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Chow groups and pseudoffective cones of complexity-one T-varieties

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Abstract. We show that the pseudoeffective cone of k-cycles on a complete complexity-one T-variety is rational polyhedral for any k, generated by classes of T-invariant subvarieties. When X is also rational, we give a presentation of the Chow groups of X in terms of generators and relations, coming from the combinatorial data defining X as a T-variety.

1. Introduction

A T-variety is a normal algebraic variety X with an effective action of an algebraic torus T, defined over an algebraically closed field of characteristic 0. The complexity of a T-variety is defined as $\dim X - \dim T$, thus T-varieties of complexity zero correspond to the toric varieties. For toric varieties there is a well-known correspondence between the geometry of a variety and combinatorial data coming from the T-action. There has in recent years been developed a similar quasi-combinatorial language for describing T-varieties of higher complexity, starting with Altmann and Hausen's paper [1]. Following this, there have been many papers studying the geometrical and combinatorial properties of T-varieties (see for instance [3,5,12,13,17,19] and the references in [4]).

Here we will study algebraic cycles on T-varieties of complexity one. Our first main result is a description of the cones of effective cycles:

Theorem 1.1. *The pseudoeffective cones $\overline{\mathrm{Eff}}_k(X)$ of a complete T-variety X of complexity one are rational polyhedral, generated by classes of invariant subvarieties.*

This generalizes Scott's results on the pseudoeffective cone of curves on a T-variety [20]. In general there are not many examples where all pseudoeffective cones of cycles are known. Our result gives a large class of examples where these are rational polyhedral. When X is rational, it is known that X is a Mori dream space [12], so the statement was previously known for the cone of curves and effective divisors. However, the result applies also in the non-rational cases, showing that while the T-varieties X are not Mori dream spaces, their cones of curves and divisors

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are still rational polyhedral. Moreover, even in the case X is rational, and thus Mori dream, it is not a priori clear that the effective cones of cycles of intermediate dimensions should be rational polyhedral, as was shown by [7, Example 6.10].

Our second main result gives a presentation for the Chow groups of X, in the case X is a rational complete complexity-one T-variety. The result is inspired by two different well-known results. First of all, for toric varieties, Fulton and Sturmfels showed that invariant subvarieties generate the Chow groups, and moreover they described the relations between these generators [9, Proposition 2]. On the other hand, for a rational complete T-variety of complexity one, Altmann and Petersen give an analogous short exact sequence describing its Picard group [5, Corollary 2.3]. These two results give some hints to how the Chow groups of a -one T-variety might look. Our second main result gives a complete description of these.

To explain the result, we first recall some basic facts about T-varieties. If X is a complete rational T-variety of one it comes equipped with a rational quotient map, that is, a rational map $f : X \dashrightarrow \mathbb{P}^1$, and another complete rational T-variety of one, $\widetilde{X}$, with a T-equivariant blow-up map $r : \widetilde{X} \to X$ which resolves f; thus there is a map $g : \widetilde{X} \to \mathbb{P}^1$ such that $f \circ r = g$ whenever defined. Following [13], we have that X corresponds to a fan Σ giving the general toric fiber of g, as well as finitely many special fibers which correspond to polyhedral complexes with tailfan Σ. We denote by P the following set: If the set of points in $\mathbb{P}^1$ such that the fiber is not general has size at least 2 then P equals this set. If this set is smaller, then P is the union of this set with (1 or 2) additional arbitrary points of $\mathbb{P}^1$. In addition we need to keep track of the data of which cones correspond to varieties contracted by r (see Section 2 for more details).

Letting the dimension of X be $n+1$ and Σ be the fan describing the general fiber of g (which is the toric variety X_Σ) we define, for a non-negative integer k, the following sets:

$R_k =$ Cones of dimension $n+1-k$ corresponding to subvarieties not contracted by r.

$V_k =$ Faces of dimension $n-k$ of polyhedral subdivisions corresponding to fibers of points in P, such that the tailcone corresponds to a subvariety not contracted by r.

$T_k =$ Cones of dimension $n-k$ corresponding to subvarieties contracted by r.

Theorem 1.2. *For a complete rational T-variety X of one there is for any* $0 \leq k \leq \dim X$ *an exact sequence*

$$\bigoplus_{F \in V_{k+1}} M(F) \bigoplus_{\tau \in R_{k+1}} (M(\tau) \oplus \mathbb{Z}^P/\mathbb{Z}) \bigoplus_{\tau \in T_{k+1}} M(\tau) \to \mathbb{Z}^{V_k} \oplus \mathbb{Z}^{R_k} \oplus \mathbb{Z}^{T_k} \to A_k(X) \to 0,$$

where for a rational polyhedral cone τ *in* $N_{\mathbb{Q}}$, $M(\tau) = \tau^\perp \cap M$. *Also* $M(F)$ *is the character lattice of the toric variety corresponding to* F.

The maps will be described below. For $k = n$ this coincides with the exact sequence of Altmann and Petersen. The above results also generalizes the results of Laface, Liendo and Moraga [17], where they give a presentation of the rational Chow ring of a complete complexity-one T-variety which is contraction-free, that is, when $X = \widetilde{X}$. Being contraction-free is however quite restrictive, for instance their rational Chow ring is generated by divisors [17, Lemma 4.1], which is not true in general. We do not have an explicit description of the ring structure of a general complexity-one T-variety, although we believe that the results in this paper will be useful in describing it. We illustrate our results by studying examples such as toric downgrades (meaning we only remember the action of a codimension one torus on a toric variety), projectivizations of rank-two toric vector bundles and the Grassmannian $\mathrm{Gr}(2, 4)$, none of which (in general) are contraction-free.

2. Preliminaries on T-varieties

The papers [1,3] give a general framework for describing T-varieties of any complexity, we briefly recall the set-up. Denote by $T \simeq (\mathbb{K}^*)^n$ a torus of dimension n and let M and N denote the lattices of characters and one-parameter subgroups of T, respectively. $\mathbb{K}$ is an algebraically closed field of characteristic 0.

Recall that any polyhedron Δ can be decomposed as a Minkowski sum $\sigma + P$, where σ is a unique polyhedral cone, called the tailcone, and P is a polytope. Fixing a polyhedral cone $\sigma \subset N_{\mathbb{Q}}$, we consider the semigroup under Minkowski addition

$$\mathrm{Pol}^+_{\mathbb{Q}}(N, \sigma) = \{\Delta \subset N_{\mathbb{Q}} | \Delta \text{ is a polyhedron with tailcone } \sigma\}$$

We also allow $\emptyset$ as an element of $\mathrm{Pol}^+_{\mathbb{Q}}(N, \sigma)$. Let Y be a normal and semiprojective variety (meaning it is projective over some affine variety) and let $\mathrm{CDiv}(Y)$ denote the group of Cartier-divisors on Y. We consider "divisors" of the form

$$\mathcal{D} = \sum_Z \Delta_Z \otimes Z,$$

where Δ_Z is an element of $\mathrm{Pol}^+_{\mathbb{Q}}(N, \sigma)$ and the Z are Q-Cartier divisor on Y, such that only finitely many Δ_Z differ from the tailcone. For $u \in \sigma^\vee \cap M$, we may consider the evaluation

$$\mathcal{D}(u) = \sum_{Z | \Delta_Z \neq \emptyset} \min\langle \Delta_Z, u\rangle Z \in \mathrm{CDiv}_{\mathbb{Q}}(Y),$$

which is a finite sum, since the minimum for any Z such that Δ_Z equals the tailcone is 0. We call $\mathcal{D}$ a p-divisor on (Y, N) if $\mathcal{D}(u)$ is semiample for all $u \in \sigma^\vee \cap M$, as well as big for u in the interior of $\sigma^\vee \cap M$. To a p-divisor $\mathcal{D}$ we can associate the sheaf of rings $\mathcal{O}_Y(\mathcal{D}) = \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_Y(\mathcal{D}(u))$. Then $X = \operatorname{Spec} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}))$ is an affine T-variety of complexity $\dim Y$. Also $\widetilde{X} = \operatorname{Spec}_Y \mathcal{O}_Y(\mathcal{D})$ is T-variety of complexity $\dim Y$ and there is an equivariant map $r : \widetilde{X} \to X$. We say that X is *contraction-free* if $X = \widetilde{X}$.

Altmann and Hausen [1] shows that any affine T-variety arises from a p-divisor in this way.

If $\mathcal{D}$ and $\mathcal{D}'$ both are p-divisors on (Y, N) we define their intersection $\mathcal{D} \cap \mathcal{D}'$ as having coefficient $\Delta_Z \cap \Delta'_Z$ on Z. We say that $\mathcal{D} \subset \mathcal{D}'$ if $\Delta_Z \subset \Delta'_Z$ for all Z. In that case we say that $\mathcal{D}$ is a face of $\mathcal{D}'$ if the induced map $X(\mathcal{D}) \to X(\mathcal{D}')$ is an open embedding; there is a technical condition [3, Proposition 3.4] for checking this, which we do not recall here.

Definition 2.1. A finite set $\mathcal{S}$ of p-divisors on (Y, N) is called a divisorial fan if the intersection of any two p-divisors is a common face of both and $\mathcal{S}$ is closed under intersections.

The condition on the intersections comes from the fact that one can glue the varieties $X(\mathcal{D})$, $X(\mathcal{D}')$ along the open set $X(\mathcal{D} \cap \mathcal{D}')$. Then Altmann, Hausen and Süss [3] show that any T-variety arises from a divisorial fan in this way.

While the arguments in this paper are mostly geometric, the perspective of a divisorial fan will be useful. Also, in complexity one, there is a different perspective, due to Ilten and Süss [13], which has the advantage of avoiding the technical condition about the open embeddings $X(\mathcal{D}) \to X(\mathcal{D}')$. We will alternate between these perspectives, depending on what is most convenient.

We now specialize to the case where Y is a curve, denoted by C, which we can assume is a smooth curve [1, Corollary 8.12]. Also we generally denote the divisor Z by p, since divisors on curves correspond to sums of points. For a p-divisor $\mathcal{D}$ on C we define its degree $\deg \mathcal{D}$ as the Minkowski sum $\sum_p \Delta_p$.

Definition 2.2. A *marked fansy divisor* on C is a formal sum $\Xi = \sum S_p \otimes [p]$ together with a fan $\Sigma \subset N_{\mathbb{Q}}$ and a subset $K \subset \Sigma$ such that

(1) Each S_p is a complete polyhedral subdivison of $N_{\mathbb{Q}}$ such that $\text{tail}(S_p) = \Sigma$ for all p.
(2) If $\sigma \in K$ has full dimension then $\mathcal{D}^\sigma = \sum \mathcal{D}_p^\sigma \otimes [p]$ is a p-divisor, where $\mathcal{D}_p^\sigma$ is the polyhedron in S_p with $\text{tail}(\mathcal{D}_p^\sigma) = \sigma$.
(3) for a full dimensional cone $\sigma \in K$ and a face τ of σ, $\tau \in K$ if and only if $\deg \mathcal{D}^\sigma \cap \tau \neq \emptyset$.
(4) If τ is a face of σ, then $\tau \in K$ implies that $\sigma \in K$.

The subsets $\deg \mathcal{D}^\sigma$ glue to a subset $\deg \Xi \subset N_{\mathbb{Q}}$.

Then [13] shows that any complete T-variety of complexity one corresponds to a marked fansy divisor. In other words, in this case we only need to remember the polyhedral subdivisions of the fibers as well as the set K which records which subvarieties are contracted by r.

Given a complete rational complexity one T-variety defined by a divisorial fan $\mathcal{S}$, one can associate a marked fansy divisor by taking the polyhedral subdivisions given by $\mathcal{S}$ and letting K consist of all cones σ such that there exists $\mathcal{D} \in \mathcal{S}$ with tailcone σ and such that no coefficients of $\mathcal{D}$ equals $\emptyset$. The variety $\widetilde{X}$ is given as a marked fansy divisor by the same subdivisions as for X but with $K = \emptyset$.

The intuition here is the following: For any point $p \in Y$ the fiber of g is the toric variety (possibly non-reduced, non-irreducible) corresponding to the polyhedral subdivison at the point p. The fan Σ defines a toric variety which is the general fiber, but at some points there might be other fibers. Nevertheless these fibers are all unions of toric varieties. The T-action on $\widetilde{X}$ restricted to a fiber of g is just the T-action on the fiber as a union of toric varieties.

Example 2.3. Figure 1 shows polyhedral subdivisions for a complexity-one T-variety X of dimension three, with three special fibers. Its tailfan is the fan Σ of the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in two points contained in $\mathbb{P}^1 \times 0$. The subdivisions alone

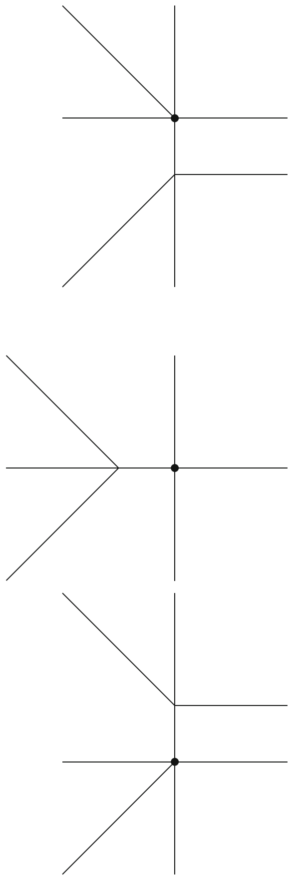

Fig. 1. Polyhedral subdivisons defining a T-variety X of dimension three

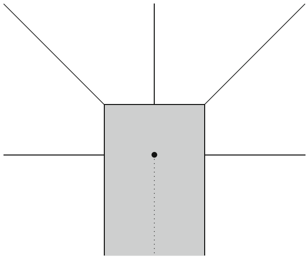

Fig. 2. Everything except the gray area is deg Ξ

do not define X, we also need to specify the subset K. If we for instance choose that K contains all maximal cones of Σ, we get from Fig. 2 that all rays except the one generated by $(0, -1)$ are also in K.

This choice of K corresponds to the projectivization of a rank-two non-split toric vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$, see Section 6.

3. Subvarieties of *T*-varieties of complexity-one

Given a fan Σ, we denote by X_Σ the associated toric variety, by B_Σ the torus-invariant boundary of X_Σ and by T_Σ the torus. For $\sigma \in \Sigma$, let $V(\sigma)$ denote the T-invariant subvariety of X_Σ of codimension $\dim \sigma$ associated to σ. It is a toric variety with torus $T(\sigma)$, whose lattice of one-parameter subgroups is $N(\sigma) = N/N_\sigma$, where N_σ is the lattice generated by $\sigma \cap N$.

Fix a complete T-variety of complexity-one X with tailfan Σ. On the variety $\widetilde{X}$ there are various subvarieties arising from the combinatorial structure defining it. For a point $p \in C$ and a face F of the polyhedral complex S_p defining the fiber $g^{-1}(p)$ there is the T-orbit $\mathrm{orb}(p, F)$ of dimension codim F, the closure is denoted by $Z_{p,F} = \overline{\mathrm{orb}}(p, F) \subset \widetilde{X}$.

We denote the generic point of C by η. For any cone $\sigma \in \Sigma$ there is a T-invariant subvariety B_σ which is given by the closure of $\mathrm{orb}(\eta, \sigma)$, in other words B_σ dominates C and in the general fiber is given by the subvariety $V(\sigma) \subset X_\Sigma$. We define $B = \cup_{\rho \in \Sigma(1)} B_\rho$. Then B is a finite union of divisors on $\widetilde{X}$, each of which themselves is a T-variety of complexity-one [12, Proof of Proposition 4.12]. This fact will be important in the proof of Theorem 3.1.

Fix any ray $\rho \in \Sigma$. Locally, $\widetilde{X}$ is given by a p-divisor $D = \sum_Z D_Z Z$ with tailcone σ. For a ray ρ, let $\pi : N_\mathbb{Q} \to N_\mathbb{Q}/\mathbb{Q}\rho$ denote the projection sending ρ to 0. Then $D_\rho = \sum_Z \pi(D_Z) Z$ is a new p-divisor on Y with tailcone $\pi(\sigma)$; this defines B_ρ as a T-variety [12, Proof of Proposition 4.12].

On X some of the subvarieties B_σ are contracted, namely those where σ is in K. In this case $r(B_\sigma)$ is contracted to a variety of dimension one less. We denote the corresponding orbit closure $r(B_\sigma)$ in X by W_σ. We also denote $r(Z_{p,F})$ by $W_{p,F}$, for any p, F. When $\sigma \notin K$ then B_σ is not contracted and we also denote the corresponding subvarieties of X by the same symbol.

We are interested in studying the pseudoeffective cones $\overline{\mathrm{Eff}}_k(X)$ inside the group $N_k(X)$ of cycles modulo numerical equivalence, which by definition is the closure of the cone of effective k-cycles.

Theorem 3.1. *The pseudoeffective cones $\overline{\mathrm{Eff}}_k(X)$ of a complete T-variety X of complexity one are rational polyhedral, generated by classes of invariant subvarieties.*

Proof. We first show the statement for the contraction-free case, in other words when $X = \widetilde{X}$. Fix a subvariety V of dimension k of $\widetilde{X}$ and choose a basis $v_1, \ldots, v_n$ of the lattice N. Let λ_i be the one-parameter subgroup of T corresponding to v_i. We have a corresponding action $\mathbb{K}^* \times \widetilde{X} \to \widetilde{X}$ for each λ_i. Let V_1 be the flat limit as t goes to zero of $\lambda_1 \cdot V$. Similarly let V_i be the flat limit as t goes to zero of $\lambda_i \cdot V_{i-1}$. Then V_n is an effective cycle numerically equivalent to V and invariant under the entire torus action. Now for any irreducible component W of V_n we have two possibilities: Either W dominates C or it is contained in a fiber $\widetilde{X}_y$.

If it is contained in a fiber $\widetilde{X}_y$, then since it is irreducible it is contained in an irreducible component of the fiber. There are only finitely many components which are different from X_Σ. In each of these there are only finitely many irreducible invariant subvarieties. Thus if W is contained in one of these special fibers it has to equal one of these finitely many subvarieties. If it is contained in a general fiber then, since the map $\widetilde{X} \to C$ is flat, it is algebraically equivalent to a cycle in one fixed general fiber. In particular it is also numerically equivalent. Thus W can be written as an effective sum of finitely many generators.

If it is not contained in a fiber, then we claim it has to be contained in the boundary divisor B. Indeed, if not, then there is a point $y \in C$ such that $\widetilde{X}_y \simeq X_\Sigma$ and such that there is x in $(\widetilde{X}_y \setminus B_y) \cap W_y = T_\Sigma \cap W_y$. Then T_Σ acting on x will be a set of dimension n in the fiber $\widetilde{X}_y$. But then W_y has to have dimension at least n, since it is invariant under the entire torus action. However by assumption it is a proper closed subset of X_Σ which has dimension n, so its dimension must be smaller than n.

Thus when W is not contained in a fiber, it is contained in B which is a finite union of T-varieties of complexity one of smaller dimension than $\widetilde{X}$. Since we are on $\widetilde{X}$ we know that no varieties are contracted by r, in particular any p-divisor must have $\emptyset$ as a coefficient. Thus any p-divisor for a component B_ρ of B, which by the above description is given by projecting the polyhedral coefficients of a p-divisor for $\widetilde{X}$, will also have an empty coefficient. In particular $B_\rho = \widetilde{B_\rho}$, in other words it is also contraction-free. A T-variety of dimension one and complexity one is simply a curve. The proposition is obviously true for any curve, thus by induction it is true for any contraction-free T-variety of complexity one.

Now we consider the case of the general X. By the same argument as above, any subvariety V of X of dimension k can be written as an effective sum of T-invariant irreducible subvarieties. By [1, Theorem 10.1] any irreducible T-invariant subvariety of X is the image of a T-invariant subvariety of $\widetilde{X}$ under r. From this we get that V can be written as an effective sum of the images of the generators of $\overline{\mathrm{Eff}}_k(\widetilde{X})$. This proves the theorem also for X. $\square$

In fact the above proof gives some information about the generators of the pseudoeffective cones of X:

Corollary 3.2. *Let X be a complete contraction-free T-variety of complexity one, in other words such that $\widetilde{X} = X$. Then the pseudoeffective cone $\overline{\mathrm{Eff}}_k(X)$ is generated by B_τ, where $\dim \tau = n + 1 - k$ and $Z_{y,F}$, where $y \in Y$ and $\dim F = n - k$.*

Proof. The proof of Theorem 3.1 implicitly describes the generators: Any effective irreducible k-cycle W of dimension less than $\dim X$ has to either be contained in some B_ρ, $\rho \in \Sigma(1)$ or in some fiber $\widetilde{X}_y$.

If it is contained in a fiber $\widetilde{X}_y$ then, since the fiber is a union of irreducible toric varieties, it has to be contained in one of them, say the one corresponding to the vertex v of S_y. After possibly replacing any compact face containing v with the corresponding cone emanating from v we have a complete fan Σ' with vertex v defining the toric variety. Now we know that as a cycle in $X_{\Sigma'}$, W can be written as an effective sum $W \equiv \sum_{F_i \in \Sigma'(n-k)} a_i V(F_i)$. Letting $j_i : V(F_i) \to \widetilde{X}$ denote the inclusion, we have that $(j_i)_* V(F_i) = Z_{v,F_i}$. Thus in this case W can be written as a positive sum of the finitely many $Z_{v,F}$.

Next we note, as in the proof of Theorem 3.1, that any boundary divisor in $\widetilde{X}$ is also contraction-free. If an effective cycle is contained in some boundary divisor B_ρ we argue by induction on $\operatorname{codim} W$. If $\operatorname{codim} W = 1$ then since W is contained in B_ρ and they are both irreducible of the same dimension, we must have $W = B_\rho$, hence we are done. If $\operatorname{codim} W > 1$ then by induction W can be written as a positive sum of $B_{p(\tau)}$ and $Z_{v,p(F)}$, where $p : N_\mathbb{Q} \to N_\mathbb{Q}/\mathbb{Q}\rho$ is the projection. Denoting by $j : B_\rho \to \widetilde{X}$ the inclusion we have that $j_*(B_{p(\tau)}) = B_\tau$ and $j_*(Z_{v,p(F)}) = Z_{v,F}$. Thus we are done by induction. □

To study the generators of the pseudoeffective cones for general X we need to also take into account the varieties that are contracted by r. First of all, it follows from [19, Proof of Proposition 3.13] that for any invariant subvariety Z of a T-variety $\widetilde{X}$ we have that

$$\dim g(Z) + \dim r(Z) - \dim(Z) \geq 0.$$

This fact, combined with the fact that we always have $\dim(Z) \geq \dim(r(Z))$, implies that if $\dim(g(Z)) = 0$ then Z is not contracted. Moreover if $\dim(g(Z)) = 1$ and Z is contracted then $\dim(r(Z)) = \dim(Z) - 1$. If B_σ has codimension k in $\widetilde{X}$ (equivalently σ is of dimension k), then if B_σ is contracted by r to W_σ (meaning $\sigma \in K$) it has to be contracted to a subvariety of codimension $k + 1$ in X.

We will describe the images of the contracted varieties. We denote by $\mu(F)$ the smallest positive integer in N/N_σ such that v_F is a lattice point.

Lemma 3.3. *Let $\sigma \in \Sigma$ and consider $B_\sigma \subset \widetilde{X}$ as a complexity-one T-variety with respect to the torus $T(\sigma)$. Then for a face $Z_{q,F}$ with tailcone σ we have that the generic stabilizer group of the corresponding divisor in B_σ is cyclic of order $\mu(F)$.*

Proof. By the proof of [12, Proposition 4.12] B_σ is described as a T-variety by projecting the coefficients from N to $N/\mathbb{Q}\sigma$. By [12, Proposition 4.11] the order of the generic stabilizer group is cyclic of the stated order. □

We can determine the difference of generic stabilizers of a general fiber $Z_{p,\sigma}$ and its image $r(Z_{p,\sigma})$. Since the map r is equivariant any $t \in T$ in the stabilizer of $Z_{p,F}$ will be in the stabilizer of the image $r(Z_{q,F})$, for any F with tailcone σ. Each $Z_{p,F}$ is a toric variety with an effective action of a corresponding torus. For a general fiber this torus has lattice of one-parameter subgroups N/N_σ, while in general it will be a lattice containing this: The lattice L_F generated by the unit vectors together with the vertex v_F in N/N_σ. By [1, Proposition 5.2 ii] the stabilizer of a point in X is given by a torus associated to a certain lattice. For a point in $Z_{p,F}$ this lattice is L_F by [1, Proposition 7.10]. Since all $Z_{p,F}$ are mapped to the same image by the proof of Lemma 3.4, the stabilizer of $r(Z_{q,\sigma})$ will thus simply be the group generated by the stabilizers of all $Z_{p,F}$. Letting L be the lattice generated by all integer linear combinations of the unit vectors together with set of v_F, for all $Z_{p,F}$ with the tailcone of F equal to σ, we obtain a lattice L containing the lattice N/N_σ. The difference of stabilizers $[\mathrm{Stab}(Z_{q,\sigma}) : \mathrm{Stab}(r(Z_{q,\sigma}))]$ will thus equal the lattice index $[L : N/N_\sigma]$. We denote this number by s_σ.

Lemma 3.4. *Assume $\sigma \in \Sigma$ and that $\sigma \in K$. Then all faces $Z_{p,F}$ with* tail $F = \sigma$ *are mapped to W_σ under the contraction map r. Moreover we have the equality $\mu(F)r_*([Z_{p,F}]) = s_\sigma[W_\sigma]$ of numerical classes, for F with tailcone σ.*

Proof. This follows from [1, Theorem 10.1]: Two faces $Z_{p,F}$ and $Z_{y,G}$ of the same polyhedral divisor $\mathcal{D}$ are identified under r if they have the same normal cone λ and if $\mathcal{D}(u) = 0$ for some u in the relative interior of λ (if $C = \mathbb{P}^1$ this is in fact equivalent to $Z_{p,F}$ being identified with $Z_{y,G}$).

Let $\sigma \in K$ be a maximal cone. There is a polyhedral divisor $\mathcal{D}^\sigma$ with tailcone σ and no empty coefficients. For any coefficient Δ_p^σ of $\mathcal{D}^\sigma$ the associated normal cone is the point 0, moreover $\mathcal{D}(0) = 0$, thus W_{p,Δ_p^σ} is identified with W_{q,Δ_q^σ} for any points $p, q \in C$. In particular, the entire B_σ in $\widetilde{X}$ is mapped to any fixed subvariety of the form W_{p,Δ_p^σ}.

If $\tau \in K$ is not maximal we have that $\cap_{\tau \preceq \sigma} \deg \mathcal{D}^\sigma \cap \tau \neq \emptyset$ by Definition 2.2 (3). This implies that in any fiber S_p the intersection $\cap_{\tau \preceq \sigma} \Delta_p^\sigma \neq \emptyset$, which implies that in any fiber there is only one face F_p with tailcone τ. Let $\mathcal{D}^\tau$ be the p-divisor having the corresponding faces F_p as coefficients. We have $F_p = \tau + Q_p$, where Q_p is some polytope in $N_\mathbb{Q}$. Letting v_p be a vertex of Q_p, we have that $F_p = \tau + v_p + (Q_p - v_p)$. Since there is only one face of S_p with tailcone τ, the polytope $Q_p - v_p$ must be contained in the linear span of τ. This implies that all F_p have the same normal cone λ. We wish to show that

$$\mathcal{D}^\tau(u) = \sum \min\langle F_p, u\rangle = 0$$

for u in the relative interior of λ. We have that $\min\langle F_p, u\rangle = \langle v_p, u\rangle$, since u by definition is normal to any point in the linear span of τ. Since $\deg \mathcal{D}^\tau = \deg \mathcal{D}^\sigma \cap \tau \subset \tau$ we must also have that $\sum v_p \in \tau$, thus $\langle \sum_p v_p, u\rangle = 0$, which is what we wanted to show. Thus Z_{p,F_p} is identified with Z_{q,F_q} for any $p, q \in C$, thus the horizontal subvariety B_τ maps to any W_{p,F_p}.

For the last claim, we have that for a general fiber q the map r restricted to $Z_{q,\sigma}$ is finite of order s_σ, thus $r_*(Z_{q,\sigma}) = s_\sigma W_\sigma$. For a subvariety $Z_{p,F}$ of a special fiber the stabilizer group will have order $\frac{s_\sigma}{\mu(F)}$, thus $r_*(Z_{p,F}) = \frac{s_\sigma}{\mu(F)} W_\sigma$, which proves the claim. □

Proposition 3.5. *Let X be a complete T-variety of complexity one. Then the pseudoeffective cone $\overline{\mathrm{Eff}}_k(X)$ is generated by the following classes:*

- *B_τ, where $\dim \tau = n+1-k$ and $\tau \notin K$*
- *$W_{y,F}$, where $y \in Y$ and $F \subset S_y$ has dimension $n-k$ and $\mathrm{tail}(F) \notin K$*
- *W_σ where σ has dimension $n-k$ and $\sigma \in K$.*

Proof. As noted earlier $\overline{\mathrm{Eff}}_k(X)$ is the image of $\overline{\mathrm{Eff}}_k(\widetilde{X})$ via r_*, thus we know that $r(B_\tau)$ and $r(Z_{p,F})$ generate $\overline{\mathrm{Eff}}_k(X)$ as above. However we can omit $r(B_\tau)$ for $\tau \in \Sigma(n+1-k) \cap K$ since $r_*(B_\tau) = 0$ in this case. Fixing $\sigma \in K$ of dimension $n-k$ we have by the lemma above that all classes $r_*(Z_{p,F})$, for any point p and F with tailcone σ, are proportional, thus it is more convenient to only remember the single representative W_σ instead of all the different $W_{p,F}$. □

4. Chow groups

We now assume X is a rational and complete T-variety of complexity one with tailfan Σ. Recall that r denotes the contraction map $\widetilde{X} \to X$. We say that a cone $\sigma \in \Sigma$ is contracted by r if $\sigma \in K$. Also recall that P is the following set: If the set of points in $\mathbb{P}^1$ such that the fiber of r is not general, has size at least 2 then P equals this set. If this set is smaller, then P is the union of this set with (1 or 2) additional arbitrary points of $\mathbb{P}^1$. Fix k and, inspired by the above result, define the following sets
R_k = Cones in Σ of dimension $n+1-k$ not contracted by r corresponding to subvarieties B_σ.
V_k = Faces of dimension $n-k$ of fibers of points in P, such that the tailcone is not contracted corresponding to subvarieties $W_{p,F}$, for $p \in P$, tail $F \notin K$.
T_k = Cones in Σ of dimension $n-k$ contracted by r corresponding to subvarieties W_σ.

Note that we will show in the proof below that invariant subvarieties corresponding to faces of points that have general fibers are in the group generated by the other subvarieties (inside the Chow-group), thus we do not require these as generators.

Then there is a surjection

$$\mathbb{Z}^{R_k \cup V_k \cup T_k} \to A_k(X) \to 0 \tag{1}$$

For $k = n$ this is the surjection for the Picard group of X given in [5] (note that T_n is always empty). Altmann and Petersen also describe the relations between the generators:

$$0 \to \mathbb{Z}^P/\mathbb{Z} \oplus M \to \mathbb{Z}^{V_n \cup R_n} \to \mathrm{Pic}(X) \to 0.$$

Recall that P is the set of points p in $\mathbb{P}^1$ where the polyhedral subdivison S_p do not equal Σ. If P has size less than 2 then we simply add points so that P has size 2. This corresponds to fixing a T-invariant structure on $\mathbb{P}^1$. Without loss of generality we assume $\infty \in P$. The first map is given by the following: generators for $\mathbb{Z}^P/\mathbb{Z}$ correspond to principal divisors $[p] - [\infty]$ on $\mathbb{P}^1$, one such generator is mapped to $\sum_{p,v} \mu(v) W_{p,v} - \sum_{\infty,v} \mu(v) W_{\infty,v}$. A character $m \in M$ is mapped to $\sum_v \mu(v)\langle m, v\rangle W_{p,v} + \sum_{R_n} \langle m, \rho\rangle B_\rho$. Here $\mu(v)$ is the smallest integer such that $\mu(v)v$ is a lattice point.

Theorem 4.1. *For a rational complete complexity-one T-variety X there is for any* $0 \leq k \leq \dim X$ *an exact sequence*

$$\bigoplus_{F\in V_{k+1}} M(F) \bigoplus_{\tau\in R_{k+1}} (M(\tau) \oplus \mathbb{Z}^P/\mathbb{Z}) \bigoplus_{\tau\in T_{k+1}} M(\tau) \to \mathbb{Z}^{V_k} \oplus \mathbb{Z}^{R_k} \oplus \mathbb{Z}^{T_k} \to A_k(X) \to 0.$$

The maps are given as follows:

Any $F \in V_{k+1}$ corresponds to an invariant subvariety of an irreducible component of a fiber p (possibly several components, if so pick one). We denote the character lattice of the corresponding toric variety by $M(F)$. Then $m \in M(F)$ maps to

$$\sum_{\substack{\dim G=n-k, F\subset G\\ \operatorname{tail} G\notin K}} \langle m, v_{F,G}\rangle Z_{p,G} + \sum_{\substack{\dim G=n-k, F\subset G\\ \operatorname{tail} G\in K}} \langle m, v_{F,G}\rangle \frac{s_{\operatorname{tail} G}}{\mu(G)} W_{\operatorname{tail} G},$$

where $v_{F,G}$ generates $N(F)/N(G)$. This is the usual notion of rational equivalence on a toric variety.

If $\tau \in R_{k+1}$ then B_τ itself corresponds to a T-variety of complexity one. The map comes from this structure, as in the exact sequence of Altmann and Petersen. Explicitly a generator of $\mathbb{Z}^P/\mathbb{Z}$ corresponds to a divisor $[p]-[\infty]$ and it is mapped to

$$\sum_{F,\operatorname{tail} F=\tau} \mu(v_F) W_{p,F} - \sum_{F,\operatorname{tail} F=\tau} \mu(v_F) W_{\infty,F},$$

where v_F is the vertex corresponding to the image of F in the divisorial fan corresponding to B_τ as a T-variety (see Section 3). A character $m \in M(\tau)$ is mapped to

$$\sum_{F\in V_k,\operatorname{tail} F=\tau} \mu(v_F)\langle m, v_F\rangle W_{p,F} + \sum_{\sigma\in R_k, \tau\subset\sigma} \langle m, \bar{\sigma}\rangle B_\sigma,$$

where $\bar{\sigma}$ is the ray which is the image of σ in N/N_τ.

If $\tau \in T_{k+1}$ then every cone $\sigma \in \Sigma$ containing τ will also be contracted by r. In particular any such σ of dimension $n-k$ lies in T_k. Thus $m \in M(\tau)$ maps to

$$\sum_{\dim \sigma=n-k, \tau\subset\sigma} \langle m, v_{\tau,\sigma}\rangle W_\sigma.$$

Again this is just the usual notion of rational equivalence on a toric variety.

Proof. As already noted Proposition 3.5 implies that the final map is surjective.

An invariant subvariety W of X corresponding to some face of a polyhedral subdivision could be non-normal. By [1, Theorem 10.1] the corresponding subvariety of $\widetilde{X}$ is the normalization $\overline{W}$. Since the map r is the normalization when restricted to any invariant subvariety, we get that r restricted to any invariant subvariety is generically $1 : 1$. This implies by [11, Ch. 5, Proposition 3.3] that the normalization is in this case bijective. By [10, Example 1.2.3] we thus have that the orders of vanishing of any rational function on W is the same as the orders of vanishing on $\overline{W}$. In particular they can be computed using the usual techniques on normal toric varieties.

By construction of the maps giving the relations, we see that they are given by choosing an invariant subvariety Z of dimension $k + 1$ and choosing a rational function on Z and taking its divisor. Thus by definition these will give relations in $A_k(X)$, thus the composition of the two maps are zero.

By [8, Theorem 1] the canonical homomorphism $A^T_k(X) \to A_k(X)$ is an isomorphism, where $A^T_k(X)$ is the T-stable Chow group of X. By definition this is the quotient $Z^T_k(X)/R^T_k(X)$ of the group $Z^T_k(X)$ generated by T-invariant subvarieties of X modulo the subgroup $R^T_k(X)$ generated by divisors of eigenfunctions on T-invariant $(k + 1)$-dimensional subvarieties of X. In particular this implies that all relations in $A_k(X)$ come from divisors on T-invariant subvarieties.

In the exact sequence above we have by construction all such subvarieties and relations, except that we omit T-invariant subvarieties of general fibers such that the tailcone is not contracted. Letting q be a point with general fiber we set $P' = P\cup\{q\}$, thus we now consider q as a special fiber. We show that every new subvariety and every new relation in the corresponding exact sequence is already generated by those in the exact sequence from P. Thus we may omit any general fiber, hence the sequence in the theorem is exact. This is similar to Altmann-Petersen's proof for the case of divisors [5, Corollary 2.3].

For any cone $\sigma \in \Sigma(n - k)$ where $\sigma \notin K$ we get the subvariety $W_{q,\sigma}$ as a summand in V_k. There will also be a new relation coming from considering $\sigma \in R_{k+1}$ and observing that $\mathbb{Z}^{P'}/\mathbb{Z}$ has rank one more than $\mathbb{Z}^P/\mathbb{Z}$. The extra relation expresses $W_{p,F}$ in terms of the other generators, thus we may omit $W_{q,\sigma}$ as a generator in V_k, as well as the corresponding relation.

There is one more class of relations coming from adding q: Each cone τ of dimension $n - k - 1$ with $\tau \in K$ gives an element in V_{k+1}. Any $m \in M(\tau)$ gives the relation

$$\sum_{\tau\preceq\sigma,\dim\sigma=n-k} \langle m, v_{\tau,\sigma}\rangle W_{q,\sigma}$$

Now we already have relations saying that

$$W_{q,\sigma} = \sum_{H\preceq S_\infty,\operatorname{tail} H=\sigma} \mu(H)W_{\infty,H}.$$

Thus we need to show that the relation

$$\sum_{\tau\preceq\sigma,\dim\sigma=n-k} \langle m, v_{\tau,\sigma}\rangle \sum_{H\preceq S_\infty,\operatorname{tail} H=\sigma} \mu(H)W_{\infty,H}$$

is in the group of relations generated by sequence in the theorem.

Pick a face G of S_∞ of dimension $n-k-1$ with tailcone τ. Then $G = \tau + Q$, where Q is a polytope. Letting v be a vertex of Q we have that $G = \tau + v + (Q - v)$. Since $\dim \tau = \dim G$ we must have that $(Q - v)$ is contained in the linear span of τ. After taking the quotient $N/\mathbb{Q}\tau$ the image of G will thus be the vertex $\bar{v}$, the image of v. Let $e_1, ..., e_k$ be a $\mathbb{Z}$-basis for $\bar{v}^\perp$ and $e_0, e_1, ..., e_k$ a $\mathbb{Z}$-basis for $M(\tau)$. We have that $M(G)$ is generated by $\mu(G)e_0, e_1, ..., e_k$. This implies that the lattice index $[N(\tau) : N(G)]$ equals $\mu(G)$. We then have the relation

$$\sum_{G \preceq H, \dim H = n-k} \langle \mu(G)e_i, v_{G,H} \rangle W_{\infty,H}.$$

For any H with tailcone σ we have that $\mu(G)v_{G,H} = \mu(H)v_{\tau,\sigma}$, by Lemma 4.2 (see below), thus the relation equals

$$\sum_{G \preceq H, \text{tail } H = \sigma} \mu(H)\langle e_i, v_{\tau,\sigma} \rangle W_{\infty,H} + \sum_{G \preceq H, \text{tail } H = \tau} \mu(G)\langle e_i, v_{G,H} \rangle W_{\infty,H},$$

where all H in the sums are of dimension $n-k$. Now, if H contains G, has dimension $n-k$ and tailcone τ then the image $\overline{H}$ in $N/\mathbb{Q}\tau$ is a compact edge e having $\overline{G}$ has one of its vertices. The other vertex also corresponds to some G' with tailcone τ and of dimension $n-k-1$. There is a similar relation to the above, corresponding to G'. Now $\mu(G)v_{G,H}$ equals a primitive generator in $N/\tau \cap N$ for the linear space spanned by e. Similarly $\mu(G')v_{G',H}$ equals a primitive generator the same linear space, but with different sign. Thus if we sum the relations from G and G' the term $W_{\infty,H}$ will cancel. Thus if we sum all the relations corresponding to all possible such G's once, we see that the resulting relation is

$$\sum_{G} \sum_{G \preceq H, \dim H = n-k, \text{tail } H = \sigma} \mu(H)\langle e_i, v_{\tau,\sigma} \rangle W_{\infty,H}$$

By grouping together terms corresponding to the same cone σ we see we can write this relation as

$$\sum_{\tau \preceq \sigma, \dim \sigma = n-k} \langle e_i, v_{\tau,\sigma} \rangle \sum_{H \preceq S_\infty, \text{tail } H = \sigma} \mu(H) W_{\infty,H}$$

which is the relation we wanted to show for $m = e_i$. Since this is true for any i it will also follow for any m by linearity. ◻

Lemma 4.2. *Assume* $\tau \preceq \sigma$ *are cones satisfying* $\dim \tau + 1 = \dim \sigma$. *Assume* $G \preceq H$ *are faces of some* S_p, *with* $\operatorname{tail} G = \tau$, $\operatorname{tail} H = \sigma$, $\dim G = \dim \tau$ *and* $\dim H = \dim \sigma$. *Then* $\mu(G)v_{G,H} = \mu(H)v_{\tau,\sigma}$.

Proof. We may assume $e_1, ..., e_k$ is a basis for $N(H)$ and $e_1, ..., e_{k+1}$ is a basis for $N(G)$. $N(\tau)$ is a sublattice of $N(H)$ and there is an upper triangular integer matrix B such that $\{b_i = Be_i\}$ is a basis for $N(\tau)$. In particular the index $\mu(G) = [N(G) : N(\tau)]$ equals the product of the diagonal entries β_i of B. We may also assume that $b_1, ..., b_k$ is a basis for $N(\sigma)$. By definition $v_{\tau,\sigma}$ is a generator of $N(\tau)/N(\sigma)$, in our chosen basis we can choose it as the image of b_{k+1} in the quotient. Similarly the image of e_{k+1} is a generator of $N(G)/N(H)$. Thus we see that $\beta_{k+1}v_{G,H} = v_{\tau,\sigma}$. We also have that $\mu(H) = [N(H) : N(\sigma)] = \beta_1 \cdots \beta_k$. In particular $\beta_{k+1} = \frac{\mu(G)}{\mu(H}$, which proves the statement. □

Notation 4.3. *We denote the number of elements of* R_k, V_k, T_k *by* r_k, v_k, t_k, *respectively.*

Example 4.4. Let $X = \mathrm{Gr}(2, 4)$ which we can identify with the quadric

$$V(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}) \subset \mathbb{P}^5.$$

The group $SL_4(\mathbb{K})$ acts on $\mathrm{Gr}(2, 4)$ and contains the subgroup of diagonal matrices which is a 3-dimensional torus and which acts effectively on $\mathrm{Gr}(2, 4)$ making it into a T-variety of complexity one. The rational quotient Y is naturally identified with the moduli space of marked genus 0 curves $\overline{M_{0,4}} \simeq \mathbb{P}^1$ (see [15]). The identification is defined as follows. We now think of $\mathrm{Gr}(2, 4)$ as the space of lines in $\mathbb{P}^3 = \mathbb{P}(V)$ where V has basis $x_1, ..., x_4$. If $l \in \mathrm{Gr}(2, 4)$ is a general line then the intersections $l \cap \{x_i = 0\}$ will give four points p_1, p_2, p_3, p_4 on l. Then the cross ratio $\mathrm{CR}(p_1, p_2, p_3, p_4)$ defines the rational map to $\mathbb{P}^1$. In this way we obtain all values in $\mathbb{P}^1$ except $\{0, 1, \infty\}$. The points $0, 1, \infty$ are obtained by the non-general lines in the sets $V(p_{14}) \cup V(p_{23})$, $V(p_{12}) \cup V(p_{34})$, $V(p_{13}) \cup V(p_{24})$, respectively.

In coordinates the map is given as follows: On the open affine set $D(p_{12})$ a point in Plucker coordinates maps to $(p_{13}p_{24} : p_{23}p_{14}) \in \mathbb{P}^1$. The indeterminancy locus is given when both coordinates equals 0, we see that this locus consists of the eight planes

$$T_k^+ = V(\{p_{ij} | k \in \{i, j\}\})$$
$$T_k^- = V(\{p_{ij} | k \notin \{i, j\}\}).$$

We can resolve the quotient map by blowing up the union of the eight planes to get a map $\widetilde{X} = \mathrm{Bl}\,\mathrm{Gr}(2, 4) \to \mathbb{P}^1$. We will reinterpret this example in the language of T-varieties.

The paper [2] exhibits a divisorial fan for $\mathrm{Gr}(2,4)$: Let $N = \mathbb{Z}^4/\mathbb{Z}$ and let e_1, e_2, e_3, e_0 denote the image of the standard basis vectors of $\mathbb{Z}^4$, thus $e_0 = -e_1 - e_2 - e_3$. The tailfan Σ is the toric threefold with maximal cones $\mathrm{Cone}(\pm e_1, \pm e_2, \pm e_3 \pm e_4 |$there are exactly 2 pluses and 2 minuses). This has 6 maximal cones, 12 cones of dimension two and 8 rays. The special fibers correspond to the boundary divisors $\overline{M_{0,4}} \setminus M_{0,4}$ of reducible genus 0 curves, of which there are three. They correspond to partitions (({1,4}, {2,3}), ({1, 2}, {3, 4}), ({1, 3}, {2, 4}), we may assume these correspond to the points $0, 1, \infty$, respectively.

The fiber over 0 corresponds to replacing the origin with the compact edge f_{23} with vertices $(0, 0, 0)$ and $(-1, -1, 0)$. Similarly in the fiber over 1 we insert the edge f_{12} with vertices $(0, 0, 0)$ and $(-1, 0, -1)$ and over ∞ the edge f_{13} with vertices $(1, 1, 1)$ and $(1, 0, 0)$ (the polyhedra which is written in [2, Theorem 4.2] is a shifted version of the above, with rational coefficients. By [2, p.8 Remark 2] the true p-divisor correspond to a shift turning all polyhedra into lattice polyhedra, which is what we have done.) The p-divisor containing the edge f_{ij} has the empty coefficient over the other two special fibers. For a cone σ of Σ the faces of the special fibers with tailcone σ all belong to the same p-divisor. See Fig. 3.

Thus we get the following numbers:

$$\begin{aligned}
r_3 &= 0, v_3 = 6, t_3 = 0\\
r_2 &= 0, v_2 = 3, t_2 = 8\\
r_1 &= 0, v_1 = 0, t_1 = 12\\
r_0 &= 0, v_0 = 0, t_0 = 6.
\end{aligned}$$

The six invariant subvarieties $V(p_{ij})$ correspond to V_3. V_2 corresponds to the subvarieties $V(p_{12}, p_{34})$, up to permutation, while $t_2 = 8$ says exactly that there are eight invariant subvarieties of codimension two which are blown up by r, they correspond to the sets $T_k^{\pm}$. Thus we get an exact sequence

$$\mathbb{Z}^{18} \to \mathbb{Z}^{11} \to A_2(\mathrm{Gr}(2,4)) \to 0$$

The $\mathbb{Z}^{11}$ has generators W_{12}, W_{13}, W_{23} corresponding to the edges f_{ij}, and $E_i^{\pm}$ corresponding to the rays $\pm e_i$. The $\mathbb{Z}^{18}$ corresponds to six copies of $\mathbb{Z}^3$, one for each vertex of a special fiber. Fixing for example the vertex $v = (0,0,0)$ of f_{13}, we have that this is the vertex of a toric variety with rays with directions

$$\begin{vmatrix}1\\0\\0\end{vmatrix}, \quad \begin{vmatrix}0\\1\\0\end{vmatrix}, \quad \begin{vmatrix}0\\0\\-1\end{vmatrix}, \quad \begin{vmatrix}1\\1\\1\end{vmatrix}, \quad \begin{vmatrix}-1\\-1\\0\end{vmatrix}.$$

The relations we get from this will be

$$\begin{aligned}
\mathrm{div}(1,0,0) &= E_1^+ + E_0^- - W_{13}\\
\mathrm{div}(0,1,0) &= E_2^+ + E_0^- - W_{13}\\
\mathrm{div}(0,0,1) &= E_3^- - E_0^-.
\end{aligned}$$

Doing this for all vertices we see that all E_i^- are identified, call this class E^-, similarly all E_i^+ are identified, call this E^+ and all W_{ij} are identified, call this W. Then this gives a presentation

$$A_2(Gr(2,4)) \simeq \mathbb{Z}(E^+, E^-, W)/(E^+ + E^- - W)$$

which we see is isomorphic to a well known presentation of this group, namely

$$\mathbb{Z}(s_{1,1}, s_2, s_1^2)/(s_{1,1} + s_2 - s_1^2)$$

Here the $s_1, s_2, s_{1,1}$ correspond to Schubert cycles in the Chow ring of $\mathrm{Gr}(2,4)$.

Similarly there is an exact sequence

$$\mathbb{Z}^{22} \to \mathbb{Z}^{12} \to A_1(Gr(2,4)) \to 0$$

The generators of $\mathbb{Z}^{12}$ correspond to the 12 two-dimensional faces of Σ, they are of the form $\mathrm{Cone}(e_i, -e_j)$. Denote the corresponding generator by $Z_{i,-j}$. The

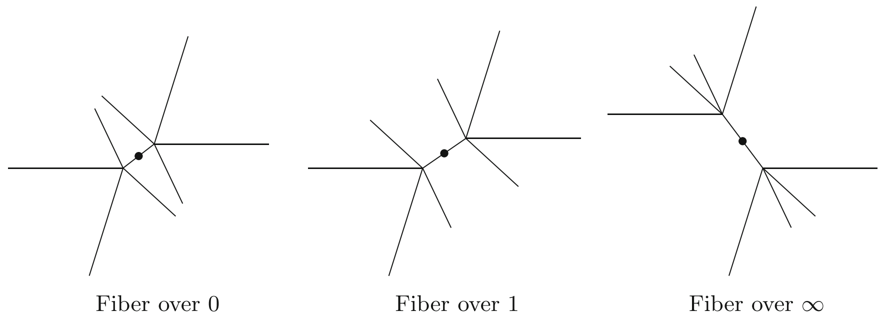

Fig. 3. Polyhedral subdivisons defining $\mathrm{Gr}(2, 4)$. From each of the non-zero vertices of the polyhedral subdivisions there is emanating a three-dimensional cone with four generators

relations come from two types: three copies of $\mathbb{Z}^2$ coming from the edges f_{ij}. Writing out the relations we see that they give $Z_{i,-j} = Z_{j,-i}$. Also there are eight copies of $\mathbb{Z}^2$ corresponding to the rays of Σ, they give relations $Z_{i,-j} = Z_{x,-k}$. Combining these we see that all $Z_{i,-j}$ are identified, thus $A_1(Gr(2,4))$ is one-dimensional, as expected.

5. Toric downgrades

In this section we study the example of downgrading a toric variety to only consider it as a T-variety of complexity one. We use the methods described in [14]. Not surprisingly the exact sequence of Theorem 4.1 coincides with the exact sequence of Fulton-Sturmfels.

We now consider a toric variety coming from a fan Σ living in $\mathbb{Z}^{n+1} \otimes \mathbb{Q}$. We choose a splitting $\mathbb{Z}^{n+1} = \mathbb{Z}^n \oplus \mathbb{Z} = N \oplus \mathbb{Z}$ and consider X_Σ only with the action of T_N. We then have an exact sequence

$$0 \to N \to N \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

The $\mathbb{Z}$ corresponds to the quotient, which for us will be $\mathbb{P}^1$. By construction there will only be two special fibers, over 0 and ∞. We denote by s the projection $N \oplus \mathbb{Z} \to N$ and by ϕ the map $N \oplus \mathbb{Z} \to \mathbb{Z}$. For a cone $\sigma \in \Sigma$ we get a polyhedral divisor with tailcone $\sigma \cap N$ and coefficient $s(\sigma \cap \phi^{-1}(1))$ over [0] and coefficient $s(\sigma \cap \phi^{-1}(-1))$ over $[\infty]$.

We consider the vector space $V = \mathbb{Q}^{n+1} = \mathbb{Z}^{n+1} \otimes \mathbb{Q}$ with basis $v_1, \dots, v_{n+1}$ and denote the last coordinate hyperplane by $H = \{v = \sum t_i v_i | t_{n+1} = 0\}$ and $H_{\geq 0} = \{v = \sum t_i v_i | t_{n+1} \geq 0\}$, $H_{>0} = \{v = \sum t_i v_i | t_{n+1} > 0\}$ and similarly for $H_{\leq 0}$, $H_{<0}$.

Lemma 5.1. *A cone $\sigma \in \Sigma$ of dimension $n - k + 1$ corresponds to*

(1) *an element of R_k if and only if $\sigma \subset H$.*
(2) *an element of V_k if and only if $\sigma \subset H_{\leq 0}$ or $\sigma \subset H_{\geq 0}$, but σ is not contained in H.*
(3) *an element of T_k if and only if σ intersects both $H_{>0}$ and $H_{<0}$.*

Proof. If $\sigma \subset H$ then we see that the tailcone of the associated polyhedral divisor is σ. Moreover we see that this divisor will have $\emptyset$ as coefficient over [0] and $[\infty]$, thus it will not be an element of K.

If $\sigma \subset H_{\leq 0}$ then the coefficient of [0] will be empty thus the tailcone will not lie in K. Moreover the coefficient over $[\infty]$ will have dimension $n - k$ as it will equal a compact polyhedron with vertices corresponding to generators having strictly positive last coordinate plus the tailcone which corresponds to rays with zero last coordinate.

If σ intersects both $H_{>0}$ and $H_{<0}$, then first of all we see that there will be no $\emptyset$ coefficients. Moreover the tailcone will be the intersection of σ with H. In particular it will have dimension $n - k$.

Since any cone σ belongs to only one of the three categories the only if statements follow as well. □

Corollary 5.2. *For a downgraded toric variety the exact sequence of Theorem 4.1 equals the exact sequence*

$$\oplus_{\sigma \in \Sigma(n-k)} M(\sigma) \to \oplus_{\sigma \in \Sigma(n+1-k)} \mathbb{Z} \to A_k(X) \to 0$$

from [9, Proposition 1.1].

Proof. This follows from the above lemma, together with the fact that for $\tau \in R_k$ the relation coming from $\mathbb{Z}^P/\mathbb{Z}$, when the number of special fibers is two, is the same as the relation coming from the last factor of the torus corresponding to $N \oplus \mathbb{Z}$, since both are saying that the fibers over [0] and [∞] of the corresponding $\mathbb{P}^1$ are equivalent. □

In particular, we see from the above that the number of elements of individual R_k, V_k, T_k can vary a lot, depending on which subtorus we choose, even if their sum is constant, equal to the number of cones in Σ of dimension $n - k + 1$.

Example 5.3. Let the fan for $\mathbb{P}^2$ be given from rays $\rho_1 = (1, 0)$, $\rho_2 = (0, 1)$, $\rho_0 = (-1, -1)$ and denote the associated divisors by D_i. Let $\mathcal{E} = D_1 \oplus 0$ and $\mathcal{F} = (D_1 + D_2) \oplus D_0$. Then $X = \mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{F})$ and it is a toric variety, however the different choices for $\mathcal{E}$ and $\mathcal{F}$ corresponds to different T-structures on X, giving different polyhedral subdivisions and marked cones defining it as a T-variety, see Figs. 4 and 5.

For $\mathbb{P}(\mathcal{E})$ the only marked cones are the ones containing the ray $(1, 0)$, while for $\mathbb{P}(\mathcal{F})$ all non-zero cones are marked. This is both an example of a toric downgrade, as well as an example of a toric vector bundle (see the next section).

6. Toric Vector bundles as *T*-varieties

A well-studied example of T-varieties are toric vector bundles. Fix a toric variety X_Σ. A vector bundle $\mathcal{E}$ on X_Σ is called toric if there is a T-action on the geometric vector bundle which is compatible with the action on X_Σ and linear on the fibers of $\mathcal{E}$. Toric vector bundles where classified by Klyachko [16]: they correspond to, for each ray, a filtration of the fiber E over the identity of the torus, indexed by integers and satisfying a compatibility condition.

Given an indecomposable toric vector bundle of rank $r+1$, the projectivization $\mathbb{P}(\mathcal{E})$ can be considered a T-variety of complexity r (if it decomposes then the complexity is lower). If $\mathcal{E}$ splits as a sum of line bundles then $\mathbb{P}(\mathcal{E})$ is in fact a toric variety and can thus be described as a complexity-one T-variety via downgrading.

Fix now a smooth toric variety X_Σ, a toric vector bundle $\mathcal{E}$ on X_Σ and fix a maximal cone σ in Σ. Any vector bundle on an affine toric variety splits as a sum of line bundles which again implies that $\mathbb{P}(\mathcal{E}|_{U_\sigma})$ is a toric variety [18, p.31]. We can describe $\mathbb{P}(\mathcal{E}|_{U_\sigma})$ as a T-variety via downgrading the torus action to only remember the action on the base. The description of $\mathbb{P}(\mathcal{E})$ as a T-variety will locally be glued from such pieces.

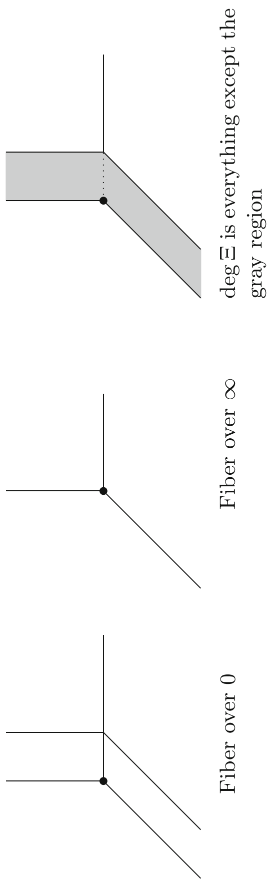

Fig. 4. Let $\mathcal{E} = D_1 \oplus 0$ on $\mathbb{P}^2$. The associated T-variety only has one special fiber over 0

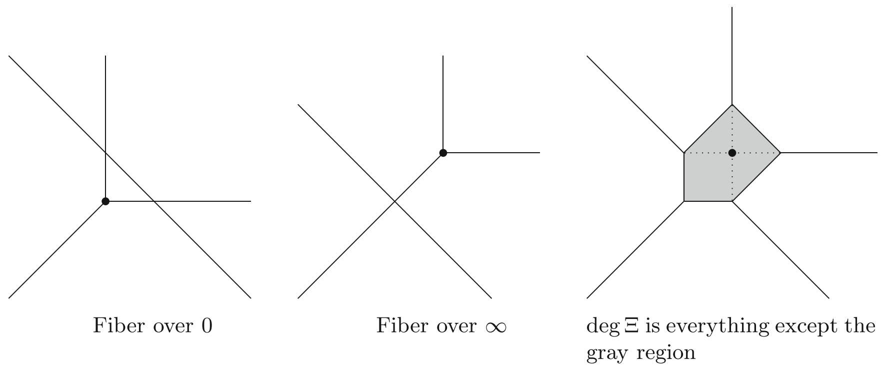

Fig. 5. Let $\mathcal{F} = (D_1 + D_2) \oplus D_0$ on $\mathbb{P}^2$. Then $\mathbb{P}(\mathcal{F})$ is isomorphic to $\mathbb{P}(\mathcal{E})$, however the different choice of T-structure gives different data as T-varieties

This description of rank-two toric vector bundles as T-varieties is given in [3, Proposition 8.4]. Fix a maximal cone σ. Then $\mathcal{E}|_{U_\sigma} = \mathcal{O}(u_1) \oplus \mathcal{O}(u_2)$, where u_1, u_2 are characters of the torus. Define the four polyhedra

$$\begin{aligned}
\Delta_1 &= \{v \in N_{\mathbb{Q}} | \langle u_1 - u_2, v\rangle \geq 1\} \cap \sigma \\
\Delta_2 &= \{v \in N_{\mathbb{Q}} | \langle u_2 - u_1, v\rangle \geq 1\} \cap \sigma \\
\nabla_1 &= \{v \in N_{\mathbb{Q}} | \langle u_1 - u_2, v\rangle \leq 1\} \cap \sigma \\
\nabla_2 &= \{v \in N_{\mathbb{Q}} | \langle u_2 - u_1, v\rangle \leq 1\} \cap \sigma
\end{aligned}$$

Fix $v_1, v_2 \in E$ corresponding to the 1-dimensional vector spaces in the Klyachko-filtration for $\mathcal{E}$ on σ. If there are no two distinct such spaces, simply choose any vectors such that v_1, v_2 is a basis, preferably v_i whose span appears in filtration for other rays (this minimizes the number of special fibers). Then

$$\begin{aligned}
\mathcal{D}_1 &= \Delta_1 \otimes v_1 + \nabla_2 \otimes v_2 \\
\mathcal{D}_2 &= \nabla_1 \otimes v_1 + \Delta_2 \otimes v_2
\end{aligned}$$

are polyhedral divisors describing $\mathbb{P}(\mathcal{E}|_{U_\sigma})$. Here v_i is considered a point in $\mathbb{P}^1 = \mathbb{P}(E)$. Thus the special fibers P correspond to the distinct one-dimensional subspaces appearing in the Klyachko filtrations.

The description above enables us to write out which cycles get contracted by the map $r : \widetilde{X} \to X$. For a cone σ there are essentially three different cases, corresponding to whether there are no one-dimensional linear spaces in the filtrations on the rays of σ, or only one distinct one-dimensional linear space, or if there are two different one-dimensional linear spaces (there cannot be more, by the compatibility condition for a toric vector bundle).

If $u_1 = u_2$ (meaning that there are no one-dimensional spaces) then σ itself is a cone of the tailfan and both polyhedral divisors have $\emptyset$ as a coefficient, hence σ is non-contracted, hence no face of σ will be contracted.

If $u_1 \geq u_2$, but $u_1 \neq u_2$ on σ (meaning there is one distinct one-dimensional space) we have that $\mathcal{D}_1$ has tailcone σ and no empty coefficients, thus σ will be contracted. We see that Δ_2 is $\emptyset$. Let $u_1 = (x_1, ..., x_n)$ and $u_2 = (y_1, ..., y_n)$ and assume $x_i - y_i > 0$ for $i = 1, ..., s$ and $= 0$ for $i = s + 1, ..., n$. Then

$$\nabla_1 = \left\{ (a_1, ..., a_n) \in N_{\mathbb{Q}} | a_i \geq 0, \sum_{i=0}^{s} a_i (x_i - y_i) \leq 1 \right\}.$$

We see that $\nabla_1 = \tau + P$ where τ is the $n - s$-dimensional cone generated by $e_{s+1}, ..., e_n$ (where $\sigma = \mathrm{Cone}(e_1, ...e_n)$) and P is the s-dimensional simplex defined by

$$P = \left\{ a_i \geq 0, \sum_{i=0}^{s} a_i (x_i - y_i) \leq 1 \right\}.$$

For a face τ of σ we have that τ is contracted if and only if $\deg \mathcal{D}_1 \cap \tau \neq \emptyset$. We see that

$$\deg \mathcal{D}_1 = \left\{ (a_1, ..., a_n) \in N_{\mathbb{Q}} | a_i \geq 0, \sum_{i=0}^{s} a_i (x_i - y_i) \geq 1 \right\}.$$

A face τ of σ is given by an arbitrary subset S of $\{1, ..., n\}$:

$$\tau = \mathrm{Cone}(e_i | i \in S \subset \{1, ..., n\}) = \{(a_1, ..., a_n) \in N_{\mathbb{Q}} | a_i \geq 0, a_i = 0 \text{ for } i \notin I\}$$

We see that τ is not contracted if and only if $S \subset \{s + 1...., n\}$.

The last case is if there exists at least one component such that $u_1 - u_2$ is positive and at least one component such that it is negative. We have that now $\mathcal{D}_1$ and $\mathcal{D}_2$ will have distinct full-dimensional tailcones σ_1 and σ_2, with $\sigma = \sigma_1 \cup \sigma_2$. There are no empty coefficients in any of the polyhedral divisors, thus all faces of σ_1, σ_2 are contracted.

The above might seem quite technical and non-illuminating, however if we go to the Klyachko perspective and reformulate the above in terms of the filtrations we get the following.

Proposition 6.1. *Let $X \simeq \mathbb{P}(\mathcal{E})$ be a toric vector bundle of rank two on a smooth toric variety X_Σ. Then there is a divisorial fan for X with tailfan Σ' being a subdivision of Σ. For a cone $\tau \in \Sigma'$ the corresponding orbit is not contracted by the map $\widetilde{X} \to X$ if and only if $\tau \in \Sigma$ and for any $v \in \tau(1)$ there doesn't appear a one-dimensional space in the filtration $E^v(j)$.*

Proof. This is just a reformulation of the above, when we recall that the condition $u_i = z_i$ is equivalent to $E^{v_i}(j)$ jumps directly from 0 to E. □

Example 6.2. The T-variety in Example 2.3 can be obtained as follows. Start with $\mathbb{P}^1 \times \mathbb{P}^1$, considered as the toric variety with rays $\pm e_i, i = 1, 2$ and consider a rank-two toric vector bundle on it. The filtrations are given by

$$E^{e_1}(j) = \begin{cases} E & \text{if } j \leq 0 \\ [0] & \text{if } 0 < j \leq 1 \\ 0 & \text{if } 1 < j \end{cases}, \quad E^{e_2}(j) = \begin{cases} E & \text{if } j \leq 0 \\ [1] & \text{if } 0 < j \leq 1 \\ 0 & \text{if } 1 < j \end{cases}$$

$$E^{-e_1}(j) = \begin{cases} E & \text{if } j \leq 0 \\ [\infty] & \text{if } 0 < j \leq 1 \\ 0 & \text{if } 1 < j \end{cases}, \quad E^{-e_2}(j) = \begin{cases} E & \text{if } j \leq 0 \\ 0 & \text{if } 1 < j \end{cases}$$

where we by $[p]$ mean the one-dimensional vector space in E for which the corresponding point in $\mathbb{P}(E)$ is p. Then one can easily check that we obtain the data given in Figs. 1 and 2.

Given $\mathcal{E}$ of rank two on X_Σ we partition the cones σ in Σ into three subsets as above. For any cone σ we denote the characters associated to $\mathcal{E}|_{U_\sigma}$ by u_1^σ and u_2^σ. We say that $\sigma \in H$ if $u_1^\sigma = u_2^\sigma$ on σ. We say that $\sigma \in I$ if $u_1^\sigma - u_2^\sigma > 0$ and $u_2^\sigma - u_1^\sigma > 0$ both intersects σ. Lastly $\sigma \in J$ if $u_1^\sigma \geq u_2^\sigma$ (or $\leq$) on σ, but they are not equal.

Proposition 6.3. *For a rank-two toric vector bundle $\mathcal{E}$ on the smooth toric variety X_Σ we have that for $X = \mathbb{P}(\mathcal{E})$ the cycles R_k, V_k, T_k for $k < n$ corresponds bijectively to the following sets*

$$R_k \leftrightarrow \Sigma(n-k+1) \cap H$$
$$V_k \leftrightarrow (\Sigma(n-k+1) \cap J) \cup (\Sigma(n-k) \cap J) \cup 2(\Sigma(n-k) \cap H)$$
$$T_k \leftrightarrow (\Sigma(n-k+1) \cap I) \cup (\Sigma(n-k) \cap J) \cup 2(\Sigma(n-k) \cap I)$$

In particular, by summing these numbers we get $r_k + v_k + t_k = \#\Sigma(n-k+1) + 2\#\Sigma(n-k)$ *for* $k < n$.

When $k = n$ *we have that the correspondences for* R_n *and* T_n *still hold, however an element of* V_n *corresponds to* $(\Sigma(1) \cap J) \cup P$ *and* $r_n + v_n + t_n = \#\Sigma(1) + \#P$.

In particular the number of generators in the surjection (1) does not depend on the bundle $\mathcal{E}$ except when $k = n$.

Proof. This is an application of Lemma 5.1 together with the description of $\mathbb{P}(\mathcal{E})$ as a T-variety, since locally a toric vector bundle is given as a toric downgrade.

The statement on R_k is immediate from Proposition 6.1.

If $\sigma \in \Sigma(n-k+1) \cap J$ then $D_2^\sigma = \nabla_1 \otimes v_1 + \Delta_2 \otimes v_2$ where ∇_1 is a sum $\tau + P$, where P is a polytope and τ a cone not contracted by r, and Δ_2 equals $\emptyset$. Recall that

$$\nabla_1 = \left\{ (a_1, ..., a_n) \in N_{\mathbb{Q}} | a_i \geq 0, \sum_{i=0}^{s} a_i(x_i - y_i) \leq 1 \right\}$$

The intersection of the hyperplane $\sum_{i=0}^{s} a_i(x_i - y_i) \leq 1$ with σ is a face of dimension $n-k$ contained in the relative interior of σ, with tailcone τ, thus giving an element of V_k. If $\sigma \in \Sigma(n-k) \cap J$ then the corresponding ∇_1 itself will give an element of V_k. If $\sigma \in \Sigma \cap H$ then there will be two different fibers, each containing σ as a face, however both will have empty coefficients somewhere, thus these fibers are special fibers, hence they contribute twice to V_k.

If $\sigma \in \Sigma(n-k+1) \cap I$ then in Σ', σ will be subdivided by intersecting with a hyperplane intersecting the interior of σ. The intersection of σ with this hyperplane is a cone of dimension $n-k$ which will be contracted by r, giving an element of T_k. If $\sigma \in \Sigma(n-k) \cap J$ then by the above σ is contracted thus it defines an element of T_k. If $\sigma \in \Sigma(n-k) \cap I$ then in Σ' it will be subdivided into two different cones of dimension $n-k$, both of which will be contracted by r.

For $k = n$ we have the short exact sequence

$$0 \to \mathbb{Z}^P/\mathbb{Z} \oplus M \to Z^{V_n \cup R_n} \to \mathrm{Pic}(\mathbb{P}(\mathcal{E})) \to 0$$

Implying that $\#\Sigma(1) + \#P = v_n + r_n$. Moreover we see that as above

$$R_n \leftrightarrow \Sigma(1) \cap H$$

However an element of V_n (which is simply any vertex of a special fiber) corresponds to either $\Sigma(1) \cap J$ (as above) or to the vertex 0 which is a vertex of S_p for any $p \in P$. □

Remark 6.4. If $\mathcal{E}$ is a direct sum of line bundles, then $\mathbb{P}(\mathcal{E})$ is itself a toric variety $X_{\Sigma'}$ and the presentation of $\mathbb{P}(\mathcal{E})$ as a T-variety is simply a toric downgrade. In this case Proposition 6.3 follows from Corollary 5.2 and the description of the fan Σ' [6, Proposition 7.3.3].

Remark 6.5. Consider once again the varieties in Example 5.3. We see that for $\mathcal{E}$ and $\mathcal{F}$ the numbers r_k, v_k, t_k are given by Table 1. We see that $r_k + v_k + t_k$ is independent of whether we use $\mathcal{E}$ or $\mathcal{F}$, as predicted by Proposition 6.3.

Table 1. The numbers r_i, v_i, t_i for Example 5.3

	r_2	v_2	t_2	$r_2+v_2+t_2$	r_1	v_1	t_1	$r_1+v_1+t_1$	r_0	v_0	t_0	$r_0+v_0+t_0$
$\mathbb{P}(\mathcal{E})$	3	0	2	5	7	1	1	9	4	2	0	6
$\mathbb{P}(\mathcal{F})$	5	0	0	5	4	5	0	9	1	5	0	6

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