VOLTERRA EQUATIONS DRIVEN BY ROUGH SIGNALS 2: HIGHER ORDER EXPANSIONS

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Abstract. We extend the recently developed rough path theory for Volterra equations from [12] to the case of more rough noise and/or more singular Volterra kernels. It was already observed in [12] that the Volterra rough path introduced there did not satisfy any geometric relation, similar to that observed in classical rough path theory. Thus, an extension of the theory to more irregular driving signals requires a deeper understanding of the specific algebraic structure arising in the Volterra rough path. Inspired by the elements of "non-geometric rough paths" developed in [11] and [14] we provide a simple description of the Volterra rough path and the controlled Volterra process in terms of rooted trees, and with this description we are able to solve rough Volterra equations driven by more irregular signals.

1. Introduction

1.1. **Background and description of the results.** Volterra equations of the second kind are typically given of the form

$$y_t = y_0 + \int_0^t k_1(t,s)b(y_s)ds + \int_0^t k_2(t,s)\sigma(y_s)dx_s, \quad y_0 \in \mathbb{R}^m \tag{1.1}$$

where b and σ are sufficiently smooth functions, $x : [0,T] \to \mathbb{R}^d$ is a α-Hölder continuous path with $\alpha \in (0,1)$, and k_1 and k_2 are two possibly singular kernels, behaving like $|t-s|^{-\gamma}$ for some $\gamma \in [0,1)$ whenever $s \to t$. Such equations frequently appear in mathematical models for natural or social phenomena which exhibits some form of memory of its own past as it evolves in time (see e.g. [3] and the references therein). Most recently, Volterra equations of this form have become very popular in the modelling of stochastic volatility for financial asset prices. In this case the kernels $k_1(t,s)$ and $k_2(t,s)$ are typically assumed to be very singular when $s \to t$, and the path x is assumed to be a sample path of a Gaussian process (see e.g. [10, 9, 4]).

Whenever the driving noise x is sampled from a Brownian motion (or some other continuous semi-martingale), one may use traditional probabilistic techniques from stochastic analysis (see e.g. [17, 18]) in order to make sense of equations like (1.1). However, for more general driving noise x with rougher regularity than a Brownian motion, very little is known about solutions to Volterra equations. Inspired by the theory of rough paths [15], it is desirable to solve equation (1.1) in a purely pathwise sense relying only on the analytic behaviour of the sample paths of x. This would in particular allow to solve such equations driven by non-martingale stochastic processes in a pathwise way. However, due to the non-local nature of the equations induced by the kernels k_1

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and k_2, the theory of rough paths can not directly be applied in order to solve singular Volterra equations of the form of (1.1). Indeed, the fundamental algebraic relations satisfied by a a classical rough path do not hold when the signal is influenced by a possibly singular kernel. Let us mention at this point a few contributions in the rough paths realm trying to overcome this obstacle:

(i) The articles [7, 8] handle some cases of rough Volterra equations thanks to an elaboration of traditional rough paths elements. However, the analysis is only valid for kernels with no singularities.

(ii) The paper [16] focuses on Volterra equations from a para-controlled calculus perspective, with possibly singular kernels and rough noise. In contrast to the methodology proposed in [12], the framework is based on Besov spaces providing a slightly more flexible framework. At the moment the framework is limited to the type of Volterra kernels of the form $k(t,s)=k(t-s)$. However, the conditions under which existence and uniqueness of the Volterra equation holds (In Hölder-Besov space) is similar to the conditions derived in [12]. Being based on the theory of para-controlled calculus, it seems challenging to lower regularity assumptions on the driving noise or singular kernel. It might be possible to do this with the recent developments of higher order para-controlled calculus proposed in [1], but we are not aware of further extensions in this direction.

(iii) The contribution [4] investigates Volterra equations through the lens of regularity structures. The Volterra equations considered there are driven by a Brownian motion, and infinite-renormalization techniques are invoked in a similar spirit as is used for singular SPDEs, in contrast to what is considered here. However, in terms of regularity assumptions on the nonlinear vector field in the Volterra equation, their results are indeed comparable to [12], and what is considered here. The framework of regularity structures is indeed promising for extensions of the rough path theory to Volterra equations with singular kernels, but requires a heavy machinery in order to deal with a problem that can otherwise be handled with more elementary arguments based on rough paths theory (as illustrated in [12]).

With those preliminary notions in mind, in the recent article [12] we initiated a rough path inspired study of singular Volterra equations, in a reduced form of (1.1) given by

$$u_t = u_0 + \int_0^t k(t,r) f(u_r) dx_r, \tag{1.2}$$

where f is a sufficiently regular function, x is a Hölder continuous path, and k is a singular kernel. To this end, we define

$$\Delta_n := \Delta_n([a,b]) = \{(x_1,\ldots,x_n) \in [a,b]^n \mid a \le x_1 < \cdots < x_n \le b\}. \tag{1.3}$$

Next we introduce a class of two parameter paths $z : \Delta_2 \to \mathbb{R}^d$, needed to capture the possible singularity and regularity imposed by the kernels k_1 and k_2 and the driving noise x in (1.1). These paths will then constitute the fundamental building blocks of the framework. The canonical example of such path is given by

$$z_t^\tau := \int_0^t k(\tau,s) dx_s, \quad \text{where} \quad t \le \tau \in [0,T]. \tag{1.4}$$

For the moment, we may assume that x is a sufficiently regular path $x:[0,T]\to\mathbb{R}^d$, and $k(t,s)$ is an integrable (but possibly singular) kernel when $s \to t$, so that the above integral makes pathwise sense. We observe in particular that $t \mapsto z_t^t$ is just a standard Volterra integral (commonly referred to as a Volterra process in stochastic analysis). Heuristically one may think that the regularity arising from the mapping $\tau \mapsto z_t^\tau$ is induced by the behavior of the kernel k while the regularity of the mapping $t \mapsto z_t^\tau$ is inherited by the regularity of x. By construction of a Volterra sewing lemma, we observed that this was indeed the case, even when x is only α-Hölder continuous for some $\alpha \in (0,1)$. In general, we thus define a class of two variables paths in terms of the regularity in its upper and lower variable. This lead us in [12] to introduce two modifications of the classical Hölder semi-norms. The corresponding processes were then called Volterra paths.

Motivated by processes of the form (1.4), we constructed Volterra signatures as a collection of iterated integrals with respect to two-parameters Volterra paths. We also introduced a convolution product $*$, playing the role as the tensor product $\otimes$ in the classical rough path signature. The signature is then given as a family three-variable functions $\{(s,t,\tau) \mapsto \mathbf{z}_{ts}^{n,\tau}\}_{n\in\mathbb{N}}$, where, in the case of smooth x, each term is given by

$$\mathbf{z}_{ts}^{n,\tau} = \int_{\Delta_n([s,t])} k(\tau, r_n) \dots k(r_2, r_1) dx_{r_1} \otimes \dots \otimes dx_{r_n}, \tag{1.5}$$

where we recall that $\Delta_n([s,t])$ is defined by (1.3). The algebraic structure associated with such iterated integrals resembles that of the tensor algebra of rough path theory, but where the tensor product is replaced by the convolution product. Together with Volterra signatures, we defined a class of controlled Volterra paths. Combining those two notions, it allowed to give a pathwise construction of solutions to Volterra equations of the form (1.1). Similarly to the theory of rough paths, the number of iterated integrals needed in order to give a pathwise definition of a rough Volterra integral is strongly dependent on the regularity of the path $x \in \mathcal{C}^\alpha([0,T];\mathbb{R}^d)$ and the singularity of the kernel k. Under the assumption that $|k(t,s)|$ behaves like $|t-s|^{-\gamma}$ when $s \to t$, the investigation in [12] was limited to the case when $\alpha - \gamma > \frac{1}{3}$, and thus only considers the first two components of the Volterra signature.

Our article [12] therefore left two important open questions, related to both the algebraic and probabilistic perspectives on rough paths theory:

(i) Algebraic aspects: Are there suitable algebraic relations describing the Volterra signature which are adaptable to prove existence and uniqueness of (1.1) in the case when $\alpha - \gamma < \frac{1}{3}$?

(ii) Probabilistic aspects: For what type of stochastic processes $\{x_t;\, t \in [0,T]\}$ and singular kernels k does there exist a collection of iterated integrals of the form of (1.5) almost surely satisfying the required algebraic and analytic relations?

The current article has to be seen as a step towards the answer of the algebraic problem mentioned above. Namely we investigate the case when $\alpha - \gamma \in (\frac{1}{4}, \frac{1}{3}]$, and leave the probabilistic problem for a future work.

The rough Volterra picture gets significantly more involved when introducing a rougher signal x or a more singular kernel k. Indeed, the main challenge lies in the fact that the Volterra signature does not satisfy any geometric type property, in contrast with the classical rough paths situation. That is, classical integration by parts does not hold for Volterra iterated integrals, and therefore we do *not* have a relation of the form

$$\mathbf{z}_{ts}^{2,\tau} + (\mathbf{z}_{ts}^{2,\tau})^T = \mathbf{z}_{ts}^{1,\tau} * \mathbf{z}_{ts}^{1,\cdot}, \tag{1.6}$$

where $(\cdot)^T$ denotes the transpose. Thus in order to consider $\alpha - \gamma$ lower than $\frac{1}{3}$, one needs to resort to different techniques than what is standard in the theory of rough paths.

Inspired by Martin Hairer's theory of regularity structures, we will in this article show that the Volterra signature is given with a Hopf algebraic type structure. Hence with the help of a description by rooted trees for the Volterra rough path, we are able to describe the necessary algebraic relations desired for the Volterra rough stochastic calculus. We will limit the scope of the current article to the case when $\alpha - \gamma > \frac{1}{4}$, and show that in order to prove existence and uniqueness of (1.1) in a "Volterra rough path" sense, one needs to introduce two more iterated integrals, as well as two more controlled Volterra derivatives than what is needed in the case $\alpha - \gamma > \frac{1}{3}$. We believe that the techniques developed here are an important stepping stone towards the goal of providing a rough paths framework for Volterra equations of the form of (1.1) in the general regime $\alpha - \gamma > 0$.

Remark 1.1. In fact, very recently (that is a few months after the first preprint of the current article was submitted), Bruned and Katsetsiadis [5] were able to extend the framework of Volterra rough paths proposed in [12] to any Hölder continuous noise with suitable (but possibly singular) Volterra kernel. Their methodology is strongly influenced by the algebraic techniques of regularity structures, but some extra care has been taken towards algebraic aspects related to the extended convolution product. The contribution [5] provides thus a rather complete picture of the algebraic structure of Volterra rough paths.

1.2. **Motivation from Volterra integral expansions.** This section is devoted to give some heuristic insight on our approach to Volterra integration in a very rough context. All the notions alluded to here will be introduced rigorously in the next sections. Namely in a similar spirit to the classical rough paths theory, Volterra rough paths rely on the interplay between controlled processes and iterated integrals. As already mentioned, in the Volterra context the notion of tensor product is replaced by a convolution product. This results in a far more complicated structure on both the controlled processes and the iterated integrals, as more parameters appear in both expressions. To motivate this, we slightly rewrite the equation (1.2) in order to make the solution a Volterra path (which is a two parameter path, see equation (1.4) and discussion below):

$$\bar{u}_t^\tau = u_0 + \int_0^t k(\tau, r) f(\bar{u}_r^r) dx_r. \tag{1.7}$$

Note that in this case $\bar{u}_t^t = u_t$ where u solves (1.2). Heuristically, for $\tau \geq t$ the regularity $\tau \to \bar{u}_t^\tau$ is inherited from the regularity of the kernel k, while the regularity of $t \mapsto \bar{u}_t^\tau$ is inherited from x. The splitting of the variables thus allows us to look for regularity contributions from the two components separately. Similarly to the theory of rough paths, we can now consider a Taylor type expansion of the integral, but where we only look at the expansion in terms of the lower parameter $t \mapsto \bar{u}_t^\tau$. To this aim, let $y : \Delta_2 \to \mathbb{R}^d$ be a smooth Volterra process and let $f \in C_b^\infty(\mathbb{R}^d)$ be a bounded smooth function. Consider the Volterra integral

$$\int_s^t k(\tau, r) f(y_r^r) dx_r. \tag{1.8}$$

An elementary expansion of $f(y_r^r)$ around the point y_s^r gives the relation

$$\int_s^t k(\tau, r) f(y_r^r) dx_r = \int_s^t k(\tau, r) f(y_s^r) dx_r + \int_s^t k(\tau, r) Df(y_s^r) y_{rs}^r dx_r + \int_s^t k(\tau, r) R_{rs}^r dx_r, \tag{1.9}$$

where R is a remainder term from the Taylor expansion and where we have set $y_{rs}^r = y_r^r - y_s^r$. Again, inspired by the rough paths formalism, suppose y is a controlled process (in the lower variable), in the sense that there exists a "derivative" $y' : \Delta_3 \to \mathcal{L}(\mathbb{R}^d; \mathbb{R}^m)$ and a function $R : \Delta_3 \to \mathbb{R}^m$ such that for all $s \le t \le \tau$

$$y^\tau_{ts} = \int_s^t k(\tau, u) y'^{,\tau,u}_s dx_u + \bar{R}^\tau_{ts}. \tag{1.10}$$

We can think of this as "for each $\tau > 0$, the increments of y^τ are controlled by two functions $(y'^{,\tau}, R^\tau)$". As seen from the expansion (1.9), for each τ the derivative process y^τ must then be a Volterra process (i.e. a two variable function), thus making it a three variable function. Inserting the relation in (1.10) into (1.9), it is readily seen that

$$\begin{aligned}\int_s^t k(\tau, r) f(y^r_r) dx_r = \int_s^t k(\tau, r) f(y^r_s) dx_r + \int_s^t k(\tau, r) Df(y^r_s) \int_s^r k(r, u) y'^{,r,u}_s dx_u dx_r \\ + \int_s^t k(\tau, r) \bar{R}^r_{rs} dx_r + \int_s^t k(\tau, r) R^r_{rs} dx_r.\end{aligned} \tag{1.11}$$

In the first two terms on the RHS we see that the lower variable is fixed, and integration is only appearing in the upper variable. As in (1.5), these considerations lead us to define the tuple $(\mathbf{z}^1, \mathbf{z}^2) : \Delta_3 \to \mathbb{R}^d \oplus \mathbb{R}^{d \times d}$ by

$$\mathbf{z}^{1,\tau}_{ts} := \int_s^t k(\tau, r) dx_r \quad \text{and} \quad \mathbf{z}^{2,\tau}_{ts} := \int_s^t \int_s^r k(\tau, r) k(r, u) dx_u dx_r. \tag{1.12}$$

If we denote by y^1 a function of 1 upper variable and by $y^{1,2}$ a function of 2 upper variables (see Notation 2.24 below), we can introduce a new notation:

$$\begin{aligned}\mathbf{z}^{1,\tau}_{ts} * f(y^1_s) &:= \int_s^t k(\tau, r) f(y^r_s) dx_r \\ \mathbf{z}^{2,\tau}_{ts} * \left(Df(y^1_s) y'^{,1,2}_s\right) &:= \int_s^t k(\tau, r) Df(y^r_s) \int_s^r k(r, u) y'^{,\tau,u}_s dx_u dx_r.\end{aligned}$$

Then the relation in (1.11) can be written as

$$\int_s^t k(\tau, r) f(y^r_r) dx_r = \mathbf{z}^{1,\tau}_{ts} * f(y^1_s) + \mathbf{z}^{2,\tau}_{ts} * \left(Df(y^1_s) y'^{,1,2}_s\right) + \int_s^t k(\tau, r) \tilde{R}^r_{rs} dx_r \tag{1.13}$$

where $\tilde{R} = R + \bar{R}$. The structure of this expansion greatly resembles the expansions from classical rough paths theory. However, the tensor product $\otimes$ is now replaced by the convolution product $*$. So far, we have assumed that all functions appearing in the above expansions are smooth, and so there is no difficulty in defining each of the terms, or the convolution product. In the rough setting, the existence of the convolution product must be justified in order that such expansions make sense.

Let us say a few words about the way to define the integral in the left hand side of (1.13). It will be based on a Volterra type sewing lemma applied to the integrand

$$\Xi^\tau_{ts} = \mathbf{z}^{1,\tau}_{ts} * f(y^1_s) + \mathbf{z}^{2,\tau}_{ts} * \left(Df(y^1_s) y'^{,1,2}_s\right). \tag{1.14}$$

Notice that Ξ^τ above is amenable to a sewing lemma only if $\mathbf{z}^\tau$ satisfies a specific Chen type relation involving convolution products. Namely we assume that for $s < t < \tau$ we have

$$\mathbf{z}^{2,\tau}_{ts} - \mathbf{z}^{2,\tau}_{tu} - \mathbf{z}^{2,\tau}_{us} = \mathbf{z}^{1,\tau}_{tu} * \mathbf{z}^{1,\cdot}_{us}, \tag{1.15}$$

where the convolution on the right hand side is in the sense that

$$\mathbf{z}^{1,\tau}_{tu} * \mathbf{z}^{1,\cdot}_{us} = \int_u^t k(\tau,r)\int_s^u k(r,u)dx_u dx_r. \tag{1.16}$$

We should also briefly discuss the definition of $\mathbf{z}^1$ and $\mathbf{z}^2$. In the setting when $t \mapsto x_t$ is α-Hölder continuous with $\alpha \le \frac{1}{2}$, the existence of $\mathbf{z}^1$ and $\mathbf{z}^2$ does not follow from simple analytical arguments in general, and thus the relation (1.15) is not obvious. Similarly to what is done in the theory of rough paths, $\mathbf{z}^1$ and $\mathbf{z}^2$ may be constructed by using probabilistic arguments in the case when the driving noise x is given as a sample path of a stochastic process. A deeper investigation of this probabilistic construction is dealt with in the forthcoming paper [13].

As explained in [12], the expansion proposed in (1.14) is only sufficient in the case when $\alpha > \frac{1}{3}$ and k is a possible singular kernel, not exploding "too fast" on the diagonal. To extend the theory to $\alpha \le \frac{1}{3}$, one needs to introduce more terms in the Taylor like expansion (1.9) to create a suitable integrand to be used in the sewing lemma. In the current article we extend the regime to the case $\alpha > \frac{1}{4}$, under suitable singularity conditions for k. To this end we have to introduce two extra levels of iterated integrals and two extra controlled processes. In order to give a flavor of what is to be expected for higher order expansions, our considerations for the third order case will be based on a family of processes indexed by trees.

The particular challenge when extending to lower regularity is that the extra controlled processes will depends on three (rather than two) upper variables, requiring extra arguments to make sure that the convolution product is well defined. Additionally, the non-geometric nature of the Volterra rough paths, as described before equation (1.6), makes the corresponding iterated integrals algebra much more complicated and requires extra care. This will be elaborated in the coming sections.

1.3. **Organization of the paper.** In Section 2 we provide the necessary assumptions and preliminary results from [12]. In particular, we give the definition of Volterra paths, recall the Volterra sewing lemma and the convolution product between Volterra paths. Those results will play a central role for our subsequent analysis. Section 3 is devoted to the extension of the sewing lemma from the previous section to the case of two singularities, and we will apply this to create a third order convolution product between Volterra rough paths. In Section 4 we motivate the use of rooted trees to describe the Volterra rough path, and give a definition of controlled Volterra processes analogously. With this definition we prove both the convergence of a rough Volterra integral with respect to controlled Volterra paths, and that compositions of (sufficiently) smooth functions with a controlled Volterra path are again controlled Volterra paths. We conclude Section 4 with a proof of existence and uniqueness of Volterra equations driven by rough signals in the rougher regime.

1.4. **Frequently used notation.** We reserve the letter E to denote a Banach space, and we let the norm on E be denoted by $|\cdot|_E$. In subsequent sections, E will typically be given as $\mathbb{R}^d$ or $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)$ (The space of linear operators from $\mathbb{R}^m$ to $\mathbb{R}^d$). We will write $a \lesssim b$, whenever there exists a constant $C > 0$ (not depending on any parameters of significance) such that $a \le Cb$. The space of continuous functions $f : X \to Y$ is denoted by $\mathcal{C}(X,Y)$. Whenever the codomain is not important, we use the shorter notation $\mathcal{C}(X)$. To denote that there exists a constant C which depends on a parameter p, we write $a \lesssim_p b$. For a one parameter path $f : [0,T] \to E$, we write $f_{ts} := f_t - f_s$, with a slight abuse of notation, we will later also use this notation for two variable functions of the form $f : [0,T]^2 \to \mathbb{R}^d$, where f_{ts} means evaluation in the point $(s,t) \in [0,T]^2$.

We believe that it will always be clear from context what is meant. For $\alpha \in (0,1)$, we denote by $\mathcal{C}^\alpha([0,T];E)$ the standard space of α-Hölder continuous functions from $[0,T]$ into E, equipped with the norm $\|f\|_{\mathcal{C}^\alpha} := |f_0|_E + \|f\|_\alpha$, where $\|f\|_\alpha$ denotes the classical Hölder seminorm given by

$$\|f\|_\alpha := \sup_{(s,t)\in\Delta_2} \frac{|f_{ts}|}{|t-s|^\alpha}. \tag{1.17}$$

Whenever the domain and codomain is otherwise clear from the context, we will use the short hand notation $\mathcal{C}^\alpha$. We recall here that the n-simplex was already defined in (1.3). Throughout the article, we will frequently use the following simple bounds: for $(s,u,t) \in \Delta_3$ and $\gamma > 0$, then

$$|t-u|^\gamma \lesssim |t-s|^\gamma \quad \text{and} \quad |t-s|^{-\gamma} \lesssim |t-u|^{-\gamma}.$$

2. Assumptions and fundamentals of Volterra Rough Paths

We will start by presenting the necessary assumptions on the Volterra kernel k, as well as the driving noise x in (1.2). A full description (together with proofs) for the results recalled in this section can be found in [12].

Let us begin to present a working hypothesis for the type of kernels k, seen in (1.2), that we will consider in this article.

Hypothesis 2.1. *Let k be a kernel $k : \Delta_2 \to \mathbb{R}$, we assume that there exists $\gamma \in (0,1)$ such that for all $(s,r,q,\tau) \in \Delta_4([0,T])$ and $\eta, \beta \in [0,1]$ we have*

$$\begin{aligned}
|k(\tau,r)| &\lesssim |\tau-r|^{-\gamma} \\
|k(\tau,r) - k(q,r)| &\lesssim |q-r|^{-\gamma-\eta} |\tau-q|^\eta \\
|k(\tau,r) - k(\tau,s)| &\lesssim |\tau-r|^{-\gamma-\eta} |r-s|^\eta \\
|k(\tau,r) - k(q,r) - k(\tau,s) + k(q,s)| &\lesssim |q-r|^{-\gamma-\beta} |r-s|^\beta \\
|k(\tau,r) - k(q,r) - k(\tau,s) + k(q,s)| &\lesssim |q-r|^{-\gamma-\eta} |\tau-q|^\eta .
\end{aligned}$$

In the sequel a kernel fulfilling condition the Hypothesis 2.1 will be called Volterra kernel of order γ.

Remark 2.2. We limit our investigations in this article to the case of real valued Volterra kernels k for conciseness. The Volterra sewing lemma, and most results relating to Volterra rough paths are however easily extended to general Volterra kernels $k : \Delta_2 \to \mathcal{L}(E)$ for some Banach space E, by appropriate change of the bounds in 2.1, see e.g. [6, 2] where the Volterra sewing lemma from [12] is readily applied in an infinite dimensional setting.

As mentioned in the introduction, one of the key ingredients in [12] is to consider processes $(t,\tau) \mapsto z_t^\tau$ indexed by Δ_2 (where we recall that the simplex Δ_n was defined in (1.3)). We begin this section with a recollection of the Hölder space containing such processes and introduce the Volterra sewing Lemma 2.13, we will then move on to introduce the convolution product and discuss its relation with the Volterra signature.

2.1. **The space of Volterra paths.**

We begin this section by recalling the topology used to measure the regularity of processes like (1.4), and give a simple motivation for the introduction of this type of space. Before defining the proper spaces quantifying this type of regularity, let us introduce a notation:

Notation 2.3. *Let* $(\alpha,\gamma)\in(0,1)^2$ *be such that* $\alpha>\gamma$. *For* $(s,t,\tau)\in\Delta_3$, *we set*

$$\psi^1_{\alpha,\gamma}(\tau,t,s)=\left[|\tau-t|^{-\gamma}|t-s|^{\alpha}\right]\wedge|t-s|^{\alpha-\gamma}. \tag{2.1}$$

Considering $(s,t,\tau',\tau)\in\Delta_4$ *and two additional parameters* $\zeta\in[0,\alpha-\gamma)$ *and* $\eta\in[\zeta,1]$, *we also set*

$$\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(\tau,\tau',t,s)=|\tau-\tau'|^{\eta}|\tau'-t|^{-(\eta-\zeta)}\left(\left[|\tau'-t|^{-\gamma-\zeta}|t-s|^{\alpha}\right]\wedge|t-s|^{\alpha-\gamma-\zeta}\right). \tag{2.2}$$

Whenever the subscript parameters and the function arguments are not of particular relevance, we will refer to these functions as ψ^1 *and* $\psi^{1,2}$ *for short.*

We are now ready to introduce some functional spaces called $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$, which are also used in the definition of $\mathcal{V}^{(\alpha,\gamma)}$ in [12]. Those spaces are natural function sets when dealing with Volterra type regularities.

Definition 2.4. *Let* E *be a Banach space, and consider four parameters* α, γ *in* $(0,1)$ *and* ζ, η *in* $[0,1]$ *satisfying*

$$\rho\equiv\alpha-\gamma>0,\qquad 0\le\zeta\le\inf(\rho,\eta) \tag{2.3}$$

We define the space of Volterra paths of index $(\alpha,\gamma,\eta,\zeta)$, *denoted by* $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2;E)$, *as the set of functions* $z:\Delta_2\to E$, *given by* $(t,\tau)\mapsto z^\tau_t$, *with the condition* $z^\tau_0=z_0\in E$ *for all* $\tau\in(0,T]$, *and satisfying*

$$\|z\|_{(\alpha,\gamma,\eta,\zeta)}=\|z\|_{(\alpha,\gamma),1}+\|z\|_{(\alpha,\gamma,\eta,\zeta),1,2}<\infty. \tag{2.4}$$

Recalling Notation 2.3, the 1-norms and (1,2)-norms in (2.4) are respectively defined as follows:

$$\|z\|_{(\alpha,\gamma),1}=\sup_{(s,t,\tau)\in\Delta_3}\frac{|z^\tau_{ts}|}{\psi^1_{\alpha,\gamma}(\tau,t,s)}, \tag{2.5}$$

$$\|z\|_{(\alpha,\gamma,\eta,\zeta),1,2}=\sup_{(s,t,\tau',\tau)\in\Delta_4}\frac{|z^{\tau\tau'}_{ts}|}{\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(\tau,\tau',t,s)}, \tag{2.6}$$

with the convention $z^\tau_{ts}=z^\tau_t-z^\tau_s$ *and* $z^{\tau\tau'}_{ts}=z^\tau_{ts}-z^{\tau'}_{ts}$. *Notice that under the mapping*

$$z\mapsto|z_0|+\|z\|_{(\alpha,\gamma,\eta,\zeta)},$$

the space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2;E)$ *is a Banach space.*

Whenever the domain and codomain is otherwise clear from the context, we will simply write $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}:=\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2;E)$. Throughout the article, we will typically let the Banach space E be given by $\mathbb{R}^d$ or $\mathcal{L}(\mathbb{R}^d)$.

Remark 2.5. As will be proved in Theorem 2.14 below, the typical example of path in $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ is given by z^τ_t defined as in (1.4), with suitable assumption on k and x. Note also that $\mathcal{C}^\alpha([0,1];\mathbb{R}^d)\subset\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2([0,1]);\mathbb{R}^d)$ for any $\gamma\in[0,1)$. Indeed, for a path $x\in\mathcal{C}^\alpha$, define $z^\tau_t=x_t$. Using that $|t-s|^\alpha\le|\tau-s|^\alpha$, it is readily checked that $|z^\tau_{ts}|\lesssim|\tau-t|^{-\gamma}|t-s|^\alpha\wedge|\tau-s|^\alpha$. Furthermore, $z^{\tau\tau'}_{ts}=0$, and thus $\|z\|_{(\alpha,\gamma,\eta,\zeta)}<\infty$ for any $\gamma\in(0,1)$.

Remark 2.6. We will also consider functions $u:\Delta_3\to\mathbb{R}^d$, which, with a slight abuse of notation, will be denoted by the mapping $(s,t,\tau)\mapsto u^\tau_{ts}$. We then define the space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^d)$ analogously as in Definition 2.4, but where the increments of the path $(t,\tau)\mapsto z^\tau_t$ in the lower variable, appearing in (2.5) and (2.6), is simply replaced by the evaluation u^τ_{ts} and $u^\tau_{ts}-u^{\tau'}_{ts}$ respectively.

Remark 2.7. Similarly as for the classical Hölder spaces, we have the following elementary embedding: for $\beta < \alpha \in (0,1)$, with $\beta - \gamma > 0$, we have $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)} \hookrightarrow \mathcal{V}^{(\beta,\gamma,\eta,\zeta)}$. Indeed, suppose $y \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$, it is readily checked that

$$|y_{ts}^{\tau}| \lesssim \left[|\tau - t|^{-\gamma}|t-s|^{\alpha}\right] \wedge |\tau - s|^{\alpha-\gamma} \leq T^{\alpha-\beta}\left(\left[|\tau - t|^{-\gamma}|t-s|^{\beta}\right] \wedge |\tau - s|^{\beta-\gamma}\right),$$

and thus $\|y\|_{(\beta,\gamma),1} \leq T^{\alpha-\beta}\|y\|_{(\alpha,\gamma),1}$. Similarly, one can also show that $\|y\|_{(\beta,\gamma,\eta,\zeta),1,2} \leq T^{\alpha-\beta}\|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}$, and thus $\|y\|_{(\beta,\gamma,\eta,\zeta)} \leq T^{\alpha-\beta}\|y\|_{(\alpha,\gamma,\eta,\zeta)}$.

Remark 2.8. While the parameters α and γ in Definition 2.4 are the two parameters that one can say truly captures the regularity of a Volterra path, the extra parameters η and ζ are needed in order to obtain full information about the regularity of the rectangular increment $z_{ts}^{\tau\tau'}$. In Proposition 2.10 we will see that when $\eta = \alpha - \gamma$ and $\zeta \leq \eta$, the restriction of the Volterra path given by $t \mapsto z_t^t$ is contained in $\mathcal{C}^{\alpha-\gamma}$. In later analysis of rough integration in Section 3 and subsequent sections, we will require that η is a parameter closer to 1 in order to define the convolution type integrals. Thus the parameters η and ζ works as "fine tuning" parameters, to obtain the as much information as possible about the regularity of the process.

The following lemma gives useful embedding results for $\mathcal{V}^{(\alpha,\gamma)}$ related to variations in the singularity parameter γ. It justifies the use of the extra parameters η, ζ in $\psi^{1,2}$.

Lemma 2.9. *Let $\alpha, \gamma \in (0,1)$, $\eta, \zeta \in [0,1]$ satisfy (2.3). Then for the spaces $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ given in Definition 2.4, the following inclusion holds true:*

$$\mathcal{V}^{(3\rho+\gamma,\gamma,\eta,\zeta)} \subset \mathcal{V}^{(3\rho+2\gamma,2\gamma,\eta,\zeta)} \subset \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}. \tag{2.7}$$

Proof. We will prove the second relation: $\mathcal{V}^{(3\rho+2\gamma,2\gamma,\eta,\zeta)} \subset \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$, the first relation being proved in a similar way. Moreover, in order to prove that $\mathcal{V}^{(3\rho+2\gamma,2\gamma,\eta,\zeta)} \subset \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$, we will show that $\|z\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)} \leq \|z\|_{(3\rho+2\gamma,2\gamma,\eta,\zeta)}$, for the $(\alpha,\gamma,\eta,\zeta)-$norms introduced in Definition 2.4. Also recall that the $(\alpha,\gamma,\eta,\zeta)-$norms are defined by (2.5) and (2.6). For sake of conciseness we will just prove that

$$\|z\|_{(3\rho+3\gamma,3\gamma),1} \leq \|z\|_{(3\rho+2\gamma,2\gamma),1}, \tag{2.8}$$

and leave the similar bound for the $(1,2)-$norm to the reader.

In order to prove (2.8), we refer again to (2.5). From this definition, it is readily checked that (2.8) can be reduced to prove the following relation:

$$\left[|\tau - t|^{-3\gamma}|t-s|^{3\rho+3\gamma}\right] \wedge |\tau - s|^{3\rho} \lesssim \left[|\tau - t|^{-2\gamma}|t-s|^{3\rho+2\gamma}\right] \wedge |\tau - s|^{3\rho}. \tag{2.9}$$

The proof of (2.9) will be split in 2 cases, according to the respective values of $|\tau - t|$ and $|t-s|$. In the sequel C_1 designates a strictly positive constant.

Case 1: $|\tau - t| \leq C_1|t-s|$. Let us write

$$|\tau - s|^{3\rho} = |\tau - s|^{3\rho+2\gamma}|\tau - s|^{-2\gamma}.$$

Then if $|\tau - t| \leq C_1|t-s|$, one has $|\tau - s|^{3\rho+2\gamma} = |\tau - t + t - s|^{3\rho+2\gamma} \lesssim |t-s|^{3\rho+2\gamma}$. Hence we get

$$|\tau - s|^{3\rho} \lesssim |t-s|^{3\rho+2\gamma}|\tau - s|^{-2\gamma} \lesssim |t-s|^{3\rho+2\gamma}|\tau - t|^{-2\gamma}. \tag{2.10}$$

Relation (2.9) is then immediately seem from (2.10).

Case 2: $|\tau - t| > C_1|t-s|$. In this case write

$$|t-s|^{3\rho+2\gamma}|\tau - t|^{-2\gamma} = |t-s|^{3\rho+3\gamma}|\tau - t|^{-3\gamma}\left(\frac{|\tau - t|}{|t-s|}\right)^{\gamma}.$$

Then resort to the fact that $|\tau-t|\geq C_1|t-s|$ in order to get $|\tau-t|^{\gamma}|t-s|^{-\gamma}\geq C_1^{\gamma}$. This yields

$$|t-s|^{3\rho+2\gamma}|t-s|^{-2\gamma}\gtrsim|t-s|^{3\rho+3\gamma}|\tau-t|^{-3\gamma},$$

from which (2.9) is readily checked.

Combining Case 1 and Case 2, we have thus finished the proof of (2.9). As mentioned above, this implies that (2.8) is true and achieves our claim (2.7). ☐

We conclude this section with a fundamental property of the Volterra paths, relating them directly to classical Hölder paths when restricted to the diagonal of Δ_2.

Proposition 2.10. *Let $\alpha,\gamma\in(0,1)$ such that $\rho=\alpha-\gamma>0$. Suppose that for $\eta=\rho-\epsilon$ for some arbitrarily small $\epsilon>0$, and $\zeta=\eta$, we have $z\in\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$, where $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ is introduced in Definition 2.4. Then the restriction of z to $[0,T]$ given by $t\mapsto z_t^t$ is contained in $\mathcal{C}^{\rho-\epsilon}$.*

Proof. By adding and subtracting z_s^t and the triangle inequality, we see that

$$|z_t^t-z_s^s|\leq|z_{ts}^t|+|z_s^{ts}|. \tag{2.11}$$

Using the norms given in Definition 2.4, it follows that the term $|z_{ts}^t|$ above satisfies

$$|z_{ts}^t|\leq\|z\|_{(\alpha,\gamma),1}|t-s|^{\rho}. \tag{2.12}$$

Also recall from Definition 2.4 that $z_0^{\tau}=0$. Therefore adding and subtracting $0=z_0^{ts}$, we can bound the second term in the right hand side of (2.11) as follows:

$$|z_s^{ts}|=|z_{s0}^{ts}|\leq\|z\|_{(\alpha,\gamma,\eta,\zeta),1,2}|t-s|^{\eta}|s-s|^{-\eta+\zeta}|s|^{\rho-\zeta}$$

where $|s-s|^{-\eta+\zeta}=1$ since $\eta=\zeta$ and we understand the inequality by continuous extension of z, i.e. $z_s^{ts}=\lim_{r\to s}z_r^{ts}$. Thus we get

$$|z_s^{ts}|\leq\|z\|_{(\alpha,\gamma,\eta,\zeta)}|t-s|^{\rho-\epsilon}|s|^{\epsilon}. \tag{2.13}$$

Combining the estimates in (2.12) and (2.13) concludes the proof. ☐

2.2. **Volterra Sewing lemma.** We begin with a recollection of the space of abstract Volterra integrands, to which the Volterra sewing Lemma 2.13 will apply. The typical path in this space exhibits different types of regularities/singularities in its arguments, similarly to Definition 2.4. As a necessary ingredient in the subsequent definition we introduce a particular notation, which will frequently be used throughout the article.

Notation 2.11. *Recall that the simplex Δ_n is defined by (1.3). For a path $g:\Delta_2\to\mathbb{R}^d$ and $(s,u,t)\in\Delta_3$, we set*

$$\delta_u g_{ts}=g_{ts}-g_{tu}-g_{us} \tag{2.14}$$

We will consider δ as an operator from $\mathcal{C}(\Delta_2)$ to $\mathcal{C}(\Delta_3)$, where $\mathcal{C}(\Delta_n)$ denotes the spaces of continuous functions on Δ_n. Notice that even for paths g possessing upper variables, the operator δ only acts on the lower variables.

Definition 2.12. *We consider 6 parameters α, γ, κ in $(0,1)$, η, ζ in $[0,1]$ and $\beta>1$ satisfying*

$$\beta-\kappa\geq\alpha-\gamma>0,\quad \text{and}\quad 0\leq\zeta\leq\inf(\rho,\eta), \tag{2.15}$$

where $\rho=\alpha-\gamma$. Denote by $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}\left(\Delta_3;\mathbb{R}^d\right)$, the space of all functions $\Xi:\Delta_3\to\mathbb{R}^d$ such that

$$\|\Xi\|_{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}=\|\Xi\|_{(\alpha,\gamma,\eta,\zeta)}+\|\delta\Xi\|_{(\beta,\kappa,\eta,\zeta)}<\infty, \tag{2.16}$$

where δ is introduced in (2.14) and the norm $\|\Xi\|_{(\alpha,\gamma,\eta,\zeta)}$ is given by (2.4) (see also Remark 2.6). Similar to Definition 2.4, the quantity $\|\delta\Xi\|_{(\beta,\kappa,\eta,\zeta)}$ is defined by

$$\|\delta\Xi\|_{(\beta,\kappa,\eta,\zeta)} = \|\delta\Xi\|_{(\beta,\kappa),1} + \|\delta\Xi\|_{(\beta,\kappa,\eta,\zeta),1,2}, \tag{2.17}$$

and the 1-norms and (1,2)-norms in (2.17) are respectively defined as follows:

$$\|\delta\Xi\|_{(\beta,\kappa),1} := \sup_{(s,m,t,\tau)\in\Delta_4} \frac{|\delta_m \Xi^\tau_{ts}|}{\psi^1_{\beta,\kappa}(\tau,t,s)}, \tag{2.18}$$

$$\|\delta\Xi\|_{(\beta,\kappa,\eta,\zeta),1,2} := \sup_{(s,m,t,\tau',\tau)\in\Delta_5} \frac{|\delta_m \Xi^{\tau\tau'}_{ts}|}{\psi^{1,2}_{\beta,\kappa,\eta,\zeta}(\tau,\tau',t,s)}, \tag{2.19}$$

where ψ^1 and $\psi^{1,2}$ are given in Notation 2.3. In the sequel the space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}$ will be our space of abstract Volterra integrands.

With these two Volterra spaces in hand, we are ready to recall the Volterra sewing Lemma which can be found, together with a full proof, in [12, Lemma 21].

Lemma 2.13. *Consider the same exponents $\alpha,\gamma,\beta,\kappa,\eta,\zeta$ as in Definition 2.12, satisfying condition (2.15). Let $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}$ and $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ be the spaces given in Definition 2.12 and Definition 2.4 respectively. Then there exists a linear continuous map $\mathcal{I} : \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}\left(\Delta_3;\mathbb{R}^d\right) \to \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}\left(\Delta_2;\mathbb{R}^d\right)$ such that the following holds true.*

(i) The quantity $\mathcal{I}(\Xi^\tau)_{ts} := \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} \Xi^\tau_{vu}$ exists for all $(s,t,\tau)\in\Delta_3$, where $\mathcal{P}$ is a generic partition of $[s,t]$ and $|\mathcal{P}|$ denotes the mesh size of the partition. Furthermore, we define $\mathcal{I}(\Xi^\tau)_t := \mathcal{I}(\Xi^\tau)_{t0}$, and we have that $\mathcal{I}(\Xi^\tau)_{ts} = \mathcal{I}(\Xi^\tau)_t - \mathcal{I}(\Xi^\tau)_s$.
(ii) Recalling the Notation 2.3 of ψ^1 and $\psi^{1,2}$, for all $(s,t,\tau)\in\Delta_3$ we have

$$|\mathcal{I}(\Xi^\tau)_{ts} - \Xi^\tau_{ts}| \lesssim \|\delta\Xi\|_{(\beta,\kappa),1}\, \psi^1_{\beta,\kappa}(\tau,t,s), \tag{2.20}$$

while for $(s,t,\tau',\tau)\in\Delta_4$ we get

$$\left|\mathcal{I}(\Xi^{\tau\tau'})_{ts} - \Xi^{\tau\tau'}_{ts}\right| \lesssim \|\delta\Xi\|_{(\beta,\kappa,\eta,\zeta),1,2}\, \psi^{1,2}_{\beta,\kappa,\eta,\zeta}(\tau,\tau',t,s). \tag{2.21}$$

Lemma 2.13 is applied in [12] in order to get the construction of the path $(t,\tau)\mapsto z^\tau_t$ introduced in (1.4). We recall this result here, since z is at the heart of our future considerations.

Theorem 2.14. *Let $x\in\mathcal{C}^\alpha$ and k be a Volterra kernel of order $-\gamma$ satisfying Hypothesis 2.1, such that $\rho=\alpha-\gamma>0$. We define an element $\Xi^\tau_{ts} = k(\tau,s)x_{ts}$. Then the following holds true:*

(i) For any exponents β,κ,η,ζ fulfilling (2.15) we have $\Xi\in\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}$, where $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}$ is given in Definition 2.12. It follows that the element $\mathcal{I}(\Xi^\tau)$ obtained in Lemma 2.13 is well defined as an element of $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ and we set $z^\tau_{ts} \equiv \mathcal{I}(\Xi^\tau)_{ts} = \int_s^t k(\tau,r)dx_r$.
(ii) According to the definition (2.1) of ψ^1, for $(s,t,\tau)\in\Delta_3$, z satisfies the bound

$$|z^\tau_{ts} - k(\tau,s)x_{ts}| \lesssim \psi^1_{\alpha,\gamma}(\tau,t,s),$$

and in particular it holds that $\|z\|_{(\alpha,\gamma),1}<\infty$.
(iii) Recall the definition (2.2) of $\psi^{1,2}$, for any $\eta\in[0,1]$ and any $(s,t,q,p)\in\Delta_4$ we have

$$|z^{pq}_{ts}| \lesssim \psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(p,q,t,s),$$

where $z^{pq}_{ts} = z^p_t - z^q_t - z^p_s + z^q_s$. In particular it holds that $\|z\|_{(\alpha,\gamma,\eta,\zeta),1,2}<\infty$.

Remark 2.15. Thanks to Theorem 2.14, we know that a typical example of a Volterra path in $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ is given by the integral $\int_s^t k(\tau,r)dx_r$, as mentioned in Remark 2.5.

2.3. **Convolution product in the rough case $\alpha-\gamma>\frac{1}{3}$.** A second crucial ingredient in the Volterra formalism put forward in [12] is the notion of convolution product. In this section we show how this mechanism is introduced for first and second order convolutions, where we recall that second order convolutions were enough to handle the case $\rho=\alpha-\gamma>\frac{1}{3}$ in [12].
Let us first introduce a piece of notation which will prevail throughout the paper.

Notation 2.16. *In the sequel we will often consider products of the form $y_s z_{ts}^\tau$, where y and z are increments lying respectively in $\mathcal{C}([0,T];\mathcal{L}(\mathbb{R}^d;\mathbb{R}^m))$ and $\mathcal{C}(\Delta_2;\mathbb{R}^d)$. For algebraic reasons due to our rough Volterra formalism, we will write this product as*

$$[(z_{ts}^\tau)^\intercal\ y_s^\intercal]^\intercal \tag{2.22}$$

For obvious notational reason, we will simply abbreviate (2.22) into

$$z_{ts}^\tau\, y_s$$

In the same way, products of 3 (or more) elements of the form $f'(y_s)y_s z_{ts}^\tau$ will be denoted as $z_{ts}^\tau y_s f'(y_s)$ without further notice.

We now recall how the convolution with respect to z^τ is obtained, borrowing the following proposition from [12, Theorem 25].

Proposition 2.17. *We consider two Volterra paths $z\in\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^d)$ and $y\in\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\mathcal{L}(\mathbb{R}^d))$ as given in Definition 2.4. On top of condition (2.3), we assume that the exponents α,η are such that $\eta>1-\alpha$. That is, our parameters α,γ,ζ,η and $\rho\equiv\alpha-\gamma$ satisfy*

$$\rho>0, \qquad 0\le\zeta\le\inf(\rho,\eta), \quad \textit{and} \quad \eta>1-\alpha. \tag{2.23}$$

Then the convolution product of the two Volterra paths y and z is a bilinear operation on $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^d)$ given by

$$z_{tu}^\tau * y_{us}^{\cdot} = \int_{t>r>u} dz_r^\tau y_{us}^r := \lim_{|\mathcal{P}|\to 0}\sum_{[u',v']\in\mathcal{P}} z_{v'u'}^\tau y_{us}^{u'}. \tag{2.24}$$

The integral in (2.24) is understood as a Volterra-Young integral for all $(s,u,t,\tau)\in\Delta_4$. Moreover, the following two inequalities holds for any $(s,u,t,\tau,\tau')\in\Delta_5$:

$$|z_{tu}^\tau * y_{us}^{\cdot}| \lesssim \|z\|_{(\alpha,\gamma),1}\|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}\,\psi^1_{(2\rho+\gamma),\gamma}(\tau,t,s), \tag{2.25}$$

$$\left|z_{tu}^{\tau'\tau} * y_{us}^{\cdot}\right| \lesssim \|z\|_{(\alpha,\gamma,\eta,\zeta),1,2}\|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}\,\psi^{1,2}_{(2\rho+\gamma),\gamma,\eta,\zeta}(\tau,\tau',t,s), \tag{2.26}$$

where ψ^1 and $\psi^{1,2}$ are given in Notation 2.3.

Remark 2.18. Up to now, we have been able to assume (e.g in condition (2.23)) the existence of a general $\rho>0$ only. In order to be restricted to a Volterra rough path of order 3 in our main considerations, from now on we will suppose that $\rho>\frac{1}{4}$. More precisely, the parameters α,γ,ζ,η satisfy

$$\frac{1}{3}>\rho\equiv\alpha-\gamma>\frac{1}{4}, \qquad 0\le\zeta\le\inf(\rho,\eta), \quad \text{and} \quad \eta>1-\alpha. \tag{2.27}$$

Remark 2.19. In the case when we have no upper index dependence, i.e. $z_t^\tau=z_t$ and $y_s^\tau=y_t$, it is readily seen that the convolution product $z_{ts}^\tau * y_s^{\cdot}$ is simply given by the pairing $z_{ts}y_s$, which is the typical integrand we see for example in Young integration.

In addition to Proposition 2.17, the rough Volterra formalism relies on a stack of iterated integrals verifying convolutional type algebraic identities. Thanks to Proposition 2.17 we can now state the main assumption about this stack of integrals, which should be seen as the equivalent of Chen's relation in our Volterra context.

Hypothesis 2.20. *Consider $\alpha, \gamma \in (0,1)$ with $\rho = \alpha - \gamma$. For an arbitrary finite integer $N \geq 1$, let $\{\zeta_k, \eta_k; 1 \leq k \leq N\}$ be a family of exponents satisfying (2.27) and such that Theorem 2.14 applies. We denote by z_{ts}^{τ} the Volterra increment constructed through Theorem 2.14. Then for n such that $(n+1)\rho + \gamma > 1$, we assume that there is a family $\{\mathbf{z}^{j,\tau}; j \leq n\}$ such that $\mathbf{z}_{ts}^{j,\tau} \in (\mathbb{R}^d)^{\otimes j}$, $\mathbf{z}^1 = z$ and verifying*

$$\delta_u \mathbf{z}_{ts}^{j,\tau} = \sum_{i=1}^{j-1} \mathbf{z}_{tu}^{j-i,\tau} * \mathbf{z}_{us}^{i,\cdot} = \sum_{i=1}^{j-1} \int_s^t d\mathbf{z}_{tr}^{j-i,\tau} \otimes \mathbf{z}_{us}^{i,r}, \tag{2.28}$$

where the right hand side of (2.28) is defined in Proposition 2.17. In addition, we suppose that the increment $\mathbf{z}^j$ sits in the following space:

$$\mathbf{z}^j \in \bigcap_{k=1}^{N} \mathcal{V}^{(j\rho+\gamma,\gamma,\eta_k,\zeta_k)}. \tag{2.29}$$

Remark 2.21. Note that (2.28) gives us Chen's relation for Volterra iterated integrals. In particular for $j = 2$, relation (2.28) tells us that

$$\mathbf{z}_{ts}^{j,\tau} - \mathbf{z}_{tu}^{j,\tau} - \mathbf{z}_{us}^{j,\tau} = \mathbf{z}_{tu}^{1,\tau} * \mathbf{z}_{us}^{1,\cdot}. \tag{2.30}$$

As discussed in Section 1.2, specifically, (1.15), for a smooth driving noise x and $z_t^{\tau} = \int_0^t k(\tau, r) dx_r$, it is readily checked that the Volterra iterated integral $\mathbf{z}^2$ should satisfy condition (2.30).

Notation 2.22. *In the sequel we will set*

$$\mathcal{A}_N = \{(\eta_k, \zeta_k); 1 \leq k \leq N\}, \tag{2.31}$$

so that (2.29) can be recast as

$$\mathbf{z}^j \in \bigcap_{(\eta,\zeta)\in\mathcal{A}_N} \mathcal{V}^{(j\rho+\gamma,\gamma,\eta,\zeta)}. \tag{2.32}$$

Remark 2.23. The fact that we require $\mathbf{z}$ to belong to an intersection of $\mathcal{V}$-spaces in (2.29) stems from several applications of the sewing lemma in our computations. Let us mention for instance:

(i) In Theorem 3.8 we let $(\alpha, \gamma, \eta_1, \zeta_1) = (\alpha, \gamma, \eta, \zeta)$ with $\alpha, \gamma, \eta, \zeta$ compatible with (2.27) and such that $3\rho + \gamma + \eta > 1$. This allows to apply the sewing lemma to the terms (3.31)-(3.34).

(ii) As mentioned above and in [12], we have to consider tuples of the form $(\alpha, \gamma, \eta_2, \zeta_2)$ with $\eta_2 = \zeta_2 = \rho$ and verifying (2.27). This ensures proper Hölder regularity for $t \mapsto z_t^t$. See in particular Proposition 2.10.

The last notation we need to recall from [12] is the concept of second order convolution product. As discussed in Section 1.2, and directly observed in (1.13), when doing Taylor type expansions of the Volterra integral we observe that the "derivative term" has two upper variables to integrate. To accommodate for this feature we will introduce a suitable class of functions with two upper parameters.

Notation 2.24. *We will denote by $u^{1,2}$ a function $u : \Delta_3 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 2}, \mathbb{R}^d)$ with two upper indices, namely,*

$$\Delta_3 \ni (s, \tau_1, \tau_2) \mapsto u_s^{\tau_2,\tau_1} \in \mathbb{R}^d.$$

The notation $u^{1,2}$ highlights the order of integration in future computations.

We now specify the kind of topology we will consider for functions of the form $u^{1,2}$.

Definition 2.25. *Let $\mathcal{W}_2^{(\alpha,\gamma,\eta,\zeta)}$ denote the space of functions $u : \Delta_3 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 2}, \mathbb{R}^d)$ with a fixed initial condition $u_0^{p,q} = u_0$ for all $q \le p \in [0,T]$, endowed with the norm*

$$\left\|u^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta)} := \left\|u^{1,2}\right\|_{(\alpha,\gamma),1} + \left\|u^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2}. \tag{2.33}$$

The right hand side of (2.33) is defined as follows, recalling the convention $\rho = \alpha - \gamma$ and the definition (2.1) of ψ^1:

$$\left\|u^{1,2}\right\|_{(\alpha,\gamma),1} := \sup_{(s,t,\tau)\in\Delta_3} \frac{|u_{ts}^{\tau,\tau}|}{\psi^1_{\alpha,\gamma}(\tau,t,s)}, \tag{2.34}$$

and

$$\left\|u^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2} := \left\|u^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,>} + \left\|u^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,<}, \tag{2.35}$$

where the norms $\|u^{1,2}\|_{(\alpha,\gamma,\eta,\zeta),1,2,>}$ and $\|u^{1,2}\|_{(\alpha,\gamma,\eta,\zeta),1,2,<}$ are respectively defined by

$$\left\|u^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,>} = \sup_{(s,t,r_1,r_2,r)\in\Delta_5} \frac{|u_{ts}^{r,r_2} - u_{ts}^{r,r_1}|}{h_{\eta,\zeta}(s,t,r_1,r_2,r)}, \tag{2.36}$$

$$\left\|u^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,<} = \sup_{(s,t,r,r_1,r_2)\in\Delta_5} \frac{|u_{ts}^{r_2,r} - u_{ts}^{r_1,r}|}{h_{\eta,\zeta}(s,t,r_1,r_2,r)}. \tag{2.37}$$

Here we define $h_{\eta,\zeta}$ as follows :

$$h_{\eta,\zeta}(s,t,r_1,r_2,r) = \begin{cases} |r_2 - r_1|^{\eta} |r - t|^{-\eta+\zeta} \left(\left[|r-t|^{-\gamma-\zeta} |t-s|^{\alpha} \right] \wedge |r-s|^{\alpha-\gamma-\zeta} \right) & \text{if } r_2 \ge r_1 \ge r \\ |r_2 - r_1|^{\eta} |r_1 - t|^{-\eta+\zeta} \left(\left[|r_1-t|^{-\gamma-\zeta} |t-s|^{\alpha} \right] \wedge |r_1-s|^{\alpha-\gamma-\zeta} \right) & \text{if } r > r_2 \ge r_1 \end{cases} \tag{2.38}$$

Remark 2.26. In the sequel we will need to estimate differences of functions $u^{\cdot,\cdot} : \Delta_3 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 2}, \mathbb{R}^d)$ of the form $|u_t^{\tau,q} - u_t^{\tau,p}|$. Those differences can be handled thanks to Definition 2.25 as follows:

$$\begin{aligned} |u_t^{\tau,q} - u_t^{\tau,p}| &\le |u_0^{\tau,q} - u_0^{\tau,p}| + |u_{t0}^{\tau,q} - u_{t0}^{\tau,p}| \\ &\le \|u\|_{(\alpha,\gamma,\eta,\zeta),1,2} |q-p|^{\eta} |p-t|^{-\eta+\zeta} \left(\left[|p-t|^{-\gamma-\zeta} |t|^{\alpha} \right] \wedge |p|^{\rho-\zeta} \right). \end{aligned} \tag{2.39}$$

Since $\zeta \in [0,\rho)$ and $\eta \in [\zeta, 1]$, then we can set $\eta = \zeta$, that is

$$|u_t^{\tau,q} - u_t^{\tau,p}| \lesssim \|u\|_{(\alpha,\gamma,\eta,\zeta),1,2} |q-p|^{\zeta} \lesssim \|u\|_{(\alpha,\gamma,\eta,\zeta),1,2}.$$

we also have, for any $\tau \in [0,T]$,

$$|u_t^{\tau,\tau} - u_0^{\tau,\tau}| \le \|u\|_{(\alpha,\gamma),1} \left[|\tau - t|^{-\gamma} |t|^{\alpha} \wedge |\tau|^{\rho} \right] \lesssim \|u\|_{(\alpha,\gamma),1}. \tag{2.40}$$

With the above definition at hand, we are now ready to recall the construction of second order convolution products in the rough case $\alpha - \gamma > \frac{1}{3}$.

Theorem 2.27. *Let $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ be as given in Definition 2.4 with $\alpha, \gamma \in (0,1)$ and $\eta, \zeta \in [0,1]$ satisfying (2.27). We assume that $\mathbf{z}$ fulfills Hypothesis 2.20 with $n = 2$. Consider a function $y : \Delta_3 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 2}, \mathbb{R}^d)$ with $\|y^{1,2}\|_{(\alpha,\gamma,\eta,\zeta),1,2} < \infty$ and $y_0^{1,2} = y_0$, for a fixed initial condition $y_0 \in \mathcal{L}((\mathbb{R}^d)^{\otimes 2}, \mathbb{R}^d)$. For all fixed $(s,t,\tau) \in \Delta_3$ we have that*

$$\mathbf{z}_{ts}^{2,\tau} * y_s^{1,2} := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbf{z}_{vu}^{2,\tau} y_s^{u,u} + (\delta_u \mathbf{z}_{vs}^{2,\tau}) * y_s^{1,2} \tag{2.41}$$

is a well defined Volterra-Young integral. It follows that $*$ is a well defined bi-linear operation between the three parameters Volterra function $\mathbf{z}^2$ and a 3-parameter path y. Moreover, the following inequality holds

$$\begin{aligned} \left|\mathbf{z}_{ts}^{2,\tau} * y_s^{1,2} - \mathbf{z}_{ts}^{2,\tau} y_s^{s,s}\right| \lesssim & \|y^{1,2}\|_{(\alpha,\gamma,\eta,\zeta),1,2} \\ & \times \left(\|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma),1} + \|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2} \|\mathbf{z}^1\|_{(\alpha,\gamma),1}\right) \psi^1_{(2\rho+\gamma),\gamma}(\tau,t,s), \end{aligned} \tag{2.42}$$

where ψ^1 is given in (2.1)

Remark 2.28. By Hypothesis 2.20, the term $(\delta_u \mathbf{z}_{vs}^{2,\tau}) * y_s^{1,2}$ in the right hand side of (2.41) can be rewritten as

$$\mathbf{z}_{vu}^{1,\tau} * \mathbf{z}_{us}^{1,\cdot} * y_s^{1,2},$$

where the convolution with $\mathbf{z}^{1,\tau}$ is defined through (2.24) and the inside integral concerns the second variable in $y^{1,2}$. As an example, if k, x are smooth functions and $\mathbf{z}_{vs}^{1,\tau} = \int_s^v k(\tau,r)dx_r$, then this convolution is understood in the following way

$$\mathbf{z}_{vu}^{1,\tau} * \mathbf{z}_{us}^{1,\cdot} * y_s^{1,2} = \int_u^v k(\tau,r_1)dx_{r_1} \otimes \int_s^u k(r_1,r_2)dx_{r_2} y_s^{r_1,r_2}.$$

Remark 2.29. Recalling that $\rho = \alpha - \gamma$, notice that Proposition 2.17 and Theorem 2.27 tell us how to define convolution products of between n Volterra paths, i.e. $f^1 * \ldots * f^n$, under the regularity condition $\rho > \frac{1}{3}$. We will follow a similar strategy to define third order convolution products and construct our solution to equation (1.2) with $\rho > \frac{1}{4}$ in the subsequent section.

3. Volterra rough paths for $\alpha - \gamma > \frac{1}{4}$

This section is devoted to the generalization of the concepts introduced in Section 2 to accommodate the case of Volterra rough paths with regularity $\rho = \alpha - \gamma > \frac{1}{4}$. As described in Section 1.2, when lowering the regularity assumption on $\alpha - \gamma$, we also need to include two more terms in the Taylor like expansion in (1.13). This complicates our considerations significantly, not only introducing two more types of iterated integrals, but also the fact that we will require three upper variables in the "second derivative" controlled process. So far we have only proven that the convolution product for functions with two upper variables is well defined. Now that we will work with integrands with three upper parameters, we must therefore extend this concept as well. To this end, we will state a version of our Volterra sewing Lemma 2.13 extended to the case of two types of Volterra singularities.

3.1. Volterra sewing lemma with two singularities.

With the aim of extending the Volterra sewing Lemma 2.13 with one singularity to an increment exhibiting two singularities, we first introduce a new space of abstract integrands. Here we allow for two distinct singularities for the integrand. As seen in the motivation from Section 1.2, when working with the Taylor type expansions for Volterra equations, the "derivative" process will depend on two "upper parameters".

Therefore, the following extension comes from the application towards integration of terms of the form $w^\tau_{ts} := z^\tau_{ts} * y^{\tau,\cdot}_s$ over a domain $[s,\tau]$. Here we have one singularity $t \to \tau$ coming from z, but also one possible singularity when $\tau \to s$. To account for these two types of singularities, we introduce a broader class of abstract integrands to be used in the Volterra sewing lemma.

Definition 3.1. *We consider 7 parameters $\alpha,\gamma,\kappa,\theta \in (0,1)$, $\eta,\zeta \in [0,1]$, and $\beta > 1$ satisfying*

$$0 < \kappa + \theta < 1, \quad \beta - \kappa - \theta \geq \alpha - \gamma > 0, \quad \text{and} \quad 0 \leq \zeta \leq \inf(\alpha - \gamma, \eta). \tag{3.1}$$

Denote by $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\theta,\eta,\zeta)}(\Delta_4;\mathbb{R}^d)$, the space of all functions of the form $\Delta_4 \ni (v,s,t,\tau) \mapsto (\Xi^\tau_v)_{ts} \in \mathbb{R}^d$ such that the following norm is finite:

$$\|\Xi\|_{\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\theta,\eta,\zeta)}} = \|\Xi\|_{(\alpha,\gamma,\eta,\zeta)} + \|\delta\Xi\|_{(\beta,\kappa,\theta,\eta,\zeta)}. \tag{3.2}$$

In equation (3.2), the operator δ is introduced in (2.14), the quantity $\|\Xi\|_{(\alpha,\gamma,\eta,\zeta)}$ is given by (2.4) and the term $\|\delta\Xi\|_{(\beta,\kappa,\theta,\eta,\zeta)}$ takes the double singularity into account. Namely we have

$$\|\delta\Xi\|_{(\beta,\kappa,\theta,\eta,\zeta)} = \|\delta\Xi\|_{(\beta,\kappa,\theta),1} + \|\delta\Xi\|_{(\beta,\kappa,\theta,\eta,\zeta),1,2},$$

where

$$\|\delta\Xi\|_{(\beta,\kappa,\theta),1} := \sup_{(v,s,m,t,\tau)\in\Delta_5} \frac{|\delta_m(\Xi^\tau_v)_{ts}|}{\phi^1_{\beta,\kappa,\theta}(\tau,t,s,v)}, \tag{3.3}$$

and the term $\|\delta\Xi\|_{(\beta,\kappa,\theta,\eta,\zeta),1,2}$ is defined by

$$\|\delta\Xi\|_{(\beta,\kappa,\theta,\eta,\zeta),1,2} := \sup_{(v,s,m,t,\tau',\tau)\in\Delta_6} \frac{\left|\delta_m(\Xi^{\tau\tau'}_v)_{ts}\right|}{\phi^{1,2}_{\beta,\kappa,\theta,\eta,\zeta}(\tau,\tau',t,s,v)}, \tag{3.4}$$

where the function $\psi^1_{\beta,\kappa,\theta}(\tau,t,s,v)$ and $\psi^{1,2}_{\beta,\kappa,\theta,\eta,\zeta}(\tau,\tau',t,s,v)$ are respectively given by

$$\phi^1_{\beta,\kappa,\theta}(\tau,t,s,v) = \left[|\tau - t|^{-\kappa}|t-s|^\beta|s-v|^{-\theta}\right] \wedge |\tau - v|^{\beta-\kappa-\theta} \tag{3.5}$$

$$\phi^{1,2}_{\beta,\kappa,\theta,\eta,\zeta}(\tau,\tau',t,s,v) = |\tau-\tau'|^\eta|\tau'-t|^{-\eta+\zeta}\left(\left[|\tau'-t|^{-\kappa-\zeta}|t-s|^\beta|s-v|^{-\theta}\right]\wedge|\tau'-v|^{\beta-\kappa-\theta-\zeta}\right). \tag{3.6}$$

Notice that we will use $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\theta,\eta,\zeta)}$ as a space of abstract Volterra integrands with a double singularity.

With this new space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\theta,\eta,\zeta)}$ at hand, we are ready to state the Volterra sewing Lemma with two singularities alluded to above.

Lemma 3.2. *Consider the same exponents $\alpha,\gamma,\beta,\kappa,\eta,\zeta$ as in Definition 3.1 satisfying condition (3.1). Let $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\theta,\eta,\zeta)}$ and $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ be the spaces given in Definition 3.1 and Definition 2.4 respectively. Then there exists a linear continuous map $\mathcal{I} : \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\theta,\eta,\zeta)}\left(\Delta_4;\mathbb{R}^d\right) \to \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}\left(\Delta_3;\mathbb{R}^d\right)$ such that the following holds true.*

(i) *The quantity $\mathcal{I}(\Xi^\tau_v)_{ts} := \lim_{|\mathcal{P}|\to 0}\sum_{[u,w]\in\mathcal{P}}(\Xi^\tau_v)_{wu}$ exists for all $(v,s,t,\tau)\in\Delta_4$, where $\mathcal{P}$ is a generic partition of $[s,t]$ and $|\mathcal{P}|$ denotes the mesh size of the partition. Furthermore, we define $\mathcal{I}(\Xi^\tau_v)_t := \mathcal{I}(\Xi^\tau_v)_{t0}$, and have $\mathcal{I}(\Xi^\tau_v)_{ts} = \mathcal{I}(\Xi^\tau_v)_{t0} - \mathcal{I}(\Xi^\tau_v)_{s0}$.*

(ii) *For all $(v,s,t,\tau)\in\Delta_4$ we have*

$$|\mathcal{I}\left(\Xi^\tau_v\right)_{ts} - (\Xi^\tau_v)_{ts}| \lesssim \|\delta\Xi\|_{(\beta,\kappa,\theta),1}\,\phi^1_{\beta,\kappa,\theta}(\tau,t,s,v), \tag{3.7}$$

while for $(v,s,t,\tau',\tau)\in\Delta_5$ *we get*

$$\left|\mathcal{I}(\Xi_v^{\tau\tau'})_{ts}-(\Xi_v^{\tau\tau'})_{ts}\right|\lesssim\|\delta\Xi\|_{(\beta,\kappa,\theta,\eta,\zeta),1,2}\,\phi^{1,2}_{\beta,\kappa,\theta,\eta,\zeta}(\tau,\tau',t,s,v),\tag{3.8}$$

where ϕ^1 *and* $\phi^{1,2}$ *are the functions given by* (3.5) *and* (3.6).

Proof. This is an extension of [12, Lemma 21]. Let us consider the n-th order dyadic partition $\mathcal{P}^n$ of $[s,t]$ where each set $[u,w]\in\mathcal{P}^n$ has length $2^{-n}|t-s|$. We define the n-th order Riemann sum of Ξ_v^τ, denoted $\mathcal{I}^n(\Xi_v^\tau)_{ts}$, as follows

$$\mathcal{I}^n(\Xi_v^\tau)_{ts}=\sum_{[u,w]\in\mathcal{P}^n}(\Xi_v^\tau)_{wu}\,.$$

Our aim is to show that the sequence $\{\mathcal{I}^n(\Xi_v^\tau);n\geq 1\}$ converges to an element $\mathcal{I}(\Xi_v^\tau)$ which fulfills relation (3.7). To this aim we begin to consider the difference $\mathcal{I}^{n+1}(\Xi_v^\tau)-\mathcal{I}^n(\Xi_v^\tau)$. A series of elementary computations reveals that

$$\mathcal{I}^{n+1}(\Xi_v^\tau)_{ts}-\mathcal{I}^n(\Xi_v^\tau)_{ts}=-\sum_{[u,w]\in\mathcal{P}^n}\delta_m(\Xi_v^\tau)_{wu},\tag{3.9}$$

where $m=\frac{w+u}{2}$ and where we recall that δ is given by relation (2.14). Plugging relation (3.3) into (3.9), it is easy to check that

$$\sum_{[u,w]\in\mathcal{P}^n}|\delta_m(\Xi_v^\tau)_{wu}|\lesssim\|\delta\Xi\|_{(\beta,\kappa,\theta),1}\sum_{[u,w]\in\mathcal{P}^n}|\tau-w|^{-\kappa}|u-v|^{-\theta}|w-u|^{\beta}.\tag{3.10}$$

We will upper bound the right hand side above. Invoking the fact that $\beta>1$ and $|w-u|=2^{-n}|t-s|$, for $u,w\in\mathcal{P}^n$ we write

$$\sum_{[u,w]\in\mathcal{P}^n}|\tau-w|^{-\kappa}|u-v|^{-\theta}|w-u|^{\beta}\leq 2^{-n(\beta-1)}|t-s|^{\beta-1}\sum_{[u,w]\in\mathcal{P}^n}|\tau-w|^{-\kappa}|u-v|^{-\theta}|w-u|.\tag{3.11}$$

With the definition of Riemann sums in mind, the term

$$\sum_{[u,w]\in\mathcal{P}^n}|\tau-w|^{-\kappa}|u-v|^{-\theta}|w-u|$$

in the right hand side of (3.11) can be dominated by the following integral:

$$\int_s^t|\tau-x|^{-\kappa}|x-v|^{-\theta}dx.$$

Note that by assumption, $v\leq s\leq t\leq\tau$, and so this integral is finite since $\kappa+\theta<1$. In addition, some elementary calculations show that the above integral can be upper bounded as follows,

$$\int_s^t|\tau-x|^{-\kappa}|x-v|^{-\theta}dx\lesssim|\tau-t|^{-\kappa}|s-v|^{-\theta}|t-s|\wedge|t-v|^{1-\kappa-\theta}.\tag{3.12}$$

Plugging the inequality (3.12) into (3.11), we thus get

$$\sum_{[u,w]\in\mathcal{P}^n}|\tau-w|^{-\kappa}|u-v|^{-\theta}|w-u|^{\beta}$$
$$\lesssim 2^{-n(\beta-1)}\left(\left[|\tau-t|^{-\kappa}|s-v|^{-\theta}|t-s|^{\beta}\right]\wedge|\tau-v|^{\beta-\kappa-\theta}\right).$$

Then taking (3.10) into account, relation (3.9) can be recast as

$$|\mathcal{I}^{n+1}(\Xi^{\tau}_{v})_{ts} - \mathcal{I}^{n}(\Xi^{\tau}_{v})_{ts}| \lesssim 2^{-n(\beta-1)}\|\delta\Xi\|_{(\beta,\kappa,\theta),1}\left(\left[|\tau-t|^{-\kappa}|s-v|^{-\theta}|t-s|^{\beta}\right]\wedge|\tau-v|^{\beta-\kappa-\theta}\right). \quad (3.13)$$

Since $\beta > 1$, then (3.13) implies that the sequence $\{\mathcal{I}^n(\Xi^\tau_v); n \geq 1\}$ is Cauchy. It thus converges to a quantity $\mathcal{I}(\Xi^\tau_v)_{ts}$ which satisfies (3.7). The rest of this proof is the same as [15, Lemma 4.2], which means that the element $\mathcal{I}(\Xi^\tau_v)$ has finite $\|\cdot\|_{(\beta,\kappa,\theta),1}$ norm. The proof of relation (3.8) is very similar to (3.7), and left to the reader for sake of conciseness. We just define an increment $\Xi^{\tau,\tau'}_v$ instead of Ξ^τ_v and then proceed as in (3.9)-(3.13). The proof is now complete. □

3.2. **Third order convolution products in the rough case $\alpha - \gamma > \frac{1}{4}$.** In this section we establish a proper definition of third order convolution products. Let us first introduce the class of integrands we shall consider for those products.

Notation 3.3. *Similarly to Notation 2.24, we denote by $u^{1,2,3}$ a function $u : \Delta_4 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 3}, \mathbb{R}^d)$ given by*

$$(s, \tau_1, \tau_2, \tau_3) \mapsto u^{\tau_3,\tau_2,\tau_1}_s.$$

To motivate the upcoming analysis and in order to get a better intuition of what is meant by third order convolution products, let us first give a definition of the third order convolution product for smooth functions, and prove a useful relation for the construction of this convolution.

Definition 3.4. *Let x be a continuously differentiable function and consider a Volterra kernel k which fulfills Hypothesis 2.1 with $\gamma < 1$. Let also $f : \Delta_4 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 3}, \mathbb{R}^d)$ be a smooth function given in Notation 3.3. Then recalling our Notation 2.16 for $\tau \geq t > s \geq v$ the convolution $\mathbf{z}^{3,\tau}_{ts} * f^{1,2,3}_v$ is defined by*

$$\mathbf{z}^{3,\tau}_{ts} * f^{1,2,3}_v := \int_{t>r_1>s} k(\tau, r_1)dx_{r_1} \otimes \int_{r_1>r_2>s} k(r_1, r_2)dx_{r_2} \otimes \int_{r_2>r_3>s} k(r_2, r_3)dx_{r_3} f^{r_1,r_2,r_3}_v. \quad (3.14)$$

Lemma 3.5. *Under the same conditions as in Definition 3.4, let $\mathbf{z}^{3,\tau}_{ts} * f^{1,2,3}_s$ be the increment given by (3.14). Consider $(s,t) \in \Delta_2$ and a generic partition $\mathcal{P}$ of $[s,t]$. Then we have*

$$\mathbf{z}^{3,\tau}_{ts} * f^{1,2,3}_s = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbf{z}^{3,\tau}_{vu} * f^{1,2,3}_s + \left(\delta_u \mathbf{z}^{3,\tau}_{vs}\right) * f^{1,2,3}_s. \quad (3.15)$$

Proof. Starting from expression (3.14), it is readily seen that

$$\mathbf{z}^{3,\tau}_{ts} * f^{1,2,3}_s = \sum_{[u,v]\in\mathcal{P}} \int_{v>r_1>u} k(\tau, r_1)dx_{r_1} \otimes \int_{r_1>r_2>s} k(r_1, r_2)dx_{r_2} \otimes \int_{r_2>r_3>s} k(r_2, r_3)dx_{r_3} f^{r_1,r_2,r_3}_s.$$

Then for each $[u,v] \in \mathcal{P}$, divide the region $\{v > r_1 > u\} \cap \{r_1 > r_2 > r_3 > s\}$ into

$$\{v > r_1 > r_2 > r_3 > u\} \cup \{v > r_1 > r_2 > u > r_3 > s\} \cup \{v > r_1 > u > r_2 > r_3 > s\}.$$

This yields a decomposition of $\mathbf{z}^{3,\tau}_{ts} * f^{1,2,3}_s$ of the form

$$\mathbf{z}^{2,\tau}_{ts} * f^{1,2,3}_s = \sum_{[u,v]\in\mathcal{P}} A^\tau_{vu} + B^\tau_{vu} + C^\tau_{vu},$$

where A^{τ}_{vu} , B^{τ}_{vu} , and C^{τ}_{vu} are respectively given by

$$A^{\tau}_{vu} = \int_{v>r_1>u} k(\tau, r_1)dx_{r_1} \otimes \int_{r_1>r_2>u} k(r_1, r_2)dx_{r_2} \otimes \int_{r_2>r_3>u} k(r_2, r_3)dx_{r_3} f^{r_1,r_2,r_3}_s$$
$$B^{\tau}_{vu} = \int_{v>r_1>u} k(\tau, r_1)dx_{r_1} \otimes \int_{r_1>r_2>u} k(r_1, r_2)dx_{r_2} \otimes \int_{u>r_3>s} k(r_2, r_3)dx_{r_3} f^{r_1,r_2,r_3}_s$$
$$C^{\tau}_{vu} = \int_{v>r_1>u} k(\tau, r_1)dx_{r_1} \otimes \int_{u>r_2>s} k(r_1, r_2)dx_{r_2} \otimes \int_{r_2>r_3>s} k(r_2, r_3)dx_{r_3} f^{r_1,r_2,r_3}_s.$$

We recognize the term A^{τ}_{vu} as the expression $\mathbf{z}^{3,\tau}_{vu} * f^{1,2,3}_s$ given by Definition 3.14. Moreover, we can check that $B^{\tau}_{vu} = \mathbf{z}^{2,\tau}_{vu} * \mathbf{z}^{1,\cdot}_{us} * f^{1,2,3}_s$, and $C^{\tau}_{vu} = \mathbf{z}^{1,\tau}_{vu} * \mathbf{z}^{2,\cdot}_{us} * f^{1,2,3}_s$. Then since $\mathbf{z}^{3,\tau}$ satisfies (2.28), we have $B^{\tau}_{vu} + C^{\tau}_{vu} = (\delta_u \mathbf{z}^{3,\tau}_{vs}) * f^{1,2,3}_s$. This finishes the proof of our claim (3.15). □

In order to generalize the notion of convolution product beyond the scope of Definition 3.4 to accommodate rough signals x, let us introduce the kind of norm we shall consider for processes with 3 upper variables of the form $u^{1,2,3}$, and in that connection introduce another Volterra-Hölder space equipped with this new norm.

Definition 3.6. *Let $\mathcal{W}^{(\alpha,\gamma,\eta,\zeta)}_3$ denote the space of functions $u : \Delta_4 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 3}, \mathbb{R}^d)$ as given in Notation 3.3 with $u^{\tau_1,\tau_2,\tau_3}_0 = u_0 \in \mathcal{L}((\mathbb{R}^d)^{\otimes 3}, \mathbb{R}^d)$ and such that $\|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta)} < \infty$, where the norm $\|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta)}$ is defined by*

$$\left\|u^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta)} := \left\|u^{1,2,3}\right\|_{(\alpha,\gamma),1} + \left\|u^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}. \tag{3.16}$$

More specifically, recalling the definition (2.1) for ψ^1,the $\|\cdot\|_{(\alpha,\gamma),1}$ and $\|\cdot\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$ norms in (3.16) are respectively defined by

$$\left\|u^{1,2,3}\right\|_{(\alpha,\gamma),1} := \sup_{(s,t,\tau)\in\Delta_3} \frac{|u^{\tau,\tau,\tau}_{ts}|}{\psi^1_{\alpha,\gamma}(\tau, t, s)}, \tag{3.17}$$

and

$$\left\|u^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} := \left\|u^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2} + \left\|u^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,3} + \left\|u^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),2,3}. \tag{3.18}$$

In the right hand side of (3.18), similarly to (2.36)-(2.37), we have set $\|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2}$ as the sum $\|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,>} + \|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,<}$, with

$$\left\|u^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,>} := \sup_{(s,t,r_3,r_1,r_2,r')\in\Delta_6} \frac{|u^{r',r_2,r_3}_{ts} - u^{r',r_1,r_3}_{ts}|}{h_{\eta,\zeta}(s, t, r_1, r_2, r_3, r')}, \tag{3.19}$$

$$\left\|u^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,<} := \sup_{(s,t,r_3,r',r_1,r_2)\in\Delta_6} \frac{|u^{r_2,r',r_3}_{ts} - u^{r_1,r',r_3}_{ts}|}{h_{\eta,\zeta}(s, t, r_1, r_2, r_3, r')}. \tag{3.20}$$

Define h as follows wherever $r = \min(r_1, r_2, r_3, r')$:

$$h_{\eta,\zeta}(s, t, r_1, r_2, r) := |r_2 - r_1|^{\eta} |r - t|^{-\eta+\zeta} \left(\left[|r - t|^{-\gamma-\zeta} |t - s|^{\alpha} \right] \wedge |r - s|^{\alpha-\gamma-\zeta} \right). \tag{3.21}$$

Moreover, the norms $\|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),2,3}$ and $\|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,3}$ in (3.18) are defined similarly to relations (3.19)-(3.20).

Remark 3.7. Notice that Definition 3.6 has been introduced so that the increments $y^{u,u,u} - y^{r,r,r}$ can be controlled by (3.18). Indeed, we have for any $\eta \in [0,1]$ and $\zeta \in [0,\rho)$

$$\begin{aligned}
|y^{u,u,u}_{ts} - y^{r,r,r}_{ts}| &= |y^{u,u,u}_{ts} - y^{u,r,r}_{ts} + y^{u,r,r}_{ts} - y^{r,r,r}_{ts}| \leq |y^{u,u,u}_{ts} - y^{u,r,r}_{ts}| + |y^{u,r,r}_{ts} - y^{r,r,r}_{ts}| \\
&\leq \left(\|y\|_{(\alpha,\gamma,\eta,\zeta),2,3} + \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}\right) |u-r|^{\eta} |r-t|^{-\eta+\zeta} \left(\left[|r-t|^{-\gamma-\zeta} |t-s|^{\alpha} \right] \wedge |r-s|^{\rho-\zeta} \right) \\
&\lesssim \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} |u-r|^{\eta} |r-t|^{-\eta+\zeta} \left(\left[|r-t|^{-\gamma-\zeta} |t-s|^{\alpha} \right] \wedge |r-s|^{\rho-\zeta} \right) \\
&\leq \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} |u-r|^{\eta} |r-t|^{-\eta+\zeta} |r-s|^{\rho-\zeta}. \qquad (3.22)
\end{aligned}$$

Hence similarly to (2.40), we let $\eta = \zeta$ and we obtain

$$|y^{u,u,u}_{ts} - y^{r,r,r}_{ts}| \lesssim \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}. \qquad (3.23)$$

Thanks to Hypothesis 2.20 and Definition 3.6, we can now state a general convolution product for functions defined on Δ_4. As mentioned above, it has to be seen as a generalization of Definition 3.4 to a rough context.

Theorem 3.8. *Let $\alpha, \gamma \in (0,1)$ such that $\rho \equiv \alpha - \gamma > \frac{1}{4}$. Consider $N \geq 1$ and η_k, ζ_k satisfying (2.27), for $k = 1, \ldots, N$. We assume that z fulfills Hypothesis 2.20 with n=3. Consider a function $y : \Delta_4 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 3}, \mathbb{R}^d)$ as given in Notation 3.3 such that $\|y^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} < \infty$ for all $(\eta,\zeta) = (\eta_k,\zeta_k), k = 1,\ldots,N$ and $y^{1,2,3}_0 = y_0$, where $\|y^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$ is defined by (3.18). Then with Notation 2.16 in mind, we have for all fixed $(s,t,\tau) \in \Delta_3$ that*

$$\mathbf{z}^{3,\tau}_{ts} * y^{1,2,3}_s := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbf{z}^{3,\tau}_{vu} y^{u,u,u}_s + \left(\delta_u \mathbf{z}^{3,\tau}_{vs}\right) * y^{1,2,3}_s. \qquad (3.24)$$

is a well defined Volterra-Young integral. It follows that $*$ is a well defined bi-linear operation between the three parameters Volterra function $\mathbf{z}^3$ and a 4-parameter path y. Moreover, we have that for all $(\eta,\zeta) = (\eta_k,\zeta_k)$

$$\begin{aligned}
\left| \mathbf{z}^{3,\tau}_{ts} * y^{1,2,3}_s - \mathbf{z}^{3,\tau}_{ts} y^{s,s,s}_s \right| \lesssim \|y^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \big(&\|\mathbf{z}^3\|_{(3\rho+\gamma,\gamma),1} + \|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2} \|\mathbf{z}^2\|_{(\alpha,\gamma),1} \\
&+ \|\mathbf{z}^2\|_{(\alpha,\gamma,\eta,\zeta),1,2} \|\mathbf{z}^1\|_{(\alpha,\gamma),1} \big) \, \psi^1_{(3\rho+\gamma),\gamma}(\tau,t,s), \qquad (3.25)
\end{aligned}$$

where ψ^1 is given in (2.1).

Remark 3.9. Similarly to Remark 2.28, the term $(\delta_u \mathbf{z}^{3,\tau}_{vs}) * y^{1,2,3}_s$ is defined thanks to the fact that (according to relation (2.28))

$$\delta_u \mathbf{z}^{3,\tau}_{vs} * y^{1,2,3}_s = \mathbf{z}^{2,\tau}_{vu} * \mathbf{z}^{1,\cdot}_{us} * y^{1,2,3} + \mathbf{z}^{1,\tau}_{vu} * \mathbf{z}^{2,\cdot}_{us} * y^{1,2,3}, \qquad (3.26)$$

and the convolutions with respect to $\mathbf{z}^{1,\tau}$, $\mathbf{z}^{2,\tau}$ in (3.26) are respectively defined by Proposition 2.17 and Theorem 2.27.

Proof of Theorem 3.8. We first prove (3.24). To this aim, for a generic partition $\mathcal{P}$ of $[s,t]$ let us denote by $\mathcal{I}_{\mathcal{P}}$ the approximation of the right hand side of (3.24). Specifically we set $\mathcal{I}_{\mathcal{P}} := \sum_{[u,v]\in\mathcal{P}} (\Xi^\tau_s)_{vu}$, where

$$(\Xi^\tau_s)_{vu} = \mathbf{z}^{3,\tau}_{vu} y^{u,u,u}_s + \left(\delta_u \mathbf{z}^{3,\tau}_{vs}\right) * y^{1,2,3}_s. \qquad (3.27)$$

We now compute $\delta_r (\Xi^\tau_s)_{vu}$ in order to check that the extended Volterra sewing Lemma 3.2 can be applied in our context. Recall that

$$\delta_r (\Xi^\tau_s)_{vu} = (\Xi^\tau_s)_{vu} - (\Xi^\tau_s)_{vr} - (\Xi^\tau_s)_{ru}, \quad \text{for all} \quad \tau > v > r > u > s.$$

Moreover, we know from Hypothesis 2.20 that

$$\delta_r \mathbf{z}^{3,\tau}_{vu} = \mathbf{z}^{2,\tau}_{vr} * \mathbf{z}^{1,\cdot}_{ru} + \mathbf{z}^{1,\tau}_{vr} * \mathbf{z}^{2,\cdot}_{ru}.$$

Therefore, a few elementary computations reveal that

$$\delta_r \left(\mathbf{z}^{3,\tau}_{vu} y^{u,u,u}_s\right) = -\mathbf{z}^{3,\tau}_{vr} \left(y^{r,r,r}_s - y^{u,u,u}_s\right) + \left(\mathbf{z}^{2,\tau}_{vr} * \mathbf{z}^{1,\cdot}_{ru} + \mathbf{z}^{1,\tau}_{vr} * \mathbf{z}^{2,\cdot}_{ru}\right) y^{u,u,u}_s \tag{3.28}$$

$$\delta_r \left(\left(\delta_u \mathbf{z}^{3,\tau}_{vs}\right) * y^{1,2,3}_s\right) = -\left(\mathbf{z}^{2,\tau}_{vr} * \mathbf{z}^{1,\cdot}_{ru} + \mathbf{z}^{1,\tau}_{vr} * \mathbf{z}^{2,\cdot}_{ru}\right) * y^{1,2,3}_s, \tag{3.29}$$

Combining (3.28) and (3.29), we thus get

$$\delta_r \left(\Xi^\tau_s\right)_{vu} = -\left(Q^1_{vru} + Q^2_{vru} + Q^3_{vru}\right), \tag{3.30}$$

where the quantities Q^1_{vru}, Q^2_{vru}, Q^3_{vru} are defined by

$$Q^1_{vru} = \mathbf{z}^{3,\tau}_{vr} \left(y^{r,r,r}_s - y^{u,u,u}_s\right)$$
$$Q^2_{vru} = \mathbf{z}^{2,\tau}_{vr} * \mathbf{z}^{1,\cdot}_{ru} * \left(y^{1,2,3}_s - y^{u,u,u}_s\right)$$
$$Q^3_{vru} = \mathbf{z}^{1,\tau}_{vr} * \mathbf{z}^{2,\cdot}_{ru} * \left(y^{1,2,3}_s - y^{u,u,u}_s\right)$$

We will bound each of the above terms separately.

Invoking the definition of $\|\mathbf{z}^3\|_{(3\rho+\gamma,\gamma),1}$ in (2.5), and using that $r \in [u,v]$ we have for any $\eta \in (1-\alpha, 1]$

$$\left|Q^1_{vru}\right| \lesssim \left\|y^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \left\|\mathbf{z}^3\right\|_{(3\rho+\gamma,\gamma),1} |u-s|^{-(\eta-\zeta)}|\tau-v|^{-\gamma}|v-u|^{3\rho+\gamma+\eta}, \tag{3.31}$$

We then choose η such that $3\rho+\gamma+\eta > 1$, which is always possible since $\rho > 0$, to obtain the desired regularity. For the term Q^2_{vru}, we invoke the bound in (2.42), and observe that

$$\begin{aligned}\left|Q^2_{vru}\right| \leq{}& \left|\mathbf{z}^{2,\tau}_{vr}\right| \left|\mathbf{z}^{1,r}_{ru} * \left(y^{r,r,3}_s - y^{u,u,u}_s\right)\right| \\ &+ \|\hat{y}\|_{(\alpha,\gamma,\eta,\zeta),1,2} \left(\|\mathbf{z}^1\|^2_{(\alpha,\gamma,\eta,\zeta)} + \|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma,\eta,\zeta)}\right) |\tau-v|^{-\gamma}|v-u|^{3\rho+\gamma} \wedge |\tau-u|^{3\rho}\end{aligned} \tag{3.32}$$

where $\hat{y}^{l,w}_{ru} = \mathbf{z}^{1,\cdot}_{ru} * \left(y^{l,w,3}_s - y^{u,u,u}_s\right)$, and we will need to find a bound for $\|\hat{y}\|_{(\alpha,\gamma,\eta,\zeta),1,2}$. Note that convolution only happens in the first term of $y^{l,w,3}_s - y^{u,u,u}_s$. By (2.26) it follows that

$$\|\hat{y}\|_{(\alpha,\gamma,\eta,\zeta),1,2} \lesssim \|\mathbf{z}\|_{(\alpha,\gamma),1}\|y^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),2,3}|v-u|^{\eta}|u-s|^{-(\eta-\zeta)}.$$

Furthermore, from (2.25) it is readily checked that

$$\left|\mathbf{z}^{1,r}_{ru} * \left(y^{r,2,3}_s - y^{u,u,u}_s\right)\right| \lesssim \|\mathbf{z}^1\|_{(\alpha,\gamma),1}\|y^{r,2,3} - y^{u,u,u}_s\|_{(\alpha,\gamma,\eta,\zeta),1,2}|r-u|^{\rho}$$

We continue to investigate the first terms in (3.32). From the above regularity estimate it follows that

$$\left|\mathbf{z}^{2,\tau}_{vr}\right| \left|\mathbf{z}^{1,\cdot}_{ru} * \left(y^{1,2,3}_s - y^{u,u,u}_s\right)\right| \lesssim \left\|\mathbf{z}^2\right\|_{(2\rho+\gamma,\gamma,\eta,\zeta)} \left\|\mathbf{z}^1\right\|_{(\alpha,\gamma,\eta,\zeta)} \|y\|_{(\alpha,\gamma)} |\tau-v|^{-\gamma}|v-u|^{3\rho+\gamma+\eta}|u-s|^{-(\eta-\zeta)}.$$

Combining our estimates for the different terms on the right hand side of (3.32), we have that

$$\left|Q^2_{vru}\right| \lesssim \left\|y^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \left\|\mathbf{z}^2\right\|_{(2\rho+\gamma,\gamma,\eta,\zeta)} \left\|\mathbf{z}^1\right\|_{(\alpha,\gamma,\eta,\zeta)} |\tau-v|^{-\gamma}|v-u|^{3\rho+\gamma+\eta}|u-s|^{-(\eta-\zeta)}, \tag{3.33}$$

By similar computations as for the bound for Q^2, we obtain a bound for Q^3 given by

$$\left|Q^3_{vru}\right| \lesssim \left\|y^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \left\|\mathbf{z}^1\right\|_{(\alpha,\gamma,\eta,\zeta)} \left\|\mathbf{z}^2\right\|_{(2\rho+\gamma,\gamma,\eta,\zeta)} |\tau-v|^{-\gamma}|v-u|^{3\rho+\gamma+\eta}|u-s|^{-(\eta-\zeta)}. \tag{3.34}$$

Plugging (3.31)-(3.34) into (3.30), we have thus obtained

$$\left|\delta_r \left(\Xi^\tau_s\right)_{vu}\right| \lesssim C_{y,\mathbf{z}}|\tau-v|^{-\gamma}|u-s|^{-(\eta-\zeta)}|v-u|^{3\rho+\gamma+\eta}, \tag{3.35}$$

where the constant $C_{y,\mathbf{z}}$ used above is given explicitly as

$$c_{y,\mathbf{z}} = \left\|y^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}\left(\|\mathbf{z}^3\|_{(3\rho+\gamma,\gamma,\eta,\zeta)} + 2\left\|\mathbf{z}^2\right\|_{(2\rho+\gamma,\gamma,\eta,\zeta)}\left\|\mathbf{z}^1\right\|_{(\alpha,\gamma,\eta,\zeta)}\right).$$

Recalling that $(s,u,r,v,t,\tau)\in\Delta_6$, we have $|u-s|^\zeta \le T$. Then the relation (3.35) yields

$$|\delta_r\left(\Xi_s^\tau\right)_{vu}| \lesssim C_{y,\mathbf{z}}|\tau-v|^{-\gamma}|u-s|^{-\eta}|v-u|^{3\rho+\gamma+\eta} \tag{3.36}$$

Starting from (3.36), one can now check that

$$\|\delta\Xi\|_{(3\rho+\gamma+\eta,\gamma,\eta),1} < \infty, \tag{3.37}$$

where the norm in the left hand side of (3.37) is defined by (3.3). In the same way, we let the patient reader check that $\|\Xi\|_{(3\rho+\gamma+\eta,\gamma,\eta,\eta,\zeta),1,2} < \infty$, where the $\|\cdot\|_{(3\rho+\gamma+\eta,\gamma,\eta,\eta,\zeta),1,2}$ norm is introduced in (3.4). Since we have chosen η such that $3\rho+\gamma+\eta>1$, we can apply Lemma 3.2 to the increment Ξ and recall the Notation 2.3 of ψ^1, $\psi^{1,2}$, which directly yields our claims (3.7) and (3.8). □

Remark 3.10. The general convolution $\mathbf{z}^{3,\tau} * y_s^{1,2,3}$ is given in (3.24), for a path y defined on Δ_4. If we wish to consider the convolution restricted to a path $y_s^{1,2}$ defined on Δ_3, a natural way to proceed is to define

$$\mathbf{z}_{ts}^{3,\tau} * y_s^{1,2} := \mathbf{z}_{ts}^{3,\tau} * \hat{y}^{1,2,3}, \quad \text{with} \quad \hat{y}^{r_1,r_2,r_3} = y^{r_2,r_3}.$$

This means that the path $\hat{y}$ has no dependence in r_1. Therefore resorting to the notations (2.34)-(2.35), and (3.17)-(3.18), it is not difficult to check that

$$\begin{aligned}
&\left\|\hat{y}^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,>} = \left\|y^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta)1,2,<}, && \left\|\hat{y}^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,<} = 0,\\
&\left\|\hat{y}^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,3,>} = \left\|y^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta)1,2,>}, && \left\|\hat{y}^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,3,<} = 0,\\
&\left\|\hat{y}^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),2,3,>} = \left\|y^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta)1,2,>}, && \left\|\hat{y}^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),2,3,<} = \left\|y^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta)1,2,<}.
\end{aligned}$$

Hence we have $\|\hat{y}^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \lesssim \|y^{1,2}\|_{(\alpha,\gamma,\eta,\zeta),1,2}$, where $\|\hat{y}^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$ is given in (3.18) and the norm $\|y^{1,2}\|_{(\alpha,\gamma,\eta,\zeta),1,2}$ is introduced in (2.35). This will be invoked for our rough path constructions in the upcoming section.

Remark 3.11. In our applications to rough Volterra equations we will consider the case $\rho=\alpha-\gamma\in(\frac{1}{4},\frac{1}{3})$. Therefore it is sufficient to show that the convolution product $*$ can be performed on the third level of a Volterra rough path.

4. Calculus for Volterra rough paths

In this section we carry out some of the main steps leading to a proper differential calculus in a Volterra context. That is, we show how to integrate a Volterra controlled process in Section 4.1, and we solve Volterra type equations in Section 4.2.

4.1. **Volterra controlled processes and rough Volterra integration.** We begin with a proper definition of rough Volterra integration in rough case $\alpha-\gamma>\frac{1}{4}$. As usual in rough integration theory, one needs to specify a proper class of processes which can be integrated with respect to the driving noise. As we will see, a non-geometric rough path type theory based on tree type expansions is needed, in order to construct a well defined rough Volterra integral. We therefore begin with some motivation for tree type expansions for iterated integrals.

4.1.1. *Tree expansions setting.* In Hypothesis 2.20, we have introduced the notion of a convolutional rough path $\mathbf{z}$ above z. While $\mathbf{z}$ satisfies the Chen type relation (2.28), it cannot be considered as a geometric rough path (see e.g. [15]). The reader might check for instance that for a path $\mathbf{z}^{1,\tau}$ given by the mapping $(t,\tau)\mapsto\int_0^t k(\tau,r)dx_r$, the component $\mathbf{z}^{2,\tau}$ will *not* satisfy the component-wise relation

$$\left(\mathbf{z}_{ts}^{2,\tau}\right)^{ii}=\frac{1}{2}\left((\mathbf{z}_{ts}^{1,\tau})^i\right)^2, \qquad i=1,\cdots,d.$$

In terms of rough paths, the second order Volterra integral is not weakly geometric. Hence in order to define a rough path type calculus of order 2 related to z^τ, we have to invoke techniques related to non-geometric settings. The standard language in this kind of context is related to the Hopf algebra of trees. We give a brief account on those notions in the current section, referring to [14] for further details.

Let $\mathcal{T}_3$ be the set of rooted trees with at most 3 vertices, whose vertices are decorated by labels from the alphabet $\{1,\dots,d\}$. A full description of the undecorated version of $\mathcal{T}_3$ is given by

$$\mathcal{T}_3=\left\{\bullet,\ \mathbf{:},\ \vdots,\ \vee\right\}. \tag{4.1}$$

In the sequel we will use the operation $[\cdot]$ on trees. Namely for $\sigma_1,\dots,\sigma_m\in\mathcal{T}_3$ and $a\in\{1,\dots,d\}$. we define $\sigma=[\sigma_1\cdots\sigma_k]_a$ as the tree for which $\sigma_1,\dots,\sigma_k$ are attached to a new root with label a. For instance in the unlabeled case we have

$$[\,1\,]=\bullet \qquad [\bullet]=\mathbf{:} \qquad [\mathbf{:}]=\vdots \qquad [\bullet\bullet]=\vee.$$

It is thus readily checked that any tree in $\mathcal{T}_3$ can be constructed iteratively from smaller trees thanks to the operation $[\cdot]$. Let us also mention that we always assume that the order of the branches in each tree does not matter, in the sense that $[\sigma_1\cdots\sigma_m]_i=[\sigma_{\pi_1}\cdots\sigma_{\pi_m}]_i$ for all permutations π of $\{1,\dots,m\}$.

The set $\mathcal{T}_3$ can be turned into a Hopf algebra when equipped with a suitable coproduct and antipode. This elegant structure is applied and discussed in detail in [14], but is not necessary in our context. However, we shall use some of the notation contained in [14] for our future computations.

Notation 4.1. *For any tree $\sigma\in\mathcal{T}_3$, the quantity $|\sigma|$ denotes the numbers of vertices in σ. We call the set $\mathcal{F}_2$ a forest consisting of elements with 2 vertices or less. Namely, $\mathcal{F}_2$ is defined by*

$$\mathcal{F}_2=\{\bullet,\ \mathbf{:},\ \bullet\bullet\}.$$

Remark 4.2. Note that the operation $[\cdot]$ sends the set $\{1\}\cup\mathcal{F}_2$ into $\mathcal{T}_3$.

4.1.2. *Tree indexed rough path and controlled processes.* We have already introduced the family $\{\mathbf{z}^{j,\tau}, j=1,2,3\}$ in Hypothesis 2.20. These objects will be identified with tree indexed objects below. On top of this family, our computations will also hinge on an additional function called $\mathbf{z}^{\vee,\tau}$. Similarly to (3.14), whenever x is a continuously differentiable function and k satisfies Hypothesis 2.1, the increment $\mathbf{z}^{\vee,\tau}$ is defined by

$$\mathbf{z}_{ts}^{\vee,\tau}=\int_s^t k(\tau,r)\left(\int_s^r k(r,l)dx_l\right)\otimes\left(\int_s^r k(r,w)dx_w\right)\otimes dx_r\,. \tag{4.2}$$

However, for a generic rough signal x we need some more abstract assumptions which are summarized below.

Hypothesis 4.3. *Let z be a third index Volterra rougher path as introduced in Hypothesis 2.20, for a given $N \geq 1$ and $(\alpha,\gamma) \in (0,1)$, $(\eta,\zeta) \in \mathcal{A}_N$ satisfying (2.27). In addition, we assume the existence of a family $\mathbf{z} = \{\mathbf{z}^{\sigma,\tau}, \sigma \in \mathcal{T}_3\}$ such that $\mathbf{z}_{ts}^{\sigma,\tau} \in (\mathbb{R}^d)^{\otimes|\sigma|}$. This family is defined by*

$$\mathbf{z}^{\bullet,\tau} = \mathbf{z}^{1,\tau}, \qquad \mathbf{z}^{\substack{\bullet\\ \bullet},\tau} = \mathbf{z}^{2,\tau}, \qquad \mathbf{z}^{\substack{\bullet\\ \bullet\\ \bullet},\tau} = \mathbf{z}^{3,\tau},$$

where $\mathbf{z}^{1,\tau}, \mathbf{z}^{2,\tau}, \mathbf{z}^{3,\tau}$ are introduced in Hypothesis 2.20. Moreover the increment $\mathbf{z}^{\substack{\bullet\,\bullet\\ \bullet},\tau}$ satisfies the algebraic relation

$$\delta_u \mathbf{z}_{ts}^{\substack{\bullet\,\bullet\\ \bullet},\tau} = 2\mathbf{z}_{tu}^{\substack{\bullet\\ \bullet},\tau} * \mathbf{z}_{us}^{\bullet,\cdot} + \mathbf{z}_{tu}^{\bullet,\tau} * (\mathbf{z}_{us}^{\bullet,\cdot})^{\otimes 2}\,, \tag{4.3}$$

where the right hand side of (4.3) is defined in Proposition 2.17. Analytically, we require each $\mathbf{z}^{\sigma,\tau}$ to be an element of $\mathcal{V}^{(|\sigma|\rho+\gamma,\gamma)}$, and we define (for all $(\eta,\zeta) \in \mathcal{A}_N$)

$$|||\mathbf{z}|||_{(\alpha,\gamma,\eta,\zeta)} := \sum_{\sigma\in\mathcal{T}^3} \|\mathbf{z}^{\sigma}\|_{(|\sigma|\rho+\gamma,\gamma,\eta,\zeta)}. \tag{4.4}$$

Remark 4.4. Note that $|||\cdot|||$ does not define a seminorm of any sort (as the collection of elements $\mathbf{z}$ does not form a linear space) but is rather meant as a convenient way to collect the seminorm terms concerning $\mathbf{z}^{\sigma}$ for $\sigma \in \mathcal{T}^3$.

Together with the elements in Hypothesis 4.3, we will also make use of the family $\{\mathbf{z}^{\delta}; \delta \in \mathcal{F}_2\}$ in the sequel. To this aim, let us now introduce the element $\mathbf{z}^{\bullet\bullet}$.

Notation 4.5. *As stated in Hypothesis 4.3, we have set $\mathbf{z}^{\bullet,\tau} = \mathbf{z}^{1,\tau}$. In addition, we also define $\mathbf{z}^{\bullet\bullet,\tau}$ as*

$$\mathbf{z}_{ts}^{\bullet\bullet,\tau} = \int_s^t k(\tau,r)dx_r \int_s^t k(\tau,l)dx_l = (\mathbf{z}_{ts}^{\bullet,\tau})^{\otimes 2}\,. \tag{4.5}$$

Therefore we can recast (4.3) as

$$\delta_u \mathbf{z}_{ts}^{\substack{\bullet\,\bullet\\ \bullet},\tau} = 2\mathbf{z}_{tu}^{\substack{\bullet\\ \bullet},\tau} * \mathbf{z}_{us}^{\bullet,\cdot} + \mathbf{z}_{tu}^{\bullet,\tau} * \mathbf{z}_{ts}^{\bullet\bullet,\tau}\,. \tag{4.6}$$

Assuming Hypothesis 4.3 holds, similarly to Theorem 3.8, we now give a convolution result for $\mathbf{z}^{\substack{\bullet\,\bullet\\ \bullet},\tau}$.

Theorem 4.6. *Let $\mathbf{z}$ be a Volterra rough path such that Hypothesis 4.3 is met, with $(\alpha,\gamma) \in (0,1)$ and a given $N \geq 1$. Recall that the set $\mathcal{A}_N$ is introduced in Notation 2.22. Consider a function $y : \Delta_4 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 3}, \mathbb{R}^d)$ as given in Notation 3.3 such that for all $(\eta,\zeta) \in \mathcal{A}_N$ we have $\|y^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} < \infty$ and $y_0^{1,2,3} = y_0$, where $\|y^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$ is defined by (3.18). Then with Notation 2.16 in mind, we have for all fixed $(s,t,\tau) \in \Delta_3$ that*

$$\mathbf{z}_{ts}^{\substack{\bullet\,\bullet\\ \bullet},\tau} * y_s^{1,2,3} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbf{z}_{vu}^{\substack{\bullet\,\bullet\\ \bullet},\tau} y_s^{u,u,u} + \left(\delta_u \mathbf{z}_{vs}^{\substack{\bullet\,\bullet\\ \bullet},\tau}\right) * y_s^{1,2,3}. \tag{4.7}$$

is a well defined Volterra-Young integral. It follows that $*$ is a well defined bi-linear operation between the three parameters Volterra function $\mathbf{z}^{\substack{\bullet\,\bullet\\ \bullet}}$ and a 4-parameter path y. Moreover, we have that

$$\left|\mathbf{z}_{ts}^{\substack{\bullet\,\bullet\\ \bullet},\tau} * y_s^{1,2,3} - \mathbf{z}_{ts}^{\substack{\bullet\,\bullet\\ \bullet},\tau} y_s^{s,s,s}\right| \lesssim \left\|y^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} |||\mathbf{z}|||_{(\alpha,\gamma)}^3\, \psi^1_{(3\rho+\gamma),\gamma}(\tau,t,s), \tag{4.8}$$

where ψ^1 is defined by (2.1).

Proof. The proof goes along the same lines as the proof of Theorem 3.8, and is omitted for sake of conciseness. ☐

We are now ready to introduce the natural class of processes one can integrate with respect to $\mathbf{z}$, called Volterra controlled processes.

Definition 4.7. *Let $\mathbf{z}$ be a Volterra rough path satisfying Hypothesis 4.3 with $(\alpha,\gamma)\in(0,1)$, $N\geq 1$ and a family $\mathcal{A}_N$ such that (2.27) is met for all $(\eta,\zeta)\in\mathcal{A}_N$ (recall that $\mathcal{A}_N$ is given by (2.31)). Consider a Volterra path $y:\Delta_2\to\mathbb{R}^m$. We assume that there exists a family $\{y^\sigma;\sigma\in\mathcal{F}_2\}$, with $\mathcal{F}_2$ as in Notation 4.1, such that the following holds true.*

(i) *y^σ is a function from $\Delta_{|\sigma|+2}$ to $\mathcal{L}((\mathbb{R}^d)^{\otimes|\sigma|},\mathbb{R}^m)$, and y^σ has $|\sigma|+1$ upper arguments. The initial conditions are respectively given by*

$$y_0^{\bullet,p,q}=y_0^{\bullet},\qquad y_0^{\overset{\bullet}{\bullet},p,q,r}=y_0^{\overset{\bullet}{\bullet}},\qquad y_0^{\bullet\bullet,p,q,r}=y_0^{\bullet\bullet},\qquad \textit{for all}\quad (r,q,p)\in\Delta_3.$$

(ii) *The family $\{y^\sigma;\sigma\in\mathcal{F}_2\}$ is related to the increments of y^τ in the following way: for $(s,t,\tau)\in\Delta_3$ we have*

$$y_{ts}^{\tau}=\mathbf{z}_{ts}^{\bullet,\tau}*y_s^{\bullet,\tau,\cdot}+\mathbf{z}_{ts}^{\overset{\bullet}{\bullet},\tau}*y_s^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot}+\mathbf{z}_{ts}^{\bullet\bullet,\tau}*y_s^{\bullet\bullet,\tau,\cdot,\cdot}+R_{ts}^{y,\tau},\tag{4.9}$$

and

$$y_{ts}^{\bullet,\tau,p}=\mathbf{z}_{ts}^{\bullet,\tau}*\left(y_s^{\overset{\bullet}{\bullet},\tau,p,\cdot}+2y_s^{\bullet\bullet,\tau,p,\cdot}\right)+R_{ts}^{\bullet,\tau,p},\tag{4.10}$$

where $y^{\overset{\bullet}{\bullet}},y^{\bullet,\bullet}\in\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$, $R^{\bullet}\in\mathcal{W}_2^{(2\rho+2\gamma,2\gamma,\eta,\zeta)}(\mathcal{L}(\mathbb{R}^m))$ and $R^y\in\mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}(\mathbb{R}^m)$ for all $(\eta,\zeta)=(\eta_k,\zeta_k)$, $k=1,\dots,N$ (recall that $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ and $\mathcal{W}_2^{(2\rho+2\gamma,2\gamma,\eta,\zeta)}$ are introduced in Definition 2.4 and Definition 2.25 respectively).

Whenever $\mathbf{y}\equiv(y,y^{\bullet},y^{\overset{\bullet}{\bullet}},y^{\bullet\bullet})$ satisfies relation (4.9)-(4.10), we say that $\mathbf{y}$ is a Volterra path controlled by $\mathbf{z}$ (or simply controlled Volterra path) and we write $\mathbf{y}\in\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}(\Delta_2;\mathbb{R}^m)$. We equip this space with a semi-norm $\|\cdot\|_{\mathbf{z},(\alpha,\gamma)}=\sum_{k=1}^N\|\cdot\|_{\mathbf{z},(\alpha,\gamma,\eta_k,\zeta_k)}$, where each $\|\cdot\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}$ is given by

$$\begin{aligned}\|\mathbf{y}\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}&=\left\|\left(y,y^{\bullet},y^{\overset{\bullet}{\bullet}},y^{\bullet\bullet}\right)\right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\\&=\|y^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)}+\|y^{\bullet\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}+\|R^y\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)}+\|R^{\bullet}\|_{(2\rho+2\gamma,2\gamma,\eta,\zeta)}.\end{aligned}\tag{4.11}$$

Observe that the norms in (4.11) are respectively defined by (2.4) and (2.33). Then a norm on $\mathcal{D}^{(\alpha,\gamma)}$ is defined as

$$\mathbf{y}=\left(y,y^{\bullet},y^{\overset{\bullet}{\bullet}},y^{\bullet\bullet}\right)\mapsto|y_0|+|y_0^{\bullet}|+|y_0^{\overset{\bullet}{\bullet}}|+|y_0^{\bullet\bullet}|+\sum_{k=1}^N\left\|\left(y,y^{\bullet},y^{\overset{\bullet}{\bullet}},y^{\bullet\bullet}\right)\right\|_{\mathbf{z},(\alpha,\gamma,\eta_k,\zeta_k)},$$

the space $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}$ is a Banach space.

Remark 4.8. It is easily seen from (4.9) and (4.10) that if for $(\eta,\zeta)\in\mathcal{A}_N$ and $\mathbf{y}\in\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}$, then $y,y^{\bullet}\in\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$. Indeed, we observe directly from (4.10) that

$$\|y^{\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}\lesssim\|\mathbf{z}^{\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}\left(|y_0^{\overset{\bullet}{\bullet}}|+|y_0^{\bullet,\bullet}|+\|y^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)}+\|y^{\bullet,\bullet}\|_{(\alpha,\gamma,\eta,\zeta)})+\|R^{\bullet}\|_{(2\rho+2\gamma,2\gamma,\eta,\zeta)}\right),$$

where the quantities on the right hand side are finite by assumption. Furthermore, by relation (4.9) we then have that

$$\begin{aligned}\|y\|_{(\alpha,\gamma,\eta,\zeta)}\leq\|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)}\big(&|y_0^{\bullet}|+|y_0^{\overset{\bullet}{\bullet}}|+|y_0^{\bullet,\bullet}|+\|y^{\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}+\|y^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)}\\&+\|y^{\bullet,\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}+\|R^y\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)}\big).\end{aligned}\tag{4.12}$$

Remark 4.9. According to Definition 4.7, the function $y^{\bullet}$ is defined on Δ_3 and has two upper variables, while $y^{\substack{\bullet\\ \bullet}}$ and $y^{\bullet\bullet}$ are defined on Δ_4 and have three upper arguments. Therefore in (4.11) the norm $\|y^{\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}$ has to be understood as a norm in $\mathcal{W}_2^{(\alpha,\gamma,\eta,\zeta)}$, while the norms $\|y^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta)}$ and $\|y^{\bullet\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}$ have to be considered as norms in $\mathcal{W}_3^{(\alpha,\gamma,\eta,\zeta)}$. The reader is referred to Definition 2.25 and 3.6 for the definition of $\mathcal{W}_2^{(\alpha,\gamma,\eta,\zeta)}$ and $\mathcal{W}_3^{(\alpha,\gamma,\eta,\zeta)}$ respectively. We stick to the notation $\|\cdot\|_{(\alpha,\gamma,\eta,\zeta)}$ for the norm on those different spaces, for notational ease.

4.1.3. *Integration of controlled processes.* Our next step is to show that we may construct a Volterra rough integral in the rough case $\alpha-\gamma>\frac{1}{4}$, and then prove that the Volterra rough integral of a controlled path with respect to a driving Hölder noise $x\in\mathcal{C}^\alpha$ is again a controlled Volterra path.

Theorem 4.10. *For $\alpha\in(0,1)$, $\mathcal{A}_N$ given by (2.31) $(\eta,\zeta)\in\mathcal{A}_N$, let $x\in\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$ and k be a Volterra kernel satisfying Hypothesis 2.1 with a parameter γ such that relation (2.27) holds. Define $z_t^\tau=\int_0^t k(\tau,r)\,dx_r$ and assume there exists a tree indexed rough path $\mathbf{z}=\{\mathbf{z}^{\sigma,\tau};\sigma\in\mathcal{T}_3\}$ above z satisfying Hypothesis 4.3. Let $M>0$ be a constant such that $\|\!|\mathbf{z}|\!\|_{(\alpha,\gamma,\eta,\zeta)}\le M$. We now consider a controlled Volterra path $\mathbf{y}\in\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}(\mathcal{L}(\mathbb{R}^d,\mathbb{R}^m))$, as introduced in Definition 4.7. Then the following holds true:*

(i) *Define $\Xi^\tau_{vu}:=\mathbf{z}^{\bullet,\tau}_{vu}*y^{\cdot}_u+\mathbf{z}^{\substack{\bullet\\ \bullet},\tau}_{vu}*y^{\bullet,\cdot,\cdot}_u+\mathbf{z}^{\substack{\bullet\\ \bullet\\ \bullet},\tau}_{vu}*y^{\substack{\bullet\\ \bullet},\cdot,\cdot,\cdot}_u+\mathbf{z}^{\substack{\bullet\,\bullet\\ \bullet},\tau}_{vu}*y^{\bullet\bullet,\cdot,\cdot,\cdot}_u$. The following limit exists for all $(s,t,\tau)\in\Delta_3$,*

$$w^\tau_{ts}=\int_s^t k(\tau,r)dx_r y^r_r:=\lim_{|\mathcal{P}|\to0}\sum_{[u,v]\in\mathcal{P}}\Xi^\tau_{vu}. \tag{4.13}$$

(ii) *Let w be defined by (4.13). Recalling the Notation 2.3 of ψ^1, $\psi^{1,2}$ and Notation 2.22 for $\mathcal{A}_N$, there exists a positive constant $C=C_{M,\alpha,\gamma}$ such that for all $(s,t,\tau)\in\Delta_3$ and $(\eta,\zeta)\in\mathcal{A}_N$ we have*

$$|w^\tau_{ts}-\Xi^\tau_{ts}|\le C\left\|\left(y,y^{\bullet},y^{\substack{\bullet\\ \bullet}},y^{\bullet\bullet}\right)\right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\|\!|\mathbf{z}|\!\|_{(\alpha,\gamma,\eta,\zeta)}\,\psi^1_{(4\rho+\gamma),\gamma}(\tau,t,s). \tag{4.14}$$

(iii) *There exists a positive constant $C=C_{M,\alpha,\gamma}$ such that for all $(s,t,p,q)\in\Delta_4$, we have*

$$|w^{qp}_{ts}-\Xi^{qp}_{vu}|\le C\left\|\left(y,y^{\bullet},y^{\substack{\bullet\\ \bullet}},y^{\bullet\bullet}\right)\right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\|\!|\mathbf{z}|\!\|_{(\alpha,\gamma,\eta,\zeta)}\,\psi^{1,2}_{(4\rho+\gamma),\gamma,\eta,\zeta}(p,q,t,s) \tag{4.15}$$

(iv) *The triple $\mathbf{w}=(w,w^{\bullet},w^{\substack{\bullet\\ \bullet}},0)$ is a controlled Volterra path in $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}(\Delta_2,\mathbb{R}^m)$, where we recall that w is defined by (4.13), and where $w^{\bullet}$, $w^{\substack{\bullet\\ \bullet}}$ are respectively given by*

$$w_t^{\bullet,\tau,p}=y_t^p,\quad \textit{and}\quad w_t^{\substack{\bullet\\ \bullet},\tau,q,p}=y_t^{\bullet,q,p}.$$

Remark 4.11. From Theorem 4.10, we also can find a bound for $\|R^w\|_{(3\alpha,3\gamma,\eta,\zeta)}$ and $\|R^{w^{\bullet}}\|_{(2\alpha,2\gamma,\eta,\zeta)}$ for all $(\eta,\zeta)\in\mathcal{A}_N$.
Specifically, according to Theorem 4.10 *(ii)* we have

$$\|R^w\|_{(3\alpha,3\gamma,\eta,\zeta)}\lesssim\left\|\left(y,y^{\bullet},y^{\substack{\bullet\\ \bullet}},y^{\bullet\bullet}\right)\right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\|\!|\mathbf{z}|\!\|_{(\alpha,\gamma,\eta,\zeta)}. \tag{4.16}$$

Moreover, thanks to Theorem 4.10 *(iv)* we have $w_t^{\bullet,\tau,p}=y_t^p$. Recalling (4.9) together with (4.16), we obtain

$$\|R^{w^{\bullet}}\|_{(2\alpha,2\gamma,\eta,\zeta)}\lesssim\left\|\left(y,y^{\bullet},y^{\substack{\bullet\\ \bullet}},y^{\bullet\bullet}\right)\right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\|\!|\mathbf{z}|\!\|_{(\alpha,\gamma,\eta,\zeta)}. \tag{4.17}$$

Proof of Theorem 4.10. Let Ξ be given as in *(i)*. Thanks to Proposition 2.17, Theorem 2.27, Theorem 3.8, and Theorem 4.6, Ξ is well-defined. Our global strategy is to show that the Volterra sewing lemma can be applied to Ξ. In order to do so, let us compute $\delta_m \Xi^{\tau}_{vu}$ for $(u,m,v,\tau)\in\Delta_4$. Owing to (2.28), as well as some elementary properties of the operator δ, we get

$$\delta_m\left(\Xi^{\tau}_{vu}\right) := A^{\tau}_{vmu} + B^{\tau}_{vmu}. \tag{4.18}$$

where the quantities A^{τ}_{vmu} and B^{τ}_{vmu} are given by

$$A^{\tau}_{vmu} = -\left(\mathbf{z}^{\bullet,\tau}_{vm} * y^{\cdot}_{mu} + \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vm} * y^{\bullet,\cdot,\cdot}_{mu} + \mathbf{z}^{\overset{\bullet}{\overset{\bullet}{\bullet}},\tau}_{vm} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{mu} + \mathbf{z}^{\overset{\bullet\bullet}{\bullet},\tau}_{vm} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{mu}\right) \tag{4.19}$$

and

$$B^{\tau}_{vmu} = \delta_m \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vu} * y^{\bullet,\cdot,\cdot}_{u} + \delta_m \mathbf{z}^{\overset{\bullet}{\overset{\bullet}{\bullet}},\tau}_{vu} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{u} + \delta_m \mathbf{z}^{\overset{\bullet\bullet}{\bullet},\tau}_{vu} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{u}. \tag{4.20}$$

Due to the assumption that $(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet}) \in \mathcal{D}^{(\alpha,\gamma)}_{\mathbf{z}}$, we have that for any $(s,t,\tau)\in\Delta_3$

$$y^{\cdot}_{ts} = \mathbf{z}^{\bullet,\cdot}_{ts} * y^{\bullet,\cdot,\cdot}_{s} + \mathbf{z}^{\overset{\bullet}{\bullet},\cdot}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{s} + \mathbf{z}^{\bullet\bullet,\cdot}_{ts} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{s} + R^{y,\cdot}_{ts},$$

and

$$y^{\bullet,\cdot,\cdot}_{ts} = \mathbf{z}^{\bullet,\cdot}_{ts} * \left(y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{s} + 2y^{\bullet\bullet,\cdot,\cdot,\cdot}_{s}\right) + R^{\bullet,\cdot,\cdot}_{ts}.$$

Plugging the above two relations into (4.19), we obtain

$$\begin{aligned} A^{\tau}_{vmu} =& -\mathbf{z}^{\bullet,\tau}_{vm} * \mathbf{z}^{\bullet,\cdot}_{mu} * y^{\bullet,\cdot,\cdot}_{u} - \mathbf{z}^{\bullet,\tau}_{vm} * \mathbf{z}^{\overset{\bullet}{\bullet},\cdot}_{mu} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{u} - \mathbf{z}^{\bullet,\tau}_{vm} * \mathbf{z}^{\bullet\bullet,\cdot}_{mu} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{u} - \mathbf{z}^{\bullet,\tau}_{vm} * R^{y,\cdot}_{mu} \\ & - \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vm} * \mathbf{z}^{\bullet,\cdot}_{mu} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{u} - 2\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vm} * \mathbf{z}^{\bullet,\cdot}_{mu} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{u} - \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vm} * R^{\bullet,\cdot,\cdot}_{mu} \\ & - \mathbf{z}^{\overset{\bullet}{\overset{\bullet}{\bullet}},\tau}_{vm} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{mu} - \mathbf{z}^{\overset{\bullet\bullet}{\bullet},\tau}_{vm} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{mu}. \end{aligned} \tag{4.21}$$

Thanks to Hypothesis 2.20 and Hypothesis 4.3, plugging in the algebraic relations from (2.28) and (4.3) into (4.20), we have

$$\begin{aligned} B^{\tau}_{vum} =& \mathbf{z}^{\bullet,\tau}_{vm} * \mathbf{z}^{\bullet,\cdot}_{mu} * y^{\bullet,\cdot,\cdot}_{u} + \mathbf{z}^{\bullet,\tau}_{vm} * \mathbf{z}^{\overset{\bullet}{\bullet},\cdot}_{mu} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{u} + \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vm} * \mathbf{z}^{\bullet,\cdot}_{mu} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{u} \\ & + 2\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vm} * \mathbf{z}^{\bullet,\cdot}_{mu} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{u} + \mathbf{z}^{\bullet,\tau}_{vm} * (\mathbf{z}^{\bullet,\cdot}_{mu})^{\otimes 2} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{u}. \end{aligned} \tag{4.22}$$

We now insert (4.21) and (4.22) into (4.18). Let us also recall that $\mathbf{z}^{\bullet\bullet,\cdot}_{mu} = (\mathbf{z}^{\bullet,\cdot}_{mu})^{\otimes 2}$ according to (4.5). Then some elementary algebraic manipulations and cancellations show that

$$\delta_m\left(\Xi^{\tau}_{vu}\right) = -\mathbf{z}^{\overset{\bullet}{\overset{\bullet}{\bullet}},\tau}_{vm} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{mu} - \mathbf{z}^{\overset{\bullet\bullet}{\bullet},\tau}_{vm} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{mu} - \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vm} * R^{\bullet,\cdot,\cdot}_{mu} - \mathbf{z}^{\bullet,\tau}_{vm} * R^{y,\cdot}_{mu}. \tag{4.23}$$

We now bound successively the 4 terms in the right hand side of (4.23). First we apply a small variant of (3.25) and (4.8), which takes into account the fact that increments of the form $y^{\overset{\bullet}{\bullet}}_{mu}$ and $y^{\bullet\bullet}_{mu}$ are considered. We also bound the terms involving $y^{\overset{\bullet}{\bullet},u,u,u}_{mu}$, $y^{\bullet\bullet,u,u,u}_{mu}$ properly in (3.25) and (4.8). Resorting to (4.11) and the definition (2.1) of ψ^1, for a generic $(\eta,\zeta)\in\mathcal{A}_N$ we get

$$\begin{aligned} &\left|\mathbf{z}^{\overset{\bullet}{\overset{\bullet}{\bullet}},\tau}_{vm} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{mu} + \mathbf{z}^{\overset{\bullet\bullet}{\bullet},\tau}_{vm} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{mu}\right| \\ &\quad\lesssim \left(\|y^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} + \|y^{\bullet\bullet}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}\right) \|\!|\!|\mathbf{z}|\!|\!|_{(\alpha,\gamma,\eta,\zeta)} \left|u-m\right|^{\rho}\left|\tau-m\right|^{-\gamma}\left|v-m\right|^{3\rho+\gamma}. \end{aligned} \tag{4.24}$$

where we recall that $\|\!|\mathbf{z}|\!\|_{(\alpha,\gamma,\eta,\zeta)}$ was defined in (4.4). Next invoking the fact that $R^y \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$ and Proposition 2.17, together with ψ^1 as given in (2.1), we obtain

$$|\mathbf{z}^{\bullet,\tau}_{vm} * R^{y,\cdot}_{mu}| \le \|R^y\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)}\|\mathbf{z}^{\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}|\tau - v|^{-\gamma}|u-m|^{4\rho+\gamma}. \tag{4.25}$$

Eventually, resorting to Theorem 2.27 and owing to the fact that $R^{\bullet} \in \mathcal{W}_2^{(2\rho+2\gamma,2\gamma,\eta,\zeta)}$, we can check that

$$\left|\mathbf{z}^{\overset{\bullet}{\underset{\bullet}{|}},\tau}_{vm} * R^{\bullet,\cdot,\cdot}_{mu}\right| \le \|R^{\bullet}\|_{(2\rho+2\gamma,2\gamma,\eta,\zeta),1,2}\|\!|\mathbf{z}|\!\|^2_{(\alpha,\gamma,\eta,\zeta)}\,|\tau - v|^{-\gamma}\,|v-u|^{4\rho+\gamma}. \tag{4.26}$$

Plugging (4.24), (4.25) and (4.26) into (4.23), we have thus obtained

$$|\delta_m \Xi^{\tau}_{vu}| \lesssim \left\|\left(y, y^{\bullet}, y^{\overset{\bullet}{\underset{\bullet}{|}}}, y^{\bullet\bullet}\right)\right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} \|\!|\mathbf{z}|\!\|^2_{(\alpha,\gamma,\eta,\zeta)}|\tau - v|^{-\gamma}|v-u|^{4\rho+\gamma}. \tag{4.27}$$

Recall that by assumption, $\rho > \frac{1}{4}$, and therefore $\beta \equiv 4\rho + \gamma > 1$. We have thus obtained that $\|\delta\Xi\|_{(\beta,\gamma),1} < \infty$. Following along the same lines above, it is readily checked that also $\|\delta\Xi\|_{(\beta,\gamma,\eta,\zeta),1,2} < \infty$. Therefore we apply directly the Volterra sewing Lemma 2.13 in order to achieve the claims in (4.13), (4.14) and (4.15).

We now proceed to prove the last claim, *(iv)* . To this aim, observe that the bound in (4.14) together with the fact that $\mathbf{z}^{\overset{\bullet}{\underset{\bullet}{|}}}, \mathbf{z}^{\overset{\bullet\,\bullet}{\vee}} \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$ for all $(\eta,\zeta) \in \mathcal{A}_N$, implies the existence of a function $R \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}(\mathbb{R}^m)$ such that

$$w^{\tau}_{ts} = \mathbf{z}^{\bullet,\tau}_{ts} * y^{\cdot}_s + \mathbf{z}^{\overset{\bullet}{\underset{\bullet}{|}},\tau}_{ts} * y^{\bullet,\cdot,\cdot}_s + R^{\tau}_{ts}. \tag{4.28}$$

From (4.28) it is readily seen that w^{τ} can be decomposed as a controlled Volterra path in $\mathcal{D}^{(\alpha,\gamma)}_{\mathbf{z}}(\Delta_2, \mathbb{R}^m)$ with $w^{\bullet,\tau,p}_t = y^p_t$, $w^{\overset{\bullet}{\underset{\bullet}{|}},\tau,q,p}_t = y^{\bullet,q,p}_t$, $w^{\bullet\bullet,\tau,q,p}_t = 0$. This finishes our proof. □

Remark 4.12. From Theorem 4.10 (d), we know that the process w defined by (4.13) satisfies

$$w^{\bullet,\tau,p}_t = y^p_t = w^{\bullet,p}_t.$$

Therefore $w^{\bullet}$ depends on two variables instead of 3 variables in the general definition (4.9). In the same way, we have

$$w^{\overset{\bullet}{\underset{\bullet}{|}},\tau,q,p}_t = y^{\bullet,q,p}_t = w^{\overset{\bullet}{\underset{\bullet}{|}},q,p}_t,$$

that is, $w^{\overset{\bullet}{\underset{\bullet}{|}}}$ depends on three variables (vs 4 variables in the general definition (4.9)). Therefore we can refine Theorem 4.10 and state that the Volterra rough integration sends $(y, y^{\bullet}, y^{\overset{\bullet}{\underset{\bullet}{|}}}, y^{\bullet\bullet}) \in \mathcal{D}^{(\alpha,\gamma)}_{\mathbf{z}}(\mathbb{R}^m)$ to a controlled process $(w, w^{\bullet}, w^{\overset{\bullet}{\underset{\bullet}{|}}}, 0) \in \hat{\mathcal{D}}^{(\alpha,\gamma)}_{\mathbf{z}}(\mathbb{R}^m)$, where the space $\hat{\mathcal{D}}^{(\alpha,\gamma)}_{\mathbf{z}}(\mathbb{R}^m)$ is defined by

$$\hat{\mathcal{D}}^{(\alpha,\gamma)}_{\mathbf{z}}(\mathbb{R}^m) := \Big\{ \left(w, w^{\bullet}, w^{\overset{\bullet}{\underset{\bullet}{|}}}, 0\right) \in \mathcal{D}^{(\alpha,\gamma)}_{\mathbf{z}}(\Delta_2; \mathbb{R}^m) \,\Big|\, w^{\bullet,\tau,p}_s = w^{\bullet,p}_s, \\ w^{\overset{\bullet}{\underset{\bullet}{|}},\tau,q,p}_s = w^{\overset{\bullet}{\underset{\bullet}{|}},q,p}_s, \text{and } w^{\bullet\bullet,\tau,q,p}_s = 0\Big\}. \tag{4.29}$$

4.1.4. *The composition of a Volterra controlled processes with a smooth function.* With Remark 4.12 in mind, we will now prove that one can compose processes in $\hat{\mathcal{D}}^{(\alpha,\gamma)}_{\mathbf{z}}$ and still get a controlled process.

Proposition 4.13. *Let $f \in C^4_b(\mathbb{R}^d; \mathbb{R}^m)$ and assume $(y, y^{\bullet}, y^{\overset{\bullet}{\underset{\bullet}{|}}}, 0) \in \hat{\mathcal{D}}^{(\alpha,\gamma)}_{\mathbf{z}}(\mathbb{R}^m)$ as given in Remark 4.12. Also recall our Notation 2.16 for matrix products. Then the composition $f(y)$ can be* *seen as a controlled path $(\phi, \phi^{\bullet}, \phi^{\overset{\bullet}{\bullet}}, \phi^{\bullet\bullet})$, where $\phi = f(y)$ and where in the decomposition (4.9) we have*

$$\phi_t^{\bullet,q,p} = y_t^{\bullet,p} f'\left(y_t^q\right), \tag{4.30}$$

$$\phi_t^{\overset{\bullet}{\bullet},r,q,p} = y_t^{\overset{\bullet}{\bullet},q,p} f'\left(y_t^r\right), \quad \textit{and} \quad \phi_t^{\bullet\bullet,r,q,p} = \frac{1}{2}(y_t^{\bullet,q}) \otimes (y_t^{\bullet,p}) f''\left(y_t^r\right). \tag{4.31}$$

Moreover, there exists a constant $C = C_{\alpha,\gamma,\|f\|_{C_b^4}} > 0$ such that for $\mathcal{A}_N$ defined by (2.31) and $(\eta,\zeta) \in \mathcal{A}_N$ we have

$$\|(\phi, \phi^{\bullet}, \phi^{\overset{\bullet}{\bullet}}, \phi^{\bullet\bullet})\|_{\mathbf{z};(\alpha,\gamma,\eta,\zeta)} \leq C(1 + \|\!|\mathbf{z}|\!\|_{(\alpha,\gamma,\eta,\zeta)})^3 \Big[\Big(|y_0^{\bullet}| + |y_0^{\overset{\bullet}{\bullet}}| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} \Big) \vee \Big(|y_0^{\bullet}| + |y_0^{\overset{\bullet}{\bullet}}| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} \Big)^3 \Big]. \tag{4.32}$$

Proof. We separate this proof into two parts: in the first step we will find the appropriate expression for $\phi^{\bullet}$, $\phi^{\overset{\bullet}{\bullet}}$ and $\phi^{\bullet\bullet}$ (namely (4.30) and (4.31)), as well as proving that $(\phi, \phi^{\bullet}, \phi^{\overset{\bullet}{\bullet}}, \phi^{\bullet\bullet}) \in \mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}$. In the second step, we will prove relation (4.32). Without loss of generality, we do the below analysis component-wise for $f(y) = (f_1(y), \ldots, f_m(y))$, where each $f_i : \mathbb{R}^d \to \mathbb{R}$ for $i = 1, \ldots, m$. With a slight abuse of notation, we drop the subscript notation, and still just write $f(y)$ representing each component.

Step 1: Expression for $\phi^{\bullet}$, $\phi^{\overset{\bullet}{\bullet}}$ and $\phi^{\bullet\bullet}$. An elementary application of Taylor's formula enables us to decompose the increment $f(y^\tau)_{ts}$ into

$$f(y^\tau)_{ts} = y_{ts}^\tau f'(y_s^\tau) + \frac{1}{2}(y_{ts}^\tau)^{\otimes 2} f''(y_s^\tau) + r_{ts}^\tau, \tag{4.33}$$

where $r_{ts}^\tau = \frac{1}{6}(y_{ts}^\tau)^{\otimes 3} \int_0^1 f^{(3)}(c_{ts}^\tau(\theta)) d\theta$, where $c_{ts}^\tau(\theta) = \theta y_s^\tau + (1-\theta) y_t^\tau$. It is readily checked from (4.33) that $r \in \mathcal{V}^{(3\alpha,3\gamma,\eta,\zeta)}$. Indeed, it follows directly that

$$\|r\|_{(3\alpha,3\gamma),1} \lesssim \|y\|_{(\alpha,\gamma),1}^3 \|f\|_{C_b^3}.$$

Furthermore, for $(s,t,\tau,\tau') \in \Delta_4$, we have

$$|r_{ts}^{\tau'\tau}| \leq 3|y_{ts}^{\tau'\tau}||(y_{ts}^{\tau'})^{\otimes 2} + (y_{ts}^\tau)^{\otimes 2}|\|f\|_{C_b^3} + \|y\|_{(\alpha,\gamma),1}^3 \|f\|_{C_b^4} (|y_s^{\tau'\tau}| + |y_t^{\tau'\tau}|)(|\tau - t|^{-\gamma}|t-s|^\alpha \wedge |\tau - s|^\rho)^3.$$

It is simply checked that the following inequality hold for all $(\eta,\zeta) \in \mathcal{A}_N$

$$|y_s^{\tau'\tau}| \lesssim |y_0| + \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2} |\tau' - \tau|^\eta |\tau - s|^{-(\eta-\zeta)} \left([|\tau - s|^{\eta-\zeta}|s|^\alpha] \wedge |\tau|^{\rho-\zeta} \right),$$

and thus it follows that

$$\|r\|_{(3\alpha,3\gamma,\eta,\zeta),1,2} \lesssim (\|y\|_{(\alpha,\gamma,\eta,\zeta),1,2} \|y\|_{(\alpha,\gamma),1}^2 + \|y\|_{(\alpha,\gamma),1}^3 (|y_0| + \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2})) \|f\|_{C_b^3}.$$

Combining the above estimates, we get

$$\|r\|_{(3\alpha,3\gamma,\eta,\zeta)} \lesssim (|y_0| + \|y\|_{(\alpha,\gamma,\eta,\zeta)})^3 \|f\|_{C_b^4}. \tag{4.34}$$

Now observe that $(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0) \in \hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma)}$ where $\hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma)}$ is the subset of the space $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}$ as defined in Remark 4.12. In particular, y, $y^{\bullet}$, $y^{\overset{\bullet}{\bullet}}$ satisfy relation (4.9). Then taking squares in the relation (4.9), we end up with

$$(y_{ts}^\tau)^{\otimes 2} = (\mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\tau,\cdot})^{\otimes 2} + \tilde{r}_{ts}^\tau, \tag{4.35}$$

where the reminder term $\tilde{r}^{\tau}_{ts}$ is defined by

$$\tilde{r}^{\tau}_{ts} = (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot}_s)^{\otimes 2} + (R^{y,\tau}_{ts})^{\otimes 2} + (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot}_s) \otimes (\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\tau,\cdot}_s) + (\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\tau,\cdot}_s) \otimes (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot}_s)$$
$$+ (\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\tau,\cdot}_s) \otimes R^{y,\tau}_{ts} + R^{y,\tau}_{ts} \otimes (\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\tau,\cdot}_s) + (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot}_s) \otimes R^{y,\tau}_{ts} + R^{y,\tau}_{ts} \otimes (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot}_s). \quad (4.36)$$

Plugging (4.35) into (4.33), we get

$$f(y^{\tau})_{ts} = y^{\tau}_{ts} f'(y^{\tau}_s) + \frac{1}{2}(\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\cdot}_s)^{\otimes 2} f''(y^{\tau}_s) + \frac{1}{2}\tilde{r}^{\tau}_{ts} f''(y^{\tau}_s) + r^{\tau}_{ts},$$

We now invoke (4.9) in order to further decompose y^{τ}_{ts} above. We end up with

$$f(y^{\tau})_{ts} = \mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\cdot}_s f'(y^{\tau}_s) + \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}_s f'(y^{\tau}_s) + \frac{1}{2}(\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\cdot}_s)^{\otimes 2} f''(y^{\tau}_s) + R^{\phi,\tau}_{ts}, \quad (4.37)$$

where the reminder R^{ϕ} is defined by

$$R^{\phi,\tau}_{ts} = R^{y,\tau}_{ts} f'(y^{\tau}_s) + \frac{1}{2}\tilde{r}^{\tau}_{ts} f''(y^{\tau}_s) + r^{\tau}_{ts}. \quad (4.38)$$

Thanks to (4.37) and the definition of the relations in (4.30)-(4.31), the proof of $(\phi, \phi^{\bullet}, \phi^{\overset{\bullet}{\bullet}}, \phi^{\bullet\bullet}) \in \mathcal{D}^{(\alpha,\gamma)}_{\mathbf{z}}$ are now reduced to proving the following two claims:

- Claim 1: The remainder term $R^{\phi,\tau}_{ts}$ in (4.38) is of order 3. Specifically, due to (4.33) and the fact that $(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0) \in \hat{\mathcal{D}}^{(\alpha,\gamma)}$, relation (4.38) this is reduced to the following claim:
$$\tilde{r} \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}. \quad (4.39)$$
- Claim 2: $\phi^{\bullet}$ fulfills relation (4.10), which can be written as
$$\phi^{\bullet,\cdot,\cdot} - \mathbf{z}^{\bullet,\cdot} * (\phi^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot} + 2\phi^{\bullet\bullet,\cdot,\cdot,\cdot}) \in \mathcal{W}^{(2\rho+2\gamma,2\gamma,\eta,\zeta)}_2, \quad (4.40)$$
where we recall that $\phi^{\bullet}$, $\phi^{\overset{\bullet}{\bullet}}$, $\phi^{\bullet\bullet}$ are defined by (4.30)-(4.31).

In the following, we will prove those two Claims separately.

Proof of Claim 1. According to relation (4.36), there are eight terms to evaluate in $\tilde{r}$. For conciseness, we can consider one of these terms, say the increment I^{τ}_{ts} defined by $I^{\tau}_{ts} = (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot}_s) \otimes (\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\tau,\cdot}_s)$, and the remaining terms will follow directly from similar considerations. To this aim, a first observation is that since $(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0) \in \hat{\mathcal{D}}^{(\alpha,\gamma)}_{\mathbf{z}}$ as given in (4.29), then both $y^{\bullet}$ and $y^{\overset{\bullet}{\bullet}}$ don't dependent on τ and we have $I^{\tau}_{ts} = (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}_s) \otimes (\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\cdot}_s)$. Moreover, due to the fact that I is part of the reminder $\tilde{r}$, we have to evaluate $\|I\|_{(3\rho+2\gamma,2\gamma,\eta,\zeta)}$. Owing to Definition 2.4, this is equivalent to evaluate

$$\|I\|_{(3\rho+2\gamma,2\gamma,\eta,\zeta)} = \|I\|_{(3\rho+2\gamma,2\gamma),1} + \|I\|_{(3\rho+2\gamma,2\gamma,\eta,\zeta),1,2}. \quad (4.41)$$

In order to upper bound the right hand side of (4.41), it suffices to estimate $|I^{\tau}_{ts}|$ and $|I^{qp}_{ts}|$. Some elementary computations reveal that

$$|I^{\tau}_{ts}| = \left|\left(\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}_s\right) \otimes (\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\cdot}_s)\right| \lesssim \left|\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}_s\right| \left|\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\cdot}_s\right|, \quad (4.42)$$

and

$$|I^{qp}_{ts}| = |I^{q}_{ts} - I^{p}_{ts}| = \left|\left(z^{\overset{\bullet}{\bullet},q}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}_s\right) \otimes (\mathbf{z}^{\bullet,q}_{ts} * y^{\bullet,\cdot}_s) - \left(\mathbf{z}^{\overset{\bullet}{\bullet},p}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}_s\right) \otimes (\mathbf{z}^{\bullet,p}_{ts} * y^{\bullet,\cdot}_s)\right|$$
$$\lesssim \left|\mathbf{z}^{\bullet,qp}_{ts} * y^{\bullet,\cdot}_s\right| \left|\mathbf{z}^{\overset{\bullet}{\bullet},q}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}_s\right| + \left|\mathbf{z}^{\overset{\bullet}{\bullet},qp}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}_s\right| \left|\mathbf{z}^{\bullet,q}_{ts} * y^{\bullet,\cdot}_s\right|. \quad (4.43)$$

To bound the right hand side of (4.42), thanks to a slight variation of Proposition 2.17 and Theorem 2.27, we have

$$|I^{\tau}_{ts}| \lesssim (|y^{\overset{\bullet}{\bullet}}_0| + |y^{\bullet}_0| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)})^2 \vert\!\vert\!\vert \mathbf{z} \vert\!\vert\!\vert^2_{(\alpha,\gamma,\eta,\zeta)}\, \psi^1_{(3\rho+2\gamma),2\gamma}(\tau, t, s).$$

Similarly, we can bound $|I^{qp}_{ts}|$ is the following way:

$$|I^{qp}_{ts}| \lesssim (|y^{\overset{\bullet}{\bullet}}_0| + |y^{\bullet}_0| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)})^2 \vert\!\vert\!\vert \mathbf{z} \vert\!\vert\!\vert^2_{(\alpha,\gamma,\eta,\zeta)}\, \psi^{1,2}_{(3\rho+2\gamma),2\gamma,\eta,\zeta}(p, q, t, s) \tag{4.44}$$

It follows by definition of the quantities $\|I\|_{(3\rho+2\gamma,2\gamma),1}$ and $\|I\|_{(3\rho+2\gamma,2\gamma,\eta,\zeta),1,2}$ as given in Definition 2.4, that

$$\|I\|_{(3\rho+2\gamma,2\gamma),1} \vee \|I\|_{(3\rho+2\gamma,2\gamma,\eta,\zeta),1,2} \lesssim (|y^{\overset{\bullet}{\bullet}}_0| + |y^{\bullet}_0| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{(\alpha,\gamma,\eta,\zeta)})^2 \vert\!\vert\!\vert \mathbf{z} \vert\!\vert\!\vert^2_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} \tag{4.45}$$

which implies $I \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$ according to Lemma 2.9. Similarly, we let the patient reader check that

$$(\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\tau,\cdot}_s) \otimes (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot}_s) \in \mathcal{V}^{(3\rho+2\gamma,2\gamma,\eta,\zeta)}, \quad (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}_s)^{\otimes 2} \in \mathcal{V}^{(4\rho+2\gamma,2\gamma,\eta,\zeta)}, \quad (R^y_{ts})^{\otimes 2} \in \mathcal{V}^{(6\rho+6\gamma,6\gamma,\eta,\zeta)}, \tag{4.46}$$

as well as

$$(\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\cdot}_s) \otimes R^y_{ts} \in \mathcal{V}^{(4\rho+4\gamma,4\gamma,\eta,\zeta)}, \quad (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}_s) \otimes R^y_{ts} \in \mathcal{V}^{(5\rho+4\gamma,4\gamma,\eta,\zeta)}. \tag{4.47}$$

In fact the appropriate norm for each of these terms is easily seen to be bounded by the product $(\vert\!\vert\!\vert \mathbf{z} \vert\!\vert\!\vert_{(\alpha,\gamma,\eta,\zeta)} + \vert\!\vert\!\vert \mathbf{z} \vert\!\vert\!\vert^2_{(\alpha,\gamma,\eta,\zeta)})(|y^{\overset{\bullet}{\bullet}}_0| + |y^{\bullet}_0| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{(\alpha,\gamma,\eta,\zeta)})^2$. Combining (4.45), (4.46) and (4.47), we have thus obtained that $\tilde{r} \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$, and it follows that

$$\|\tilde{r}\|_{(3\rho+3\gamma,\gamma,\eta,\zeta)} \lesssim (\vert\!\vert\!\vert \mathbf{z} \vert\!\vert\!\vert_{(\alpha,\gamma,\eta,\zeta)} + \vert\!\vert\!\vert \mathbf{z} \vert\!\vert\!\vert^2_{(\alpha,\gamma,\eta,\zeta)})(|y^{\overset{\bullet}{\bullet}}_0| + |y^{\bullet}_0| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)})^2 \tag{4.48}$$

Proof of Claim 2. Before proving relation (4.40), we will give some algebraic insight on the terms of ϕ^{σ} for $\sigma \in \mathcal{F}_2$. Indeed, resorting to (4.37), we can safely set

$$\phi^{\bullet,q,p}_t = y^{\bullet,p}_t f'(y^q_t), \tag{4.49}$$

as stated in (4.30). According to (4.37), we also let

$$\phi^{\overset{\bullet}{\bullet},r,q,p}_t = y^{\overset{\bullet}{\bullet},q,p}_t f'(y^r_t), \qquad \phi^{\bullet\bullet,r,q,p}_t = \frac{1}{2}(y^{\bullet,q}_t) \otimes (y^{\bullet,p}_t) f''(y^r_t). \tag{4.50}$$

With relation (4.9) in mind, we can rewrite (4.37) as

$$f(y^{\tau}_t) - f(y^{\tau}_s) = \mathbf{z}^{\bullet,\tau}_{ts} * \phi^{\bullet,\tau,\cdot}_s + \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * \phi^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot}_s + (\mathbf{z}^{\bullet,\tau}_{ts})^{\otimes 2} * \phi^{\bullet\bullet,\tau,\cdot,\cdot}_s + R^{\phi,\tau}_{ts} \tag{4.51}$$

Let us briefly give a few details regarding the expressions on the right hand side of (4.51). Specifically, we will explain how to compute $\mathbf{z}^{\bullet,\tau}_{ts} * \phi^{\bullet,\tau,\cdot}_s = \mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\cdot}_s f'(y^{\tau}_s)$. Referring to Notation 2.16, the expression $\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\cdot}_s f'(y^{\tau}_s)$ can be rewritten as $[(\mathbf{z}^{\bullet,\tau}_{ts})^{\intercal} * (y^{\bullet,\cdot}_s f'(y^{\tau}_s))^{\intercal}]^{\intercal} = f'(y^{\tau}_s) y^{\bullet,\cdot}_s * \mathbf{z}^{\bullet,\tau}_{ts}$, where we have used also that $y^{\bullet,\cdot}_s f'(y^{\tau}_s)$ can be rewritten as $[(y^{\bullet,\cdot}_s)^{\intercal}(f'(y^{\tau}_s))^{\intercal}]^{\intercal} = f'(y^{\tau}_s) y^{\bullet,\cdot}_s$. In addition, notice that $f'(y_s) \in \mathbb{R}^m$, $y^{\bullet}_s \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m)$ and $\mathbf{z}^{\bullet,\tau}_{ts} \in \mathbb{R}^d$. Therefore the quantity $\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\cdot}_s f'(y^{\tau}_s)$ has to be interpreted as an inner product, and we let the patient reader perform the same kind of manipulation for the term $z^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet}} f'(y^{\tau}_s)$. In the end we get that both the left hand side and the right hand side of (4.51) are real-valued.

Now we are ready to prove (4.40). To this aim we set

$$J^{\tau,\cdot}_{ts} := \phi^{\bullet,\tau,\cdot}_{ts} - \mathbf{z}^{\bullet,\tau}_{ts} * (\phi^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot}_s + 2\phi^{\bullet\bullet,\tau,\cdot,\cdot}_s). \tag{4.52}$$

Our claim (4.40) amounts to show that $J \in \mathcal{W}_2^{(2\rho+2\gamma,\gamma,\eta,\zeta)}$, with $\mathcal{W}_2^{(2\rho+2\gamma,2\gamma,\eta,\zeta)}$ given in Definition 2.25. Thanks to (4.49) and (4.50), we first write

$$J_{ts}^{\tau,\cdot} = y_t^{\bullet,\cdot}\left(f'(y_t^\tau) - f'(y_s^\tau)\right) + y_{ts}^{\bullet,\cdot} f'(y_s^\tau) - \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot} f'(y_s^\tau) - \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\cdot} f''\left(y_s^\tau\right). \tag{4.53}$$

We now invoke (4.10), recalling that $y^{\bullet\bullet} = 0$ since we have assumed that $y \in \hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma)}$. Plugging this information in (4.53), we end up with

$$J_{ts}^{\tau,\tau} = y_t^{\bullet,\tau}\left(f'(y_t^\tau) - f'(y_s^\tau)\right) + R_{ts}^{\bullet,\tau} f'(y_s^\tau) - \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\tau} f''(y_s^\tau). \tag{4.54}$$

Let us apply a Taylor expansion to the first term of right hand side of (4.54). Specifically we write

$$f'(y_t^\tau) - f'(y_s^\tau) - y_{ts}^\tau f''(y_s^\tau) = F_{ts}^{(2),\tau},$$

where the term $F_{ts}^{(2),\tau} = (F_{ts}^{(2),\tau,1}, \dots, F_{ts}^{(2),\tau,d})$ is defined as a reminder in a Taylor expansion. Namely consider multi-indices $\beta = (\beta_1, \dots, \beta_d)$ with $\beta_i \in \{0,1,2\}$. We set $|\beta| = \sum_{j=1}^d \beta_j$ and $|\beta|! = \Pi_{j=1}^d \beta_j!$. Then for $i = 1, \dots, d$, $F_{ts}^{(2),\tau,i}$ is given by

$$F_{ts}^{(2),\tau,i} = 2 \sum_{|\beta|=2} \frac{(y_{ts}^\tau)^{\otimes|\beta|}}{\beta!} \int_0^1 (1-r)\partial^\beta\left(\partial_i f(y_s^\tau + r y_{st}^\tau)\right) dr. \tag{4.55}$$

With expression (4.55) in hand and recalling (4.54), we thus get

$$\begin{aligned} J_{ts}^{\tau,\tau} &= y_t^{\bullet,\tau}\left(f'(y_t^\tau) - f'(y_s^\tau) - y_{ts}^\tau f''(y_s^\tau)\right) + y_t^{\bullet,\tau} y_{ts}^\tau f''(y_s^\tau) - \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\tau} f''(y_s^\tau) + R_{ts}^{\bullet,\tau} f'(y_s^\tau) \\ &= y_t^{\bullet,\tau} F_{ts}^{(2),\tau} + R_{ts}^{\bullet,\tau} f'(y_s^\tau) + y_t^{\bullet,\tau} y_{ts}^\tau f''(y_s^\tau) - \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\tau} f''(y_s^\tau), \end{aligned} \tag{4.56}$$

Furthermore, we plug in identity (4.9) in the above expansion in order to expand the term $y_t^{\bullet,\tau} y_{ts}^\tau f''(y_s^\tau)$ in (4.56). This yields

$$\begin{aligned} J_{ts}^{\tau,\tau} &= y_t^{\bullet,\tau} F_{ts}^{(2),\tau} + R_{ts}^{\bullet,\tau} f'(y_s^\tau) + y_t^{\bullet,\tau} \mathbf{z}_{ts}^{\overset{\bullet}{\bullet},\tau} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot} f''(y_s^\tau) + y_t^{\bullet,\tau} R_{ts}^\tau f''(y_s^\tau) \\ &\quad + y_t^{\bullet,\tau} \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} f''(y_s^\tau) - \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\tau} f''(y_s^\tau). \end{aligned} \tag{4.57}$$

Next we resort to the forthcoming identity (4.70) in order to handle the term $\mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\tau} f''(y_s^\tau)$ above. One obtains that the last two terms in (4.57) combine into one term $y_{ts}^{\bullet,\tau} \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} f''(y_s^\tau)$. We end up with

$$\begin{aligned} J_{ts}^{\tau,\tau} =& y_t^{\bullet,\tau} F_{ts}^{(2),\tau} + R_{ts}^{\bullet,\tau} f'(y_s^\tau) + y_t^{\bullet,\tau} \mathbf{z}_{ts}^{\overset{\bullet}{\bullet},\tau} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot} f''(y_s^\tau) \\ &+ y_t^{\bullet,\tau} R_{ts}^\tau f''(y_s^\tau) + y_{ts}^{\bullet,\tau} \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} f''(y_s^\tau). \end{aligned} \tag{4.58}$$

In the same way, we let the patient reader check that we can rewrite $J_{ts}^{q,q} - J_{ts}^{p,p}$ as

$$J_{ts}^{q,q} - J_{ts}^{p,p} = J_1^{qp} + J_2^{qp} + J_3^{qp} + J_4^{qp} + J_5^{qp}, \tag{4.59}$$

where the terms J_1^{qp}, J_2^{qp}, J_3^{qp}, J_4^{qp}, and J_5^{qp} are defined respectively by

$$\begin{aligned} J_1^{qp} &= y_t^{\bullet,qp} F_{ts}^{(2),q} + y_t^{\bullet,p}\left(F_{ts}^{(2),q} - F_{ts}^{(2),p}\right) \\ J_2^{qp} &= \left(y_t^{\bullet,p} \mathbf{z}_{ts}^{\overset{\bullet}{\bullet},qp} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot} + y_t^{\bullet,qp} \mathbf{z}_{ts}^{\overset{\bullet}{\bullet},q} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}\right) f''(y_s^q) + \left(y_t^{\bullet,p} \mathbf{z}_{ts}^{\overset{\bullet}{\bullet},p} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot}\right)\left(f''(y_s^q) - f''(y_s^p)\right) \\ J_3^{qp} &= \left(y_t^{\bullet,p} R_{ts}^{qp} + y_t^{\bullet,qp} R_{ts}^q\right) f''(y_s^q) + y_t^{\bullet,p} R_{ts}^p \left(f''(y_s^q) - f''(y_s^p)\right), \\ J_4^{qp} &= \left(y_{ts}^{\bullet,qp} \mathbf{z}_{ts}^{\bullet,q} * y_s^{\bullet,\cdot} + y_{ts}^{\bullet,p} \mathbf{z}_{ts}^{\bullet,qp} * y_s^{\bullet,\cdot}\right) f''(y_s^q) + y_{ts}^{\bullet,p} \mathbf{z}_{ts}^{\bullet,p} * y_s^{\bullet,\cdot}\left(f''(y_s^q) - f''(y_s^p)\right) \\ J_5^{qp} &= R_{ts}^{\bullet,qp} f'(y_s^q) + R_{ts}^{\bullet,p}\left(f'(y_s^q) - f'(y_s^p)\right). \end{aligned} \tag{4.60}$$

With (4.58)-(4.60) at hand, and recalling Definition 2.25 for the spaces $\mathcal{W}$, it is readily checked, using the information of the regularities in the different terms of J_i for $i = 1, \dots, 5$ that $J \in \mathcal{W}_2^{(2\rho+2\gamma,2\gamma,\eta,\zeta)}$. We omit further details, as the arguments follows directly along the same lines as in previous computations in the proof of claim 1.

Summarizing our analysis so far, we have now proved both Claim 1 and Claim 2 above. Therefore we obtain that $(\phi, \phi^{\bullet}, \phi^{\stackrel{\bullet}{\bullet}}, \phi^{\bullet\bullet})$ is an element of $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}$.

Step 2: Proof of relation (4.32). According to the definition (4.11) for the norm in $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}$, we have

$$\|(\phi, \phi^{\bullet}, \phi^{\stackrel{\bullet}{\bullet}}, \phi^{\bullet\bullet})\|_{\mathbf{z};(\alpha,\gamma,\eta,\zeta)} = \|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)} + \|\phi^{\bullet\bullet}\|_{(\alpha,\gamma,\eta,\zeta)} + \|R^{\phi}\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)} + \|R^{\phi^{\bullet}}\|_{(2\rho+2\gamma,2\gamma,\eta,\zeta)}. \quad (4.61)$$

In the following, we will bound four terms in the right hand side of (4.61) separately.

We begin to handle the term $\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)}$ in (4.61). We recall that $\phi^{\stackrel{\bullet}{\bullet}}$ is given by (4.31), and its $(\alpha, \gamma, \eta, \zeta)$-norm is introduced in Definition 3.6. According to this definition, it is thus enough to bound $\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma),1}$ and $\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$. Towards this aim, we write

$$\begin{aligned} \left|\phi_{ts}^{\stackrel{\bullet}{\bullet},\tau,\tau,\tau}\right| &= \left|y_t^{\stackrel{\bullet}{\bullet},\tau,\tau} f'(y_t^{\tau}) - y_s^{\stackrel{\bullet}{\bullet},\tau,\tau} f'(y_s^{\tau})\right| = \left|y_t^{\stackrel{\bullet}{\bullet},\tau,\tau} \left(f'(y_t^{\tau}) - f'(y_s^{\tau})\right) + y_{ts}^{\stackrel{\bullet}{\bullet},\tau,\tau} f'(y_s^{\tau})\right| \\ &\lesssim \|f\|_{C_b^2} (|y_0^{\stackrel{\bullet}{\bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2})\, \psi^1_{\alpha,\gamma}(\tau, t, s), \end{aligned} \quad (4.62)$$

where ψ^1 as given in (2.1). This yields

$$\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma),1} \lesssim \|f\|_{C_b^2} (|y_0^{\stackrel{\bullet}{\bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2}). \quad (4.63)$$

We now wish to handle the norm $\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$ in (3.16). Otherwise stated, we wish to bound the terms in the right hand side of (3.18) for $\phi^{\stackrel{\bullet}{\bullet}}$. For the term $\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2,>}$, we thus write

$$\begin{aligned} \left|\phi_{ts}^{\stackrel{\bullet}{\bullet},p',p_2,p} - \phi_{ts}^{\stackrel{\bullet}{\bullet},p',p_1,p}\right| &= \left|y_t^{\stackrel{\bullet}{\bullet},p_2,p} f'(y_t^{p'}) - y_s^{\stackrel{\bullet}{\bullet},p_2,p} f'(y_s^{p'}) - y_t^{\stackrel{\bullet}{\bullet},p_1,p} f'(y_t^{p'}) + y_s^{\stackrel{\bullet}{\bullet},p_1,p} f'(y_s^{p'})\right| \\ &\le \left|\left(y_{ts}^{\stackrel{\bullet}{\bullet},p_2,p} - y_{ts}^{\stackrel{\bullet}{\bullet},p_1,p}\right) f'(y_t^{p'})\right| + \left|\left(y_s^{\stackrel{\bullet}{\bullet},p_2,p} - y_s^{\stackrel{\bullet}{\bullet},p_1,p}\right)\left(f'(y_t^{p'}) - f'(y_s^{p'})\right)\right|. \end{aligned}$$

In addition, owing to Remark 4.9 and (3.21) and since $\mathbf{y} \in \hat{D}_{\mathbf{z}}^{(\alpha,\gamma)}$, we have $y^{\stackrel{\bullet}{\bullet}} \in \mathcal{W}_2^{(\alpha,\gamma,\eta,\zeta)}$. Due to the fact that y is also an element of $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ according to Remark 4.8, we get

$$\left|\phi_{ts}^{\stackrel{\bullet}{\bullet},p',p_2,p} - \phi_{ts}^{\stackrel{\bullet}{\bullet},p',p_1,p}\right| \lesssim \|f\|_{C_b^2} (|y_0^{\stackrel{\bullet}{\bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2})\, \psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(p_2, p_1, t, s), \quad (4.64)$$

where $\psi^{1,2}$ as given in (2.2). We thus have

$$\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2} \lesssim \|f\|_{C_b^2} (|y_0^{\stackrel{\bullet}{\bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2}). \quad (4.65)$$

Moreover, it is easily seen that $\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,3}$ and $\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),2,3}$ are bounded exactly in the same way as (4.65). Hence we get the following bound for $\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$:

$$\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \lesssim \|f\|_{C_b^2} (|y_0^{\stackrel{\bullet}{\bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2}). \quad (4.66)$$

Eventually, plugging (4.63) and (4.66) into (3.16), we obtain the desired bound for $\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)}$:

$$\|\phi^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)} \lesssim \|f\|_{C_b^2} (|y_0^{\stackrel{\bullet}{\bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\stackrel{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2}). \quad (4.67)$$

We let the reader check that the term $\|\phi^{\bullet\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}$ in (4.61) can be treated in a similar way. Indeed, $\phi^{\bullet\bullet}$ has to be considered as a process in $\mathcal{W}_3$, exactly like $\phi^{\stackrel{\bullet}{\bullet}}$. Therefore owing to the definition (4.31) of $\phi^{\bullet\bullet}$ and to the definition (3.18) of the $(1,2,3)$-norm in $\mathcal{W}_3$, we get the following bound along the same lines as (4.62)-(4.67):

$$\|\phi^{\bullet\bullet}\|_{(\alpha,\gamma,\eta,\zeta)} \lesssim \|f\|_{C_b^2}(|y_0^{\substack{\bullet\\\bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\substack{\bullet\\\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2}). \tag{4.68}$$

We are now ready to bound the fourth term $\|R^{\phi^{\bullet}}\|_{(2\rho+2\gamma,2\gamma,\eta,\zeta)}$ in the right hand side of (4.61). To this aim, recall that according to (4.10) we have

$$R_{ts}^{\bullet,\tau,p} = y_{ts}^{\bullet,\tau,p} - \mathbf{z}_{ts}^{\bullet,\tau} * \left(y_s^{\substack{\bullet\\\bullet},\tau,p,\cdot} + 2y_s^{\bullet\bullet,\tau,p,\cdot}\right).$$

Comparing this expression to (4.52), we get $R^{\phi^{\bullet}} = J$. Now recall that J has been analyzed through a decomposition in (4.58)-(4.60). Note that all the terms appearing in the decomposition are directly bounded due to the fact that $(y, y^{\bullet}, y^{\substack{\bullet\\\bullet}}, 0) \in \hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma)}$. It is therefore readily checked that

$$\|R^{\phi^{\bullet}}\|_{(2\rho+2\gamma,2\gamma,\eta,\zeta)} \leq C(1 + \|\!|\mathbf{z}|\!\|_{(\alpha,\gamma,\eta,\zeta)})^3 \Big[\left(|y_0^{\bullet}| + |y_0^{\substack{\bullet\\\bullet}}| + \|(y, y^{\bullet}, y^{\substack{\bullet\\\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\right) \\ \vee \left(|y_0^{\bullet}| + |y_0^{\substack{\bullet\\\bullet}}| + \|(y, y^{\bullet}, y^{\substack{\bullet\\\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\right)^3 \Big].$$

Eventually, we handle the term $\|R^{\phi}\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$ in (4.61). Recall that R^{ϕ} is given by (4.38), and that we have already bounded the term r and $\tilde{r}$ in (4.34) and (4.48) respectively. Furthermore, it follows directly that $\|R^{y}\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)} \leq \|(y, y^{\bullet}, y^{\substack{\bullet\\\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma.\eta,\zeta)}$. Combining the above considerations, we see that

$$\|R^{\phi}\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)} \lesssim (1 + \|\!|\mathbf{z}|\!\|_{(\alpha,\gamma,\eta,\zeta)})^3 \Big[\left(|y_0| + |y_0^{\bullet}| + |y_0^{\substack{\bullet\\\bullet}}| + \|(y, y^{\bullet}, y^{\substack{\bullet\\\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\right) \\ \vee \left(|y_0| + |y_0^{\bullet}| + |y_0^{\substack{\bullet\\\bullet}}| + \|(y, y^{\bullet}, y^{\substack{\bullet\\\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\right)^3 \Big].$$

Gathering the bounds found above, it is now evident that

$$\|(\phi, \phi^{\bullet}, \phi^{\substack{\bullet\\\bullet}}, \phi^{\bullet\bullet})\|_{\mathbf{z};(\alpha,\gamma,\eta,\zeta)} \lesssim \left(1 + \|\!|\mathbf{z}|\!\|_{(\alpha,\gamma,\eta,\zeta)}\right)^3 \Bigg[\left(|y_0| + |y_0^{\bullet}| + |y_0^{\substack{\bullet\\\bullet}}| + \|(y, y^{\bullet}, y^{\substack{\bullet\\\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\right) \\ \vee \left(|y_0| + |y_0^{\bullet}| + |y_0^{\substack{\bullet\\\bullet}}| + \|(y, y^{\bullet}, y^{\substack{\bullet\\\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\right)^3 \Bigg],$$

where the hidden constant depends on $\|f\|_{C_b^4}$, α, and γ. The above relation is exactly (4.32), which concludes our proof. □

Remark 4.14. In Proposition 4.13 we have obtained useful bounds on the composition map from $\hat{D}_{\mathbf{z}}^{(\alpha,\gamma)}(\Delta_2([0,T]); \mathbb{R}^d)$ to $D_{\mathbf{z}}^{(\alpha,\gamma)}(\Delta_2([0,T]); \mathbb{R}^m)$. Let us now choose a parameter β such that $\beta < \alpha$ and we still have $\beta - \gamma > \frac{1}{4}$. We will in the next section consider the composition map from $\hat{\mathcal{D}}_{\mathbf{z}}^{(\beta,\gamma)}(\Delta_2([0,T]); \mathbb{R}^d)$ to $\mathcal{D}_{\mathbf{z}}^{(\beta,\gamma)}(\Delta_2([0,T]); \mathbb{R}^m)$. Due to Remark 2.7, it is readily checked that there exists a constant $C = C_{M,\alpha,\beta,\gamma,\eta,\zeta,\|f\|_{C_b^5}}$ such that,

$$\|(\phi, \phi^{\bullet}, \phi^{\substack{\bullet\\\bullet}}, \phi^{\bullet\bullet})\|_{\mathbf{z};(\beta,\gamma,\eta,\zeta)} \leq C\left(1 + \|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)}\right)^3 \left(\left[|y_0^{\bullet}| + |y_0^{\substack{\bullet\\\bullet}}| + \|(y, y^{\bullet}, y^{\substack{\bullet\\\bullet}}, 0)\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}\right]\right) \\ \vee \left(\left[|y_0^{\bullet}| + |y_0^{\substack{\bullet\\\bullet}}| + \|(y, y^{\bullet}, y^{\substack{\bullet\\\bullet}}, 0)\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}\right]^3\right) T^{\alpha-\beta}. \tag{4.69}$$

We close this section by presenting a technical result which leads to some useful cancellations in the rough path expansion (4.57).

Lemma 4.15. *Let $f \in C_b^4(\mathbb{R}^m)$ and assume $(y, y^{\bullet}, y^{\substack{\bullet\\ \bullet}}, 0) \in \hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma)}(\mathbb{R}^m)$ as given in Remark 4.12. Also recall our Notation 2.16 for matrix products. Then for any $(s,t,\tau) \in \Delta_3$, we have*

$$y_s^{\bullet,\tau}\mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} f''(y_s^{\tau}) = \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\tau} f''(y_s^{\tau}) \tag{4.70}$$

Proof. Let L (respectively M) be the left hand side (respectively right hand side) of (4.70) . Recalling the dimension considerations after equation (4.51), notice that both L and M are elements of $\mathbb{R}^{m\times m}$. For $a \in \mathbb{R}^m$, we consider the matrix products aL and aM in the sense of Notation 2.16. In particular, our Notation 2.16 implies that aL has to be interpreted as $f''(y_s^{\tau})y_s^{\bullet,\cdot} * \mathbf{z}_{ts}^{\bullet,\tau} y_s^{\bullet,\tau} a$. Expressing this in coordinates we get

$$\begin{aligned} aL &= \sum_{i,j=1}^{m} f''(y_s^{\tau})^{ij} \sum_{i_1=1}^{m} y_s^{\bullet,\cdot,ii_1}\mathbf{z}_{ts}^{\bullet,\tau,i_1} \sum_{j_1=1}^{m} y_s^{\bullet,\tau,jj_1}a^{j_1} \\ &= \sum_{i,j,i_1,j_1=1}^{m} f''(y_s^{\tau})^{ij} y_s^{\bullet,\cdot,ii_1}\mathbf{z}_{ts}^{\bullet,\tau,i_1} y_s^{\bullet,\tau,jj_1}a^{j_1}. \end{aligned} \tag{4.71}$$

Similarly, the product aM can be expressed as

$$aM = f''(y_s^{\tau})^{ij} y_s^{\bullet,\tau} \otimes y_s^{\bullet,\tau} * \mathbf{z}_{ts}^{\bullet,\cdot} \cdot a = \sum_{i,j,i_1,j_1=1}^{m} f''(y_s^{\tau})^{ij} y_s^{\bullet,\cdot,ii_1} y_s^{\bullet,\tau,jj_1}\mathbf{z}_{ts}^{\bullet,\tau,i_1}a^{j_1}. \tag{4.72}$$

Comparing (4.71) and (4.72), it is clear that $aL = aM$ for any $a \in R^m$. Thus $L = M$, which finishes the proof. ☐

4.2. Rough Volterra Equations.

In this section we gather all the element of stochastic calculus put forward in Sections 3.2-4.1, in order to achieve one of main goals in this paper. Namely we will solve Volterra type equations in a very rough setting. We start by introducing a new piece of notation.

Notation 4.16. *Let us define a new space $\mathcal{D}_{\mathbf{z};\mathbf{y}_0}^{(\beta,\gamma)}\left(\Delta_2^T\left([0,\bar{T}]\right);\mathbb{R}^m\right)$, where $\mathbf{y}_0$ is of the form $(y_0, y_0^{\bullet}, y_0^{\substack{\bullet\\ \bullet}}, y_0^{\bullet\bullet})$. For $0 \le a < b \le T$ we define a simplex type set $\Delta_2^T([a,b])$ as follows,*

$$\Delta_2^T([a,b]) = \left\{(s,\tau) \in [a,b]\times[0,T] \,\middle|\, a \le s \le \tau \le T\right\}. \tag{4.73}$$

Note that the first component of $(s,\tau) \in \Delta_2^T([a,b])$ is restricted to $[a,b]$ while the second component is allowed to vary in the whole interval $[0,T]$. Without loss of generality, we assume that $\|\mathbf{z}\|_{(\alpha,\gamma)} \le M \in \mathbb{R}_+$. As in Remark 4.14, we choose a parameter $\beta < \alpha$ but still satisfying $\beta - \gamma > 1/4$. Let us also consider a time horizon $\bar{T} \le T$ (this $\bar{T}$ will be made small enough to perform a contraction argument later on). We will work on a space $\mathcal{D}_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma)}(\Delta_2^T([0,\bar{T}]);\mathbb{R}^m)$ defined by

$$\mathcal{D}_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma)}(\Delta_2^T([0,\bar{T}]);\mathbb{R}^m) = \Big\{ \left(y, y^{\bullet}, y^{\substack{\bullet\\ \bullet}}, y^{\bullet\bullet}\right) \in \mathcal{D}_{\mathbf{z}}^{(\beta,\gamma)}\left(\Delta_2^T\left([0,\bar{T}]\right);\mathbb{R}^m\right) \Big| \\ \mathbf{y}_0 = \{y_0^{\tau}, y_0^{\bullet,\tau}, y_0^{\substack{\bullet\\ \bullet},\tau,\tau}, y_0^{\bullet\bullet,\tau,\tau}\} = \{y_0, y_0^{\bullet}, y_0^{\substack{\bullet\\ \bullet}}, y_0^{\bullet\bullet}\}\Big\}. \tag{4.74}$$

Notice that the norm on $D_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma)}$ is still defined by (4.11). The only difference between $D_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma)}$ and $D_{\mathbf{z}}^{(\beta,\gamma)}$ in Definition 4.7 is that $D_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma)}$ has an affine space structure, in contrast with the Banach space nature of $D_{\mathbf{z}}^{(\beta,\gamma)}$.

Remark 4.17. As the reader can see, the main aim of the definition encoded by (4.74) is to have fixed initial conditions for controlled processes. This slight variation on the notion of controlled path will be beneficial in order to set up our fixed point argument below. Notice that it is a standard practice in rough paths theory to have fixed initial conditions, as addressed in [15, Section 8.5].

We are now ready to solve Volterra type equations in the rough case $\alpha-\gamma>\frac{1}{4}$.

Theorem 4.18. *Consider a path $x\in\mathcal{C}^{\alpha}([0,T];\mathbb{R}^d)$ and a Volterra kernel $k:\Delta_2\to\mathbb{R}$ of order γ. And consider another two parameters $\eta,\zeta\in[0,1]$ such that relation (2.27) holds. Define $z\in\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2;\mathbb{R}^d)$ by $z_t^\tau=\int_0^t k(\tau,r)dx_r$ and assume there exists a tree indexed Volterra rough path $\mathbf{z}=\{\mathbf{z}^{\sigma,\tau};\sigma\in\mathcal{T}_3\}$ above z satisfying Hypothesis 4.3. Additionally, suppose $f\in\mathcal{C}_b^5(\mathbb{R}^m;\mathcal{L}(\mathbb{R}^d;\mathbb{R}^m))$. Then there exists a unique solution in $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}(\mathbb{R}^m)$ to the Volterra equation*

$$y_t^\tau=y_0+\int_0^t k(\tau,r)\,dx_r f(y_r^r),\quad (t,\tau)\in\Delta_2([0,T]),\quad y_0\in\mathbb{R}^m, \tag{4.75}$$

where the integral is understood as a rough Volterra integral according to Theorem 4.10.

Proof. We will proceed in a classical way by (i) Establishing a fixed point argument on a small interval. (ii) Patching the solutions obtained on the small intervals. Since this procedure is standard, we will skip some details.

We wish to solve (4.75) in a class of controlled processes. This means that the right hand side of (4.75) has to be understood according to Theorem 4.10. In particular referring to Theorem 4.10*(iv)*, the controlled process $\mathbf{y}$ will be of the form $\mathbf{y}=\{y,y^{\bullet},y^{\ddagger},0\}$. In the remainder of the proof, we will consider a controlled path $\mathbf{y}\in\mathcal{D}_{\mathbf{z};\mathbf{y}_0}^{(\beta,\gamma)}\left(\Delta_2^T\left([0,\bar{T}]\right);\mathbb{R}^m\right)$ as given in (4.74), that is a controlled processes $\mathbf{y}$ starting from an initial value $\mathbf{y}_0=(y_0,f(y_0),f(y_0)f'(y_0),0)$. As in Remark 4.14, we consider a parameter β such that for all $(\eta,\zeta)\in\mathcal{A}_N$ we have

$$\beta<\alpha,\ \ \beta-\gamma>\frac{1}{4},\ \ \zeta<\beta-\gamma,\ \text{ and }\ \eta>1-\beta. \tag{4.76}$$

In addition, we introduce a mapping

$$\mathcal{M}_{\bar{T}}:\mathcal{D}_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma)}\left(\Delta_2^T\left([0,\bar{T}]\right);\mathbb{R}^m\right)\to\mathcal{D}_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma)}\left(\Delta_2^T\left([0,\bar{T}]\right);\mathbb{R}^m\right), \tag{4.77}$$

such that for all $\left(y,y^{\bullet},y^{\ddagger},0\right)\in\mathcal{D}_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma)}(\mathbb{R}^m)$, we have

$$\begin{aligned}\mathcal{M}_{\bar{T}}&\left(y,y^{\bullet},y^{\ddagger},0\right)_t^\tau\\&=\left\{\left(y_0+\int_0^t k(\tau,r)\,dx_r f(y_r^r),\ f(y_t^\tau),f(y_t^\tau))\,f'(y_t^\tau),0\right)\Big|\,(t,\tau)\in\Delta_2^T\left([0,\bar{T}]\right)\right\}.\end{aligned} \tag{4.78}$$

We are now ready to implement the first piece (i) of the general strategy described above.

Step 1: Invariant ball on a small interval. In this step, our goal is to show that there exists a ball of radius 1 in $\mathcal{D}_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma)}(\Delta_2^T([0,\bar{T}]);\mathbb{R}^m)$ which is left invariant by $\mathcal{M}_{\bar{T}}$ provided that $\bar{T}$ is small enough. To this aim, we introduce some additional notation. Namely for $\mathbf{y}$ as in (4.78) we define a controlled process $\mathbf{w}$ in the following way:

$$(s,t,\tau)\mapsto\mathbf{w}_{ts}^\tau=\left(w_{ts}^\tau,w_{ts}^{\bullet;\tau},w_{ts}^{\ddagger;\tau},0\right)=\mathcal{M}_{\bar{T}}\left(y,y^{\bullet},y^{\ddagger},0\right)_{ts}^\tau, \tag{4.79}$$

where we recall that $\mathcal{M}_{\bar{T}}$ is defined by (4.78). Next consider the unit ball $\mathcal{B}_{\bar{T}}$ within the space $\mathcal{D}^{(\beta,\gamma)}_{\mathbf{z},\mathbf{y}_0}(\Delta_2^T([0,\bar{T}]);\mathbb{R}^m)$, defined by

$$\mathcal{B}_{\bar{T}} = \left\{ \left(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0\right) \in \mathcal{D}^{(\beta,\gamma)}_{\mathbf{z},\mathbf{y}_0}\left(\Delta_2^T\left([0,\bar{T}]\right);\mathbb{R}^m\right) \,\middle|\, \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq 1 \right\}. \tag{4.80}$$

In order to bound the process defined by (4.79), notice that $\mathcal{M}_{\bar{T}}$ is given as the Volterra type integral of $\phi = f(y)$. Hence according to (4.69) there exists a constant C such that for all $(\eta,\zeta) \in \mathcal{A}_N$ we have

$$\|(\phi, \phi^{\bullet}, \phi^{\overset{\bullet}{\bullet}}, \phi^{\bullet\bullet})\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \lesssim \left(1 + \|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)}\right)^3 \left(1 + Q^3\right) \bar{T}^{\alpha-\beta}, \tag{4.81}$$

where we have set

$$Q = |f(y_0)| + |f(y_0)f'(y_0)| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}. \tag{4.82}$$

In addition, our process $\mathbf{w}$ is defined in (4.79) as

$$w^{\tau}_{ts} = \int_s^t k(\tau, r) dx_r \phi^r_r.$$

Thus an easy extension of (4.14)-(4.17) to a process $\phi \in \mathcal{D}^{(\beta,\gamma)}_{\mathbf{z}}$ with β satisfying (4.76) yields

$$\|\mathbf{w}\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq C\|(\phi, \phi^{\bullet}, \phi^{\overset{\bullet}{\bullet}}, \phi^{\bullet\bullet})\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)} \leq C\left(1 + \|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)}\right)^4 \left(1 + Q^3\right) \bar{T}^{\alpha-\beta}, \tag{4.83}$$

for a universal constant which can change from line to line. Furthermore, since we have assumed that $\|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)} \leq M$, one can recast (4.83) as

$$\|\mathbf{w}\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq C\left(1 + M^4\right)\left(1 + Q^3\right) \bar{T}^{\alpha-\beta}. \tag{4.84}$$

Considering $\bar{T} \leq \left(C\left(1+M^4\right)\left(1+Q^3\right)\right)^{\frac{1}{\alpha-\beta}}$ and back to our definition (4.79), it is now easily seen that $\mathcal{B}_{\bar{T}}$ in (4.80) is left invariant by the map $\mathcal{M}_{\bar{T}}$. This completes the proof of step 1.

Next, we handle the second piece (ii) of the general strategy described above.

Step 2: $\mathcal{M}_{\bar{T}}$ is contractive. The aim of this step is to prove that $\mathcal{M}_{\bar{T}}$ is a contraction mapping on $\mathcal{D}^{(\alpha,\gamma)}_{\mathbf{z},\mathbf{y}_0}(\Delta_2^T([0,\bar{T}]);\mathbb{R}^m)$. That is, we will show that there exists a small $\hat{T} \leq \bar{T}$ and a constant $0 < q < 1$ such that for two paths $\mathbf{y} = (y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)$ and $\tilde{\mathbf{y}} = (\tilde{y}, \tilde{y}^{\bullet}, \tilde{y}^{\overset{\bullet}{\bullet}}, 0)$ in $\mathcal{D}^{(\beta,\gamma)}_{\mathbf{z},\mathbf{y}_0}(\Delta_2^T([0,\hat{T}]);\mathbb{R}^m)$ we have (for $(\eta,\zeta) \in \mathcal{A}_N$)

$$\left\|\mathcal{M}_{\bar{T}}\left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\overset{\bullet}{\bullet}} - \tilde{y}^{\overset{\bullet}{\bullet}}, 0\right)\right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq q \left\|\left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\overset{\bullet}{\bullet}} - \tilde{y}^{\overset{\bullet}{\bullet}}, 0\right)\right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}. \tag{4.85}$$

To this aim, we set $F = f(y) - f(\tilde{y})$, and consider the controlled path $\mathbf{F} = (F, F^{\bullet}, F^{\overset{\bullet}{\bullet}}, F^{\bullet\bullet}) \in \mathcal{D}^{(\beta,\gamma)}_{\mathbf{z}}(\Delta_2^T([0,\hat{T}]);\mathbb{R}^m)$ defined through Proposition 4.13. According to expression (4.78), we have

$$\mathcal{M}_{\hat{T}}\left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\overset{\bullet}{\bullet}} - \tilde{y}^{\overset{\bullet}{\bullet}}, 0\right)^{\tau}_{ts} = \left\{ \left(\int_s^t k(\tau, r)\, dx_r F^r_r,\ F^{\tau}_{st},\ F^{\bullet;\tau,\tau}_{ts}, 0\right) \,\middle|\, (s,t,\tau) \in \Delta_3^T\left([0,\hat{T}]\right) \right\}. \tag{4.86}$$

Hence in order to prove (4.85), it is sufficient to bound the right hand side of (4.86). Now similarly to Step 1, thanks to Remark 4.11 and upper bounds (4.14)-(4.15), we obtain

$$\left\|\mathcal{M}_{\hat{T}}\left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\overset{\bullet}{\bullet}} - \tilde{y}^{\overset{\bullet}{\bullet}}, 0\right)\right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq C\|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)} \left\|\left(F, F^{\bullet}, F^{\overset{\bullet}{\bullet}}, F^{\bullet\bullet}\right)\right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \hat{T}^{\alpha-\beta}. \tag{4.87}$$

In the following, we will bound $\|(F, F^{\bullet}, F^{\stackrel{\bullet}{\bullet}}, F^{\bullet\bullet})\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}$, that is, we need to find a bound for $\|(F, F^{\bullet}, F^{\stackrel{\bullet}{\bullet}}, F^{\bullet\bullet})\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}$ with respect to $\|(y-\tilde{y}, y^{\bullet}-\tilde{y}^{\bullet}, y^{\stackrel{\bullet}{\bullet}}-\tilde{y}^{\stackrel{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}$. Recalling that $F = f(y) - f(\tilde{y})$ and the definition (4.30)-(4.31), we can rewrite $\mathbf{F}$ as

$$\mathbf{F} = \Big(f(y) - f(\tilde{y}), f(y)f'(y) - f(\tilde{y})f'(\tilde{y}), f(y)f'(y)f'(y) - f(\tilde{y})f'(\tilde{y})f'(\tilde{y}),$$
$$\frac{1}{2}f(y)f(y)f''(y) - \frac{1}{2}f(\tilde{y})f(\tilde{y})f''(\tilde{y})\Big). \quad (4.88)$$

The strategy to bound $\|\mathbf{F}\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} = \|(F, F^{\bullet}, F^{\stackrel{\bullet}{\bullet}}, F^{\bullet\bullet})\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}$ as given in (4.88) is very similar to the classical rough path case as explained in [15]. Due to the fact that both $\mathbf{y}$ and $\tilde{\mathbf{y}}$ sit in the ball $\mathcal{B}_T$ defined by (4.80), we let the patient reader to check that there exists a constant $\tilde{C} = \tilde{C}_{M,\alpha,\gamma,\|f\|_{C_b^5}}$ such that

$$\left\| \Big(F, F^{\bullet}, F^{\stackrel{\bullet}{\bullet}}, F^{\bullet\bullet} \Big) \right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq \tilde{C} \left\| \Big(y-\tilde{y}, y^{\bullet}-\tilde{y}^{\bullet}, y^{\stackrel{\bullet}{\bullet}}-\tilde{y}^{\stackrel{\bullet}{\bullet}}, 0 \Big) \right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}. \quad (4.89)$$

Reparting (4.89) into (4.87), we thus get the existence of a constant C such that

$$\left\| \mathcal{M}_{\hat{T}} \Big(y-\tilde{y}, y^{\bullet}-\tilde{y}^{\bullet}, y^{\stackrel{\bullet}{\bullet}}-\tilde{y}^{\stackrel{\bullet}{\bullet}}, 0 \Big) \right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq CM \left\| \Big(y-\tilde{y}, y^{\bullet}-\tilde{y}^{\bullet}, y^{\stackrel{\bullet}{\bullet}}-\tilde{y}^{\stackrel{\bullet}{\bullet}}, 0 \Big) \right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \hat{T}^{\alpha-\beta}. \quad (4.90)$$

By choosing $\hat{T}$ small enough such that $q \equiv CM\hat{T}^{\alpha-\beta} < 1$, we can recast (4.90) as

$$\left\| \mathcal{M}_{\hat{T}} \Big(y-\tilde{y}, y^{\bullet}-\tilde{y}^{\bullet}, y^{\stackrel{\bullet}{\bullet}}-\tilde{y}^{\stackrel{\bullet}{\bullet}}, 0 \Big) \right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq q \left\| \Big(y-\tilde{y}, y^{\bullet}-\tilde{y}^{\bullet}, y^{\stackrel{\bullet}{\bullet}}-\tilde{y}^{\stackrel{\bullet}{\bullet}}, 0 \Big) \right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}.$$

It follows that $\mathcal{M}_{\bar{T}}$ is contractive on $\mathcal{D}_{\mathbf{z}}^{(\beta,\gamma)}(\Delta_2^T([0,\bar{T}]);\mathbb{R}^m)$, which completes the proof of Step 2. Combining Step 1 and Step 2, we have proved that if a small enough $\hat{T}$ is chosen then $\mathcal{M}_{\hat{T}}$ admits a unique fixed point $\mathbf{y} = (y, y^{\bullet}, y^{\stackrel{\bullet}{\bullet}}, 0)$ in the ball $\mathcal{B}_{\hat{T}}$ defined by (4.80). This fixed point is the unique solution to (4.75) in $\mathcal{B}_{\hat{T}}$. In addition, owing to (4.82) plus the fact that f, f' are uniformly bounded, it is easily proved that the choice of $\hat{T}$ can again be done uniformly in the starting point y_0. Hence the solution on $[0,T]$ is constructed iteratively on intervals $[k\hat{T},\, (k+1)\hat{T}]$. The proof of Theorem 4.18 is now finished. □

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