REMARKS ON CLASSICAL NUMBER THEORETIC ASPECTS OF MILNOR–WITT K-THEORY

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ABSTRACT. We record a few observations on number theoretic aspects of Milnor–Witt K-theory, focusing on generalizing classical results on reciprocity laws, Hasse's norm theorem and K_2 of number fields and rings of integers.

1. INTRODUCTION

The algebraic K-groups of number fields and rings of integers are known to encode deep arithmetic information. This is witnessed already by the computation of the zeroth and first K-groups of the ring of integers $\mathcal{O}_F$ in a number field F: indeed, the torsion subgroup of $K_0(\mathcal{O}_F)$ is precisely the ideal class group of F, while $K_1(\mathcal{O}_F)$ is the group of units in $\mathcal{O}_F$. In the 70's, Tate discovered that the second K-group of F is inherently related to reciprocity laws on F [Tat71]. More precisely, Tate found that in the case when $F = \mathbf{Q}$ we have

$$K_2(\mathbf{Q}) \cong \mathbf{Z}/2 \oplus \bigoplus_{p \text{ prime}} \mathbf{F}_p^\times.$$

Tate's proof method follows essentially Gauss' first proof of the quadratic reciprocity law involving an induction over the primes. One can show that Tate's structure theorem for $K_2(\mathbf{Q})$ gives rise to the product formula for Hilbert symbols over $\mathbf{Q}$, which is an equivalent formulation of the law of quadratic reciprocity. See [Gra03, II §7] for details.

Besides the algebraic K-groups there are several other important invariants attached to a number field F. A noteworthy example is the Witt ring $W(F)$ of F, which subsumes much of the theory of quadratic forms over F. For instance, the celebrated Hasse–Minkowski's local-global principle can be formulated in terms of the Witt ring by stating that an element of $W(\mathbf{Q})$ is trivial if and only if it maps to zero in $W(\mathbf{R})$ and in $W(\mathbf{Q}_p)$ for each prime p [MH73, IV Corollary 2.4]. Another example is given by the *Milnor* K-*groups of F*, $K_n^M(F)$, introduced by Milnor in his 1970 paper [Mil70]. By definition, the Milnor K-groups of F coincide with the algebraic K-groups of F in degrees 0 and 1, while Matsumoto's theorem on K_2 of fields [Mat69] implies that also $K_2^M(F) \cong K_2(F)$. In higher degrees, however, these groups are in general different. On the other hand, Milnor proved in [Mil70] that the Milnor K-groups of F are intimately linked with the Witt ring of F. The understanding of this connection between Milnor K-theory and quadratic forms was greatly enhanced in the wake of Morel and Voevodsky's introduction of motivic homotopy theory [MV99], and in particular by Orlov, Vishik and Voevodsky's solution of Milnor's conjecture on quadratic forms [OVV07]. In fact, both the Milnor K-groups and the Witt ring was set in new light in the context of motivic homotopy groups. More precisely, Hopkins and Morel introduced the so-called *Milnor–Witt* K-*groups* $K_*^{MW}(F)$ of F, and Morel showed in [Mor04a, Theorem 6.4.1] that for any integer n, there is a canonical isomorphism

$$\pi_{n,n}\mathbf{1} \cong K_{-n}^{MW}(F). \tag{1.1}$$

Here $\pi_{n,n}\mathbf{1}$ denotes the motivic homotopy group of the motivic sphere spectrum $\mathbf{1}$ over F in bidegree (n,n). We refer the reader to, e.g., [IØ20] for a survey on motivic homotopy groups. The Milnor–Witt K-groups of F are equipped with forgetful maps to the Milnor K-groups as well as to the Witt ring, and can therefore be considered as an enhancement of the Milnor K-groups of F which also takes into account information coming from quadratic forms defined over F.

Below we investigate what number theoretic information the lower Milnor–Witt K-groups carry. In particular, we define Hilbert symbols and idèle class groups in this setting; we consider the connection between $\mathrm{K}_2^{\mathrm{MW}}(F)$ and reciprocity laws; we compute $\mathrm{K}_2^{\mathrm{MW}}(\mathcal{O}_F)$ in a few explicit examples; and we show a Hasse type norm theorem for $\mathrm{K}_2^{\mathrm{MW}}$. Since the Milnor–Witt K-groups contain information coming from quadratic forms, we obtain analogs of classical results that are more sensitive to the infinite real places of the number field than the ordinary K-groups.

1.1. **Outline.** In Section 2 we start by recalling the definition and basic properties of Milnor–Witt K-theory, before we move on to computing Milnor–Witt K-groups of the rationals as well as some local fields. We finish this preliminary section by defining valuations in the setting of Milnor–Witt K-theory, lifting the classical valuations on a number field.

In Section 3 we put a topology on the first Milnor–Witt K-group $\mathrm{K}_1^{\mathrm{MW}}(F_v)$ of a completion of the number field F, in such a way that $\mathrm{K}_1^{\mathrm{MW}}(F_v)$ becomes a covering of $F_v^\times$. This is used in Section 4 where we define idèles and idèle class groups in the setting of Milnor–Witt K-theory. We show that the associated volume zero idèle class group $\widetilde{C}_F^0$ is again a compact topological group extending the classical compact group C_F^0. See Proposition 4.9 for more details.

In Section 5 we shift focus from $\mathrm{K}_1^{\mathrm{MW}}$ to $\mathrm{K}_2^{\mathrm{MW}}$. We define Hilbert symbols on Milnor–Witt K-groups and show a Moore reciprocity sequence in this setting; see Proposition 5.8. We then move on to the study of $\mathrm{K}_2^{\mathrm{MW}}$ of rings of integers in Section 6. Finally, in Section 7 we show an analog of Hasse's norm theorem similar to the generalizations of Bak–Rehmann [BR84] and Østvær [Øst03].

1.2. **Conventions and notation.** Throughout we let F denote a number field of signature (r_1, r_2), and we let Pl_F denote the set of places of F. For any $v \in \mathrm{Pl}_F$, we let F_v denote the completion of F at the place v, and we let $i_v\colon F \hookrightarrow F_v$ denote the embedding of F into F_v. By Ostrowski's theorem, Pl_F decomposes as a disjoint union $\mathrm{Pl}_F = \mathrm{Pl}_0 \cup \mathrm{Pl}_\infty$ of the finite and infinite places of F, respectively. The set Pl_∞ of infinite places of F decomposes further into the sets $\mathrm{Pl}_\infty^{\mathrm{r}}$ and $\mathrm{Pl}_\infty^{\mathrm{c}}$ of real and complex infinite places, respectively. Finally, let $\mathrm{Pl}_F^{\mathrm{nc}}$ denote the set of noncomplex places of F.

In order to streamline the notation with the literature, we will often use the notations $\mathrm{K}_n^{\mathrm{M}}(F)$ and $\mathrm{K}_n(F)$ interchangeably whenever $n \in \{0, 1, 2\}$.

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2. Preliminaries

2.1. **Milnor–Witt K-theory.** We start out by providing some generalities on Milnor–Witt K-groups, mostly following Morel's book [Mor12].

Definition 2.1 (Hopkins–Morel)**.** The *Milnor–Witt* K-*theory* $\mathrm{K}_*^{\mathrm{MW}}(F)$ of F is the graded associative $\mathbf{Z}$-algebra with one generator $[a]$ of degree $+1$ for each unit $a \in F^\times$, and one generator η of degree -1, subject to the following relations:

(I) $[a][1 - a] = 0$ for any $a \in F^\times \setminus \{1\}$ (Steinberg relation).
(II) $[ab] = [a] + [b] + \eta[a][b]$ (twisted η-logarithmic relation).
(III) $\eta[a] = [a]\eta$ (η-commutativity).
(IV) $(2 + \eta[-1])\eta = 0$ (hyperbolic relation).

We let $K_n^{MW}(F)$ denote the n-th graded piece of $K_*^{MW}(F)$. The product $[a_1] \cdots [a_n] \in K_n^{MW}(F)$ may also be denoted by $[a_1, \ldots, a_n]$; by [Mor12, Lemma 3.6 (1)], these symbols generate the abelian group $K_n^{MW}(F)$.

2.1.1. *Milnor–Witt K-theory and quadratic forms.* Let us explain the relationship between Milnor–Witt K-theory and quadratic forms. Recall for instance from [MH73; Sch85] that a *symmetric bilinear form*[1] over F is a finite dimensional F-vector space V together with a nondegenerate symmetric bilinear map $\beta \colon V \times V \to F$. The group completion of the semiring of isomorphism classes of symmetric bilinear forms over F is called the *Grothendieck–Witt ring* of F, denoted $GW(F)$. Any unit u of F defines a symmetric bilinear form $\langle u \rangle$ whose underlying vector space is just F, and whose bilinear map β is given by $\beta(x, y) = uxy$. In fact, the forms $\langle u \rangle$ for $u \in F^\times$ additively generate the Grothendieck–Witt ring of F [Mor12, Lemma 3.9]. Sending the element $1 + \eta[u] \in K_0^{MW}(F)$ to $\langle u \rangle \in GW(F)$ gives a well defined ring homomorphism from $K_0^{MW}(F)$ to $GW(F)$ which is in fact an isomorphism [Mor12, Chapter 3]. In light of this isomorphism we will, for any $u \in F^\times$, denote the element $1 + \eta[u] \in K_0^{MW}(F)$ also by $\langle u \rangle$.

The addition and multiplication in the ring $GW(F)$ stems from direct sum and tensor product of vector spaces over F. The form $H := \langle 1 \rangle + \langle -1 \rangle \in GW(F)$ is called the *hyperbolic plane*, and this form generates an ideal which is isomorphic to $\mathbf{Z}$. The resulting quotient ring $W(F) := GW(F)/(H)$ is called the *Witt ring* of F. In the defining relation (IV) of Milnor–Witt K-theory above, the element $2 + \eta[-1] = 1 + \langle -1 \rangle \in K_0^{MW}(F)$ corresponds to $H \in GW(F)$. Thus the hyperbolic relation essentially implies that multiplication by η on the negative Milnor–Witt K-groups becomes an isomorphism, identifying $K_n^{MW}(F)$ with $W(F)$ for all $n < 0$; see [Mor12, Lemma 3.10] for details.

Taking the rank of forms over F defines a ring homomorphism $GW(F) \to \mathbf{Z}$, which descends to a homomorphism $W(F) \to \mathbf{Z}/2$. The kernel of the map $W(F) \to \mathbf{Z}/2$ consists of the even-dimensional forms in the Witt ring, and is referred to as the *fundamental ideal* of F, denoted $I(F)$. The powers $I^n(F)$ of the fundamental ideal are generated by the so-called *Pfister forms* $\langle\langle a_1, \ldots, a_n \rangle\rangle := (\langle 1 \rangle - \langle a_1 \rangle) \cdots (\langle 1 \rangle - \langle a_n \rangle)$. For $n \geq 1$ there is a group homomorphism from $K_n^{MW}(F)$ to $I^n(F)$ given by mapping $[a_1, \ldots, a_n]$ to the Pfister form $\langle\langle a_1, \ldots, a_n \rangle\rangle$. We can use this map to define, for each infinite real place v of F, a *signature homomorphism* $\widetilde{\mathrm{sgn}}_v \colon K_n^{MW}(F) \to \mathbf{Z}$ as the composition

$$\widetilde{\mathrm{sgn}}_v \colon K_n^{MW}(F) \to K_n^{MW}(F_v) \to I^n(F_v) \xrightarrow{\cong} \mathbf{Z}. \tag{2.1}$$

Here the last homomorphism is given by the signature of quadratic forms [MH73, p. 62]: for example, for $a \in F_v^\times$, the signature of $\langle a \rangle$ is $+1$ if a is positive, and -1 otherwise. Thus the signature of $\langle\langle -1, \ldots, -1 \rangle\rangle \in I^n(F_v)$ is 2^n, and this defines an isomorphism $I^n(F_v) \cong \mathbf{Z}$ by [MH73, III Corollary 2.7], carrying the generator $\langle\langle -1, \ldots, -1 \rangle\rangle$ to $1 \in \mathbf{Z}$.

As the name suggests, Milnor–Witt K-theory is also related to Milnor K-theory. Indeed, for any $n \geq 0$ there is a surjective homomorphism $p \colon K_n^{MW}(F) \to K_n^M(F)$ determined by killing η and sending $[a]$ to $\{a\} \in K_1^M(F)$. Its kernel is $I^{n+1}(F)$, the $(n+1)$-th power of the fundamental ideal in the Witt ring of F. The above discussion is subsumed by the following pullback square,

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[1]The connection between symmetric bilinear forms and quadratic forms is as follows. Any symmetric bilinear form β gives rise to a quadratic form q by setting $q(x) := \beta(x, x)$. Moreover, if F is of characteristic different from 2, then any quadratic form over F arises uniquely in this way from a symmetric bilinear form over F.

which is proved in [Mor04b, Theorem 5.3]:

$$
\begin{array}{ccc}
\mathrm{K}_n^{\mathrm{MW}}(F) & \xrightarrow{\ p\ } & \mathrm{K}_n^{\mathrm{M}}(F) \\
\downarrow & & \downarrow \\
\mathrm{I}^n(F) & \longrightarrow & \mathrm{I}^n(F)/\mathrm{I}^{n+1}(F)
\end{array}
\tag{2.2}
$$

2.1.2. *The residue map.* In Milnor K-theory, there is a residue map, or *tame symbol*, $\partial_v \colon \mathrm{K}_*^{\mathrm{M}}(F) \to \mathrm{K}_{*-1}^{\mathrm{M}}(k(v))$ defined for each finite place v of F. These homomorphisms assemble to a *total residue map*

$$
\partial \colon \mathrm{K}_*^{\mathrm{M}}(F) \to \bigoplus_{v \in \mathrm{Pl}_0} \mathrm{K}_{*-1}^{\mathrm{M}}(k(v))
$$

given as $\partial = \bigoplus_{v \in \mathrm{Pl}_0} \partial_v$. Morel shows in [Mor12, Theorem 3.15] that the same is true for Milnor–Witt K-theory. More precisely, for each uniformizer π_v for v, there is a unique homomorphism $\partial_v^{\pi_v}$ giving rise to a graded homomorphism

$$
\partial = \bigoplus_{v \in \mathrm{Pl}_0} \partial_v^{\pi_v} \colon \mathrm{K}_*^{\mathrm{MW}}(F) \to \bigoplus_{v \in \mathrm{Pl}_0} \mathrm{K}_{*-1}^{\mathrm{MW}}(k(v))
$$

which commutes with η and satisfies $\partial_v^{\pi_v}([\pi_v, u_1, \ldots, u_n]) = [\overline{u}_1, \ldots, \overline{u}_n]$ whenever the u_i's are units modulo π_v. In contrast to the case for Milnor K-theory, the maps $\partial_v^{\pi_v}$ depend on the choice of uniformizer π_v; this stems from the relation $[u\pi_v] = [u] + [\pi_v] + \eta[u, \pi_v]$. One can however define a twisted version of Milnor–Witt K-theory in order to make the maps $\partial_v^{\pi_v}$ canonical. Indeed, for any field k and any one-dimensional k-vector space V, let $\mathrm{K}_*^{\mathrm{MW}}(k, V) := \mathrm{K}_*^{\mathrm{MW}}(k) \otimes_{\mathbf{Z}[k^\times]} \mathbf{Z}[V \setminus \{0\}]$, where $\mathbf{Z}[k^\times]$ acts by $u \mapsto \langle u \rangle$ on $\mathrm{K}_*^{\mathrm{MW}}(k)$ and by multiplication on $\mathbf{Z}[V \setminus \{0\}]$. Then the map $\partial_v \colon \mathrm{K}_*^{\mathrm{MW}}(F) \to \mathrm{K}_{*-1}^{\mathrm{MW}}(k(v), (\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee)$ given by $\partial_v([\pi_v, u_1, \ldots, u_n]) = [\overline{u}_1, \ldots, \overline{u}_n] \otimes \overline{\pi}_v$ is independent of the choice of uniformizer [Mor12, Remark 3.21].

2.1.3. *A few exact sequences.* Let us collect some short exact sequences involving Milnor–Witt K-groups that will be used later in the text. First of all, the square (2.2) gives a short exact sequence

$$
0 \to \mathrm{I}^{n+1}(F) \to \mathrm{K}_n^{\mathrm{MW}}(F) \xrightarrow{p} \mathrm{K}_n^{\mathrm{M}}(F) \to 0.
\tag{2.3}
$$

Here the map $\mathrm{I}^{n+1}(F) \to \mathrm{K}_n^{\mathrm{MW}}(F)$ is defined by sending the Pfister form $\langle\langle a_1, \ldots, a_{n+1} \rangle\rangle$ to $\eta[a_1 \ldots, a_{n+1}]$. There is a similar sequence with the fundamental ideal on the right hand-side; it takes the form [HT10, p. 6]

$$
0 \to 2\mathrm{K}_n^{\mathrm{M}}(F) \to \mathrm{K}_n^{\mathrm{MW}}(F) \to \mathrm{I}^n(F) \to 0.
\tag{2.4}
$$

The left hand-side homomorphism is here given by mapping $2\{a_1, \ldots, a_n\}$ to $h[a_1, \ldots, a_n]$, where $h := 1 + \langle -1 \rangle$ is the hyperbolic plane.

On the other hand, there is a fundamental computation by Morel [Mor12, Theorem 3.24] (which follows Milnor's computation in the case of Milnor K-theory [Mil71]) showing that there is a split short exact sequence

$$
0 \to \mathrm{K}_n^{\mathrm{MW}}(F) \to \mathrm{K}_n^{\mathrm{MW}}(F(t)) \xrightarrow{\partial} \bigoplus_{x \in (\mathbf{A}_F^1)^{(1)}} \mathrm{K}_{n-1}^{\mathrm{MW}}(k(x)) \to 0.
\tag{2.5}
$$

Here x runs over all closed points of $\mathbf{A}_F^1$.

2.1.4. *Transfers in Milnor–Witt* K-*theory.* If L/F is an extension of number fields, recall that there exist *norm maps*, or *transfer maps* in Milnor K-theory [Kat78],

$$\tau_{L/F} \colon \mathrm{K}_n^{\mathrm{M}}(L) \to \mathrm{K}_n^{\mathrm{M}}(F),$$

which generalize the classical norm $\mathrm{N}_{L/F} \colon L^\times \to F^\times$. Similar maps exist also for Milnor–Witt K-theory, and are constructed in the same way as for the Milnor K-groups. We briefly recall this construction. Let α be a primitive element for the extension of number fields L/F, so that $L = F(\alpha)$. Let $P \in F[t]$ be the minimal polynomial of α. The transfer map

$$\tau_{L/F} \colon \mathrm{K}_n^{\mathrm{MW}}(L) \to \mathrm{K}_n^{\mathrm{MW}}(F)$$

is defined by using the split short exact sequence (2.5) as follows. Choose a section $s := \bigoplus_x s_x$ of ∂, and let $y \in (\mathbf{A}_F^1)^{(1)}$ be the closed point corresponding to P. Then we define $\tau_{L/F}$ as the composition

$$\tau_{L/F} \colon \mathrm{K}_n^{\mathrm{MW}}(L) \xrightarrow{\cong} \mathrm{K}_n^{\mathrm{MW}}(k(y)) \xrightarrow{s_y} \mathrm{K}_{n+1}^{\mathrm{MW}}(F(t)) \xrightarrow{-\partial_\infty^{-1/t}} \mathrm{K}_n^{\mathrm{MW}}(F),$$

where $\partial_\infty^{-1/t}$ is the residue map corresponding to the valuation on $F(t)$ with uniformizer $-1/t$. It is a difficult theorem, proved by Morel in [Mor12, Theorem 4.27], that the map $\tau_{L/F}$ does not depend on the choices made.

2.2. **Milnor–Witt** K-**theory of finite and local fields.** We compute some Milnor–Witt K-groups of finite and local fields. The results follow readily from the structure of the corresponding Milnor K-groups along with some knowledge about fundamental ideals.

Proposition 2.2.
   (i) *If* $\mathbf{F}$ *is a finite field, then* $\mathrm{K}_n^{\mathrm{MW}}(\mathbf{F}) \cong \mathrm{K}_n^{\mathrm{M}}(\mathbf{F})$ *for all* $n \geq 1$.
   (ii) *Let* $v \in \mathrm{Pl}_F$ *be a place of* F. *We have isomorphisms of abelian groups*

$$\mathrm{K}_n^{\mathrm{MW}}(F_v) \cong \begin{cases} \mathrm{K}_n^{\mathrm{M}}(F_v), & v \in \mathrm{Pl}_0, \quad n \geq 2 \\ \mathbf{Z} \oplus A_v, & v \in \mathrm{Pl}_\infty^{\mathrm{r}}, \quad n \geq 1 \\ \mathrm{K}_n^{\mathrm{M}}(F_v), & v \in \mathrm{Pl}_\infty^{\mathrm{c}}, \quad n \geq 0, \end{cases}$$

   where the A_v's *are uniquely divisible abelian groups.*

Remark 2.3. The Milnor K-groups of local fields are known: if v is a finite place of F, then $\mathrm{K}_2^{\mathrm{M}}(F_v) \cong \mu(F_v) \oplus A_v$, where A_v is a uniquely divisible group [Mer83], while $\mathrm{K}_n^{\mathrm{M}}(F_v)$ is uniquely divisible for $n \geq 3$ [Siv85]. If v is an infinite real place of F, then for all $n \geq 1$, $\mathrm{K}_n^{\mathrm{M}}(F_v)$ is the direct sum of a cyclic group of order 2 generated by $\{-1, \ldots, -1\}$ and a uniquely divisible group [Wei13, Example 7.2]. Finally, if v is a complex place of F then $\mathrm{K}_n^{\mathrm{M}}(F_v)$ is uniquely divisible for all $n \geq 1$ [Wei13, Example 7.2].

Proof of Proposition 2.2. The first claim follows from the exact sequence (2.3) since $\mathrm{I}^2(\mathbf{F}) = 0$ for any finite field $\mathbf{F}$ [MH73, p. 81].

For (ii), first assume $v \in \mathrm{Pl}_\infty^{\mathrm{c}}$. Then the statement follows from the fact that $\mathbf{C}$ is quadratically closed [Mor12, Proposition 3.13]. If $v \in \mathrm{Pl}_0$ we have $\mathrm{I}^3(F_v) = 0$ by [MH73, p. 81], and hence the exact sequence (2.3) yields $\mathrm{K}_n^{\mathrm{MW}}(F_v) \cong \mathrm{K}_n^{\mathrm{M}}(F_v)$ for each $n \geq 2$. Finally, suppose $v \in \mathrm{Pl}_\infty^{\mathrm{r}}$. Then we have $\mathrm{I}^{n+1}(F_v) \cong \mathbf{Z}$, generated by the Pfister form $\langle\langle -1, \ldots, -1 \rangle\rangle$ [MH73, p. 81]. Furthermore, by Remark 2.3, $\mathrm{K}_n^{\mathrm{M}}(F_v) \cong \mathbf{Z}/2 \oplus A$, where A is a uniquely divisible abelian group. Using that $2\mathrm{K}_n^{\mathrm{M}}(F_v) \cong A$, the sequence (2.4) above reduces in this case to

$$0 \to A \to \mathrm{K}_n^{\mathrm{MW}}(F_v) \to \mathbf{Z} \to 0.$$

The right hand-side being free, this sequence splits and the result follows. $\qquad\square$

Remark 2.4. In the case of finite places and $n = 2$, the isomorphisms appearing in Proposition 2.2 are given by the classical local Hilbert symbols $(-, -)_v$ [Neu99, V §3]. On the other hand, if $v \in \mathrm{Pl}_\infty^r$, we can think of the signature map $\mathrm{K}_2^{\mathrm{MW}}(\mathbf{R}) \to \mathbf{Z}$ as a "$\mathbf{Z}$-valued Hilbert symbol" extending the classical $\mathbf{Z}/2$-valued Hilbert symbol on $\mathbf{R}$. We will return to this point of view in Section 5.1.

2.3. **Milnor–Witt K-theory of the rationals.** The Witt- and Grothendieck–Witt ring of $\mathbf{Q}$ is determined for instance in [MH73, IV §2]. Thus we know $\mathrm{K}_n^{\mathrm{MW}}(\mathbf{Q})$ for $n \leq 0$. The the remaining groups are given as follows:

Proposition 2.5. *For each $n \geq 1$, the residue map ∂ defined in Section 2.1.2 induces an isomorphism of abelian groups*
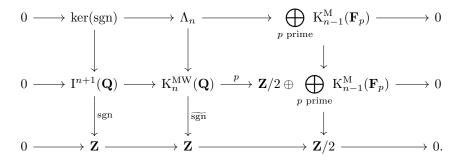
$$\mathrm{K}_n^{\mathrm{MW}}(\mathbf{Q}) \cong \mathbf{Z} \oplus \bigoplus_{p \text{ prime}} \mathrm{K}_{n-1}^{\mathrm{MW}}(\mathbf{F}_p).$$

In particular,

$$\mathrm{K}_1^{\mathrm{MW}}(\mathbf{Q}) \cong \mathbf{Z} \oplus \bigoplus_{p \text{ prime}} \mathrm{GW}(\mathbf{F}_p); \quad \mathrm{K}_2^{\mathrm{MW}}(\mathbf{Q}) \cong \mathbf{Z} \oplus \bigoplus_{p \text{ prime}} \mathbf{F}_p^\times,$$

while $\mathrm{K}_n^{\mathrm{MW}}(\mathbf{Q}) \cong \mathbf{Z}$ for $n \geq 3$.

Proof. For each $n \geq 1$ let Λ_n denote the kernel of the signature homomorphism $\widetilde{\mathrm{sgn}} \colon \mathrm{K}_n^{\mathrm{MW}}(\mathbf{Q}) \to \mathbf{Z}$ defined in (2.1). Since the target of $\widetilde{\mathrm{sgn}}$ is free, we have $\mathrm{K}_n^{\mathrm{MW}}(\mathbf{Q}) \cong \mathbf{Z} \oplus \Lambda_n$, and it remains to identify Λ_n. Since $\mathrm{K}_n^{\mathrm{M}}(\mathbf{Q}) \cong \mathbf{Z}/2 \oplus \bigoplus_{p \text{ prime}} \mathrm{K}_{n-1}^{\mathrm{M}}(\mathbf{F}_p)$ for $n \geq 1$ [BT73], then using the sequence (2.3) it follows that we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\mathrm{sgn}) & \longrightarrow & \Lambda_n & \longrightarrow & \bigoplus_{p \text{ prime}} \mathrm{K}_{n-1}^{\mathrm{M}}(\mathbf{F}_p) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{I}^{n+1}(\mathbf{Q}) & \longrightarrow & \mathrm{K}_n^{\mathrm{MW}}(\mathbf{Q}) & \overset{p}{\longrightarrow} & \mathbf{Z}/2 \oplus \bigoplus_{p \text{ prime}} \mathrm{K}_{n-1}^{\mathrm{M}}(\mathbf{F}_p) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{sgn}} & & \downarrow{\scriptstyle \widetilde{\mathrm{sgn}}} & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 0.
\end{array}
$$

Here sgn denotes the signature map on quadratic forms [MH73, p. 62]. From knowledge about the fundamental ideal of $\mathbf{Q}$ [MH73, IV §2] we conclude the following:

- For $n = 1$ the upper short exact sequence reads

$$0 \to \bigoplus_{p > 2 \text{ prime}} \mathbf{Z}/2 \to \Lambda_1 \to \bigoplus_{p \text{ prime}} \mathbf{Z} \to 0.$$

  The right hand-side being free, this sequence splits, and we conclude that

$$\Lambda_1 \cong \mathbf{Z} \oplus \bigoplus_{p > 2 \text{ prime}} (\mathbf{Z} \oplus \mathbf{Z}/2) \cong \bigoplus_{p \text{ prime}} \mathrm{GW}(\mathbf{F}_p).$$

- If $n \geq 2$, then the map sgn is an isomorphism and the claim follows.

This finishes the proof. $\qquad\square$

2.4. **Valuations.** Let v be a finite place of F. Classically, the v-adic discrete valuation on the local field F_v is a homomorphism $\mathrm{ord}_v \colon F_v^\times \to \mathbf{Z}$ which, by definition, coincides with the residue map $\partial_v \colon \mathrm{K}_1^{\mathrm{M}}(F_v) \to \mathrm{K}_0^{\mathrm{M}}(k(v)) = \mathbf{Z}$ on Milnor K-theory. The units $\mathcal{O}_v^\times$ of the corresponding valuation ring $\mathcal{O}_v$ then coincides with the kernel of ord_v. Following [Gra03, I Definition 1.5] we extend this picture to the real and complex places of F. Thus, if v is an infinite real place of F we denote by ord_v the homomorphism $F_v^\times \to \mathbf{Z}/2$ under which $x \in F_v^\times \cong \mathbf{R}^\times$ maps to 0 if $x > 0$, and 1 if $x < 0$. On the other hand, we set $\mathrm{ord}_v := 0$ whenever v is a complex infinite place. In any case, we let $\mathcal{O}_v^\times$ denote the kernel of ord_v.

Definition 2.6. Let v be a place of F.

(1) If $v \in \mathrm{Pl}_0$, we define
$$\widetilde{\mathrm{ord}}_v \colon \mathrm{K}_1^{\mathrm{MW}}(F_v) \to \mathrm{GW}(k(v), (\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee)$$
by $\widetilde{\mathrm{ord}}_v := \partial_v$, where ∂_v is the residue map on Milnor–Witt K-theory.

(2) If $v \in \mathrm{Pl}_\infty^{\mathrm{r}}$, we define the homomorphism
$$\widetilde{\mathrm{ord}}_v \colon \mathrm{K}_1^{\mathrm{MW}}(F_v) \to \mathbf{Z}$$
as the signature homomorphism (2.1).

(3) For $v \in \mathrm{Pl}_\infty^{\mathrm{c}}$, we let $\widetilde{\mathrm{ord}}_v$ be the trivial homomorphism on $\mathrm{K}_1^{\mathrm{MW}}(F_v)$.

In any case, we let $\mathrm{K}_1^{\mathrm{MW}}(\mathcal{O}_v)$ denote the kernel of $\widetilde{\mathrm{ord}}_v$.

Remark 2.7. For the infinite real places of F we are in Definition 2.6 not really taking $\mathrm{K}_1^{\mathrm{MW}}$ of a field or ring: the notation $\mathrm{K}_1^{\mathrm{MW}}(\mathcal{O}_v)$ is only suggestive in order to obtain analogs of the unit groups attached to each place of F [Gra03, I §1]. For a finite place or complex infinite place v of F, however, the above definition is the same as Morel's definition of Milnor–Witt K-groups of valuation rings [Mor12, §3]. Morel shows in [Mor12, Theorem 3.22] that with this definition, the Milnor–Witt K-theory of $\mathcal{O}_v$ is generated as a ring by η along with the symbols $[u]$ for u a unit of $\mathcal{O}_v$. It follows that this definition coincides with other definitions of Milnor–Witt K-theory of rings given in the literature, for example that of Schlichting in [Sch17, §4].

Lemma 2.8. *Let $v \in \mathrm{Pl}_F$ be a place of F.*

- *If v is either an infinite place or a nondyadic finite place, then $\mathrm{K}_1^{\mathrm{MW}}(\mathcal{O}_v) \cong \mathcal{O}_v^\times$ (where, by definition, $\mathcal{O}_v^\times := \mathbf{R}_{>0}^\times$ for $v \in \mathrm{Pl}_\infty^{\mathrm{r}}$ and $\mathcal{O}_v^\times := \mathbf{C}^\times$ for $v \in \mathrm{Pl}_\infty^{\mathrm{c}}$).*
- *If v is a dyadic place, then there is a short exact sequence*
$$0 \to \mathrm{I}^2(F_v) \to \mathrm{K}_1^{\mathrm{MW}}(\mathcal{O}_v) \to \mathcal{O}_v^\times \to 0.$$

Proof. The statement is clear for the infinite places by Proposition 2.2. Let $v \in \mathrm{Pl}_0$ be a finite place of F, and consider the commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{K}_1^{\mathrm{MW}}(\mathcal{O}_v) & \longrightarrow & \mathrm{K}_1^{\mathrm{MW}}(F_v) & \xrightarrow{\partial_v} & \mathrm{K}_0^{\mathrm{MW}}(k(v)) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p''} & & \\
0 & \longrightarrow & \mathcal{O}_v^\times & \longrightarrow & F_v^\times & \xrightarrow{\mathrm{ord}_v} & \mathbf{Z} & \longrightarrow & 0.
\end{array}
$$

If $v \nmid 2$, then the induced map $\ker p \cong \mathrm{I}^2(F_v) \cong \mathbf{Z}/2 \to \ker p'' \cong \mathrm{I}(k(v)) \cong \mathbf{Z}/2$ is an isomorphism and so we conclude by the snake lemma. If $v \mid 2$, then $k(v)$ is quadratically closed and hence the map $p'' \colon \mathrm{K}_0^{\mathrm{MW}}(k(v)) \to \mathbf{Z}$ is an isomorphism. In this case the snake lemma applied to the above diagram yields $\ker p' \cong \ker p \cong \mathrm{I}^2(F_v)$. $\qquad\square$

3. Topology on $K_1^{MW}(F_v)$

We now aim to put a topology on $K_1^{MW}(F_v)$, for each place v of F, in such a way that the homomorphism $p \colon K_1^{MW}(F_v) \to F_v^\times$ becomes a covering map.

3.0.1.   In general, suppose that we are given an abelian group G (written additively) along with a subgroup H of G which is a topological group. We can then extend the topology on H to a topology on G as follows. Choose a set theoretical section s of the quotient map $\pi \colon G \to G/H$, and let G/H have the discrete topology. Then s defines a partition

$$G = \coprod_{x \in G/H} (s(x) + H)$$

of G, and there is a natural topology on each coset $(s(x) + H) \cong H$ coming from that on H. We can then declare a subset $U \subseteq G$ to be open if and only if $U \cap (s(x) + H)$ is open for every $x \in G/H$. This turns G into a topological group.

3.0.2.   Now let v be a place of F. Then, by taking $n = 1$ in the short exact sequence (2.4) we get the exact sequence

$$0 \to F_v^{\times 2} \to K_1^{MW}(F_v) \to I(F_v) \to 0.$$

In the notations above, we can then let $G := K_1^{MW}(F_v)$ and $H := F_v^{\times 2}$. The quotient map $\pi \colon G \to G/H$ is given as $[a] \mapsto \langle\!\langle a \rangle\!\rangle$. Using the set theoretical section $s(\langle\!\langle x \rangle\!\rangle) := [x]$ of π along with the fact that $F_v^{\times 2}$ is a topological group, we obtain by the above a topology on $K_1^{MW}(F_v)$.

Proposition 3.1. *For any $v \in \mathrm{Pl}_F$, the map $p \colon K_1^{MW}(F_v) \to F_v^\times$ is a covering of topological groups. Furthermore, if v is a finite place of F, then we have the following:*

  (1) *The map p is proper.*
  (2) *The space $K_1^{MW}(F_v)$ is locally compact and totally disconnected, and $K_1^{MW}(\mathcal{O}_v)$ is compact in $K_1^{MW}(F_v)$.*

Proof. If v is a complex place, then by Proposition 2.2 there is nothing to show. So we may assume that $v \in \mathrm{Pl}_F^{\mathrm{nc}}$. We have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_v^{\times 2} & \longrightarrow & K_1^{MW}(F_v) & \longrightarrow & I(F_v) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle p} & & \downarrow & & \\
0 & \longrightarrow & F_v^{\times 2} & \longrightarrow & F_v^\times & \longrightarrow & I(F_v)/I^2(F_v) & \longrightarrow & 0.
\end{array}
$$

That p is a covering map follows from this diagram along with the fact that the quotient topology on $F_v^\times / F_v^{\times 2} \cong I(F_v)/I^2(F_v)$ is the discrete topology.

Now let v be a finite place of F. Then $I^2(F_v) \cong \mathbf{Z}/2$, so that we have an exact sequence

$$0 \to \mathbf{Z}/2 \to K_1^{MW}(F_v) \xrightarrow{p} F_v^\times \to 0.$$

Thus p is in this case a two sheeted covering map, hence proper. The claim (2) follows from Lemma 2.8 along with the properties of the topology on $F_v^\times$. $\qquad\square$

4. Idèles

Recall that the idèle group $\mathbf{J}_F$ of F is the restricted product of the groups of units of the completions $F_v^\times$ with respect to the compact subgroups $\mathcal{O}_v^\times$, with the restricted product topology. Equivalently, $\mathbf{J}_F$ can be defined as the direct limit $\varinjlim_S \mathbf{J}_F(S)$, where S is a set of places of F and

$$\mathbf{J}_F(S) := \prod_{v \in S} F_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times.$$

Definition 4.1. For any finite set S of places of F, put

$$\widetilde{\mathbf{J}}_F(S) := \prod_{v \in S} \mathrm{K}_1^{\mathrm{MW}}(F_v) \times \prod_{v \notin S} \mathrm{K}_1^{\mathrm{MW}}(\mathcal{O}_v).$$

The *Milnor–Witt idèle group* $\widetilde{\mathbf{J}}_F$ of F is defined as the direct limit

$$\widetilde{\mathbf{J}}_F := \varinjlim_S \widetilde{\mathbf{J}}_F(S),$$

where S ranges over all finite subsets of Pl_F.

Proposition 4.2. *There is a short exact sequence*

$$0 \to \bigoplus_{v \in \mathrm{Pl}_F} \mathrm{I}^2(F_v) \to \widetilde{\mathbf{J}}_F \xrightarrow{p} \mathbf{J}_F \to 0,$$

where the map p is induced by the projection maps from Milnor–Witt K-theory to Milnor K-theory.

Proof. For the definition of the homomorphism p, note that we have projection maps $\widetilde{\mathbf{J}}_F(S) \to \mathbf{J}_F(S)$ for any finite set of places of F. Here $\mathbf{J}_F(S) := \prod_{v \in S} F_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$. The map p is then the induced morphism on the colimit, which we notice is surjective.

Let $\mathrm{Pl}_2 := \{v \in \mathrm{Pl}_F : v \mid 2\}$ denote the dyadic places of F. It follows from Lemma 2.8 along with the sequence (2.3) that the kernel of the projection map $\widetilde{\mathbf{J}}_F(S) \to \mathbf{J}_F(S)$ is $\bigoplus_{v \in S \cup \mathrm{Pl}_2} \mathrm{I}^2(F_v)$. Passing to the colimit as S varies, we thus find that the kernel of the map $\widetilde{\mathbf{J}}_F \to \mathbf{J}_F$ is $\bigoplus_{v \in \mathrm{Pl}_F} \mathrm{I}^2(F_v)$. $\qquad\square$

Definition 4.3. Let S be a finite set of places of F. We define a topology on $\widetilde{\mathbf{J}}_F(S)$ by taking the topology generated by the sets

$$W_S := \prod_{v \in S} U_v \times \prod_{v \notin S} \mathrm{K}_1^{\mathrm{MW}}(\mathcal{O}_v),$$

where the U_v's are open subsets of $\mathrm{K}_1^{\mathrm{MW}}(F_v)$ for each $v \in S$. This defines a topology on $\widetilde{\mathbf{J}}_F$ via the direct limit topology.
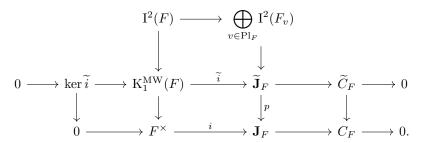
Lemma 4.4. *The Milnor–Witt idèle group $\widetilde{\mathbf{J}}_F$ is a locally compact topological group.*

Proof. This follows from the fact that $\widetilde{\mathbf{J}}_F$ is the restricted product of the groups $\mathrm{K}_1^{\mathrm{MW}}(F_v)$ with respect to the subgroups $\mathrm{K}_1^{\mathrm{MW}}(\mathcal{O}_v)$, which are compact for almost all v by Proposition 3.1. $\quad\square$

Definition 4.5. Define the map $\widetilde{i} \colon \mathrm{K}_1^{\mathrm{MW}}(F) \to \widetilde{\mathbf{J}}_F$ by $\widetilde{i}([x]) := ([i_v(x)])_{v \in \mathrm{Pl}_F}$. We refer to the cokernel $\widetilde{C}_F := \mathrm{coker}\, \widetilde{i}$ as the *Milnor–Witt idèle class group of F.*

Lemma 4.6. *The map $\widetilde{i}$ is injective. Hence $\widetilde{C}_F = \widetilde{\mathbf{J}}_F / \widetilde{i}(\mathrm{K}_1^{\mathrm{MW}}(F))$.*

Proof. By Proposition 4.2, the kernel of the map $\widetilde{\mathbf{J}}_F \to \mathbf{J}_F$ is $\bigoplus_{v \in \mathrm{Pl}_F} \mathrm{I}^2(F_v)$. Hence there is a commutative diagram

$$
\begin{array}{ccccccccc}
 & & \mathrm{I}^2(F) & \longrightarrow & \bigoplus_{v \in \mathrm{Pl}_F} \mathrm{I}^2(F_v) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \ker \widetilde{i} \longrightarrow \mathrm{K}_1^{\mathrm{MW}}(F) & \xrightarrow{\ \widetilde{i}\ } & \widetilde{\mathbf{J}}_F & \longrightarrow & \widetilde{C}_F & \longrightarrow & 0 \\
 & & \downarrow \qquad\qquad \downarrow & & \downarrow {\scriptstyle p} & & \downarrow & & \\
0 & \longrightarrow & F^\times & \xrightarrow{\ i\ } & \mathbf{J}_F & \longrightarrow & C_F & \longrightarrow & 0.
\end{array}
$$

By the Hasse–Minkowski theorem, the map $I^2(F) \to \bigoplus_{v \in \mathrm{Pl}_F} I^2(F_v)$ is injective. It follows from this and a diagram chase that $\ker \widetilde{i} = 0$. $\qquad\square$

4.0.1. *Idèles of volume zero.* Recall that we have a volume map $\mathrm{vol} \colon \mathbf{J}_F \to \mathbf{R}_{>0}^\times$ defined as $\mathrm{vol}(x) := \prod_{v \in \mathrm{Pl}_F} |x_v|_{F_v}$ [Neu99, p. 361]. Here $x = (x_v)_{v \in \mathrm{Pl}_F} \in \mathbf{J}_F$, and $|\cdot|_{F_v}$ is the v-adic absolute value on F_v. We let $\mathbf{J}_F^0$ denote the kernel of the volume map. By the classical product formula for absolute values [Neu99, Chapter III, Proposition 1.3] we have $F^\times \subseteq \mathbf{J}_F^0$. The fact that the resulting quotient group $C_F^0 := \mathbf{J}_F^0 / F^\times$ is compact is an equivalent formulation of Dirichlet's unit theorem and the finiteness of the ideal class group [Gra03, I Proposition 4.2.7].

Definition 4.7. Let $p \colon \widetilde{\mathbf{J}}_F \to \mathbf{J}_F$ denote the projection map. We define a volume map $\widetilde{\mathrm{vol}} \colon \widetilde{\mathbf{J}}_F \to \mathbf{R}_{>0}^\times$ on $\widetilde{\mathbf{J}}_F$ by $\widetilde{\mathrm{vol}} := \mathrm{vol} \circ p$, and put $\widetilde{\mathbf{J}}_F^0 := \ker(\widetilde{\mathrm{vol}})$. Let $\widetilde{C}_F^0$ be the quotient group
$$\widetilde{C}_F^0 := \widetilde{\mathbf{J}}_F^0 / \widetilde{i}(\mathrm{K}_1^{\mathrm{MW}}(F)).$$

Remark 4.8. By Lemma 4.6 along with the product formula for absolute values we have $\widetilde{i}(\mathrm{K}_1^{\mathrm{MW}}(F)) \subseteq \widetilde{\mathbf{J}}_F^0$, justifying the definition of $\widetilde{C}_F^0$.

Proposition 4.9. *There is a short exact sequence*
$$0 \to \mathbf{Z}/2 \to \widetilde{C}_F^0 \to C_F^0 \to 0.$$

Hence $\widetilde{C}_F^0$ is a compact topological group.

Proof. By definition, the kernel $\bigoplus_{v \in \mathrm{Pl}_F} I^2(F_v)$ of the projection map $p \colon \widetilde{\mathbf{J}}_F \to \mathbf{J}_F$ is contained in $\widetilde{\mathbf{J}}_F^0$. Consider the commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{K}_1^{\mathrm{MW}}(F) & \xrightarrow{\ \widetilde{i}\ } & \widetilde{\mathbf{J}}_F^0 & \longrightarrow & \widetilde{C}_F^0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle p} & & \downarrow & & \\
0 & \longrightarrow & F^\times & \xrightarrow{\ i\ } & \mathbf{J}_F^0 & \longrightarrow & C_F^0 & \longrightarrow & 0.
\end{array}
$$

By the snake lemma it suffices to show that the cokernel of the map $I^2(F) \to \bigoplus_{v \in \mathrm{Pl}_F} I^2(F_v)$ is $\mathbf{Z}/2$. But this follows from the snake lemma applied to the commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I^3(F) & \longrightarrow & I^2(F) & \longrightarrow & I^2(F)/I^3(F) & \longrightarrow & 0 \\
& & {\scriptstyle\cong}\downarrow{\scriptstyle\iota} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{v \in \mathrm{Pl}_F} I^3(F_v) & \longrightarrow & \bigoplus_{v \in \mathrm{Pl}_F} I^2(F_v) & \longrightarrow & \bigoplus_{v \in \mathrm{Pl}_F} I^2(F_v)/I^3(F_v) & \longrightarrow & 0,
\end{array}
$$

using the fact that the left hand vertical map ι is an isomorphism by Lemma 4.10 below, and that the cokernel of the right hand vertical map is $\mathbf{Z}/2$ by [Mil70, Lemma A.1]. $\qquad\square$

Lemma 4.10. *For any number field F, the canonical map $\iota \colon I^3(F) \to \bigoplus_{v \in \mathrm{Pl}_F} I^3(F_v)$ induced by the embeddings $i_v \colon F \hookrightarrow F_v$ is an isomorphism.*

Proof. The map is injective by the Hasse–Minkowski theorem. We must show that it is surjective.

If $v \in \mathrm{Pl}_0 \cup \mathrm{Pl}_\infty^c$, then $I^3(F_v) = 0$ and there is nothing to show. On the other hand, if v is an infinite real place, then $I^3(F_v) \cong \mathbf{Z}$ by [MH73, III Corollary 2.7]. By strong approximation [Neu99, p. 193] we can find an element $a \in F$ which is negative in the i-th ordering on F and positive otherwise. Then $\langle\!\langle -1, -1, a \rangle\!\rangle \in I^3(F)$ maps to the i-th unit vector in $\bigoplus_{v \in \mathrm{Pl}^r} I^3(F_v)$. $\qquad\square$

Remark 4.11. One can speculate on whether there is a variant of the abelianized étale fundamental group of $\mathrm{Spec}(\mathcal{O}_F)$ which is the recipient of a reciprocity map defined on the Milnor–Witt idèle class group $\widetilde{C}_F$ and which lifts the classical Artin reciprocity map $\rho \colon C_F \to \mathrm{Gal}(L/F)^{\mathrm{ab}}$.

5. A Moore reciprocity sequence for Milnor–Witt K-theory

The classical result of Moore on uniqueness of reciprocity laws states that there is an exact sequence

$$0 \to \mathrm{WK}_2(F) \to \mathrm{K}_2(F) \xrightarrow{h} \bigoplus_{v \in \mathrm{Pl}_F^{\mathrm{nc}}} \mu(F_v) \xrightarrow{\pi} \mu(F) \to 0.$$

Here h denotes the global Hilbert symbol, and the group $\mathrm{WK}_2(F)$ is known as the *wild kernel* [Gra03, II §7]. Moreover, the map π is defined as

$$\pi((\zeta_v)_v) := \prod_{v \in \mathrm{Pl}_F^{\mathrm{nc}}} \zeta_v^{m_v/m},$$

where $m_v := \#\mu(F_v)$ and $m := \#\mu(F)$. In this section we will show that also $\mathrm{K}_2^{\mathrm{MW}}(F)$ fits into a similar exact sequence.

5.1. **Hilbert symbols.** In order to obtain a Moore reciprocity sequence for $\mathrm{K}_2^{\mathrm{MW}}$, we first need to define Hilbert symbols in the setting of Milnor–Witt K-theory. These should be particular instances of maps of the following type:

Definition 5.1. Let A be a $\mathrm{GW}(F)$-module. A *Milnor–Witt symbol on F with values in A* is a $\mathrm{GW}(F)$-bilinear map

$$(-,-) \colon \mathrm{K}_1^{\mathrm{MW}}(F) \times \mathrm{K}_1^{\mathrm{MW}}(F) \to A$$

satisfying $([a],[1-a]) = 0$ for all $a \in F^\times \setminus \{1\}$.

Remark 5.2. Note that any abelian group A is also a $\mathrm{GW}(F)$-module via the rank map $\mathrm{rk} \colon \mathrm{GW}(F) \to \mathbf{Z}$. Thus we should think of the definition of a Milnor–Witt symbol as a lift of the classical notion of a *symbol*, i.e., a $\mathbf{Z}$-bilinear map $(-,-) \colon F^\times \times F^\times \to A$ satisfying the Steinberg relation [Tat71]. Just as $\mathrm{K}_2(F)$ is the universal object with respect to symbol maps, it follows from [Mor12, Remark 3.2] that $\mathrm{K}_2^{\mathrm{MW}}(F)$ is the universal object with respect to Milnor–Witt symbols.

Definition 5.3. For any place v of F, let B_v be the group

$$B_v := \begin{cases} \mu(F_v), & v \in \mathrm{Pl}_0 \\ \mathbf{Z}, & v \in \mathrm{Pl}_\infty^{\mathrm{r}} \\ 0, & v \in \mathrm{Pl}_\infty^{\mathrm{c}}. \end{cases}$$

Furthermore, define a map $q_v \colon B_v \to \mu(F_v)$ by letting q_v be the identity if $v \in \mathrm{Pl}_0$; reduction modulo 2 if $v \in \mathrm{Pl}_\infty^{\mathrm{r}}$; or the trivial homomorphism if $v \in \mathrm{Pl}_\infty^{\mathrm{c}}$. Finally, write $q := \bigoplus_{v \in \mathrm{Pl}_F^{\mathrm{nc}}} q_v$.

5.1.1. By Proposition 2.2 we have $\mathrm{K}_2^{\mathrm{MW}}(F_v) \cong B_v \oplus A_v$ for each $v \in \mathrm{Pl}_F^{\mathrm{nc}}$. Thus we can define, for any noncomplex place v of F, the homomorphism $b_v \colon \mathrm{K}_2^{\mathrm{MW}}(F_v) \to B_v$ as the composition of the isomorphism $\mathrm{K}_2^{\mathrm{MW}}(F_v) \cong B_v \oplus A_v$ followed by the projection $B_v \oplus A_v \to B_v$. Thus, if v is a finite place of F then b_v is just the classical local Hilbert symbol, while for $v \in \mathrm{Pl}_\infty^{\mathrm{r}}$ the map b_v is the signature homomorphism (2.1).

Definition 5.4. For each $v \in \mathrm{Pl}_F^{\mathrm{nc}}$, let $h_v^{\mathrm{MW}} \colon \mathrm{K}_2^{\mathrm{MW}}(F) \to B_v$ denote the composite

$$\mathrm{K}_2^{\mathrm{MW}}(F) \xrightarrow{i_v} \mathrm{K}_2^{\mathrm{MW}}(F_v) \xrightarrow{b_v} B_v,$$

where the first map is induced by the embedding $i_v \colon F \hookrightarrow F_v$. Moreover, let

$$h^{\mathrm{MW}} \colon \mathrm{K}_2^{\mathrm{MW}}(F) \to \bigoplus_{v \in \mathrm{Pl}_F^{\mathrm{nc}}} B_v$$

be the map $h^{\mathrm{MW}} := \bigoplus_{v \in \mathrm{Pl}_F^{\mathrm{nc}}} h_v^{\mathrm{MW}}$.

Definition 5.5. Let $v \in \mathrm{Pl}_v^{\mathrm{nc}}$ be a noncomplex place of F. We define the *local Milnor–Witt Hilbert symbol at v*, denoted $(-,-)_v^{\mathrm{MW}}$, as the composite

$$(-,-)_v^{\mathrm{MW}} \colon \mathrm{K}_1^{\mathrm{MW}}(F) \times \mathrm{K}_1^{\mathrm{MW}}(F) \to \mathrm{K}_2^{\mathrm{MW}}(F) \xrightarrow{h_v^{\mathrm{MW}}} B_v.$$

Here the first map is multiplication on Milnor–Witt K-theory.

Lemma 5.6. *For any $v \in \mathrm{Pl}_F^{\mathrm{nc}}$, we have a commutative diagram*

$$
\begin{array}{ccc}
\mathrm{K}_1^{\mathrm{MW}}(F) \times \mathrm{K}_1^{\mathrm{MW}}(F) & \xrightarrow{(-,-)_v^{\mathrm{MW}}} & B_v \\
{\scriptstyle p \times p}\big\downarrow & & \big\downarrow{\scriptstyle q_v} \\
F^\times \times F^\times & \xrightarrow[(-,-)_v]{} & \mu(F_v).
\end{array}
$$

Proof. The statement is clear for the finite places, so let v be an infinite real place of F. Recall that $(a,b)_v \in \mathbf{Z}/2$ is defined as 0 if $i_v(a)X^2 + i_v(b)Y^2 = 1$ has a solution in $F_v \cong \mathbf{R}$, and 1 otherwise [Mil71, p. 104]. On the other hand, the map (2.1) carries $[a,b]$ to 0 if any of $i_v(a)$ or $i_v(b)$ is positive in the ordering v, and to 1 otherwise. So $([a],[b])_v^{\mathrm{MW}} \equiv (a,b)_v \pmod{2}$. $\qquad\square$

5.1.2. *Wild kernels.* Using the maps h_v^{MW} we can define wild kernels for Milnor–Witt K-theory similarly as in the classical case:

Definition 5.7. Let $\mathrm{WK}_2^{\mathrm{MW}}(F)$ denote the kernel of the map $h^{\mathrm{MW}} = \bigoplus_{v \in \mathrm{Pl}_F^{\mathrm{nc}}} h_v^{\mathrm{MW}}$.

5.1.3. We are now ready to formulate a Moore reciprocity sequence for Milnor–Witt K-theory:

Proposition 5.8. *For any number field F, there is an isomorphism $\mathrm{WK}_2^{\mathrm{MW}}(F) \cong \mathrm{WK}_2(F)$. Moreover, there is an exact sequence*

$$0 \to \mathrm{WK}_2(F) \to \mathrm{K}_2^{\mathrm{MW}}(F) \xrightarrow{h^{\mathrm{MW}}} \bigoplus_{v \in \mathrm{Pl}_F^{\mathrm{nc}}} B_v \to \mu(F) \to 0,$$

where v runs over all noncomplex places of F, and where the B_v's are the groups defined in Definition 5.3.

Proof. Consider the following commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
\ker p' & \longrightarrow & \mathrm{I}^3(F) & \xrightarrow[\cong]{\iota} & \bigoplus_{v \in \mathrm{Pl}_F^{\mathrm{nc}}} \mathrm{I}^3(F_v) & \longrightarrow & \ker q' \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
0 \longrightarrow \mathrm{WK}_2^{\mathrm{MW}}(F) & \longrightarrow & \mathrm{K}_2^{\mathrm{MW}}(F) & \xrightarrow{h^{\mathrm{MW}}} & \bigoplus_{v \in \mathrm{Pl}_F^{\mathrm{nc}}} B_v & \longrightarrow & \mathrm{coker}\, h^{\mathrm{MW}} \longrightarrow 0 \\
{\scriptstyle p'}\big\downarrow & & {\scriptstyle p}\big\downarrow & & {\scriptstyle q}\big\downarrow & & {\scriptstyle q'}\big\downarrow \\
0 \longrightarrow \mathrm{WK}_2(F) & \longrightarrow & \mathrm{K}_2(F) & \xrightarrow{h} & \bigoplus_{v \in \mathrm{Pl}_F^{\mathrm{nc}}} \mu(F_v) & \xrightarrow{\pi} & \mu(F) \longrightarrow 0.
\end{array}
$$

Here p' and q' are induced from p and q, respectively. According to Lemma 4.10, the map $\iota \colon \mathrm{I}^3(F) \to \bigoplus_{v \in \mathrm{Pl}_F^{\mathrm{nc}}} \mathrm{I}^3(F_v)$ is an isomorphism. It follows from this and a diagram chase that p' and q' are isomorphisms. $\qquad\square$

Example 5.9. We see from Proposition 2.5 that in the case $F = \mathbf{Q}$, the exact sequence of Proposition 5.8 reads

$$0 \to \mathbf{Z} \oplus \bigoplus_{p \text{ prime}} \mathbf{F}_p^\times \to \mathbf{Z} \oplus \mathbf{Z}/2 \oplus \bigoplus_{p \text{ prime}} \mathbf{F}_p^\times \to \mathbf{Z}/2 \to 0.$$

Here we have used that $\mathrm{WK}_2(\mathbf{Q}) = 0$.

6. Regular kernels and Milnor–Witt K-theory of rings of integers

6.1. We will now consider various kernels of the Hilbert symbols and the tame symbols. Recall that in classical K-theory, there are three subgroups of $\mathrm{K}_2(F)$ of particular interest:

$$\mathrm{K}_2(\mathcal{O}_F) = \ker\left(\partial\colon \mathrm{K}_2(F) \to \bigoplus_{v \in \mathrm{Pl}_0} k(v)^\times\right)$$

$$\cup\mathsf{I}$$

$$\mathrm{K}_2^+(\mathcal{O}_F) := \ker\left(\bigoplus_{v \in \mathrm{Pl}_\infty^r} h_v\big|_{\mathrm{K}_2(\mathcal{O}_F)}\colon \mathrm{K}_2(\mathcal{O}_F) \to \bigoplus_{v \in \mathrm{Pl}_\infty^r} \mu(F_v)\right)$$

$$\cup\mathsf{I}$$

$$\mathrm{WK}_2(F) := \ker\left(h\colon \mathrm{K}_2(F) \to \bigoplus_{v \in \mathrm{Pl}_F^{nc}} \mu(F_v)\right).$$

We have already encountered the wild kernel $\mathrm{WK}_2(F)$, and it is a classical result of Quillen [Qui73, §5] that $\mathrm{K}_2(\mathcal{O}_F)$ is the kernel of the tame symbols ∂. The group $\mathrm{K}_2^+(\mathcal{O}_F)$ was introduced by Gras in [Gra86] and is referred to as the *regular kernel*, or the *narrow K_2-group*. It is a modification of $\mathrm{K}_2(\mathcal{O}_F)$ that takes into account also the real places of F.

Definition 6.1. For any $n \geq 1$, let $\mathrm{K}_n^{\mathrm{MW}}(\mathcal{O}_F)$ denote the kernel of

$$\partial := \bigoplus_{v \in \mathrm{Pl}_0} \partial_v^{\pi_v}\colon \mathrm{K}_n^{\mathrm{MW}}(F) \to \bigoplus_{v \in \mathrm{Pl}_0} \mathrm{K}_{n-1}^{\mathrm{MW}}(k(v)).$$

Here π_v is any choice of uniformizer for the discrete valuation v. Moreover, we let

$$\mathrm{K}_n^{\mathrm{MW},+}(\mathcal{O}_F) := \ker\left(\mathrm{K}_n^{\mathrm{MW}}(\mathcal{O}_F) \to \mathbf{Z}^{r_1}\right); \quad \mathrm{K}_n^{\mathrm{MW},+}(F) := \ker\left(\mathrm{K}_n^{\mathrm{MW}}(F) \to \mathbf{Z}^{r_1}\right),$$

where the maps are given by the signatures with respect to the orderings on F, as defined in (2.1).

Remark 6.2. By [Mor12, Proposition 3.17 (3)], the kernel of $\partial_v^{\pi_v}$ is independent of the choice of uniformizer π_v.

Example 6.3. By Proposition 2.5 we have $\mathrm{K}_n^{\mathrm{MW}}(\mathbf{Z}) \cong \mathbf{Z}$ for all $n \geq 1$.

Remark 6.4. If S is a set of places of F containing the infinite places, note that we can also define Milnor–Witt K-theory of the ring of S-integers $\mathcal{O}_{F,S}$ in F as

$$\mathrm{K}_n^{\mathrm{MW}}(\mathcal{O}_{F,S}) := \ker\left(\bigoplus_{v \notin S} \partial_v^{\pi_v}\colon \mathrm{K}_n^{\mathrm{MW}}(F) \to \bigoplus_{v \notin S} \mathrm{K}_{n-1}^{\mathrm{MW}}(k(v))\right).$$

Here π_v is a uniformizer for the place v. For $n = 2$, the groups $\mathrm{K}_2^{\mathrm{MW}}(\mathcal{O}_{F,S})$ were also considered in [Hut16]. More precisely, Hutchinson defines a subgroup $\widetilde{\mathrm{K}}_2(2, \mathcal{O}_{F,S})$ of the second unstable K-group $\mathrm{K}_2(2, F)$ by

$$\widetilde{\mathrm{K}}_2(2, \mathcal{O}_{F,S}) := \ker\left(\mathrm{K}_2(2, F) \to \bigoplus_{v \notin S} k(v)^\times \right).$$

By [Hut16, Proposition 3.12] there is a natural isomorphism $\mathrm{K}_2(2, F) \xrightarrow{\cong} \mathrm{K}_2^{\mathrm{MW}}(F)$, hence $\widetilde{\mathrm{K}}_2(2, \mathcal{O}_{F,S}) \cong \mathrm{K}_2^{\mathrm{MW}}(\mathcal{O}_{F,S})$. The main theorem of [Hut16] states that if S is a set of places of $\mathbf{Q}$ containing 2 and 3, then $\mathrm{K}_2^{\mathrm{MW}}(\mathbf{Z}_S) \cong \mathrm{H}_2(\mathrm{SL}_2(\mathbf{Z}_S), \mathbf{Z})$ (where $\mathbf{Z}_S$ denotes the localization of $\mathbf{Z}$ at the primes of S). In particular, $\mathrm{K}_2^{\mathrm{MW}}(\mathbf{Z}[1/m]) \cong \mathrm{H}_2(\mathrm{SL}_2(\mathbf{Z}[1/m]), \mathbf{Z})$ provided $6 \mid m$. It is a conjecture that this isomorphism holds for any even m.

Proposition 6.5.

(i) *We have a short exact sequence*

$$0 \to \mathbf{Z}^{r_1} \to \mathrm{K}_2^{\mathrm{MW}}(\mathcal{O}_F) \xrightarrow{q} \mathrm{K}_2(\mathcal{O}_F) \to 0,$$

where the homomorphism q is induced by the forgetful map from Milnor–Witt K-theory to Milnor K-theory.

(ii) *We have $\mathrm{K}_2^{\mathrm{MW},+}(\mathcal{O}_F) \cong \mathrm{K}_2^+(\mathcal{O}_F)$ and $\mathrm{K}_2^{\mathrm{MW},+}(F) \cong \mathrm{K}_2^+(F)$.*

(iii) *The groups $\mathrm{K}_2^{\mathrm{MW}}(\mathcal{O}_F)$ and $\mathrm{K}_2^{\mathrm{MW}}(F)$ decompose as*

$$\mathrm{K}_2^{\mathrm{MW}}(\mathcal{O}_F) \cong \mathrm{K}_2^+(\mathcal{O}_F) \oplus \mathbf{Z}^{r_1}; \quad \mathrm{K}_2^{\mathrm{MW}}(F) \cong \mathrm{K}_2^+(F) \oplus \mathbf{Z}^{r_1}.$$

In particular, by Garland's theorem on the finiteness of $\mathrm{K}_2(\mathcal{O}_F)$ [Gar71], the group $\mathrm{K}_2^{\mathrm{MW}}(\mathcal{O}_F)$ is a finitely generated abelian group of rank r_1.

Proof. For the first point, consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{K}_2^{\mathrm{MW}}(\mathcal{O}_F) & \longrightarrow & \mathrm{K}_2^{\mathrm{MW}}(F) & \xrightarrow{\partial} & \bigoplus_{v \in \mathrm{Pl}_0} \mathrm{K}_1^{\mathrm{MW}}(k(v)) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \mathrm{K}_2(\mathcal{O}_F) & \longrightarrow & \mathrm{K}_2(F) & \xrightarrow{\partial} & \bigoplus_{v \in \mathrm{Pl}_0} k(v)^\times & \longrightarrow & 0.
\end{array}
$$

By Proposition 2.2 (i), the right hand vertical map is an isomorphism. Using Lemma 4.10, the claim follows.

For the second point, first note that we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{K}_2^{\mathrm{MW},+}(\mathcal{O}_F) & \longrightarrow & \mathrm{K}_2^{\mathrm{MW}}(\mathcal{O}_F) & \longrightarrow & \mathbf{Z}^{r_1} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{K}_2^+(\mathcal{O}_F) & \longrightarrow & \mathrm{K}_2(\mathcal{O}_F) & \longrightarrow & (\mathbf{Z}/2)^{r_1} & \longrightarrow & 0.
\end{array}
\tag{6.1}
$$

Here the upper right hand-side map is given by the signature homomorphisms (2.1); commutativity of the right hand square follows similarly as the proof of Lemma 5.6. The first claim of (ii) then follows from (i) along with the snake lemma applied to the diagram (6.1); the second claim of (ii) follows similarly.

For (iii), we use (ii) along with the observation that since the right hand-side in the upper short exact sequence of (6.1) is free, the sequence splits. A similar argument works for $\mathrm{K}_2^{\mathrm{MW}}(F)$. $\quad\square$

Remark 6.6. In contrast to point (iii) of Proposition 6.5, the corresponding short exact sequence for $K_2(\mathcal{O}_F)$,

$$0 \to K_2^+(\mathcal{O}_F) \to K_2(\mathcal{O}_F) \to (\mathbf{Z}/2)^{r_1} \to 0,$$

need not split. For example, [Keu89, Example 3.10] shows that if $F = \mathbf{Q}(\sqrt{14})$, then $\mathrm{rk}_2(K_2^+(\mathcal{O}_F)) = 1$ while $\mathrm{rk}_2(K_2(\mathcal{O}_F)) = 2$. On the other hand, the sequence

$$0 \to K_2^+(F) \to K_2(F) \to (\mathbf{Z}/2)^{r_1} \to 0$$

is always split (see [Keu89, §2.1]).

6.2. **Sample computations of $K_2^{\mathrm{MW}}(\mathcal{O}_F)$.** Using Proposition 6.5 along with similar results for $K_2(\mathcal{O}_F)$ we deduce some computations of $K_2^{\mathrm{MW}}(\mathcal{O}_F)$ for various fields F. Below we make use of the calculations of [BG04] to determine $K_2^{\mathrm{MW}}(\mathcal{O}_F)$.

In the following table, we consider the number fields $F = \mathbf{Q}[x]/(P)$ defined by the polynomial P. We let Δ_F denote the discriminant of F, and (r_1, r_2) the signature.

P	Δ_F	(r_1, r_2)	$K_2(\mathcal{O}_F)$	$K_2^{\mathrm{MW}}(\mathcal{O}_F)$
$x^3 + x^2 - 2x - 1$	49	$(3, 0)$	$(\mathbf{Z}/2)^3$	$\mathbf{Z}^3$
$x^3 + x^2 + 2x + 1$	-23	$(1, 1)$	$\mathbf{Z}/2$	$\mathbf{Z}$
$x^3 + x^2 + 3$	-255	$(1, 1)$	$\mathbf{Z}/6$	$\mathbf{Z}/3 \oplus \mathbf{Z}$
$x^4 - x - 1$	-283	$(2, 1)$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}^2$
$x^5 - x^3 - x^2 + x + 1$	1609	$(1, 2)$	$\mathbf{Z}/2$	$\mathbf{Z}$

7. Hasse's norm theorem for K_2^{MW}

Recall that the classical norm theorem of Hasse states that if L is a cyclic extension of the number field F, then a nonzero element of F is a local norm at every place if and only if it is a global norm. We can think of this result as a norm theorem for K_1.

In [BR84], Bak and Rehmann extended Hasse's norm theorem to K_2. More precisely, they showed that if L/F is any finite extension of number fields, then an element of $K_2(F)$ lies in the image of the transfer map $\tau_{L/F} \colon K_2(L) \to K_2(F)$ if and only if its image in each $K_2(F_v)$ lies in the image of the map

$$\bigoplus_{w|v} \tau_{L_w/F_v} \colon \bigoplus_{w|v} K_2(L_w) \to K_2(F_v);$$

see [BR84, Theorem 1]. By [BR84, p. 4], this result can be reformulated as the exactness of the sequence

$$K_2(L) \xrightarrow{\tau_{L/F}} K_2(F) \xrightarrow{\bigoplus_{v \in \Sigma_{L/F}} h_v} \bigoplus_{v \in \Sigma_{L/F}} \mu(F_v) \to 0. \tag{7.1}$$

Here $\Sigma_{L/F}$ denotes the set of infinite real places of F that are complexified in the extension L/F, and h_v denotes the classical local Hilbert symbol at v.

7.0.1. The aim of this section is to show that a similar result as (7.1) holds also for K_2^{MW}:

Proposition 7.1. *Let L/F be an extension of number fields. Then an element of $K_2^{\mathrm{MW}}(F)$ lies in the image of the transfer map $\tau_{L/F} \colon K_2^{\mathrm{MW}}(L) \to K_2^{\mathrm{MW}}(F)$ if and only if its image in $K_2^{\mathrm{MW}}(F_v)$ lies in the image of $\bigoplus_{w|v} \tau_{L_w/F_v}$ for all $v \in \mathrm{Pl}_F$.*

7.0.2. We note that Proposition 7.1 is equivalent to the following assertion:

Proposition 7.2. *Let L/F be an extension of number fields. Denote by $\Sigma_{L/F}$ the set of infinite real places of F that are complexified in the extension L/F. Then there is an exact sequence*

$$\mathrm{K}_2^{\mathrm{MW}}(L) \xrightarrow{\tau_{L/F}} \mathrm{K}_2^{\mathrm{MW}}(F) \xrightarrow{\oplus_{v \in \Sigma_{L/F}} h_v^{\mathrm{MW}}} \bigoplus_{v \in \Sigma_{L/F}} \mathbf{Z} \to 0,$$

where $\tau_{L/F}$ denotes the transfer map on Milnor–Witt K-theory as defined in (2.1.4), and the right hand-side map is given by the local Milnor–Witt Hilbert symbols.

To show the equivalence of Propositions 7.1 and 7.2, notice first that Proposition 7.1 is equivalent to the assertion that the map $\mathrm{coker}(\tau_{L/F}) \to \prod_{v \in \mathrm{Pl}_F} \mathrm{coker}\left(\bigoplus_{w|v} \tau_{L_w/F_v}\right)$ is injective. It is therefore enough to show that $\mathrm{coker}(\tau_{L_w/F_v})$ is trivial except for $v \in \Sigma_{L/F}$, in which case it is $\mathbf{Z}$. But this follows from Proposition 2.2 along with the corresponding statement for K_2 proved in [BR84, p. 4].

7.0.3. Let us proceed with the proof of Proposition 7.2. We follow closely the strategy of Bak and Rehmann [BR84]. Recall from Definition 5.3 the definition of the groups B_v for $v \in \mathrm{Pl}_F$.

Definition 7.3. If v is any place of F and $w \mid v$ is a place of L above v, we define a homomorphism

$$b_{w|v} \colon B_w \to B_v$$

by

$$b_{w|v} := \begin{cases} 0, & v \in \mathrm{Pl}_{F,\infty}^{\mathrm{r}} \ \text{and} \ w \in \mathrm{Pl}_{L,\infty}^{\mathrm{c}} \\ \mathrm{id}, & v \in \mathrm{Pl}_{F,\infty}^{\mathrm{r}} \ \text{and} \ w \in \mathrm{Pl}_{L,\infty}^{\mathrm{r}} \\ n_w/m_v, & v \in \mathrm{Pl}_{F,0} \,. \end{cases}$$

Here $n_w := \#\mu(L_w)$ and $m_v := \#\mu(F_v)$.

Lemma 7.4. *For any place v of F, the diagram*

$$\begin{array}{ccc} \mathrm{K}_2^{\mathrm{MW}}(L) & \xrightarrow{\oplus_{w|v} h_w^{\mathrm{MW}}} & \bigoplus\limits_{w|v} B_w \\ {\scriptstyle \tau_{L/F}} \downarrow & & \downarrow {\scriptstyle \oplus_{w|v} b_{w|v}} \\ \mathrm{K}_2^{\mathrm{MW}}(F) & \xrightarrow{h_v^{\mathrm{MW}}} & B_v \end{array}$$

is commutative.

Proof. Since $\mathrm{K}_2^{\mathrm{MW}}(F_v) \cong \mathrm{K}_2(F_v)$ for each $v \notin \mathrm{Pl}_\infty^{\mathrm{r}}$ it suffices by [BR84, Proof of Proposition 2] to note that $\tau_{L_w/F_v} = \mathrm{id} \colon \mathrm{K}_2^{\mathrm{MW}}(\mathbf{R}) \to \mathrm{K}_2^{\mathrm{MW}}(\mathbf{R})$ whenever v and w both are infinite real places. Indeed, this follows since L_w/F_v is then the trivial extension. $\square$

Lemma 7.5. *Let $x \in \mathrm{WK}_2^{\mathrm{MW}}(F)$, and let p be a rational prime. Then there is an element $z \in \mathrm{K}_2^{\mathrm{MW}}(F)$ such that $pz = x$.*

Proof. By Proposition 5.8 we have $\mathrm{WK}_2^{\mathrm{MW}}(F) \cong \mathrm{WK}_2(F)$. Moreover, $\mathrm{K}_2^{\mathrm{MW}}(F) \cong \mathrm{K}_2^+(F) \oplus \mathbf{Z}^{r_1}$ by Proposition 6.5. Then, since $\mathrm{WK}_2^{\mathrm{MW}}(F) \subseteq \mathrm{K}_2^+(F)$, the statement follows from the fact that any element in the classical wild kernel has a p-th root in $\mathrm{K}_2(F)$ [Tat71]. $\square$

Lemma 7.6. *Suppose that $x \in \ker\left(\bigoplus_{v \in \Sigma_{L/F}} h_v^{\mathrm{MW}}\right)$. If $nz = x$ for some $z \in \mathrm{K}_2^{\mathrm{MW}}(F)$ and some integer $n \geq 1$, then there is an element $y \in \mathrm{K}_2^{\mathrm{MW}}(L)$ such that $x + \tau_{L/F}(y) \in n\mathrm{WK}_2^{\mathrm{MW}}(F)$.*

Proof. Using the Moore reciprocity sequence 5.8 along with the fact that the homomorphism $K_2^{MW}(F) \to \bigoplus_{v \in \Sigma_{L/F}} \mathbf{Z}$ is split, the same proof as that of [BR84, Lemma 1] goes through. $\square$

Proof of Proposition 7.2. The desired result now follows from Lemma 7.6 in an identical manner as the proof of [BR84, Theorem 2]. $\square$

REFERENCES

[BG04]   K. Belabas and H. Gangl. Generators and relations for $K_2(\mathcal{O}_F)$. In: *K-Theory* 31.3 (2004), pp. 195–231.

[BR84]   A. Bak and U. Rehmann. K_2-analogs of Hasse's norm theorems. In: *Comment. Math. Helv.* 59.1 (1984), pp. 1–11.

[BT73]   H. Bass and J. Tate. The Milnor ring of a global field. In: *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Seattle, Wash., Battelle Memorial Inst., 1972).* Springer, Berlin, 1973, 349–446. Lecture Notes in Math., Vol. 342.

[Gar71]   H. Garland. A finiteness theorem for K_2 of a number field. In: *Ann. of Math. (2)* 94 (1971), pp. 534–548.

[Gra03]   G. Gras. Class field theory. Springer Monographs in Mathematics. From theory to practice, Translated from the French manuscript by Henri Cohen. Springer-Verlag, Berlin, 2003, pp. xiv+491.

[Gra86]   G. Gras. Remarks on K_2 of number fields. In: *J. Number Theory* 23.3 (1986), pp. 322–335.

[HT10]   K. Hutchinson and L. Tao. Homology stability for the special linear group of a field and Milnor-Witt K-theory. In: *Doc. Math.* Extra volume: Andrei A. Suslin sixtieth birthday (2010), pp. 267–315.

[Hut16]   K. Hutchinson. The second homology of SL_2 of S-integers. In: *J. Number Theory* 159 (2016), pp. 223–272.

[IØ20]   D. C. Isaksen and P. A. Østvær. Motivic stable homotopy groups. In: *Handbook of homotopy theory.* CRC Press/Chapman Hall Handb. Math. Ser. CRC Press, Boca Raton, FL, 2020, pp. 757–791.

[Kat78]   K. Kato. A generalization of local class field theory by using K-groups. II. In: *Proc. Japan Acad. Ser. A Math. Sci.* 54.8 (1978), pp. 250–255.

[Keu89]   F. Keune. On the structure of the K_2 of the ring of integers in a number field. In: *Proceedings of Research Symposium on K-Theory and its Applications (Ibadan, 1987).* Vol. 2. 5. 1989, pp. 625–645.

[Mat69]   H. Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. In: *Ann. Sci. École Norm. Sup. (4)* 2 (1969), pp. 1–62.

[Mer83]   A. S. Merkurjev. On the torsion in K_2 of local fields. In: *Ann. of Math. (2)* 118.2 (1983), pp. 375–381.

[MH73]   J. Milnor and D. Husemoller. Symmetric bilinear forms. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73. Springer-Verlag, New York-Heidelberg, 1973, pp. viii+147.

[Mil70]   J. Milnor. Algebraic K-theory and quadratic forms. In: *Invent. Math.* 9 (1970), pp. 318–344.

[Mil71]   J. Milnor. Introduction to algebraic K-theory. Annals of Mathematics Studies, No. 72. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971, pp. xiii+184.

[Mor04a] F. Morel. On the motivic π_0 of the sphere spectrum. Vol. 131. NATO Sci. Ser. II Math. Phys. Chem. Kluwer Acad. Publ., Dordrecht, 2004, pp. 219–260.

[Mor04b] F. Morel. Sur les puissances de l'idéal fondamental de l'anneau de Witt. In: *Comment. Math. Helv.* 79.4 (2004), pp. 689–703.

[Mor12]   F. Morel. $\mathbf{A}^1$-algebraic topology over a field. Vol. 2052. Lecture Notes in Mathematics. Springer, Heidelberg, 2012, pp. x+259.

[MV99]   F. Morel and V. Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. In: *Inst. Hautes Études Sci. Publ. Math.* 90 (1999), 45–143 (2001).

[Neu99]   J. Neukirch. Algebraic number theory. Vol. 322. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. Springer-Verlag, Berlin, 1999, pp. xviii+571.

[OVV07]   D. Orlov, A. Vishik, and V. Voevodsky. An exact sequence for $K_*^M/2$ with applications to quadratic forms. In: *Ann. of Math. (2)* 165.1 (2007), pp. 1–13.

[Qui73]   D. Quillen. Higher algebraic K-theory. I. In: *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972).* 1973, 85–147. Lecture Notes in Math., Vol. 341.

[Sch17]   M. Schlichting. Euler class groups and the homology of elementary and special linear groups. In: *Adv. Math.* 320 (2017), pp. 1–81.

[Sch85]   W. Scharlau. Quadratic and Hermitian forms. Vol. 270. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1985, pp. x+421.

[Siv85]   I. Y. Sivitskii. Torsion in Milnor K-groups for a local field. In: *Mat. Sb. (N.S.)* 126(168).4 (1985), pp. 576–583.

[Tat71]   J. Tate. Symbols in arithmetic. In: *Actes du Congrès International des Mathématiciens (Nice, 1970),*
            Tome 1. Gauthier-Villars, Paris, 1971, pp. 201–211.
[Wei13]   C. A. Weibel. The K-book. Vol. 145. Graduate Studies in Mathematics. An introduction to algebraic
            K-theory. American Mathematical Society, Providence, RI, 2013, pp. xii+618.
[Øst03]   P. A. Østvær. A norm principle in higher algebraic K-theory. In: *Bull. London Math. Soc.* 35.4 (2003),
            pp. 491–498.

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