

## Research Article

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# A topological proof of Sklar's theorem in arbitrary dimensions

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**Abstract:** Copulas are appealing tools in multivariate probability theory and statistics. Nevertheless, the transfer of this concept to infinite dimensions entails some nontrivial topological and functional analytic issues, making a deeper theoretical understanding indispensable toward applications. In this short work, we transfer the well-known property of compactness of the set of copulas in finite dimensions to the infinite-dimensional framework. As an application, we prove Sklar's theorem in infinite dimensions via a topological argument and the notion of inverse systems.

**Keywords:** Copulas, Sklar's theorem, topological inverse limits, infinite dimensions, compactness

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## 1 Introduction

Copulas are widely used and well-known concepts in the realm of statistics and probability theory. This is not least due to the advantages that go along with their often intuitive and flexible handling. To some extent, such practicality is lost in infinite dimensions, as consistency problems may occur and constructions via copulas in topological vector spaces culminate in general in cylindrical, rather than actual probability measures (c.f. [10] and [2]). Building up a functional analytic and topological theory for copulas becomes pivotal in order to make the copula approach fully applicable to general stochastic processes.

The keystone of the theory is Sklar's theorem and there is a vast literature solely focusing on different proofs of this fundamental result. Among others there are proofs based on the distributional transform in [13] and [4] and earlier already in [11], based on mollifiers in [6] or the constructive approach by the extension of subcopulas, as it was proved for the bivariate case in [14] and for the general multivariate case in [17] or [3].

The naive transfer of the subcopula approach to an infinite-dimensional setting appears to be challenging, since, after the extension of the subcopulas corresponding to the finite-dimensional laws of an infinite-dimensional distribution, one would also have to check that this construction meets the necessary consistency conditions. In contrast, and besides the approach via distributional transforms (as extended to an infinite-dimensional setting in [2]), a nonconstructive proof based on topological arguments in [7] is naturally in tune with an infinite-dimensional setting.

In this article, we will therefore adopt this ansatz. In contrast to the proof in [2], we prove Sklar's theorem on the level of probability measures and not on the level of random variables. The argument carries the steps in [7] over to an infinite-dimensional setting:

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- (i) Show that the set of copula measures is compact with respect to the topology of convergence of the finite dimensional distributions (Definition 4.1).
- (ii) Prove the second part of Sklar's theorem (that every copula measure can be merged with any family of marginals to a probability measure). This step is straightforward and as in [2].
- (iii) Prove that the operation of merging a copula measure with marginals is a continuous mapping and use the compactness of the set of copulas to conclude that this map has closed image. The second part of Sklar's theorem follows by arguing that this image is also dense in the space of probability measures.

In finite dimensions, the compactness of copulas is described as “folklore” in [6] and it implies some useful applications. Some of these results carry over to the infinite-dimensional setup (c.f. Section 5).

## 2 Short primer on topological inverse systems

We will frequently use the notation  $\bar{\mathbb{R}}$  for the extended real line  $[-\infty, \infty]$ . For any measure  $\mu$  on a measurable space  $(B, \mathcal{B})$  and a measurable function  $f : (B, \mathcal{B}) \rightarrow (A, \mathcal{A})$  into another measurable space  $(A, \mathcal{A})$  we denote by  $f_*\mu$  the pushforward measure with respect to  $f$  given by  $f_*\mu(S) := \mu(f^{-1}(S))$  for all  $S \in \mathcal{A}$ . For  $I$  an arbitrary index set,  $B = \bar{\mathbb{R}}^I$  and  $\mathcal{B} = \otimes_{i \in I} \mathcal{B}(\bar{\mathbb{R}})$ , we use the shorter notations  $\pi_J_*\mu := \mu_J$  for a subset  $J \subseteq I$  and  $\pi_{\{i\}}_*\mu := \mu_i$  for an element  $i \in I$ , where  $\pi_J$  denotes the canonical projection on  $\bar{\mathbb{R}}^I$ . If  $J \subseteq I$  is finite, we denote the corresponding finite-dimensional cumulative distribution functions by  $F_{\mu_J}$  or  $F_{\mu_i}$ , respectively, where in the latter we used  $J = \{i\}$ . We use the notation  $\mathcal{I}$  for the set consisting of all finite subsets of  $I$ . Moreover, for a one-dimensional Borel measure  $\mu_i$  on  $\mathbb{R}$ , we use the notation  $F_{\mu_i}^{[-1]}$  for the quantile functions

$$F_{\mu_i}^{[-1]}(u) := \inf\{x \in (-\infty, \infty) : F_{\mu_i}(x) \geq u\}. \quad (2.1)$$

We will refer to the one-dimensional distributions  $\mu_i, i \in I$  and equivalently  $F_{\mu_i}, i \in I$  as marginals of the measure  $\mu$ . We denote the set of all probability measures on  $(\bar{\mathbb{R}}^I, \otimes_{i \in I} \mathcal{B}(\bar{\mathbb{R}}))$  by  $\mathcal{P}(\bar{\mathbb{R}}^I)$ . Moreover, for two topological spaces  $X, Y$  we write  $X \cong Y$  if they are homeomorphic.

The remainder of the section is mainly based on [12]. Let  $X_J$  be a set for each  $J \in \mathcal{I}$  and

$$(P_{J_1, J_2} : X_{J_2} \rightarrow X_{J_1}) \quad \text{for } J_1 \subseteq J_2, \text{ with } J_1, J_2 \in \mathcal{I}$$

a family of mappings, also called projections, such that

- (i)  $P_{J, J} = id_J$  is the identity mapping for all  $J \in \mathcal{I}$ , and
- (ii)  $P_{J_1, J_3} = P_{J_1, J_2} \circ P_{J_2, J_3}$  for all  $J_1 \subseteq J_2 \subseteq J_3$  in  $\mathcal{I}$ .

The system

$$(X_J, P_{J_1, J_2}, \mathcal{I}) := ((X_J)_{J \in \mathcal{I}}, ((P_{J_1, J_2} : X_{J_2} \rightarrow X_{J_1})_{\substack{J_1 \subseteq J_2 \\ J_1, J_2 \in \mathcal{I}}}))$$

is called an *inverse system* (over the partially ordered set  $\mathcal{I}$ ). If  $(X_J, \tau_J)$  are topological spaces for each  $J \in \mathcal{I}$  and  $(P_{J_1, J_2})$  are continuous for all  $J_1 \subseteq J_2$  with  $J_1, J_2 \in \mathcal{I}$ , we call

$$(X_J, \tau_J, P_{J_1, J_2}, J \in \mathcal{I}) := ((X_J, \tau_J)_{J \in \mathcal{I}}, ((P_{J_1, J_2} : X_{J_2} \rightarrow X_{J_1})_{\substack{J_1 \subseteq J_2 \\ J_1, J_2 \in \mathcal{I}}}))$$

a *topological inverse system*. A *topological inverse limit* of this inverse system is a space  $X$  together with continuous mappings  $P_J : X \rightarrow X_J, J \in \mathcal{I}$ , such that  $P_{J_1, J_2} P_{J_2} = P_{J_1}$  for all  $J_1 \subseteq J_2$  in  $\mathcal{I}$  (i.e., the mappings are *compatible*) and the following *universal property* holds: Whenever there is a topological space  $Y$ , such that there are continuous mappings  $(\psi_J : Y \rightarrow X_J)_{J \in \mathcal{I}}$  which are compatible, i.e.,  $P_{J_1, J_2} \psi_{J_2} = \psi_{J_1}$  for all  $J_1 \subseteq J_2$  in  $\mathcal{I}$ , then there exists a unique continuous mapping

$$\Psi : Y \rightarrow X, \quad (2.2)$$

with the property  $P_J\Psi = \psi_J$  for all  $J \in \mathcal{I}$ . We have that

$$\{x = (x_j)_{j \in \mathcal{I}} \in \prod_{J \in \mathcal{I}} X_J : P_{J_1, J_2}(\pi_{J_2}(x)) = \pi_{J_1}(x) \text{ for } J_1 \subseteq J_2\} \subseteq \prod_{J \in \mathcal{I}} X_J \quad (2.3)$$

equipped with the subspace topology with respect to the product topology is an inverse limit of the topological inverse system, induced by the canonical projections  $\pi_{J'}((x_j)_{j \in \mathcal{I}}) = x_{J'}$ . Each topological inverse limit is homeomorphic to this space and therefore to every topological inverse limit (see the proof of Theorem 1.1.1 in [12]). We write  $\lim_{\leftarrow} X_J \subseteq \prod_{J \in \mathcal{I}} X_J$  for the inverse limit as a subset of the product space and we equip it throughout with the induced subspace topology.

**Lemma 2.1.** *Let  $(X_J, \tau_J, \pi_{J_1, J_2})$  be a topological inverse system (over the poset  $\mathcal{I}$ ) of Hausdorff spaces. Then  $\lim_{\leftarrow} X_J$  is a closed subset of  $\prod_{J \in \mathcal{I}} X_J$  with respect to the product topology.*

**Proof.** See [12, Lemma 1.1.2]. □

**Lemma 2.2.** *Let  $X$  be a compact Hausdorff space and  $(X_J, \tau_J, \pi_{J_1, J_2})$  be a topological inverse system of compact Hausdorff spaces. Let  $\psi_J : X \rightarrow X_J, J \in \mathcal{I}$  be a family of compatible surjections and  $\Psi$  the induced mapping. Then either  $\lim_{\leftarrow} X_J = \emptyset$  or  $\Psi(X)$  is dense in  $\lim_{\leftarrow} X_J$ .*

**Proof.** See [12, Corollary 1.1.7]. □

### 3 Copulas and Sklar's theorem

As they are cumulative distribution functions, copulas in finite dimension have a one-to-one correspondence to probability measures. In infinite dimensions we will therefore work with the notion of copula measures as introduced in [2].

**Definition 3.1.** A copula measure (or simply copula) on  $\bar{\mathbb{R}}^I$  is a probability measure  $C \in \mathcal{P}(\bar{\mathbb{R}}^I)$ , such that its marginals  $C_i$  are uniformly distributed on  $[0, 1]$ . We will denote the space of copula measures on  $\bar{\mathbb{R}}^I$  by  $C(\bar{\mathbb{R}}^I)$ .

Sklar's theorem as stated below was proved in [2] by following the arguments for the finite-dimensional assertion in [13]. Here we give an alternative proof for the infinite-dimensional setting using a topological argument as in [7].

**Theorem 3.2.** (Sklar's theorem) *Let  $\mu \in \mathcal{P}(\bar{\mathbb{R}}^I)$  be a probability measure with marginal one-dimensional distributions  $\mu_i, i \in I$ . There exists a copula measure  $C$ , such that for each  $J \in \mathcal{I}$ , we have*

$$F_C((F_{\mu_i}(x_j))_{j \in J}) = F_{\mu}((x_j)_{j \in J}) \quad (3.1)$$

for all  $(x_j)_{j \in J} \in \bar{\mathbb{R}}^J$ . Moreover,  $C$  is unique if  $F_{\mu_i}$  is continuous for each  $i \in I$ . Vice versa, let  $C$  be a copula measure on  $\bar{\mathbb{R}}^I$  and let  $(\mu_i)_{i \in I}$  be a collection of (one-dimensional) Borel probability measures over  $\bar{\mathbb{R}}$ . Then there exists a unique probability measure  $\mu \in \mathcal{P}(\bar{\mathbb{R}}^I)$ , such that (3.1) holds.

### 4 Topological properties of Copulas and a proof of Sklar's theorem

The collection  $(\mathcal{P}(\bar{\mathbb{R}}^J), J \in \mathcal{I})$ , where each  $\mathcal{P}(\bar{\mathbb{R}}^J)$  is considered as a topological space with the topology of weak convergence, is a topological inverse system with the projections  $\pi_{J_1, J_2}(\mu_{J_2}) = (\mu_{J_2})_{J_1}$  for  $\mu_{J_2} \in \mathcal{P}(\bar{\mathbb{R}}^{J_2})$

and  $J_1, J_2 \in \mathcal{I}$ ,  $J_1 \subseteq J_2$ . Moreover, observe that each  $\mathcal{P}(\bar{\mathbb{R}}^J)$  is a Hausdorff space, since it is metrizable by the Prokhorov metric (c.f. [15, Theorem 4.2.5]). The space  $\lim_{\leftarrow} \mathcal{P}(\bar{\mathbb{R}}^J) \subset \prod_{J \in \mathcal{I}} \mathcal{P}(\bar{\mathbb{R}}^J)$  of consistent families of probability measures is a topological inverse limit, equipped with the corresponding inverse limit topology. The space of probability measures on  $\otimes_{i \in \mathcal{I}} \mathcal{B}(\mathbb{R})$  has via its finite-dimensional distributions a one-to-one correspondence with this family of consistent finite-dimensional distributions, and hence there is a natural bijection between  $\lim_{\leftarrow} \mathcal{P}(\bar{\mathbb{R}}^J)$  and  $\mathcal{P}(\bar{\mathbb{R}}^I)$ .

We equip the space  $\mathcal{P}(\bar{\mathbb{R}}^I)$  with the topology of *weak convergence of the finite-dimensional distributions*, which we define as follows:

**Definition 4.1.** The topology of convergence of the finite-dimensional distributions on  $\mathcal{P}(\bar{\mathbb{R}}^I)$  is defined as the topology such that  $\mathcal{P}(\bar{\mathbb{R}}^I) \cong \lim_{\leftarrow} \mathcal{P}(\bar{\mathbb{R}}^J)$ .

$\mathcal{P}(\bar{\mathbb{R}}^I)$  with this topology is by definition a topological inverse limit. Define also  $\lim_{\leftarrow} C(\bar{\mathbb{R}}^J) := \lim_{\leftarrow} \mathcal{P}(\bar{\mathbb{R}}^J) \cap \prod_{J \in \mathcal{I}} C(\bar{\mathbb{R}}^J)$ . Certainly, we have

$$C(\bar{\mathbb{R}}^I) \cong \lim_{\leftarrow} C(\bar{\mathbb{R}}^J) \quad (4.1)$$

with the corresponding topologies.

The following result contains among other things the topological proof of Sklar's theorem 3.2.

**Theorem 4.2.** *The following statements hold.*

- (1)  $\mathcal{P}(\bar{\mathbb{R}}^I)$  with the topology of weak convergence of the finite-dimensional distributions is a Hausdorff space.
- (2) The space of copula measures  $C(\bar{\mathbb{R}}^I)$  is compact with respect to the topology of convergence of finite-dimensional distributions.
- (3) For a copula measure  $C$  on  $\bar{\mathbb{R}}^I$  and (one-dimensional) Borel probability measures  $(\mu_i)_{i \in I}$  over  $\bar{\mathbb{R}}$  the push-forward measure

$$\mu := \left( (F_{\mu_i}^{[-1]})_{i \in I} \right)_* C \quad (4.2)$$

satisfies (3.1).

- (4) If we equip  $C(\bar{\mathbb{R}}^I) \times \prod_{i \in I} \mathcal{P}(\bar{\mathbb{R}})$  with the product topology of weak convergence on each  $\mathcal{P}(\bar{\mathbb{R}})$  and the topology of convergence of the finite-dimensional distributions on  $C(\bar{\mathbb{R}}^I)$  and  $\mathcal{P}(\bar{\mathbb{R}}^I)$ , then the mapping  $\Phi : C(\bar{\mathbb{R}}^I) \times \prod_{i \in I} \mathcal{P}(\bar{\mathbb{R}}) \rightarrow \mathcal{P}(\bar{\mathbb{R}}^I)$  given by

$$\Phi(C, (\mu_i)_{i \in I}) := \left( (F_{\mu_i}^{[-1]})_{i \in I} \right)_* C$$

is continuous and surjective. In particular, Sklar's theorem holds.

**Proof.** (1) Since products of Hausdorff spaces are Hausdorff and  $\mathcal{P}(\bar{\mathbb{R}}^I)$  is homeomorphic to a subset of a product of Hausdorff spaces, it is Hausdorff.

(2) We know by [6, Thm. 3.3] that every  $C(\bar{\mathbb{R}}^J)$  is compact with respect to the topology of weak convergence on  $\mathcal{P}(\bar{\mathbb{R}}^J)$ . Tychonoff's theorem guarantees also that  $\prod_{J \in \mathcal{I}} C(\bar{\mathbb{R}}^J)$  is compact with respect to the product topology on  $\prod_{J \in \mathcal{I}} \mathcal{P}(\bar{\mathbb{R}}^J)$ . Therefore, as  $\lim_{\leftarrow} \mathcal{P}(\bar{\mathbb{R}}^J)$  is closed by Lemma 2.1, we obtain that  $C(\bar{\mathbb{R}}^I)$  is compact, since it is homeomorphic to an intersection of a closed and a compact set in the product topology.

(3) This corresponds to the second part of Sklar's theorem and the proof can be conducted analogously to the one in [2]. Therefore, it is enough to see that

$$([0, F_{\mu_j}(x_j)]_{j \in J}) \setminus ((F_{\mu_j}^{[-1]})^{-1}(-\infty, x_1]_{j \in J})$$

is a  $C_J$ -nullset for all  $(x_j)_{j \in J} \in \bar{\mathbb{R}}^J$ ,  $J \in \mathcal{I}$ , since then we immediately obtain

$$C_J(((F_{\mu_j}^{[-1]})^{-1}(-\infty, x_1]_{j \in J})) = C_J \left( \left( [0, F_{\mu_j}(x_j)]_{j \in J} \right) \right) = F_{C_J}(F_{\mu_j}((x_j)_{j \in J})).$$

(4) Define  $\phi_J : C(\bar{\mathbb{R}}^J) \times \prod_{i \in I} \mathcal{P}(\bar{\mathbb{R}}) \rightarrow \mathcal{P}(\bar{\mathbb{R}}^J)$  by

$$\phi_J(C, (\mu_i)_{i \in I}) := \Phi(C, (\mu_i)_{i \in I})_J,$$

which is well defined by (3). Since the finite-dimensional distributions of a law are consistent,  $(\phi_J, J \in \mathcal{I})$  forms a compatible family. Define analogously for  $J \in \mathcal{I}$  also  $\tilde{\phi}_J : C(\bar{\mathbb{R}}^J) \times \prod_{j \in J} \mathcal{P}(\bar{\mathbb{R}}) \rightarrow \mathcal{P}(\bar{\mathbb{R}}^J)$  by

$$\tilde{\phi}_J(C_J, (\mu_j)_{j \in J}) = \left( (F_{\mu_j}^{[-1]})_{j \in J} \right)_* C_J.$$

This is by Sklar's theorem in finite-dimensions surjective and by [16, Thm. 2] also continuous. Hence,  $\phi_J = \tilde{\phi}_J \pi_J$  is continuous and surjective, since both  $\tilde{\phi}_J$  and  $\pi_J$  are.  $\Phi$  must be the uniquely induced continuous mapping by the family  $(\phi_J, J \in \mathcal{I})$  by the universality property of the inverse limit. Moreover, since by [15, Corollary 4.2.6]  $\mathcal{P}(\bar{\mathbb{R}})$  is compact and by (2) also  $C(\bar{\mathbb{R}}^J)$  is compact, we have that  $C(\bar{\mathbb{R}}^J) \times \prod_{i \in I} \mathcal{P}(\bar{\mathbb{R}})$  is compact by Tychonoff's theorem. The continuity of  $\Phi$  implies therefore that  $\Phi(C(\bar{\mathbb{R}}^J) \times \prod_{i \in I} \mathcal{P}(\bar{\mathbb{R}}))$  is compact, hence closed. Since moreover Lemma 2.2 implies that  $\Phi(C(\bar{\mathbb{R}}^J) \times \prod_{i \in I} \mathcal{P}(\bar{\mathbb{R}}))$  is dense, we obtain that  $\Phi$  is surjective and therefore also the first part of Sklar's theorem holds. The uniqueness of the copulas in the case of continuous marginals follows immediately by Sklar's theorem in finite dimensions via the uniqueness of the finite-dimensional distribution of the corresponding copula measure.  $\square$

## 5 Applications of the compactness of copulas in infinite dimensions

Apart from the alternative proof of Sklar's theorem, the compactness of the family of copulas has some useful implications.

Observe that  $\mathcal{P}(\bar{\mathbb{R}}^J)$  is a subset of a locally convex Hausdorff space. Indeed,  $\mathcal{P}(\bar{\mathbb{R}}^J)$  once equipped with the topology of convergence of the finite-dimensional distributions is topologically embedded in  $C_b(\bar{\mathbb{R}}^J)^*$  equipped with the weak\*-topology, where  $C_b(\bar{\mathbb{R}}^J)^*$  is the topological dual of the space of bounded continuous functions equipped with the topology induced by the uniform norm. Thus, with respect to the topology of weak convergence for each  $J \in \mathcal{I}$ , we obtain that also the inverse limit  $\mathcal{P}(\bar{\mathbb{R}}^J) \cong \lim_{\leftarrow} \mathcal{P}(\bar{\mathbb{R}}^J)$  is topologically embedded in a locally convex Hausdorff space, as it is isomorphic to a subset of the product  $\prod_{J \in \mathcal{I}} \mathcal{P}(\bar{\mathbb{R}}^J) \hookrightarrow \prod_{J \in \mathcal{I}} C_b(\bar{\mathbb{R}}^J)^*$ . Since  $C(\bar{\mathbb{R}}^J) \hookrightarrow \mathcal{P}(\bar{\mathbb{R}}^J)$  is convex, we obtain the following result by the Krein–Milman theorem [5, Thm.V.8.4], as mentioned for instance in [8, p. 30] for the finite-dimensional case:

**Lemma 5.1.**  *$C(\bar{\mathbb{R}}^J)$  is the closure of the convex hull of its extremal points with respect to the topology of weak convergence of finite-dimensional distributions.*

As mentioned in [1] this implies for instance that

$$\sup_{C \in C(\bar{\mathbb{R}}^J)} g(C) = \sup_{C \in \text{ext}(C(\bar{\mathbb{R}}^J))} g(C),$$

where  $\text{ext}(C(\bar{\mathbb{R}}^J))$  denotes the set of extremal points of  $C(\bar{\mathbb{R}}^J)$  and  $g : C(\bar{\mathbb{R}}^J) \rightarrow \mathbb{R}$  is a convex function.

The compactness of copulas might also be of interest for proving limit theorems. In fact, by the compactness of copulas in finite dimensions, we obtain that every sequence  $(C^n)_{n \in \mathbb{N}}$  of multivariate copulas (of fixed dimension) has a convergent subsequence. This was for instance used in [9]. If  $(C^n)_{n \in \mathbb{N}} \subset C(\bar{\mathbb{R}}^J)$  is a sequence of copula measures and  $I$  is an infinite index set, this also implies that all finite distributions  $F_{C^n}$  for  $J \subset I$  finite have a convergent subsequence that converges weakly. However, it is not clear if this subsequence can be chosen uniformly for all finite  $J \in I$ , i.e., for all finite dimensional distributions. Thus, this result is intricate to transfer to the infinite dimensional setting especially since the notions of compactness and sequential compactness may not coincide and one has to appeal to the notion of nets instead of sequences. For countable index sets we have at least

**Lemma 5.2.** *If  $I$  is a countable index set, then  $C(\bar{\mathbb{R}}^I)$  is sequentially compact.*

**Proof.**  $\bar{\mathbb{R}}^I$  is a product of polish spaces and hence polish with respect to the product topology. Thus, the Lévy–Prokhorov metric makes  $\mathcal{P}(\bar{\mathbb{R}}^I)$  a metric space, whose topology coincides with the topology of convergence in distribution with respect to the product topology on  $\bar{\mathbb{R}}^I$ , which itself coincides with the topology of weak convergence of the finite dimensional distributions. As a compact set in a metrizable space,  $C(\bar{\mathbb{R}}^I)$  is sequentially compact.  $\square$

With this lemma it might be easy to prove convergence criteria in some topological vector spaces. Recall that the  $p$ -Wasserstein space  $\mathcal{W}_p(E)$  over a separable Banach space  $E$  is given by

$$\mathcal{W}_p(E) = \left\{ \nu : \nu \text{ is a Borel law on } E, \int_E \|x\|_E^p \nu(dx) < \infty \right\}.$$

Let  $E = l^p$  be the sequence space

$$l^p := \left\{ (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \|(x_n)_{n \in \mathbb{N}}\|_p := \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}$$

for some  $p \in [1, \infty)$ . Recall that for the case  $p = 2$  this class of spaces contains (by isomorphy) all separable Hilbert spaces.

Define by  $\pi_i$  for  $i \in \mathbb{N}$  the projection onto the  $i$ th component, i.e.,  $\pi_i(x_1, x_2, \dots) = x_i \quad \forall x \in l^p$ . Any  $\mu \in \mathcal{W}_p(l^p)$  is uniquely specified by the family  $\mu_{i_1, \dots, i_d}$  of finite dimensional distributions given by

$$\mu_{i_1, \dots, i_d} := \mu \circ (\pi_{i_1}, \dots, \pi_{i_d})^{-1}$$

for any finite subset  $\{i_1, \dots, i_d\} \subset \mathbb{N}$  of the natural numbers. We write  $m_{\mu_i}^p$  for the  $p$ th absolute moment of the univariate measures  $\mu_i$ ,  $i \in \mathbb{N}$ , i.e.,

$$m_{\mu_i}^p := \int_{\mathbb{R}} |x|^p \mu_i(dx).$$

**Corollary 5.3.** *Assume  $(\mu^n)_{n \in \mathbb{N}} \subset \mathcal{W}_p(l^p)$ . If all for all  $i \in \mathbb{N}$  there is a  $\mu_i^\infty \in \mathcal{W}_p(l^p)$  such that*

$$\sum_{i=1}^{\infty} m_{\mu_i^\infty}^p < \infty,$$

*then there is a subsequence  $(\mu^{n_k})_{k \in \mathbb{N}}$  of Borel laws that converges with respect to the topology of weak convergence of finite dimensional distributions to some Borel law  $\mu$  in  $\mathcal{W}_p(l^p)$ , such that*

$$\mu_i^\infty = \mu_i \quad \forall i \in \mathbb{N}. \quad (5.1)$$

**Proof.** Recall from Section 3.1 in [2] that a basis copula corresponding to a measure  $\nu \in \mathcal{W}_1(l^p)$  is the copula measure  $C_\nu$  in  $\mathbb{R}^n$ , such that its finite dimensional distributions are given by the copula measures associated with the finite dimensional distributions  $\nu_{i_1, \dots, i_d}$ . Let  $(C_n)_{n \in \mathbb{N}}$  denote the sequence of basis copulas associated with the sequence  $(\mu_n)_{n \in \mathbb{N}}$ . Then by Lemma 5.2, there is a subsequence  $(C_{n_k})_{k \in \mathbb{N}}$  that converges to a copula measure  $C \in C(\bar{\mathbb{R}}^I)$  with respect to the topology of convergence of the finite dimensional distributions. Theorem 2 in [16] tells us that for multivariate random variables weak convergence follows by convergence of the marginal distributions and the weak convergence of the associated copulas. By assumption, this yields the existence of a Borel law  $\mu$  in  $\mathbb{R}^{\mathbb{N}}$  such that

$$\mu^n \longrightarrow \mu,$$

Equation (5.1) holds and  $\mu$  has  $C$  as its underlying copula measure. That indeed  $\mu \in \mathcal{W}_p(l^p)$  holds follows from Corollary 4 in [2].  $\square$

**Remark 5.4.** Certainly, we can identify elements of Banach spaces with a Schauder basis uniquely with elements in  $\mathbb{R}^{\mathbb{N}}$ . In that way, transferring the assertion of Corollary 5.3 to this more general situation is possible for instance by appealing to Corollary 3 in [2].

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