$ is surjective.

Set

$$\mathcal{X}_k := \overline{\{(P, P') \mid P \in X_k, P' \in N_{X, P}^k\}} \subset X \times \mathbb{P}(V).$$

Let $p_k : \mathcal{X}_k \rightarrow X$ and $q_k : \mathcal{X}_k \rightarrow \mathbb{P}(V)$ denote the projections. The dimension of \mathcal{X}_k is $n - m_k + m$. The sheaf $p_{k*} q_k^* \mathcal{O}_{\mathbb{P}(V)}(1)$ is generically locally free, with rank $n - m_k + 1$. Let $\nu_k : \tilde{X}^k \rightarrow X$ be the ‘‘Nash transform’’ such that $\nu_k^* p_{k*} q_k^* \mathcal{O}_{\mathbb{P}(V)}(1)$ admits a locally free rank $n - m_k + 1$ quotient bundle \mathcal{E}^k . By the definition of the normal spaces, it follows that \mathcal{E}^k restricted to X_k splits as $(\mathcal{K}^k)^\vee|_{X_k} \oplus \mathcal{O}_{X_k}(1)$, and that, by replacing if necessary $\nu^k : \tilde{X}^k \rightarrow X$ and $\nu_k : \tilde{X}^k \rightarrow X$ by a further Nash transform (by abuse of notation, also denoted $\nu^k : \tilde{X}^k \rightarrow X$), the map $V_{\tilde{X}^k} \rightarrow (\mathcal{K}^k)^\vee \oplus \nu^{k*} \mathcal{O}_X(1)$ factors via the surjection $V_{\tilde{X}^k} \rightarrow \mathcal{E}^k$. We call \mathcal{E}^k the *kth Euclidean normal bundle* to X . The difference between \mathcal{E}^k and $(\mathcal{K}^k)^\vee \oplus \nu^{k*} \mathcal{O}_X(1)$ measures where $V_{\tilde{X}^k} \rightarrow (\mathcal{K}^k)^\vee \oplus \nu^{k*} \mathcal{O}_X(1)$ is not surjective.

Proposition 8. *For a generic choice of a hyperplane H_∞ and a quadric $Q_\infty \subset H_\infty$, the k th Euclidean normal bundle \mathcal{E}^k splits as $(\mathcal{K}^k)^\vee \oplus \nu^{k*} \mathcal{O}_X(1)$ on an appropriate Nash modification $\nu^k : \tilde{X}^k \rightarrow X$.*

Proof. We need to identify the points $P \in \tilde{X}^k$ such that $\alpha : V_{\tilde{X}^k} \rightarrow (\mathcal{K}^k)^\vee \oplus \nu^{k*} \mathcal{O}_X(1)$ is not surjective at P . First of all, by genericity of H_∞ , we may assume that H_∞ intersects X transversally, i.e., H_∞ intersects each Whitney stratum of X transversally. This means that H_∞ does not contain a tangent space, nor a limit tangent space, to X at any point of $X \cap H_\infty$. (This is the content of the surjectivity of $V_{\tilde{X}^1}^{\prime \vee} \rightarrow (\mathcal{K}^1)^\vee$.) This implies that H_∞ does not contain a k th order osculating space, nor a limit of such a space, at any point of $X \cap H_\infty$. Hence α is surjective at all points of \tilde{X}^k above $X \setminus H_\infty$. Now consider $P \in \tilde{X}^k$ mapping to $X \cap H_\infty$. The map α is not surjective at P only if $\nu^k(P) \in (\tilde{O}_P \cap H_\infty)^\perp$, where \tilde{O}_P is the

limit k th osculating space to X at $\nu^k(P)$ corresponding to P . But this implies that $\nu^k(P) \in \nu^k(P)^\perp$, hence $\nu^k(P) \in Q_\infty$. Therefore $\nu^k(P)^\perp = T_{Q_\infty, \nu^k(P)}$, and hence $T_{Q_\infty, \nu^k(P)} \supseteq \tilde{O}_P \cap H_\infty \supseteq \tilde{T}_P \cap H_\infty$. But we may assume this does not happen: by [12, Cor. 4, p. 291], a general quadric $Q_\infty \subset H_\infty$ intersects each of the Whitney strata of $X \cap H_\infty$ transversally. \square

The morphism $\varphi_k: \tilde{X}^k \rightarrow \text{Grass}_{m_k+1}(V)$ corresponding to the quotient $V_{\tilde{X}^k} \rightarrow \mathcal{P}^k$ is the k th order associated map of $X \subset \mathbb{P}(V)$. The k th order associated normal map is the morphism

$$\psi_k: \tilde{X}^k \rightarrow \text{Grass}_{n-m_k+1}(V),$$

defined by the quotient $V_{\tilde{X}^k} \rightarrow \mathcal{E}^k$.

5. HIGHER ORDER RECIPROCAL POLAR LOCI

Instead of imposing conditions on the osculating spaces of a variety, one can similarly impose conditions on the higher order Euclidean normal spaces. In what follows, we assume that H_∞ and Q_∞ are general with respect to X .

For each $i = 0, \dots, m$ and $k \geq 1$, let $L^{k,i} \subset \mathbb{P}(V)$ be a general linear space of codimension $n - m_k + i$ and define the i th reciprocal polar locus of order k with respect to $L^{k,i}$ to be

$$M_{k,i}^\perp(L^{k,i}) := \overline{\{P \in X_k \mid N_{X,P}^k \cap L^{k,i} \neq \emptyset\}}.$$

Note that, with $p_k: \mathcal{X}_k \rightarrow X$ and $q_k: \mathcal{X}_k \rightarrow \mathbb{P}(V)$ as in the previous section, $M_{k,i}^\perp(L^{k,i}) = p_k q_k^{-1}(L^{k,i})$, and $M_{k,0}^\perp(L^{k,0}) = X$ for all $k \geq 1$.

Theorem 9. *The class of $M_{k,i}^\perp(L^{k,i})$ is independent of $L_{k,i}$, for general $L_{k,i}$, and is given by*

$$[M_{k,i}^\perp] = \nu_*^k(\{c(\mathcal{P}^k)s(\nu^{k*}\mathcal{O}_X(1))\}_i \cap [\tilde{X}^k]).$$

Proof. Let $\Sigma_{k,i}(L^{k,i}) \subset \text{Grass}_{n-m_k+1}(V)$ denote the special Schubert variety consisting of the set of $(n - m_k)$ -planes that meet the $(m_k - i)$ -plane $L^{k,i}$. Then $M_{k,i}^\perp(L^{k,i}) = \psi_k^{-1}(\Sigma_{k,i}(L^{k,i}))$. The first statement follows, with the same reasoning as in [23, Prop. (1.2), p. 253].

Let $L^{k,i} = \mathbb{P}(W)$ and set $W' = \text{Ker}(V \rightarrow W)$. The condition $N_{X,P}^k \cap L^{k,i} \neq \emptyset$ means that the rank of the composed map $W'_{\tilde{X}^k} \rightarrow \mathcal{E}^k$ is $\leq n - m_k$. By Porteous' formula (see [8, Thm. 14.4, p. 254]), $M_{k,i}^\perp(L^{k,i})$ has class

$$[M_{k,i}^\perp] = \nu_*^k(s_i(\mathcal{E}^k) \cap [\tilde{X}^k]) = \nu_*^k(\{s((\mathcal{K}^k)^\vee)s(\nu^{k*}\mathcal{O}_X(1))\}_i \cap [\tilde{X}^k]).$$

Since the Segre class $s((\mathcal{K}^k)^\vee)$ is equal to the Chern class $c(\mathcal{P}^k)$, the formula follows. \square

Corollary 10. *Let $h := c_1(\mathcal{O}_X(1))$ denote the hyperplane class. We have*

$$[M_{k,i}^\perp] = \sum_{j=0}^i h^{i-j} \cap [M_{k,j}],$$

and hence

$$\deg[M_{k,i}^\perp] = \sum_{j=0}^i \deg[M_{k,j}].$$

Proof. This follows, using Theorem 2, since

$$s(\mathcal{O}_{\tilde{X}^k}(1)) = 1 + c_1(\mathcal{O}_{\tilde{X}^k}(1)) + c_1(\mathcal{O}_{\tilde{X}^k}(1))^2 + \cdots$$

and $\mathcal{O}_{\tilde{X}^k}(1) = \nu^{k*} \mathcal{O}_X(1)$. □

Remark 11. The (generic) *Euclidean distance degree* of $X \subset \mathbb{P}(V)$, introduced in [7], can be interpreted as the degree of $[M_{1,m}^\perp]$. The above formula says that this degree is equal to the sum of the degrees of the polar classes of X , as stated in [7, Thm. 5.4, p. 126] (see also [26]).

6. CURVES

Let $X \subset \mathbb{P}(V)$ be a non-degenerate curve. At a general point $P \in X$ we have a complete flag

$$\{P\} \subseteq T_{X,P} = \text{Osc}_{X,P}^1 \subset \text{Osc}_{X,P}^2 \subset \cdots \subset \text{Osc}_{X,P}^{n-1} \subset \mathbb{P}(V).$$

In this case $\nu^k : \tilde{X}^k = \tilde{X} \rightarrow X$ is the normalization of X , $m_k = k$, and $\dim L_{k,1} = n - k - 1$.

Note that the locus $M_{k,1}(L_{k,1})$ maps to k th order hyperosculating points on the image of X under the linear projection $f: \mathbb{P}(V) \dashrightarrow \mathbb{P}(V')$ with center $L_{k,1}$. Namely, let $P \in M_{k,1}(L_{k,1})$. Then $\text{Osc}_{X,P} \cap L_{k,1} \neq \emptyset$, and hence $f(\text{Osc}_{X,P})$ has dimension $< k$. This means that the k th jet map of $f(X)$ has rank $< k + 1$ at the point $f(P)$. For example, for $k = 1$, $f(P)$ is a cusp on $f(X)$, and for $k = 2$, $f(P)$ is an inflection point.

Recall that the k th rank r_k of the curve X is defined as the degree of the k th osculating developable of X , i.e., as the number of k th order osculating spaces intersecting a given, general linear space of dimension $n - k - 1$ [2, pp. 199–200]. Hence $r_k = \deg \nu_*^k(c_1(\mathcal{P}^k) \cap [\tilde{X}])$, and r_k is also equal to the degree of the k th associated curve of X (see [22, Prop. (3.1), p. 480] and [28], [21]).

Let $d := r_0$ denote the degree of X . Since $[M_{k,1}] = \nu_*^k(c_1(\mathcal{P}^k) \cap [\tilde{X}])$, we get $\deg[M_{k,1}] = r_k$. We have $r_k = (k+1)(d+k(g-1)) - \sum_{j=0}^{k-1} (k-j)\kappa_j$, where g is the genus of \tilde{X} and κ_j is the j th stationary index of X (see [22, Thm. (3.2), p. 481]). It follows that $\deg[M_{k,1}^\perp] = \deg[M_{k,0}] + \deg[M_{k,1}] = d + r_k$. For more on ranks, duality, projections, and sections, see [22]. For example, the ranks of the *strict dual curve* $X^{(n-1)} \subset \mathbb{P}(V)^\vee$ satisfy $r_k(X^{(n-1)}) = r_{n-k-1}(X)$.

Example 12. If $X \subset \mathbb{P}(V)$ is a *rational normal curve* of degree n , then $\deg[M_{k,1}] = r_k = (k+1)(n-k)$ and $\deg[M_{k,1}^\perp] = n + r_k = n + (k+1)(n-k)$.

Note that X is *toric* and $(n-1)$ -*self dual*: $X^{(n-1)} \subset \mathbb{P}(V)^\vee$ is a rational normal curve of degree n .

7. SCROLLS

Ruled varieties – scrolls – are examples of varieties that are not generically k -regular, for $k \geq 2$. Hence we cannot hope to use the bundles of principal parts to compute the degrees of the higher polar classes for such varieties. However, in several cases we have results. In particular, since the degree of the “top” k th order polar class is the same as the degree of the k th dual variety, we can consider the following situations.

7.1. Rational normal scrolls. Let $X = \mathbb{P}(\oplus_{i=1}^m \mathcal{O}_{\mathbb{P}^1}(d_i)) \subset \mathbb{P}(V)$ be a rational normal scroll of type (d_1, \dots, d_m) , with $m \geq 2$, $0 < d_1 \leq \dots \leq d_m$, and $n = \sum_{i=1}^m (d_i + 1) - 1$. Let $d := d_1 + \dots + d_m$ denote the degree of X . The higher order dual varieties of rational normal scrolls were studied in [27]. For example, for k such that $k \leq d_1$, we have

$$\deg[M_{k,m}] = \deg X^{(k)} = kd - k(k-1)m,$$

where the first equality follows from Theorem 6 and the second was computed in [6, Prop. 4.1, p. 389]. In the case $k = d_1 = \dots = d_m$, this gives

$$\deg[M_{k,m}] = \deg X^{(k)} = kd - k(k-1)m = md_1^2 - d_1(d_1 - 1)m = md_1 = d.$$

Indeed, in this case X is k -selfdual: $X^{(k)} \subset \mathbb{P}(V)^\vee$ is a rational normal scroll of the same type as X .

We refer to [27] and [6] for other cases.

7.2. Elliptic normal surface scrolls. Higher order dual varieties of elliptic normal scrolls were studied in [16]. Let C be a smooth elliptic curve and \mathcal{E} a rank two bundle on C . Assume $H^0(C, \mathcal{E}) \neq 0$, but that $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for all invertible sheaves \mathcal{L} with $\deg \mathcal{L} < 0$. Then let $e := \deg \mathcal{E}$ denote the Atiyah invariant. There are two cases. Either \mathcal{E} is decomposable – then $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ and $e = -\deg \mathcal{L} \geq 0$, or \mathcal{E} is indecomposable, in which case $e = 0$ or $e = -1$. Now let \mathcal{M} be an invertible sheaf on C of degree d . If $d \geq e + 3$, then $\mathcal{O}_{\mathbb{P}(\mathcal{E} \otimes \mathcal{M})}(1)$ yields an embedding of $\mathbb{P}(\mathcal{E} \otimes \mathcal{M})$ as a linearly normal scroll $X \subset \mathbb{P}(V) \cong \mathbb{P}^{2d-e-1}$ of degree $2d - e$. The following holds [16, Thm. 1, Thm. 2, p. 150].

If \mathcal{E} is decomposable:

- (i) if $e = 0$, then $\deg[M_{d-1,2}] = \deg X^{(d-1)} = 2d(d-1)$;
- (ii) if $e = 1$, then $\deg[M_{d-2,2}] = \deg X^{(d-2)} = 2d^2 - 5d + 2$;
- (iii) if $e \geq 2$, then $\deg[M_{d-2,2}] = \deg X^{(d-2)} = d(d-1)$.

If \mathcal{E} is indecomposable:

- (i) if $e = -1$, then $\deg[M_{d-1,2}] = \deg X^{(d-1)} = 2d^2 - 3$;
- (ii) if $e = 0$, then $\deg[M_{d-1,2}] = \deg X^{(d-1)} = 2d^2 - d - 2$.

7.3. Scrolls over smooth curves. Consider now the more general situation where $X \subset \mathbb{P}(V)$ is a smooth scroll of dimension m and degree d over a smooth curve C of genus g . We showed in [14] that in this case, for k such that $m_k = km$, the k th jet map j_k factors via a bundle \mathcal{P}^k of rank $km + 1$. Moreover, the Chern classes of \mathcal{P}^k can be expressed in terms of d, m, k, g , the class of a fiber of $X \rightarrow C$, and the class of a hyperplane section of X . Thus we can get a formula for the “top” k th order polar class in terms of these numbers and the Segre classes of the inflection loci of X , see [14] for details.

7.4. Scrolls over smooth varieties. When we replace the curve C by a higher-dimensional smooth projective variety of dimension r , the situation gets more complicated, but it is again possible to find a k th osculating bundle \mathcal{P}^k of rank $(m-r) \binom{r+k-1}{r} + \binom{r+k}{r}$, whose Chern classes can be computed, see [15].

8. TORIC VARIETIES

The Schwartz–MacPherson Chern class of a toric variety X with torus orbits $\{X_\alpha\}_\alpha$ is equal to, by Ehlers' formula (see [9, Lemma, p. 109], [3, Thm., p. 188], [18, Thm. 1.1; Cor. 1.2 (a)], [1, Thm. 4.2, p. 410]),

$$c^{\text{SM}}(X) = \sum_\alpha [\overline{X}_\alpha].$$

This implies (proof by the definition of $c^{\text{SM}}(X)$ and induction on $\dim X$) that the Mather Chern class of a toric variety X is equal to

$$c^{\text{M}}(X) = \sum_\alpha \text{Eu}_X(X_\alpha) [\overline{X}_\alpha],$$

where $\text{Eu}_X(X_\alpha)$ denotes the value of the local Euler obstruction of X at a point in the orbit X_α .

Therefore the polar classes of a toric variety X of dimension m are

$$[M_i] = \sum_{j=0}^i (-1)^j \binom{m-j+1}{m-i+1} h^{i-j} \cap \sum_\alpha \text{Eu}_X(X_\alpha) [\overline{X}_\alpha],$$

and the reciprocal polar classes

$$\begin{aligned} [M_i^\perp] &= \sum_{\ell=0}^i h^{i-\ell} \sum_{j=0}^\ell (-1)^j \binom{m-j+1}{m-\ell+1} h^{\ell-j} \cap \sum_\alpha \text{Eu}_X(X_\alpha) [\overline{X}_\alpha] \\ &= \sum_{j=0}^i (-1)^j \sum_{\ell=0}^{i-j} \binom{m-j+1}{\ell} h^{i-j} \cap \sum_\alpha \text{Eu}_X(X_\alpha) [\overline{X}_\alpha], \end{aligned}$$

where the second sum in each expression is over α such that $\text{codim } X_\alpha = j$.

It follows that if $X = X_\Pi$ is a projective toric variety corresponding to a convex lattice polytope Π , then (cf. [17, Thm. 1.4, p. 2042])

$$\deg[M_i] = \sum_{j=0}^i (-1)^j \binom{m-j+1}{m-i+1} \text{EVol}^j(\Pi),$$

and

$$\deg[M_i^\perp] = \sum_{j=0}^i (-1)^j \sum_{\ell=0}^{i-j} \binom{m-j+1}{\ell} \text{EVol}^j(\Pi),$$

where $\text{EVol}^j(\Pi) := \sum_\alpha \text{Eu}_X(X_\alpha) \text{Vol}(F_\alpha)$ denotes the sum of the lattice volumes of the faces F_α of Π of codimension j weighted by the local Euler obstruction $\text{Eu}_X(X_\alpha)$ of X at a point of X_α , where X_α is the torus orbit of X corresponding to the face F_α . In particular, we get

$$\deg[M_m^\perp] = \sum_{j=0}^m (-1)^j (2^{m-j+1} - 1) \text{EVol}^j(\Pi),$$

(cf. [11, Thm. 1.1, p. 215]).

Example 13. Nødland [20, 4.1] studied weighted projective threefolds. In particular he showed the following. Assume a, b, c are positive, pairwise relatively prime integers. The weighted projective threefold $\mathbb{P}(1, a, b, c)$ is the toric variety corresponding to the lattice polyhedron $\Pi := \text{Conv}\{(0, 0, 0), (bc, 0, 0), (0, ac, 0), (0, 0, bc)\}$. It has isolated singularities at the three points corresponding to the three vertices other than $(0, 0, 0)$. Let $\text{Vol}^j(\Pi)$ denote the sum of the lattice volumes of the faces of Π of codimension j . We have $\text{Vol}^0(\Pi) = a^2 b^2 c^2$, $\text{Vol}^1(\Pi) = abc(1 + a + b + c)$, $\text{Vol}^2(\Pi) = a + b + c + bc + ac + ab$, and $\text{Vol}^3(\Pi) = 4$. The algorithm given in [20, A.2] can be used to compute the local Euler obstruction at the singular points. Nødland gave several examples of integers a, b, c such that the local Euler obstruction at each singular point is 1, thus providing counterexamples to a conjecture of Matsui and Takeuchi [17, p. 2063]. For example this holds for $a = 2, b = 3, c = 5$, so in this case $\text{EVol}^j(\Pi) = \text{Vol}^j(\Pi)$ and we can compute

$$\deg[M_0] = 900, \deg[M_1] = 3270, \deg[M_2] = 4451, \deg[M_3] = 2688,$$

and hence

$$\deg[M_3^\perp] = 11309.$$

In general, formulas for the degrees of higher order polar classes and reciprocal polar classes of toric varieties are not known. However, in some cases, they can be found. As we have seen, for smooth varieties (not necessarily toric), if the k th jet map is surjective (i.e., the embedded variety is k -regular), then the classes $[M_{k,i}]$ can be expressed in terms of Chern classes of the sheaf of principal parts $\mathcal{P}_X^k(1)$. Hence, in the case of a k -regular toric variety, they can be expressed in terms of lattice volumes of the faces of the corresponding polytope. The following two examples were worked out in [5, Rmk. 3.5, p. 385; Thm. 3.7, p. 387].

Example 14. Let $\Pi \subset \mathbb{R}^2$ be a smooth lattice polygon with edge lengths $\geq k$. Then X_Π is k -regular and

$$\begin{aligned} \deg[M_{k,1}] &= \binom{k+2}{2} \text{Vol}^0(\Pi) - \binom{k+2}{3} \text{Vol}^1(\Pi), \\ \deg[M_{k,2}] &= \binom{k+3}{4} (3 \text{Vol}^0(\Pi) - 2k \text{Vol}^1(\Pi) - \frac{1}{3}(k^2 - 4) \text{Vol}^2(\Pi) + 4(k^2 - 1)). \end{aligned}$$

Example 15. Let $\Pi \subset \mathbb{R}^3$ be a smooth lattice polyhedron with edge lengths ≥ 2 . Then X_Π is 2-regular and

$$\begin{aligned} \deg[M_{2,1}] &= 4 \text{Vol}^0(\Pi) - \text{Vol}^1(\Pi), \\ \deg[M_{2,2}] &= 36 \text{Vol}^0(\Pi) - 27 \text{Vol}^1(\Pi) + 6 \text{Vol}^2(\Pi) + 18 \text{Vol}^0(\Pi_0) + 9 \text{Vol}^1(\Pi_0), \\ \deg[M_{2,3}] &= 62 \text{Vol}^0(\Pi) - 57 \text{Vol}^1(\Pi) + 28 \text{Vol}^2(\Pi) - 8 \text{Vol}^3(\Pi) \\ &\quad + 58 \text{Vol}^0(\Pi_0) + 51 \text{Vol}^1(\Pi_0) + 20 \text{Vol}^2(\Pi_0), \end{aligned}$$

where $\Pi_0 := \text{Conv}(\text{int}(\Pi) \cap \mathbb{Z}^3)$ is the convex hull of the interior lattice points of Π .

Recall [6, Def. 1.1, p. 1760] that a variety $X \subset \mathbb{P}(V)$ is said to be k -selfdual if there exists a linear isomorphism $\phi : \mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V)^\vee$ such that $\phi(X) = X^{(k)}$. If $m = \dim X$, then $\deg[M_{k,m}]$ is the degree of the k -dual variety $X^{(k)}$. So if X is k -selfdual, then $\deg[M_{k,m}] = \deg X$. We refer to [6] for examples of toric k -selfdual varieties.

References

- [1] P. Aluffi, *Classes de Chern des variétés singulières, revisitées*, C. R. Math. Acad. Sci. Paris **342** (2006), no. 6, 405–410.
- [2] H. F. Baker, *Principles of geometry. Volume 5. Analytical principles of the theory of curves*, Cambridge Library Collection, Cambridge University Press, Cambridge, 2010. Reprint of the 1933 original.
- [3] G. Barthel, J.-P. Brasselet, and K.-H. Fieseler, *Classes de Chern des variétés toriques singulières*, C. R. Acad. Sci. Paris Sér. I Math. **315** (1992), no. 2, 187–192.
- [4] F. Catanese and C. Trifogli, *Focal loci of algebraic varieties. I*, Comm. Algebra **28** (2000), no. 12, 6017–6057.
- [5] A. Dickenstein, S. Di Rocco, and R. Piene, *Higher order duality and toric embeddings*, Ann. Inst. Fourier (Grenoble) **64** (2014), no. 1, 375–400.
- [6] A. Dickenstein and R. Piene, *Higher order selfdual toric varieties*, Ann. Mat. Pura Appl. (4) **196** (2017), no. 5, 1759–1777.
- [7] J. Draisma, E. Horobeț, G. Ottaviani, B. Sturmfels, and R. R. Thomas, *The Euclidean Distance Degree of an Algebraic Variety*, Found. Comput. Math. **16** (2016), no. 1, 99–149.
- [8] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984.

- [9] ———, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.
- [10] C. Haubrichs dos Santos, *Étienne Bobillier (1798–1840): parcours mathématique, enseignant et professionnel*, Université de Lorraine, 2015. Thesis (Ph.D.)
- [11] M. Helmer and B. Sturmfels, *Nearest points on toric varieties*, Math. Scand. **122** (2018), no. 2, 213–238.
- [12] S. L. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287–297.
- [13] ———, *Tangency and duality*, Proceedings of the 1984 Vancouver conference in algebraic geometry, CMS Conf. Proc., vol. 6, Amer. Math. Soc., Providence, RI, 1986, pp. 163–225.
- [14] A. Lanteri, R. Mallavibarrena, and R. Piene, *Inflectional loci of scrolls*, Math. Z. **258** (2008), no. 3, 557–564.
- [15] ———, *Inflectional loci of scrolls over smooth, projective varieties*, Indiana Univ. Math. J. **61** (2012), no. 2, 717–750.
- [16] R. Mallavibarrena and R. Piene, *Duality for elliptic normal surface scrolls*, Enumerative algebraic geometry (Copenhagen, 1989), Contemp. Math., vol. 123, Amer. Math. Soc., Providence, RI, 1991, pp. 149–160.
- [17] Y. Matsui and K. Takeuchi, *A geometric degree formula for A -discriminants and Euler obstructions of toric varieties*, Adv. Math. **226** (2011), no. 2, 2040–2064.
- [18] L. G. Maxim and J. Schürmann, *Characteristic classes of singular toric varieties*, Comm. Pure Appl. Math. **68** (2015), no. 12, 2177–2236.
- [19] H. C. Mork and R. Piene, *Polars of real singular plane curves*, Algorithms in algebraic geometry, IMA Vol. Math. Appl., vol. 146, Springer, New York, 2008, pp. 99–115.
- [20] B. I. U. Nødland, *Local Euler obstructions of toric varieties*, J. Pure Appl. Algebra **222** (2018), no. 3, 508–533.
- [21] R. Piene, *Plücker formulas*, ProQuest LLC, Ann Arbor, MI, 1976. Thesis (Ph.D.)—Massachusetts Institute of Technology.
- [22] ———, *Numerical characters of a curve in projective n -space*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 475–495.
- [23] ———, *Polar classes of singular varieties*, Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 2, 247–276.
- [24] ———, *Cycles polaires et classes de Chern pour les variétés projectives singulières*, Introduction à la théorie des singularités, II, Travaux en Cours, vol. 37, Hermann, Paris, 1988, pp. 7–34.
- [25] ———, *A note on higher order dual varieties, with an application to scrolls*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 335–342.
- [26] ———, *Polar varieties revisited*, Computer algebra and polynomials, Lecture Notes in Comput. Sci., vol. 8942, Springer, Cham, 2015, pp. 139–150.
- [27] R. Piene and G. Sacchiero, *Duality for rational normal scrolls*, Comm. Algebra **12** (1984), no. 9–10, 1041–1066.
- [28] W. F. Pohl, *Differential geometry of higher order*, Topology **1** (1962), 169–211.
- [29] M. Raynaud and L. Gruson, *Critères de platitude et de projectivité. Techniques de “platification” d’un module*, Invent. Math. **13** (1971), 1–89.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, PO BOX 1053 BLINDERN, NO-0316 OSLO, NORWAY

E-mail address: ragnip@math.uio.no