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A note on the temporal and spatial attenuation of ocean waves

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ABSTRACT

We study attenuation of free waves in the homogeneous ocean. Using the Cauchy–Riemann relations and then energy equation, we derive a novel equation that relates temporal and spatial decay to the energy dissipation in the fluid. In particular, for spatially damped waves we find that the attenuation coefficient is inversely proportional to the group velocity. We discuss the consequences of this for continental shelf waves where the group velocity may tend to zero in wave number space, and hence the spatial attenuation coefficient may become infinitely large. We show that generally for a travelling wave solution, the spatial damping will remain unchanged in a frame of reference moving with the group velocity, revealing the different physics behind temporal wave damping and spatial wave damping.

1. Introduction

In an elegant note on hydrodynamic stability, Gaster (1962) compares temporally growing disturbances and spatially growing disturbances having the same wave number. He shows that for small amplification rates, the frequencies are equal to a high order of approximation, and that the temporal growth is equal to the spatial growth times the group velocity of the unstable wave. The same results apply to small decay rates, i.e. the temporal decay is equal to the spatial decay times the group velocity of the damped wave. Although Gaster's pioneering note only contains three references, the study of spatio-temporal wavy disturbances in hydrodynamic stability has a long history; see for example the extensive account in Schmid and Henningson (2001).

In his derivation, Gaster (1962) refers to instability governed by the Orr–Sommerfeld equation. However, his result is not restricted to disturbances in shear flow. It is equally valid in fluids originally at rest where the wavy perturbation attenuates due to friction. One such phenomenon is ocean waves subject to viscous dissipation, which is the topic studied here. Along the line of Gaster, we apply the Cauchy–Riemann relations and extend the analysis to periodic ocean waves where the amplitude decays slowly in space *and* time. This yields a novel relation between temporal damping, spatial damping and the dissipation in the wave motion.

This note is organized as follows: In Section 2 we apply the Cauchy–Riemann relations and the energy equation to derive the general result, relating the temporal and spatial damping to the energy dissipation. Furthermore, we display an explicit form of this relation for surface gravity waves with viscosity when the ocean depth is finite. In Section 3, we consider frictionally damped continental shelf waves, and in Section 4, we discuss briefly the case when the group velocity may tend to zero, as may happen in continental shelf waves. Finally, Section 5 contains some concluding remarks, including a discussion of the different physical nature of temporal wave damping and spatial wave damping.

2. General theory

In the ocean, freely propagating waves suffer attenuation due to friction. For two-dimensional motion in an incompressible fluid with waves propagating along the x-axis, we introduce a stream function Ψ such that

$$\Psi = \Psi_0 \exp i(\kappa x - nt), \tag{1}$$

where

$$\kappa = k + i\alpha, \tag{2}$$

$$n = \omega - i\beta \tag{3}$$

Here, k, ω are the real wave number and angular frequency, respectively, while α, β are the real (and small) spatial and temporal damping rates due to friction. The dispersion relation for the wave problem can be written in general terms as

$$n = G(\kappa, T), \tag{4}$$

where T is a constant that may depend on parameters like topography, the earth's rotation etc. We assume that n is an analytic function of κ. Then, from the Cauchy–Riemann relations we have, using the notation (2), (3), that

$$\partial\omega/\partial k = -\partial\beta/\partial\alpha, \tag{5}$$

$$\partial\omega/\partial\alpha = \partial\beta/\partial k. \tag{6}$$

Here $\partial\omega/\partial k = c_g$ is the group velocity. In the inviscid case, the dispersion relation is $\omega = G(k, T)$. In our problem the attenuation rates α and β are assumed to be small, and (5) implies that $O(\alpha) \sim O(\beta)$ if

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$c_g \neq 0$ and finite.[1] Furthermore, from (6) we find

$$\omega = G(k,T) + O(\alpha^2, k, T). \tag{7}$$

We then obtain from (5) that

$$\beta + c_g\alpha = Q, \tag{8}$$

where Q is a small quantity that does not depend on α, and c_g is the group velocity to the lowest approximation. Actually, Q is determined by the energy dissipation in the system, which is seen as follows: Let E be the energy density of a monochromatic wave subjected to temporal and spatial decay. The equation for the energy balance of the fluctuating wave motion, in the absence of mean currents, is (Phillips, 1977):

$$\partial E/\partial t + \partial(c_g E)/\partial x = -\varepsilon. \tag{9}$$

Here $c_g E$ is the advection of mean wave energy by the group velocity and ε is the rate of energy dissipation per unit area due to friction in the fluid. Since $E \sim \exp(-2\alpha x - 2\beta t)$, we obtain from (9) that

$$\beta + c_g\alpha = \varepsilon/(2E). \tag{10}$$

Hence, in (8),

$$Q = \varepsilon/(2E), \tag{11}$$

as postulated above.

From (8), we note that if $\alpha = 0$ (temporal attenuation), then $\beta = Q$. If $\beta = 0$ (spatial attenuation), then $\alpha = Q/c_g$. Accordingly, when comparing the two different cases, we have $\beta = c_g\alpha$, as shown by Gaster (1962).

Specific examples of (10) may be found in the literature; see for example Jenkins (1986) for surface gravity waves in an infinitely deep viscous ocean. In the case of finite depth H, we find from Weber (2019), Eq. (34), when we neglect the atmospheric stress at the ocean surface:

$$\beta + c_g\alpha = \omega[k/\gamma + 1/(2\sinh 2kH)](k/\gamma), \tag{12}$$

where

$$c_g = [\omega/(2k)][1 + 2kH/\sinh 2kH], \tag{13}$$

and

$$\gamma = \omega^{1/2}/(2\nu)^{1/2}. \tag{14}$$

Here ν is the small eddy viscosity. The quantity γ^{-1} represents the viscous boundary-layer thickness. The analysis assumes that $k/\gamma \ll 1$. The first term in the parenthesis on the right-hand side of (12) arises from the viscous boundary layer at the free surface, while the second term is due to a no-slip condition at the ocean bottom. In the limit $H \to \infty$, (12) was first derived by Jenkins (1986).

We note that for spatial damping in a fluid with dissipation ($Q \neq 0, \beta = 0$), we obtain from (8) that

$$\alpha = Q/c_g. \tag{15}$$

For gravity waves this result poses no problems, since c_g is non-zero and finite. On the other hand, for a wave-type like continental shelf waves (Buchwald and Adams, 1968), we have the possibility that $c_g \to 0$ for a certain wave number k. Then, from (15), it follows that $\alpha \to \infty$. We discuss this intriguing problem in Section 4. Before that, we derive the complex dispersion relation for continental shelf waves subject to linear bottom friction. This is done to display the explicit form of (10) for this case.

[1] Actually, for continental shelf waves, we may have $c_g = 0$ for a certain value of k. We discuss this interesting problem in Section 4.

3. Continental shelf waves

Continental shelf waves are fundamentally different from gravity waves. In these waves, the motion is basically governed by the conservation of potential vorticity (Longuet-Higgins, 1965). Since the classic papers of Buchwald and Adams (1968), and Gill and Schumann (1974), work on continental shelf waves have been steadily increasing. Some updated references can be found in Drivdal et al. (2016). We here idealize the shelf geometry, as in Buchwald and Adams (1968), and place the horizontal axes at the ocean surface, with the x-axis along the coast and the y-axis towards the sea. The bottom profile is given by $z = -H(y)$, where

$$H = \begin{cases} H_0 \exp(2by), & 0 \le y \le B \\ H_0 \exp(2bB), & y \ge B. \end{cases} \tag{16}$$

Here b is a constant describing the steepness of the shelf slope, and B is the shelf width.

We assume that $B^2/a_0^2 \ll 1$, where a_0 is the barotropic Rossby radius, defined by mid-depth. This means that we can make the rigid lid approximation (Gill and Schumann, 1974). The continuity equation then allows for the introduction of a stream function ψ such that $uH = -\partial\psi/\partial y$ and $vH = \partial\psi/\partial x$, where u and v are the horizontal velocity components. We take that the waves are so long that the pressure is hydrostatic in the vertical direction. The effect of bottom friction on the barotropic wave field is discussed at some length in Weber and Drivdal (2012). A robust formulation is obtained by taking the bottom stress to vary linearly with the horizontal fluxes. The mathematical derivation is standard, and we give some details for didactic reasons.

The linearized momentum equations for this problem can be written

$$-\partial^2\psi/\partial t\partial y - f\partial\psi/\partial x = -gH\partial\eta/\partial x + r\partial\psi/\partial y, \tag{17}$$

$$\partial^2\psi/\partial t\partial x - f\partial\psi/\partial y = -gH\partial\eta/\partial y - r\partial\psi/\partial x. \tag{18}$$

Here η is the surface elevation, f is the constant Coriolis parameter and r is a small bottom friction coefficient. We take that the coast is impermeable. Hence,

$$\psi = 0, \quad y = 0. \tag{19}$$

At the edge of the shelf, $y = B$, we must generally have continuity of pressure (here surface elevation) and normal fluxes. Utilizing that the deep ocean has a flat bottom, it easy to show that the continuity conditions imply for the stream function over the shelf that (Weber and Drivdal, 2012),

$$\partial\psi/\partial y + \kappa\psi = 0, \quad y = B, \tag{20}$$

where κ is the complex wave number in the direction along the coast.

It is convenient to write ψ as (Gill, 1982):

$$\psi = H^{1/2}\phi(y)\exp i(\kappa x - nt). \tag{21}$$

Introducing (21) into (17)–(18), and utilizing (16), we find by performing the curl that

$$\phi'' + L^2\phi = 0, \tag{22}$$

where the prime denotes derivation with respect to y. Furthermore, the cross-shelf wave number is given

$$L^2 = 2fb\kappa/(n + ir) - b^2 - \kappa^2. \tag{23}$$

In terms of ϕ in (21), the boundary conditions become

$$\phi = 0, \quad y = 0, \tag{24}$$

$$\phi' + (b + \kappa)\phi = 0, \quad y = B. \tag{25}$$

Using (24), the solution of (22) is

$$\phi = C\sin Ly, \tag{26}$$

where the constant C is arbitrary. Inserting from (26), we then find from (25) that

$$(b+\kappa)\sin BL + L\cos BL = 0. \quad (27)$$

The spatial and temporal damping coefficients in (2) and (3) are small quantities of $O(r)$ in this problem. In Weber and Drivdal (2012), it was assumed that the cross-shelf wave number L in (23) was real. This actually corresponds to placing an impermeable wall at $y = B$. More generally, it is seen from (23) that L also contains a small imaginary part. We can then write

$$L = l + i\delta, \quad (28)$$

where the imaginary part δ is of $O(r)$. The presence of damping will also modify the frequency. The modification will be of $O(r^2)$ (to be verified later); see also (7).

3.1. The dispersion relation

The complex equations (23) and (27), arising from the governing equation for the wave motion and the boundary condition, respectively, form an eigen-value problem that determine the dispersion properties for these waves. From (23), the real part to lowest order yields the well-known dispersion relation

$$\omega = 2fbk/(b^2 + l^2 + k^2), \quad (29)$$

where l is determined from the boundary condition (27), as shown below. The imaginary part of (23) becomes to $O(r)$,

$$\delta = [\left(b^2 + l^2 + k^2\right)(\beta - r) + c\left(b^2 + l^2 - k^2\right)\alpha]/(2\omega l), \quad (30)$$

where $c = \omega/k$ is the lowest order wave speed. It is then easily seen from the real part of (23) that the second-order correction to ω is of $O(r^2)$, as anticipated.

From the real part of (27) we find to lowest order that

$$\tan lB = -l/(b+k), \quad (31)$$

as shown by Buchwald and Adams (1968). For given bottom topography, this transcendental equation yields infinitely many discrete values of l (transverse eigen-modes) for each k.

In general, $(n - 1/2)\pi \le l_n B \le n\pi, n = 1, 2, 3..$ (Buchwald and Adams, 1968).

Finally, from the imaginary part of (27), we obtain for the spatial damping rate to $O(r)$,

$$\alpha = [b + k + (b+k)^2 B + l^2 B](\delta/l). \quad (32)$$

We remark that (32) is a linear relation between α and δ, while (30) relates α, β and δ. By eliminating δ between (30) and (32), we finally arrive at

$$\beta + \frac{c}{(b^2 + l^2 + k^2)}[b^2 + l^2 - k^2 - 2kl^2/\{b + k + (b+k)^2 B + l^2 B\}]\alpha = r. \quad (33)$$

From Weber and Drivdal (2012), their Eq. (A.12), we can write the group velocity c_g for continental shelf waves, when the bottom topography is given by (16), as

$$c_g/c = [b^2 + l^2 - k^2 - 2kl^2/\{b + k + (b+k)^2 B + l^2 B\}]/(b^2 + l^2 + k^2). \quad (34)$$

Comparing (33) and (34), we see right away that

$$\beta + c_g\alpha = r, \quad (35)$$

as we expect from (10). We note that $Q = \varepsilon/(2E) = r$ in this case.

4. Spatial damping

An intriguing aspect of the wave-damping problem was mentioned in Section 2. In many cases of practical interest, it is natural to consider

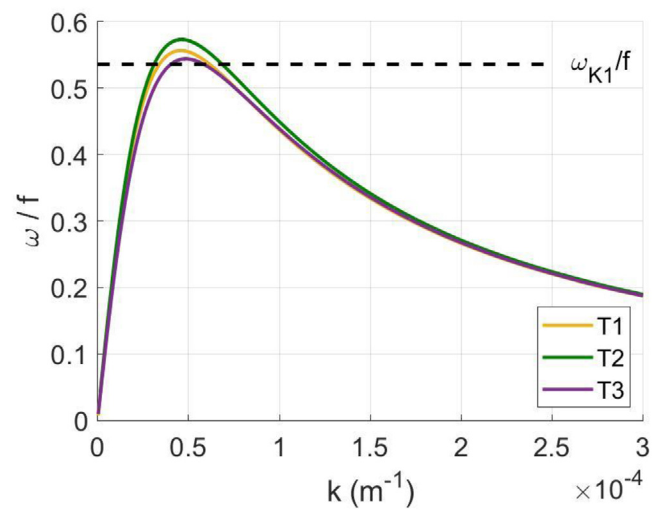

Fig. 1. Dispersion diagrams for continental shelf waves (first mode) at three transects T1, T2, T3 outside Vesterålen, showing non-dimensional frequency ω/f *vs* wave number k. Here $f = 1.3 \cdot 10^{-4}\,s^{-1}$ is the Coriolis parameter. The upper horizontal dashed line is the non-dimensional frequency for the tidal K_1 component.

spatial damping. An example here is the operation of a wave maker at a given frequency at one end of a wave tank in experimental studies. For continental shelf waves we have from (34) that $c_g \to 0$ for certain wave numbers k. Then, from (15), it follows that $\alpha \to \infty$. Obviously, our calculations assuming a small damping rate is not valid here, but this singular behaviour indicates that wave energy accumulates in the region where c_g is close to zero.

Such energy accumulation appears to occur on the narrow, steep continental slope north-west of Norway for continental shelf waves with the diurnal tidal frequency K_1. At this frequency, the local group velocity is close to zero; see the dispersion diagrams in Fig. 1 at three different locations outside Vesterålen, north Norway (Weber and Børve, 2022).

The intersection between the dashed and full lines in Fig. 1 shows that continental shelf waves can be generated by the action of the diurnal tidal K_1 component outside Vesterålen in a wave number region centred around zero group velocity. A study of this problem when $c_g \to 0$ is found in Weber and Børve (2021, 2022), where the interested reader also may find references to similar studies of tidally forced continental shelf waves. In the two cited papers, the wave damping is not due to bottom friction, but is caused by the energy loss in the exchange of water between the shelf and the inner island archipelago through a permeable coastline. This confirms that the damping relation (10) is general, and not dependent on how we specifically model energy loss in the problem.

5. Concluding remarks

Inserting from (8), we may write the stream function (1) as

$$\Psi = \Psi_0 \exp\{-\alpha(x - c_g t) - Qt\}\exp\{i(kx - \omega t)\}. \quad (36)$$

We thus see that for a travelling wave solution, the spatial damping will remain unchanged in a frame of reference moving with the group velocity. The different physical nature of temporal and spatial damping should be noted. Whatever value of the group velocity, the temporal attenuation of the wave field remains small. Thus, temporal damping is related to the kinematics of the waves. If we have wave motion at a fixed location in space, the temporal damping is just the measured decrease of wave amplitude at that location. The spatial attenuation, on the other hand, is related to the wave energetics. As we have seen, the spatial attenuation increases without bounds as the group velocity approaches zero. In this case, the wave energy will accumulate in the region of wave generation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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