UiO : **University of Oslo**

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Topics in the geometry of spaces of symmetric tensors

Thesis submitted for the degree of Philosophiae Doctor

Department of Mathematics
Faculty of Mathematics and Natural Sciences

UNIVERSITAS OSLOENSIS · MDCCCXI ·

2022

Series of dissertations submitted to the
Faculty of Mathematics and Natural Sciences, University of Oslo
No. 2491

ISSN 1501-7710

Cover: Hanne Baadsgaard Utigard.
Print production: Reprosentralen, University of Oslo.

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of Philosophiae Doctor (Ph.D.) at the University of Oslo. The research presented here was conducted under the supervision of professor Kristian Ranestad.

Acknowledgements

First and foremost, I would like to express my sincere gratitude to my advisor, Kristian Ranestad for his invaluable guidance and support throughout my time as a PhD student. I am especially thankful for his time and patience, and for the way he let me work with autonomy, at my own pace.

Secondly, I would like to thank my coadvisor, professor John Christian Ottem, for his encouragement and his numerous comments and suggestions, which have helped to improve the exposition of my work.

Special thanks go to my co-authors Laura, Roser and Bjørn, with whom I had the pleasure of facing very interesting mathematical challenges.

Both collaborations were part of projects involving many researchers from different universities. Therefore, I would like to express my appreciation to the organizers and participants in the working groups, for the opportunity to share stimulating ideas and keep motivation alive, especially during the pandemic period.

I am also very grateful to Enrico and Maurizio, for reading my thesis and giving a lot of feedback.

Furthermore, I would like to mention my other colleagues at the University of Oslo, who made my time as a PhD student more memorable, including Andrea, Cédric, Claudio, Edvard, Felix, Gard, Giovanni, Håkon, Lorenzo, Luca, Martin, (ny) Martin, Matilde, Michele, Ola, Sabrina and Silvia.

Spending three years abroad can be a challenging experience, especially during a pandemic. I have been lucky enough to be surrounded by people who have always made me feel at home, and for that I would like to thank Beate, Olav, Sissel and Willy.

The support of my friends has been an important constant throughout my life, and for that I wish to thank Alessia, Debora, Elisabetta, Jessica and all the Albedo Team.

Finally, I would like to express my gratitude to my parents, Lucia and Lorenzo, and my sister Marta. Thank you for always being there for me.

Elisa Cazzador
Oslo, February 2022

Contents

Introduction

This thesis is placed in the field of classical algebraic geometry, with particular focus on the study of spaces of symmetric tensors, namely the spaces of homogeneous forms of a fixed degree in a given number of variables.

The problems that will be addressed in the forthcoming chapters revolve around recurring topics such as reciprocal varieties, apolarity, secant varieties to Veronese embeddings, resolution of rational maps and generalized notions of rank for symmetric tensors.

All the chapters in Part I explore these themes with both theoretical and numerical results: in Chapters 1 and 2, we examine the so-called reciprocal varieties to spaces of catalecticant matrices associated with forms in two variables of any degree, and with forms in three variables of degree four, respectively; in Chapter 3, we study the natural rank and the local cactus rank of general forms in three variables of any degree.

Part II of the thesis consists solely of Chapter 4, which is concerned with the study of a certain variety that is naturally associated with general cubic forms in four variables. Specifically, we consider the image closure of a rational map defined via the action of the general linear group on cubic surfaces and we explain the first steps towards its resolution, which in turn provide a way to compute the degree of the image closure.

The content of Chapter 2 is an adaptation and extension of the published paper:

> *Homs, R., Cazzador, E., and Brustenga i Moncusí, L., "Inverting catalecticants of ternary quartics",* Le Matematiche (Catania),

appearing as a contribution in the *Special Issue on Linear Spaces of Symmetric Matrices*, see [33].

The content of Chapter 4 is the published paper:

> *Cazzador, E. and Skauli, B., "Towards the degree of the* $\mathrm{PGL}(4)$*-orbit of a cubic surface",* Le Matematiche (Catania),

appearing as a contribution in the *Special Issue on Twenty-Seven Questions about the Cubic Surface*, see [17].

The main definitions, ideas and theorems are presented below, following the ordering of the chapters. Along with that, we give a bit of context to where these results appear.

Reciprocal varieties

To any algebraic subvariety of a space of square matrices, we can naturally associate a variety in terms of its set of inverse matrices. These varieties, called *reciprocal varieties*, have recently aroused interest in the context of algebraic statistics see e.g. [5, 22, 41, 42, 50, 51, 52, 56] and [16], where linear subspaces of symmetric matrices (LSSM) are considered.

Explicitly, when $\mathcal{L}$ is a LSSM in the space $\mathbb{S}^m$ of $m \times m$ complex symmetric matrices, its reciprocal variety $\mathcal{L}^{-1}$ is defined as the Zariski closure of its set of inverses:

$$\mathcal{L}^{-1} := \overline{\{A^{-1} \mid A \in \mathcal{L}, \det(A) \neq 0\}} \subseteq \mathbb{S}^m.$$

When $\mathcal{L}$ is defined by real linear equations, the intersection of $\mathcal{L}^{-1}$ with the cone of positive definite symmetric matrices, is a *linear concentration model*, that is, a centered Gaussian statistical model, where the covariance matrices of normal distributions are defined by linear constraints on the entries of their inverses, see [51].

Therefore, from a statistical perspective, one reason to study these reciprocal varieties is to measure how well these models fit the data. This information is encoded by the *maximum likelihood degree* (ML-degree), that is the number of complex solutions to the critical equations of the *log-likelihood function*, defined by

$$A \mapsto \log\det(A) - \operatorname{tr}(SA),$$

where S is a random sample covariance matrix of sample data vectors.

From an algebro-geometric point of view, reciprocal varieties have also their own interest, especially when the chosen LSSM is associated with classically known geometric objects, as in the case study of Chapters 1 and 2 of this thesis, namely, spaces of Hankel/catalecticant matrices. Of these varieties, one typically would like to know their geometric properties, such as their dimension, degree, rank stratifications, singularities, etc.

In 2016, Michałek, Sturmfels et al. [42] proved the following proposition:

Proposition 0.1 ([42], Proposition 7.2)**.** *The reciprocal variety of the Hankel space associated with binary forms of degree $2k$ is projectively equivalent to the Grassmannian of lines $G(2, k+2)$ in its Plücker embedding. In particular, it is smooth of degree $\frac{1}{k+1}\binom{2k}{k}$.*

The key idea of their proof is based on the fact that inverses of Hankel matrices are Bézoutian matrices. In Chapter 1, we give an accurate analysis of the geometry of reciprocal varieties to Hankel spaces, starting from Theorem 1.2.1, where we provide an alternative proof for Proposition 0.1. This is done by simply using the fact that, for a fixed degree, Hankel spaces associated with different orders of contractions have the same ideal of r-minors, see [34].

In the same paper [42, Problem 7.5], the authors point out that not much is known about reciprocal varieties to catalecticant spaces associated with forms in more than two variables. The first unknown case is the one of ternary quartics, which we study in Chapter 2. Here, it is more challenging to obtain the defining equations of the reciprocal variety: first, the strategy applied for the binary case cannot be naturally adapted; second, one could make a more direct attempt and find the equations that the variety needs to satisfy by saturating a suitable ideal (as we explain in Proposition 1.3.1), but unfortunately, this procedure is unfeasible with computer algebra systems.

Giving up finding exact equations, we instead focus on computing the most relevant quantities following a numerical approach. Specifically, we use tools implemented in the `Julia` package `HomotopyContinuation.jl` [12], and obtain:

Theorem 0.2 (Theorem 2.1.3)**.** *The reciprocal variety of the catalecticant space associated with ternary forms of degree* 4 *is a* 14*-dimensional variety of degree* 85*, containing a* 27*-dimensional linear space of cubic generators in its defining ideal.*

A bridge between the statistical and the geometrical viewpoint comes from the interpretation of the ML-degree as the degree of a suitable projection map defined in terms of the *orthogonal space* to the LSSM, (cfr. Definition 1.1.21). With this equivalent definition, it is possible to see that the ML-degree of a linear concentration model is a lower bound for the degree of the reciprocal variety, where equality is reached if and only if the intersection between the orthogonal and the reciprocal variety is empty, see [42, Theorem 5.5].

For a generic LSSM, these two invariants are equal [51, Theorem 1] and in this sense we may say that Hankel spaces have a general behaviour:

Proposition 0.3 ([42], Proposition 7.4)**.** *For the Hankel space of binary forms of degree* $2k$*, the intersection between the orthogonal and the reciprocal variety is empty. In particular, the ML-degree of the associated linear concentration model is* $\frac{1}{k+1}\binom{2k}{k}$*.*

As for the previous result, we provide an alternative proof of this fact, see Proposition 1.4.3. Instead, in the case of ternary quartics, we observe a more special behaviour, giving the first instance of a catalecticant space for which degree and ML-degree do not coincide:

Theorem 0.4 (Theorem 2.1.3, Proposition 2.3.8)**.** *For the catalecticant space of ternary quartics, the intersection between the orthogonal and the reciprocal variety is set-theoretically a Veronese surface* $\nu_2(\mathbb{P}^2)$*. Moreover, the ML-degree of the associated linear concentration model is* 36*.*

The ML-degree is computed with numerical methods using the `Julia` package `LinearCovarianceModels.jl`, [55], while an explicit description of the intersection is obtained via a more theoretical approach.

A standard setting for the proofs of the results explained so far, as well as the more technical ones that are to come, consists of viewing reciprocal varieties as the image closure of linear projective spaces $\mathbb{P}\mathcal{L} \subseteq \mathbb{P}\mathbb{S}^m$ via a suitable rational map between projective spaces of symmetric matrices. Here we consider the *adjugate map*, defined by

$$\mathrm{Adj}_m \colon \mathbb{P}\mathbb{S}^m \dashrightarrow \mathbb{P}(\wedge^{m-1}\mathbb{S}^m) \qquad\qquad [A] \mapsto [\wedge^{m-1}A],$$

that is, by sending matrices to their cofactor matrices. This map clearly extends the operation of taking the inverse, and therefore it is birational. In particular, the reciprocal variety of any LSSM is irreducible of the same dimension of the original space.

The adjugate map has a base locus, consisting set-theoretically of matrices of corank at least two. By observing that a pair of degenerate matrices (A, B) in the graph of the adjugate map satisfies $AB = 0$, we see that its regularization sends rank-r matrices to corank-r matrices (cfr. [43, Proposition 12]).

Computing the degree of the reciprocal variety is a priori a considerably harder problem. Indeed, this would require finding a good compactification for the definition locus, together with a regularization of the original map, usually by repeatedly blowing up components in the base locus. Then, computing the degree would amount to understand the class of this compactification in the intersection ring of the blow-up. A compactification that has been proven to work in many examples is the one of complete quadrics (see [40] and the recent preprint [21]), obtained by subsequently blowing up the loci of symmetric matrices of rank $1, 2$, etc.

For certain classes of LSSMs though, this procedure is not necessary, and one could as well compute the relevant invariants by exploiting the geometrical peculiarities of the chosen space.

This is partially the case of our special instance of LSSMs, namely the spaces of square *catalecticant matrices* (see Definition 1.1.2). First introduced by Sylvester in [53], they arise in apolarity theory as matrices associated with linear maps of *contractions*, see [34]. Specifically, given a homogeneous form of degree d in n variables, we have a linear map from the space of operators of order k to the one of forms of degree $d - k$, defined by derivations. When $d = 2k$ is even, catalecticant matrices are square matrices of order $\binom{k+n}{n}$.

In the binary case, catalecticant matrices are often referred to as *Hankel matrices* and they are characterized by having constant skew-diagonals. Linear spaces of Hankel matrices are also called *Hankel spaces*.

When studying reciprocal varieties of Hankel spaces and catalecticant spaces of ternary quartics, there is a crucial property that makes the two cases treatable and somehow comparable: the locus of matrices of rank at most r coincides with $\sigma_r(\nu_d(\mathbb{P}^n))$, namely the *$r$-secant variety* of the *d-uple embedding* of $\mathbb{P}^n$, when $n = 1$ and $n = 2, d = 4$, see [38]. The dimensions of these secant varieties are well known, thanks to a theorem of Alexander and Hirschowitz (see [1] for the original reference and [11] for a modern proof).

This characterization of the rank loci allows us to use Terracini's Lemma [54] several times when we need to compute relevant dimensions of objects defined in terms of rank conditions. For arbitrary values of n and d, the above property almost never holds (failing already for ternary sextics) giving a partial obstacle to generalize our theory to any space of catalecticant matrices.

We now continue our presentation of the main results, emphasizing the comparison between the two cases of study. As we already mentioned, the adjugate map has a base locus, so one might wonder how base points (small-rank matrices) are blown up and what their image via the regularization is.

Equivalently, if we denote by $\Gamma_{\mathcal{L}}$ the graph closure of $\mathrm{Adj}_m|_{\mathbb{P}\mathcal{L}}$ in the product $\mathbb{P}\mathbb{S}^m \times \mathbb{P}(\wedge^{m-1}\mathbb{S}^m)$, and we let π_1, π_2 be the two projection maps from this product, then, for any point $A \in \mathbb{P}\mathcal{L}$ we are interested in studying

$$F_{\mathcal{L}}(A) := \pi_2(\pi_1^{-1}(A) \cap \Gamma_{\mathcal{L}}) \subseteq \mathbb{P}\mathcal{L}^{-1},$$

which we call the *reciprocal set* of A.

For Hankel matrices, reciprocal sets can be understood recursively by studying matrices corresponding to points belonging to most-degenerate secant spaces to rational normal curves, namely hyperosculating spaces. We obtain:

Proposition 0.5 (Proposition 1.4.5)**.** *For a degenerate rank-r matrix in the Hankel space of binary forms of degree $2k$, its reciprocal set is projectively equivalent to a Grassmannian of lines $G(2, k-r+2)$.*

The importance of this proposition is twofold: on the one hand, it constitutes the starting point for our alternative proof of Proposition 0.3. On the other hand, it allows to compute the dimension of the rank loci in the reciprocal variety. For these loci it is also possible to compute the degree. This is done by comparison with dual varieties to secant varieties of rational normal curves, which are known to be *coincident root loci*, see [31] and [39]. These results are summarized in

Theorem 0.6 (Theorem 1.4.8)**.** *For the Hankel space of binary forms of degree $2k$, the locus of matrices of corank at least r in the reciprocal variety is a variety of dimension $2k-r$, union of Grassmannians $G(2, k-r+2)$ over the r-secant variety of $\nu_{2k}(\mathbb{P}^1)$. Its degree is:*

$$\frac{2^r}{k+1-r}\binom{2k-r}{r}\binom{2k-2r}{k-r}.$$

When moving to the case of ternary quartics, we do not have an analogous description of the rank loci in terms of coincident roots. Nevertheless, we succeed in fully describing reciprocal sets of points. This time, we reduce to study matrices corresponding to points in most general secant spaces, and we prove that their reciprocal sets are linear sections of secant varieties to Grassmannians of lines. More precisely:

Theorem 0.7 (Theorem 2.2.7)**.** *For a degenerate rank-r matrix in the catalecticant space of ternary quartics, its reciprocal set is:*

(1) *An 11-fold of degree 14 in $\mathbb{P}^{14}$, defined by the cubic Pfaffians of a 7×7 skew-symmetric matrix, when $r = 1$;*

(2) *A cubic hypersurface in $\mathbb{P}^9$, defined by the cubic Pfaffian of a 6×6 skew-symmetric matrix, when $r = 2$;*

(3) *A linear projective space of dimension 5, when $r = 3$;*

(4) *A projective plane, when $r = 4$;*

(5) *A single point, when* $r = 5$.

The fact that Pfaffian equations show up establishes a common pattern with the binary case. One interesting difference though, besides the obvious ones, is that reciprocal sets of points not only vary depending on their rank r, but also on the kind of r-secant space they lie on (see Proposition 2.2.10).

By knowing the behaviour of single points, we can both explain the intersection that happens in the above Theorem 0.4 and use Terracini's Lemma to compute the dimension of the rank loci in the reciprocal variety:

Proposition 0.8 (Proposition 2.3.1)**.** *For the catalecticant space of ternary quartics, and for* $r = 1, \dots, 4$*, the locus of matrices of corank at least* r *in the reciprocal variety is a variety of dimension* $2r + 13 - \dim \sigma_r(\nu_4(\mathbb{P}^2))$*. The locus of matrices of corank at least* 5 *is the* 5*-fold* $\nu_2(\mathbb{P}^5)$*.*

As we pointed out above, intersection theory on complete quadrics offers a good setting for computing the degree of the reciprocal variety in terms of the exceptional divisor classes resulting from the blow-up of the rank loci of symmetric matrices. In particular, a closed formula can be given as soon as we can specify the class of the proper transform of the LSSM, intersected with the exceptional divisors. In this formula, only some classes give actual contibution to the degree, namely the ones that are not contracted by the regularization of the adjugate map.

When the LSSM in question is the catalecticant space of ternary quartics, the above Proposition 0.8 implies that only one of these classes give actual contribution, namely the one of the blown-up rank-1 locus. Recalling how the adjugate map affects the rank of points in the image, we deduce:

Theorem 0.9 (Theorem 2.3.4)**.** *For the catalecticant space of ternary quartics, the degree of the reciprocal variety is equal to the degree of its rank-*5 *locus, divided by* 6*.*

One final important result that marks the difference between the two study cases is about singularities.

Theorem 0.10 (Theorem 2.4.1)**.** *The reciprocal variety to the catalecticant space of ternary quartics is singular along its locus of rank at most* 2*.*

The proof of this fact uses a hybrid between geometric and numerical techniques: for rank-1 points we show that the tangent cones to the reciprocal variety are too much big (in fact, they span the entire space of symmetric matrices). For for rank-2 points we use instead a numerical procedure.

Considering all the information gathered from our analysis, especially the similarities between Proposition 0.5 and Theorem 0.7, we conclude with the following conjecture:

Conjecture 0.11 (Conjecture 2.4.9)**.** *The reciprocal variety to the catalecticant space of ternary quartics is defined by exactly* 27 *cubic equations which are Pfaffians of at least two* 7×7 *skew-symmetric matrices.*

Natural rank, local cactus rank

The common thread connecting the first two chapters with Chapter 3 is that of catalecticant matrices and secant varieties of d-uple embeddings, with a focus on apolarity and diverse notions of rank for symmetric tensors, namely the natural rank and the local cactus rank.

To motivate and understand the interest for the study of these invariants, we take a step back to the more classical notion of tensor rank. For a symmetric tensor, or equivalently, a homogeneous form F of degree d in $n+1$ variables, its *tensor rank*, is the minimal integer r such that we can write

$$F = l_1^d + l_2^d + \cdots + l_r^d,$$

where $l_1, \ldots, l_r$ are linear forms.

Symmetric tensors of rank at most r are parametrized by the r-secant variety to the d-uple embedding $\nu_d(\mathbb{P}^n)$, and the rank of a general symmetric tensor is the minimal order of secant variety filling up the ambient space. The problem of determining the dimension of these secant varieties is classical and solved by Alexander-Hirschowitz's theorem.

The notion of r-secant variety has been generalized in [15] by the one of r-*cactus variety*, defined for any projective variety as the closure of the union of the linear spaces spanned by its subschemes of length r. The *cactus rank* of a point is the minimal length of a subscheme whose linear span contains it. If the minimum is taken over local subschemes, namely those that are supported at a single point, we talk about the *local cactus rank*.

For d-uple embeddings, cactus varieties can be better understood via the *apolarity* action (see Definition 1.1.1), that is, the action of polynomial algebras to their dual algebras, induced by the contractions associated with catalecticant matrices.

Then, for any given polynomial, its *apolar ideal* is the annihilator with respect to the apolarity action, and its *apolar* schemes are those whose defining ideal contain the apolar ideal.

The relation between all these notions is explained by Apolarity Lemma [34], stating that the cactus rank of a homogeneous form is equal to the minimal length of a zero-dimensional subscheme apolar to it.

For d-uple embeddings of $\mathbb{P}^2$, cactus and secant varieties coincide [15, Theorem 1.6], hence for general ternary forms the cactus rank is equal to the rank. Note that this is not always the case: for forms in more variables, the cactus rank may be less than the rank [6, Theorem 1] and possibly equal to the local cactus rank.

In Chapter 3, we show that for general ternary forms the local cactus rank is strictly greater than the rank. Our investigation starts by observing as in [6] that a natural class of zero-dimensional local apolar schemes to a form is described by the apolar ideals of its dehomogenizations at different linear forms. We call *natural rank* the minimal length of such schemes.

The first question we address is: *What is the natural rank of a general form in a given number of variables and fixed degree?*

For general binary forms of degree d, the natural rank is equal to d. The proof is straightforward and it is simply based on the fact that zero sets of these forms consist of distinct points (cfr. Proposition 3.2.1).

For forms in more than two variables, only some partial results are known, see [32] and Proposition 3.2.2, where the difficulty in generalizing the procedure leads back to finding normal forms for equations of plane curves.

In this thesis we offer a different approach to the same question, which yields a definitive answer in the case of ternary forms:

Theorem 0.12 (Theorem 3.3.1). *The natural rank of a general ternary form F of degree d is:*

$$\operatorname{nat}(F) = \left\lfloor \frac{d(d+4)}{4} \right\rfloor .$$

When $d = 2k+1$ is odd, the natural rank is realized at $\frac{3(k+1)(k+2)(3k^2+3k+1)}{2}$ linear forms. When $d = 2k$ is even, the natural rank is realized at a curve of degree $3k(k+1)$ in the space of linear forms.

A starting point that allows us to build a good theoretical setup is based on associating catalecticant block matrices to inhomogeneous forms. More precisely, for a homogeneous form $F \in k[x, y, z]$ of degree d, we may consider its dehomogenization $f \in k[x, y]$ with respect to z. Then, the matrix we are attaching to f is the square catalecticant matrix of the form $z^d F$, of degree $2d$.

Using the correspondence that Macaulay established between polynomials and Artinian local Gorenstein algebras, the problem of computing the length of a local apolar scheme translates into to knowing the values of the corresponding Hilbert function.

With the Key Lemma 3.1.11, we show that these Hilbert functions are determined by the rank of suitable submatrices of the above catalecticant block matrix. As a consequence, we can give equations for the varieties of inhomogeneous polynomials with a given Hilbert function.

To prove Theorem 0.12, we reduce to showing that the dehomogenization of a general ternary form has Hilbert function of length equal to the maximal, decreased by one. This in turn allows us to reformulate the problem in terms of bundle maps that are fiberwise catalecticant.

The count of the linear forms realizing the natural rank is made by computing classes for the degeneracy loci of these bundle maps via Porteous' Theorem [45].

Afterwards, we proceed with the question: *What is the local cactus rank of a general form in a given number of variables and a fixed degree?* This is inspired by [7], where the same problem has been solved for quaternary cubics.

Our main result concerns the ternary case, giving a closed formula for small degrees:

Proposition 0.13 (Proposition 3.4.1). *The local cactus rank of a general ternary form F of degree $d \leq 5$ is:*

$$\operatorname{lcr}(F) = \left\lceil \frac{d(d+3)}{4} \right\rceil .$$

The complexity of the required computations is a partial obstacle to make the theory more systematic and generalize the result, but in fact we conjecture that the above formula holds for any degree, see Conjecture 3.4.15.

A result that comes useful when studying problems of this kind is provided by [7, Proposition 4]: for a homogeneous form F of degree d, the zero-dimensional local apolar schemes of minimal length are affine apolar schemes of higher degree polynomials whose degree $\leq d$ part (tail) is a dehomogenization of F.

We are particularly interested in the tails coming from polynomials with *unitary* Hilbert function, that is, equal to $(1, 1, \ldots, 1)$. Indeed, thanks to the Key Lemma, we can assume without loss of generality that:

Proposition 0.14 (Proposition 3.4.2)**.** *The local cactus rank of a homogeneous form is computed by polynomials with unitary Hilbert function.*

This motivates Lemma 3.4.5, which gives recursive explicit equations for the varieties of polynomials in two variables with unitary Hilbert function and with a given space of linear partial derivatives. The indeterminates for these equations are the coefficients of polynomials. Having very manageable equations, it is immediate to deduce the dimension of these varieties.

Moreover, considering inhomogeneous polynomials in two variables as dehomogenizations of ternary forms with respect to some linear form, we can count the choice for this linear forms, as well as the choice for the space of linear partial derivatives, obtaining:

Proposition 0.15 (Proposition 3.4.7)**.** *The variety of homogeneous ternary forms of degree e admitting a dehomogenization with unitary Hilbert function has dimension $2e + 2$.*

The degree-d tails of polynomials in these varieties are obtained by projecting them to the space of polynomials of degree $d \leq e$. The local cactus rank of a general ternary form of degree d is therefore the minimal e such that this map is dominant.

One way to understand whether the projection is dominant or not, is to consider the recursive explicit equations mentioned above and eliminate all the indeterminates corresponding to coefficients of higher degree monomials. This can be reasonably accomplished for small degrees, bringing the following:

Proposition 0.16 (Proposition 3.4.11)**.** *When $d \leq e \leq 10$, the variety of homogeneous ternary forms of degree d admitting as dehomogenization a tail of a degree-e polynomial with unitary Hilbert function has dimension equal to*

$$\min\left\{\binom{d+2}{2} - 1,\, 2e + 2\right\}.$$

This immediately implies Proposition 0.13. For example, when $d = 5$, the minimal value of e for which the map is dominant on the $\mathbb{P}^{20}$ of ternary quintics is $e = 9$. The length of the local apolar scheme of a polynomial of degree 9 and unitary Hilbert function is 10, which is also the number predicted by the closed formula for the local cactus rank.

Linear orbits of cubic surfaces

The last chapter of this thesis moves the focus on forms of degree three in four variables, namely polynomials whose zero sets define cubic surfaces.

The interest in the geometry of these objects has a very long history, especially since 1849, when Cayley and Salmon proved that every smooth cubic surface in the complex projective 3-space contains exactly 27 lines.

In the special issue [46], a list of questions on the cubic surface is presented, aiming to revisit its geometry from a traditional to a more advanced perspective.

The study carried out in Chapter 4 addresses the first of these questions: *"Given a generic homogeneous cubic F in x, y, z, w, what can we say about the orbit closure $\overline{\mathrm{PGL}(4) \cdot F}$? What is the degree of this variety in $\mathbb{P}^{19}$? Can we determine some of its defining polynomial equations?"*

The question about determining the degree of the orbit closure of a general form of degree d in $n+1$ variables was already addressed by Enriques and Fano in 1897, who solved the problem for $n = 1$ (points in the projective line) in some simple cases [25]. Their work was completed later in the 90's, in a paper from Aluffi and Faber [2], who next provided an answer also for the case $n = 2$ (plane curves), see [4] and [3].

The next interesting case is $n = d = 3$. For a given cubic surface $V(F)$, we consider the map

$$\phi \colon \mathbb{P}\mathrm{Hom}(\mathbb{C}^4, \mathbb{C}^4) \dashrightarrow \mathbb{P}\mathrm{Sym}^3(\mathbb{C}^4)^*,$$

which is induced by the action of the linear group and defined by pre-composition. Its image closure is precisely $\overline{\mathrm{PGL}(4)} \cdot F$ and a way to compute its degree is to find an explicit resolution of the base locus.

We outline the first steps towards this resolution of ϕ by adapting the techniques developed by Aluffi and Faber. Each step consists in successively describing the support of the base locus and then blowing up its components obtaining a new induced rational map.

The findings in this chapter quickly become quite technical, so we mention here only one result, which also makes clear how the role of the 27 lines shows up in the description of the base locus of ϕ:

Proposition 0.17 (Proposition 4.3.1)**.** *The base locus of ϕ is supported at the union of two closed components B and C, with $B \simeq \mathbb{P}^3 \times V(F)$ and $C \simeq \cup_{i=1}^{27} C_i$, where the C_i's are the irreducible components of C and each C_i is isomorphic to $\mathbb{P}^7$.*

The problem of computing the degree of the orbit closure of a general cubic surface was also considered by Brustenga i Moncusí, Timme, and Weinstein [14], who obtained the number 96120 using numerical techniques.

More recently, in 2021, Deopurkar, Patel and Tseng settled a definitive answer, confirming the numerically computed number using equivariant geometry [20].

Part I

Reciprocal varieties and rank problems

Chapter 1

Inverting Hankel matrices

In this chapter, we introduce reciprocal varieties and study the case of spaces of square Hankel matrices. In [42], it is proven that reciprocal varieties of these spaces are projectively equivalent to Grassmannian of lines. We give a new proof of this fact (Theorem 1.2.1). We also provide a new proof for the ML-degree of Hankel spaces (Proposition 1.4.3). This relies on the fact that reciprocal sets of degenerate points (Definition 1.1.12) are also Grassmannian of lines, of increasingly smaller dimension, as the rank of the point increases. The relation with duality of coincident root loci, allows us to give closed formulas for both dimension and degree of the rank loci in the reciprocal variety (Theorem 1.4.8).

In Section 1.1, we set all the definitions, in Sections 1.2 and 1.3 we study the reciprocal variety and the reciprocal sets of points for Hankel matrices. Finally, in Section 1.4 we study the geometry of the rank loci in the reciprocal variety.

1.1 Preliminaries

In this section, we fix all definitions and notation for the rest of the chapter. First we introduce apolarity and catalecticant matrices, also known as Hankel matrices when specializing to the case of binary forms. Secondly, we define reciprocal varieties via rational maps. For points in the base locus, we define their reciprocal sets by taking their preimage in the graph and projecting to the second factor. Thirdly, we recall the basic facts about projective duality and use this to compare the graphs of our rational maps with conormal varieties. Finally, we define an inner product on symmetric matrices: orthogonal spaces induce certain projection maps and, in turn, give the geometric definition of ML-degree.

1.1.1 Apolarity and catalecticant matrices

Let V be an $(n+1)$-dimensional vector space over $\mathbb{C}$. We may choose a basis $x_0, \dots x_n$ for V and a dual basis $\partial_0, \dots, \partial_n$ for the dual vector space V^*. The graded symmetric algebra $S(V^*) = \mathbb{C}[\partial_0, \dots, \partial_n]$ acts on $S(V) = \mathbb{C}[x_0, \dots, x_n]$ via the following action:

Definition 1.1.1. The *apolarity action* is defined as:

$$\begin{array}{ccc} S^kV^* \otimes S^dV & \to & S^{d-k}V \\ (\partial^\alpha, x^\beta) & \mapsto & \partial^\alpha \circ x^\beta := \begin{cases} x^{\beta-\alpha} & \text{if } \beta \geq \alpha \\ 0 & \text{otherwise,} \end{cases} \end{array}$$

where $\partial = (\partial_0, \dots, \partial_n)$, $x = (x_0, \dots, x_n)$, with multi-indices $\alpha = (\alpha_0, \dots, \alpha_n)$, $\beta = (\beta_0, \dots, \beta_n)$ satisfying $\sum_{i=0}^n \alpha_i = k$ and $\sum_{i=0}^n \beta_i = d$.

Apolarity action, also commonly known as *contraction*, was first introduced by Sylvester [53] and it has been widely studied as it is the key tool in the development of the theory of apolar schemes and tensor decomposition (see [8] for a nice overview).

When a homogeneous form $F \in S^dV$ is fixed, the apolarity action induces a linear map for every $k \leq d$, which we call the k-th *catalecticant morphism* of F:

$$\begin{array}{cccc} \gamma_{k,F}\colon & S^kV^* & \to & S^{d-k}V \\ & D & \mapsto & D \circ F. \end{array}$$

For every choice of ordered bases $\mathcal{B}_{\partial,k}$, $\mathcal{B}_{x,d-k}$ for S^kV^* and $S^{d-k}V$, there is an associated matrix to $\gamma_{k,F}$. Once the bases for V and V^* are fixed, then their symmetric powers canonically inherit monomial bases. Our standard choice for the ordering will be the reverse lexicographical one, that is:

$$\begin{aligned} \mathcal{B}_{\partial,k} &= \{\partial_0^k, \partial_0^{k-1}\partial_1, \partial_0^{k-2}\partial_2, \dots, \partial_{n-1}\partial_n^{k-1}, \partial_n^k\} \\ \mathcal{B}_{x,d-k} &= \{x_0^{d-k}, x_0^{d-k-1}x_1, x_0^{d-k-2}x_2, \dots, x_{n-1}x_n^{d-k-1}, x_n^{d-k}\}. \end{aligned} \tag{1.1.1}$$

Definition 1.1.2. For any homogeneous form $F \in S^dV$, its k-th *catalecticant matrix*, denoted by $\mathrm{Cat}(k, F)$, is the matrix associated with the catalecticant morphism $\gamma_{k,F}$, with respect to the standard bases (1.1.1).

Remark 1.1.3. When F has degree d, its k-th catalecticant matrix has size $m' \times m$, with $m = \binom{k+n}{n}$ and $m' = \binom{d-k+n}{n}$. If we denote with $i = (i_0, \dots, i_n)$ and $j = (j_0, \dots, j_n)$ the multiindices for the monomial bases (1.1.1), we see that the entries of $\mathrm{Cat}(k, F)$ are simply the coefficients of F, arranged so that in position (i, j) we find the coefficient for the monomial $x_0^{i_0+j_0} \cdots x_n^{i_n+j_n}$. If d is even, by taking $k = d/2$, we obtain a square catalecticant matrix, which in particular is a symmetric matrix.

Example 1.1.4. For the binary cubic form $F = x_0^3 + 2x_0^2x_1 - x_0x_1^2$, the second catalecticant matrix is

$$\mathrm{Cat}(2, F) = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ -1 & 0 \end{bmatrix}.$$

♢

Remark 1.1.5. It is also common to define apolarity and catalecticant matrices in terms of standard differentiation. With this theory, the variety of forms with rank-1 catalecticant matrix is a Veronese variety and consists of all forms which are powers of linear forms. Using contraction, we have an equivalent theory, where the rank-1 locus is still projectively equivalent to a Veronese variety, but it does not describe powers of linear forms anymore.

From now on, every reasoning about the tensor rank of a form – namely, the number of summands in its minimal expression as sum of powers of linear forms – will tacitly imply that we are operating under this identification. This is needed mainly in Lemma 1.4.6.

Let us now consider a degree-d form in $n+1$ variables with indeterminate coefficients

$$F = \sum_{i_0+\cdots+i_n=d} a_{(i_0,\ldots,i_n)} x_0^{i_0} \cdots x_n^{i_n}. \tag{1.1.2}$$

Then the k-th catalecticant morphism of F is associated with a catalecticant matrix with indeterminate entries $a_{(i_0,\ldots,i_n)}$, parametrizing the linear space of all catalecticant matrices in the family F.

Definition 1.1.6. The *k-catalecticant space* of $(n+1)$-ary forms of degree d is

$$\mathrm{Cat}(k, d-k; n+1) := \{\mathrm{Cat}(k, F) \mid F \in S^d V\}.$$

When d is even and $k = d/2$, we simply write $\mathrm{Cat}(k, n+1)$ instead of $\mathrm{Cat}(k, k; n+1)$. This is a linear subspace in the space of $m \times m$ symmetric matrices.

In this exposition, the main focus is on catalecticant matrices of binary forms of even degree. Their square catalecticants are often also known as *Hankel matrices*, characterized as those matrices with constant skew-diagonals. Note that this characterizarion is correct when chosing the standard bases $\mathcal{B}_{\partial,k}$ and $\mathcal{B}_{x,d-k}$ as in (1.1.1). We are going to see that in general, what remains invariant is the rank filtration of the Hankel space, namely the filtration determined by its loci of matrices of rank at most r.

Let us denote with $I_{r+1}(\mathrm{Cat}(k, d-k; n+1))$ the ideal generated by the $(r+1)\times(r+1)$ minors of $\mathrm{Cat}(k, n-k; n+1)$. The following lemma allows us to compare the rank loci of k-catalecticant matrices associated with binary forms of a fixed degree d, whenever the size of the matrices is good enough:

Lemma 1.1.7 ([28], Lemma 2.3)**.** *Let $2r \leq d$ and let u, v be integers which satisfy $r \leq u \leq d-u$ and $r \leq v \leq d-v$. Then*

$$I_{r+1}\mathrm{Cat}(u, d-u; 2) = I_{r+1}\mathrm{Cat}(v, d-v; 2).$$

In some cases, ideals of minors of catalecticant matrices define secant varieties of Veronese embeddings. This is certainly the case of Hankel matrices of binary forms, see Proposition 1.1.8 below (and [38] for a complete account). Here, by *d-th Veronese embedding* of $\mathbb{P}^n$, we mean the image of $\mathbb{P}^n$ via the map

$$\begin{array}{rccc} \nu_d\colon & \mathbb{P}^n & \longrightarrow & \mathbb{P}^{\binom{d+n}{n}-1} \\ & [x_0 : \cdots : x_n] & \longmapsto & [x_0^d : x_0^{d-1}x_1 : \cdots : x_{n-1}x_n^{d-1} : x_n^d]. \end{array}$$

When $n=1$, the image $\nu_d(\mathbb{P}^1)$ is also referred to as the *rational normal curve* of degree d.

For a projective variety $X \subseteq \mathbb{P}^N$, its *r-secant variety*, denoted by $\sigma_r(X)$, is the Zariski closure of the union of all linear spaces spanned by r points lying on X, in symbols:

$$\sigma_r(X) := \overline{\bigcup_{x_1,\ldots,x_r \in X} \langle x_1, \ldots, x_r \rangle} \subseteq \mathbb{P}^N.$$

The following result was already known to Sylvester (see [23, Proposition 1.4.3] for a modern proof):

Proposition 1.1.8. *Let $C = \nu_d(\mathbb{P}^1) \subset \mathbb{P}^d$ be the rational normal curve of degree d and let $2r \leq d$. Then $I_{r+1}(\mathrm{Cat}(k, d-k; 2))$ is the graded ideal of the r-secant variety $\sigma_r(C)$.*

Later in the next sections we will study properties related to secant varieties of rational normal curves. Many times we will reduce to prove the statements in the case of the most degenerate secant spaces, as they carry a very natural geometric feature:

Remark 1.1.9. Let C be as above and let $S \subset \mathbb{P}^d$ an r-secant space to C. If $S \cap C$ is supported at a single point P, then S is the r-th osculating space to C at P.

The following lemma is a classical result due to Terracini [54], and as a key tool it will be repeatedly referred to throughout this presentation.

Lemma 1.1.10. *(Terracini's Lemma) Let $X \subseteq \mathbb{P}^n$ be an irreducible projective variety and let $p_1, \ldots, p_r$ be general linearly independent points of X. Then for a general point $p \in \langle p_1, \ldots, p_r \rangle$ on the secant variety $\sigma_r(X)$, we have*

$$T_p\sigma_r(X) = \langle T_{p_1}X, \ldots, T_{p_r}X \rangle.$$

1.1.2 The reciprocal variety

We use the following compact notation for the linear space of $m \times m$ symmetric matrices and its dual:

$$\mathbb{S}^m := S^2(\mathbb{C}^m) \qquad\qquad (\mathbb{S}^m)^* := S^2(\mathbb{C}^m)^*,$$

viewed as affine linear spaces of dimension $\frac{m(m+1)}{2}$.

For every linear subspace $\mathcal{L} \subseteq \mathbb{S}^m$, we would like to describe the Zariski closure of its set of inverses, namely:

$$\mathcal{L}^{-1} := \overline{\{A^{-1} \mid A \in \mathcal{L}, \det(A) \neq 0\}} \subseteq \mathbb{S}^m. \tag{1.1.3}$$

We formulate the problem in a projective setting. We first observe that, for every invertible matrix A, we have $\mathrm{Adj}(A) = \det(A)A^{-1}$, where $\mathrm{Adj}(A)$ is the adjugate matrix of A. In particular, if $\det(A) \neq 0$, the classes $[A^{-1}]$ and $[\mathrm{Adj}(A)]$ represent the same point in the projective space $\mathbb{PS}^m$. This motivates the following definitions:

Definition 1.1.11. For every integer m, the *adjugate map* of $m \times m$ symmetric matrices is the rational map

$$\begin{array}{cccc} \mathrm{Adj}_m\colon & \mathbb{PS}^m & \dashrightarrow & (\mathbb{PS}^m)^\vee \\ & [A] & \mapsto & [\wedge^{m-1}A]. \end{array}$$

The target space $(\mathbb{P}\mathbb{S}^m)^\vee$ of the adjugate map is the projective dual of $\mathbb{P}\mathbb{S}^m$, see the next section for more details on dual spaces. Its place in Definition 1.1.11 makes sense, since there is a canonical isomorphism $\mathbb{P}\big(S^2(\bigwedge^{m-1}\mathbb{C}^m)\big) \simeq (\mathbb{P}\mathbb{S}^m)^\vee$. Explicitly, for every $i = 0, \ldots, m$, we have a pairing

$$\begin{array}{ccccc} \bigwedge^{m-i}(\mathbb{C}^m) \times \bigwedge^{i}(\mathbb{C}^m) & \to & \bigwedge^{m}(\mathbb{C}^m) & \simeq & \mathbb{C} \\ (\alpha, \beta) & \mapsto & \alpha \wedge \beta, & & \end{array} \tag{1.1.4}$$

where the right-hand side isomorphism is determined by the choice of a non-zero constant. The pairing (1.1.4) is perfect, which means that we have isomorphism $\wedge^{m-i}(\mathbb{C}^m) \simeq \wedge^i(\mathbb{C})^*$. Taking $i = 1$ and applying the functor of symmetric powers, we obtain isomorphism $S^2(\wedge^{m-1}\mathbb{C}^m) \simeq S^2(\mathbb{C}^m)^*$, that is canonical modulo constants.

Note that the adjugate map extends the operation of inverting matrices: it is well-defined not only for full-rank matrices, but also for corank-1 matrices. Moreover, for every full rank matrix A, we have $\mathrm{Adj}_m(\mathrm{Adj}_m(A)) = \det(A)^{m-2}A$, so the adjugate map is in fact a birational map. We denote its inverse by

$$\mathrm{Adj}_m^\vee : (\mathbb{P}\mathbb{S}^m)^\vee \dashrightarrow \mathbb{P}\mathbb{S}^m.$$

For the sake of brevity, from now on we will drop the square brackets in $[A]$ and we will simply write A for a point in the projective space.

Let $\mathbb{P}\mathcal{L}$ be a projective linear subspace of $\mathbb{P}\mathbb{S}^m$. We can define the projective analogous of (1.1.3) as a subvariety of $(\mathbb{P}\mathbb{S}^m)^\vee$:

Definition 1.1.12. let $\mathbb{P}\mathcal{L}$ be a linear subspace of $\mathbb{P}\mathbb{S}^m$, with generic element of rank m. The *reciprocal variety* of $\mathbb{P}\mathcal{L}$ is defined to be:

$$\mathbb{P}\mathcal{L}^{-1} := \overline{\mathrm{Adj}_m(\mathbb{P}\mathcal{L}^\circ)} \subseteq (\mathbb{P}\mathbb{S}^m)^\vee,$$

where $\mathbb{P}\mathcal{L}^\circ$ denotes the definition locus of Adj_m in $\mathbb{P}\mathcal{L}$.

As subvarieties of spaces of matrices, $\mathbb{P}\mathcal{L}$ and $\mathbb{P}\mathcal{L}^{-1}$ inherit a rank filtration. Let $D^r_{\mathbb{S}^m}$ and $D^r_{(\mathbb{S}^m)^*}$ denote the *determinantal varieties* of matrices of rank at most r in $\mathbb{P}\mathbb{S}^m$ and $(\mathbb{P}\mathbb{S}^m)^\vee$ respectively. Studying the geometry of $\mathbb{P}\mathcal{L}^{-1} \cap D^r_{(\mathbb{S}^m)^*}$ requires a deeper understanding of the fibers in the graph of Adj_m. So let us denote by $\Gamma_\mathcal{L}$ the closure of the graph of $\mathrm{Adj}_m|_{\mathbb{P}\mathcal{L}}$ in the product $\mathbb{P}\mathbb{S}^m \times (\mathbb{P}\mathbb{S}^m)^\vee$ and let π_1, π_2 be the two projection maps from this product. Then, for every rank-r matrix $A \in \mathbb{P}\mathcal{L}$, we define the set

$$F_\mathcal{L}(A) := \pi_2(\pi_1^{-1}(A) \cap \Gamma_\mathcal{L}) \subseteq \mathbb{P}\mathcal{L}^{-1},$$

which is closed since π_2 is a closed map. When $r = m$, then $F_\mathcal{L}(A) = F_{\mathbb{S}^m}(A) = \mathrm{Adj}_m(A)$ is a point, whose product with A is (proportional to) the identity. More generally:

Definition 1.1.13. For a closed irreducible subvariety $X \subseteq \mathbb{P}\mathcal{L}$, its $\mathcal{L}$-*reciprocal set* is defined as:

$$F_\mathcal{L}(X) := \overline{\bigcup_{\substack{A \in X \\ \text{generic rank}}} F_\mathcal{L}(A)} \subseteq \mathbb{P}\mathcal{L}^{-1}. \tag{1.1.5}$$

With this definition we have $F_{\mathcal{L}}(\mathcal{L}) = \mathbb{P}\mathcal{L}^{-1}$.

Remark 1.1.14. As a function of $\mathcal{L}$, the operator $F_{\mathcal{L}}(X)$ preserves inclusions: for every $\mathcal{L}_1 \subseteq \mathcal{L}_2$ and for every $X \subset \mathcal{L}_1$, we have $F_{\mathcal{L}_1}(X) \subseteq F_{\mathcal{L}_2}(X)$, where equality can possibly hold also when the inclusion of $\mathcal{L}_1$ in $\mathcal{L}_2$ is strict.

1.1.3 Relation with the conormal variety

Let V be an $(n+1)$-dimensional vector space over $\mathbb{C}$ and $\mathbb{P}^n = \mathbb{P}V$ its associated projective space. Its *dual projective space*, denoted by $(\mathbb{P}^n)^\vee := \mathbb{P}V^*$, parametrises hyperplanes in $\mathbb{P}^n$ and viceversa.

More generally, the linear span of points in $(\mathbb{P}^n)^\vee$ corresponds by duality to the intersection of the corresponding hyperplanes of $\mathbb{P}^n$. In particular, the projective dual of an r-dimensional linear space of $\mathbb{P}^n$ is a linear space of dimension $n-r-1$ in $(\mathbb{P}^n)^\vee$.

Projective duality is defined more generally for closed irreducible subvarieties $X \subset \mathbb{P}^n$. A hyperplane $H \subset \mathbb{P}^n$ is said to be *tangent* to X if there exists a smooth point $x \in X$ such that $x \in H$ and we have containment $T_xH \supset T_xX$ of the corresponding tangent spaces.

Definition 1.1.15. The *projective dual variety* of X, denoted by $X^\vee$ is the Zariski closure of the set of all points $[H] \in (\mathbb{P}^n)^\vee$ such that H is tangent to X.

Definition 1.1.16. For any closed irreducible variety $X \subset \mathbb{P}^n$, its *conormal variety*, is defined to be

$$\mathrm{C}(X) := \overline{\{\, (x, [H]) \in \mathbb{P}^n \times (\mathbb{P}^n)^\vee \mid H \text{ is tangent to } X \text{ at } x)\}}.$$

If we denote with π_1, π_2 the two projection maps from $\mathbb{P}^n \times (\mathbb{P}^n)^\vee$, then from Definition 1.1.15, we have $\pi_2(\mathrm{C}(X)) = X^\vee$.

Projective duality has been studied for the determinantal loci of spaces of matrices (see [26] for the general case and [43] for the symmetric case).

For symmetric matrices, we clearly have $\mathbb{P}(\mathbb{S}^m)^{-1} = (\mathbb{P}\mathbb{S}^m)^\vee$, so the graph of the adjugate map is given by

$$\Gamma_{\mathbb{S}^m} = \{(A, B) \in \mathbb{P}\mathbb{S}^m \times (\mathbb{P}\mathbb{S}^m)^\vee \mid AB = \lambda \cdot \mathrm{Id}_m, \text{ for some } \lambda \in \mathbb{C}\}. \tag{1.1.6}$$

The constant of proportionality λ can possibly be zero. For example, when X is a closed irreducible subvariety of $\mathbb{P}\mathbb{S}^m$ of generic rank $r < m$, then the matrices B satisfying the equations in (1.1.6) have rank $m-r$ and the preimage of X in the graph is given by

$$\pi_1^{-1}(X) \cap \Gamma_{\mathbb{S}^m} = \{(A, B) \in X \times D^{m-r}_{(\mathbb{S}^m)^*} \mid AB = 0\}. \tag{1.1.7}$$

When $X = D^r_{\mathbb{S}^m}$, these preimages are conormal varieties:

Proposition 1.1.17 ([43], Proposition 12)**.** *For every $r < m$, the irreducible variety*

$$\{(A, B) \in D^r_{\mathbb{S}^m} \times D^{m-r}_{(\mathbb{S}^m)^*} \mid AB = 0\}$$

coincides with the conormal variety $\mathrm{C}(D^r_{\mathbb{S}^m})$*. In particular, the projective dual variety* ${D^r_{\mathbb{S}^m}}^\vee \subset (\mathbb{P}\mathbb{S}^m)^\vee$ *is isomorphic to* $D^{m-r}_{\mathbb{S}^m}$*.*

Remark 1.1.18. Proposition 1.1.17 allows to understand the geometry of the reciprocal sets $F_{\mathbb{S}^m}(X)$: if X is a closed irreducible subvariety of $\mathbb{P}\mathbb{S}^m$ of generic rank $r < m$, then (1.1.7) implies

$$F_{\mathbb{S}^m}(X) = \{B \in D^{m-r}_{(\mathbb{S}^m)^*} \mid AB = 0 \text{ for some } A \in X\}.$$

Equivalently, $F_{\mathbb{S}^m}(X)$ is the closure of the set of hyperplanes in $\mathbb{P}\mathbb{S}^m$ tangent to $D^r_{\mathbb{S}^m}$ at smooth points of X.

If X is a point of rank r, then $F_{\mathbb{S}^m}(X)$ is a projective linear subspace of $(\mathbb{P}\mathbb{S}^m)^\vee$ of dimension $\frac{(m-r)(m-r+1)}{2} - 1$ and generic rank $m - r$.

If $X = D^r_{\mathbb{S}^m}$, we have $F_{\mathbb{S}^m}(D^r_{\mathbb{S}^m}) = (D^r_{\mathbb{S}^m})^\vee = D^{m-r}_{(\mathbb{S}^m)^*}$.

Corollary 1.1.19. *For every* $r = 1, \dots, m$ *the following set-theoretical equality holds:*

$$F_{\mathcal{L}}(\mathbb{P}\mathcal{L} \cap D^r_{\mathbb{S}^m}) = \mathbb{P}\mathcal{L}^{-1} \cap D^{m-r}_{(\mathbb{S}^m)^*}.$$

Proof. The inclusion from the left to the right-hand side of the equation follows by the definition given in (1.1.5) and Remark 1.1.18. The other inclusion is a consequence of Biduality Theorem ([26, Theorem 1.1]). ■

1.1.4 Orthogonality

The space of symmetric matrices is equipped with an inner product, defined for any two matrices $X_1, X_2 \in \mathbb{S}^m$ as

$$X_1 \bullet X_2 := \mathrm{tr}(X_1 X_2).$$

This is in fact the standard component-wise inner product, after identifying X_1, X_2 with two points in $\mathbb{C}^{m^2}$. The choice of this product fixes an isomorphism $\phi\colon \mathbb{S}^m \xrightarrow{\sim} (\mathbb{S}^m)^*$, so, for every $X \in \mathbb{S}^m$ and $Y \in (\mathbb{S}^m)^*$, we define

$$X \bullet Y \ := X \bullet \phi^{-1}(Y).$$

Then, for every linear subspace $\mathbb{P}\mathcal{L} \subseteq \mathbb{P}\mathbb{S}^m$, we can regard its *orthogonal space* as a subspace of $(\mathbb{P}\mathbb{S}^m)^\vee$:

$$\mathbb{P}\mathcal{L}^\perp := \{Y \in (\mathbb{P}\mathbb{S}^m)^\vee \mid X \bullet Y = 0, \forall X \in \mathbb{P}\mathcal{L}\}.$$

With this definition, the orthogonal space $\mathbb{P}\mathcal{L}^\perp \subseteq (\mathbb{P}\mathbb{S}^m)^\vee$ is the projective dual to $\mathbb{P}\mathcal{L} \subset \mathbb{P}\mathbb{S}^m$, parametrizing hyperplanes of $\mathbb{P}\mathbb{S}^m$ containing $\mathbb{P}\mathcal{L}$. In particular, if $\dim(\mathbb{P}\mathcal{L}) = d$, then $\mathbb{P}\mathcal{L}^\perp$ has codimension $d+1$ in $(\mathbb{P}\mathbb{S}^m)^\vee$.

For linear spaces, we will keep using the orthogonal notation $\mathbb{P}\mathcal{L}^\perp$ to indicate the projective dual to $\mathbb{P}\mathcal{L}$, so that we can distinguish it from the dual projective, which instead will be denoted by $(\mathbb{P}\mathcal{L})^\vee$.

Remark 1.1.20. When $\mathcal{L}$ is a 1-dimensional linear space, then $A = \mathbb{P}\mathcal{L}$ is a point in $(\mathbb{P}\mathbb{S}^m)^\vee$, and it is easily seen from the definitions that $F_{\mathbb{S}^m}(A) \subseteq A^\perp$.

The inclusion of a $(d+1)$-dimensional linear subspace $\mathcal{L} \subset \mathbb{S}^m$ induces by duality a canonical isomorphism $\mathcal{L}^* \simeq (\mathbb{S}^m)^*/\mathcal{L}^\perp$. We would like to consider the natural projection map

$$\pi_{\mathcal{L}} \colon (\mathbb{P}\mathbb{S}^m)^\vee \dashrightarrow \mathbb{P}((\mathbb{S}^m)^*/\mathcal{L}^\perp) \simeq \mathbb{P}\mathcal{L}^\vee, \tag{1.1.8}$$

which sends hyperplanes of $\mathbb{P}\mathbb{S}^m$ to hyperplane sections of $\mathbb{P}\mathcal{L}$. After suitable identifications, $\pi_{\mathcal{L}}$ can be understood as the projection with center $\mathbb{P}\mathcal{L}^\perp$ to any d-dimensional linear subspace of $(\mathbb{P}\mathbb{S}^m)^\vee$ disjoint from it.

The map $\pi_{\mathcal{L}}$ is well-defined in the set-theoretical complement $(\mathbb{P}\mathcal{L}^\perp)^c$ and by surjectivity its restriction to $\mathbb{P}\mathcal{L}^{-1}$ is generically finite-to-one.

Definition 1.1.21. Given a linear subspace $\mathbb{P}\mathcal{L} \subseteq \mathbb{P}\mathbb{S}^m$, its *maximum likelihood degree*, denoted by $\text{ML-}\deg(\mathbb{P}\mathcal{L})$, is the degree of the generic fiber of $\pi_{\mathcal{L}}$ restricted to $\mathbb{P}\mathcal{L}^{-1} \cap (\mathbb{P}\mathcal{L}^\perp)^c$.

The relation with the reciprocal degree is explained in [42] in the more general context of exponential varieties. In our case, we may say:

Theorem 1.1.22 ([42], Theorem 5.5)**.** *For every linear space of symmetric matrices $\mathbb{P}\mathcal{L}$, we have*

$$\text{ML-}\deg(\mathbb{P}\mathcal{L}) \leq \deg(\mathbb{P}\mathcal{L}^{-1})$$

and equality holds if and only if $\mathbb{P}\mathcal{L}^{-1} \cap \mathbb{P}\mathcal{L}^\perp = \emptyset$.

Remark 1.1.23. For any linear subspace $\mathbb{P}\mathcal{L} \subseteq \mathbb{P}\mathbb{S}^m$, the orthogonal space $\mathbb{P}\mathcal{L}^\perp$ does not contain any full-rank point of $\mathbb{P}\mathcal{L}^{-1}$. Indeed, if B was a matrix of rank m in $\mathbb{P}\mathcal{L}^{-1} \cap \mathbb{P}\mathcal{L}^\perp$, then by definition, it would satisfy both the relations $AB = \mathrm{Id}_m$ and $\mathrm{tr}(AB) = 0$ for some matrix $A \in \mathbb{P}\mathcal{L}$ of rank r, which is not possible.

1.2 Grassmannians as reciprocal varieties: a new proof

Our main object of study is the projective space of catalecticant matrices $\mathbb{P}\mathrm{Cat}(m-1,2) \subset \mathbb{P}\mathbb{S}^m$ associated with binary forms of degree $d = 2m-2$, that is, forms of the kind

$$F = \sum_{i=0}^{d} a_{(d-i,i)} x_0^i x_1^{d-i}.$$

Reciprocal varieties of these spaces are Grassmannians of lines. This is proven in [42, Proposition 7.2] using Bézout matrices. We give here an alternative proof of the same result:

Theorem 1.2.1. *The reciprocal variety of $\mathbb{P}\mathrm{Cat}(m-1,2)$ is a Grassmannian $G(2, m+1)$ in its Plücker embedding. In particular, its degree is $\frac{1}{m}\binom{2m-2}{m-1}$.*

Proof. The linear space $\mathrm{Cat}(m-1,2)$ consists of all the $m \times m$ catalecticant matrices associated with binary forms of degree $d = 2(m-1)$. By Lemma 1.1.7,

$I_{m-1}(\mathrm{Cat}(m-1,2)) = I_{m-1}(\mathrm{Cat}(m-2,m;2))$ so the reciprocal variety of $\mathrm{Cat}(m-1,2)$ can be equivalently defined as the image closure of

$$\mathbb{P}\mathrm{Cat}(m-2,m;2) \dashrightarrow (\mathbb{PS}^m)^\vee, \qquad A \mapsto \wedge^{m-1}A.$$

This defines a d-dimensional irreducible subvariety of the Grassmannian $G(m-1,m+1)$ in its Plücker embedding, which is also irreducible of dimension d, so they coincide. The statement follows from identifying $G(m-1,m+1) \simeq G(2,m+1)$ by duality. The degree formula is classical and due to Schubert [48]. ■

Example 1.2.2 (Binary sextics). When $m=4$, we have the catalecticant space $\mathrm{Cat}(3,2)$ associated with binary forms of degree 6. Matrices in this space have parametric form

$$\begin{bmatrix} a_{(6,0)} & a_{(5,1)} & a_{(4,2)} & a_{(3,3)} \\ a_{(5,1)} & a_{(4,2)} & a_{(3,3)} & a_{(2,4)} \\ a_{(4,2)} & a_{(3,3)} & a_{(2,4)} & a_{(1,5)} \\ a_{(3,3)} & a_{(2,4)} & a_{(1,5)} & a_{(0,6)} \end{bmatrix}$$

The reciprocal variety is the image closure of

$$\mathbb{P}\mathrm{Cat}(3,2) \dashrightarrow (\mathbb{PS}^4)^\vee, \qquad A \mapsto \wedge^3(A),$$

but since $I_3(\mathrm{Cat}(3,2)) = I_3(\mathrm{Cat}(2,4;2))$, we may equivalently parametrize it with the 3-minors of:

$$\begin{bmatrix} a_{(6,0)} & a_{(5,1)} & a_{(4,2)} & a_{(3,3)} & a_{(2,4)} \\ a_{(5,1)} & a_{(4,2)} & a_{(3,3)} & a_{(2,4)} & a_{(1,5)} \\ a_{(4,2)} & a_{(3,3)} & a_{(2,4)} & a_{(1,5)} & a_{(0,6)} \end{bmatrix},$$

which means studying the image of

$$\mathbb{P}\mathrm{Cat}(2,4;2) \dashrightarrow (\mathbb{PS}^4)^\vee, \qquad A \mapsto \wedge^3(A).$$

In particular, matrices in $\mathrm{Cat}(2,4;2)$ have size 3×5, so the above map produces a subvariety of $G(3,5)$ in its Plücker embedding. We have two irreducible varieties of dimension 6, one contained in to the other, so they must be the same. ♢

1.3 Reciprocal sets of points

In this section we collect a series of technical lemmas regarding the reciprocal sets $F_{\mathcal{L}}(A)$ of catalecticant matrices of given rank. The mantra is that most of the statements can be proved recursively by induction on the size of the matrices. This applies especially to Proposition 1.4.5, which shows that also the reciprocal set of a single matrix is a Grassmannian of lines.

For any linear subspace of symmetric matrices, one can obtain the defining equations for the reciprocal variety via saturation of certain ideals. Although these operations become soon unfeasible with symbolic computation programs such as `Macaulay2`, it is possible to use this result to verify statements on reciprocal varieties and reciprocal sets of points in small examples (binary quartics, sextics, octics).

Proposition 1.3.1. *Let $\mathbb{P}\mathcal{L} \subseteq \mathbb{P}\mathbb{S}^m$ be any linear subspace of matrices whose generic rank is m. Then*

$$\mathbb{P}\mathcal{L}^{-1} = \overline{F_{\mathbb{S}^m}(\mathbb{P}\mathcal{L}) \setminus D^{m-1}_{(\mathbb{S}^m)^*}},$$

where $F_{\mathbb{S}^m}(\mathbb{P}\mathcal{L})$ and $D^{m-1}_{(\mathbb{S}^m)^*}$ denote the $\mathbb{S}^m$-reciprocal set of $\mathbb{P}\mathcal{L}$ and the locus of degenerate matrices in $(\mathbb{P}\mathbb{S}^m)^\vee$, respectively.

Before proving the statement, we observe that $\mathbb{P}\mathcal{L}^{-1} = F_{\mathcal{L}}(\mathbb{P}\mathcal{L})$, and that $F_{\mathbb{S}^m}(\mathbb{P}\mathcal{L})$ can also be identified with the pull-back of $\mathbb{P}\mathcal{L}$ via the dual map $\mathrm{Adj}_m^\vee$. If we denote with $I = I(\mathbb{P}\mathcal{L}^{-1})$ and $J = I(F_{\mathbb{S}^m}(\mathbb{P}\mathcal{L}))$ the defining ideals of these two reciprocal sets, and with $\det_{\mathbb{S}}$ the determinantal polynomial defining $D^{m-1}_{(\mathbb{S}^m)^*}$, then Proposition 1.3.1 gives a recipe to compute the equations of $\mathbb{P}\mathcal{L}^{-1}$, via saturation:

$$I = (J : \det_{\mathbb{S}}^\infty). \tag{1.3.1}$$

Proof of Proposition 1.3.1. Clearly $F_{\mathcal{L}}(\mathbb{P}\mathcal{L}) = \overline{F_{\mathcal{L}}(\mathbb{P}\mathcal{L}) \setminus D^{m-1}_{(\mathbb{S}^m)^*}}$ and for every matrix $A \in \mathbb{P}\mathcal{L}$ of rank m, $F_{\mathcal{L}}(A) = F_{\mathbb{S}^m}(A)$. ■

The following lemma will be proved later in Chapter 2 (cfr. Lemma 2.2.2). Its importance is twofold: on the one side, it provides a procedure to obtain parametrizations for reciprocal sets. On the other side, it explains how Terracini's Lemma is involved in the study of reciprocal varieties.

Lemma 1.3.2. *For any linear subspace $\mathbb{P}\mathcal{L} \subset \mathbb{P}\mathbb{S}^m$ and every matrix $A \in \mathbb{P}\mathcal{L}$, we have*

$$F_{\mathcal{L}}(A) = \overline{\{\lim_{t\to 0} \mathrm{Adj}_m(A + tX) \mid X \in \mathbb{P}\mathcal{L},\, \det(X) \neq 0\}} \subseteq (\mathbb{P}\mathbb{S}^m)^\vee.$$

Moreover, if $L_1, \dots L_m$ are the rank loci of $\mathbb{P}\mathcal{L}$ and A is a rank-r smooth point in L_r, the above set of limits only depends on the normal space $N_A L_r$ to L_r at A. In particular, $\dim F_{\mathcal{L}}(A) \leq \dim N_A L_r - 1$.

Note that the limit notation is used a bit freely: what we mean is that, for any parametric matrix $M(t)$, its limit for $t \to 0$ is a matrix whose entries are obtained by dividing the entries of $M(t)$ by their greatest common factor, and then setting $t = 0$.

It is a classical fact that, when $\mathbb{P}\mathcal{L}$ is the space $\mathbb{P}\mathrm{Cat}(m-1, 2) \subseteq \mathbb{P}\mathbb{S}^m$, namely the space of square catalecticant matrices of binary forms of degree $d = 2m - 2$, the locus of matrices of rank at most r is

$$\mathbb{P}\mathrm{Cat}(m-1, 2) \cap D^r_{\mathbb{S}^m} = \sigma_r(\nu_d(\mathbb{P}^1)),$$

namely the r-th secant variety to the d-th Veronese embedding of $\mathbb{P}^1$.

There are other instances of spaces of catalecticant matrices whose r-minors define r-secant varieties, such as quadratic forms – where catalecticant matrices are simply symmetric matrices and $D^r_{\mathbb{S}^m} = \sigma_r(\nu_2(\mathbb{P}^{m-1}))$ – and ternary forms of degree four (treated in Chapter 2). See [38] for an account of the known cases.

In the binary case, we also know that a matrix is smooth in $\sigma_r(\nu_d(\mathbb{P}^1))$ if and only if it has rank equal to r (see [34, Theorem 1.45]). Therefore, for any catalecticant matrix $A \in \mathbb{P}\mathrm{Cat}(m-1,2)$ of rank equal to r, the normal space to the rank-r locus is well-defined, so the bound of Lemma 1.3.2 for the dimension of $F_{\mathcal{L}}(A)$ applies to any such matrix A. In fact, this bound is an equality, as we will see in Proposition 1.4.5.

Lemma 1.3.3. *Let $\mathbb{P}\mathcal{L} = \mathbb{P}\mathrm{Cat}(m-1,2)$ be the space of catalecticant matrices associated with binary forms of degree $d = 2m-2$. For any two distinct general matrices $A_1, A_2 \in \mathbb{P}\mathcal{L}$ of rank equal to $r < m$, the following are equivalent:*

(1) $F_{\mathcal{L}}(A_1) = F_{\mathcal{L}}(A_2)$;

(2) *A_1 and A_2 belong to the same r-secant space to $\nu_d(\mathbb{P}^1)$;*

(3) *There exists a matrix $B \in F_{\mathcal{L}}(A_1) \cap F_{\mathcal{L}}(A_2)$ of rank $m-r$.*

Proof. The implication (1) $\Rightarrow$ (3) is trivial.

Implication (2) $\Rightarrow$ (1): If A_1 and A_2 belong to the same r-secant space, then by Terracini's Lemma they have the same tangent space (hence the same normal space) to $C_r := \sigma_r(\nu_d(\mathbb{P}^1))$. Moreover, by Lemma 1.3.2, the reciprocal sets $F_{\mathcal{L}}(A_1)$ and $F_{\mathcal{L}}(A_2)$ depend only on the normal spaces to C_r at A_1 and A_2, respectively.

Implication (3) $\Rightarrow$ (2): As previously observed in Remark 1.1.18, a matrix $B \in F_{\mathcal{L}}(A_1) \cap F_{\mathcal{L}}(A_2)$ corresponds to a hyperplane $H_B \subseteq \mathbb{P}\mathbb{S}^m$ tangent to $D^r_{\mathbb{S}^m}$ both at A_1 and A_2. In particular, H_B contains both the tangent spaces $T_{A_1}C_r$ and $T_{A_2}C_r$. Moreover, since B has rank exactly $m-r$, then H_B cannot be tangent to any $D^{r'}_{\mathbb{S}^m}$ for $r' > r$.

If A_1 and A_2 belong to two different r-secant spaces, then $T_{A_1}C_r \neq T_{A_2}C_r$ (see also Remark 1.3.4 below). This implies that, also for the tangent spaces of rank-r symmetric loci, we have $T_{A_1}D^r_{\mathbb{S}^m} \neq T_{A_2}D^r_{\mathbb{S}^m}$, otherwise they would both intersect $\nu_d(\mathbb{P}^1)$ in the same r-tuple of points.

Therefore, by Terracini's Lemma there exists a point $P \in \nu_2(\mathbb{P}^{m-1}) = D^1_{\mathbb{S}^m}$ such that $T_P D^1_{\mathbb{S}^m} \nsubseteq T_{A_2}D^r_{\mathbb{S}^m}$. Then, the linear span $\langle T_P D^1_{\mathbb{S}^m}, T_{A_2}D^r_{\mathbb{S}^m}\rangle$ is a tangent space to $D^{r+1}_{\mathbb{S}^m}$ at a point of rank $r+1$, a contradiction. ∎

Remark 1.3.4. The last part of the proof of Lemma 1.3.3 uses the fact that, in the binary case, general symmetric tensors of subgeneric rank are *identifiable*, namely they can be written essentially in a unique way as sum of powers of linear forms. Therefore, by Terracini's Lemma, the corresponding tangent spaces to the r-secant varieties must be uniquely determined by the r points (in general position) on the rational normal curve. For more information on identifiablity of general symmetric tensors of given rank, see [19].

For catalecticants of binary forms, the reciprocal set of every degenerate matrix is contained in that of a rank-1 matrix:

Lemma 1.3.5. *Let A be any rank-r matrix in $\mathbb{P}\mathcal{L} = \mathbb{P}\mathrm{Cat}(m-1, 2)$, let $d = 2m-2$, and let S_r be any r-secant space to $\nu_d(\mathbb{P}^1)$ containing A, possibly spanned by r points that are not in general position. Let $P_1, \ldots, P_s$, $s \leq r$ be the points in the support of $S_r \cap \nu_d(\mathbb{P}^1)$. Then $F_\mathcal{L}(A) \subseteq F_\mathcal{L}(P_i)$, for every $i = 1, \ldots, s$.*

Proof. Let us denote for short $C_r := \sigma_r(\nu_d(\mathbb{P}^1))$. The matrix A is a smooth point of C_r and the points $P_1, \ldots, P_s$ are on the curve C_1. For every $i = 1, \ldots, s$, we have inclusions of tangent spaces $T_A C_r \supset T_{P_i} C_1$, hence inclusion of normal spaces $N_A C_r \subset N_{P_i} C_1$. Therefore, the statement follows from Lemma 1.3.2. ■

Corollary 1.3.6. *Let $\mathbb{P}\mathcal{L}$ be as above and let $C_r = \sigma_r(\nu_d(\mathbb{P}^1))$ be the r-secant varietiey of $\nu_d(\mathbb{P}^1)$. Then, for every $r = 2, \ldots, m-1$, we have:*

$$F_\mathcal{L}(C_r) \subset F_\mathcal{L}(C_{r-1}).$$

Before continuing with a more precise analysis of reciprocal sets of points of Hankel matrices, we observe a useful property that holds for reciprocal points with respect to any LSSM.

Lemma 1.3.7. *For any linear subspace $\mathbb{P}\mathcal{L} \subseteq \mathbb{P}\mathbb{S}^m$ and for any rank-r matrix $A \in \mathbb{P}\mathcal{L}$ the reciprocal set $F_\mathcal{L}(A)$ is projectively equivalent to the reciprocal variety of a linear subspace of $\mathbb{P}\mathbb{S}^{m-r}$.*

Proof. Let $A \in \mathbb{P}\mathcal{L}$ be a rank-r matrix. Recall that $F_\mathcal{L}(A)$ is the Zariski closure of a set of limits, in the sense of Lemma 1.3.2. Equivalently, we can compute limits of their translates. More specifically, let $H \in \mathrm{GL}(m)$ be such that $H^{-1}AH = T$, where T a triangular matrix whose only non-zero entries are in the top left $r \times r$ block. Let X be any matrix in $\mathbb{P}\mathcal{L}$ of maximal rank m and write $X' = H^{-1}XH$. We claim that

$$M := \lim_{t \to 0} \mathrm{Adj}_m[H^{-1}(A + tX)H] = \lim_{t \to 0} \mathrm{Adj}_m[T + tX'].$$

is an $m \times m$ matrix whose first r rows and columns are set to zero, while the remaining entries are the cofactors of the $(m-r) \times (m-r)$ submatrix of X' obtained by erasing the first r rows and columns. In particular, these cofactors are linear combination of $(m-r-1)$-minors of the original X.

Indeed, $\mathrm{Adj}_m[T + tX']$ is a matrix with polynomial entries in t and the limit operation consists of dividing these entries by the least common power of t and then setting $t = 0$. With computations analogous to the ones in the proof of the forthcoming Lemma 1.3.8, one sees that such least common power is t^{m-r} and that the entries in the first r rows and columns are all divisible by t^{m-r+1}.

Taking the translate with respect to H commutes both with the limit and the adjugate map, so HMH^{-1} is a symmetric matrix of rank $m-r$ whose non-zero entries are linear combinations of $(m-r-1)$-minors of X. ■

Lemma 1.3.8. *Let $\mathbb{P}\mathcal{L} = \mathbb{P}\mathrm{Cat}(m-1, 2)$, let $d = 2m-2$, and let A be a rank-r degenerate matrix in $\mathbb{P}\mathcal{L}$ corresponding to a point lying on an r-osculating space to $\nu_d(\mathbb{P}^1)$. Then the reciprocal set $F_\mathcal{L}(A)$ is projectively equivalent to the reciprocal* *variety of a Hankel space of $(m-r)\times(m-r)$ matrices. In particular, $F_{\mathcal{L}}(A)$ is projectively equivalent to a Grassmannian $G(2,m-r+1)$ and $\langle F_{\mathcal{L}}(A)\rangle = F_{\mathbb{S}^m}(A)$.*

Proof. We prove that the parametrization is the same. By Lemma 1.3.2, we know a parametrization for $F_{\mathcal{L}}(A)$. This is obtained in three steps: first, we compute $\mathrm{Adj}_m(A+tX)$, where X is the generic catalecticant matrix. Then, since we are working projectively, we divide by the greatest power of t dividing all the entries of $\mathrm{Adj}_m(A+tX)$ and finally we set $t=0$.

By hypothesis, the point A belongs to the most degenerate orbit of rank-r matrices in $\mathbb{P}\mathrm{Cat}(m-1,2)$, that is, the orbit of r-tuple of points supported at one single point of $\nu_d(\mathbb{P}^1)$. Without loss of generality, we may assume that the point in the support is $P=\nu_d(1:0)$.

By Remark 1.1.9, the secants that we are considering are r-osculating planes to the rational normal curve. Points in the r-th osculating space to P correspond to matrices of the form

$$A=\begin{bmatrix} s_1 & \cdots & s_{r-1} & s_r & 0 & \cdots & 0\\ s_2 & \cdots & s_r & 0 & \cdots & \cdots & 0\\ \vdots & \iddots & \iddots & & & \iddots & \vdots\\ s_r & 0 & & & \iddots & & \vdots\\ 0 & \vdots & & \iddots & & & \vdots\\ \vdots & \vdots & \iddots & & & & \vdots\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

We focus on the case $r=2$, the other cases can be treated similarly. Describing $\mathrm{Adj}_m(A+tX)$ requires to compute the cofactors of the following matrix:

$$A+tX=\begin{bmatrix} s_1+ta_{(2m-2,0)} & s_2+ta_{(2m-3,1)} & ta_{(2m-4,2)} & \cdots & ta_{(m-1,m-1)}\\ s_2+ta_{(2m-3,1)} & ta_{(2m-4,2)} & ta_{(2m-5,3)} & \cdots & ta_{(m-2,m)}\\ ta_{(2m-4,2)} & ta_{(2m-5,3)} & ta_{(2m-6,4)} & \cdots & ta_{(m-3,m+1)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ ta_{(m-1,m-1)} & ta_{(m-2,m)} & ta_{(m-3,m+1)} & \cdots & ta_{(0,2m-2)} \end{bmatrix}.$$

There are three kinds of cofactors: first, the $(1,1)$-cofactor is a bihomogenous polynomial in t and $a_{(2m-2-i,i)}$, of bidegree $(m-1,m-1)$:

$$\mathrm{Cof}_{(1,1)}=\begin{vmatrix} ta_{(2m-4,2)} & ta_{(2m-5,3)} & \cdots & ta_{(m-2,m)}\\ ta_{(2m-5,3)} & ta_{(2m-6,4)} & \cdots & ta_{(m-3,m+1)}\\ \vdots & \vdots & \ddots & \vdots\\ ta_{(m-2,m)} & ta_{(m-3,m+1)} & \cdots & ta_{(0,2m-2)} \end{vmatrix}.$$

Second, we have (i,j)-cofactors (equivalently, (j,i)-cofactors), with $i>1, j\leq 2$. They are sum of two bihomogeneous polynomials of bidegree $(m-1,m-1)$

and $(m-2, m-2)$ respectively. For example, the $(1,2)$-cofactor is

$$\mathrm{Cof}_{(1,2)} = -\begin{vmatrix} s_2 + ta_{(2m-3,1)} & ta_{(2m-5,3)} & \cdots & ta_{(m-2,m)} \\ ta_{(2m-4,2)} & ta_{(2m-6,4)} & \cdots & ta_{(m-3,m+1)} \\ \vdots & \vdots & \ddots & \vdots \\ ta_{(m-1,m-1)} & ta_{(m-3,m+1)} & \cdots & ta_{(0,2m-2)} \end{vmatrix}$$

and the claim on the bihomogenous terms follows immediately by expanding à la Laplace.

Similarly, for all the other cofactors we get a sum of 3 polynomials of bidegree $(m-1, m-1)$, $(m-2, m-2)$ and $(m-3, m-3)$ respectively. For example, the $(3,3)$-cofactor is

$$\mathrm{Cof}_{(3,3)} = \begin{vmatrix} s_1 + ta_{(2m-2,0)} & s_2 + ta_{(2m-3,1)} & ta_{(2m-5,3)} & \cdots & ta_{(m-1,m-1)} \\ s_2 + ta_{(2m-3,1)} & ta_{(2m-4,2)} & ta_{(2m-6,4)} & \cdots & ta_{(m-2,m)} \\ ta_{(2m-5,3)} & ta_{(2m-6,4)} & ta_{(2m-7,5)} & \cdots & ta_{(m-3,m+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ta_{(m-1,m-1)} & ta_{(m-2,m)} & ta_{(m-4,m+2)} & \cdots & ta_{(0,2m-2)} \end{vmatrix}. \tag{1.3.2}$$

In particular, the bihomogenous polynomial of bidegree $(m-3, m-3)$ is the sub-maximal minor of (1.3.2) obtained by erasing the first two rows and the first two columns.

Altogether, the greatest power of t dividing each cofactor of $A + tX$ is t^{m-3}. After dividing by this power of t and setting $t = 0$, only the $(m-3, m-3)$-bihomogeous terms survive, so the resulting parametrizing matrix is

$$\begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \mathrm{Cof}_{(3,3)} & \mathrm{Cof}_{(3,4)} & \cdots & \mathrm{Cof}_{(3,m)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \mathrm{Cof}_{(m,3)} & \mathrm{Cof}_{(m,4)} & \cdots & \mathrm{Cof}_{(m,m)} \end{bmatrix}, \tag{1.3.3}$$

which identifies with the parametrization of the image of $(m-2) \times (m-2)$ Hankel matrices of type

$$\begin{bmatrix} a_{(2m-6,4)} & a_{(2m-7,5)} & \cdots & a_{(m-3,m+1)} \\ a_{(2m-7,5)} & a_{(2m-8,6)} & \cdots & a_{(m-3,m+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(m-3,m+1)} & a_{(m-4,m+2)} & \cdots & a_{(0,2m-2)} \end{bmatrix},$$

therefore $F_{\mathcal{L}}(A)$ is a Grassmannian $G(2, m-r+1)$.

Finally, the inclusion $\langle F_{\mathcal{L}}(A) \rangle \subseteq F_{\mathbb{S}^m}(A)$ is projectively equivalent to the linear span of $G(2, m-r+1)$, whose dimension coincides with that of $F_{\mathbb{S}^m}(A)$. ∎

Remark 1.3.9. From Lemma 1.3.8 we observe that: when $\mathrm{rk}(A) = m-1$, then $F_{\mathcal{L}}(A) = G(2,2)$, that is a point; when $\mathrm{rk}(A) = m-2$, then $F_{\mathcal{L}}(A) = G(2,3)$, that is a $\mathbb{P}^2$.

Example 1.3.10 (binary quartics)**.** Let $\mathbb{PL} = \mathbb{P}\mathrm{Cat}(2,2)$. The matrices in this catalecticant space have parametric form

$$\begin{bmatrix} a_{(4,0)} & a_{(3,1)} & a_{(2,2)} \\ a_{(3,1)} & a_{(2,2)} & a_{(1,3)} \\ a_{(2,2)} & a_{(1,3)} & a_{(0,4)} \end{bmatrix},$$

while for the space of 3×3 symmetric matrices $\mathbb{PS}^3$ and its dual $(\mathbb{PS}^3)^\vee$ we choose coordinates

$$\begin{bmatrix} y_{(1,1)} & y_{(1,2)} & y_{(1,3)} \\ y_{(1,2)} & y_{(2,2)} & y_{(2,3)} \\ y_{(1,3)} & y_{(2,3)} & y_{(3,3)} \end{bmatrix} \qquad \begin{bmatrix} y^*_{(1,1)} & y^*_{(1,2)} & y^*_{(1,3)} \\ y^*_{(1,2)} & y^*_{(2,2)} & y^*_{(2,3)} \\ y^*_{(1,3)} & y^*_{(2,3)} & y^*_{(3,3)} \end{bmatrix},$$

so that $\mathbb{PL}$ is a hyperplane in $\mathbb{PS}^3$ cut out by $y_{(2,2)} - y_{(1,3)} = 0$. We consider the adjugate map and its inverse

$$\mathrm{Adj}_3 \colon \mathbb{PS}^3 \dashrightarrow (\mathbb{PS}^3)^\vee \qquad \mathrm{Adj}_3^\vee \colon (\mathbb{PS}^3)^\vee \dashrightarrow \mathbb{PS}^3.$$

By Proposition 1.3.1, the defining equations of $\mathbb{PL}^{-1}$ are given by saturating the ideal J of the pull-back of $\mathbb{PL}$ via $\mathrm{Adj}_3^\vee$, with the determinant polynomial of symmetric matrices in $(\mathbb{PS}^3)^\vee$.

The ideal J is monomial, where the homogeneous generator is obtained by setting equality between the cofactors $\mathrm{Cof}_{(1,3)}$ and $\mathrm{Cof}_{(2,2)}$ of the parametric matrix for $(\mathbb{PS}^3)^\vee$, that is:

$$y^*_{(1,3)}(y^*_{(1,3)} - y^*_{(2,2)}) + y^*_{(1,2)}y^*_{(2,3)} - y^*_{(1,1)}y^*_{(3,3)}, \tag{1.3.4}$$

and it is possible to see that it is already saturated with respect to the determinant. In fact, the equation in (1.3.4) is a quadric defining a $G(2,4) \subset \mathbb{P}^5 = (\mathbb{PS}^3)^\vee$. More precisely, that is the quadratic Pfaffian of the following 4×4 skew-symmetric matrix:

$$\begin{bmatrix} 0 & y^*_{(1,1)} & y^*_{(1,2)} & y^*_{(1,3)} \\ -y^*_{(1,1)} & 0 & y^*_{(2,2)} - y^*_{(1,3)} & y^*_{(2,3)} \\ -y^*_{(1,2)} & y^*_{(1,3)} - y^*_{(2,2)} & 0 & y^*_{(3,3)} \\ -y^*_{(1,3)} & -y^*_{(2,3)} & -y^*_{(3,3)} & 0 \end{bmatrix}.$$

The adjugate map is well-defined on matrices of rank 2 and 3, while for a rank-1 matrix A we may consider the set $F_{\mathcal{L}}(A)$. For instance, when A is the matrix corresponding to the form x_0^4, then the corresponding $F_{\mathcal{L}}(A)$ has parametric form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & y^*_{(2,2)} & y^*_{(2,3)} \\ 0 & y^*_{(2,3)} & y^*_{(3,3)} \end{bmatrix},$$

which can be identified with a $\mathbb{P}^2$ of 2×2 symmetric matrices by forgetting the first row and column. Indeed, recalling Lemma 1.3.8, we need to compute the reciprocal variety of $\mathbb{P}\mathrm{Cat}(1,2)$, which in this case already coincides with $\mathbb{PS}^3$. ♢

Example 1.3.11 (Binary sextics)**.** The matrices in the space $\mathbb{P}\mathrm{Cat}(3,2)$ have parametric form

$$\begin{bmatrix} a_{(6,0)} & a_{(5,1)} & a_{(4,2)} & a_{(3,3)} \\ a_{(5,1)} & a_{(4,2)} & a_{(3,3)} & a_{(2,4)} \\ a_{(4,2)} & a_{(3,3)} & a_{(2,4)} & a_{(1,5)} \\ a_{(3,3)} & a_{(2,4)} & a_{(1,5)} & a_{(0,6)} \end{bmatrix},$$

while for the space of 4×4 symmetric matrices $\mathbb{P}\mathbb{S}^4$ and its dual $(\mathbb{P}\mathbb{S}^4)^\vee$ we choose coordinates

$$\begin{bmatrix} y_{(1,1)} & y_{(1,2)} & y_{(1,3)} & y_{(1,4)} \\ y_{(1,2)} & y_{(2,2)} & y_{(2,3)} & y_{(2,4)} \\ y_{(1,3)} & y_{(2,3)} & y_{(3,3)} & y_{(3,4)} \\ y_{(1,4)} & y_{(2,4)} & y_{(3,4)} & y_{(4,4)} \end{bmatrix} \qquad \begin{bmatrix} y^*_{(1,1)} & y^*_{(1,2)} & y^*_{(1,3)} & y^*_{(1,4)} \\ y^*_{(1,2)} & y^*_{(2,2)} & y^*_{(2,3)} & y^*_{(2,4)} \\ y^*_{(1,3)} & y^*_{(2,3)} & y^*_{(3,3)} & y^*_{(3,4)} \\ y^*_{(1,4)} & y^*_{(2,4)} & y^*_{(3,4)} & y^*_{(4,4)} \end{bmatrix}.$$

The catalecticant space has codimension 3 in $\mathbb{P}\mathbb{S}^4$ and is defined by the equations

$$y_{(2,2)} - y_{(1,3)} = 0, \quad y_{(2,3)} - y_{(1,4)} = 0, \quad y_{(3,3)} - y_{(2,4)} = 0 \tag{1.3.5}$$

The ideal J of the pull-back is generated by the three relations obtained by setting equality between the cofactors of the parametric matrix for $(\mathbb{P}\mathbb{S}^4)^\vee$, corresponding to the equations in (1.3.5):

$$\mathrm{Cof}_{(2,2)} - \mathrm{Cof}_{(1,3)}, \qquad \mathrm{Cof}_{(2,3)} - \mathrm{Cof}_{(1,4)}, \qquad \mathrm{Cof}_{(3,3)} - \mathrm{Cof}_{(2,4)}.$$

After saturating with the determinant polynomial, we get the equations for the reciprocal variety $\mathbb{P}\mathrm{Cat}(3,2)^{-1}$, and it is easily verified that they are the quadric Pfaffians of the following 5×5 skew-symmetric matrix:

$$\begin{bmatrix} 0 & y^*_{(1,1)} & y^*_{(1,2)} & y^*_{(1,3)} & y^*_{(1,4)} \\ -y^*_{(1,1)} & 0 & y^*_{(2,2)} - y^*_{(1,3)} & y^*_{(2,3)} - y^*_{(1,4)} & y^*_{(2,4)} \\ -y^*_{(1,2)} & y^*_{(1,3)} - y^*_{(2,2)} & 0 & y^*_{(3,3)} - y^*_{(2,4)} & y^*_{(3,4)} \\ -y^*_{(1,3)} & y^*_{(1,4)} - y^*_{(2,3)} & -y^*_{(3,3)} & 0 & y^*_{(4,4)} \\ -y^*_{(1,4)} & -y^*_{(2,4)} & -y^*_{(3,4)} & -y^*_{(4,4)} & 0 \end{bmatrix}.$$

In particular, they define a Grassmannian $G(2,5) \subset (\mathbb{P}\mathbb{S}^4)^\vee$.

The adjugate map is well-defined on matrices of rank 4 and 3. Moreover, by Lemma 1.3.8 we know that for a rank-2 matrix A lying on a tangent line, the corresponding $F_{\mathcal{L}}(A)$ is a $\mathbb{P}^2$ and, similarly, when $\mathrm{rk}(A) = 1$ then $F_{\mathcal{L}}(A)$ is a $G(2,4)$.

As we will see later (cfr. Proposition 1.4.5), also for a rank-2 matrix A on a proper secant line, we have that $F_{\mathcal{L}}(A)$ is a $\mathbb{P}^2$. For instance, when A is the catalecticant associated to the rank-2 form $x_0^6 + x_1^6$, then the elements of $F_{\mathcal{L}}(A)$ are of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & y^*_{(2,2)} & y^*_{(2,3)} & 0 \\ 0 & y^*_{(2,3)} & y^*_{(3,3)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If A is associated with the rank-1 form x_0^6, then Lemma 1.3.8 implies that $F_{\mathcal{L}}(A)$ is computed by the adjugate map of 3×3 matrices, restricted to a sub-catalecticant space of $\mathbb{P}\mathrm{Cat}(3,2)$ obtained by forgetting the first row and column:

$$\left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & a_{(4,2)} & a_{(3,3)} & a_{(2,4)} \\ 0 & a_{(3,3)} & a_{(2,4)} & a_{(1,5)} \\ 0 & a_{(2,4)} & a_{(1,5)} & a_{(0,6)} \end{array}\right] \overset{\mathrm{Adj}_3}{\dashrightarrow} \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & y^*_{(2,2)} & y^*_{(2,3)} & y^*_{(2,4)} \\ 0 & y^*_{(2,3)} & y^*_{(3,3)} & y^*_{(3,4)} \\ 0 & y^*_{(2,4)} & y^*_{(3,4)} & y^*_{(4,4)} \end{array}\right].$$

We get the reciprocal variety of a $\mathbb{P}\mathrm{Cat}(2,2)$, which is a $G(2,4)$ in the space of 3×3 symmetric matrices obtained from $(\mathbb{P}\mathbb{S}^4)^\vee$ by again forgetting the first row and column. In other words, $F_{\mathcal{L}}(A)$ is defined by the four linear equations $y_{(1,1)} = \cdots = y_{(1,4)} = 0$ plus the quadric is the Pfaffian of

$$\begin{bmatrix} 0 & y^*_{(2,2)} & y^*_{(2,3)} & y^*_{(2,4)} \\ -y^*_{(2,2)} & 0 & y^*_{(3,3)} - y^*_{(2,4)} & y^*_{(3,4)} \\ -y^*_{(2,3)} & y^*_{(2,4)} - y^*_{(3,3)} & 0 & y^*_{(4,4)} \\ -y^*_{(2,4)} & -y^*_{(3,4)} & -y^*_{(4,4)} & 0 \end{bmatrix}.$$

$\diamondsuit$

Motivated by the previous examples, we would like to formalize how to find skew-symmetric matrices whose quadratic Pfaffians define the reciprocal varieties. We use coordinates $y_{(i,j)}$ for the spaces of symmetric matrices $\mathbb{P}\mathbb{S}^m$ and $y^*_{(i,j)}$ for its dual $(\mathbb{P}\mathbb{S}^m)^\vee$.

Remark 1.3.12. Examples 1.3.10 and 1.3.11 suggest a general procedure to find skew-symmetric matrices whose quadratic Pfaffians define the reciprocal varietiy of a given Hankel space.

More precisely, with usual choice of coordinates, when $\mathbb{P}\mathcal{L} = \mathbb{P}\mathrm{Cat}(m-1,2)$, the reciprocal variety $\mathbb{P}\mathcal{L}^{-1}$ is cut out by the quadratic Pfaffians of the $(m+1) \times (m+1)$ skew-symmetric matrix $(S_{ij})_{i,j=1,\ldots,m+1}$ defined by:

$$S_{ij} = \begin{cases} y^*_{(1,j-1)} & i = 1 \\ y^*_{(i,m)} & j = m+1 \\ y^*_{(i,j-1)} - y^*_{(i-1,j)} & i < j < m+1 \\ 0 & i = j. \end{cases}$$

This is in fact a rephrasing of the proof of [42, Proposition 7.2], where a linear change of coordinates is used to express Plücker coordinates in terms of the entries of symmetric Bézoutians $\Sigma = (\Sigma_{ij})_{i,j=1,\ldots,m}$, where

$$\Sigma_{ij} = S_{i,j+1}, \quad \text{for } j \geq i = 1, \ldots, m,$$

extended by symmetry for $j < i$.

1.4 Rank loci in the reciprocal variety

From now on, we will denote by $\mathbb{P}\mathcal{L}$ the catalecticant space $\mathbb{P}\mathrm{Cat}(m-1,2)$ associated with binary forms of degree $d = 2m-2$, while for $r = 1,\dots,m$, we will denote by C_r the locus $D^r_{\mathbb{S}^m} \cap \mathbb{P}\mathcal{L}$, that is, the secant variety $\sigma_r(\nu_d(\mathbb{P}^1))$.

Recalling that matrices in $\mathbb{P}\mathcal{L}$ have constant skew-diagonals, it is easy to see that the orthogonal space $\mathbb{P}\mathcal{L}^\perp$ is cut out by linear equations $l_1,\dots,l_{2m-1}$, where l_k is the sum of all the entries in the k-th skew diagonal of the generic matrix $(y^*_{(i,j)})_{i,j=1,\dots m}$ parametrizing $(\mathbb{P}\mathbb{S}^m)^\vee$. Explicitly:

$$l_k = \begin{cases} 2\sum_{i=1}^{k/2} y^*_{(i,k-i+1)} & \text{if } k \text{ is even} \\ y^*_{(\frac{k+1}{2},\frac{k+1}{2})} + 2\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} y^*_{(i,k-i+1)} & \text{if } k \text{ is odd.} \end{cases} \tag{1.4.1}$$

As a consequence of this fact, we see that the intersection between $\mathbb{P}\mathcal{L}^\perp$ and the linear spans of reciprocal sets of points is recursively orthogonal to a smaller catalecticant space:

Lemma 1.4.1. *Let $A \in \mathbb{P}\mathcal{L}$ be a rank-r degenerate matrix corresponding to a point lying on an r-osculating space to $\nu_d(\mathbb{P}^1)$ and consider the linear isomorphism $F_\mathcal{L}(A) \xrightarrow{\sim} (\mathbb{P}\mathcal{L}')^{-1}$, where $\mathbb{P}\mathcal{L}'$ is the subspace $\mathbb{P}\mathrm{Cat}(m-r-1,2) \subseteq \mathbb{P}\mathbb{S}^{m-r}$. Then, the intersection $\langle F_\mathcal{L}(A)\rangle \cap \mathbb{P}\mathcal{L}^\perp$ is sent to $(\mathbb{P}\mathcal{L}')^\perp$, which is a linear space of dimension $\binom{m-r+1}{2} - 2(m-r-1) - 2$.*

Proof. Without loss of generality, we may assume that A corresponds to $\nu_d(1:0)$. By Lemma 1.3.8, $F_\mathcal{L}(A)$ is projectively equivalent to the reciprocal variety of a $\mathbb{P}\mathcal{L}' = \mathbb{P}\mathrm{Cat}(m-r-1,2)$ and its linear span $\langle F_\mathcal{L}(A)\rangle = F_{\mathbb{S}^m}(A)$ has dimension $\binom{m-r+1}{2} - 1$ and it is defined by $y^*_{(i,j)} = 0$ for $i \le r$ (cfr. Remark 1.1.18).

The linear system of these latter equations, together with the ones in (1.4.1) simplifies to the system of linear equations $l'_1,\dots,l'_{2(m-r)-1}$ defining $(\mathbb{P}\mathcal{L}')^\perp$. ■

Using the explicit description of the defining equations for $\mathbb{P}\mathcal{L}^{-1}$ and $\mathbb{P}\mathcal{L}^\perp$, it is possible to prove that their intersection is empty. In [42, Proposition 7.4], this is done by studying linear system of parameters modulo Stanley-Reisner ideals. Here we give an alternative proof of this fact, based on an inductive argument.

First, we prove that the orthogonal space does not intersect reciprocal sets of points lying on hyperosculating spaces.

Lemma 1.4.2. *Let $A \in \mathbb{P}\mathcal{L}$ be a matrix of rank $r < m$ corresponding to a point lying on an r-osculating space to $\nu_d(\mathbb{P}^1)$. Then we have $F_\mathcal{L}(A) \cap \mathbb{P}\mathcal{L}^\perp = \emptyset$.*

Proof. We prove the statement by induction on m. The base case is $m = 3$, binary quartics: here $\mathbb{P}\mathcal{L}^\perp$ is a point and it easily seen that its coordinates do not satisfy the defining equations of $F_\mathcal{L}(A)$ (cfr. Example 1.3.10).

By induction, let us assume that the statement holds for every $m' < m$, that is, for every catalecticant space $\mathbb{P}\mathrm{Cat}(m-r,2)$, with $r = 2,\dots,m-2$.

Let now $\mathbb{P}\mathcal{L} = \mathbb{P}\mathrm{Cat}(m-1,2)$ and assume by contradiction that there exists a matrix $B \in F_\mathcal{L}(A) \cap \mathbb{P}\mathcal{L}^\perp$, of rank at most $m-r$. By Lemma 1.3.8, the reciprocal set $F_{\mathcal{L}}(A)$ is projectively equivalent to $(\mathbb{P}\mathcal{L}')^{-1}$, with $\mathcal{L}' = \mathbb{P}\mathrm{Cat}(m-r-1,2)$. Then,

$$B \in F_{\mathcal{L}}(A) \cap \mathbb{P}\mathcal{L}^{\perp} \subseteq F_{\mathbb{S}^m}(A) \cap \mathbb{P}\mathcal{L}^{\perp} \simeq (\mathbb{P}\mathcal{L}')^{\perp} \subseteq (\mathbb{P}\mathbb{S}^{m-r})^{\vee},$$

where the latter linear isomorphism is given by Lemma 1.4.1.

Therefore, B can be identified with a $(m-r) \times (m-r)$ symmetric matrix $B' \in (\mathbb{P}\mathcal{L}')^{-1} \cap (\mathbb{P}\mathcal{L}')^{\perp}$. Recalling Remark 1.1.23, the rank of B' (hence that of B) must be strictly smaller than $m-r$.

But then B' belongs to the reciprocal set $F_{\mathcal{L}'}(A')$ of a degenerate matrix of $\mathbb{P}\mathcal{L}'$. By Lemma 1.3.5, we may assume that $\mathrm{rk}(A') = 1$. We have found a matrix

$$B' \in F_{\mathcal{L}'}(A') \cap (\mathbb{P}\mathcal{L}')^{\perp},$$

with A' a point on the curve $\nu_{2(m-r)}(\mathbb{P}^1)$ (trivially, on its 1-osculating space), contradicting the induction hypothesis. ∎

Proposition 1.4.3. *We have $\mathbb{P}\mathcal{L}^{-1} \cap \mathbb{P}\mathcal{L}^{\perp} = \emptyset$. In particular, the restriction of $\pi_{\mathcal{L}} \colon (\mathbb{P}\mathbb{S}^m)^{\vee} \to (\mathbb{P}\mathcal{L})^{\vee}$ to $\mathbb{P}\mathcal{L}^{-1}$ is a regular surjective map of degree equal to $\deg G(2, m+1)$.*

Proof. We prove the statement fiberwise, namely that for any matrix $A \in \mathbb{P}\mathcal{L}$ of rank-r, we have $F_{\mathcal{L}}(A) \cap \mathbb{P}\mathcal{L}^{\perp} = \emptyset$. The statement is trivial when $r = m$, by Remark 1.1.23.

Let now $r < m$, and let S_r be any r-secant space containing A. Then S_r corresponds to a point of a certain orbit in the space of r-uples of points on $\mathbb{P}^1$, under the action of $\mathrm{PGL}(2)$. The closure of any orbit contains the closed orbit of the r-fold point, see [2, Proposition 2.1]. Points in this closed orbit correspond to r-secant spaces that are r-osculating spaces to the rational normal curve.

Now, having intersection with $\mathbb{P}\mathcal{L}^{\perp}$ is a closed condition. Therefore, if we had $F_{\mathcal{L}}(A) \cap \mathbb{P}\mathcal{L}^{\perp} \neq \emptyset$, then the same would hold for rank-r matrices on r-osculating spaces, which Lemma 1.4.2 does not allow. ∎

Corollary 1.4.4. *The projection map $\pi_{\mathcal{L}}$, restricted to $F_{\mathcal{L}}(C_r)$, is a finite-to-one surjection to $C_r^{\vee} \subseteq \mathbb{P}\mathrm{Cat}(m-1,2)^{\vee}$.*

Proof. Recall, for any $A \in \mathbb{P}\mathcal{L} \subseteq \mathbb{P}\mathbb{S}^m$, we have $F_{\mathcal{L}}(A) \subseteq F_{\mathbb{S}^m}(A)$. By Remark 1.1.18, points in $F_{\mathbb{S}^m}(A)$ are hyperplanes of $\mathbb{P}\mathbb{S}^m$ tangent to $D^r_{\mathbb{S}^m}$ at A, so it is clear that $\pi_{\mathcal{L}}$ sends these points to hyperplanes of $\mathbb{P}\mathcal{L}$ tangent to C_r at A, in symbols: $\pi_{\mathcal{L}}(F_{\mathcal{L}}(C_r)) \subseteq C_r^{\vee}$.

Now, by Proposition 1.4.3, the reciprocal variety $\mathbb{P}\mathcal{L}^{-1}$ is mapped surjectively to $\mathbb{P}\mathcal{L}^{\vee}$. Moreover, for a general $P \in C_r^{\vee}$, the points in the fiber $\pi^{-1}_{\mathcal{L}|_{F_{\mathcal{L}}(C_r)}}(P)$ have rank $m-r$, hence by Lemma 1.3.3 they are contained in a unique $F_{\mathcal{L}}(A)$, for some rank-r matrix $A \in \mathbb{P}\mathcal{L}$. Therefore, also $F_{\mathcal{L}}(C_r)$ is mapped surjectively to $C_r^{\vee}$. ∎

We can now describe reciprocal sets of arbitrary rank-r degenerate matrices $A \in \mathbb{P}\mathcal{L} = \mathbb{P}\mathrm{Cat}(m-1,2)$, which, as we show below, behave as those lying on hyperosculating spaces. Explicitly, we have that $F_{\mathcal{L}}(A)$ is projectively equivalent to the reciprocal variety of a Hankel space of $(m-r)\times(m-r)$ matrices, hence $F_{\mathcal{L}}(A)$ is projectively equivalent to a Grassmannian $G(2, m-r+1)$ and $\langle F_{\mathcal{L}}(A)\rangle = F_{\mathbb{S}^m}(A)$.

Proposition 1.4.5. *Lemma 1.3.8 holds for every rank-r matrix $A \in \mathbb{P}\mathcal{L}$.*

Proof. First, we prove that, for any matrix $A \in \mathbb{P}\mathcal{L}$ of rank r, the dimension of its reciprocal set $F_{\mathcal{L}}(A)$ is the same as that of a matrix on an osculating space, namely $\dim G(2, m-r+1) = 2(m-r-1)$. This certainly holds if A is *general*. Indeed, by Corollary 1.4.4 the dimension of $F_{\mathcal{L}}(A)$ is equal to the dimension of the space of hyperplanes of $\mathbb{P}\mathcal{L}$ that are tangent to C_r at A, namely:

$$\dim F_{\mathcal{L}}(A) = \dim \mathbb{P}\mathcal{L} - (2r-1) - 1 = 2(m-r-1).$$

Note that this is also the dimension of the normal space $\dim N_A C_r$. For a special matrix A of rank r, the dimension of $F_{\mathcal{L}}(A)$ could only be bigger, compared to the general one, which the bound in Lemma 1.3.2 does not allow.

Second, the reciprocal sets $F_{\mathcal{L}}(A)$ define a fibration over the projective space $\mathbb{P}^r \simeq \mathbb{P}(S^r(\mathbb{C}^2))$ of r-tuples of points on $\mathbb{P}^1$. In this fibration, all fibers have the same dimension. Moreover, the fiber over r-fold points are Grassmannians, which are rigid. Therefore, the fibration is flat, hence all the reciprocal sets of rank-r matrices are projectively equivalent to $G(2, m-r+1)$. ∎

For rational normal curves of degree d, the projective dual to the secant variety can be described as a coincident root locus. More precisely, let $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\sum_{i=1}^n \lambda_i = d$ and let $m_i, \ldots, m_k$ count the multiplicities for each occurring integer, namely $m_j := |\{i : \lambda_i = j\}|$.

The *coincident root locus* $\Delta_\lambda \subset \mathbb{P}^d = \mathbb{P}(\mathbb{C}[x_0, x_1]_d)$ is the set of binary forms F of degree d which admit a factorization

$$F = \prod_{i=1}^n L_i^{\lambda_i},$$

for some linear forms L_i.

A degree formula for the coincident root loci of binary forms is classical and due to Hilbert [31]. Duality for Δ_λ has been studied in [39]. In particular:

$$\deg(\Delta_\lambda) = \frac{n!}{m_1! m_2! \cdots m_k!} \lambda_1 \lambda_2 \cdots \lambda_n. \tag{1.4.2}$$

Lemma 1.4.6. *The variety $C_r^\vee \subseteq \mathbb{P}\mathcal{L}^\vee$ is the coincident root locus $\Delta_{(1^{2m-2r-2}, 2^r)}$. In particular, it has degree*

$$\deg C_r^\vee = 2^r \binom{2m-r-2}{r}.$$

Proof. By [39, Proposition 3.1], we have equality $\sigma_r(\nu_d(\mathbb{P}^1)) = (\Delta_\lambda)^\vee$, with $\lambda = (1^{d-2r}, 2^r)$. Then the statement follows by applying the Biduality Theorem (see [26, Theorem 1.1]) and the degree formula (1.4.2). ■

A more explicit proof is obtained by observing the relation between duality and apolarity. Let $S = \mathbb{C}[x_0, x_1], S^* = \mathbb{C}[\partial_0, \partial_1]$, so that $\mathbb{P}\mathcal{L} = \mathbb{P}(S_d)$ and $\mathbb{P}\mathcal{L}^\vee = \mathbb{P}(S_d^*)$, where $d = 2m - 2$.

Alternative proof. Let $A \in \mathbb{P}\mathcal{L}$ be a rank-r matrix associated with the binary form $L_1^d + \cdots + L_r^d$, where $L_i \in S_1$.

For every $i = 1, \ldots, r$, the apolar ideal of L_i is generated by two operators, of degree 1 and $d+1$ respectively. Let D_i be the degree-1 apolar operator and let us complete to bases $\{L_i, L_i'\}$ and $\{D_i, D_i'\}$ for S_1 and S_1^* respectively.

Then the tangent space $T_{L_1} C_r \subset \mathbb{P}\mathcal{L}$ parametrizes forms of type $L_i^d + L_i^{d-1} L_i'$. In particular, $(D_i')^2$ is apolar to all such forms. Altogether, for every operator G in the ideal $I = (D_1' \cdots D_r')^2$ and for every F parametrized by $T_A C_r$, we have $G \circ F = 0$.

Equivalently, the dual space $T_A C_r^\vee \subseteq \mathbb{P}\mathcal{L}^\vee$ parametrizes all homogeneous binary operators in $\mathbb{P}(I_d^2)$, namely operators of degree d vanishing at the r roots of D_i with multiplicity 2. As A varies in C_r, we obtain all the degree-d dual operators with roots of multiplicity $(1^{d-2r}, 2^r)$. ■

Lemma 1.4.7. *The dual variety $C_r^\vee$ is a $\mathbb{P}^{s-1}$-scroll over $\mathbb{P}^r$, where s is the fiber dimension of the normal bundle of $\sigma_r(\nu_d(\mathbb{P}^1))$ in $\mathbb{P}^d = \mathbb{P}\mathrm{Cat}(m-1, 2)$. The minimal desingularization of this scroll is a projective bundle $\mathbb{P}\mathcal{E}$, with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^r}(2)^{\oplus s}$.*

Proof. Recalling Lemma 1.4.6 above, we can write

$$C_r^\vee = \{f \mid f = gh^2, \deg(g) = d - 2r, \deg(h) = r\}.$$

This is a scroll over the $\mathbb{P}^r$ of degree-r forms in $\mathbb{P}^1$. For the resolution, we need to find a free bundle $\mathbb{P}\mathcal{E}$ projecting to $C_r^\vee$ and such that there exist global sections $\sigma_i \colon \mathbb{P}^r \to \mathbb{P}\mathcal{E}$, $i = 1, \ldots, s$ inducing independent maps $\mathbb{P}^r \to C_r^\vee$ by composition

$$\begin{tikzcd} & \mathbb{P}\mathcal{E} \arrow[dl, "\pi"'] & \\ C_r^\vee & & \mathbb{P}^r \arrow[ul, "\sigma_i"'] \arrow[ll, "\pi\circ\sigma_i"] \end{tikzcd}.$$

Every form g of degree $d - 2r$ determines a map $\mathbb{P}^r \to C_r^\vee$, defined by $h \mapsto gh^2$ and factoring through $\mathcal{O}_{\mathbb{P}^r}(2)$. Its image is a Veronese variety

$$V_g := \{f \mid f = gh^2\} \simeq \nu_2(\mathbb{P}^r).$$

We need to show that, if $g_1 \neq g_2$ are two generic forms of degree $d - 2r$, then $V_{g_1} \cap V_{g_2} = \emptyset$. The claim in the lemma will follow since the dimension of the linear space of degree-$(d-2r)$ forms is $d - 2r + 1 = s$.

Let us assume by contradiction that there exists an element $\eta \in V_{g_1} \cap V_{g_2}$. Then:

$$\eta = g_1 h_1^2 = g_2 h_2^2, \tag{1.4.3}$$

for some $h_1, h_2 \in \mathbb{P}^r$. Without loss of generality, we may assume that $(g_1, g_2) = 1$, hence $g_2 | h_1$ and $g_1 | h_2$ and we can write

$$\eta = g_1^2 g_2^2 h', \tag{1.4.4}$$

for some h'. Comparing (1.4.3) and (1.4.4), we get

$$g_1 g_2^2 h' = h_1^2, \qquad g_1^2 g_2 h' = h_2^2,$$

which are squares. In particular, $g_1 | h'$ and $g_2 | h'$, so $\eta = g_1^3 g_2^3 h''$, for some h''. Proceeding this way, we can continue until we obtain

$$\eta = g_1^{d/2} g_2^{d/2} = g_1 (g_1^{d/2-1} g_2^{d/2}) = g_2 (g_1^{d/2} g_2^{d/2-1}).$$

But $g_1^{d/2-1} g_2^{d/2}$ and $g_1^{d/2} g_2^{d/2-1}$ are not squares, contradicting (1.4.3). ■

For the following theorem we will work with the projection map $\pi_{\mathcal{L}}$. We have seen in Lemma 1.4.1 that $\langle F_{\mathcal{L}}(A) \rangle \cap \mathbb{P}\mathcal{L}^{\perp} = (\mathbb{P}\mathcal{L}')^{\perp}$ for a smaller catalecticant space $\mathbb{P}\mathcal{L}'$ for which the reciprocal variety is exactly $F_{\mathcal{L}}(A)$. This means that we have equality

$$\pi_{\mathcal{L}|_{F_{\mathcal{L}}(A)}} = \pi_{\mathcal{L}'|_{F_{\mathcal{L}}(A)}}, \tag{1.4.5}$$

or, equivalently, the map $\pi_{\mathcal{L}}$ restricted to $F_{\mathcal{L}}(A)$ is identified with a projection from a smaller projection center $(\mathbb{P}\mathcal{L}')^{\perp}$. In particular, it is again a finite-to-one map.

Theorem 1.4.8. *Let $\mathbb{P}\mathcal{L}$ be the catalecticant space $\mathbb{P}\mathrm{Cat}(m-1, 2) \subseteq \mathbb{P}\mathbb{S}^m$, with rank loci $C_r = D^r_{\mathbb{S}^m} \cap \mathbb{P}\mathcal{L}$, for $r = 1, \ldots, m$. When $r < m$, the reciprocal set $F_{\mathcal{L}}(C_r)$ is the image of the incidence variety*

$$\{(A, B) \in C_r \times (\mathbb{P}\mathbb{S}^m)^{\vee} \mid B \in F_{\mathcal{L}}(A)\} \twoheadrightarrow F_{\mathcal{L}}(C_r) \subseteq (\mathbb{P}\mathbb{S}^m)^{\vee}$$

via its projection to the second factor, where all the fibers are projectively equivalent to a Grassmannian $G(2, m-r+1)$ and are distinct if and only if they come from points on distinct r-secant spaces. In particular, $F_{\mathcal{L}}(C_r)$ is a subvariety of $D^{m-r}_{(\mathbb{S}^m)^*}$ of dimension $2m-r-2$ and degree

$$\deg F_{\mathcal{L}}(C_r) = \frac{2^r}{m-r} \binom{2m-r-2}{r} \binom{2m-2r-2}{m-r-1}. \tag{1.4.6}$$

Proof. First, by Lemma 1.3.2 and Terracini's Lemma, the operator $F_{\mathcal{L}}(-)$ is constant on secant spaces to the rational normal curve C_1. For every r-secant space S we may take a rank-r matrix A_S representing it, so that we can write $F_{\mathcal{L}}(C_r)$ as the union of the $F_{\mathcal{L}}(A_S)$, where every A_S uniquely depends on the choice of r points (with multiplicity) on C_1. From Proposition 1.4.5, each $F_{\mathcal{L}}(A_S)$

is a $G(2, m-r+1)$, so the first part of the theorem is proved. The dimension count is justified by Lemma 1.3.3, for which a point in $F_{\mathcal{L}}(C_r)$ generically belongs to only one $F_{\mathcal{L}}(A_S)$.

Recall from Corollary 1.4.4 that $\pi_{\mathcal{L}}$, restricted to $F_{\mathcal{L}}(C_r)$, is a finite-to-one map. Then we define

$$\delta := \#(\pi^{-1}_{\mathcal{L}|_{F_{\mathcal{L}}(C_r)}}(P)) = \#(\pi^{-1}_{\mathcal{L}|_{F_{\mathcal{L}}(A)}}(P)),$$

which is finite as observed after equation (1.4.5).

Recalling Lemma 1.4.7, the projection $\pi_{\mathcal{L}}$ maps $F_{\mathcal{L}}(C_r)$ surjectively onto $C_r^\vee$, whose degree is given in Lemma 1.4.6. Hence we have:

$$\deg F_{\mathcal{L}}(C_r) = \delta \cdot \deg(C_r^\vee) = \delta \cdot 2^r \binom{2m-r-2}{r}.$$

Finally, $F_{\mathcal{L}}(A)$ is mapped $\delta : 1$ to a linear space of the same dimension (see Lemma 1.4.1), so

$$\delta = \deg F_{\mathcal{L}}(A) = \deg G(2, m-r-1) = \frac{1}{m-r}\binom{2m-2r-2}{m-r-1},$$

which proves (1.4.6). ■

Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^r}(2)^{\oplus s}$ be as in Lemma 1.4.7 and consider the projecion $\mathbb{P}\mathcal{E} \xrightarrow{\pi} \mathbb{P}^r$. When r and s are small, the degree of $\mathbb{P}\mathcal{E}$ (hence the degree of $C_r^\vee$ and $F_{\mathcal{L}}(C_r)$) is easily computed by hand, even without assuming Lemma 1.4.6.

If $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \in \mathrm{CH}(\mathbb{P}\mathcal{E})$ denotes the first Chern class, we have

$$\deg(\mathbb{P}\mathcal{E}) = \int_{\mathbb{P}\mathcal{E}} \zeta^{s+r-1}. \tag{1.4.7}$$

This can be expressed in terms of the classes $c_i := c_i(\mathcal{E}) \in \mathrm{CH}(\mathbb{P}^r)$. Indeed, considering the tautological sequence of bundles

$$0 \to S \to \pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \to 0$$

and applying Whitney's formula we obtain

$$\zeta^s = c_1\zeta^{s-1} - c_2\zeta^{s-2} + \cdots - (-1)^r c_r \zeta^{s-r} = 0.$$

With further algebraic manipulations, it is possible to write ζ^{s+r-1} as a product $\zeta^{s-1} f(c_i)$, for some polynomial f. Substituting into (1.4.7), we get

$$\deg(\mathbb{P}\mathcal{E}) = \int_{\mathbb{P}^r} f(c_1, \ldots, c_r). \tag{1.4.8}$$

Example 1.4.9 (binary quartics)**.** Let $\mathbb{P}\mathcal{L} = \mathbb{P}\mathrm{Cat}(2,2) \subset \mathbb{P}\mathbb{S}^3$. Then:

- $F_{\mathcal{L}}(C_1)$ is a $\mathbb{P}^2$-scroll of degree 6 over $\mathbb{P}^1$, and it has empty intersection with the point $\mathbb{P}\mathcal{L}^{\perp}$. Therefore, $\pi_{\mathcal{L}|_{F_{\mathcal{L}}(C_1)}}$ is a $1:1$ map onto its image, and $\deg(C_1^{\vee}) = \deg(F_{\mathcal{L}}(C_1))$. A free resolution for $C_1^{\vee}$ is given by $\mathbb{P}\mathcal{E}$, with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}$, so its degree can be equivalently computed as

$$\deg(\mathbb{P}\mathcal{E}) = \int_{\mathbb{P}\mathcal{E}} \zeta^2 c_1 = \int_{\mathbb{P}^1} c_1 = 3\int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(2)) = 6.$$

  In the notation of (1.4.8), we have $f(c_1) = c_1$.

- $F_{\mathcal{L}}(C_2)$ is the double embedding of $\mathbb{P}^2$. This is $\mathbb{P}\mathcal{E}$, with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2)$. Here $f(c_1) = c_1^2$ and we have $\deg(F_{\mathcal{L}}(C_2)) = \deg(C_2^{\vee}) = \deg(\mathbb{P}\mathcal{E}) = 4$.

$\diamondsuit$

Example 1.4.10 (binary sextics)**.** Let $\mathbb{P}\mathcal{L} = \mathbb{P}\mathrm{Cat}(3,2) \subset \mathbb{P}\mathbb{S}^4$. Then:

- $F_{\mathcal{L}}(C_1)$ is a variety of degree 20, union of $G(2,4)$ over $\mathbb{P}^1$. Each $G(2,4)$ intersects the plane $\mathbb{P}\mathcal{L}^{\perp}$ in a point, therefore $\pi_{|F_{\mathcal{L}}(C_1)}$ is a $2:1$ map onto its image, so $\deg(C_1^{\vee}) = \frac{\deg(F_{\mathcal{L}}(C_1))}{2} = 10$. A free resolution for $C_1^{\vee}$ is given by $\mathbb{P}\mathcal{E}$, with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 5}$, so its degree can be equivalently computed as

$$\deg(\mathbb{P}\mathcal{E}) = \int_{\mathbb{P}\mathcal{E}} \zeta^4 c_1 = \int_{\mathbb{P}^1} c_1 = 5\int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(2)) = 10.$$

  As in the previous example, we have $f(c_1) = c_1$.

- $F_{\mathcal{L}}(C_2)$ is a $\mathbb{P}^2$-scroll of degree 24 on $\mathbb{P}^2$, and it has empty intersection with the orthogonal plane. Therefore, $\deg(F_{\mathcal{L}}(C_2)) = \deg(C_2^{\vee})$. The associated bundle is $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2)^{\oplus 3}$ and

$$\deg(\mathbb{P}\mathcal{E}) = \int_{\mathbb{P}\mathcal{E}} \zeta^2(c_1^2 - c_2) = \int_{\mathbb{P}^2} 9c_1^2(\mathcal{O}_{\mathbb{P}^2}(2)) - c_1^3(\mathcal{O}_{\mathbb{P}^2}(2)) = 24.$$

  Here we have $f(c_1, c_2) = c_1^2 - c_2$.

- $F_{\mathcal{L}}(C_3)$ is the double embedding of $\mathbb{P}^3$. This is $\mathbb{P}\mathcal{E}$, with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(2)$. Here $f(c_1) = c_1^3$ and we have $\deg(F_{\mathcal{L}}(C_3)) = \deg(C_3^{\vee}) = \deg(\mathbb{P}\mathcal{E}) = 8$.

$\diamondsuit$

Chapter 2

Inverting catalecticants of ternary quartics

The present chapter[1] is devoted to study the reciprocal variety to the linear subspace of symmetric matrices (LSSM) of square catalecticants

$$\mathrm{Cat}(2,3) = \left\{ \begin{bmatrix} a_{(4,0,0)} & a_{(3,1,0)} & a_{(3,0,1)} & a_{(2,2,0)} & a_{(2,1,1)} & a_{(2,0,2)} \\ a_{(3,1,0)} & a_{(2,2,0)} & a_{(2,1,1)} & a_{(1,3,0)} & a_{(1,2,1)} & a_{(1,1,2)} \\ a_{(3,0,1)} & a_{(2,1,1)} & a_{(2,0,2)} & a_{(1,2,1)} & a_{(1,1,2)} & a_{(1,0,3)} \\ a_{(2,2,0)} & a_{(1,3,0)} & a_{(1,2,1)} & a_{(0,4,0)} & a_{(0,3,1)} & a_{(0,2,2)} \\ a_{(2,1,1)} & a_{(1,2,1)} & a_{(1,1,2)} & a_{(0,3,1)} & a_{(0,2,2)} & a_{(0,1,3)} \\ a_{(2,0,2)} & a_{(1,1,2)} & a_{(1,0,3)} & a_{(0,2,2)} & a_{(0,1,3)} & a_{(0,0,4)} \end{bmatrix} : a_{(i,j,k)} \in \mathbb{C} \right\} \tag{2.0.1}$$

associated with ternary quartics

$$\begin{aligned} a_{(4,0,0)}x^4 + a_{(3,1,0)}x^3y + a_{(3,0,1)}x^3z + a_{(2,2,0)}x^2y^2 + a_{(2,1,1)}x^2yz + a_{(2,0,2)}x^2z^2 \\ + a_{(1,3,0)}xy^3 + a_{(1,2,1)}xy^2z + a_{(1,1,2)}xyz^2 + a_{(1,0,3)}xz^3 + a_{(0,4,0)}y^4 \\ + a_{(0,3,1)}y^3z + a_{(0,2,2)}y^2z^2 + a_{(0,1,3)}yz^3 + a_{(0,0,4)}z^4. \end{aligned}$$

In Section 2.1, we explain how to use numerical tools to obtain the degree of the reciprocal variety of $\mathbb{P}\mathrm{Cat}(2,3)$, as well as the ML-degree of the linear concentration model represented by it. These two numbers are computed to be 85 and 36, respectively (Theorem 2.1.3). The fact that these two numbers do not coincide already marks a difference with the case of binary forms.

On the other hand, some similarities are highligthed in Section 2.2, where we analyze reciprocal sets of points and show that their defining equations are Pfaffians (Theorem 2.2.7).

In Section 2.3, we prove that only the rank-1 locus of the catalecticant space, namely the Veronese surface $\nu_4(\mathbb{P}^2)$, contributes to the degree of the reciprocal variety (Theorem 2.3.4). Moreover, we provide a geometric explanation of why the two invariants of Theorem 2.1.3 are different. We do this by studying the intersection between the orthogonal space $\mathbb{P}\mathrm{Cat}(2,3)^\perp$ and the rank loci of the reciprocal variety (Proposition 2.3.8).

Finally, in Section 2.4 we show that the reciprocal variety is singular (Theorem 2.4.1).

Throughout this chapter, we shorten the notation for the catalecticant space $\mathrm{Cat}(2,3)$, using the symbol $\mathcal{C}$ instead.

[1]The content of this chapter is an adaptation and extension of the paper Homs, R., Cazzador, E., and Brustenga i Moncusí, L. "Inverting catalecticants of ternary quartics". In: *Le Matematiche (Catania)* vol. 76, no. 2 (2021), pp. 517–533.

2.1 A numerical approach

In this section, we present the computation of some invariants by means of numerical methods which will give some insight in the forthcoming sections: the degree of $\mathbb{P}\mathcal{C}^{-1}$, the number of linearly independent equations for the reciprocal variety in low degrees and the ML-degree of the linear concentration model represented by $\mathcal{C}$. The results are summarized in Theorem 2.1.3.

2.1.1 Degree and number of equations

Recall, Proposition 1.3.1 shows a natural approach to obtain the equations for reciprocal varieties, based on the fact that $\mathrm{Adj}_m \circ \mathrm{Adj}_m^\vee = \mathrm{Id}_{\mathbb{P}\mathbb{S}^m}$, as rational maps.

Namely, if we denote with $I = I(\mathbb{P}\mathcal{C}^{-1})$ and $J = I(F_{\mathbb{S}^6}(\mathbb{P}\mathcal{C}))$ and with $\det_{\mathbb{S}}$ the determinantal polynomial cutting the variety $D^5_{(\mathbb{S}^6)^*}$ of degenerate 6×6 symmetric matrices, we have

$$I = (J : \det_{\mathbb{S}}^\infty). \tag{2.1.1}$$

Computing this saturation turns out to be computationally infeasible for ternary quartics. Nevertheless, equation (2.1.1) allows us to use the ideal J to compute the degree of $\mathbb{P}\mathcal{C}^{-1}$ by means of numerical methods. Indeed, a general linear space $L \subseteq (\mathbb{P}\mathbb{S}^6)^\vee$ of codimension 14, intersects $F_{\mathbb{S}^6}(\mathbb{P}\mathcal{C})$ in points of full rank, hence the number of such points is equal to $\deg(\mathbb{P}\mathcal{C}^{-1})$.

We use the monodromy method, implemented in `HomotopyContinuation.jl`, see [12], setting as parameters the coefficients of L and we compute

$$\deg(\mathbb{P}\mathcal{C}^{-1}) = 85.$$

Codes for this computation may be found in Appendix A.2.3.

An invariant we may numerically estimate is, for a fixed degree d, the number $H(d)$ of linearly independent forms of degree d in I. For example, for $d = 3$, the space Ω of forms in $\dim((\mathbb{S}^6)^*) = 21$ variables of degree 3 has dimension $\binom{3+21-1}{3} = 1771$. Given a point $B \in (\mathbb{P}\mathbb{S}^6)^\vee$, the set of forms $F \in \Omega$ for which $F(B) = 0$ gives rise to a hyperplane $H_B \subseteq \Omega$ defined by the linear form with coefficients these 1771 monomials evaluated at B. Thus, for a generic set of points $\{B_1, \ldots, B_{1771}\}$, we have

$$\bigcap_{i=1}^{1771} H_{B_i} = \{\mathrm{pt}\}.$$

Instead, for a generic set of points $\{P_1, \ldots, P_{1771}\}$ of $\mathbb{P}\mathcal{C}^{-1}$, if there are forms of degree 3 in I, the hyperplanes H_{P_i} will exhibit some linear dependence, coming from the fact that there would be a linear space of positive dimension lying in the intersection of all of them, hence

$$H(3) = \dim\left(\bigcap_{i=1}^{1771} H_{P_i}\right).$$

Therefore, we may compute $H(3)$ as the rank of the 1771×1771 matrix of coefficients of the linear forms defining H_{P_i}.

Using `Julia`, we sampled points in $\mathbb{P}\mathcal{C}^{-1}$ as inverses of randomly generated catalecticant matrices, and we obtained $H(d) = 0, 27, 510$ for $d = 2, 3, 4$, see Appendix A.2.4.

2.1.2 Maximum likelihood degree

In Chapter 1, Section 1.1 we defined the ML-degree of an LSSM $\mathcal{L}$ as the degree of a suitable projection map. An alternative interpretation of it is inspired by statistics. Let $\mathbb{S}^m_{>0}$ be the cone of real positive definite $m \times m$ matrices. When $\mathcal{L}$ is defined by real linear equations, the intersection $\mathcal{L}^{-1} \cap \mathbb{S}^m_{>0}$ is a centered Gaussian statistical model. In other words, it encodes a set of multivariate normal distributions $\mathcal{N}(0, \Sigma)$ where the covariance matrix $\Sigma \subset \mathbb{S}^m_{>0}$ is defined by linear constrains on the entries of its inverse $K = \Sigma^{-1}$. This type of models are known as *linear concentration models*, see [51].

Definition 2.1.1. The *maximum likelihood degree* (ML-degree) of a linear concentration model is defined as the number of complex solutions to the critical equations of the *log-likelihood function*

$$\ell(K) = \log \det K - \operatorname{trace}(SK)\,, \tag{2.1.2}$$

where S is the sample covariance matrix of sample data vectors, see [22, Definition 2.1.4] for more details and [42, Definition 5.4] for an equivalent algebro-geometric definition.

Note that (2.1.2) depends on the random data encoded in the sample covariance matrix S. Nevertheless, this notion of ML-degree is well-defined because the number of complex solutions to the critical equations is preserved for general data vectors, see [5, Remark 2.1].

The ML-degree of the linear concentration model represented by an LSSM $\mathcal{L}$ gives a lower bound for the degree of the reciprocal variety $\mathcal{L}^{-1}$ and equality is reached if and only if $\mathcal{L}^{-1} \cap \mathcal{L}^{\perp} = \emptyset$. The two invariants coincide in the case of both generic LSSMs and spaces $\operatorname{Cat}(k, 2)$ of catalecticant matrices associated to binary forms, see [51, Theorem 2.3] and Proposition 1.4.3, respectively.

Example 2.1.2. To compute the ML-degree of the model $\operatorname{Cat}(2, 2) \cap \mathbb{S}^3_{>0}$ we replace K and S in the log-likelihood function (2.1.2) by the matrix parametrizing the catalecticant space of binary quartics and the sample covariance matrix $S = \frac{1}{3}XX^t$, where X is a 3×3 matrix whose columns are random data vectors. Its critical equations can be computed using the `Macaulay2` [27] code below

```
I=ideal{jacobian(matrix{{det K}})-(det K)*jacobian(matrix{{trace(S*K)}})}
J=saturate(I,det K)
```

and indeed the degree of the zero-dimensional ideal J coincides with the degree of $\mathbb{P}\operatorname{Cat}(2, 2)^{-1} = G(2, 4)$ (which is 2). $\diamondsuit$

The symbolic computation of the critical points of (2.1.2) in the case of catalecticants of ternary quartics has already a too high computational cost in `Macaulay2`. Performing it with `Magma` [9], we obtain[2] that the ML-degree of the model represented by $\mathrm{Cat}(2,3)$ is 36.

Note that, for a sample covariance matrix S associated to general data vectors, all complex solutions to (2.1.2) are different, i.e. they have multiplicity one, see [5, Remark 2.1]. Therefore, a numerical approach is possible and it is already offered by the `Julia` package `LinearCovarianceModels.jl` [55]. Note that our desired value corresponds to the dual ML-degree of $\mathcal{L}$ defined in [50].

The code used in `Julia` to compute ML-degrees of LSSMs of catalecticant matrices can be found in Appendix A.2.1.

We summarize all the numerical results as follows:

Theorem 2.1.3. *For the space* $\mathbb{P}\mathrm{Cat}(2,3)$ *of catalecticant matrices associated with ternary quartics, the reciprocal variety has degree* 85*, whereas the ML-degree of the linear concentration model represented by it is* 36*. Moreover, there is a a* 27*-dimensional linear space of cubic generators in the defining ideal of the reciprocal variety.*

Remark 2.1.4. The numerical approach for ternary forms of higher degree is not feasible for short-time computations. At the level of ternary quartics, the degree (and ML-degree) of a generic 15-dimensional LSSM of 6×6 matrices is 1016, see [40] and [49] for the general theory and the implementation of the algorithm to compute the degree of the reciprocal variety of a generic LSSM. For ternary sextics, the degree of a generic 28-dimensional LSSM of 10×10 matrices is 17.429.229.428. The ML-degree of $\mathrm{Cat}(3,3)$ is already lower bounded by 180.000.

2.2 Reciprocal sets of points

For every $r = 1, \dots, 6$, let us denote with C_r the locus of matrices of rank at most r in $\mathbb{P}\mathcal{C}$. As for binary forms, the locus C_r has a well-known geometric structure: it the r-th secant variety of the Veronese surface $\nu_4(\mathbb{P}^2)$ (see [47, Theorem 2.3]).

The dimensions of each C_r are known by Alexander-Hirshowitz Theorem (see [1] for the original reference and [11] for a modern proof) and they equal to $2, 5, 8, 11, 13$ for $r = 1, 2, 3, 4, 5$, respectively. This already indicates that the catalecticant space is far from being general: for a generic 14-dimensional linear subspace of $\mathbb{P}\mathbb{S}^6$, its rank loci would have dimension -1 (the emptyset), $4, 8, 11$ and 13, respectively.

The fact that r-th secant varieties of Veronese embeddings coincide with rank-r loci of catalecticant spaces does not always hold (cfr. [15] and [38]). These determinantal varieties may strictly contain secant varieties, and may even be of bigger dimension (see also [18, Proposition 4.1]). Therefore, this is a partial obstacle to the generalization of our results to any catalecticant space.

[2] These computations took an average of 5 days and were performed with Magma V2.25-5 on a Dual Core Intel(R) Xeon(R) (2.20 GHz).

Recalling Chapter 1, we defined reciprocal sets of points and, more generally, of closed subvarieties $X \subset \mathbb{P}\mathcal{C}$:

$$F_{\mathcal{C}}(A) = \pi_2(\pi_1^{-1}(A) \cap \Gamma_{\mathcal{C}}) \qquad\qquad F_{\mathcal{C}}(X) = \overline{\bigcup_{\substack{A \in X \\ \text{generic rank}}} F_{\mathcal{C}}(A)},$$

where $\Gamma_{\mathcal{C}}$ is the graph closure of $\mathrm{Adj}_6|_{\mathbb{P}\mathcal{C}}$ in $\mathbb{P}\mathbb{S}^6 \times (\mathbb{P}\mathbb{S}^6)^\vee$ and π_1, π_2 the two projection maps from this product.

Remark 2.2.1. The reciprocal set $F_{\mathcal{C}}(X)$ can also be understood in the following way: we may regularize Adj_6 by a (finite) sequence of blow-ups along smooth centers, and consider the strict transform of X by such a sequence of blow-ups. Its image via the regularized map will coincide with $F_{\mathcal{C}}(X)$. In Section 2.3, we will see that a natural way of resolving the indeterminacy locus of Adj_6 is by blowing-up the locus of rank-1 symmetric matrices, then blowing-up the strict transform of rank-2 symmetric matrices and so on.

In Chapter 1, Lemma 1.3.2, we described reciprocal sets of points as closures of sets of limits. For points belonging to smooth loci of rank-r loci, we also gave a bound for the dimension of their reciprocal sets. We now provide the proof of that result:

Lemma 2.2.2. *For any linear subspace $\mathbb{P}\mathcal{L} \subset \mathbb{P}\mathbb{S}^m$ and every matrix $A \in \mathbb{P}\mathcal{L}$, we have*

$$F_{\mathcal{L}}(A) = \overline{\{\lim_{t\to 0} \mathrm{Adj}_m(A + tX) \mid X \in \mathbb{P}\mathcal{L},\, \det X \neq 0\}} \subseteq (\mathbb{P}\mathbb{S}^m)^\vee. \tag{2.2.1}$$

Moreover, if $L_1, \dots, L_m$ are the rank loci of $\mathbb{P}\mathcal{L}$ and A is a rank-r smooth point in L_r, the above set of limits only depends on the normal space $N_A L_r$ to L_r at A. In particular, $\dim F_{\mathcal{L}}(A) \leq \dim N_A L_r - 1$.

Proof. By continuity, the points in $F_{\mathcal{L}}(A)$ are limits of images of points approaching A along directions where Adj_m is well-defined. For any $X \in \mathbb{P}\mathcal{L}$ with $\det(X) \neq 0$, and for t small enough, the matrices in the line $\{A + tX \mid t \in \mathbb{C}\}$ are also invertible. In particular, their adjugate is well-defined, and we obtain (2.2.1).

By definition, $F_{\mathcal{L}}(A) = \pi_2(\pi_1^{-1}(A) \cap \Gamma)$. Now, recalling Remark 2.2.1, let us consider the composition of the blow-ups of $\mathbb{P}\mathbb{S}^m$ along its rank-1 locus $D^1_{\mathbb{S}^m}$ and then along the strict transform of the rank-2 locus $D^2_{\mathbb{S}^m}$, and so on until the strict transform of $D^{m-1}_{\mathbb{S}^m}$.

Let us denote by $(\widetilde{\mathbb{P}\mathbb{S}^m}, \pi)$ the blow-ups composition, and by $\widetilde{\mathrm{Adj}}_m$ the regularization of Adj_m. Moreover, for every $r = 1, \dots, m-1$, let us denote by E_r the exceptional divisor over $D^r_{\mathbb{S}^m}$. We summarize the notation in the following diagram:

$$\begin{tikzcd}
E_r \arrow[r, hook] \arrow[d, "\pi_{|E_r}"] & \widetilde{\mathbb{P}\mathbb{S}^m} \arrow[r, "\widetilde{\mathrm{Adj}}_m"] \arrow[d, "\pi"] & (\mathbb{P}\mathbb{S}^m)^\vee \arrow[d, equal] \\
D^r_{\mathbb{S}^m} \arrow[r, hook] & \mathbb{P}\mathbb{S}^m \arrow[r, dashed, "\mathrm{Adj}_m"] & (\mathbb{P}\mathbb{S}^m)^\vee
\end{tikzcd}.$$

Let $\widetilde{\mathbb{P}\mathcal{L}} \subseteq \widetilde{\mathbb{P}\mathbb{S}^m}$ be the proper transform of $\mathbb{P}\mathcal{L}$. For any rank-r point $A \in \mathbb{P}\mathcal{L}$, we have:

$$\pi_2(\pi_1^{-1}(A) \cap \Gamma) = \widetilde{\mathrm{Adj}}_m(\pi^{-1}(A) \cap \widetilde{\mathbb{P}\mathcal{L}}) = \widetilde{\mathrm{Adj}}_m(\pi^{-1}(A) \cap \widetilde{\mathbb{P}\mathcal{L}} \cap E_r).$$

The intersection $\widetilde{\mathbb{P}\mathcal{L}} \cap E_r$ is isomorphic to the projectivized normal cone of L_r in $\mathbb{P}\mathcal{L}$. When A belongs to the smooth locus of L_r, the fiber of this cone at $\pi^{-1}(A)$ is isomorphic to the projectivized normal space $\mathbb{P}(N_A L_r)$. In particular, they have the same dimension, which might drop after composing with $\widetilde{\mathrm{Adj}}_m$. ■

2.2.1 Dimension of reciprocal sets of points

The second part of Lemma 2.2.2 holds for *smooth* points in the rank-r locus. Determining which points are smooth in the rank loci is in general a hard problem. For the catalecticant space of ternary quartics, we have:

Lemma 2.2.3. *The singular locus of C_r contains C_{r-1}, and equality holds when $r \leq 3$.*

Proof. The inclusion of C_{r-1} in the singular locus of C_r is standard (and in fact it holds more generally for all r-secant varieties to Veronese varieties, see [58, Corollary 1.8]). Equality for the case $r = 3$ is [29, Theorems 2.12-2.14], while the case $r = 2$ is [36, Theorem 3.3]. ■

Note that it is currently not known whether the equality in Lemma 2.2.3 holds also for $r = 4, 5$. It certainly does not hold for quaternary quartics: indeed, the singular locus of $\sigma_3(\nu_4(\mathbb{P}^3))$ has two irreducible components, namely $\sigma_2(\nu_4(\mathbb{P}^3))$, of dimension 7, and the 8-dimensional variety of binary quartics, given by quartic surfaces consisting of 4 planes meeting in the same line (see [29, Theorem 2.1]).

Let $A \in \mathbb{P}\mathcal{C}$ be a matrix of rank r and recall that $F_{\mathcal{C}}(A) \subseteq F_{\mathbb{S}^6}(A)$. When A is a smooth point of C_r, then the bound in Lemma 2.2.2 applies, so:

$$\begin{aligned} \dim F_{\mathcal{C}}(A) &\leq \min\{\dim F_{\mathbb{S}^6}(A), \dim N_A C_r - 1\} \\ &= \min\left\{\tfrac{(6-r)(7-r)}{2} - 1, \dim N_A C_r - 1\right\}. \end{aligned} \tag{2.2.2}$$

By Lemma 2.2.3, this is always the case when $r \leq 3$. When $A \in \mathbb{P}\mathcal{C}$ has rank equal to 4, we do not know whether it is a smooth point of C_4, so we can only say

$$\dim F_{\mathcal{C}}(A) \leq \dim F_{\mathbb{S}^6}(A) = 2. \tag{2.2.3}$$

We now proceed with a more precise description of reciprocal sets of points in $\mathbb{P}\mathcal{C}$. First, we compute their dimension; then we give defining equations. The strategy applied here is slightly different from that of Chapter 1. For the binary case, we reduced to study rank-r points on most degenerate r-secant spaces. Here, we do the opposite: we study the most general r-secant spaces, namely those spanned by r points in general position.

We have two reasons for pursuing this strategy. First, as we will see, for a rank-r point on a general secant, its reciprocal set is very easy to describe, and in most cases it is simply a linear space. Second, when $r \leq 4$, we can freely move between any two rank-r points lying on general r-secant spaces. More precisely:

Remark 2.2.4. The natural action of $\mathrm{PGL}(3)$ on $\mathbb{P}^2$ induces an action on the Hilbert scheme of points $\mathrm{Hilb}_r(\mathbb{P}^2)$, which has an open orbit when $r \leq 4$, namely the orbit of r points in general position.

Moreover, the implication (2) $\Rightarrow$ (1) of Lemma 1.3.3 holds also for the catalecticant space $\mathbb{P}\mathcal{C}$ of ternary quartics, which means that:

Remark 2.2.5. When S_r is a general r-secant space to $C_1 = \nu_4(\mathbb{P}^2)$ and $A_1, A_2 \in S_r$ are two distinct rank-r matrices, we have: $F_{\mathcal{C}}(A_1) = F_{\mathcal{C}}(A_2) = F_{\mathcal{C}}(S_r)$.

Lemma 2.2.6. *Let $A \in \mathbb{P}\mathcal{C}$ be a catalecticant matrix of rank $r = 4$ (resp. $3, 2, 1$) . Then $F_{\mathcal{C}}(A)$ is projectively equivalent to the reciprocal variety of a linear subspace $\mathbb{P}\mathcal{L}_A \subset \mathbb{P}\mathbb{S}^{6-r}$ of dimension 2 (resp. $5, 8, 11$), whose generic element has rank $m - r$.*

Proof. By Lemma 1.3.7, we know that $F_{\mathcal{C}}(A)$ is projectively equivalent to some reciprocal variety of a $\mathbb{P}\mathcal{L}_A \subseteq \mathbb{P}\mathbb{S}^{6-r}$. We need to determine the dimension and the generic rank of $\mathbb{P}\mathcal{L}_A$.

The result follows as soon as it is checked on a particular point on a general r-secant space. Indeed, as noted in Remark 2.2.5, $F_{\mathcal{C}}(-)$ is constant on a secant space and Remark 2.2.4 implies that the reciprocal sets of any two general r-secant spaces are projectively equivalent. Moreover, the dimension $F_{\mathcal{C}}(A)$ can only increase at special points, which the bounds (2.2.2) and (2.2.3) do not allow.

We consider the points

$$P_1 = \nu_4(1:0:0) \quad P_2 = \nu_4(0:0:1) \quad P_3 = \nu_4(0:1:0) \quad P_4 = \nu_4(1:1:0),$$

and we denote with A_r the matrix in $\mathbb{P}\mathcal{C}$ corresponding to $\sum_{i=1}^{r} P_i$ for $r = 1, \ldots, 4$, namely:

$$A_1 = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} 2 & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}, \tag{2.2.4}$$

where dots are written instead of zero entries for the sake of readability. In particular, A_r belongs to the proper r-secant space $\langle P_1, \ldots, P_r \rangle$.

A parametrization for $F_{\mathcal{C}}(A_r)$ can be obtained as in the proof of Lemma 1.3.8. More explicitly, let us use coordinates $y_{(i,j)}$ and $y^*_{(i,j)}$ for the spaces of matrices $\mathbb{P}\mathbb{S}^6$ and $(\mathbb{P}\mathbb{S}^6)^\vee$, and coordinates $a_{(i,j,k)}$ for the catalecticant space $\mathbb{P}\mathcal{C}$, as in (2.0.1).

The reciprocal set $F_{\mathcal{C}}(A_1)$ is parametrized by the cofactors of the 5×5 matrix obtaind by erasing the first row and column of the generic catalecticant (2.0.1).

This matrix depends on 12 parameters, so we are computing the reciprocal variety of an 11-dimensional linear subspace of $\mathbb{PS}^5$:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_{(2,2,0)} & a_{(2,1,1)} & a_{(1,3,0)} & a_{(1,2,1)} & a_{(1,1,2)} \\ \cdot & a_{(2,1,1)} & a_{(2,0,2)} & a_{(1,2,1)} & a_{(1,1,2)} & a_{(1,0,3)} \\ \cdot & a_{(1,3,0)} & a_{(1,2,1)} & a_{(0,4,0)} & a_{(0,3,1)} & a_{(0,2,2)} \\ \cdot & a_{(1,2,1)} & a_{(1,1,2)} & a_{(0,3,1)} & a_{(0,2,2)} & a_{(0,1,3)} \\ \cdot & a_{(1,1,2)} & a_{(1,0,3)} & a_{(0,2,2)} & a_{(0,1,3)} & a_{(0,0,4)} \end{bmatrix} \overset{\mathrm{Adj}_5}{\dashrightarrow} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & y^*_{(2,2)} & y^*_{(2,3)} & y^*_{(2,4)} & y^*_{(2,5)} & y^*_{(1,6)} \\ \cdot & y^*_{(2,3)} & y^*_{(3,3)} & y^*_{(3,4)} & y^*_{(3,5)} & y^*_{(2,6)} \\ \cdot & y^*_{(2,4)} & y^*_{(3,4)} & y^*_{(4,4)} & y^*_{(4,5)} & y^*_{(3,6)} \\ \cdot & y^*_{(2,5)} & y^*_{(3,5)} & y^*_{(4,5)} & y^*_{(5,5)} & y^*_{(4,6)} \\ \cdot & y^*_{(2,6)} & y^*_{(3,6)} & y^*_{(4,6)} & y^*_{(5,6)} & y^*_{(6,6)} \end{bmatrix}. \tag{2.2.5}$$

Analogously, the parametrization for $F_{\mathcal{C}}(A_2)$ is given by the cofactors of the 4×4 matrix obtained by erasing the first and last row and the first and last column of the generic catalecticant. This matrix depends on 9 parameters, so we are computing the reciprocal variety of an 8-dimensional linear subspace of $\mathbb{PS}^4$:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_{(2,2,0)} & a_{(2,1,1)} & a_{(1,3,0)} & a_{(1,2,1)} & \cdot \\ \cdot & a_{(2,1,1)} & a_{(2,0,2)} & a_{(1,2,1)} & a_{(1,1,2)} & \cdot \\ \cdot & a_{(1,3,0)} & a_{(1,2,1)} & a_{(0,4,0)} & a_{(0,3,1)} & \cdot \\ \cdot & a_{(1,2,1)} & a_{(1,1,2)} & a_{(0,3,1)} & a_{(0,2,2)} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \overset{\mathrm{Adj}_4}{\dashrightarrow} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & y^*_{(2,2)} & y^*_{(2,3)} & y^*_{(2,4)} & y^*_{(2,5)} & \cdot \\ \cdot & y^*_{(2,3)} & y^*_{(3,3)} & y^*_{(3,4)} & y^*_{(3,5)} & \cdot \\ \cdot & y^*_{(2,4)} & y^*_{(3,4)} & y^*_{(4,4)} & y^*_{(4,5)} & \cdot \\ \cdot & y^*_{(2,5)} & y^*_{(3,5)} & y^*_{(4,5)} & y^*_{(5,5)} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \tag{2.2.6}$$

The linear subspace associated with $F_{\mathcal{C}}(A_3)$ is obtained by additionally erasing the third row and column

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_{(2,2,0)} & a_{(2,1,1)} & \cdot & a_{(1,2,1)} & \cdot \\ \cdot & a_{(2,1,1)} & a_{(2,0,2)} & \cdot & a_{(1,1,2)} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_{(1,2,1)} & a_{(1,1,2)} & \cdot & a_{(0,2,2)} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \overset{\mathrm{Adj}_3}{\dashrightarrow} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & y^*_{(2,2)} & y^*_{(2,3)} & \cdot & y^*_{(2,5)} & \cdot \\ \cdot & y^*_{(2,3)} & y^*_{(3,3)} & \cdot & y^*_{(3,5)} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & y^*_{(2,5)} & y^*_{(3,5)} & \cdot & y^*_{(5,5)} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

and finally for $F_{\mathcal{C}}(A_4)$ we erase also the second row and column:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_{(2,0,2)} & \cdot & a_{(1,1,2)} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_{(1,1,2)} & \cdot & a_{(0,2,2)} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \overset{\mathrm{Adj}_2}{\dashrightarrow} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & y^*_{(3,3)} & \cdot & y^*_{(3,5)} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & y^*_{(3,5)} & \cdot & y^*_{(5,5)} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

■

2.2.2 Defining equations for reciprocal sets of points

The following theorem is the analogous of Proposition 1.4.5 for catalecticants of ternary quartics. Reciprocal sets of points are not Grassmannians anymore, but they are still defined by Pfaffians:

Theorem 2.2.7. *Let $A \in \mathbb{P}\mathcal{C}$ be any catalecticant matrix of rank r.*

- *When $r \geq 3$, then $F_{\mathcal{C}}(A) = F_{\mathbb{S}^6}(A)$.*
- *When $r = 2$, then $F_{\mathcal{C}}(A)$ is a cubic hypersuface of $F_{\mathbb{S}^6}(A)$, defined by the cubic Pfaffian of a 6×6 skew-symmetric matrix.*
- *When $r = 1$, then $F_{\mathcal{C}}(A)$ is an 11-fold of degree 14 in $F_{\mathbb{S}^6}(A)$, defined by the cubic Pfaffians of a 7×7 skew-symmetric matrix.*

Proof. The statement is trivial for $r = 5, 6$.

When $r = 3, 4$ the inclusion $F_{\mathcal{C}}(A) \subseteq F_{\mathbb{S}^6}(A)$ is an equality, since by Lemma 2.2.6 the two reciprocal sets have the same dimension.

When $r = 1, 2$ the above inclusion is strict but, again by Lemma 2.2.6, we know that $F_{\mathcal{C}}(A)$ is projectively equivalent to the reciprocal variety of some $\mathbb{P}\mathcal{L}_A \subset \mathbb{P}\mathbb{S}^{6-r}$ of dimension 11 and 8 respectively. The defining equations of $\mathbb{P}\mathcal{L}_A^{-1}$ can be found using Proposition 1.3.1. Specifically, we saturate the ideal of the pull-back of $\mathbb{P}\mathcal{L}_A$ with the determinant ideal of $(\mathbb{P}\mathbb{S}^{6-r})^\vee$.

For $r = 1$ it is enough to check the statement for a particular rank-1 matrix, while for $r = 2$ we need to check it both for a point on a proper secant line and a point on a tangent line.

We now show the computations, which can be performed with `Macaulay2` using the codes in Appendix A.1.3. Let $A = A_1$ be the rank-1 matrix in (2.2.4). First, $F_{\mathbb{S}^6}(A)$ is cut out by $y^*_{(1,j)} = 0$ for $j = 1, \ldots, 6$. Then $F_{\mathcal{C}}(A) \subset F_{\mathbb{S}^6}(A)$ is the reciprocal variety of the 11-dimensional space $\mathbb{P}\mathcal{L}_A \subseteq \mathbb{P}\mathbb{S}^5$ in (2.2.5). More precisely, if we use coordinates $(y_{(i,j)})_{i,j=2,\ldots,6}$ for $\mathbb{P}\mathbb{S}^5$ and $(y^*_{(i,j)})_{i,j=2,\ldots,6}$ for $(\mathbb{P}\mathbb{S}^5)^\vee = F_{\mathbb{S}^6}(A)$, then $\mathbb{P}\mathcal{L}_A$ is defined by

$$y_{(2,6)} = y_{(3,5)}, \qquad y_{(2,5)} = y_{(3,4)}, \qquad y_{(4,6)} = y_{(5,5)},$$

and the pull-back of $\mathbb{P}\mathcal{L}_A$ via $\mathrm{Adj}_5^\vee$ is defined by the corresponding linear relation among the cofactors of the generic symmetric matrix in $(\mathbb{P}\mathbb{S}^5)^\vee$:

$$\mathrm{Cof}_{(2,6)} = \mathrm{Cof}_{(3,5)}, \quad \mathrm{Cof}_{(2,5)} = \mathrm{Cof}_{(3,5)}, \qquad \mathrm{Cof}_{(4,6)} = \mathrm{Cof}_{(5,5)},$$

where $\mathrm{Cof}_{(i,j)}$ is the cofactor relative to $y^*_{(i,j)}$. The ideal generated by these three quartic relations is not saturated with respect to the determinant polynomial of $(\mathbb{P}\mathbb{S}^5)^\vee$. After saturating, we obtain 7 cubics, which are verified to be equal to the cubics Pfaffians of the following skew-symmetric matrix:

$$S_1 = \begin{bmatrix} 0 & y^*_{(6,6)} & y^*_{(5,6)} & y^*_{(4,6)} & y^*_{(3,6)} & y^*_{(2,6)} & 0 \\ -y^*_{(6,6)} & 0 & -y^*_{(4,6)}+y^*_{(5,5)} & y^*_{(4,5)} & -y^*_{(2,6)}+y^*_{(3,5)} & y^*_{(2,5)} & -y^*_{(3,6)} \\ -y^*_{(5,6)} & y^*_{(4,6)}-y^*_{(5,5)} & 0 & y^*_{(4,4)} & -y^*_{(2,5)}+y^*_{(3,4)} & y^*_{(2,4)} & -y^*_{(3,5)} \\ -y^*_{(4,6)} & -y^*_{(4,5)} & -y^*_{(4,4)} & 0 & -y^*_{(2,4)} & 0 & -y^*_{(3,4)} \\ -y^*_{(3,6)} & y^*_{(2,6)}-y^*_{(3,5)} & y^*_{(2,5)}-y^*_{(3,4)} & y^*_{(2,4)} & 0 & y^*_{(2,2)} & -y^*_{(3,3)} \\ -y^*_{(2,6)} & -y^*_{(2,5)} & -y^*_{(2,4)} & 0 & -y^*_{(2,2)} & 0 & -y^*_{(2,3)} \\ 0 & y^*_{(3,6)} & y^*_{(3,5)} & y^*_{(3,4)} & y^*_{(3,3)} & y^*_{(2,3)} & 0 \end{bmatrix}. \qquad (2.2.7)$$

Let now $A = A_2$ be the rank-2 matrix in (2.2.4). Then $F_{\mathbb{S}^6}(A)$ is defined by $y^*_{(1,j)} = 0$ for $j = 1, \ldots, 6$ and $y^*_{(i,6)} = 0$ for $1 = 1, \ldots, 6$, and $F_{\mathcal{C}}(A)$ is the reciprocal variety of the 8-dimensional space $\mathbb{P}\mathcal{L}_A \subset \mathbb{P}\mathbb{S}^4$ in (2.2.6). If we use coordinates $(y_{(i,j)})_{i,j=2,\dots,5}$ for $\mathbb{P}\mathbb{S}^4$ and $(y^*_{(i,j)})_{i,j=2,\dots,5}$ for $(\mathbb{P}\mathbb{S}^4)^\vee = F_{\mathbb{S}^6}(A)$, then $\mathbb{P}\mathcal{L}_A$ is the hyperplane defined by $y_{(2,5)} = y_{(3,4)}$ and the pull-back via $\mathrm{Adj}_4^\vee$ is defined by the single relation $\mathrm{Cof}_{(2,5)} = \mathrm{Cof}_{(3,4)}$ among the cofactors of $(\mathbb{P}\mathbb{S}^4)^\vee$. This defines a monomial ideal, which is already saturated with respect to the determinant polynomial of $(\mathbb{P}\mathbb{S}^4)^\vee$. This is verified to be the cubic Pfaffian of

$$S_2 = \begin{bmatrix} 0 & 0 & y^*_{(2,2)} & y^*_{(2,3)} & y^*_{(2,4)} & y^*_{(2,5)} \\ 0 & 0 & y^*_{(2,3)} & y^*_{(3,3)} & y^*_{(3,4)} & y^*_{(3,5)} \\ -y^*_{(2,2)} & -y^*_{(2,3)} & 0 & y^*_{(3,4)}-y^*_{(2,5)} & y^*_{(4,4)} & y^*_{(4,5)} \\ -y^*_{(2,3)} & -y^*_{(3,3)} & y^*_{(2,5)}-y^*_{(3,4)} & 0 & y^*_{(4,5)} & y^*_{(5,5)} \\ -y^*_{(2,4)} & -y^*_{(3,4)} & -y^*_{(4,4)} & -y^*_{(4,5)} & 0 & 0 \\ -y^*_{(2,5)} & -y^*_{(3,5)} & -y^*_{(4,5)} & -y^*_{(5,5)} & 0 & 0 \end{bmatrix}. \tag{2.2.8}$$

Finally, we consider the rank-2 matrix $A = A_2'$ for which $a_{(4,0,0)} = a_{(3,1,0)} = 1$ and the remaining entries are zero. This corresponds to a point on a tangent line to $\nu_4(1:0:0)$. With analogous computations to the ones above, one shows that $F_{\mathbb{S}^6}(A)$ is defined by $y^*_{(i,j)} = 0$ for $i = 1, 2$ and $j = 1, \dots, 6$. Using coordinates $(y_{(i,j)})_{i,j=3,\dots,6}$ for $\mathbb{P}\mathbb{S}^4$ and $(y^*_{(i,j)})_{i,j=3,\dots,6}$ for $(\mathbb{P}\mathbb{S}^4)^\vee = F_{\mathbb{S}^6}(A)$, the hyperplane $\mathbb{P}\mathcal{L}_A$ is defined by $y_{(3,6)} = y_{(5,5)}$ and the pull-back via $\mathrm{Adj}_4^\vee$ is defined by the cubic equation $\mathrm{Cof}_{(3,6)} = \mathrm{Cof}_{(5,5)}$. Again, the corresponding monomial ideal is saturated with respect to the determinant and it is verified to be the cubic Pfaffian of

$$S_2' = \begin{bmatrix} 0 & 0 & y^*_{(5,6)} & y^*_{(6,6)} & y^*_{(3,6)} & y^*_{(4,6)} \\ 0 & 0 & y^*_{(3,5)} & y^*_{(3,6)} & y^*_{(3,3)} & y^*_{(3,4)} \\ -y^*_{(5,6)} & -y^*_{(3,5)} & 0 & y^*_{(4,6)}-y^*_{(5,5)} & y^*_{(3,4)} & y^*_{(4,4)} \\ -y^*_{(6,6)} & -y^*_{(3,6)} & y^*_{(5,5)}-y^*_{(4,6)} & 0 & y^*_{(3,5)} & y^*_{(4,5)} \\ -y^*_{(3,6)} & -y^*_{(3,3)} & -y^*_{(3,4)} & -y^*_{(3,5)} & 0 & 0 \\ -y^*_{(4,6)} & -y^*_{(3,4)} & -y^*_{(4,4)} & -y^*_{(4,5)} & 0 & 0 \end{bmatrix}. \tag{2.2.9}$$

■

Proposition 2.2.8. *Let A_r be any matrix in $\mathbb{P}\mathcal{C}$ of rank $r < 6$, and let S_r be an r-secant space to $\nu_4(\mathbb{P}^2)$ containing A_r, spanned by r points $P_1, \dots, P_r$ of $\nu_4(\mathbb{P}^2)$ in general position. If A_r is in the smooth locus of C_r, then for every $i = 1, \dots, r$ we have*

$$F_{\mathcal{C}}(A_r) = F_{\mathcal{C}}(P_i) \cap F_{\mathbb{S}^6}(A_r). \tag{2.2.10}$$

Proof. When $r = 5$, the equality in (2.2.10) follows from $F_{\mathcal{C}}(A_r) = F_{\mathbb{S}^6}(A_r)$. For the remaining cases, we have an open orbit (see Remark 2.2.4), so without loss of generality, we may reduce to check the statement for $A_1, \dots, A_4$ as in (2.2.4).

We use the computations in the proof of Theorem 2.2.7. The reciprocal set $F_{\mathcal{C}}(A_1)$ is the intersection $V(\mathrm{Pf}_3(S_1)) \cap F_{\mathbb{S}^6}(A_1)$, and using explicit equations one can directly see that

$$\begin{aligned} V(\mathrm{Pf}_3(S_1)) \cap F_{\mathbb{S}^6}(A_1) \cap F_{\mathbb{S}^6}(A_2) &= V(\mathrm{Pf}_3(S_1)) \cap F_{\mathbb{S}^6}(A_2) \\ &= V(\mathrm{Pf}_3(S_2)) \cap F_{\mathbb{S}^6}(A_2) \\ &= F_{\mathcal{C}}(A_2), \end{aligned}$$

with S_1 as in (2.2.7) and S_2 as (2.2.8).

Analogously, when $r = 3, 4$, we have

$$V(\mathrm{Pf}_3(S_1)) \cap F_{\mathbb{S}^6}(A_1) \cap F_{\mathbb{S}^6}(A_r) = F_{\mathbb{S}^6}(A_r) = F_{\mathcal{C}}(A_r).$$

■

Corollary 2.2.9. *For $r = 2, \ldots, 5$, we have inclusions $F_{\mathcal{C}}(C_r) \subseteq F_{\mathcal{C}}(C_{r-1})$.*

Proof. The natural inclusions $F_{\mathbb{S}^6}(A_r) \subseteq F_{\mathbb{S}^6}(A_{r-1})$, applied to Proposition 2.2.8 give

$$\{F_{\mathcal{C}}(A_r) \mid A_r \in C_r \text{ generic}\} \subseteq \{F_{\mathcal{C}}(A_{r-1}) \mid A_{r-1} \in C_{r-1} \text{ generic}\},$$

and the statement follows by taking the Zariski closure. ■

Corollary 2.2.9 is the analogous of Corollary 1.3.6. Note that this property is very special: for a generic $\mathbb{P}\mathcal{L} \subset \mathbb{P}\mathbb{S}^m$, the reciprocal sets of rank loci would all have the same dimension.

2.2.3 Relation with secant varieties of Grassmannians

An equivalent way to read Theorem 2.2.7 is that the $\mathcal{C}$-reciprocal sets of points are linear sections of secant varieties of Grassmannians. More precisely, if A_r is a rank-r catalecticant matrix, there is a linear isomorphism

$$F_{\mathcal{C}}(A_r) \simeq \sigma_2(G(2, 8-r)) \cap \mathbb{P}^{N_r}, \tag{2.2.11}$$

with $\mathbb{P}^{N_r} = F_{\mathbb{S}^6}(A_r)$, of dimension $N_r = \binom{7-r}{2} - 1$.

Given a skew-symmetric matrix S, let us denote with $\mathrm{Pf}_k(S)$ the ideal of the degree-k Pfaffians of S. For every n, it is known that the Grassmannian of lines in $\mathbb{P}^n$ and its secant are defined by Pfaffians:

$$G(2, n) = V(\mathrm{Pf}_2(S)), \qquad \sigma_2(G(2, n)) = V(\mathrm{Pf}_3(S)),$$

where S is a generic $n \times n$ skew-symmetric matrix with linear polynomial entries. Moreover, the singular locus of the 2-secant is exactly

$$\mathrm{Sing}(\sigma_2(G(2, n))) = G(2, n).$$

The linear sections in (2.2.11) are not generic with respect to the singularity property.

Proposition 2.2.10. *Let $A_2, A_2' \in \mathbb{P}\mathcal{C}$ be two rank-2 matrices corresponding to a point on a secant line and a point on a tangent line and let S_2, S_2' be 6×6 skew-symmetric matrices whose cubic Pfaffians define $F_{\mathcal{C}}(A_2)$ and $F_{\mathcal{C}}(A_2')$. Then:*

(1) *The singular locus of $F_{\mathcal{C}}(A_2)$ properly contains $V(\mathrm{Pf}_2(S_2)) \cap F_{\mathbb{S}^6}(A_2)$, and similarly for A_2';*

(2) *The reciprocal sets* $F_{\mathcal{C}}(A_2)$ *and* $F_{\mathcal{C}}(A_2')$ *are not projectively equivalent;*

(3) *There exists another* 6×6 *matrix* T_2*, non-equivalent to* S_2*, whose cubic Pfaffian defines* $F_{\mathcal{C}}(A_2)$*.*

Proof. Without loss of generality, we may assume that A_2, A_2' and S_2, S_2' are as in the proof of Theorem 2.2.7. All the following claims can be verified with `Macaulay2`.

To prove (1), we compute the irreducible decomposition of the singular loci and check that only some of the components show up in the decomposition of the quadric Pfaffians.

For (2), it is enough to check that the components appearing in the tangent and secant case are not isomorphic.

In the secant case we have:

$$\mathrm{Sing}(F_{\mathcal{C}}(A_2)) = \gamma_1 \cup \gamma_2 \cup \gamma_3,$$

where

- $\gamma_1 = \nu_2(\mathbb{P}^3)$ is defined by the 2-minors of

$$\begin{bmatrix} y^*_{(2,2)} & y^*_{(2,3)} & y^*_{(2,4)} & y^*_{(2,5)} \\ y^*_{(2,3)} & y^*_{(3,3)} & y^*_{(3,4)} & y^*_{(3,5)} \\ y^*_{(2,4)} & y^*_{(3,4)} & y^*_{(4,4)} & y^*_{(4,5)} \\ y^*_{(2,5)} & y^*_{(3,5)} & y^*_{(4,5)} & y^*_{(5,5)} \end{bmatrix}$$

  plus the linear equations of $F_{\mathbb{S}^6}(A_2)$,

- $\gamma_2 \simeq \gamma_3$ are two rational normal $\mathbb{P}^2$-scrolls of degree 6, defined by the 2-minors of

$$\left[\begin{array}{cc|cc|cc} y^*_{(2,2)} & y^*_{(2,3)} & y^*_{(2,4)} & y^*_{(3,4)} & y^*_{(3,3)} & y^*_{(3,5)} \\ y^*_{(2,3)} & y^*_{(3,3)} & y^*_{(3,4)} & y^*_{(4,5)} & y^*_{(3,5)} & y^*_{(5,5)} \end{array}\right]$$

  and

$$\left[\begin{array}{cc|cc|cc} y^*_{(2,2)} & y^*_{(2,4)} & y^*_{(2,3)} & y^*_{(3,4)} & y^*_{(4,4)} & y^*_{(4,5)} \\ y^*_{(2,4)} & y^*_{(4,4)} & y^*_{(3,4)} & y^*_{(3,5)} & y^*_{(4,5)} & y^*_{(5,5)} \end{array}\right]$$

  respectively, plus the linear equations of $F_{\mathbb{S}^6}(A_2)$ and the additional linear equation $y^*_{(3,4)} - y^*_{(2,5)} = 0$.

On the other hand we only have

$$V(\mathrm{Pf}_2(S_2)) \cap F_{\mathbb{S}^6}(A_2) = \gamma_1 \cup \gamma_3. \tag{2.2.12}$$

In the tangent case we have irreducible decomposition

$$\mathrm{Sing}(F_{\mathcal{C}}(A_2')) = \gamma_1' \cup \gamma_2' \cup \gamma_3',$$

where

- $\gamma_1' = \nu_2(\mathbb{P}^3)$ is defined by the 2-minors of

$$\begin{bmatrix} y^*_{(3,3)} & y^*_{(3,4)} & y^*_{(3,5)} & y^*_{(3,6)} \\ y^*_{(3,4)} & y^*_{(4,4)} & y^*_{(4,5)} & y^*_{(4,6)} \\ y^*_{(3,5)} & y^*_{(4,5)} & y^*_{(5,5)} & y^*_{(5,6)} \\ y^*_{(3,6)} & y^*_{(4,6)} & y^*_{(5,6)} & y^*_{(6,6)} \end{bmatrix},$$

  plus the linear equations of $F_{\mathbb{S}^6}(A_2)$.

- $\gamma_2' = G(2,4)$ is defined by the quadric Pfaffian of

$$\begin{bmatrix} 0 & y^*_{(4,4)} & y^*_{(4,5)} & y^*_{(4,6)} \\ -y^*_{(4,4)} & 0 & y^*_{(5,5)}-y^*_{(4,6)} & y^*_{(5,6)} \\ -y^*_{(4,5)} & y^*_{(4,6)}-y^*_{(5,5)} & 0 & y^*_{(6,6)} \\ -y^*_{(4,6)} & y^*_{(3,5)} & -y^*_{(5,6)} & 0 \end{bmatrix},$$

  plus the linear equations of $F_{\mathbb{S}^6}(A_2)$ and the additional linear equations $y^*_{(3,j)} = 0$ for $j = 1, \dots, 6$.

- γ_3' is an embedded component of γ_2'. Set-theoretically, it is defined by the same equations of γ_2' plus the determinant of

$$\begin{bmatrix} y^*_{(4,4)} & y^*_{(4,5)} & y^*_{(4,6)} \\ y^*_{(4,5)} & y^*_{(5,5)} & y^*_{(5,6)} \\ y^*_{(4,6)} & y^*_{(5,6)} & y^*_{(6,6)} \end{bmatrix}.$$

  On the other hand, we have

$$V(\mathrm{Pf}_2(S_2')) \cap F_{\mathbb{S}^6}(A_2') = \gamma_1' \cup \gamma_2'. \tag{2.2.13}$$

Note that $\dim(\gamma_2') = 4$, so the above equation (2.2.13) is an improper intersection between a $G(2,6)$ and a $\mathbb{P}^9$.

Finally, to prove (3), we need to find a matrix T_2 such that

$$F_{\mathcal{C}}(A_2) = V(\mathrm{Pf}_3(T_2)) \cap F_{\mathbb{S}^6}(A_2)$$

with decomposition on the quadric Pfaffians

$$V(\mathrm{Pf}_2(T_2)) \cap F_{\mathbb{S}^6}(A_2) = \gamma_1 \cup \gamma_2.$$

As suggested by Proposition 2.2.8, a way to find both S_2 and T_2 requires knowing the 7×7 skew-symmetric matrices associated with rank-1 matrices. For example, consider the A_1 above corresponding to $\nu_4(1:0:0)$, as well as the S_1 in (2.2.7).

Then, imposing the equations of $\mathbb{P}^9 = F_{\mathbb{S}^6}(A_2)$, we obtain a new matrix from S_1, whose first row and column are set to zero, so it can be identified with a 6×6 skew-symmetric matrix. This matrix is in fact equivalent to S_2, so we have

$$F_{\mathcal{C}}(A_2) = V(\mathrm{Pf}_3(S_1)) \cap F_{\mathbb{S}^6}(A_2), \tag{2.2.14}$$

and the quadric Pfaffians decompose as $\gamma_1 \cup \gamma_3$.

Similarly, we can consider another rank-1 matrix A'_1 corresponding to the second point $\nu_4(0:0:1)$. Then

$$S'_1 = \begin{bmatrix} 0 & 0 & y^*_{(5,5)} & y^*_{(4,5)} & y^*_{(3,5)} & y^*_{(2,5)} & y^*_{(1,5)} \\ 0 & 0 & y^*_{(4,5)} & y^*_{(4,4)} & y^*_{(3,4)} & y^*_{(2,4)} & y^*_{(1,4)} \\ -y^*_{(5,5)} & -y^*_{(4,5)} & 0 & -y^*_{(2,5)}+y^*_{(3,4)} & y^*_{(3,3)} & -y^*_{(1,5)}+y^*_{(2,3)} & y^*_{(1,3)} \\ -y^*_{(4,5)} & -y^*_{(4,4)} & y^*_{(2,5)}-y^*_{(3,4)} & 0 & y^*_{(2,3)} & -y^*_{(1,4)}+y^*_{(2,2)} & y^*_{(1,2)} \\ -y^*_{(3,5)} & -y^*_{(3,4)} & -y^*_{(3,3)} & -y^*_{(2,3)} & 0 & -y^*_{(1,3)} & 0 \\ -y^*_{(2,5)} & -y^*_{(2,4)} & y^*_{(1,5)}-y^*_{(2,3)} & y^*_{(1,4)}-y^*_{(2,2)} & y^*_{(1,3)} & 0 & y^*_{(1,1)} \\ -y^*_{(1,5)} & -y^*_{(1,4)} & -y^*_{(1,3)} & -y^*_{(1,2)} & 0 & -y^*_{(1,1)} & 0 \end{bmatrix}$$

satisfies

$$F_{\mathcal{C}}(A'_1) = V(\mathrm{Pf}_3(S'_1)) \cap F_{\mathbb{S}^6}(A'_1),$$

and imposing the equations of the same $\mathbb{P}^9 = F_{\mathbb{S}^6}(A_2)$, we obtain a new matrix from S'_1, whose last row and column are set to zero, so it can be identified with a 6×6 skew-symmetric matrix T_2.

Then

$$F_{\mathcal{C}}(A_2) = V(\mathrm{Pf}_3(S'_1)) \cap F_{\mathbb{S}^6}(A_2) = V(\mathrm{Pf}_3(T_2)) \cap F_{\mathbb{S}^6}(A_2),$$

but the quadric Pfaffians decompose into $\gamma_1 \cup \gamma_2$. ■

Note that a similar behavior for points on tangent lines is not to be expected: the components γ'_i above do not present the same kind of symmetry and in fact they can only depend on one rank-1 point, namely the tangency point.

We conclude the section with the following proposition, which gives an indication of why reciprocal sets of points are defined by Pfaffians.

Proposition 2.2.11. *For any point $A \in C_1$, its reciprocal set $F_{\mathcal{C}}(A)$ is a union of Grassmannians $G(2,6)$.*

Proof. By Lemma 2.2.6, $F_{\mathcal{C}}(A)$ is computed by the restriction of $\mathrm{Adj}_5 : \mathbb{P}\mathbb{S}^5 \dashrightarrow (\mathbb{P}\mathbb{S}^5)^\vee$ to an 11-dimensional projective linear subspace $\mathbb{P}\mathcal{L}_A \subset \mathbb{P}\mathbb{S}^5$.

We prove the statement by showing that $\mathbb{P}\mathcal{L}_A$ is a union of $\mathbb{P}^8$s that are linear spans of rational normal curves of degree 8. Then the result follows by recalling that every such $\mathbb{P}^8$ parametrizes a catalecticant space of binary octics, so its image via Adj_5 is a $G(2,6)$ (cfr. Theorem 1.2.1).

Using the action of PGL(3), we may assume without loss of generality that A is the matrix corresponding to $\nu_4(P)$, with $P = [1:0:0]$. Then matrices in $\mathbb{P}\mathcal{L}_A$ have parametric form.

$$\begin{bmatrix} a_{(2,2,0)} & a_{(2,1,1)} & a_{(1,3,0)} & a_{(1,2,1)} & a_{(1,1,2)} \\ a_{(2,1,1)} & a_{(2,0,2)} & a_{(1,2,1)} & a_{(1,1,2)} & a_{(1,0,3)} \\ a_{(1,3,0)} & a_{(1,2,1)} & a_{(0,4,0)} & a_{(0,3,1)} & a_{(0,2,2)} \\ a_{(1,2,1)} & a_{(1,1,2)} & a_{(0,3,1)} & a_{(0,2,2)} & a_{(0,1,3)} \\ a_{(1,1,2)} & a_{(1,0,3)} & a_{(0,2,2)} & a_{(0,1,3)} & a_{(0,0,4)} \end{bmatrix}.$$

The rank-1 locus of $\mathbb{P}\mathcal{L}_A$ is a surface S of degree 12. More specifically, S is the image of $\mathbb{P}^2$ via

$$\lambda \colon \mathbb{P}^2 \xrightarrow{|\mathcal{O}_{\mathbb{P}^2}(4)-2P|} \mathbb{P}^{11},$$

and a degree check shows that the remaining rank loci $\mathbb{P}\mathcal{L}_A \cap D^r_{\mathbb{S}^5}$ are the secant varieties $\sigma_r(S)$.

There are exactly two families of curves in $\mathbb{P}^2$ that are mapped to rational normal octics in $\mathbb{P}\mathcal{L}_A$ via λ. The first, denoted by $\mathcal{F}_1$, is the $\mathbb{P}^5$ of plane conics. The second, denoted by $\mathcal{F}_2$, is the $\mathbb{P}^6$ of cubics singular at P.

We claim that $\mathbb{P}\mathcal{L}_A = \bigcup_{\gamma \in \mathcal{F}_1} \langle \lambda(\gamma) \rangle$. Indeed, $\sigma_5(S) = \mathbb{P}\mathcal{L}_A$, so every point $Q \in \mathbb{P}\mathcal{L}_A$ belongs to at least one 5-secant space to S. We may choose one such secant, say $\langle P_1, \ldots, P_5 \rangle$, where $P_i \in S$. The points P_i are the image via λ of 5 points in the plane, determining a conic $\gamma \in \mathcal{F}_1$ hence $Q \in \langle \lambda(\gamma) \rangle$. ■

2.3 Rank loci in the reciprocal variety

We can now move to compute the dimension of the reciprocal sets of the rank loci. This, combined with basic intersection theory of complete quadrics, allows us to prove that the degree of the reciprocal variety depends only on the degree of $F_{\mathcal{C}}(C_1)$ (Theorem 2.3.4).

The numerical results obtained in Section 2.1, imply that there must be non-empty intersection between $\mathbb{P}\mathcal{C}^{-1}$ and the orthogonal space $\mathbb{P}\mathcal{C}^{\perp}$. In Proposition 2.3.8 we explain this phenomenon from a geometric perpective, by studying the intersection of each $F_{\mathcal{C}}(C_r)$ with the orthogonal space.

Proposition 2.3.1. *For $r = 1, \ldots, 4$, we have $\dim F_{\mathcal{C}}(C_r) = 2r + 13 - \dim C_r$, while $\dim F_{\mathcal{C}}(C_5) = 5$. In particular, only $F_{\mathcal{C}}(C_1)$ is a hypersurface in the reciprocal variety.*

Proof. By Theorem 2.2.7, for any rank-r matrix $A \in C_r$ we have $\dim F_{\mathcal{C}}(A) = 13 - \dim C_r$. Moreover,

$$\begin{aligned} F_{\mathcal{C}}(C_r) &= \overline{\{F_{\mathcal{C}}(A) \mid A \in C_r \text{ generic }\}} \\ &= \overline{\{F_{\mathcal{C}}(S_r) \mid S_r = \langle P_1, \ldots, P_r \rangle, P_i \in \nu_4(\mathbb{P}^2)\}}, \end{aligned}$$

where the last equality is justified by Remark 2.2.5. An r-secant space is determined by the choice of r points on the surface, so altogether $\dim F_{\mathcal{C}}(C_r) = 2r + 13 - \dim C_r$.

The case $r = 5$ is special because 5 points on $\nu_4(\mathbb{P}^2)$ uniquely determine the tangent space to C_5 and this tangent space is a hyperplane in $\mathbb{P}\mathcal{C}$, which intersects the surface in a curve that is the image, via the ν_4, of a degree 4 curve in $\mathbb{P}^2$. In particular, such curve is singular at those 5 points, so it must be the image of a double conic. Therefore, for $r = 5$ the choice of a tangent space is uniquely determined by the choice of a point in the $\mathbb{P}^5$ of plane quadrics. ■

Corollary 2.3.2. *Set-theoretically, $F_{\mathcal{C}}(C_5) = D^1_{(\mathbb{S}^6)^*}$ and $F_{\mathcal{C}}(C_4) = D^2_{(\mathbb{S}^6)^*}$.*

Proof. By Remark 1.1.18, we have a natural inclusion of $F_{\mathcal{C}}(C_5)$ (resp. $F_{\mathcal{C}}(C_4)$) in the locus of matrices of $(\mathbb{PS}^6)^\vee$ of rank at most 1 (resp. 2), which is irreducible of dimension 5 (resp. 10). ■

A natural set up for understanding the inversion map and its regularization comes from complete quadrics. In this setting, we can analyze how the rank loci in the reciprocal variety contribute to its degree.

We give a quick overview of the needed definitions and properties in the case of catalecticants of ternary quartics. For a more general and detailed treatment of this topic, see [40, Section 3] as well as the references cited therein.

Set $V = \mathbb{C}^6$ and recall that $\mathbb{P}(S^2(V))$ is the space $\mathbb{PS}^6$ of symmetric matrices while $\mathbb{P}(S^2(\wedge^5 V)) = (\mathbb{PS}^6)^\vee$.

The space of *complete quadrics* on V, denoted $\mathrm{CQ}(V)$ is the Zariski closure of the image of

$$\psi\colon \mathbb{P}(S^2(V)) \dashrightarrow \mathbb{P}(S^2(V)) \times \mathbb{P}\Big(S^2\Big(\bigwedge^2 V\Big)\Big) \times \cdots \times \mathbb{P}\Big(S^2\Big(\bigwedge^5 V\Big)\Big)$$

given by

$$[A] \mapsto ([A], [\wedge^2 A], \ldots, [\wedge^5 A]).$$

We consider two kinds of divisor classes in the intersection ring of $\mathrm{CQ}(V)$. Classes of the first kind, denoted by $\mu_1, \ldots, \mu_5$, are the pull-backs of hyperplane classes via the natural projection maps $\pi_1, \ldots, \pi_5$ on the five factors.

Classes of the second kind, denoted by $\delta_1, \ldots, \delta_5$, correspond to exceptional divisors coming from the blow-up of the rank loci. We are going to use the following relation between divisor classes:

$$\mu_5 = \frac{1}{6} \cdot \sum_{r=1}^{5} r \cdot \delta_r. \tag{2.3.1}$$

Remark 2.3.3. Complete quadrics can be equivalently thought as a particular compactification of the space of smooth quadrics in $\mathbb{P}V$. This is done by blowing-up $\mathbb{PS}^6$ first along its rank-1 locus, then the strict transform of its rank-2 locus, and so on up to the rank-5 locus. In particular, π_5 is a regularization of $\mathrm{Adj}_6 \colon \mathbb{PS}^6 \dashrightarrow (\mathbb{PS}^6)^\vee$.

Let us denote with $\widetilde{\mathbb{P}\mathcal{C}}$ the proper transform of $\mathbb{P}\mathcal{C}$ inside $\mathrm{CQ}(V)$, and with $\widetilde{C}_r$ the intersection of $\widetilde{\mathbb{P}\mathcal{C}}$ with the exceptional divisor over the rank-r symmetric locus, so that $[\widetilde{C}_r] = [\widetilde{\mathbb{P}\mathcal{C}}] \cdot \delta_r$. The projection map π_5 restricted to $\widetilde{\mathbb{P}\mathcal{C}}$ is a regularization of $\mathrm{Adj}_6 \colon \mathbb{P}\mathcal{C} \dashrightarrow (\mathbb{PS}^6)^\vee$.

Theorem 2.3.4. *The degree of the reciprocal variety of $\mathbb{P}\mathcal{C}$ only depends on the degree of $F_{\mathcal{C}}(C_1)$. More precisely,*

$$\deg(\mathbb{P}\mathcal{C}^{-1}) = \frac{\deg F_{\mathcal{C}}(C_1)}{6}.$$

Proof. The degree of $\mathbb{P}\mathcal{C}^{-1}$ is by definition the number of points in its intersection with 14 general hyperplanes of $(\mathbb{PS}^6)^\vee$. Since Adj_6 is birational, its regularization has degree 1. Therefore, by pulling back classes in the space of complete quadrics, the degree of the reciprocal variety can be equivalently computed as $[\widetilde{\mathbb{P}\mathcal{C}}] \cdot \mu_5^{14}$. According to (2.3.1) , we can rewrite the last expression as

$$\frac{\sum_{r=1}^5 r \cdot [\widetilde{\mathbb{P}\mathcal{C}}] \cdot \delta_r \cdot \mu_5^{13}}{6} = \frac{\sum_{r=1}^5 r \cdot [\widetilde{C_r}] \cdot \mu_5^{13}}{6}.$$

The product $[\widetilde{C_r}] \cdot \mu_5^{13}$ counts the number of points in the intersection of $F_\mathcal{C}(C_r)$ with 13 general hyperplanes of $(\mathbb{PS}^6)^\vee$. By Proposition 2.3.1, we have $\dim(F_\mathcal{C}(C_r)) < 13$ when $r = 2, \ldots, 5$, so the resulting intersection number is 0 and the unique contribution to the above sum is $[\widetilde{C_1}] \cdot \mu_5^{13} = \deg F_\mathcal{C}(C_1)$. ∎

Remark 2.3.5. From Section 2.1 we numerically computed the degree of $\mathbb{P}\mathcal{C}^{-1}$ to be 85, therefore the degree of the reciprocal set $F_\mathcal{C}(C_1)$ is equal to 510.

For ternary quartics and binary forms of even degree, only $F_\mathcal{C}(C_1)$ contributes to the degree of the reciprocal variety. This might not be true for higher catalecticants, where the rank loci are not necessarily secant varieties to Veronese embeddings.

Question 2.3.6. *Does the degree of* $\mathbb{P}\mathrm{Cat}(k, n+1)^{-1}$ *depend only on the degree of* $F_\mathcal{C}(C_1)$*, for every* (k, n)*?*

The orthogonal space $\mathbb{P}\mathcal{C}^\perp$ is a 5-dimensional projective linear space. We want to study its intersection with $\mathbb{P}\mathcal{C}^{-1}$ and the first step towards this is the analysis of its rank loci.

Remark 2.3.7. A computation with `Macaulay2` (see Appendix A.1.4) shows that the generic rank of $\mathbb{P}\mathcal{C}^\perp$ is 6 and that it is empty in rank 1 and 2. On the other hand, the rank-3 locus is a Veronese surface $\nu_2(\mathbb{P}^2)$ while the loci of rank 4 and 5 are the cubic hypersurface defining the secant variety $\sigma_2(\nu_2(\mathbb{P}^2))$. The $\mathrm{PGL}(3)$ action in Remark 2.2.4 ensures that the only possibilities for $F_\mathcal{C}(C_r) \cap \mathbb{P}\mathcal{C}^\perp$ are the rank loci of the orthogonal space.

Proposition 2.3.8. *The orthogonal* $\mathbb{P}\mathcal{C}^\perp$ *does not contain the image of any full-rank point and intersects the varieties* $F_\mathcal{C}(C_r)$ *in the emptyset for* $r = 3, 4, 5$*, and set-theoretically in a Veronese surface* $\nu_2(\mathbb{P}^2)$ *for* $r = 1, 2$*. In particular:*

$$\mathbb{P}\mathcal{C}^{-1} \cap \mathbb{P}\mathcal{C}^\perp \neq \emptyset.$$

Proof. The orthogonal space does not contain full-rank points of the reciprocal variety as we already pointed out in Remark 1.1.23.

Let now A have rank $r \leq 5$. When $r = 4, 5$, the statement follows from Remark 2.3.7: indeed, points on $F_\mathcal{C}(C_4)$ and $F_\mathcal{C}(C_5)$ have rank at most 2 and 1 respectively, while $\mathbb{P}\mathcal{C}^\perp$ is empty in rank 1 and 2.

The cases $r = 1, 2, 3$ can be checked fiberwise using Theorem 2.2.7 (see also the codes in Appendix A.1.5). If A is any rank-3 point, or a rank-2 point lying on a proper secant line to C_1, we have $F_{\mathcal{C}}(A) \cap \mathbb{P}\mathcal{C}^{\perp} = \emptyset$. On the other hand, if A is a rank-2 point lying on a tangent line to C_1 or a rank-1 point, then $F_{\mathcal{C}}(A) \cap \mathbb{P}\mathcal{C}^{\perp}$ in one single point belonging to its Veronese surface $\nu_2(\mathbb{P}^2)$. This intersection point varies inside the surface as we move the tangent line (the point on C_1). We then conclude by Remark 2.3.7. ■

Remark 2.3.9. The non-empty intersection between the orthogonal and the reciprocal variety of $\mathbb{P}\mathcal{C}$ not only explains that $\deg(\mathbb{P}\mathcal{C}^{-1}) \neq \text{ML-}\deg(\mathbb{P}\mathcal{C})$. It also shows that the problem of computing the degree $F_{\mathcal{C}}(C_1)$ cannot be easily dealt with as for Hankel matrices (cfr. Theorem 1.4.8) since now the projection $\pi_{\mathcal{C}}$ is not regular.

2.4 Singularities

The main result of this section is Theorem 2.4.1, which gives information on the singularities of $\mathbb{P}\mathcal{C}^{-1}$. We provide both a numerical and a theoretical proof, with different levels of insight.

Theorem 2.4.1. *The reciprocal variety $\mathbb{P}\mathcal{C}^{-1}$ of the catalecticant space of ternary quartics is singular along the locus $D^2_{(\mathbb{S}^m)^*}$ of dual symmetric matrices of rank at most* 2*.*

Numerical proof of Theorem 2.4.1. Recalling Theorem 2.1.3, we have 27 cubics numerically computed to be in the set of minimal generators for the ideal of $\mathbb{P}\mathcal{C}^{-1}$. Let J denote the Jacobian of these cubics. This is a 21×27 matrix, whose generic rank is 6. We check that the rank drops on a single rank-1 symmetric matrix and on a single rank-2 symmetric matrix. Then the statement follows because by Corollary 2.3.2, $\mathbb{P}\mathcal{C}^{-1}$ contains all rank-1 and rank-2 symmetric matrices and we can move from one point to another using the action of PGL(6) on $(\mathbb{P}\mathbb{S}^6)^{\vee}$.

For a point $B \in (\mathbb{P}\mathbb{S}^6)^{\vee}$ of rank r we compute with `Julia`:

$$\operatorname{rk} J(B) = \begin{cases} 3 & \text{if } r = 2 \\ 0 & \text{if } r = 1. \end{cases}$$

The statement for rank-1 points can also be proven by computing degree and dimension of the variety defined by the the zero locus of the entries of J. With `HomotopyContinuation.jl`, we obtain a variety of projective dimension 5 and degree 32, which matches with $\nu_2(\mathbb{P}^5) = D^1_{(\mathbb{S}^6)^*}$. Codes for these computations can be found in Appendix A.2.5. ■

Remark 2.4.2. The case of rank-1 points is very special: indeed, the numerical computations show that the tangent cone of the reciprocal variety at rank-1 points spans the whole $(\mathbb{P}\mathbb{S}^6)^{\vee}$ or, equivalently, that rank-1 points are singular for each of the 27 cubics.

We proceed with an alternative approach for the proof of Theorem 2.4.1, exploiting our knowledge on the geometry of the rank loci in the reciprocal variety. We start with some preparatory results.

Definition 2.4.3. For any $\mathbb{P}\mathcal{L} \subseteq \mathbb{P}\mathbb{S}^m$ and for any rank-r matrix $B \in \mathbb{P}\mathcal{L}^{-1}$, we define the *$\mathcal{L}$-reciprocal preimage* of B to be

$$F_{\mathcal{L}}^{-1}(B) := \overline{\{A \in \mathbb{P}\mathcal{L} \cap D_{\mathbb{S}^m}^{m-r} \mid B \in F_{\mathcal{C}}(A)\}}.$$

Note that, with this definition we have: $F_{\mathcal{L}}^{-1}(B) = F_{\mathbb{S}^m}^{-1}(B) \cap \mathbb{P}\mathcal{L}$

Lemma 2.4.4. *For any matrix $B \in (\mathbb{P}\mathbb{S}^6)^\vee$ of rank-1 (resp. $2, 3, 4, 5, 6$), the intersection $F_{\mathcal{C}}^{-1}(B)$ is a projective linear space of dimension 8 (resp. $3, 2, 1, 0, 0$).*

Proof. Let $r = \mathrm{rk}(B)$. When $r = 5, 6$, the statement is trivial. In the remaining cases, we observe that $F_{\mathbb{S}^m}^{-1}(B)$ is linear in $\mathbb{P}\mathbb{S}^6$, so it remains linear when intersecting with $\mathbb{P}\mathcal{C}$. The dimensions are given by $13 - \dim F_{\mathcal{C}}(C_{6-r})$, that is 8 (resp. $3, 2, 1$) when $r = 1$ (resp. $2, 3, 4$). ■

Remark 2.4.5 (The role of the fiber $\mathbb{P}^8$)**.** The above Lemma 2.4.4 gives also an important relation between reciprocal preimages and secant spaces.

Indeed, when $\mathrm{rk}(B) = 2$ (resp. $3, 4$), then $F_{\mathcal{C}}^{-1}(B)$ contains a 4 (resp. $3, 2$)-secant space to $\nu_4(\mathbb{P}^2)$ and they must coincide because they are both linear and of the same dimension.

The case $\mathrm{rk}(B) = 1$ is different: there is not only a single 5-secant space in the preimage, but many of them, and altogether they form a $\mathbb{P}^8$. Recall, the tangent spaces to C_5 at its smooth points are embedded in $\mathbb{P}\mathcal{C}$ as 11-dimensional projective spaces that cut $\nu_4(\mathbb{P}^2)$ in a curve, image of a double conic via ν_4. This is therefore a rational curve of degree 8, the composition $\nu_4(\nu_2(\mathbb{P}^1))$. Its linear span is the above $\mathbb{P}^8$.

Since $F_{\mathcal{C}}(C_5) = F_{\mathbb{S}^m}(D_{\mathbb{S}^6}^5)$ (cfr. Corollary 2.3.2), then for every fixed B of rank 1, the associated $\mathbb{P}^8$ is simply given by

$$F_{\mathcal{C}}^{-1}(B) = \{A \in \mathbb{P}\mathcal{C} \mid AB = 0\}.$$

The Jacobian of $\mathrm{Adj}_6 \colon \mathbb{P}\mathbb{S}^6 \dashrightarrow (\mathbb{P}\mathbb{S}^6)^\vee$ is a 21×21 matrix with polynomial entries of degree 4. The restriction of the Jacobian of Adj_6 to $\mathbb{P}\mathcal{C}$, which we denote with $J_{\mathcal{C}}$, is a 21×15 matrix. For every $A \in \mathbb{P}\mathcal{C}$, the evaluation $J_{\mathcal{C}}(A)$ represents a linear map of vector spaces $\mathcal{C} \to (\mathbb{S}^6)^*$. If Adj_6 is defined on A, we have the induced tangent map on the quotients

$$T_A\mathbb{P}\mathcal{C} = \frac{\mathcal{C}}{\langle A\rangle} \xrightarrow{\tau_A} \frac{(\mathbb{S}^6)^*}{\langle \mathrm{Adj}_6(A)\rangle} = T_{\mathrm{Adj}_6(A)}(\mathbb{P}\mathbb{S}^6)^\vee. \tag{2.4.1}$$

Geometrically, τ_A sends a point $P \in T_A\mathbb{P}\mathcal{C}$ to the tangent line at $\mathrm{Adj}_6(A)$ at the quintic curve $\mathrm{Adj}_6(\langle P, A\rangle)$.

Lemma 2.4.6. *Let $A \in \mathbb{P}\mathcal{C}$ be a general rank-r catalecticant matrix. Then:*

$$\operatorname{rk} J_{\mathcal{C}}(A) = \begin{cases} 15 & \text{if} \quad r = 6 \\ 7 & \text{if} \quad r = 5 \\ 3 & \text{if} \quad r = 4 \\ 0 & \text{if} \quad r \leq 3. \end{cases}$$

Proof. The maximal rank of $J_{\mathcal{C}}$ is 15 and it is attained at the points on which Adj_6 is a well-defined bijective map, namely rank-6 points.

On rank-5 points, Adj_6 is still well-defined but not injective. We have

$$\ker(\tau_A) = \left\{ P \in \frac{\mathcal{C}}{\langle A \rangle} \,\middle|\, J_{\mathcal{C}}(A)(P) \in \langle \mathrm{Adj}_6(A) \rangle \right\} \simeq F_{\mathcal{C}}^{-1}(\mathrm{Adj}_6(A)),$$

which by Lemma 2.4.4 is an 8-dimensional space. Moreover, Adj_6 is defined at A, so $J_{\mathcal{C}}(A) \cdot A \neq 0$ and $\operatorname{rk} J_{\mathcal{C}}(A) = 15 - 8 = 7$.

On rank-4 points, $J_{\mathcal{C}}$ is well-defined but Adj_6 is not. In particular, we cannot use the tangent map in (2.4.1). The analogous map to consider is

$$\frac{\mathcal{C}}{\langle A \rangle} \xrightarrow{\tau'_A} \frac{(\mathbb{S}^6)^*}{V},$$

where V is the 3-dimensional vector space for which $F_{\mathcal{C}}(A) = \mathbb{P}^2 = \mathbb{P}(V)$. By construction, the map τ'_A is identically zero, and since Adj_6 is not defined at A, we have $J_{\mathcal{C}}(A) \cdot A = 0$.

Altogether, this implies that the image of $J_{\mathcal{C}}(A)$ is contained in V, hence $\operatorname{rk} J_{\mathcal{C}}(A) \leq 3$. Equality is attained at points A belonging to the orbit of points on bitangents 4-secant spaces (see Example 2.4.7 below), hence it is attained at general rank-4 points.

Finally, a point A of rank at most 3 is a base point for $J_{\mathcal{C}}$, hence $J_{\mathcal{C}}(A) = 0$. ∎

Example 2.4.7. We are going to see that when A lies on a degenerate 4-secant space that is bitangent to $\nu_4(\mathbb{P}^2)$, the rank of $J_{\mathcal{C}}(A)$ is equal to 3 and its 11-dimensional kernel is the linear span of the limits of tangent planes to points on the Veronese surface.

For example, let us consider the rank-4 matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix},$$

corresponding to a point on the $\mathbb{P}^3$ bitangent to $\nu_4(\mathbb{P}^2)$ at $P_1 = \nu_4(1:0:0)$ and $P_2 = \nu_4(0:0:1)$. Via the identification $T_A\mathbb{P}\mathcal{C} \simeq \mathbb{P}\mathcal{C}$, the kernel of $J_{\mathcal{C}}(A)$ is the linear subspace defined by the equations $a_{(2,0,2)} = a_{(1,2,1)} = a_{(0,4,0)} = 0$.

This also coincides with the linear span of the two $\mathbb{P}^5$ of matrices that are obtained as limits of spans of tangent planes to points approaching P_1 and P_2. Explicitly, these two 5-dimensional linear spaces are parametrized by

$$\begin{bmatrix} a_{(4,0,0)} & a_{(3,1,0)} & a_{(3,0,1)} & a_{(2,2,0)} & a_{(2,1,1)} & \cdot \\ a_{(3,1,0)} & a_{(2,2,0)} & a_{(2,1,1)} & a_{(1,3,0)} & \cdot & \cdot \\ a_{(3,0,1)} & a_{(2,1,1)} & \cdot & \cdot & \cdot & \cdot \\ a_{(2,2,0)} & a_{(1,3,0)} & \cdot & \cdot & \cdot & \cdot \\ a_{(2,1,1)} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_{(1,1,2)} \\ \cdot & \cdot & \cdot & \cdot & a_{(1,1,2)} & a_{(1,0,3)} \\ \cdot & \cdot & \cdot & \cdot & a_{(0,3,1)} & a_{(0,2,2)} \\ \cdot & \cdot & a_{(1,1,2)} & a_{(0,3,1)} & a_{(0,2,2)} & a_{(0,1,3)} \\ \cdot & a_{(1,1,2)} & a_{(1,0,3)} & a_{(0,2,2)} & a_{(0,1,3)} & a_{(0,0,4)} \end{bmatrix}.$$

$\diamondsuit$

Remark 2.4.8. The proof of Lemma 2.4.6 in fact works without generality assumption for the matrix A, except for the case $\mathrm{rk}(A) = 4$.

Moreover, Example 2.4.7 suggests that, when A is general of rank 4, the kernel of $J_{\mathcal{C}}(A)$ is equal to the tangent space $T_A(C_4)$.

With the knowledge we now have about the Jacobian matrix evaluated at different points, we are ready to revisit part of the argument proving that $\mathbb{P}\mathcal{C}^{-1}$ is singular.

Proof of Theorem 2.4.1 revisited. We show that $F_{\mathcal{C}}(C_5) = D^1_{(\mathbb{S}^6)^*}$ is contained in the singular locus of $\mathbb{P}\mathcal{C}^{-1}$. Let A be a general matrix of rank 5 in $\mathbb{P}\mathcal{C}$ and let $B = \mathrm{Adj}_6(A)$. It is enough to prove that B is singular for $\mathbb{P}\mathcal{C}^{-1}$. Then, thanks to the action of $\mathrm{PGL}(6)$ on $(\mathbb{P}\mathbb{S}^6)^\vee$, every other rank-1 symmetric matrix will satisfy the same property.

A sufficient condition for B to be singular is that the tangent cone of $\mathbb{P}\mathcal{C}^{-1}$ at B spans all $(\mathbb{P}\mathbb{S}^6)^\vee$. To see this, we proceed as follows (computations can be done with `Macaulay2`, see also Appendix A.1.6).

First, by Lemma 2.4.4 and Remark 2.4.5, the preimage $F_{\mathcal{C}}^{-1}(B)$ is a $\mathbb{P}^8$, union of 5-secant spaces. So, let us fix a 5-secant space $S_5 \subset C_5$ and a matrix $A \in S_5$. By Lemma 2.4.6, we have $\mathrm{rk}\, J_{\mathcal{C}}(A) = 7$. Equivalently, using the identifications in (2.4.1), the tangent space of $\mathrm{Adj}_6(T_A\mathbb{P}\mathcal{C})$ at B is embedded as a $\mathbb{P}^6$ in $(\mathbb{P}\mathbb{S}^6)^\vee$. More precisely, $\mathbb{P}^6$ is the linear span $\langle T_B\mathrm{Adj}_6(L_1), \ldots, T_B\mathrm{Adj}_6(L_{15})\rangle$, where $L_i = \langle A, P_i\rangle$ is a line through A and P_i, and the 15 points P_i are in general position.

Second, we move the matrix $A \in S_5$. For every such matrix, we obtain a $\mathbb{P}^6_A \subset (\mathbb{P}\mathbb{S}^6)^\vee$. Picking $A_1, \ldots, A_5 \in S_5$ in general position, the linear span $\langle \mathbb{P}^6_{A_1}, \ldots, \mathbb{P}^6_{A_5}\rangle$ is a $\mathbb{P}^{10}$.

Finally, we move the 5-secant $S_5 \in \mathbb{P}^8$. For every such secant we obtain a $\mathbb{P}^{10}_{S_5} \subset (\mathbb{P}\mathbb{S}^6)^\vee$. A choice of a 5-secant corresponds to the choice of a 5-tuple of points on the embedding of a plane conic via ν_4 (cfr. Remark 2.4.5). Picking five generic 5-tuples on the same plane conic, we obtain five different secants $S_{5,1}, \ldots, S_{5,5}$ and we get equality $\langle \mathbb{P}^{10}_{S_{5,1}}, \ldots, \mathbb{P}^{10}_{S_{5,5}}\rangle = (\mathbb{P}\mathbb{S}^6)^\vee$. $\blacksquare$

We now might wonder if the reciprocal variety itself is defined by the cubic Pfaffians of an 8×8 skew-symmetric matrix. Although the codimension would be the correct one, the degree would be at most 84 and the number of cubic generators would be 28, both numbers differing from the numerical results obtained in Section 2.1. What we can still conjecture is:

Conjecture 2.4.9. *The reciprocal variety* $\mathbb{P}\mathcal{C}^{-1}$ *is defined by exactly* 27 *cubic equations which are Pfaffians of at least two* 7×7 *skew-symmetric matrices.*

Chapter 3

The natural rank and the local cactus rank of ternary forms

In this chapter, we investigate the natural rank and the local catcus rank of general ternary forms of any degree. These notions of rank, introduced in [6] and [7], approximate that of cactus rank, first presented in [15].

In Section 3.1, we set all the definitions and notation. A crucial notion will be that of a catalecticant matrix associated with inhomogeneous polynomials. The importance of these matrices is highlighted in Key Lemma 3.1.11.

In Section 3.2, we explore some first easy examples, where the two ranks can be computed straightforwardly by understanding the geometry of plane curves.

In Section 3.3, we explain a general approach for computing the natural rank, yielding the closed formula of Theorem 3.3.1.

In Section 3.4, we develop generalized procedures for computing the local cactus rank, obtaining the closed formula in Proposition 3.4.1 holding for small degrees. We then conjecture that the formula holds for any degree.

We conclude with an extended example for the case of ternary quintics, which is illustrated in Section 3.5.

3.1 Preliminaries

In Chapter 1, Section 1.1, we introduced apolarity and catalecticant matrices. For any $(n+1)$-dimensional $\mathbb{C}$-vector space V generated by $x_0, \dots, x_n$, we consider the graded symmetric algebra $\bar{S} := S(V) = \mathbb{C}[x_0, \dots, x_n]$ and we denote with $\bar{S}_d$ its d-graded part. For any $F \in \bar{S}_d$ and $l \in \bar{S}_1$ we have a *dehomogenization* of F with respect to l, defined as the residue class of F in the quotient $\bar{S}/(l-1)$ and usually denoted by F_l.

In the main applications, we consider dehomogenizations with respect to $l = x_0$, so we identify the quotient space with $S = \mathbb{C}[x_1, \dots, x_n]$ and the projection map with

$$\begin{array}{rccc} \pi_{x_0}\colon & \bar{S} & \to & S \\ & F(x_0, x_1 \dots, x_n) & \mapsto & F(1, x_1, \dots, x_n). \end{array}$$

We denote with $\bar{T} := S(V^*) = \mathbb{C}[\partial_0, \dots, \partial_n]$ the dual graded algebra of differential operators, acting on $\bar{S}$ by contraction. This induces contraction on dehomogenized spaces, that is an action of $T := \mathbb{C}[\partial_1, \dots, \partial_n]$ on S. We call the *order* of an operator $D \in T$, the smallest degree of a non-zero homogeneous term of D.

Definition 3.1.1. Given a homogeneous form $F \in \bar{S}$, its *apolar ideal*, denoted by $F^\perp$, is the ideal in $\bar{T}$ defined by the annihilator of F, i.e.

$$F^\perp := \{D \in \bar{T} \mid D(F) = 0\}.$$

Analogously, given an inhomogeneous polynomial $f \in S$, its apolar ideal is the annihilator of f in T and it is denoted by $f^\perp$.

The quotient ring $T_f := T/f^\perp$ is an Artinian Gorenstein graded local ring.

Since we are working in characteristic 0, apolarity is equivalent to standard differentiation.

Definition 3.1.2. For any $f \in S$ and for any $D \in T$, we call $D(f)$ a *partial* of f. The *order* of a partial g of $f \in S$ is the largest order of an operator $D \in T$ such that $g = D(f)$.

For every f in S, let us denote with $\mathrm{Diff}(f) = \{D(f) \mid D \in T\}$ the space of all partials of f. We have isomorphism of T-modules

$$\begin{array}{rcc} \tau \colon T_f & \xrightarrow{\sim} & \mathrm{Diff}(f) \\ D & \mapsto & D(f). \end{array}$$

In particular, we may interpret the Hilbert function of T_f in terms of a filtration of $\mathrm{Diff}(f)$. More precisely, let $\mathfrak{m}$ be the maximal ideal of T_f. The $\mathfrak{m}$-adic filtration

$$T_f = \mathfrak{m}^0 \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots \supset \mathfrak{m}^d \supset \mathfrak{m}^{d+1} = 0, \tag{3.1.1}$$

where $d = \deg(f)$, defines an associated graded ring

$$T_f^* = \bigoplus_{i=0}^{d} \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

Definition 3.1.3. For any $f \in S$, the *Hilbert function* of f, denoted by H_f, is the Hilbert function of the graded ring T_f^*.

We also have a filtration

$$T_f \supset (0 : \mathfrak{m}^{d+1}) \supset (0 : \mathfrak{m}^d) \supset \cdots \supset (0 : \mathfrak{m}^2) \supset (0 : \mathfrak{m}), \tag{3.1.2}$$

inducing a sequence of ideals

$$C_k := \bigoplus_{i=0}^{d-k} C_{k,i} \subset T_f^*,$$

where $k = 0, 1, 2 \dots$ and each graded piece is defined to be

$$C_{k,i} := \frac{(0 : \mathfrak{m}^{d+1-k-i}) \cap \mathfrak{m}^i}{(0 : \mathfrak{m}^{d+1-k-i}) \cap \mathfrak{m}^{i+1}}.$$

We then consider the quotient modules

$$Q_k := C_k/C_{k+1}, \qquad k = 0, 1, 2, \dots$$

with Hilbert functions

$$\Delta_{Q,k} := H(Q_k).$$

The decomposition of C_k induces a decomposition of Q_k as a direct sum $Q_k = \oplus_{i=0}^{d} Q_{k,i}$, where $Q_{k,i} = C_{k,i}/C_{k,i+1}$. When $d \geq 2$ and $k > d-2$, we have $C_k = Q_k = 0$.

Proposition 3.1.4 ([35], Theorem 1.5)**.** *The T_f^*-modules Q_k satisfy the reflexivity condition*

$$Q_{k,i}^{\vee} \simeq Q_{k,d-k-i}.$$

In particular, the Hilbert function $\Delta_{Q,k} = H(Q_k)$ is symmetric around $(d-k)/2$. Therefore the Hilbert function of T_f^* has a symmetric decomposition

$$H(T_f^*) = \sum_k \Delta_{Q,k}.$$

Let us denote with $\mathrm{Diff}(f)_i \subset \mathrm{Diff}(f)$ the subspace of partials of f of degree at most i. The isomorphism τ sends $(0 : \mathfrak{m}^i)$ to $\mathrm{Diff}(f)_{i-1}$, so (3.1.2) induces a *degree filtration*

$$\mathbb{C} = \mathrm{Diff}(f)_0 \subset \mathrm{Diff}(f)_1 \subset \mathrm{Diff}(f)_2 \subset \cdots \subset \mathrm{Diff}(f)_d = \mathrm{Diff}(f). \tag{3.1.3}$$

Since we have isomorphism

$$(0 : \mathfrak{m}^i)/(0 : \mathfrak{m}^{i-1}) \simeq (\mathfrak{m}^{i-1}/\mathfrak{m}^i)^{\vee},$$

the Hilbert function $H_f = H(T_f^*)$ can be expressed as

$$H_f(0) = 1, \qquad H_f(i) = \dim_{\mathbb{C}} \mathrm{Diff}(f)_i - \dim_{\mathbb{C}} \mathrm{Diff}(f)_{i-1},$$

where $i = 1, \dots, d$.

On the other hand (3.1.1) induces an *order filtration* on $\mathrm{Diff}(f)$. Indeed, the image $\tau(\mathfrak{m}^i) \subseteq \mathrm{Diff}(f)_{d-i}$ is the space of partials of order at least i.

So let us denote with $\mathrm{Diff}(f)_i^k \subset \mathrm{Diff}(f)$ the space of partials of degree at most i and order at least $d-i-k$. By Proposition 3.1.4, we have

$$Q_{k,i}^{\vee} \simeq \frac{\mathrm{Diff}(f)_i^k}{(\mathrm{Diff}(f)_i^{k-1} + \mathrm{Diff}(f)_{i-1}^{k+1})},$$

so we may think of $Q_{k,i}^{\vee}$ as the space parametrizing the partials of f of degree equal to i and order equal to $d-k-i$. By Proposition 3.1.4, the sequences

$$\Delta_{f,k}(i) := \dim(Q_{k,i}^{\vee})$$

satisfy $\Delta_{f,k}(i) = \Delta_{f,k}(d-k-1)$ and they induce a symmetric decomposition $H_f = \sum_{k=0}^{d-2} \Delta_{f,k}$ for the Hilbert function of f.

Example 3.1.5. Let $f = x_1^5 + x_1^4 + x_1x_2^2$. To determine the generators for the spaces of partials $Q_k^\vee$, it is enough to compute its partials with respect to bases for the graded parts T_k. We may use $\partial_1^k\partial_2^{k-1}(f)$ for every $k = 0, \dots, 5$ and $i = 0, \dots, k$. We obtain

$$\begin{array}{lll} \partial_1^4(f) = x_1 + 1 & \partial_1^2(f) = x_1^3 + x_1^2 & \partial_1(f) = x_1^4 + x_1^3 + x_1^2 \\ \partial_1^3(f) = x_1^2 + x_1 & \partial_1\partial_2(f) = x_2 & \partial_2(f) = x_1x_2 \\ & \partial_2^2(f) = x_1, & \end{array}$$

where operators yielding constant partials are omitted. Therefore we have generators of spaces of partials

degree	0	1	2	3	4	5
$Q_0^\vee$	1	x_1	x_1^2	x_1^3	x_1^4	f
$Q_1^\vee$	0	0	0	0	0	
$Q_2^\vee$	0	x_2	x_1x_2	0		

and the associated symmetric decomposition for the Hilbert function of f:

degree	0	1	2	3	4	5
$\Delta_{f,0}$	1	1	1	1	1	1
$\Delta_{f,1}$	0	0	0	0	0	
$\Delta_{f,2}$	0	1	1	0		
H_f	1	2	2	1	1	1

.

$\diamondsuit$

Every partial sum $\sum_{\alpha=0}^{k} \Delta_{f,k}(i)$ is the Hilbert function of a $\mathbb{C}$-algebra generated in degree 1 (cfr. [34], Section 5B). In particular, they have positive values and satisfy the *Macaulay growth conditions*: If, for fixed i, k, we have binomial expansion

$$\sum_{i=0}^{k} \Delta_{f,k}(i) = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j} \tag{3.1.4}$$

with $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$, then

$$\sum_{i=0}^{k} \Delta_{f,k}(i+1) \leq \binom{m_i+1}{i+1} + \binom{m_{i-1}+1}{i} + \cdots + \binom{m_j+1}{j+1}. \tag{3.1.5}$$

We will later be interested in characterizing spaces of polynomials f with $H_f(i) = 1$ for every $i = 0, \dots, d$. For these polynomials, the generators for the spaces of partials are essentially determined by the choice of a linear form. We denote with

$$\mathrm{Lin}(f) =: \mathrm{Diff}(f) \cap S_1 \tag{3.1.6}$$

the space of linear forms that are partials of f. In Example 3.1.5, $\mathrm{Lin}(f) = \langle x_1, x_2 \rangle = S_1$ is maximal.

In [7], Bernardi, Jelisiejew, Marques and Ranestad characterize polynomials with given Hilbert function via "standard" and "exotic" forms. Here we use a different approach, based on the study of the catalecticant matrix associated with an inhomogeneous polynomial. We now give the formal definition of this matrix, which will play a central role throughout the rest of this chapter.

Definition 3.1.6. Let $f = \sum_{i=0}^{d} f_i$ be the decomposition in homogeneous summands of a degree d polynomial in S. The *catalecticant matrix* of f is defined to be

$$C(f) := \begin{bmatrix} \mathrm{Cat}(0, f_0) & \mathrm{Cat}(0, f_1) & \cdots & \mathrm{Cat}(0, f_{d-1}) & \mathrm{Cat}(0, f_d) \\ \mathrm{Cat}(1, f_1) & \mathrm{Cat}(1, f_2) & \cdots & \mathrm{Cat}(1, f_d) & 0 \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ \mathrm{Cat}(d-1, f_{d-1}) & \mathrm{Cat}(d-1, f_d) & 0 & \cdots & 0 \\ \mathrm{Cat}(d, f_d) & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Remark 3.1.7. The matrix $C(f)$ is a $(d+1) \times (d+1)$ symmetric block matrix. The rows in

$$[\,\mathrm{Cat}(k, f_k) \quad \mathrm{Cat}(k, f_{k+1}) \quad \mathrm{Cat}(k, f_d) \quad 0 \quad \cdots \quad 0\,]$$

correspond to partials of f of order k. More precisely, the (i,j)-th entry of $\mathrm{Cat}(k, f_{k+l})$ is the coefficient of $x_1^{l-j} x_2^{j}$ for the partial $\partial_1^{k-i} \partial_2^{i}(f)$.

For every $k = 0, \ldots, d$, the matrix $\mathrm{Cat}(k, f_d)$ gives the coefficients of partials of f of order equal to k and degree equal to $d-k$. Hence we have a relation with the first symmetric component of the Hilbert function of f:

$$\Delta_{f,0}(i) = \mathrm{rk}(\mathrm{Cat}(i, f_d)).$$

Remark 3.1.8. The matrix $C(f)$ is the square catalecticant matrix of a homogeneous ternary form of degree $2d$, namely $x_0^d F$, where F is the homogenization of f with respect to x_0:

$$F = f_d + x_0 f_{d-1} + x_0^2 f_{d-2} + \cdots + x_0^d f_0.$$

Example 3.1.9. Let $f = x_1^2 x_2 + x_1^2 + x_2$. Then the catalecticant matrix of f is

$$C(f) = \left[\begin{array}{c|cc|ccc|cccc} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \hline \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right].$$

$\diamondsuit$

In addition Remark 3.1.7, we can observe a further relation between $C(f)$ and the Hilbert function of f. For every $k = 0, \dots, d$, let us consider the submatrices of $C(f)$ defined by its last $k+1$ rows of blocks:

$$C_k(f) := \begin{bmatrix} \mathrm{Cat}(d-k, f_{d-k}) & \mathrm{Cat}(d-k, f_{d-k-1}) & \cdots & \mathrm{Cat}(d-k, f_{d-1}) & \mathrm{Cat}(d-k, f_d) \\ \mathrm{Cat}(d-k+1, f_{d-k+1}) & \mathrm{Cat}(d-k+1, f_{d-k+2}) & \cdots & \mathrm{Cat}(d-k+1, f_d) & 0 \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ \mathrm{Cat}(d-1, f_{d-1}) & \mathrm{Cat}(d-1, f_d) & 0 & \cdots & 0 \\ \mathrm{Cat}(d, f_d) & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

In other words, the rows of $C_k(f)$ correspond to partials in $\mathrm{Diff}(g)^0_k$. In particular, for every $k \leq d$, the matrix $C_k(f)$ is a submatrix of $C_{k+1}(f)$ and we have a strictly increasing sequence of ranks

$$\mathrm{rk}(C_0(f)) < \mathrm{rk}(C_1(f)) < \cdots < \mathrm{rk}(C_d(f)). \tag{3.1.7}$$

Definition 3.1.10. Let $f \in S$ be a polynomial of degree d, with Hilbert function $H_f = (H_f(i_0), \dots, H_f(i_d))$. We define the *length* of H_f to be

$$\mathrm{len}(H_f) := \sum_{a=0}^{d} H_f(i_a).$$

The set of Hilbert functions admits a (trivial) total ordering: let f, g be two polynomials of possibly different degree, with Hilbert functions H_f and H_g respectively. Then we say that $H_f \leq H_g$ if and only if $\mathrm{len}(H_f) \leq \mathrm{len}(H_g)$.

Note that this ordering is "coarse", since it does not compare all the single values of Hilbert functions, but only their sum. Therefore, two Hilbert functions with different values may have equal length.

We have the following key result:

Lemma 3.1.11 (Key lemma)**.** *Let f be a polynomial of degree d. Then the following are equivalent:*

(1) *For every $0 \leq i \leq d$ we have $\sum_{j=0}^{i} H_f(j) \leq \sum_{j=0}^{i} n_j$;*

(2) *For every $0 \leq i \leq d$ we have $\mathrm{rk}(C_i(f)) \leq \sum_{j=0}^{i} n_j$.*

In particular, the length of H_f is equal to the rank of $C(f)$.

Proof. It follows from:

$$\mathrm{rk}(C_i(f)) = \dim(\mathrm{Diff}(f)^0_i) = \sum_{j=0}^{i}\sum_{k=0}^{j} \dim(Q^\vee_{k,j}) = \sum_{j=0}^{i}\sum_{k=0}^{j} \Delta_{f,k}(j) = \sum_{j=0}^{i} H_f(j).$$

$\blacksquare$

Remark 3.1.12. The equivalent conditions of Key Lemma allow us to describe the set of polynomials of degree d with a given Hilbert function by means of homogeneous ideals, giving the set a subscheme structure in the projective space $\mathbb{P}(\oplus_{i=0}^{d} S_i)$, see example Example 3.1.13 and Lemma 3.4.4.

Moreover, the conditions of Key Lemma explain how total ordering on the set of Hilbert functions induces a partial ordering (by inclusion) on varieties of polynomials with Hilbert function of fixed length. Explicitly, for any two degrees $d \leq e$ and for any positive integers $\lambda \leq \mu$, we may consider the two varieties

$$\mathcal{F} = \left\{ f \in \mathbb{P}(\oplus_{i=0}^{d} S_i) \mid \mathrm{len}(H_f) \leq \lambda \right\} \subseteq \mathbb{P}(\oplus_{i=0}^{d} S_i)$$

and

$$\mathcal{G} = \{ g \in \mathbb{P}(\oplus_{i=0}^{e} S_i) \mid \mathrm{len}(H_g) \leq \mu \} \subseteq \mathbb{P}(\oplus_{i=0}^{e} S_i).$$

The natural inclusion $\mathbb{P}(\oplus_{i=0}^{d} S_i) \hookrightarrow \mathbb{P}(\oplus_{i=0}^{e} S_i)$ makes the two matrices $C(f)$ and $C(g)$ comparable, as catalecticant matrices of degree-e polynomials. The conditions of Lemma 3.1.11 for the length of Hilbert functions of elements in $f \in \mathcal{F}$ and $g \in \mathcal{G}$ imply that generically

$$\lambda = \mathrm{rk}(C(f)) \leq \mathrm{rk}(C(g)) = \mu,$$

hence $\mathcal{F} \subseteq \mathcal{G}$, as subvarieties of $\mathbb{P}(\oplus_{i=0}^{e} S_i)$.

Varieties of polynomials of fixed degree and with Hilbert function of fixed length are usually reducible. Among the small-dimensional components, one typically finds subvarieties of polynomials of smaller degree.

Example 3.1.13. The variety of cubic polynomials f in two variables with $\mathrm{len}(H_f) \leq 5$ is defined by the determinantal equation on the generic catalecticat matrix:

$$\mathrm{rk} \begin{bmatrix} a_{(0,0)} & a_{(1,0)} & a_{(0,1)} & a_{(2,0)} & a_{(1,1)} & a_{(0,2)} & a_{(3,0)} & a_{(2,1)} & a_{(1,2)} \\ a_{(1,0)} & a_{(2,0)} & a_{(1,1)} & a_{(3,0)} & a_{(2,1)} & a_{(1,2)} & \cdot & \cdot & \cdot \\ a_{(0,1)} & a_{(1,1)} & a_{(0,2)} & a_{(2,1)} & a_{(1,2)} & a_{(0,3)} & \cdot & \cdot & \cdot \\ a_{(2,0)} & a_{(3,0)} & a_{(2,1)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{(1,1)} & a_{(2,1)} & a_{(1,2)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{(0,2)} & a_{(1,2)} & a_{(0,3)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{(3,0)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{(2,1)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{(1,2)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{(0,3)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \leq 5.$$

This is a subvariety of $\mathbb{P}(\oplus_{i=0}^{3} S_i) \simeq \mathbb{P}^9$ of dimension 7, degree 9, which is defined by 48 sextics. As a scheme, it has two irreducible components, and it is not reduced.

The first component, of dimension 7 and degree 9, is defined by the square of the ideal

$$(a_{(2,1)}^2 - a_{(1,2)} a_{(3,0)},\ a_{(1,2)} a_{(2,1)} - a_{(0,3)} a_{(3,0)},\ a_{(1,2)}^2 - a_{(0,3)} a_{(2,1)}),$$

that is the ideal defining the cubic polynomials whose cubic term is a pure power of a linear form. The general element of this component has Hilbert function $(1, 2, 1, 1)$ and in fact its defining equations are given by imposing $H_f(2) \leq 1$.

The second one is an embedded (fat) component of dimension 5 and degree 50, whose radical ideal is simply

$$(a_{(3,0)}, a_{(2,1)}, a_{(1,2)}, a_{(0,3)}),$$

that is the ideal cutting the linear $\mathbb{P}^5$ of quadrics in the space of cubics. ♢

In Section 3.4, we will be interested in maximal-dimensional components of varieties of polynomials of degree e and with Hilbert function of length at most $e+1$. The next Proposition 3.1.14 shows that among these maximal components we find one whose generic element f has *unitary* Hilbert function, namely it satisfies $H_f = (1, 1, \dots, 1)$.

Proposition 3.1.14. *Let $S = \mathbb{C}[x_1, \dots, x_n]$, let $d \leq e$ be two positive integers and consider the following subvarieties of polynomials:*

$$\mathcal{F} = \left\{ f \in \mathbb{P}(\oplus_{i=0}^{d} S_i) \mid \operatorname{len}(H_f) \leq e+1 \right\}$$
$$\mathcal{G} = \left\{ g \in \mathbb{P}(\oplus_{i=0}^{e} S_i) \mid H_g(i) \leq 1, \ \forall i = 0 \dots, e \right\},$$

Then $\mathcal{F} \subseteq \mathcal{G}$, as subvarieties of $\mathbb{P}(\oplus_{i=0}^{e} S_i)$.

Proof. Any element $f \in \mathcal{F}$ can be regarded as a polynomial of degree e via the natural inclusion $\iota \colon \mathbb{P}(\oplus_{i=0}^{d} S_i) \hookrightarrow \mathbb{P}(\oplus_{i=0}^{e} S_i)$. By Key Lemma 3.1.11, we need to show that $\phi := \iota(f)$ satisfies

$$\operatorname{rk}(C_i(\phi)) \leq i+1, \qquad \text{for every} \quad i = 0, \dots, e,$$

which, in terms of the original catalecticant matrix becomes

$$\operatorname{rk}(C_i(f)) \leq i+1+e-d, \qquad \text{for every} \quad i = 0, \dots, d. \tag{3.1.8}$$

We now recall from (3.1.7) that we have a strictly increasing sequence of ranks of submatrices $C_i(f)$. If we assume that there exists $i_0 \leq d$ such that the inequality (3.1.8) is not satisfied, say

$$\operatorname{rk}(C_{i_0}(f)) = i_0 + 2 + e - d,$$

then for $i = d$ we get

$$\operatorname{len}(H_f) = \operatorname{rk}(C(f)) = \operatorname{rk}(C_d(f)) \geq 2 + e - d,$$

a contradiction. ∎

Note that a priori there may be several components whose general element has unitary Hilbert function and these components may not be all of maximal dimension. Proposition 3.1.14 only shows that we find at least one maximal component of this kind. We will see later in Proposition 3.4.7 that for polynomials in two variables, this is the only maximal component.

3.1.1 Apolar schemes, natural rank, cactus rank

We have seen that to any (in)homogeneous polynomial we can associate a natural scheme by quotienting the ring of operators with the apolar ideal.

Definition 3.1.15. Let $F \in \bar{S}$ be any homogeneous form and let $l \in \bar{S}_1$ be linear. The *natural apolar scheme* to F *at* l is $\operatorname{Spec}(T/F_l^{\perp})$.

The *natural rank* of F *at* l is the length of the natural apolar scheme to F at l or, equivalently, the length of H_{F_l}.

The *natural rank* of F is

$$\operatorname{nat}(F) := \min\left\{\operatorname{len}(H_{F_l}) \mid l \in \mathbb{P}(S_1)\right\}.$$

The notion of apolar scheme can be extended:

Definition 3.1.16. A subscheme $Z \subset \operatorname{Proj}(\bar{T})$ is an *apolar scheme* to F if its ideal $I(Z)$ is contained in $F^{\perp}$ and analogously for inhomogeneous forms $f \in S$.

As a consequence, also the definition of natural rank is extended to the following:

Definition 3.1.17. The *cactus rank* of a homogeneous form $F \in \bar{S}$ is the minimal length of a zero-dimensional scheme apolar to F:

$$\operatorname{cr}(F) = \min\left\{\operatorname{len}(Z) \;\middle|\; \begin{array}{c} Z \subset \operatorname{Proj}(\bar{T}), \quad \dim(Z) = 0, \\ I(Z) \subset F^{\perp} \end{array}\right\}.$$

The *local cactus rank* of a homogeneous form $F \in \bar{S}$ is the minimal length of a zero-dimensional scheme apolar to F supported at a single point:

$$\operatorname{lcr}(F) = \min\left\{\operatorname{len}(Z) \;\middle|\; \begin{array}{c} Z \subset \operatorname{Proj}(\bar{T}), \quad \dim(Z) = 0, \\ \operatorname{Supp}(Z) = \{\mathrm{pt}\}, \quad I(Z) \subset F^{\perp} \end{array}\right\}.$$

By definition, for any $F \in \bar{S}$ we have $\operatorname{lcr}(F) \leq \operatorname{nat}(F)$. In Definition 3.1.15 we explain how to compute the natural rank in terms of Hilbert functions of dehomogenizations. Similarly, we may compute the local cactus rank in terms of the Hilbert function of a suitable quotient ring by using the following:

Proposition 3.1.18. *Let $F \in \bar{S}_d$ and let $f = F_{x_0}$ be its dehomogenization. Suppose that Γ is a zero-dimensional scheme of minimal length among the ones apolar to F that are supported at the point $[x_0] = [1 : 0 : \cdots : 0] \in \bar{T}$. Then Γ is the apolar scheme of an affine polynomial g whose degree-d tail $g_{\leq d}$ equals f.*

A typical way to use Proposition 3.1.18 is the following: let F be homogeneous of degree d and let us assume that the natural bound for the local cactus rank is given, say $\operatorname{lcr}(F) \leq k = \operatorname{nat}(F)$. The bound is strict if and only if there exists a polynomial g in two variables such that

- $\deg(g) > \deg(F)$
- there exists a linear form l such that $F_l = g_{\leq d}$;

- $\operatorname{len}(H_g) < k$.

These conditions immediately give some restrictions for the Hilbert function of g, for which we have a finite number of cases (thanks to Macaulay's growth conditions (3.1.4)-(3.1.5)).

In Section 3.2, we directly apply the strategy explained above for the case of binary forms and ternary forms of degree 3. Explicit computations can be easily given for two main reasons: first, the set of possible Hilbert functions H_g to check is very small; second, the conditions for a polynomial to be the tail of such g has a well-understood geometric meaning.

For a more general and systematic treatment we proceed with Section 3.3 and Section 3.4.

Throughout this chapter, polynomials in n variables of degree d and homogeneous $(n+1)$-ary forms of degree d will be treated as points in projective spaces, so we will denote them with $f \in \mathbb{P}(\oplus_{i=0}^{d} S_i)$ and $F \in \mathbb{P}(\bar{S}_d)$, respectively.

3.2 First examples

In this section we compute the natural rank and the local cactus rank for general binary forms of any degree and for general ternary cubics.

3.2.1 Binary forms

The case of binary forms is classical and very simple. So let $\bar{S} = \mathbb{C}[x_0, x_1]$ and let $F \in \mathbb{P}(\bar{S}_d)$ be a general degree-d form. For any $l \in \mathbb{P}(\bar{S}_1)$, the dehomogenization F_l is a polynomial in one variable. Then we have:

Proposition 3.2.1. *For a general binary form of degree d, the natural rank and the local cactus rank are both equal to d.*

Proof. Every binary form of degree d can be written as

$$F = \prod_{\substack{i=1,\\ m_1+\cdots+m_k=d}}^{k} (\alpha_i x_0 + \beta_i x_1)^{m_i},$$

whose zero locus $V(F)$ is a collection of points $P_i = [\beta_i : -\alpha_i]$, each one counted with multiplicity m_i. For a general F, we have $m_i = 1$ for every i.

For any $l \in \mathbb{P}(\bar{S}_1)$, the dehomogenization F_l is a polynomial in one variable whose degree can be either $d-1$ or d, depending on whether l divides F or not. The associated Hilbert function will be unitary in both cases, i.e. $H_{F_l} = (1, \ldots, 1)$, so the minimal one will have length d.

This computes the local cactus rank as well: if there was another affine form $g \in \mathbb{C}[x_1]$ defining a zero dimensional apolar scheme of minimal length, it would be of degree $\deg(g) \geq d+1$. Its Hilbert function, also unitary, will have length at least $d+1$, contradicting minimality. ∎

3.2.2 Ternary cubics

Let $\bar{S} = \mathbb{C}[x_0, x_1, x_2]$ and consider a general form $F = \sum_{i+j\le d} a_{(i,j)} x_1^i x_2^j x_0^{3-i-j}$ in $\mathbb{P}(\bar{S}_3)$. Its zero locus is a cubic plane curve, so for every linear form $l \in \mathbb{P}(\bar{S}_1)$, the intersection $V(F,l)$ can only have one of the following configurations: $3P$, $2P+Q$ or $P+Q+R$, as elements of the free abelian group generated by the points of $V(l)$.

Proposition 3.2.2. *The natural rank and the local cactus rank of a general ternary cubic are both equal to* 5*.*

First we examine the possibilities for the natural rank by analyzing the geometry of the three possible configurations of points.

Lemma 3.2.3. *If $V(F) \cap V(l) = 3P$, then either F is smooth at P and $H_{F_l} = (1,2,1,1)$, or $H_{F_l} = (1,1,1,1)$.*

Proof. Without loss of generality, we may assume that $l = x_0$ and $P = [0:0:1]$. Let us write

$$F = f_3 + x_0 f_2 + x_0^2 f_1 + x_0^3 f_0, \tag{3.2.1}$$

where each f_i is a polynomial of degree i in the variables x_1, x_2. Imposing multiplicity 3 at P, we must have $f_3 = a_{(3,0)} x_1^3$ (and $a_{(3,0)} \neq 0$). Let us for short denote the dehomogenized form F_l with $f = \sum_{i=0}^3 f_i$.

The space of partial derivatives of degree 2 is generated by $\partial_1(f)$, while in degree 1 the generators are $\{\partial_1^2(f), \partial_2(f)\}$. In particular:

$$\begin{aligned} \partial_1^2(f) &= a_{(3,0)} x_1 + \partial_1^2(f_2) && \text{of degree } = 1 \\ \partial_2(f) &= a_{(1,1)} x_1 + a_{(0,2)} x_2 + \partial_2(f_1) && \text{of degree } \le 1. \end{aligned}$$

Therefore, the dimension of this space is either 1 or 2, depending on whether $a_{(0,2)}$ is zero or not. Geometrically, this condition translates into asking if $V(F)$ is singular at P. So, if P is a (smooth) flex point of $V(F)$ and $V(l)$ is its tangent line, then $H_{F_l} = (1,2,1,1)$, while if P is a singular point of $V(F)$ and the multiplicity of $V(F) \cap V(l)$ at P is 3, then $H_{F_l} = (1,1,1,1)$. ■

Lemma 3.2.4. *If $V(F) \cap V(l) = 2P + Q$, with $P \neq Q$, then $H_{F_l} = (1,2,2,1)$.*

Proof. Again, without loss of generality we may assume that $l = x_0$, $P = [0:0:1]$ and $Q = [0:1:0]$. Keeping the same notation as (3.2.1), we must have $f_3 = a_{(2,1)} x_1^2 x_2$ (and $a_{(2,1)} \neq 0$). Then the spaces of partials of degree 1 and 2 have both maximal dimension, with bases $\{\partial_2\partial_1(f), \partial_1^2(f)\}$ and $\{\partial_2(f), \partial_1(f)\}$ respectively. Indeed, in all these expressions, the leading term has non-vanishing coefficient:

$$\begin{aligned} \partial_1^2(f) &= a_{(2,1)} x_2 + \partial_1^2(f_2) & \partial_1(f) &= a_{(2,1)} x_2 x_1 + \partial_1(f_2) + \partial_1(f_1) \\ \partial_2\partial_1(f) &= a_{(2,1)} x_1 + \partial_2\partial_1(f_2) & \partial_2(f) &= a_{(2,1)} x_1^2 + \partial_2(f_2) + \partial_2(f_1). \end{aligned}$$

So, independently on P being smooth or not for $V(F)$, we have $H_{F_l} = (1,2,2,1)$. ■

Lemma 3.2.5. *If $V(F) \cap V(l) = P + Q + R$, with P, Q, R, distinct, then $H_{F_l} = (1, 2, 2, 1)$.*

Proof. Both the configuration of points and the Hilbert function are the general ones. ■

Proof of Proposition 3.2.2. Let F be a general smooth cubic ternary form. The natural rank is computed as the least number among the lengths of the three Hilbert functions studied in Lemmas 3.2.3 to 3.2.5.

Every general smooth plane cubic curve admits 9 flexes. So taking P to be one of the flex points of $V(F)$ and l the form defining the tangent line at P, we end up in the (smooth) case of Lemma 3.2.3, yielding natural rank 5.

As a consequence, $\mathrm{lcr}(F) \leq 5$. If the inequality was strict, then there would exist a polynomial g in two variables with $\deg(g) \geq 4$, whose cubic tail would coincide with F_l for some l and such that $\mathrm{len}(H_g) \leq 4$. But being g of degree at least 4, its Hilbert function would be at least $H_g = (1, 1, 1, 1, 1)$, a contradiction. ■

3.3 The natural rank of ternary forms

The main result of this section gives a closed formula for computing the natural rank of a general degree-d ternary form.

Theorem 3.3.1. *The natural rank of a general ternary form $F \in \mathbb{P}(\bar{S}_d)$ of degree d is:*

$$\mathrm{nat}(F) = \left\lfloor \frac{d(d+4)}{4} \right\rfloor .$$

When $d = 2k+1$ is odd, the natural rank is realized at $\frac{3(k+1)(k+2)(3k^2+3k+1)}{2}$ points of $\mathbb{P}(S_1)$. When $d = 2k$ is even, the natural rank is realized at a degree $3k(k+1)$ curve of $\mathbb{P}(S_1)$.

The claimed expression for the natural rank coincides with the length of the maximal Hilbert function for a polynomial of degree d in two variables, decreased by one. Explicitly, for a polynomial $f \in \mathbb{P}(\oplus_{i=0}^{d} S_i)$ the maximal Hilbert function is

$$\begin{array}{lll} \left(1, 2, \ldots, \frac{d-1}{2}, \frac{d+1}{2}, \frac{d+1}{2}, \frac{d-1}{2}, \ldots, 2, 1\right) & \text{if} & d = 2k+1 \\ \left(1, 2, \ldots, \frac{d}{2}, \frac{d}{2}+1, \frac{d}{2}, \ldots, 2, 1\right) & \text{if} & d = 2k. \end{array}$$

If we denote its length by

$$\lambda_d := \left\lfloor \frac{(d+2)^2}{4} \right\rfloor ,$$

the above Theorem 3.3.1 says that the natural rank of a general ternary form of degree d is equal to $\lambda_d - 1$.

The following lemmas characterize those polynomials f whose Hilbert function has length $\lambda_d - 1$ and $\lambda_d - 2$.

Lemma 3.3.2. *Let $S = \mathbb{C}[x_1, x_2]$ and let $f = \sum_{i=0}^{d} f_i$ be a degree-d polynomial in $\mathbb{P}(\oplus_{i=0}^{d} S_i)$. Setting $k = \lfloor \frac{d}{2} \rfloor$, we have the following equivalent conditions:*

(1) *The length of H_f is at most $\lambda_d - 1$;*

(2) *The rank of $\mathrm{Cat}(k, f_d)$ is at most k;*

(3) *The point $[f_d]$ belongs to $\sigma_k(\nu_d(\mathbb{P}^1))$.*

Proof. When $d = 2k+1$ is odd, the unique Hilbert function of length $\lambda_d - 1$ decomposes as:

degree	0	1	$\cdots$	$\frac{d-3}{2}$	$\frac{d-1}{2}$	$\frac{d+1}{2}$	$\frac{d+3}{2}$	$\cdots$	$d-1$	d
$\Delta_{f,0}$	1	2	$\cdots$	$\frac{d-1}{2}$	$\frac{d-1}{2}$	$\frac{d-1}{2}$	$\frac{d-1}{2}$	$\cdots$	2	1
$\Delta_{f,1}$	0	0	$\cdots$	0	1	0	0	$\cdots$	0	
H_f	1	2	$\cdots$	$\frac{d-1}{2}$	$\frac{d-1}{2}$	$\frac{d-1}{2}$	$\frac{d-1}{2}$	$\cdots$	2	1

. (3.3.1)

Therefore, $\mathrm{len}(H_f) \leq \lambda_d - 1$ if and only if $H_f\left(\frac{d-1}{2}\right) \leq \frac{d-1}{2}$, if and only if $\Delta_{f,0}(k) \leq k$.

When $d = 2k$ is even, the unique Hilbert function of length $\lambda_d - 1$ is symmetric and it decomposes as:

degree	0	1	$\cdots$	$\frac{d}{2}-1$	$\frac{d}{2}$	$\frac{d}{2}+1$	$\cdots$	$d-1$	d
$\Delta_{f,0}$	1	2	$\cdots$	$\frac{d}{2}$	$\frac{d}{2}$	$\frac{d}{2}$	$\cdots$	2	1
H_f	1	2	$\cdots$	$\frac{d}{2}$	$\frac{d}{2}$	$\frac{d}{2}$	$\cdots$	2	1

.

Therefore, $\mathrm{len}(H_f) \leq \lambda_d - 1$ if and only if $H_f\left(\frac{d}{2}\right) \leq \frac{d}{2}$, if and only if $\Delta_{f,0}(k) \leq k$. The equivalence of (1) and (2) follows from recalling that for every $0 \leq m \leq d$ we have $\Delta_{f,0}(m) = \mathrm{rk}(\mathrm{Cat}(m, f_d))$ (cfr. Remark 3.1.7).

The equivalence of (2) and (3) is trivial. ∎

In the same hypotheses of Lemma 3.3.2, we have a similar statement for Hilbert functions of length $\lambda_d - 2$. The odd and even case are studied separately.

Lemma 3.3.3. *When $d = 2k$ is even, the following are equivalent:*

(1) *The length of H_f is at most $\lambda_d - 2$;*

(2) *The rank of $\mathrm{Cat}(k, f_d)$ is at most $k-1$;*

(3) *The point $[f_d]$ belongs to $\sigma_{k-1}(\nu_d(\mathbb{P}^1))$.*

Proof. The unique Hilbert function of length $\lambda_d - 2$ decomposes as:

degree	0	1	$\cdots$	$\frac{d}{2}-1$	$\frac{d}{2}$	$\frac{d}{2}+1$	$\cdots$	$d-1$	d
$\Delta_{f,0}$	1	2	$\cdots$	$\frac{d}{2}-1$	$\frac{d}{2}-1$	$\frac{d}{2}-1$	$\cdots$	2	1
$\Delta_{f,1}$	0	0	$\cdots$	1	1	0	$\cdots$	0	
H_f	1	2	$\cdots$	$\frac{d}{2}$	$\frac{d}{2}$	$\frac{d}{2}-1$	$\cdots$	2	1

.

Therefore, $\operatorname{len}(H_f) \leq \lambda_d - 2$ if and only if $H_f\left(\frac{d}{2}\right) \leq \frac{d}{2} - 1$, if and only if $\Delta_{f,0}(k) \leq k - 1$.

The equivalence with (3) is trivial. ■

Lemma 3.3.4. *When $d = 2k+1$ is odd, the following are equivalent:*

(1) *The length of H_f is at most $\lambda_d - 2$;*

(2) *The rank of $\operatorname{Cat}(k, f_d)$ is at most k and furthermore*

$$\operatorname{rk}\begin{bmatrix}\operatorname{Cat}(k, f_{d-1}) & \operatorname{Cat}(k, f_d)\\ \operatorname{Cat}(k+1, f_d) & 0\end{bmatrix} \leq 2k. \tag{3.3.2}$$

Proof. The unique Hilbert function of length $\lambda_d - 2$ decomposes symmetrically as:

degree	0	1	$\cdots$	$\frac{d-3}{2}$	$\frac{d-1}{2}$	$\frac{d+1}{2}$	$\frac{d+3}{2}$	$\cdots$	$d-1$	d
$\Delta_{f,0}$	1	2	$\cdots$	$\frac{d-1}{2}$	$\frac{d-1}{2}$	$\frac{d-1}{2}$	$\frac{d-1}{2}$	$\cdots$	2	1
H_f	1	2	$\cdots$	$\frac{d-1}{2}$	$\frac{d-1}{2}$	$\frac{d-1}{2}$	$\frac{d-1}{2}$	$\cdots$	2	1

.

Compared to (3.3.1), we have an additional condition. Explicitly: $\operatorname{len}(H_f) \leq \lambda_d - 2$ if and only if $\Delta_{f,0}(k) \leq k$ and $H_{f,1}\left(\frac{d-1}{2}\right) = 0$. The first condition is equivalent to $\operatorname{rk}(\operatorname{Cat}(k, f_d)) \leq k$, so by performing elementary operations on the columns of $\operatorname{Cat}(k, f_d)$, and symmetrically on the rows of $\operatorname{Cat}(k+1, f_d) = \operatorname{Cat}(k, f_d)^t$, we obtain:

$$\begin{bmatrix}\operatorname{Cat}(k, f_{d-1}) & \operatorname{Cat}(k, f_d)\\ \operatorname{Cat}(k+1, f_d) & 0\end{bmatrix} \simeq \left[\begin{array}{cc|ccc} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & B & 0 & 0\\ & & 0 & 0 & 0\\ \hline B^t & 0 & & & \\ 0 & 0 & & 0 & \\ 0 & 0 & & & \end{array}\right], \tag{3.3.3}$$

where A and B are square matrices of size $(k+1)$ and k respectively, generically of full rank.

The second condition is equivalent to saying that there are no partials of degree k and order $d - k - 1$. Once the first is satisfied, this is equivalent to saying that the contribution to the rank of the matrix in (3.3.3) only comes from B and B^t, that is $\operatorname{rk}\begin{bmatrix} A & B \\ B^t & 0 \end{bmatrix} \leq 2k$. ■

Remark 3.3.5. The right-hand side of (3.3.3) is identified with a square matrix of size $2k+1$. So, the rank-$2k$ condition defines a hypersurface in the space of polynomials f with $\operatorname{rk}(\operatorname{Cat}(k, f_d)) \leq k$.

We consider the incidence variety associated with dehomogenizations with Hilbert function of colength 1:

$$\mathcal{I}_1 = \left\{(F,l) \in \mathbb{P}(\bar{S}_d) \times \mathbb{P}(\bar{S}_1) \mid \operatorname{len}(H_{F_l}) \leq \lambda_d - 1\right\},$$

and analogously for colength 2:

$$\mathcal{I}_2 = \left\{(F,l) \in \mathbb{P}(\bar{S}_d) \times \mathbb{P}(\bar{S}_1) \mid \operatorname{len}(H_{F_l}) \leq \lambda_d - 2\right\}.$$

We have the natural inclusion $\mathcal{I}_2 \subseteq \mathcal{I}_1$, therefore a commutative diagram

$$\begin{tikzcd}
 & \mathbb{P}(\bar{S}_d) \times \mathbb{P}(\bar{S}_1) \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \mathcal{I}_1 \arrow[u, hook] \arrow[dl, "p_1"'] \arrow[dr, "q_1"] & \\
\mathbb{P}(\bar{S}_d) & \mathcal{I}_2 \arrow[u, hook] \arrow[l, "p_2"] \arrow[r, "q_2"'] & \mathbb{P}(\bar{S}_1)
\end{tikzcd}.$$

Keeping Lemmas 3.3.2 to 3.3.4 in mind, let I_r denote the ideal of r-minors of $\operatorname{Cat}(k, f_d)$ and let J denote the determinantal ideal giving the condition (3.3.2).

Corollary 3.3.6. *Let d be any positive integer and let $k = \lfloor \frac{d}{2} \rfloor$. For every $l \in \mathbb{P}(\bar{S}_1)$ we have*

$$q_1^{-1}(l) \simeq \mathbb{P}(\oplus_{i=0}^{d} S_i / I_{k+1}) \tag{3.3.4}$$

and

$$q_2^{-1}(l) \simeq \begin{cases} \mathbb{P}(\oplus_{i=0}^{d} S_i / I_k) & d = 2k \\ \mathbb{P}(\oplus_{i=0}^{d} S_i / (J + I_{k+1})) & d = 2k+1. \end{cases} \tag{3.3.5}$$

In particular, $q_2^{-1}(l)$ is a subvariety of $q_1^{-1}(l)$ of codimension 2 (resp. 1) when $d = 2k$ (resp. $d = 2k+1$).

Proof. Without loss of generality, we can reduce to study the fiber of $l = x_0$. Then the expression in (3.3.4) follows from Lemma 3.3.2 while the two cases in (3.3.5) follow from Lemma 3.3.3 and Lemma 3.3.4.

Moreover, when $d = 2k$, the codimension of $q_2^{-1}(l) \subseteq q_1^{-1}(l)$ is equal to the codimension of $\sigma_{k-1}(\nu_d(\mathbb{P}^1)) \subseteq \sigma_k(\nu_d(\mathbb{P}^1))$, that is 2. When $d = 2k+1$, codimension 1 follows from Remark 3.3.5. ■

The two following propositions altogether prove Theorem 3.3.1. Their proofs can be found at Pages 77 and 80.

Proposition 3.3.7. *For every d, the incidence variety $\mathcal{I}_1$ dominates $\mathbb{P}(\bar{S}_d)$. More precisely:*

- *If $d = 2k+1$, then p_1 is generically finite of degree $\frac{3(k+1)(k+2)(3k^2+3k+1)}{2}$;*
- *If $d = 2k$, then the fiber of p_1 is generically a curve of degree $3k(k+1)$.*

Proposition 3.3.8. *For every d, the incidence variety $\mathcal{I}_2$ does not dominate $\mathbb{P}(\bar{S}_d)$.*

Saying that p_1 is dominant for every d, means that the equivalent conditions of Lemma 3.3.2 generically hold. If this is the case, then $\lambda_d - 1$ is a bound for the natural rank of the general ternary form of degree d.

We address this problem by reducing to a Chern class computation. Let us denote with $V \simeq \mathbb{C}^3$ the vector space of linear forms in 3 variables, so that $\mathbb{P}(\bar{S}_1) \simeq \mathbb{P}V$. Writing $\mathcal{O} = \mathcal{O}_{\mathbb{P}V}$ for short, we have the standard Euler sequence:

$$0 \to \mathcal{O}(-1) \to V \otimes \mathcal{O} \to \mathcal{Q} \to 0.$$

More generally, for every positive integer m, we have a short exact sequence (cfr. [10, Ch. III, §6.2 Proposition 4]):

$$0 \to S^{m-1}V \otimes \mathcal{O}(-1) \to S^m V \otimes \mathcal{O} \to S^m \mathcal{Q} \to 0,$$

whose exactness is preserved by twists

$$0 \to S^{m-1}V \otimes \mathcal{O}(n-1) \xrightarrow{\alpha_m} S^m V \otimes \mathcal{O}(n) \xrightarrow{\beta_m} S^m \mathcal{Q}(n) \to 0 \tag{3.3.6}$$

and by dualization

$$0 \to S^m \mathcal{Q}^* \xrightarrow{\beta_m^*} S^m V^* \otimes \mathcal{O} \xrightarrow{\alpha_m^*} S^{m-1}V^* \otimes \mathcal{O}(1) \to 0. \tag{3.3.7}$$

The fibers of $S^m V \otimes \mathcal{O}(n)$ are homogeneous degree-m ternary forms. Their restriction to lines give fibers of $S^m \mathcal{Q}(n)$. In other words, for every linear form $l \in \mathbb{P}V$, the fiber of $S^m \mathcal{Q}(n)$ over l consists of the sets of m points on $V(l)$.

Example 3.3.9. Let $\{v_0, v_1, v_2\}$ be a basis for V and let $m = n = 2$. Then $S^2 V$ decomposes as $\langle v_0^{\otimes 2} \rangle \oplus \cdots \oplus \langle v_2^{\otimes 2} \rangle$. For every linear form $l = c_0 x_0 + c_1 x_1 + c_2 x_2$ in $\mathbb{P}(\bar{S}_1)$, the sequence (3.3.6) induces a short exact sequence on fibers

$$0 \to V \otimes \mathcal{O}(1)_{|_l} \xrightarrow{\alpha_{|_l}} S^2 V \otimes \mathcal{O}(2)_{|_l} \xrightarrow{\beta_{|_l}} S^2 \mathcal{Q}(2)_{|_l} \to 0,$$

where $\alpha_{|_l}(v_i) = v_i \otimes (c_0 v_0 + c_1 v_1 + c_2 v_2)$. Thus the fiber $S^2 \mathcal{Q}(2)_{|_l}$ consists of the degree-2 forms of the plane modulo l, that is, the space of quadrics on the line $\{l = 0\}$. ♢

In this section, we are concerned with maps between symmetric powers of $\mathcal{Q}$. Specifically, recall the notation from Chapter 1, Section 1.1: for every ternary form F of degree d and for every $k \leq d$, we have a linear map of k-contractions $\gamma_{k,F}\colon S^kV^* \to S^{d-k}V$, defined by $D \mapsto D(F)$ and represented by the catalecticant matrix $\mathrm{Cat}(k,F)$.

This induces a map $S^kV^* \to S^{d-k}V \otimes \mathcal{O}(d)$, constant on each fiber, so with slight abuse of notation we keep calling it $\gamma_{k,F}$. This in turn gives a map $\rho(\gamma_{k,F})\colon S^k\mathcal{Q}^* \to S^{d-k}\mathcal{Q}(d)$, defined by the composition

$$\rho(\gamma_{k,F}) := \beta_{d-k} \circ \gamma_{k,F} \circ \beta_k^*,$$

and whose fibers are defined by restriction on lines. Explicitly, if we write $F = \sum_{i=0}^{d} l^i f_{d-i}$, the map $\rho(\gamma_{k,F})_{|l}$ is represented by the catalecticant matrix $\mathrm{Cat}(k, f_d)$.

Example 3.3.10. Let us consider the forms $F = \sum_{i+j\leq d} a_{(i,j)} x_1^i x_2^j x_0^{d-i-j}$ and $l = x_0$. The homogeneous part of degree d in the dehomogenization F_{x_0} is $f_d = \sum_{i+j=d} a_{(i,j)} x_1^i x_2^{d-i}$. We fix standard bases

$$\mathcal{B}_{x,d-k} = \{x_0^{d-k},\, x_0x_1^{d-k-1}, \ldots, x_2^{d-k}\} \qquad \mathcal{B}_{\partial,k} := \{\partial_0^k,\, \partial_0^{k-1}\partial_1, \ldots, \partial_2^k\}$$

for ternary forms of degree $d-k$, and dual operators of order k respectively. With these bases, the linear map $\gamma_{k,F}$ is represented by the catalecticant matrix

$$\mathrm{Cat}(k,F) = \left[\begin{array}{cccc|ccc|}
a_{(0,0)} & a_{(1,0)} & a_{(0,1)} \ \cdots & & a_{(d-k,0)} & a_{(d-k-1,1)} \ \cdots & a_{(0,d-k)} \\
a_{(1,0)} & & & & \vdots & & \vdots \\
a_{(0,1)} & & & & \vdots & & \vdots \\
\vdots & & & & \vdots & & \vdots \\ \cline{5-7}
a_{(k,0)} & \cdots & \cdots & \cdots & a_{(d,0)} & a_{(d-1,1)} \ \cdots & a_{(k,d-k)} \\
a_{(k-1,1)} & \cdots & \cdots & \cdots & a_{(d-1,1)} & & \vdots \\
\vdots & & & & \vdots & & \vdots \\
a_{(0,k)} & \cdots & \cdots & \cdots & a_{(k,d-k)} & \cdots \ \cdots & a_{(0,d)} \\ \cline{5-7}
\end{array}\right].$$

The fiber $\rho(\gamma_{k,F})_{|x_0}$ is represented by the highlighted submatrix itself, that is $\mathrm{Cat}(k, f_d)$. ♢

The maps $\gamma_{k,F}$ and $\rho(\gamma_{k,F})$ can be equivalently regarded as sections of $S^kV \otimes S^{d-k}V(d)$ and $S^k\mathcal{Q} \otimes S^{d-k}\mathcal{Q}(d)$ respectively. The spaces of these sections are identified with the image of the maps ι and $\rho \circ \iota$ below:

$$\begin{tikzcd}
H^0(S^kV \otimes S^{d-k}V(d)) \arrow[r, "\rho"] & H^0(S^k\mathcal{Q} \otimes S^{d-k}\mathcal{Q}(d)) \\
\mathrm{Cat}(k, d-k; 3) \arrow[u, hook, "\iota"] \arrow[ur, "\rho\circ\iota"] &
\end{tikzcd}.$$

Let us denote with S the image of $\rho \circ \iota$. Note that S can be identified with the space $\mathrm{Cat}(k, d-k; 2)$ of catalecticant matrices of degree-d binary forms. In view of proving that the conditions of Lemma 3.3.2 generically hold, we need to understand where the general element of S fails to have full rank. This is a classical question in intersection theory and the answer is a consequence of Porteous' Theorem (for the general statement see [24, Theorem 12.4] or Porteous' original reference [45]).

Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles of ranks e and f on a smooth variety X. For any map $\phi\colon \mathcal{E} \to \mathcal{F}$ we denote with $D_k(\phi)$ the subscheme of X where $\mathrm{rk}(\phi) \leq k$. We say that $D_k(\phi)$ has *expected codimension* if $\mathrm{codim}_X(D_k(\phi)) = (e-k)(f-k)$.

Theorem 3.3.11 (Porteous' Formula). *If $D_{e-1}(\phi)$ has the expected codimension $f-e+1$, then its class is given by*

$$[D_{e-1}(\phi)] = \left\{ \frac{c(\mathcal{F})}{c(\mathcal{E})} \right\}_{f-e+1},$$

where the right-hand side denotes the class of codimension $f-e+1$ of the ratio of the Chern characters of $\mathcal{E}$ and $\mathcal{F}$.

We would like to apply Porteous' formula to the the general element of S. To satisfy the hypotheses of the theorem, we need to make sure that its k-th degeneracy locus has expected codimension, namely 2 when $d = 2k+1$, and 1 when $d = 2k$. For this we are going to use a Bertini-type argument for vector bundles. The proof of the following proposition is taken almost verbatim from [44, Theorem 2.6]:

Proposition 3.3.12. *The subset $S \subset H^0(S^k\mathcal{Q} \otimes S^{d-k}\mathcal{Q}(d))$ is base-point-free. In particular, if $\phi\colon S^k\mathcal{Q}^* \to S^{d-k}\mathcal{Q}(d)$ is a general element of S, then for any $r \leq k$, we have that $D_r(\phi)$ is either empty or it has expected codimension and* $\mathrm{Sing}(D_r(\phi)) \subseteq D_{r-1}(\phi)$.

Proof. The base locus of S is

$$\begin{aligned}\mathrm{Bs}(S) &= \{l \in \mathbb{P}(\bar{S}_1) \mid \rho(\gamma_{k,F})_{|l} = 0, \forall F \in \mathbb{P}(\bar{S}_d)\} \\ &= \{l \in \mathbb{P}(\bar{S}_1) \mid V(l) \subset V(F), \forall F \in \mathbb{P}(\bar{S}_d)\},\end{aligned}$$

which is empty, since there is no line $V(l)$ contained in every degree-d plane curve.

Let us consider the subvariety of $\mathbb{P}(\bar{S}_1) \times S$ defined by

$$\Sigma_r := \{(l, \phi) \mid \mathrm{rk}(\phi_l) \leq r\}$$

and let us denote with p and q the restriction to Σ_r of the canonical projection maps to $\mathbb{P}(\bar{S}_1)$ and S respectively.

For any $(l, \phi) \in \Sigma_r$, we have

$$p^{-1}(l) \simeq \{\phi \in S \mid \mathrm{rk}(\phi_l) \leq r\} \simeq \{A \in \mathrm{Cat}(k, d-k; 2) \mid \mathrm{rk}(A) \leq r\}.$$

Moreover,

$$q^{-1}(\phi) \simeq \left\{l \in \mathbb{P}(\bar{S}_1) \mid \mathrm{rk}(\phi_l) \leq r\right\} = D_r(\phi).$$

Therefore

$$\dim(\Sigma_r) = 2 + \dim \sigma_r(\nu_d(\mathbb{P}^1)) = 2r + 1,$$

so for a general ϕ we have $\dim(q^{-1}(\phi)) = 2r + 1 - d$. This implies that only $D_k(\phi)$ is non-empty. When $d = 2k + 1$ (resp. when $d = 2k$), its codimension is 2 (resp. 1), which coincides with the expected one.

Finally, note that the smooth locus of Σ_r is

$$\mathrm{Sm}(\Sigma_r) = \Sigma_r \setminus \Sigma_{r-1}.$$

The restriction $q_{|\mathrm{Sm}(\Sigma_r)}$ has dense image, so $D_r(\phi) \setminus D_{r-1}(\phi)$ is smooth by the Generic Smoothness theorem (see [30], § III, Corollary 10.7). ∎

Corollary 3.3.13. *The hypotheses of Porteous' Theorem are satisfied by a general element $\phi \in S$. Moreover, $D_k(\phi)$ is smooth and $D_{k-1}(\phi)$ is empty.*

We are now ready to give the proofs of the two Propositions 3.3.7 and 3.3.8. Computations in the first proof are based on the following:

Proposition 3.3.14 ([24], Proposition 5.17)**.** *If $\mathcal{E}$ is a vector bundle of rank r and $\mathcal{L}$ is a line bundle, then*

$$c_k(\mathcal{E} \otimes \mathcal{L}) = \sum_{i=0}^{k} \binom{r-k+i}{i} c_1(\mathcal{L})^i c_{k-i}(\mathcal{E}).$$

Proof of Proposition 3.3.7. Let us begin with the case $d = 2k + 1$. To count the number of lines in the generic fiber of p_1, we need to apply Porteous' formula to $D_{k-1}(\phi)$ for the general map

$$\phi\colon S^k\mathcal{Q}^* \longrightarrow S^{k+1}\mathcal{Q} \otimes \mathcal{O}(2k+1).$$

Let us denote with h the hyperplane class of $\mathbb{P}V$. We have

$$c(\mathcal{O}(d)) = 1 + dh, \qquad c(\mathcal{Q}) = 1 + h + h^2 \qquad c(\mathcal{Q}^*) = 1 - h + h^2.$$

Since $\mathcal{Q}$ has rank 2, the characteristic classes of its symmetric powers may be computed using the splitting principle by assuming that $\mathcal{Q} = \mathcal{L} \oplus \mathcal{M}$. Under these assumptions, we have

$$-c_1(\mathcal{Q}^*) = c_1(\mathcal{Q}) = c_1(\mathcal{L}) + c_1(\mathcal{M}), \qquad c_2(\mathcal{Q}^*) = c_2(\mathcal{Q}) = c_1(\mathcal{L})c_1(\mathcal{M}),$$

as well as the splitting

$$S^k\mathcal{Q} = \mathcal{L}^{\otimes k} \oplus (\mathcal{L}^{\otimes k-1} \otimes \mathcal{M}) \oplus \cdots \oplus (\mathcal{L} \otimes \mathcal{M}^{\otimes k-1}) \oplus \mathcal{M}^{\otimes k},$$

and similarly for the dual. Therefore, the characteristic class of $S^k\mathcal{Q}^*$ is

$$\begin{aligned}
c(S^k\mathcal{Q}^*) &= \prod_{j=0}^{k}(1+(k-j)c_1(\mathcal{L}^*)+jc_1(\mathcal{M}^*))\\
&= \prod_{j=0}^{k}(1-(k-j)c_1(\mathcal{L})-jc_1(\mathcal{M}))\\
&= 1-\sum_{j=0}^{k} jc_1(\mathcal{Q})+\sum_{j=0}^{k-1}\sum_{i=j+1}^{k}(k-j)(k-i)(c_1(\mathcal{L})^2+c_1(\mathcal{M})^2)\\
&\quad+\sum_{j=0}^{k-1}\sum_{i=j+1}^{k}[j(k-i)+(k-j)i]c_1(\mathcal{L})c_1(\mathcal{M})\\
&= 1-\frac{k(k+1)}{2}c_1(\mathcal{Q})+\sum_{j=0}^{k-1}\sum_{i=j+1}^{k}(k-j)(k-i)(c_1(\mathcal{L})^2+c_1(\mathcal{M})^2)\\
&\quad+\sum_{j=0}^{k-1}\sum_{i=j+1}^{k}(ik-2ij+jk)c_2(\mathcal{Q}).
\end{aligned}$$

By completing squares we get:

$$\begin{aligned}
c(S^k\mathcal{Q}^*) &= 1-\frac{k(k+1)}{2}c_1(\mathcal{Q})+\sum_{j=0}^{k-1}\sum_{i=j+1}^{k}(k-j)(k-i)c_1(\mathcal{Q})^2\\
&\quad-2\sum_{j=0}^{k-1}\sum_{i=j+1}^{k}(k-j)(k-i)c_2(\mathcal{Q})+\sum_{j=0}^{k-1}\sum_{i=j+1}^{k}(ik-2ij+jk)c_2(\mathcal{Q})\\
&= 1-\frac{k(k+1)}{2}h+\sum_{j=0}^{k-1}\sum_{i=j+1}^{k}(2ik+2jk-3ij-k^2)h^2\\
&= 1-\frac{k(k+1)}{2}h+\frac{k^4+2k^3+3k^2+2k}{8}h^2.
\end{aligned}\tag{3.3.8}$$

With analogous computations, we obtain

$$\begin{aligned}
c(S^{k+1}\mathcal{Q}) &= 1+\frac{(k+1)(k+2)}{2}h\\
&\quad+\sum_{j=0}^{k}\sum_{i=j+1}^{k+1}[2i(k+1)+2j(k+1)-3ij-(k+1)^2]h^2\\
&= 1+\frac{(k+1)(k+2)}{2}h+\frac{k^4+6k^3+15k^2+18k+8}{8}h^2.
\end{aligned}$$

We now apply Proposition 3.3.14 to obtain the relevant classes for the tensor with $\mathcal{O}(2k+1)$:

$$\begin{aligned} c_1(S^{k+1}\mathcal{Q}\otimes\mathcal{O}(2k+1)) &= \sum_{i=0}^{1}\binom{k+1+i}{i}c_1(\mathcal{O}(2k+1))^i c_{1-i}(S^{k+1}\mathcal{Q}) \\ &= c_1(S^{k+1}\mathcal{Q}) + (k+2)c_1(\mathcal{O}(2k+1)) \\ &= \frac{(k+2)(5k+3)}{2}h. \end{aligned} \tag{3.3.9}$$

and

$$\begin{aligned} c_2(S^{k+1}\mathcal{Q}\otimes\mathcal{O}(2k+1)) &= \sum_{i=0}^{2}\binom{k+i}{i}c_1(\mathcal{O}(2k+1))^i c_{2-i}(S^{k+1}\mathcal{Q}) \\ &= c_2(S^{k+1}\mathcal{Q}) + (k+1)c_1(\mathcal{O}(2k+1))c_1(S^{k+1}\mathcal{Q}) \\ &\quad + \frac{(k+1)(k+2)}{2}c_1(\mathcal{O}(2k+1))^2 \\ &= \frac{25k^4+106k^3+155k^2+98k+24}{8}h^2. \end{aligned} \tag{3.3.10}$$

By Porteous' Theorem we are then reduced to compute:

$$\left\{\frac{c\big(S^{k+1}\mathcal{Q}\otimes\mathcal{O}(2k+1)\big)}{c(S^k\mathcal{Q}^*)}\right\}_2,$$

which means that we need to find the degree-2 term in the ratio. If we express the numerator as $1+ah+bh^2$ and the denominator as $1+ch+dh^2$, the term is $(c^2-ac+b-d)h^2$.

We substitute the coefficients a,b,c,d from Equations (3.3.8) to (3.3.10), and obtain:

$$\begin{aligned} &\left[\frac{k^2(k+1)^2}{4}+\frac{k(k+1)(k+2)(5k+3)}{4}+\frac{25k^4+106k^3+155k^2+98k+24}{8}\right. \\ &\qquad \left. -\frac{k^4+6k^3+15k^2+18k+8}{8}\right]h^2 \\ &=\left[\frac{9k^4+36k^3+48k^2+27k+6}{2}\right]h^2. \end{aligned}$$

This completes the proof for the case $d=2k+1$.

When $d=2k$, we need to look at the degeneracy locus $D_{k-1}(\phi)$ for the general map

$$\phi\colon S^k\mathcal{Q}^* \longrightarrow S^k\mathcal{Q}\otimes\mathcal{O}(2k),$$

which by Porteous' Theorem is

$$\left\{\frac{c(S^k\mathcal{Q}\otimes\mathcal{O}(2k))}{c(S^k\mathcal{Q}^*)}\right\}_1$$

If we express the numerator as $1+ah+bh^2$ and the denominator as $1+ch+dh^2$, the degree-1 term in the ratio is $(a-c)h$. The first Chern class of $S^k\mathcal{Q}^*$ was computed in (3.3.8) while

$$\begin{aligned} c_1(S^k\mathcal{Q}\otimes\mathcal{O}(2k)) &= \sum_{i=0}^{1}\binom{k+i}{i}c_1(\mathcal{O}(2k))^i c_{1-i}(S^k\mathcal{Q}) \\ &= c_1(S^k\mathcal{Q}) + (k+1)c_1(\mathcal{O}(2k)) \\ &= \frac{5k(k+1)}{2}h. \end{aligned}$$

Altogether, $(a-c)h = [3k(k+1)]h$. ■

Proof of Proposition 3.3.8. When $d=2k$, the incidence variety $\mathcal{I}_2$ dominates if and only if the equivalent conditions of Lemma 3.3.3 are satisfied. In terms of vector bundles, we need to look at the class of $(k-1)$-st degeneracy locus of $\phi\colon S^k\mathcal{Q}\to S^k\mathcal{Q}(2k)$, which is empty by Corollary 3.3.13.

When $d=2k+1$, the incidence variety $\mathcal{I}_2$ dominates if and only if the equivalent conditions of Lemma 3.3.4 are satisfied. We assume by contradiction that p_2 is dominant. Then let C be an irreducible component of $\mathcal{I}_2$ dominating $\mathbb{P}(\bar{S}_d)$. But $\mathcal{I}_2\subseteq\mathcal{I}_1$ and p_1 is generically finite by Proposition 3.3.7. Therefore C is also an irreducible component of $\mathcal{I}_1$, of dimension equal to the one of $\mathbb{P}(\bar{S}_d)$. By Corollary 3.3.6, for every $(F,l)\in C$ we have that $q_2^{-1}(l)=q_1^{-1}(l)\cap\mathcal{I}_2$ is a hypersurface in $q_1^{-1}(l)$. On the other hand, $q_1^{-1}(l)$ is contained in C, so it is equal to $q_2^{-1}(l)$, a contradiction. ■

3.4 The local cactus rank of ternary forms

In this section, we describe a procedure to compute the local cactus rank of a general degree-d ternary form. For small degrees, we obtain a closed formula:

Proposition 3.4.1. *The local cactus rank of a general ternary form $F\in\mathbb{P}(\bar{S}_d)$ of degree $d\leq 5$ is:*

$$\operatorname{lcr}(F) = \left\lceil \frac{d(d+3)}{4} \right\rceil.$$

As we will see at the end of this section, an obstacle to the generalization of this statement to higher degrees is to make our procedure more systematic. Nevertheless, our conjecture is that this formula in fact holds for any d (Conjecture 3.4.15).

The starting point goes back to Proposition 3.1.18, from which we know that the local cactus rank of a degree-d form F is computed by the local apolar scheme of a polynomial g of degree $e\geq d$ whose tail $g_{\leq d}$ coincides with a dehomogenization of F.

This essentially says that we need to compute the dimension of the families of tails of polynomials with Hilbert function of fixed lenght.

To prove Proposition 3.4.1, it is enough to provide the dimension of the spaces of tails of polynomials g with unitary Hilbert function, that is $H_g = (1, 1, \dots, 1)$. These dimensions are computed in the forthcoming Proposition 3.4.11.

The fact that we can reduce to the case of unitary Hilbert functions is explained by the following proposition.

Proposition 3.4.2. *The local cactus rank of a homogeneous ternary form is computed by polynomials with unitary Hilbert function.*

Proof. Let us denote with $\bar{S} = \mathbb{C}[x_0, x_1, x_2]$ and $S = \mathbb{C}[x_1, x_2]$. Let $F \in \mathbb{P}(\bar{S}_d)$ be a homogeneous form of degree d and let $g \in \mathbb{P}(\oplus_{i=0}^{e} S_i)$ be a polynomial of degree $e \geq d$ computing $\operatorname{lcr}(F) = r + 1$.

We need to show that there exists another polynomial g' of degree $r \geq e$ with $H_{g'} = (1, 1, \dots, 1)$ and such that $g'_{\leq e} = g$. If we consider the subvarieties

$$\begin{aligned} \mathcal{G}_1 &:= \{g \in \mathbb{P}(\oplus_{i=0}^{e} S_i) \mid \operatorname{len}(H_g) \leq r + 1\} \\ \mathcal{G}_2 &:= \{g' \in \mathbb{P}(\oplus_{i=0}^{r} S_i) \mid H_{g'}(i) \leq 1,\, \forall i = 0, \dots, r\}\,, \end{aligned}$$

it is enough to prove that $\mathcal{G}_1 \subseteq \mathcal{G}_2$ and that $\mathcal{G}_2$ is irreducible. The inclusion is provided by Proposition 3.1.14, while irreducibility is proved later in Proposition 3.4.7. ■

With this motivation in mind, we consider the incidence variety:

$$\mathcal{G}_e := \left\{ (G, l, l') \in \mathbb{P}(\bar{S}_e) \times \mathbb{P}(\bar{S}_1) \times \mathbb{P}(\bar{S}_1) \;\middle|\; \begin{array}{c} H_{G_l}(i) \leq 1,\, \forall i = 0, \dots, e \\ \operatorname{Lin}(G_l) = \langle l' \mod l \rangle \end{array} \right\}, \tag{3.4.1}$$

where $\bar{S} = \mathbb{C}[x_0, \dots, x_n]$ and $\operatorname{Lin}(G_l)$ denotes the space of linear partials of G_l, modulo additive constants (cfr. Equation (3.1.6)). We denote with π_1, π_2 and π_3 the three projection maps going from $\mathcal{G}_e$ to $\mathbb{P}(\bar{S}_e)$, $\mathbb{P}(\bar{S}_1)$ and $\mathbb{P}(\bar{S}_1)$ respectively.

Remark 3.4.3. By definition, $\mathcal{G}_e$ is projected via π_1 to the variety of homogeneous forms of degree e in $n+1$ variables admitting a dehomogenization with unitary Hilbert function. Moreover, the condition on the linear partials, restricts the choice of l' as a linear form in $\mathbb{P}(\bar{S}/l)_1$.

We have

$$\pi_1(\mathcal{G}_e) = \bigcup_{\substack{l \in \mathbb{P}(\bar{S}_1) \\ l' \in \mathbb{P}(\bar{S}/l)_1}} \pi_1[\pi_2^{-1}(l) \cap \pi_3^{-1}(l')].$$

As l, l' vary, the fiber intersections $\pi_2^{-1}(l) \cap \pi_3^{-1}(l')$ are all isomorphic. In particular, we have

$$\dim(\pi_1(\mathcal{G}_e)) = \min\left\{2n - 1 + \dim(\pi_2^{-1}(l) \cap \pi_3^{-1}(l')),\, \dim \mathbb{P}(\bar{S}_e)\right\}. \tag{3.4.2}$$

Our goal is to compute this dimension. Without loss of generality, we will reduce to the case $l = x_0$ and $l' = x_1 \in \mathbb{P}(\bar{S}/x_0)_1 \simeq \mathbb{P}(S_1)$.

3.4.1 Dimension of polynomials with unitary Hilbert function

We are going to use auxiliary families of inhomogeneous polynomials. The fiber $\pi_2^{-1}(x_0)$ is isomorphic to

$$\mathcal{G}_{e,x_0} := \left\{ (g, l_1) \in \mathbb{P}(\oplus_{i=0}^{e} S_i) \times \mathbb{P}(S_1) \;\middle|\; \begin{array}{c} H_g(i) \leq 1, \forall i = 0, \dots, e \\ \mathrm{Lin}(g) = \langle l_1 \rangle \end{array} \right\}.$$

Analogously, the intersection $\pi_2^{-1}(x_0) \cap \pi_3^{-1}(x_1)$ is isomorphic to

$$\mathcal{G}_{e,x_0,x_1} := \left\{ g \in \mathbb{P}(\oplus_{i=0}^{e} S_i) \;\middle|\; \begin{array}{c} H_g(i) \leq 1, \forall i = 0, \dots, e \\ \mathrm{Lin}(g) = \langle x_1 \rangle \end{array} \right\}.$$

The projection to the first factor $\pi_1(\mathcal{G}_{e,x_0})$ is the variety of polynomials of degree e with unitary Hilbert function. Its dimension is

$$\dim \pi_1(\mathcal{G}_{e,x_0}) = \min \left\{ \dim \pi_1(\mathcal{G}_{e,x_0,x_1}) + n - 1, \dim \mathbb{P}(\bar{S}_e) \right\}. \tag{3.4.3}$$

As we saw in Key Lemma 3.1.11, all these varieties are defined by determinantal equations. Between $\mathcal{G}_{e,x_0}$ and $\mathcal{G}_{e,x_0,x_1}$, the latter is more manageble when it comes to write down explicit defining equations. To this purpose, we give the following lemma:

Lemma 3.4.4. *Let $S = \mathbb{C}[x_1, \dots, x_n]$ and let $g \in \mathbb{P}(\oplus_{i=0}^{e} S_i)$. Then the following are equivalent:*

(1) *For every $i \leq e$, we have $H_g(i) \leq 1$;*

(2) *For every $i \leq e$ we have $\Delta_{g,0}(i) = 1$, while for every $1 \leq k \leq e-2$ and $i \leq e-k$ we have $\Delta_{g,k}(i) = 0$;*

(3) *There exists $l \in S_1$ such that $Q_{0,i}^{\vee} = \langle l^i \rangle$ for every $i \leq e$ and $Q_{k,1}^{\vee} = 0$ for every $1 \leq k \leq e-2$;*

(4) *There exists $l \in S_1$ such that $\mathrm{Diff}(g)_i^0 = \langle 1, l, \dots, l^i \rangle$ for every $i \leq e$;*

(5) *For every $0 \leq k \leq e-2$, we have $\mathrm{rk}(C_{k+1}(g)) \leq k+2$.*

Proof. Equivalence of (1) and (2). Follows from the decomposition $H_g(i) = \sum_{k=0}^{e-2} \Delta_{g,k}(i)$, where $\Delta_{g,k}(i) = \dim Q_{k,i}^{\vee}$.

Equivalence of (2) and (3). The condition $\Delta_{g,0}(i) = 1$ means that for every degree i, the polynomial g admits exactly one partial of degree i and order $e-i$. This is only possible when the degree-e part of g is a pure power of a linear form, that is $g_e = l^e$ for some $l \in S_1$. Therefore, this is equivalent to have $Q_{0,i}^{\vee} = \langle l^i \rangle$ for every $i \leq e$. Moreover, the vanishing of the remaining $\Delta_{g,k}(i)$ imply in particular that $\Delta_{g,k}(1) = 0$ hence $Q_{k,1}^{\vee} = 0$.

Viceversa, if $\Delta_{g,k}(1) = 0$ for every $1 \leq k \leq e-2$ and $\Delta_{g,0}(i) = 1$ for every $i \leq e$, then for any given k we have $\sum_{j=0}^{k} \Delta_{g,j}(1) = 1 = \binom{1}{1}$. Macaulay growth conditions imply that for every $i \geq 2$ we have $\sum_{j=0}^{k} \Delta_{g,j}(i) \leq \binom{i}{i} = 1$. In particular, $\Delta_{g,k}(i) = 0$ for every $k = 1, \ldots, e-2$ and $i = 2, \ldots, e-k$.

Equivalence of (3) and (4). We have previously seen that when $Q^{\vee}_{0,i} = 1$ for $i \leq e$ and $Q^{\vee}_{k,1} = 0$ for every $1 \leq k \leq e-2$, then the remaining quotient spaces vanish too. In particular, $Q^{\vee}_{1,i} = 0$ for $i \leq e-1$. Let us assume that (3) holds. Then:

$$\operatorname{Diff}(g)^0_i = \langle l^i \rangle + \operatorname{Diff}(g)^1_{i-1}, \tag{3.4.4}$$

$$\operatorname{Diff}(g)^1_i = \operatorname{Diff}(g)^2_{i-1} + \operatorname{Diff}(g)^0_i. \tag{3.4.5}$$

We show by induction that $\operatorname{Diff}(g)^0_i = \langle 1, l, \ldots, l^i \rangle$. First by definition, $\operatorname{Diff}(g)^0_0 = \langle 1 \rangle$. Let us assume that $\operatorname{Diff}(g)^0_{i-1} = \langle 1, l, \ldots, l^{i-1} \rangle$. Then (3.4.4) and (3.4.5) imply,

$$\operatorname{Diff}(g)^0_i = \langle l^i \rangle + \operatorname{Diff}(g)^2_{i-2} + \langle 1, l, \ldots, l^{i-1} \rangle.$$

Moreover, by definition

$$\operatorname{Diff}(g)^2_{i-2} \simeq \bigoplus_{j=0}^{i-2} \bigoplus_{k=0}^{i-j} Q^{\vee}_{k,j}.$$

By hypothesis, the quotient modules surviving in the direct sum are those with $k = 0$, so $\operatorname{Diff}(g)^2_{i-2} = \operatorname{Diff}(g)^0_{i-2} = \langle 1, l, \ldots, l^{i-2} \rangle$, hence the thesis.

Viceversa, let us assume that $\operatorname{Diff}(g)^0_i = \langle 1, l, \ldots, l^i \rangle$, for every $i \leq e$. Then $Q^{\vee}_{0,i} = \langle 1, l, \ldots, l^i \rangle / \operatorname{Diff}(g)^1_{i-1}$ and the sequence of inclusions

$$\langle 1, l, \ldots, l^{i-1} \rangle = \operatorname{Diff}(g)^0_{i-1} \subseteq \operatorname{Diff}(g)^1_{i-1} \subseteq \operatorname{Diff}(g)^0_i = \langle 1, l, \ldots, l^i \rangle,$$

gives $\operatorname{Diff}(g)^1_{i-1} = \operatorname{Diff}(g)^0_{i-1}$, and $Q^{\vee}_{0,i} = \langle l^i \rangle$.

Moreover, $\operatorname{Diff}(g)^k_1 = \langle 1, l \rangle$ for every $1 \leq k \leq e-2$. We use again induction. We have just seen that $\operatorname{Diff}(g)^1_1 = \operatorname{Diff}(g)^0_1$. Let us assume that $\operatorname{Diff}(g)^{k-1}_1 = \langle 1, l \rangle$. By induction we have

$$\langle 1, l \rangle = \operatorname{Diff}(g)^{k-1}_1 \subseteq \operatorname{Diff}(g)^k_1 \subseteq \operatorname{Diff}(g)^0_{k+1} = \langle 1, l, \ldots, l^{k+1} \rangle,$$

so $\operatorname{Diff}(g)^0_{k+1} = \langle 1, l \rangle$. In particular, $Q^{\vee}_{k,1} = 0$.

Equivalence of (1) and (5). Follows from Lemma 3.1.11. ∎

The next result provides an explicit description of $\mathcal{G}_{e,x_0,x_1}$ in the case of polynomials in two variables.

Lemma 3.4.5. *Let $S = \mathbb{C}[x_1, x_2]$ and let $g = \sum_{i+j \leq e} a_{(i,j)} x_1^i x_2^j$ be a polynomial in $\mathbb{P}(\oplus_{i=0}^{e} S_i)$. Then g belongs to the locally closed subset*

$$U := \{a_{(e,0)} = 1\} \cap \mathcal{G}_{e,x_0,x_1}$$

if and only if

$$a_{(i,e-i)} = 0 \quad \textit{for every} \quad 0 \leq i \leq e-1 \tag{3.4.6}$$

$$a_{(i,e-i-1)} = 0 \quad \textit{for every} \quad 0 \leq i \leq e-3 \tag{3.4.7}$$

and the following determinantal equations are satisfied for every $2 \leq k \leq e-2$ *and* $0 \leq i \leq e-k-2$*:*

$$\begin{vmatrix} a_{(i,e-k-i)} & a_{(i+2,e-k-i-1)} & a_{(i+3,e-k-i-1)} & \cdots & a_{(i+k-1,e-k-i-1)} & a_{(i+k,e-k-i-1)} \\ a_{(e-k,1)} & a_{(e-k+2,0)} & a_{(e-k+3,0)} & \cdots & a_{(e-1,0)} & 1 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ a_{(e-3,1)} & a_{(e-1,0)} & 1 & 0 & \cdots & 0 \\ a_{(e-2,1)} & 1 & 0 & \cdots & \cdots & 0 \end{vmatrix} = 0 \tag{3.4.8}$$

Proof. By Lemma 3.4.4, a polynomial $g \in \mathbb{P}(\oplus_{i=0}^{e} S_i)$ belongs to $\mathcal{G}_{e,x_0,x_1}$ if and only if $\mathrm{rk}(C_k(g)) \leq k+1$ for every $0 \leq k \leq e$.

$$Q_{0,i}^{\vee} = \langle x_1^i \rangle \qquad \text{for every} \qquad 0 \leq i \leq e \tag{3.4.9}$$

$$Q_{k,1}^{\vee} = 0 \qquad \text{for every} \qquad 1 \leq k \leq e-2. \tag{3.4.10}$$

In particular, (3.4.9) is equivalent to saying that the leading term of g is the rank-1 form $a_{e,0}x_1^e$, therefore $a_{i,e-i} = 0$ for every $0 \leq i \leq e-1$, which is (3.4.6).

Next, we are going to translate (3.4.10) in terms of rank conditions on catalecticant matrices. Let us assume that $a_{e,0} = 1$. Then for every $k = 0, \ldots, e$, we have

$$\partial_1^{e-k}(g) = x_1^k + \sum_{j=0}^{k-1} [a_{(e-k+1+j,0)} x_1^j] + a_{(e-k,1)} x_2 + g_k, \tag{3.4.11}$$

where $g_k \in \mathbb{C}[x_1, x_2]$ is a polynomial of degree at least 2 and divisible by x_2. In particular, the catalecticant blocks $\mathrm{Cat}(e-m, g_e)$ have all entries equal to zero, except for the top left one, which is equal to 1. In particular, $\mathrm{rk}(C_{k+1}(g)) \geq k+2$ for every $k \leq e-2$. Lemma 3.4.4 implies that $g \in \mathcal{G}_{e,x_0,x_1}$ if and only if

$$\mathrm{rk}(C_{k+1}(g)) = k+2 \qquad \text{for every} \qquad k \leq e. \tag{3.4.12}$$

When $k = 1$, we need $Q_{1,1}^{\vee} = 0$, that is, no partials of degree equal to 1 and order equal to $d-2$. Equivalently, $\mathrm{rk}(C_2(g)) = 3$, where

$$C_2(g) = \begin{bmatrix} a_{(e-2,0)} & a_{(e-1,0)} & a_{(e-2,1)} & 1 & 0 & \cdots & 0 \\ a_{(e-3,1)} & a_{(e-2,1)} & a_{(e-3,2)} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ a_{(0,e-2)} & a_{(1,e-2)} & a_{(0,e-1)} & 0 & \cdots & \cdots & 0 \\ a_{(e-1,0)} & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_{(e-2,1)} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & & & & & \vdots \\ a_{(0,e-1)} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

This is equivalent to asking the highlighted submatrix of $C_2(g)$ to be of rank 2, that is

$$\begin{vmatrix} a_{(0,e-1)} & 0 \\ a_{(e-2,1)} & 1 \end{vmatrix} = \cdots = \begin{vmatrix} a_{(e-3,2)} & 0 \\ a_{(e-2,1)} & 1 \end{vmatrix} = 0,$$

which is (3.4.7).

More generally, let us assume that $Q^\vee_{1,1} = \cdots = Q^\vee_{k-2,1} = 0$, so that $\mathrm{rk}(C_{k-1}(g)) = k$. The further condition $Q^\vee_{k-1,1} = 0$, is asking no partials of degree equal to 1 and order equal to $e-k$. This holds if and only if $\mathrm{rk}(C_k(g)) = k+1$. Therefore, let us consider the first $e-k+1$ rows of $C_k(g)$, which are the ones corresponding to the partials of g of order equal to $e-k$. They are of the form

$$\partial_1^{e-k-i}\partial_2^i(g) = \sum_{j=0}^{k}[a_{(i+1+j,e-k-i-1)}x_1^j] + a_{(i,e-k-i)}x_2 + h_{i,k},$$

for every $0 \le i \le e-k$, where $h_{i,k} \in \mathbb{C}[x_1,x_2]$ is a polynomial of degree at least 2 and divisible by x_2 .

When $i = 0$, we have the partial $\partial_1^{e-k}(g)$, whose degree is exactly k, so it cannot contribute to $Q^\vee_{k-1,1}$. For every other i, the only way we can possibly obtain a partial of degree equal to 1 is by taking linear combinations of $\partial_1^{e-k-i}\partial_2^i(g)$ with the partials in (3.4.11), which explains the determinantal condition in (3.4.8). ■

Remark 3.4.6. By applying recursively equations (3.4.6), (3.4.7) and (3.4.8), it follows that the variety $\mathcal{G}_{e,x_0,x_1}$ is cut out by linear equations

$$a_{(i,e-k-i)} = 0 \qquad 0 \le k \le \lfloor \tfrac{e-1}{2} \rfloor, \quad 0 \le i \le e-2k-1 \tag{3.4.13}$$

and higher degree equations of the form

$$a_{(i,e-k-i)} = f_{i,e-k-i} \qquad 2 \le k \le e-2, \quad e-2k \le i \le e-k-2, \tag{3.4.14}$$

where $f_{i,e-k-i}$ is a polynomial of degree at least 2 depending on the variables $a_{(e-2,1)}, \ldots, a_{(e-k-i+2,1)}$ and $a_{(e-1,0)}, \ldots, a_{(e-k-1+i,0)}$.

It is now straightforward to determine the dimension of $\mathcal{G}_{e,x_0,x_1}$, hence the dimension of $\pi_1(\mathcal{G}_e)$, that is, the dimension of the family of homogeneous ternary forms of degree e admitting a dehomogenization with unitary Hilbert function.

Proposition 3.4.7. *The variety $\mathcal{G}_{e,x_0,x_1} \subseteq \mathbb{P}(\oplus_{i=0}^e S_i)$ is isomorphic to the $(2e-1)$-dimensional affine space*

$$\mathrm{Spec}\left(\mathbb{C}\left[a_{(i,0)}, a_{(j,1)} \mid 0 \le i \le e-1, \quad 0 \le j \le e-2\right]\right). \tag{3.4.15}$$

In particular, $\pi_1(\mathcal{G}_{e,x_0})$ is irreducible of dimension

$$\dim(\pi_1(\mathcal{G}_{e,x_0})) = \min\left\{2e, \dim \mathbb{P}(\bar{S}_e)\right\}$$

and $\pi_1(\mathcal{G}_e)$ is irreducible of dimension

$$\dim(\pi_1(\mathcal{G}_e)) = \min\left\{2e+2, \dim \mathbb{P}(\bar{S}_e)\right\}.$$

Proof. The defining equations of $\mathcal{G}_{e,x_0,x_1}$ are given in Lemma 3.4.5, which in Remark 3.4.6 are noted to be each one linear in a different variable of $\mathbb{P}(\oplus_{i=0}^{e} S_i)$, where all the variables are covered, except for those listed in (3.4.15). The dimensions of $\pi_1(\mathcal{G}_{e,x_0,x_1})$ and $\pi_1(\mathcal{G}_e)$ follow from formulas (3.4.3) and (3.4.2). ∎

Example 3.4.8 ($e = 5$)**.** We want to write the defining equations of $\mathcal{G}_{5,x_0,x_1}$. Formulas (3.4.6) and (3.4.7) give

$$a_{(0,5)} = a_{(1,4)} = a_{(2,3)} = a_{(3,2)} = a_{(4,1)} = 0$$
$$a_{(0,4)} = a_{(1,3)} = a_{(2,2)} = 0.$$

In the next cases, we use (3.4.8). When $k = 2$, we have:

$$\begin{vmatrix} a_{(0,3)} & a_{(2,2)} \\ a_{(3,1)} & 1 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_{(1,2)} & a_{(3,1)} \\ a_{(3,1)} & 1 \end{vmatrix} = 0,$$

from which we obtain $a_{(0,3)} = 0$ and $a_{(1,2)} = f_{1,2}(a_{(3,1)}) = a_{(31)}^2$.

When $k = 3$, we have:

$$\begin{vmatrix} a_{(0,2)} & a_{(2,1)} & a_{(3,1)} \\ a_{(2,1)} & a_{(4,0)} & 1 \\ a_{(3,1)} & 1 & 0 \end{vmatrix} = 0,$$

which gives $a_{(0,2)} = f_{0,2}(a_{(3,1)}, a_{(2,1)}, a_{(4,0)}) = 2a_{(2,1)}a_{(3,1)} - a_{(3,1)}^2 a_{(4,0)}$. ◇

Remark 3.4.9. Let $Q = [1:0:\cdots:0]$ be the point in $\mathbb{P}(\oplus_{i=0}^{e} S_i)$ corresponding to the polynomial x_1^e. The tangent space of $T_Q\mathcal{G}_{e,x_0,x_1}$ is defined by equations

$$\begin{aligned} a_{(i,e-i)} &= 0 && \text{for} \quad 0 \le i \le e-1 \\ a_{(i,e-k-i)} &= 0 && \text{for} \quad 1 \le k \le e-2, \quad 0 \le i \le e-k-2. \end{aligned} \tag{3.4.16}$$

For any polynomial $g \in T_Q\mathcal{G}_{e,x_0,x_1}$, we may consider its homogenization G with respect to x_0, that is an element of $\mathbb{P}(\bar{S}_e)$. Its zero locus $V(G)$ is a degree-e plane curve, intersecting the line $V(x_0)$ in the point $[0:0:1]$. As a divisor of the line, this is identified with the point $P = [0:1]$.

Then, the equations (3.4.16) imply that a form $G \in \mathbb{P}(\bar{S}_e)$ is the homogenization of a $g \in T_Q\mathcal{G}_{e,x_0,x_1}$ if and only if we have multiplicities

$$m_P(V(G) \cap V(x_0)) \ge e$$

and

$$m_P(\mathcal{P}_P^i(G) \cap V(x_0)) \ge e - i - 1 \qquad \text{for} \qquad 1 \le i \le e-2,$$

where $\mathcal{P}_P^i(G)$ denotes the i-th polar of $V(G)$ with respect to P.

3.4.2 Dimension of tails

The next step to compute the local cactus rank requires to determine whether the tails of forms in $\pi_1(\mathcal{G}_e)$ fill up the space of degree-d forms.

More precisely, we consider the incidence variety

$$\mathcal{X}_d^e := \left\{(F,l,l') \in \mathbb{P}(\bar{S}_d)\times\mathbb{P}(\bar{S}_1)\times\mathbb{P}(\bar{S}_1) \mid F_l = (G_l)_{\le d}, \text{ for some } (G,l,l') \in \mathcal{G}_e\right\},$$

where $\bar{S} = \mathbb{C}[x_0,\dots,x_n]$ and $\mathcal{G}_e$ is defined as in (3.4.1). We denote with p_1, p_2 and p_3 the projection maps going from $\mathcal{X}_d^e$ to $\mathbb{P}(\bar{S}_d), \mathbb{P}(\bar{S}_1)$ and $\mathbb{P}(\bar{S}_1)$ respectively.

Remark 3.4.10. By definition, $\mathcal{X}_d^e$ is projected via p_1 to the variety of homogeneous forms of degree d in $n+1$ variables admitting a dehomogenization that is a tail of a polynomial of degree e in two variables, with unitary Hilbert function.

Analogously to Remark 3.4.3, for any choice of l, l', we have:

$$\dim(p_1(\mathcal{X}_d^e)) = \min\left\{2n-1+\dim(p_2^{-1}(l)\cap p_3^{-1}(l')),\ \dim\mathbb{P}(\bar{S}_d)\right\}. \quad (3.4.17)$$

As before, we define auxiliary families, so that the fiber $p_2^{-1}(x_0)$ is isomorphic to

$$\mathcal{X}_{d,x_0}^e := \left\{(f,l_1) \in \mathbb{P}(\oplus_{i=0}^d S_i) \times \mathbb{P}(S_1) \mid f = g_{\le d},\ \text{ for some } (g,l_1) \in \mathcal{G}_{e,x_0}\right\},$$

and the intersection $p_2^{-1}(x_0) \cap p_3^{-1}(x_1)$ is isomorphic to

$$\mathcal{X}_{d,x_0,x_1}^e := \left\{f \in \mathbb{P}(\oplus_{i=0}^d S_i) \mid f = g_{\le d},\ \text{ for some } g \in \mathcal{G}_{e,x_0,x_1}\right\}.$$

Our goal is to compute the dimension of $\mathcal{X}_{d,x_0,x_1}^e$, hence the dimension of $p_1(\mathcal{X}_d^e)$ via (3.4.17).

Proposition 3.4.11. *When $\bar{S} = \mathbb{C}[x_0,x_1,x_2]$ and $e \le 10$, we have:*

$$\dim(\mathcal{X}_{d,x_0,x_1}^e) = \begin{cases} \binom{d+2}{2} - 2 & \text{if } e = \frac{1}{2}\binom{d+2}{2} \\ \min\left\{\binom{d+2}{2} - 1,\ \dim(\mathcal{G}_{e,x_0,x_1})\right\} & \text{otherwise.} \end{cases}$$

In particular,

$$\dim(p_1(\mathcal{X}_d^e)) = \min\left\{\binom{d+2}{2} - 1,\ 2e+2\right\}.$$

The proof of this proposition is developed in the next few pages through a case-by-case analysis. The computations are at the same time elementary and elaborate, so we first outline a strategy that can be applied in all generality and then we make it explicit for the cases $e \le 10$.

Strategy and computations in small degrees

For every $e \ge d$, we have a map

$$\begin{array}{cccc} \pi_{e,d}\colon & \mathbb{P}(\oplus_{i=0}^e S_i) & \dashrightarrow & \mathbb{P}(\oplus_{i=o}^d S_i) \\ & \sum_{i=0}^e g_i & \mapsto & \sum_{i=0}^d g_i, \end{array}$$

sending a degree e polynomial to its degree-d tail. We want to determine the dimension of $\pi_{e,d}(\mathcal{G}_{e,x_0,x_1}) = \mathcal{X}^e_{d,x_0,x_1}$ or, equivalently, the dimension of the generic fiber for the restriction of $\pi_{e,d}$ to $\mathcal{G}_{e,x_0,x_1}$. A way to tackle this problem is via elimination. Indeed, the defining equations of $\mathcal{X}^e_{d,x_0,x_1}$ are obtained from the ones of $\mathcal{G}_{e,x_0,x_1}$ by eliminating the variables

$$a_{\geq d+1} := \left\{a_{(i,j)} \mid i+j \geq d+1\right\}.$$

We may do this by essentially applying the implicit function theorem. Specifically, we proceed as follows:

(1) We homogenize the equations of $\mathcal{G}_{e,x_0,x_1}$ given in Lemma 3.4.5 with respect to $a_{(e,0)}$.

(2) We identify the number – say n – of parameters $a_{\geq d+1}$ that we need to eliminate. According to Remark 3.4.6 (3.4.13), this can be done trivially for some parameters by using the linear equations of $\mathcal{G}_{e,x_0,x_1}$. We remain with just the $a_{(k-i,i)}$ with $d+1 \leq k \leq e$ and $0 \leq i \leq e-k$, therefore we may assume that

$$n = \begin{cases} \frac{e^2+e(1-2d)+d^2-d}{2} & \text{if } d \geq \lfloor \frac{e-1}{2} \rfloor \\ \frac{e^2+4e-2d^2-6d}{4} & \text{if } d < \lfloor \frac{e-1}{2} \rfloor. \end{cases} \tag{3.4.18}$$

To eliminate these n parameters we are now restricted to use a number m of non-linear defining equations. By Remark 3.4.6 (3.4.14), this number is

$$m = \begin{cases} \frac{e^2-4e+3}{4} & \text{if } e \text{ odd} \\ \frac{e^2-4e+4}{4} & \text{if } e \text{ even .} \end{cases} \tag{3.4.19}$$

(3) Let $J = (J_{a_{\leq d}} \mid J_{a_{\geq d+1}})$ be the Jacobian matrix of $\mathcal{G}_{e,x_0,x_1}$, where $J_{a_{\geq d+1}}$ denotes the $n \times m$ submatrix of the Jacobian associated with the partial derivatives with respect to the variables $a_{\geq d+1}$ (and similarly for $J_{a_{\leq d}}$).

For a general point $x \in \mathcal{X}^e_{d,x_0,x_1}$, we have:

$$\dim(\pi^{-1}_{e,d}(x)) = n - \operatorname{rk}(J_{a \geq k+1}).$$

In the next examples we are going to see that $J_{a_{\geq d+1}}$ has always maximal rank. We do so by identifying a suitable full-rank maximal submatrix J' of $J_{a \geq d+1}$ or, equivalently, by choosing n non-linear defining equations of $\mathcal{G}_{e,x_0,x_1}$.

(4) We use the equations Lemma 3.4.5 to re-write the entries of J' only in terms of independent parameters, namely the $a_{(i,j)}$ with $j = 0, 1$.

(5) We find values for the independent parameters at which the evaluation of J' has maximal rank.

Below, we examine the cases $e = 2, \dots, 10$. We essentially observe two kinds of situations for the restriction of $\pi_{e,d}$ to $\mathcal{G}_{e,x_0,x_1}$: it is either a finite-to-one birational map, or a dominant map with positive-dimensional fibers.

Exceptions occur only when

$$\dim(\mathcal{G}_{e,x_0,x_1}) = \dim \mathbb{P}(\oplus_{i=0}^{d} S_i),$$

in which case, the restriction of $\pi_{e,d}$ has 1-dimensional fibers. Table 3.1 summarizes all the dimensions.

d \ e	10	9	8	7	6	5	4	3	2
2	5	5	5	5	5	5	5	4	3
3	9	9	9	9	9	8	7	5	
4	14	14	14	13	11	9	7		
5	19	17	15	13	11	9			
6	19	17	15	13	11				
7	19	17	15	13					
8	19	17	15						
9	19	17							
10	19								

Table 3.1: Dimension of $\pi_{e,d}(\mathcal{G}_{e,x_0,x_1})$.

Note that it is enough to check only the highlighted cases among the ones displayed in Table 3.1. The remaining ones can be obtained by applying the following lemma:

Lemma 3.4.12. *Let d and e be fixed integers and $S = \mathbb{C}[x_1, x_2]$. Then:*

- *For any $d_1 \leq d_2 \leq e$ and for any subset $X \subseteq \mathbb{P}(\oplus_{i=0}^{e} S_i)$ we have*

$$\pi_{e,d_1}(X) \subset \pi_{e,d_2}(X).$$

- *For any $e_2 \geq e_1 \geq d$ we have*

$$\pi_{e_1,d}(\mathcal{G}_{e_1,x_0,x_1}) \subseteq \pi_{e_2,d}(\mathcal{G}_{e_2,x_0,x_1}).$$

Proof. The first claim follows from the definition of $\pi_{e,d}$. For the second claim, it is enough to observe that by Proposition 3.1.14, the variety $\mathcal{G}_{e_1,x_0,x_1}$ is an embedded component of the irreducible variety $\mathcal{G}_{e_2,x_0,x_1}$. ■

Unitary quadrics. The variety $\mathcal{G}_{2,x_0,x_1}$ is a 3-dimensional linear subspace of $\mathbb{P}(\oplus_{i=0}^{2} S_i) \simeq \mathbb{P}^5$, defined by $a_{(0,2)} = a_{(1,1)} = 0$.

Tails of unitary cubics. The variety $\mathcal{G}_{3,x_0,x_1}$ is a 5-dimensional linear subspace of $\mathbb{P}(\oplus_{i=0}^{3} S_i) \simeq \mathbb{P}^9$, defined by

$$a_{(0,3)} = a_{(1,2)} = a_{(2,1)} = 0 \qquad a_{(0,2)} = 0.$$

In particular, $\pi_{3,2}(\mathcal{G}_{3,x_0,x_1})$ is a hyperplane in $\mathbb{P}(\oplus_{i=0}^{2} S_i) \simeq \mathbb{P}^5$.

Tails of unitary quartics. The variety $\mathcal{G}_{4,x_0,x_1}$ is a 7-dimensional subvariety of $\mathbb{P}(\oplus_{i=0}^{4} S_i) \simeq \mathbb{P}^{14}$, defined by homogeneous equations

$$a_{(0,3)} = a_{(1,2)} = 0 \qquad a_{(0,4)} = \cdots = a_{(3,1)} = 0$$
$$a_{(2,1)}^2 - a_{(0,2)}a_{(4,0)} = 0.$$

Projecting to $\mathbb{P}(\oplus_{i=0}^{3} S_i)$, is equivalent to eliminating the variable $a_{(4,0)}$, which is done by using the quadric equation. Therefore, the projection $\pi_{4,3}$ is a $2:1$ when restricted to $\mathcal{G}_{4,x_0,x_1}$.

When projecting to the space of quadrics $\mathbb{P}(\oplus_{i=0}^{2} S_i)$, we need to eliminate the variables $a_{(4,0)}, a_{(3,0)}, a_{(2,1)}$. We only have one equation available, so the fiber is 2-dimensional and $\mathcal{G}_{4,x_0,x_1}$ dominates the $\mathbb{P}^5$ of quadrics.

Tails of unitary quintics. The variety $\mathcal{G}_{5,x_0,x_1}$ is a 9-dimensional subvariety of $\mathbb{P}(\oplus_{i=0}^{5} S_i) \simeq \mathbb{P}^{20}$, defined by homogeneous equations

$$a_{(0,5)} = \cdots = a_{(4,1)} = 0 \qquad a_{(0,4)} = \cdots = a_{(2,2)} = 0 \qquad a_{(0,3)} = 0 \tag{3.4.20}$$
$$a_{(4,0)}a_{(1,2)} - 2a_{(3,1)}a_{(2,1)} + a_{(0,2)}a_{(5,0)} = 0 \qquad a_{(3,1)}^2 - a_{(1,2)}a_{(5,0)} = 0. \tag{3.4.21}$$

Similarly to the previous example, the projection $\pi_{5,4}$ is finite-to-one when restricted to $\mathcal{G}_{5,x_0,x_1}$.

When projecting to the space of cubics, we need to eliminate the variables

$$a_{(5,0)},\ a_{(4,0)},\ a_{(3,1)}. \tag{3.4.22}$$

We only have 2 equations available, so we consider the submatrix J' of the Jacobian of $\mathcal{G}_{5,x_0,x_1}$ corresponding to the partial derivatives of the two quadrics in (3.4.21) with respect to variables (3.4.22).

We have

$$J' = \begin{bmatrix} a_{(0,2)} & a_{(1,2)} & -2a_{(2,1)} \\ -a_{(1,2)} & 0 & 2a_{(3,1)} \end{bmatrix}.$$

Using again the equations (3.4.21), we may re-write its entries only in terms of the independent variables, obtaining

$$J' = \begin{bmatrix} \frac{a_{(4,0)}a_{(3,1)}^2 - 2a_{(5,0)}a_{(3,1)}a_{(2,1)}}{a_{(5,0)}^2} & \frac{a_{(3,1)}^2}{a_{(5,0)}} & -2a_{(2,1)} \\ -\frac{a_{(3,1)}^2}{a_{(5,0)}} & 0 & 2a_{(3,1)} \end{bmatrix}.$$

This matrix has generically maximal rank, hence the restriction of $\pi_{(5,3)}$ to $\mathcal{G}_{e,x_0,x_1}$ has generic fiber of dimension 1 (and $\mathcal{G}_{5,x_0,x_1}$ is mapped onto to hyperplane $a_{0,3} = 0$).

Tails of unitary polynomials from sextics to decimics. The remaining cases $e = 6, \dots, 10$ are analogous. In each of these situations, $\mathcal{G}_{e,x_0,x_1}$ is defined by the equations as in Remark 3.4.6, where the nonlinear homogeneous equations can be simplified as

$$Q_{i,j} \colon a_{(e,0)}a_{(i,j)} - g_{i,j} = 0 \qquad 2 \leq j \leq \lfloor \tfrac{e}{2} \rfloor \qquad 0 \leq i \leq e - 2j,$$

where $g_{i,j}$ has degree 2. Referring to the strategy explained at Page 88, we will just specify the equations $Q_{i,j}$ detemining the submatrix J' of $J_{a_{\geq d}}$. In these examples, the evaluation of J' at

$$\begin{array}{lll} a_{(i,0)} = 1 & \text{for} & 2 \leq i \leq e \\ a_{(i,1)} = (-1)^i & \text{for} & 2 \leq i \leq e-1 \end{array}$$

will always give a matrix of maximal rank.

- ($e = 6$) To determine the fiber dimension of $\pi_{6,4}$, we need to eliminate the variables $a_{\geq 5}$. After using the linear relations, we are reduced to eliminate 3 of them. For those, we can choose the 3 non-linear equations $Q_{i,j}$, with $(i,j) \neq (0,2)$. Rank maximality for the associated matrix J' implies that the restriction $\pi_{6,4}$ is finite-to-one.

  When projecting to the space of cubics, we reduce to eliminate 6 of the variables $a_{\geq 4}$. Here we choose all the 4 available equations $Q_{i,j}$. Rank maximality for the associated J' implies that the restriction $\pi_{6,3}$ has fibers of codimension 2.

- ($e = 7$) We have only one case to check, namely the projection $\pi_{7,4}$. We reduce to eliminate 6 of the variables $a_{\geq 5}$ and to do so we use all the 6 available equations $Q_{i,j}$. We obtain that the restriction of $\pi_{7,4}$ is a finite-to-one map.

- ($e = 8$) To determine the dimension of $\pi_{8,5}(\mathcal{G}_{8,x_0,x_1})$, we reduce to eliminate 6 of the variables $a_{\geq 6}$. To do so, we select 6 among the 9 non-linear equations $Q_{i,j}$ subject to $(i,j) \neq (0,2), (0,3), (1,2)$, and we obtain that the restriction of $\pi_{8,5}$ is finite-to-one.

  When projecting to the space of quartics, we reduce to eliminate 10 of the variables $a_{\geq 5}$. We select all of the 9 non-linear equations $Q_{i,j}$ and obtain that the restriction of $\pi_{8,4}$ has fibers of codimension 1.

- ($e = 9$) We only have to check the projection $\pi_{9,5}$. We reduce to eliminate 10 of the variables $a_{\geq 6}$ and we do so by selecting the 10 equations $Q_{i,j}$ with $(i,j) \neq (0,2), (0,3)$. The restriction of $\pi_{9,5}$ is a finite-to-one map.

- ($e = 10$) We only have to check the projection $\pi_{10,5}$. We reduce to eliminate 15 of the variables $a_{\geq 6}$ and we do so by selecting the 15 equations $Q_{i,j}$ satisfying $(i,j) \neq (0,2)$. The restriction of $\pi_{10,5}$ is a finite-to-one map.

Remark 3.4.13. The above cases not only complete the proof of the closed formula presented in Proposition 3.4.11, but also, together with (3.4.18) and (3.4.19), suggest that the same formula actually holds for any degree, which we conjecture below. An obstacle to prove this conjecture is to show that, using the strategy explained above, the rank maximality condition is always satisfied.

Conjecture 3.4.14. *When $\bar{S} = \mathbb{C}[x_0, x_1, x_2]$, we have:*

$$\dim(\mathcal{X}^e_{d,x_0,x_1}) = \begin{cases} \binom{d+2}{2} - 2 & \text{if } e = \frac{1}{2}\binom{d+2}{2} \\ \min\left\{\binom{d+2}{2} - 1,\, \dim(\mathcal{G}_{e,x_0,x_1})\right\} & \text{otherwise.} \end{cases}$$

In particular,

$$\dim(p_1(\mathcal{X}^e_d)) = \min\left\{\binom{d+2}{2} - 1,\, 2e+2\right\}.$$

If Conjecture 3.4.14 holds, then we have a closed formula for computing the local cacus rank:

Conjecture 3.4.15. *The local cactus rank of a general ternary form $F \in \mathbb{P}(\bar{S}_d)$ of degree d is:*

$$\operatorname{lcr}(F) = \left\lceil \frac{d(d+3)}{4} \right\rceil.$$

Indeed, the local cactus rank is the minimum length of the the apolar scheme associated with a higher degree polynomial. By Proposition 3.4.2, we may assume that higher degree polynomials have unitary Hilbert function. The minimal length of these schemes is $e+1$, where $e \geq d$ is the minimum degree such that $p_1(\mathcal{X}^e_d)$ dominates $\mathbb{P}(\oplus_{i=0}^d S_i)$.

Conjecture 3.4.14 implies that e needs to be the minimum degree such that $2e + 2 \geq \binom{d+2}{2} - 1$, that is,

$$e = \left\lceil \frac{d^2 + 3d - 4}{4} \right\rceil.$$

Remark 3.4.16. Recall from Theorem 3.3.1 that the natural rank of a general ternary form of degree d is $\lfloor \frac{d(d+4)}{4} \rfloor$. Conjecture 3.4.15 implies that the local cactus rank is equal to the natural rank only when $d \leq 3$.

3.5 Ternary quintics: an extensive example

In this final section we take a closer look at the case of ternary quintics by retracing the ideas illustrated in the previous sections. Regarding the natural rank, the computations described in the proof of Theorem 3.3.1 will be made explicit. For the local cactus rank, on the one hand we rely on the results and examples illustrated in Section 3.4; on the other hand we explain some alternative, more direct and ad hoc strategies. Specifically, we will give an account of the possible Hilbert functions of apolar schemes, excluding those that for purely geometric reasons cannot compute the local cactus rank.

The natural rank

The natural rank of a general quintic is 11. To see this, we may begin with excluding certain Hilbert functions for the dehomogenization:

Lemma 3.5.1. *For a general quintic $F \in \mathbb{P}(\bar{S}_5)$ and any linear form l, we have $H_{F_l} \geq (1,2,2,2,2,1)$.*

Proof. Without loss of generality we may assume that $l = x_0$. Let us write $F_l = \sum_{i=0}^{5} f_i$. If there was a linear form l such that $H_{F_l}(4) = 1$, then f_5 would be a pure power of a linear form. The zero locus $V(f_5) = V(F,l)$ would then be one point, counted with multiplicity 5. The curve $V(F)$ would have a hyperinflection point, which is not a general property. ∎

Alternative proof. We can alternatively see this by looking at the codimension of the 1-st degeneracy locus of the general catalecticant element

$$S^1\mathcal{Q}^* \xrightarrow{\phi} S^4\mathcal{Q} \otimes \mathcal{O}(5),$$

which by Proposition 3.3.12 is empty. ∎

We are remained with a few possibilities to check:

$$(1,2,2,2,2,1) \qquad (1,2,3,2,2,1) \qquad (1,2,3,3,2,1).$$

The right-most one is the general one with $\operatorname{len}(1,2,3,3,2,1) = 12 = \lambda_5$. We focus on the remaining two, of length 10 and 11.

The two Propositions 3.3.7 and 3.3.8 say that for a general ternary quintic there exist 342 linear forms such that the Hilbert function of the dehomogenization is at most $(1,2,3,2,2,1)$ and that there are no linear forms such that the Hilbert function of the dehomogenization is at most $(1,2,2,2,2,1)$.

We are going to see this by running through the computations explained in the Section 3.3.

We need to compute the degeneracy class $D_2(\phi)$ for the general element

$$\phi\colon S^2\mathcal{Q}^* \to S^3\mathcal{Q} \otimes \mathcal{O}(5),$$

where $\mathcal{O} = \mathcal{O}_{\mathbb{P}V}$ and $V \simeq \mathbb{C}^3$.

We have

$$c(\mathcal{O}(5)) = 1 + 5h, \qquad c(\mathcal{Q}) = 1 + h + h^2 \qquad c(\mathcal{Q}^*) = 1 - h + h^2.$$

Assuming that $\mathcal{Q} = \mathcal{L} \oplus \mathcal{M}$, with $\mathcal{L}, \mathcal{M}$ line bundles and setting $a := c_1(\mathcal{L}), b := c_1(\mathcal{M})$, we have:

$$\begin{aligned} c(S^2\mathcal{Q}^*) &= (1-2a)(1-a-b)(1-2b) \\ &= 1 - 3(a+b) + (2a^2 + 8ab + 2b^2) \\ &= 1 - 3c_1(\mathcal{Q}) + (2c_1(\mathcal{Q})^2 + 4c_2(\mathcal{Q})) \\ &= 1 - 3h + 6h^2. \end{aligned}$$

Analogously, we compute:

$$\begin{aligned}c(S^3\mathcal{Q}) &= (1+3a)(1+2a+b)(1+a+2b)(1+3b)\\&= 1+6(a+b)+32ab+11(a^2+b^2)\\&= 1+6c_1(\mathcal{Q})+11c_1(\mathcal{Q})^2+10c_2(\mathcal{Q})\\&= 1+6h+21h^2.\end{aligned}$$

So, applying Proposition 3.3.14, we obtain

$$c(S^3\mathcal{Q}\otimes\mathcal{O}(5)) = 1+26h+261h^2.$$

Then by Porteous' Theorem, the class of $D_2(\phi)$ is:

$$\begin{aligned}[D_2(\phi)] &= \left\{\frac{c(S^3\mathcal{Q}\otimes\mathcal{O}(5))}{c(S^2(\mathcal{Q}^*))}\right\}_2\\&= \{(1+26h+261h^2)(1-(6h^2-3h)+(6h^2-3h)^2\}_2\\&= \{(1+26h+261h^2)(1+3h+3h^2)\}_2\\&= \{1+29h+342h^2\}_2\\&= 342h^2.\end{aligned}$$

The local cactus rank

The local cactus rank of a general ternary quintic form is bounded by its natural rank, that is 11. We need to check whether there exist a polynomial g in two variables of degree greater than 5 satisfying $\operatorname{len}(H_g) < 11$ and such that the quintic tail $g_{\leq 5}$ can be obtained as the dehomogenization of a general quintic.

Polynomials g of this kind clearly must have degree between 6 and 9. All the possibilities for their Hilbert functions are listed in Table 3.2.

Length	**Degree**	**Hilbert function**
7	6	$(1,1,1,1,1,1,1)$
8	6	$(1,2,1,1,1,1,1)$
	7	$(1,1,1,1,1,1,1,1)$
9	6	$(1,2,2,1,1,1,1)$
	7	$(1,2,1,1,1,1,1,1)$
	8	$(1,1,1,1,1,1,1,1,1)$
10	6	$(1,2,2,2,1,1,1)$
	7	$(1,2,2,1,1,1,1,1)$
	8	$(1,2,1,1,1,1,1,1,1)$
	9	$(1,1,1,1,1,1,1,1,1,1)$

Table 3.2: Possible H_g with $\deg(g) > 5$ and $\operatorname{len}(H_g) \leq 10$.

Note that if g has degree 6 and its Hilbert function is one of those listed in Table 3.2, then $H_g(5) = H_g(4) = 1$.

Lemma 3.5.2. *The local catus rank of a general quintic is at least* 8.

Proof. It is enough to show that the local cactus rank of a general quintic is not computed by polynomials of degree 6. Let $g \in \mathbb{C}[x_1, x_2]$ be of degree 6 such that $H_g(5) = 1$. The space of degree-5 partials is one-dimensional, which means that, after a suitable change of coordinates, we can assume that g is of the form

$$g = x_1^6 + \sum_{i+j \leq 5} a_{(i,j)} x_1^i x_2^j.$$

Let us denote with ∂_1, ∂_2 the dual operators of x_1 and x_2. Then, both $\partial_2(g)$ and $\partial_1^2(g)$ are partials of degree 4. If we additionally assume that $H_g(4) = 1$, then such partials must have leading term proportional to x_1^4. Equivalently, we can see this by writing the matrix of coefficients of $\partial_2(g)$ and $\partial_1^2(g)$ in degree 4:

$$\begin{array}{c c} & \begin{array}{ccccc} x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{array} \\ \begin{array}{c} \partial_2 \\ \partial_1^2 \end{array} & \begin{bmatrix} a_{(4,1)} & a_{(3,2)} & a_{(2,3)} & a_{(4,1)} & a_{(0,5)} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

We need this matrix to have rank 1, that is:

$$a_{(3,2)} = a_{(2,3)} = a_{(1,4)} = a_{(0,5)} = 0.$$

The polynomial g is then of the form:

$$g = x_1^6 + a_{(5,0)} x_1^5 + a_{(4,1)} x_1^4 x_2 + \sum_{i+j \leq 4} a_{(i,j)} x_1^i x_2^j.$$

In particular, the quintic tail of g has a hyperinflection point in $(0, 1)$, so a general homogenous quintic cannot have a dehomogenization of this kind. ∎

We then pass to consider polynomials g of degree 7 whose Hilbert function is again one of those listed in Table 3.2. In all the cases we have $H_g(6) = H_g(5) = H_g(4) = H_g(3) = 1$.

Lemma 3.5.3. *The local cactus rank of a general quintic is at least* 9.

Proof. It is enough to show that the local cactus rank of a general quintic is not computed by forms of degree 7. Let $g \in \mathbb{C}[x_1, x_2]$ be of degree 7. If $H_g(6) = H_g(5) = 1$, then we may argue as for Lemma 3.5.2 and assume that, up to a suitable change of coordinates, we have

$$g = x_1^7 + a_{(6,0)} x_1^6 + a_{(5,1)} x_1^5 x_2 \sum_{i+j \leq 5} a_{(i,j)} x_1^i x_2^j.$$

The space of partials of degree 4 is generated by $\partial_1\partial_2(g), \partial_1^3(g)$ and the linear combination $(\partial_2 - a_{(5,1)}\partial_1^2)(g)$. Under the condition $H_g(4) = 1$, all such partials must have proportional leading term. We are asking the matrix

$$\begin{array}{c} \\ \partial_2 - a_{(5,1)}\partial_1^2 \\ \partial_1\partial_2 \\ \partial_1^3 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{array} \\ \begin{bmatrix} a_{(4,1)} - a_{(6,0)}a_{(5,1)} & a_{(3,2)} - a_{(5,1)}^2 & a_{(2,3)} & a_{(1,4)} & a_{(0,5)} \\ a_{(5,1)} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

to have rank 1, namely

$$a_{(2,3)} = a_{(1,4)} = a_{(0,5)} = 0, \qquad a_{(3,2)} - a_{(5,1)}^2 = 0.$$

Finally, we observe that both $\partial_1^4(g)$ and the combination

$$D := [\partial_2 - a_{(5,1)}\partial_1^2 - (a_{(4,1)} - a_{(6,0)}a_{(5,1)})\partial_1^3](g)$$

are partials of degree 3. The condition $H_g(3) = 1$, then again we asks rank 1 for

$$\begin{array}{c} \\ D \\ \partial_1^4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{array} \\ \begin{bmatrix} * & * & * & a_{(0,4)} \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

which gives the condition $a_{(0,4)} = 0$. The quintic tail of g is then of the form

$$\begin{aligned} g_{\leq 5} = a_{(5,0)}x_1^5 + a_{(4,1)}x_1^3x_2 + a_{(3,2)}x_1^2x_2^2 + a_{(2,3)}x_1^2x_2^3 \\ + a_{(4,0)}x_1^4 + a_{(3,1)}x_1^3x_2 + a_{(2,2)}x_1^2x_2^2 + a_{(1,3)}x_1^3x_2 + \sum_{i+j\leq 4} a_{(i,j)}x_1^ix_2^j, \end{aligned}$$

which is singular at $(0, 1)$ and therefore it cannot be the dehomogenization of a general ternary quintic. ∎

In general, when the degree of g is close to the degree d of the tail, it is easy to understand the geometric properties we are asking for (singularities, hyperinflection points, etc. see also Section 3.2).

On the other hand, when the degree of the tail is low compared to the degree of g, these properties are not so immediate. This happens also in the case of quintics, when analyzing the remaining cases of Table 3.2. To conclude that the local cactus rank of a general quintic is equal to 10, we may instead use the more computational approach outlined in Section 3.4.2.

Part II

On the PGL(4)-orbit of a quaternary cubic

Chapter 4

Towards the degree of the PGL(4)-orbit of a cubic surface

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Published in *Le Matematiche (Catania)*, vol.75, no.2 (2020), pp. 439-456.

Abstract

We study the action of the group PGL(4) on the parameter space $\mathbb{P}^{19}$ of complex cubic surfaces. Specifically, we look at how the techniques used by Aluffi and Faber in [4] can be extended to compute the degree of the orbit closure $\overline{O}$ of a general cubic surface. We study the base locus of the induced rational map $\mathbb{P}^{15} \dashrightarrow \overline{O} \subset \mathbb{P}^{19}$, and the first steps in resolving this rational map by successively blowing up the reduced base locus.

4.1 Introduction

A complex cubic surface $\mathcal{S}$ in $\mathbb{P}^3$ is the vanishing locus of a homogenous degree-3 form of the type

$$F(\mathbf{x}) = a_0x_0^3 + a_1x_0^2x_1 + \cdots + a_{19}x_3^3.$$

It is clear that cubic surfaces are parametrized by $\mathbb{P}\mathrm{Sym}^3(\mathbb{C}^4)^* \simeq \mathbb{P}^{19}$. However, two isomorphic surfaces correspond to different points in $\mathbb{P}^{19}$ and the simplest way this can happen is when changing coordinates. A natural question would then be: *Given a fixed $\mathcal{S}$ as above, which other cubic surfaces arise from $\mathcal{S}$ by coordinate change?* In other words, we are asking to describe the orbit O of $\mathcal{S}$ under the action of the group PGL(4) on the parameter space $\mathbb{P}^{19}$.

We would like to study the geometry of O: since this just forms a locally closed subset in $\mathbb{P}^{19}$, we will rather consider its closure $\overline{O}$. A first step in this direction is to compute its degree. The latter will depend on the choice of the surface $\mathcal{S}$ and in this paper we will primarily focus on the case where $\mathcal{S}$ is chosen to be general.

In the special case of the Cayley cubic, a surface with four distinct nodes, the degree of the orbit closure is already known. In [57], this number is computed to be 305, based on counting cubic surfaces with 4 distinct double points passing through 15 general points.

When $\mathcal{S}$ is general, the degree of the orbit closure will be significantly higher and different techniques will be needed. One could start by looking at the map $\phi\colon \mathrm{PGL}(4) \to \mathbb{P}^{19}$, sending the class of a matrix to its pre-composition with F. The image of this map is the orbit O and computing the degree of its closure would amount to count the number of points in the intersection of $\overline{O}$ with a general linear subspace of complementary dimension.

We can count the number of such points using intersection theory by finding a pair $(\widetilde{\mathcal{V}}, \widetilde{\phi})$ such that $\widetilde{\mathcal{V}}$ is a compactification of PGL(4) and $\widetilde{\phi}$ a dominant regular morphism from $\widetilde{\mathcal{V}}$ to $\overline{O}$ extending ϕ and the intersections of the pull-back of a hyperplane class by $\widetilde{\phi}$ is transversal. Then we can simply compute $\widetilde{\phi}^* c_1(\mathcal{O}_{\mathbb{P}^{19}}(1))^{15}$.

The first naïve compactification one could think of is $\mathbb{P}\mathrm{Hom}(\mathbb{C}^4, \mathbb{C}^4) \simeq \mathbb{P}^{15}$, which can be as well equipped with the pre-composition map which naturally extends ϕ. Unfortunately, this pair is not good enough since the given map is not regular. From a computational viewpoint, issues come from the fact that the pull-back classes we are considering will intersect in positive dimension.

The strategy that we would like to pursue here is to find an explicit resolution of ϕ where it is possible to keep track of how the intersections change. This approach was already considered by Aluffi and Faber who studied the case of plane curves of any degree. What we are going to do in the present paper is to adapt many of the ideas contained in there. In particular, we decide to regularize ϕ by a sequence of blow-ups at smooth centers. We will start by describing the support of the base locus $\mathrm{Bs}(\phi)$ from a set-theoretical point of view. We will then study the first steps towards the resolution of ϕ by successively blowing up the reduced components of the base locus.

Four of these steps will be analyzed, though currently it is not clear if they will be sufficient to give the desired resolution. This difficulty reflects an important difference from the case of plane curves: here the base locus of ϕ has many components, and this is a consequence of the fact that a general cubic surface contains 27 distinct lines. More specifically, we will see that problems can possibly arise from those morphisms in $\mathbb{P}^{15}$ whose image is spanned by a point contained in one of these lines.

The aim of this paper is to present a report of an on-going project, where the remaining work that needs to be done regards not only proving or disproving the existence of further components to blow up. Indeed, as mentioned above, there is also a computational aspect, namely showing how the different steps in the resolution contribute in finding the degree of $\overline{O}$. These computations will not be analyzed here, since the results would currently be very partial. They are hopefully going to appear in a future paper, as the natural conclusion of the work illustrated here. For this second part as well, we believe that a considerable inspiration could be taken by the techniques developed in [4].

An alternative approach to the same problem has been recently explored by Brustenga i Moncusí, Timme and Weinstein in [14]. There is however a significant difference between the methods. Indeed, in their paper the computation of the degree of the orbit closure is treated from a more numerical perspective. The idea is to count the number of solutions of a system of polynomial equations in an affine variety using homotopy continuation and monodromy methods. As a result, for a general $\mathcal{S}$, this number turns out to be 96120. On the other hand, applying intersection theory in the context of resolutions of singularities gives a more geometric flavor and we believe that this will help to shed some light on a complete understanding on the studied phenomena.

The problem was firstly introduced to us from the *27 Questions on Cubic Surfaces* (see [46]), in view of the First Meeting on Cubic Surfaces, that was held in Oslo on May 13, 2019. We would like to thank: Kristian Ranestad and Corey Harris for the valuable discussions and the patience with the many questions, Paolo Aluffi for very nice explanations about his paper [4], Maddie Weinstein for stimulating conversations, the anonymous referees for all the corrections and suggestions.

4.2 Setup

In this section we will first describe the action of PGL(4) on the parameter space of cubic surfaces. This will naturally produce rational maps

$$\mathbb{P}^{15} \simeq \mathbb{P}\mathrm{Hom}(W, W) \dashrightarrow \mathbb{P}\mathrm{Sym}^3(W^*) \simeq \mathbb{P}^{19},$$

one for every fixed cubic. If the latter is chosen to be general, it will be possible to illustrate how this map can be used to compute the degree of the orbit closure. Throughout the paper we will work over the field $\mathbb{C}$ of complex numbers.

4.2.1 The action of PGL(4)

Let us denote with W the 4-dimensional vector space $\mathbb{C}^4$. A complex cubic surface $\mathcal{S} \subset \mathbb{P}W$ is the zero set of a homogeneous degree-3 polynomial in four variables, say

$$F(\mathbf{x}) = a_0 x_0^3 + a_1 x_0^2 x_1 + \cdots + a_{18} x_2 x_3^2 + a_{19} x_3^3,$$

which corresponds to a point $[F] := [a_0 : a_1 : \cdots : a_{19}]$ in the parameter space $\mathcal{F} := \mathbb{P}\mathrm{Sym}^3(W^*)$. The group PGL(4) acts on $\mathcal{F}$ by pre-composition (or, equivalently, by coordinate change):

$$\begin{array}{ccc} \mathrm{PGL}(4) \times \mathcal{F} & \to & \mathcal{F} \\ (\alpha, [F(\mathbf{x})]) & \mapsto & [F(\alpha(\mathbf{x}))]. \end{array}$$

For a fixed $\mathcal{S}$ (hence for a fixed F), this yields a map $\phi\colon \mathrm{PGL}(4) \to \mathcal{F}$, whose image is by definition the orbit O of F. Moreover, the fiber $\phi^{-1}(F)$ is the set of automorphisms of W leaving F unchanged, so it corresponds to group of linear automorphisms of $\mathcal{S}$. Our object of study is the degree of $\overline{O}$ in $\mathcal{F}$: to this purpose, we first need to understand the dimension of O and the degree of ϕ.

Let us denote with $\mathcal{V}$ the space $\mathbb{P}\mathrm{Hom}(W, W)$ of nonzero endomorphisms of W up to projective equivalence, which is also canonically isomorphic to the space of matrices $\mathbb{P}(W^* \otimes W)$.

Lemma 4.2.1. *Let $\mathcal{S}$ be a cubic surface with finite group of linear automorphisms. Then* $\dim \overline{O} = 15$.

Proof. By hypothesis ϕ is a finite map, so $\dim O = \dim \mathrm{PGL}(4)$. But $\dim O = \dim \overline{O}$ and PGL(4) embeds as an open subset $\mathcal{V}$, whose dimension is 15. ■

From now on we will consider $\mathcal{S}$ to be *general*, meaning that its corresponding point in $\mathcal{F}$ lies in some proper Zariski open subset.

Lemma 4.2.2. *If $\mathcal{S}$ is a general cubic surface, the above map ϕ has degree* 1.

Proof. We will prove that each fiber of ϕ consists of a single point. Suppose that there exist two points α_1, α_2 of PGL(4) with the property that $F(\alpha_1(\mathbf{x})) = F(\alpha_2(\mathbf{x}))$: then the composite $\alpha_1^{-1} \circ \alpha_2$ would be a linear automorphism of $\mathcal{S}$. But a general cubic surface has no nontrivial linear automorphisms (see [37]), so $\alpha_1 = \alpha_2$. ■

4.2.2 How to compute the degree of $\overline{O}$

As we noticed in the proof of Lemma 4.2.1, we can see PGL(4) as an open subset of $\mathcal{V}$; in particular the map ϕ can be understood as a rational map $\mathcal{V} \dashrightarrow \mathcal{F}$, which, by abuse of notation, we will keep calling ϕ. The strategy from [4] that we want to apply here is to resolve ϕ by a sequence of blow-ups in $\mathcal{V}$ and finally get a regular map $\widetilde{\phi}\colon \widetilde{\mathcal{V}} \to \mathcal{F}$, where $\widetilde{\mathcal{V}}$ is a smooth compactification of PGL(4) and $\operatorname{im}\widetilde{\phi} = \overline{O}$. The blow-ups will also produce a morphism $\pi\colon \widetilde{\mathcal{V}} \to \mathcal{V}$, such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{PGL}(4) \arrow[r, hook] \arrow[d, equal] & \widetilde{\mathcal{V}} \arrow[r, "\widetilde{\phi}"] \arrow[d, "\pi"] & \mathcal{F} \arrow[d, equal] \\ \mathrm{PGL}(4) \arrow[r, hook] & \mathcal{V} \arrow[r, dashed, "\phi"] & \mathcal{F} \end{tikzcd} .$$

Given this construction, we can compute the degree d of $\overline{O}$ as follows: let $\widetilde{\phi}_*\colon \mathrm{CH}(\widetilde{\mathcal{V}}) \to \mathrm{CH}(\mathcal{F})$ be the push-forward map between the corresponding Chow rings and let us recall that $\overline{O}$ is a 15-dimensional subvariety of $\mathcal{F}$.

Then by definition $d = \int_{\mathcal{F}} [\overline{O}] \cdot H^{15}$, where $\int_{\mathcal{F}}(\cdot)$ denotes the degree of the 0-dimensional part, while H denotes the hyperplane class in $\mathrm{CH}(\mathcal{F}) \simeq \mathbb{Z}[H]/H^{16}$. On the other hand, by construction $\widetilde{\mathcal{V}}$ dominates $\overline{O}$, so $\deg\widetilde{\phi} \cdot [\overline{O}] = \widetilde{\phi}_*(1)$. Then, using the projection formula, we find:

$$\deg\widetilde{\phi} \cdot d = \int_{\mathcal{F}} \widetilde{\phi}_*(1 \cdot \widetilde{\phi}^* H^{15}) = \int_{\widetilde{\mathcal{V}}} \widetilde{\phi}^* H^{15}. \tag{4.2.1}$$

Definition 4.2.3. With notation as above, we define the *predegree* of $\overline{O}$ to be $\int_{\widetilde{\mathcal{V}}} (\widetilde{\phi}^*(H))^{15}$.

Note that, even when $\mathcal{S}$ has nontrivial linear automorphisms, it is possible to use equation (4.2.1) to find the degree of $\overline{O}$ by dividing the predegree by the order of the group of linear automorphisms. In the general case we have the following result:

Proposition 4.2.4. *For a general cubic surface $\mathcal{S}$, the degree of $\overline{O}$ equals its predegree.*

Proof. Indeed, thanks to Lemma 4.2.2, we know that if $\mathcal{S}$ is general, $\deg \widetilde{\phi} = 1$; then the expression (4.2.1) gives the desired equality. ■

The first step towards the resolution of ϕ is to understand its base locus $\mathrm{Bs}(\phi)$. To this purpose we note that the linear system defining ϕ is spanned by a certain set of hypersurfaces having a nice geometric interpretation.

Definition 4.2.5. Let $\mathcal{S} = V(F)$ be a cubic surface in $\mathbb{P}W$. For every $p \in \mathbb{P}W$, the *point condition* P_p is defined as:

$$P_p = \{\alpha \in \mathcal{V} \mid F(\alpha(p)) = 0\},$$

namely the zero locus of $F(\alpha(p))$ as a polynomial in α.

Since the point conditions span the linear system defining ϕ, the base locus $\mathrm{Bs}(\phi)$ can be identified with the intersection $\bigcap_{p\in\mathbb{P}W} P_p$. After blowing up this locus in $\mathcal{V}$, we will get a new rational map, whose base locus will be described by the intersection of the proper transforms of the point conditions, and so on. Moreover, if we denote by $\widetilde{P_p}$ the proper transform of P_p in $\widetilde{\mathcal{V}}$, we see that $d = \int_{\widetilde{\mathcal{V}}}[\widetilde{P_p}]^{15}$.

Although the main focus of this paper is to illustrate the several steps needed to resolve ϕ, we would like to mention here a very important proposition, which can be (repeatedly) used to tell how the various blow-ups contribute in the computation of the degree of $\overline{O}$.

Proposition 4.2.6 ([4, Proposition 3.2])**.** *Let $i\colon B \to V$ be an inclusion of non-singular projective varieties, and let $X \subset V$ be a codimension-1 subvariety, smooth along B. Let $\widetilde{V}$ be the blow-up of V along B, and let $\widetilde{X}$ be the proper transform of X. Then*

$$\int_{\widetilde{V}} [\widetilde{X}]^{\dim V} = \int_V [X]^{\dim V} - \int_B \frac{([B] + i^*[X])^{\dim V}}{c(N_{B/V})},$$

where $c(N_{B/V})$ denotes the total Chern class of the normal bundle of B in V.

In our situation, the role of V and X will be played by $\mathcal{V}$ and P_p, while B will represent each time a component of the reduced base locus that we are blowing up. Since the point-conditions are cubic hypersurfaces in $\mathcal{V}$, we have $\int_{\mathcal{V}}[P_p]^{15} = 3^{15}$. Then the degree of $\overline{O}$ will be $3^{15} - n_1 - \cdots - n_k$, where the n_i's are the contributions of the blown up loci that can be explicitly computed using Proposition 4.2.6.

At each step, the most difficult part will be to compute $c(N_{B/\mathcal{V}})$ in the Chow ring $\mathrm{CH}(B)$. This motivates us to look for a resolution, by picking a suitable sequence of blow-ups that allows to handle this computation easily.

The contributions coming from the sequence of blow-ups is left for a future paper, that is thought to be the natural continuation of the present one.

4.3 Towards the resolution of ϕ

In this section, we will describe the first steps necessary to regularize ϕ according to the strategy described in Section 4.2.2. It is not yet clear if these are enough or if more blow-ups are required. An important difference from the case of plane curves studied in [4] is that the base locus $\mathrm{Bs}(\phi)$ has not only one, but many components, reflecting the fact that a general cubic surface contains 27 distinct lines.

4.3.1 The base locus of ϕ

With the next proposition we are going to describe $\mathrm{Bs}(\phi)$ as a set. To this purpose, we look at $\mathcal{V}$ as the space of matrices $\mathbb{P}(W^* \otimes W)$, together with the Segre embedding

$$\mathbb{P}W^* \times \mathbb{P}W \hookrightarrow \mathbb{P}(W^* \otimes W)$$

given by

$$([k_0 : \cdots : k_3], [q_0 : \cdots : q_3]) \mapsto \begin{bmatrix} k_0q_0 & k_1q_0 & k_2q_0 & k_3q_0 \\ k_0q_1 & k_1q_1 & k_2q_1 & k_3q_1 \\ k_0q_2 & k_1q_2 & k_2q_2 & k_3q_2 \\ k_0q_3 & k_1q_3 & k_2q_3 & k_3q_3 \end{bmatrix},$$

where $k^\perp := \{\mathbf{x} \in \mathbb{P}W \mid k_0x_0 + \cdots + k_3x_3 = 0\}$ is the kernel of such a matrix and $q := [q_0 : \cdots : q_3]$ its image.

Proposition 4.3.1. *Let $\mathcal{S} = V(F)$ be a general smooth cubic surface in $\mathbb{P}W$. Let ϕ be the map defined above. Then $\mathrm{Bs}(\phi)$ is supported at the union of two closed components B and C, with:*

(i) $B \simeq \mathbb{P}W^* \times \mathcal{S}$;

(ii) $C \simeq \cup_{i=1}^{27} C_i$,

where the C_i's are the irreducible components of C and each C_i is isomorphic to $\mathbb{P}^7$.

Proof. The map ϕ is not defined over the set

$$\{\alpha \in \mathcal{V} \mid F(\alpha(\mathbf{x})) \equiv 0\} = \{\alpha \in \mathcal{V} \mid \mathrm{im}\,\alpha \subset V(F)\}.$$

Since $\mathcal{S}$ is taken to be general, its linear subspaces are points in $\mathcal{S}$ and the 27 lines, that we denote by $\ell_1, \ldots, \ell_{27}$. We can write the base locus as

$$B \cup C,$$

where

$$B := \{\alpha \in \mathcal{V} \mid \mathrm{rk}\,\alpha = 1, \mathrm{im}\,\alpha \in \mathcal{S}\},$$
$$C := \{\alpha \in \mathcal{V} \mid \mathrm{rk}\,\alpha \leq 2, \mathrm{im}\,\alpha \subseteq \ell_i \text{ for some } i\}.$$

Then:

(i) The matrices in B are parametrized by the choice of a point in $\mathcal{S}$ and the choice of a 4-tuple of coefficients in $\mathbb{P}W^*$ (indeed each column must be a multiple of the chosen point). Hence $B \simeq \mathbb{P}W^* \times \mathcal{S}$.

(ii) Regarding C, it consists of 27 components $\{C_i\}_{i=1}^{27}$, where C_i is the space of matrices whose image is spanned by ℓ_i. So for every i we can make the identification $C_i \simeq \mathbb{P}\operatorname{Hom}(W, U)$, where U is the 2-dimensional subspace of W for which $\mathbb{P}(U) = \ell_i$. This is a 7-dimensional projective linear space and in particular we get $C \simeq \cup_{i=1}^{27} \mathbb{P}^7$.

■

Remark 4.3.2. Alternatively, one can see the elements of a fixed C_i as the sum of two rank-1 matrices parametrized by the choice of a point on the given line and the choice of a 4-tuple of coefficients in $\mathbb{P}W^*$. In other words, C_i is the union of the span of all pairs of points in $\mathbb{P}W^* \times \ell_i$ (including the degenerate case in which the two points coincide), so we are describing the secant variety $\sigma_2(\mathbb{P}W^* \times \ell_i) \simeq \sigma_2(\mathbb{P}^3 \times \mathbb{P}^1)$, which is a $\mathbb{P}^7$.

Remark 4.3.3. The subset, $\mathrm{PGL}(4) \subset \mathcal{V}$ does not intersect $\mathrm{Bs}(\phi)$, so as we resolve the rational map ϕ, we still get compactifications of $\mathrm{PGL}(4)$.

Remark 4.3.4. The above proof actually says more: the two components B and C intersect in

$$B \cap C = \{\alpha \in \mathcal{V} \mid \operatorname{rk}\alpha = 1, \operatorname{im}\alpha \text{ is a point on } \ell_i \text{ for some } i\}.$$

In particular, this implies the following Corollary.

Corollary 4.3.5. *Let $C_i, i = 1, \dots, 27$ be the components of C, each isomorphic to $\mathbb{P}^7$. Then*

$$B \cap C_i \simeq \mathbb{P}W^* \times \ell_i.$$

Moreover, for $i \neq j$ we have

$$C_i \cap C_j \simeq \begin{cases} \mathbb{P}W^* & \text{if } \ell_i \cap \ell_j \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

As we have mentioned at the end of Section 4.2, since $\mathrm{Bs}(\phi)$ has many components, there are many ways of resolving the map. The following order of blow-ups at smooth centers is suited for relating the base loci of the induced maps to properties of point conditions in $\mathcal{V}$.

We start by blowing up $\mathcal{V}$ along the component $B \simeq \mathbb{P}W^* \times \mathcal{S}$: this produces a morphism $\pi_1 : \mathcal{V}_1 \to \mathcal{V}$ and an exceptional divisor $E_1 \subset \mathcal{V}_1$. After blowing up B, the proper transforms of the point condition, denoted by $P_p^{(1)}$, will define a new rational map $\phi_1 : \mathcal{V}_1 \dashrightarrow \mathcal{F}$. Note that $B \cap \mathrm{PGL}(4) = \emptyset$ in $\mathcal{V}$, so $\mathcal{V}_1$ contains an open dense subset isomorphic to $\mathrm{PGL}(4)$ and with a little abuse of notation we will indicate it using the same symbol. Let us denote with $C_i^{(1)}$ the proper transform of C_i in $\mathcal{V}_1$ for every i.

$$\begin{tikzcd}
G_1,\dots,G_{27} \arrow[d] \arrow[rrrr, hook, bend right] & E_3^{(1)} \arrow[d] & E_2^{(2)} \arrow[d] & E_1^{(3)} \arrow[r, hook] \arrow[d] & \mathcal{V}_4 \arrow[r, dashed, "\phi_4"] \arrow[d, "\pi_4"'] & \mathcal{F} \arrow[d, equal] \\
C_1^{(3)},\dots,C_{27}^{(3)} \arrow[d] & E_3 \arrow[d] & E_2^{(1)} \arrow[d] & E_1^{(2)} \arrow[r, hook] \arrow[d] & \mathcal{V}_3 \arrow[r, dashed, "\phi_3"] \arrow[d, "\pi_3"'] & \mathcal{F} \arrow[d, equal] \\
C_1^{(2)},\dots,C_{27}^{(2)} \arrow[d] & B_2 \arrow[r, hook] & E_2 \arrow[d] & E_1^{(1)} \arrow[r, hook] \arrow[d] & \mathcal{V}_2 \arrow[r, dashed, "\phi_2"] \arrow[d, "\pi_2"'] & \mathcal{F} \arrow[d, equal] \\
C_1^{(1)},\dots,C_{27}^{(1)} \arrow[d] & & B_1 \arrow[r, hook] & E_1 \arrow[r, hook] \arrow[d] & \mathcal{V}_1 \arrow[r, dashed, "\phi_1"] \arrow[d, "\pi_1"'] & \mathcal{F} \arrow[d, equal] \\
C_1,\dots,C_{27} \arrow[rrrr, hook, bend right] & & & B \arrow[r, hook] & \mathcal{V} \arrow[r, dashed, "\phi"] & \mathcal{F}
\end{tikzcd}$$

Figure 4.1: The sequence of blow-ups

Claim 4.3.6. *The base locus* $\mathrm{Bs}(\phi_1)$ *is supported on the* 27 *components* $C_i^{(1)}$*'s, which are disjoint, plus a further component, denoted by* B_1*, contained in the exceptional divisor* E_1*, intersecting the* $C_i^{(1)}$*'s.*

We will choose B_1 to be the center of the second blow-up. As before, this will produce a new morphism $\pi_2 \colon \mathcal{V}_2 \to \mathcal{V}_1$, together with an exceptional divisor $E_2 \subset \mathcal{V}_2$. Again, the proper transforms of the point conditions, denoted by $P_p^{(2)}$, will define a rational map $\phi_2 \colon \mathcal{V}_2 \dashrightarrow \mathcal{F}$.

Claim 4.3.7. *The support of* $\mathrm{Bs}(\phi_2)$ *contains the* 27 *pairwise disjoint proper transforms* $C_i^{(2)}$*'s and a subvariety, denoted by* B_2*, which has a dominant* $2\colon 1$ *map to* B*.*

Note that it is not clear whether the subvariety B_2 is irreducible or not. What we will prove is that it must consist of either 1 or 2 components. Moreover, we need to observe that Claim 4.3.7 refers to an inclusion, but not an equality, so there might be some other components in $\mathrm{Bs}(\phi_2)$, namely the ones dominating the intersections $B \cap C_i \simeq \mathbb{P}W^* \times \ell_i$.

If we assume that we have exactly the components listed in above, we can proceed by blowing up B_2. We get as usual a map $\pi_3 \colon \mathcal{V}_3 \to \mathcal{V}_2$, an exceptional divisor $E_3 \subset \mathcal{V}_3$ and a rational map $\phi_3 \colon \mathcal{V}_3 \dashrightarrow \mathcal{V}$ induced by the proper transforms of point conditions. We expect no component of the base locus of ϕ_3 to dominate B. In fact, one might hope that the only components of $\mathrm{Bs}(\phi_3)$ are the $C_i^{(3)}$, and that blowing up these components resolves the rational map.

We summarize the construction in Figure 4.1, which also fixes notation for the rest of the section.

4.3.2 The base locus after blowing up B

We now aim to prove Claim 4.3.6; in particular, we are interested in giving the set-theoretical description of $B_1 := E_1 \cap \mathrm{Bs}(\phi_1)$.

To this purpose, we recall that B is embedded in $\mathcal{V}$ via the Segre embedding. In particular, for every $\alpha \in \mathcal{V}$, we may identify the space $T_\alpha\mathcal{V}$ with the quotient $(W^* \otimes W)/\alpha\mathbb{C}$. Let $\alpha = (k, q)$ be a point in B and let us denote with $\sigma = T_q\mathcal{S}$ the tangent space of $\mathcal{S}$ at the point q.

Lemma 4.3.8. *With the identification $T_\alpha\mathcal{V} \simeq (W^* \otimes W)/\alpha\mathbb{C}$, we have:*

(i) $T_\alpha B = \{\tau \in W^* \otimes W \mid \operatorname{im}\tau \subset \sigma,\ \tau(k^\perp) \subset q\}/\alpha\mathbb{C}$.

(ii) $T_\alpha(\mathbb{P}W^* \times \ell_i) = \{\tau \in W^* \otimes W \mid \operatorname{im}\tau \subset \ell_i, \tau(k^\perp) \subset q\}/\alpha\mathbb{C}$.

(iii) *The point condition P_p is non-singular at α and*

$$T_\alpha P_p = \{\tau \in W^* \otimes W \mid \tau(p) \subset \sigma\}/\alpha\mathbb{C}.$$

Proof. The ideas in this proof are essentially the same of [4, Lemma 2.1].

(i) The (5-dimensional) tangent space of B at α is

$$\begin{aligned} T_\alpha B &= T_k(\mathbb{P}W^* \times \{q\}) \oplus T_q(\{k\} \times \mathcal{S}) \\ &= \frac{\{k' \otimes q \in W^* \otimes W \mid k' \in \mathbb{P}W^*\}}{k\mathbb{C}} \oplus \frac{\{k \otimes q' \in W^* \otimes W \mid q' \in \sigma\}}{q\mathbb{C}} \\ &= \frac{\{\tau \in W^* \otimes W \mid \operatorname{im}\tau = q\}}{k\mathbb{C}} \oplus \frac{\{\tau \in W^* \otimes W \mid \ker\tau = k^\perp, \operatorname{im}\tau \subset \sigma\}}{q\mathbb{C}}. \end{aligned}$$

The two spaces in the direct sum decomposition are both contained in the space

$$\frac{\{\tau \in W^* \otimes W \mid \operatorname{im}\tau \subset \sigma,\ \tau(k^\perp) \subset q\}}{(k \otimes q)\mathbb{C}},$$

which is also of dimension 5, so they coincide.

(ii) Similarly we obtain the description for $T_\alpha(\mathbb{P}W^* \times \ell_i)$.

(iii) A line passing through α can be written as $\gamma_\alpha(s) = \alpha + \tau s$, for some $\tau \in \mathcal{V}$. Note that since $\operatorname{im}\alpha = q \in \mathcal{S}$, then $F(\gamma_\alpha(0)(p)) = F(\alpha(p)) = 0$. The intersection multiplicity $m_\alpha(P_p \cdot \gamma)$ is by definition the order of vanishing

$$\mathrm{ord}_{t=0}[F((\alpha + \tau s)(p))],$$

so the line γ is tangent to P_p if and only if that order is greater or equal than 2. By taking the Taylor expansion we get

$$F((\alpha + \tau s)(p)) = F(\alpha(p)) + \sum_{i=0}^{3} \left(\frac{\partial F}{\partial x_i}\right)_{\alpha(p)} \tau_i(p)s + \dots,$$

where $\tau_i(p)$ denotes the i-th coordinate of $\tau(p)$. Hence we need the constant and the linear term of this expression to vanish. We already know that $F(\alpha(p)) = 0$, while $\sum_i \left(\frac{\partial F}{\partial x_i}\right)_q \tau_i(p) = 0$ if and only if $\tau(p) \subset \sigma$, that is exactly the condition we claimed. The above computation says more: if $\tau(p) \not\subset \sigma$, then the line $\alpha + \tau s$ intersects P_p with multiplicity 1 at α, so P_p is non-singular at α.

■

We will also need a similar lemma describing various tangent spaces at points of C_i.

Lemma 4.3.9. *For every point $\alpha \in C_i$, we have:*

(i) $T_\alpha C_i = \{\tau \in W^* \otimes W \mid \operatorname{im} \tau \subset \ell_i\}/\alpha\mathbb{C}$.

(ii) $T_\alpha P_p = \{\tau \in W^* \otimes W \mid \tau(p) \subset T_{\alpha(p)}\mathcal{S}\}/\alpha\mathbb{C}$.

Proof.

(i) Since each $C_i \simeq \mathbb{P}^7$ is embedded in $\mathcal{V}$ as a linear space, if we identify the C_i with nonzero matrices with image in ℓ_i, then the tangent space to this linear space at any point is simply the linear space itself, namely the matrices with image in ℓ_i.

(ii) Exactly as the proof of Lemma 4.3.8 (iii).

■

Lemma 4.3.10. *After blowing up B_1, the $C_i^{(1)}$ are all disjoint in $\mathcal{V}_1$.*

Proof. Recall that E_1 is defined as $\mathbb{P}(N_{B/\mathcal{V}})$, with $N_{B/\mathcal{V}} \simeq T\mathcal{V}/TB$. Then the intersection $C_i^{(1)} \cap E_1$ is the projectivization of the image of TC_i via the composition

$$TC_i \hookrightarrow T\mathcal{V} \to T\mathcal{V}/TB.$$

We need to prove that if C_i and C_j intersect in $\mathcal{V}$, then $C_i^{(1)}$ and $C_j^{(1)}$ are disjoint in the blow-up $\mathcal{V}_1$. We can check this fiberwise and show that for every $\alpha \in C_i \cap C_j$, the images of $T_\alpha C_i$ and $T_\alpha C_j$ in $T_\alpha\mathcal{V}/T_\alpha B$ do not intersect.

First observe that blowing up $\mathcal{V}$ along B affects C_i as if it was blown up along $\mathbb{P}W^* \times \ell_i$, producing an exceptional divisor $F_i := \mathbb{P}\left(\frac{TC_i}{T(\mathbb{P}W^* \times \ell_i)}\right)$, embedded in E_1. We may therefore instead prove that the image of $C_j^{(1)}$ in F_i is the empty set, i.e. that $T_\alpha C_i \cap \langle T_\alpha C_j, T_\alpha(\mathbb{P}W^* \times \ell_i)\rangle$ is contained in $T_\alpha(\mathbb{P}W^* \times \ell_i)$. Write q for the intersection $\ell_i \cap \ell_j$ and σ for $T_q\mathcal{S}$. Knowing the description of the tangent spaces in Lemma 4.3.8 and Lemma 4.3.9 and recalling that the two lines ℓ_i, ℓ_j span σ, we obtain

$$\langle T_\alpha C_j, T_\alpha(\mathbb{P}W^* \times \ell_i)\rangle = \{\tau \in W^* \otimes W \mid \operatorname{im} \tau \subset \sigma,\, \tau(k^\perp) \subset \ell_j\}/\alpha\mathbb{C}.$$

Intersecting this span with $T_\alpha C_i$ we obtain exactly the tangent space $T_\alpha(\mathbb{P}W^* \times \ell_i)$, so $C_i^{(1)}$ and $C_j^{(1)}$ are disjoint in the blow-up. ■

The tangent spaces appearing in Lemma 4.3.8 are also essential to describe the base locus of ϕ_1.

Proposition 4.3.11. *The base locus* $\mathrm{Bs}(\phi_1)$ *of the rational map* $\phi_1 \colon \mathcal{V}_1 \dashrightarrow \mathcal{F}$ *is supported on*

$$B_1 \cup \{C_1^{(1)}, \ldots, C_{27}^{(1)}\},$$

where B_1 *is a* $\mathbb{P}^5$*-subbundle of* E_1*. Moreover,* $B_1 = (\bigcap_{p \in \mathbb{P}W} P_p^{(1)}) \cap E_1$ *both set and scheme-theoretically.*

Proof. This result refers to [4, Proposition 2.2]. As observed earlier, the base locus of ϕ_1 is set-theoretically $\bigcap_p P_p^{(1)}$. In particular, a point $\alpha_1 \in E_1$ lying in the fiber of $\alpha \in B$ is also in $\mathrm{Bs}(\phi_1)$ if it is determined by a vector in $\bigcap_p T_\alpha P_p$ which is normal to B. Thanks to Lemma 4.3.8 (iii), we see that the intersection of all tangent spaces to the point conditions at α is given by the 11-dimensional space $\Sigma_\alpha := \{\tau \in W^* \otimes W \mid \operatorname{im} \tau \subset \sigma\}/\alpha\mathbb{C}$. This contains $T_\alpha B$ (see again Lemma 4.3.8) and the quotient $\Sigma_\alpha / T_\alpha B$ is a 6-dimensional subspace of the fiber of $N_{B/\mathcal{V}}$ over α. Moving α, we get a rank-6 subbundle of $N_{B/\mathcal{V}}$, so a $\mathbb{P}^5$-subbundle of $E_1 = \mathbb{P}(N_{B/\mathcal{V}})$, as we wanted. The $C_i^{(1)}$'s are also base loci, since the corresponding C_i's were so.

The second statement can be proved fiberwise: indeed, the fiber of B_1, a linear subspace, is cut out by fibers of the various $P_p^{(1)} \cap E_1$, which are linear spaces themselves. ■

Corollary 4.3.12. *The component* B_1 *can be globally described as* $\mathbb{P}\left(\frac{\bigcap_p T_\alpha P_p}{TB}\right)$ *and its intersection with* $C_i^{(1)}$ *is the bundle over* $\mathbb{P}W \times \ell_i$ *given by:*

$$C_i^{(1)} \cap B_1 = \mathbb{P}\left(\frac{TC_i}{T(\mathbb{P}W^* \times \ell_i)}\right).$$

Proof. The global description of B_1 is straightforward from Proposition 4.3.11. Regarding the intersection $C_i^{(1)} \cap B_1$, this coincides with $C_i^{(1)} \cap E_1$. The description then holds by the same arguments used for Lemma 4.3.10. ■

4.3.3 The base locus after blowing up B_1

We now address Claim 4.3.7: although a complete description of the components of $\mathrm{Bs}(\phi_2)$ will not be given, we will show which of those components are the ones dominating $B \simeq \mathbb{P}W^* \times \mathcal{S}$.

So, let us denote with B_2 the closed subvariety of $\mathrm{Bs}(\phi_2) \cap E_2$ dominating B. In order to understand B_2 we will need to look at the intersection of $\mathcal{S}$ with its tangent planes. We will focus on the points of $\mathcal{S}$ lying in the subset $\mathcal{S}_0 := \mathcal{S} \setminus \bigcap_{i=1}^{27} \ell_i$. Note that for every $q \in \mathcal{S}_0$, the plane cubic curve $T_q\mathcal{S} \cap \mathcal{S}$

is either a node or a cuspidal curve and if ℓ is a line in the tangent cone of such a cubic at its singular point q, then the intersection multiplicity is $m_q(\ell \cdot (T_q\mathcal{S} \cap \mathcal{S})) = 3$.

Definition 4.3.13. A line of matrices $\alpha + \tau s$ in $\mathcal{V}$, with $\alpha = (k, q) \in B$ is called a *special line* if $q \in \mathcal{S}_0$, $\tau(k^\perp) \not\subset q$ and the image of τ is contained in a line tangent to the cubic curve $T_q\mathcal{S} \cap \mathcal{S}$ at q.

We would like to translate properties of points in B_2 to properties of points in B and as we will soon see it will be useful to observe the following:

Lemma 4.3.14. *The base locus* $\mathrm{Bs}(\phi_2)$ *is disjoint from* $E_1^{(1)}$.

Proof. This is just a rephrase of the second part of Proposition 4.3.11, thanks to which we know that the point conditions $P_p^{(1)}$ intersect E_1 transversely in $\mathcal{V}_1$. ■

Proposition 4.3.15. *Let α_2 be a point of E_2 and let us denote with $\alpha = (k, q)$ its image in B via the composite map $\pi_1 \circ \pi_2$. Suppose also that $q \in \mathcal{S}_0$. Then α_2 is in B_2 if and only if it can be written as the intersection of E_2 with the proper transform in $\mathcal{V}_2$ of a special line in $\mathcal{V}$. Moreover, the set of such α_2 is dense in B_2.*

Proof. By definition, $\alpha_2 \in \mathrm{Bs}(\phi_2)$ if and only if it is contained in the proper transform of a general point condition $P_p^{(2)}$. In particular, if α_2 is in B_2, it must represent a direction normal to B_1 and tangent to a general point condition $P_p^{(1)}$ at $\alpha_1 := \pi_2(\alpha_2)$. We can identify this direction with a smooth curve germ γ_{α_1} around α_1 in $\mathcal{V}_1$, satisfying normality to B_1 and the tangency condition:

$$m_{\alpha_1}(\gamma_{\alpha_1} \cdot P_p^{(1)}) \geq 2, \quad \text{for a general } p \in \mathbb{P}W.$$

Note that, using the above identification, we can write $\alpha_2 = E_2 \cap \gamma_{\alpha_1}^{(1)}$.

Thanks to Lemma 4.3.14 we can rephrase everything in terms of curve germs in $\mathcal{V}$: indeed, γ_{α_1} turns out to be not only normal to B_1, but to the whole of E_1, so we can think of it as the proper transform of a line $\gamma_\alpha = \alpha + \tau s \subset V$, which is normal to B and intersects a general point condition P_p with multiplicity greater or equal than 3.

Denoting as usual with σ the tangent plane $T_q\mathcal{S}$, we can equivalently say that:

$$\alpha_2 \in B_2 \iff \alpha = E_2 \cap (\alpha + \tau t)^{(2)},$$

with $\mathrm{im}\,\tau \subset \sigma$ and $\tau(k^\perp) \not\subset q$ (see Lemma 4.3.8), such that for a general p we have $m_\alpha((\alpha + \tau s) \cdot P_p) \geq 3$.

This description reduces to study a special class of lines through α in $V = \mathcal{V}$: we divide in 3 cases, depending on the rank of τ, that can be either $1, 2$ or 3.

If $\mathrm{rk}\,\tau = 3$, then $\mathrm{im}\,\tau = \sigma$. In particular, for a general $p \in \mathbb{P}W$, we have $\tau(p) = q_p$, where q_p is a point varying on σ and different from q. Then the span

$$\langle \alpha(p) = q, \tau(p) = q_p \rangle$$

is a general line λ_p in σ passing through q and $(\alpha + \tau s)(p)$ is a parametrization of such a line. Then for a general p we have

$$\begin{aligned} m_\alpha((\alpha + \tau s) \cdot P_p) &= \operatorname{ord}_{t=0}(F((\alpha + \tau s)(p))) \\ &= m_q(\lambda_p \cdot \mathcal{S}) \\ &= m_q(\lambda_p \cdot (\mathcal{S} \cap \sigma)) = 2 < 3, \end{aligned}$$

so in this case α_2 is not in the base locus.

If $\operatorname{rk} \tau = 2$, then $\operatorname{im} \tau = \ell$, where ℓ is a line in the tangent plane σ. Again, for a general $p \in \mathbb{P}W$, the span of $\alpha(p)$ and $\tau(p)$ is a line through q and we are interested in computing $m_q(\lambda_p \cdot (\mathcal{S} \cap \sigma))$. There are two possibilities: if $q \notin \ell$, then for every two distinct points p_1 and p_2 in $\mathbb{P}W$ the lines λ_{p_1} and λ_{p_2} are distinct. In particular, for a general p, the above multiplicity will be 2, so in this case as well, α_2 is not in the base locus.

On the other hand, if $q \in \ell$, then for a general p we constantly have $\lambda_p = \ell$ and α_2 is in the base locus precisely when $m_q(\ell \cdot (\mathcal{S} \cap \sigma)) = 3$, i.e. when ℓ is one of the two tangent lines at the node q (or the double tangent line in the degenerate case). Note the multiplicity computation makes sense since we are assuming that $q \in \mathcal{S}_0$.

Finally, if $\operatorname{rk} \tau = 1$, then $\operatorname{im} \tau = q'$, a point in σ different from q (otherwise this would contradict $\tau(k^\perp) \not\subset q$). Then, arguing as above, for a general p, the span of $\alpha(p) = q$ and $\tau(p) = q'$ is a constant line ℓ and α_2 is in the base locus if and only if $m_q(\ell \cdot (\mathcal{S} \cap \sigma)) = 3$. Note that the rank-1 matrices τ satisfying this property come from taking the closure of the space of rank-2 matrices described at the previous step.

The density statement is a consequence of the fact that $\mathcal{S}_0$ is dense in $\mathcal{S}$, since a component dominates B if and only if it dominates $\mathbb{P}W^* \times \mathcal{S}_0 \subset B$. ■

Our knowledge about the components of the base locus of ϕ_2 can be summarized in the following:

Proposition 4.3.16. *The components of the support of* $\operatorname{Bs}(\phi_2)$ *that dominate a component of the original base locus* $\operatorname{Bs}(\phi)$ *are* $C_1^{(2)}, \dots, C_{27}^{(2)}$ *and the irreducible components of* B_2*. Moreover, the map* $(\pi_1 \circ \pi_2)_{|B_2}$ *is a double cover of* B*, i.e.* B_2 *consists of at most* 2 *irreducible components.*

Proof. We just need to observe that B_2 is obtained by taking the closure of a subset of E_2 whose fibers over B correspond to two special lines of $\mathcal{V}$ (counted with multiplicity). ■

Remark 4.3.17. While the $C_i^{(2)}$'s are clearly irreducible, we are still left lo understand if also B_2 is.

4.3.4 The base locus after blowing up the $C_i^{(3)}$

The last part of the paper is devoted to proving the following result:

Proposition 4.3.18. *After blowing up one of the components $C_i^{(3)}$, corresponding to matrices with image contained in a line, there will be no remaining base locus over the points in C_i corresponding to matrices of rank 2.*

Since, up to this point, the centers of all blow-ups have been away from matrices of rank 2, we will for simplicity consider the base locus after blowing up C_i in $\mathcal{V}$ instead of $C_i^{(3)}$ in $\mathcal{V}_2$.

We now wish to study the intersection of the tangent spaces of all point conditions. To this end, we will study the image of matrices contained in the intersection of all the tangent spaces. From Lemma 4.3.9 (ii) we see that for every $\alpha \in C_i$, the intersection of all the tangent spaces $T_\alpha P_p$ is:

$$\bigcap_{p \in W} \{\tau \in W^* \otimes W \mid \tau(p) \subset T_{\alpha(p)}\mathcal{S}\}/\alpha\mathbb{C}. \tag{4.3.1}$$

In fact, as we will prove now, this condition will imply that the image of the matrix τ must be contained in ℓ_i.

The proof relies on pencils of hyperplanes. The hyperplanes in W containing ℓ_i are parametrized by $\mathcal{H} \simeq \mathbb{P}^1$. A pencil of hyperplanes containing ℓ_i will be a morphism $\mathbb{P}^1 \to \mathcal{H}$, and the degree of the pencil is the degree of this morphism (if it is nonconstant).

In this and the following lemma, we will work with the affine space W instead of $\mathbb{P}W$.

Lemma 4.3.19. *Let $\alpha \in C_i$ be a point corresponding to a rank-2 matrix with image ℓ_i, and let $\tau \in W^* \otimes W$ be such that the image of τ in $T_\alpha\mathcal{V} \simeq W^* \otimes W/\alpha\mathbb{C}$ is in $\bigcap_p T_\alpha P_p$. Then for any two-dimensional subspace $U \subset W$ such that $\alpha(U) = \ell_i$, we have $\tau(U) \subseteq \ell_i$.*

Proof. From the two-dimensional subspace U we can construct a degree-two pencil $\mathcal{P}_1$ of hyperplanes in W containing ℓ_i by assigning to $u \in U$ the hyperplane defined by the equation

$$\sum_{i=0}^{3} \left(\frac{\partial F}{\partial x_i}\right)_{\alpha(u)} = 0,$$

where F is the general degree three polynomial defining the cubic surface $\mathcal{S}$. We think of $\mathcal{P}_1$ as assigning to $u \in U$ the tangent plane of $\mathcal{S}$ at $\alpha(s)$. This defines a map from $\mathbb{P}(U) \simeq \mathbb{P}^1$ to $\mathcal{H}$. This pencil will have degree two, as it is defined by degree two polynomials.

Assume for contradiction that $\tau(U) \not\subseteq \ell_i$. There are three cases: $\tau(U)$ is either a one-dimensional space not contained in ℓ_i, a two-dimensional space with one-dimensional intersection with ℓ_i, or a two-dimensional space with zero-dimensional intersection with ℓ_i. In all cases, we construct a second pencil $\mathcal{P}_2$ of hyperplanes containing ℓ_i, by assigning to $u \in U$ the hyperplane spanned by $\tau(u)$ and ℓ_i. This defines a map $\mathbb{P}(U) \dashrightarrow \mathcal{H}$ which is a priori at least rational, but extends to a morphism $\mathbb{P}(U) \to \mathcal{H}$ since the domain is a curve.

The condition (4.3.1) states that $\mathcal{P}_1$ and $\mathcal{P}_2$ are equal. Indeed, condition (4.3.1) requires that $\mathcal{P}_2(p) = \langle p, \ell_i \rangle$ is mapped to $T_{\alpha(p)}\mathcal{S}$ in $T_{\alpha(p)}\mathcal{V}$. But this can only happen if $\mathcal{P}_2(p) = \mathcal{P}_1(p)$. However, this cannot be true in any of the three cases, as we will see in the following:

- If $\tau(U)$ is a one-dimensional space not contained in ℓ_i, then $\mathcal{P}_2$ is constant, and therefore not equal to $\mathcal{P}_1$.
- In the case where $\tau(U)$ is two-dimensional and intersects ℓ_i in the one-dimensional space $q\mathbb{C}$, $\mathcal{P}_2(p)$ will be the hyperplane spanned by ℓ_i and $\tau(U)$ for any p, so the pencil is constant.
- If $\tau(U)$ is a two-dimensional space intersecting ℓ_i only in 0, then $\mathcal{P}_2$ is a pencil of degree 1. Therefore, again, it cannot be equal to $\mathcal{P}_1$.

■

From this lemma we can deduce that in fact the image of τ must be in ℓ_i.

Lemma 4.3.20. *With notation as above, let α be a matrix of rank 2 in C_i. If $\tau \in \bigcap_p T_\alpha P_p$, then τ is in the tangent space $T_\alpha C_i$.*

Proof. Let τ' be any element of $\in W^* \otimes W$ that is mapped to τ. For any vector $u \in W \setminus \ker \alpha$, it is possible to find a 2-dimensional subspace U containing u such that $\alpha(U) = \ell_i$. Then, thanks to Lemma 4.3.19, we have $\tau'(u) \in \ell_i$. But since u was arbitrarily chosen in $W \setminus \ker \alpha$ and this latter set spans W, we must have $\operatorname{im} \tau' \subset \ell_i$. ■

Putting all this together we find that after blowing up a component of the base locus corresponding to matrices with image in a certain line, the remaining base locus is supported in the fibers over the rank-1 matrices.

Proposition 4.3.21. *Let*

$$\begin{tikzcd}
\mathrm{PGL}(4) \arrow[r, hook] \arrow[d, equal] & \mathcal{V}' \arrow[r, dashed, "\phi'"] \arrow[d, "\pi"] & \mathcal{F} \arrow[d, equal] \\
\mathrm{PGL}(4) \arrow[r, hook] & \mathcal{V} \arrow[r, dashed, "\phi"] & \mathcal{F}
\end{tikzcd}$$

be the the diagram associated to the blow-up of $\mathcal{V}$, along one of the components $C_i \simeq \mathbb{P}^7$ and let $\phi' \colon \mathcal{V}' \dashrightarrow \mathcal{F}$ be the induced rational map. If we denote by G_i the exceptional divisor over C_i and by $\mathrm{Bs}(\phi')$ the base locus of ϕ', then $\pi(\mathrm{Bs}(\phi') \cap G_i)$ is contained in the $\mathbb{P}W^* \times \ell_i \subset \mathbb{C}_i$ consisting of rank-1 matrices.

Proof. We will prove the statement fiberwise. Let $\alpha \in C_i$ be a rank-2 matrix. Then we must show that $\mathrm{Bs}(\phi') \cap \pi^{-1}(\alpha)$ is empty. The fiber $\pi^{-1}(\alpha)$ is the projectivization of $(N_{C_i/\mathcal{V}})_\alpha$, the fiber of the normal bundle of C_i at α. If we denote with P_p the stict transform of a point condition, then $P_p \cap \pi^{-1}(\alpha)$ is the projectivization of the quotient $T_\alpha P_p / T_\alpha C_i$. Therefore $\mathrm{Bs}(\phi') \cap \pi^{-1}(\alpha)$, is obtained by projectivizing $\bigcap_p T_\alpha P_p / T_\alpha C_i$. But by Lemma 4.3.20 we know that $\bigcap_p T_\alpha P_p$ is actually contained in $T_\alpha C_i$, so the quotient described above must be trivial. After projectivizing, we see that $\mathrm{Bs}(\phi') \cap \pi^{-1}(\alpha)$ must be empty. ■

Remark 4.3.22. In our resolution of $\phi\colon \mathcal{V} \dashrightarrow \mathcal{F}$, we actually want to blow up the proper transforms $C_i^{(3)}$ of the C_i in $\mathcal{V}_2$. However, over the matrices of rank 2, the blow-down $\mathcal{V}_2 \to \mathcal{V}$ is an isomorphism. We can therefore conclude from Proposition 4.3.21 that also in this case there is no further base locus over the rank-2 matrices.

Having Proposition 4.3.21 been proved, the natural question to ask is:

Question 4.3.23. *After blowing up the $C_i^{(3)}$'s, is there any base locus over the subset of points that projects down to the locus of rank-1 matrices?*

Appendices

Appendix A

Auxiliary Codes

In this appendix, we gather all the auxiliary codes needed for the computations in Chapter 2.

In Section A.1, we collect the Macaulay2 codes used for Theorem 2.2.7, Remark 2.3.7, Proposition 2.3.8 and Theorem 2.4.1, while in Section A.2 we collect the Julia codes used for Theorem 2.1.3 and Theorem 2.4.1.

Most of the material presented here is an adaptation of the scripts in the repository [13].

A.1 Macaulay2 codes

We begin with defining the basic functions, then we proceed with their application to each of the above mentioned results.

A.1.1 Basic functions

Adjugate matrix $\mathrm{Adj}_m(A)$ of an $m \times m$ matrix A:

```
adjugate = A -> (
  m := numcols A;
  adjugateA := for i to m-1 list (
        for j to m-1 list (-1)^(i+j)*det(submatrix'(A,{j},{i}))
        );
  matrix adjugateA
  )
```

Matrix parametrizing the elements of $\mathbb{P}\mathrm{Cat}(k,n)$, namely the linear space of catalecticant matrices of $(n+1)$-ary forms of degree $2k$; the parameters are taken in a ring X:

```
genericCatalecticantMatrix = (k,n,X) -> (
  N := #(gens X)-1;
  a := symbol a;
  x := symbol x;
  R := QQ[a_0..a_N, x_0..x_n];
  aList := drop(gens R, -(n+1));
  xList := drop(gens R, N+1);
  dBasis := flatten entries basis(2*k, R, Variables=>xList);
  form := sum(for i to #aList-1 list aList_i*dBasis_i);
```

```
  kBasis := flatten entries basis(k, R, Variables=>xList);
  CM := contract(matrix{kBasis}, form);
  (M,C) := coefficients(CM, Variables=>xList, Monomials=>kBasis);
  f := map(X, R, join(gens X, apply(xList, i->0)));
  matrix entries f(C)
  )
```

List of entries in the upper triangular part of a given $m \times m$ symmetric matrix A:

```
symmetricEntries = (A,m) -> (
    flatten(for i to m-1 list (for j from i to m-1 list A_(i,j)))
    )
```

Coordinates of $\nu_d(P)$, namely, the d-uple embedding of a point $P \in \mathbb{P}^n$:

```
veronese = (P,d) -> (
  l := length(P)-1;
  u := symbol u;
  U := QQ[u_0..u_l];
  dBasis := first entries basis(d,U);
  for monom in dBasis list (sub(monom, for i to l list u_i=>P_i))
  )
```

Catalecticant matrix associated with the d-uple embedding of a point P:

```
toCat = (P,d) -> (
  k := d//2;
  n := length(P)-1;
  x := symbol x;
  N := binomial(n+2*k, 2*k)-1;
  X := QQ[x_0..x_N];
  cat := genericCatalecticantMatrix(k,n,X);
  veron := veronese(P,d);
  sub(cat, for i to N list x_i=>veron_i)
  )
```

Defining equations of the orthogonal space $L^\perp$ of a given linear subspace L of $m \times m$ symmetric matrices, using a coordinate ring S:

```
orthogonal = (L,S) -> (
  varsL := support L;
  N := #(varsL)-1;
  varsS := support S;
```

```
    M := #(varsS)-1;
    R := QQ[varsS][varsL];
    L' := sub(L,R);
    S' := sub(S,R);
    traceProduct := trace(L'*S');
    use R;
    ortho := trim ideal(for i to N list(coefficient(varsL_i, traceProduct)));
    sub(ortho, ring S)
    )
```

A.1.2 Setup

We write the two matrices parametrizing the space $(\mathbb{PS}^6)^\vee \simeq \mathbb{P}^{20}$ of 6×6 dual symmetric matrices and the space $\mathbb{P}\mathrm{Cat}(2,3) \simeq \mathbb{P}^{14}$ of square catalecticants associated with ternary quartics. We use

$$Y = \mathbb{Q}[y_{(1,1)}, y_{(1,2)} \ldots, y_{(6,6)}] \qquad \text{and} \qquad A = \mathbb{Q}[a_{(4,0,0)}, a_{(3,1,0)}, \ldots, a_{(0,0,4)}]$$

as their coordinate rings.

```
n = 2
k = 2
d = 2*k
m = binomial(k+n,k)

catCoord = reverse flatten(for i to d list (for j to d-i list a_(i,j,d-i-j)))
A = QQ[catCoord]

symCoord = flatten(for i from 1 to m list (for j from i to m list y_(i,j)))
Y = QQ[symCoord]

cat = genericCatalecticantMatrix(k,n,A)
sym = genericSymmetricMatrix(Y,m)
```

A.1.3 Reciprocal sets of points

This is the auxiliary code for Theorem 2.2.7, cases of rank $r = 1, 2$. First, we let $P_1 = \nu_4(1:0:0)$ and $P_2 = \nu_4(0:0:1)$. We fix a rank-1 matrix A_1 associated with P_1, a rank-2 matrix A_2 associated with a point on the secant line through P_1 and P_2, and a rank-2 matrix A_2' associated with a point on a tangent line through P_1.

```
P1 = {1,0,0}
P2 = {0,0,1}
A1 = toCat(P1,4)
```

```
A2 = toCat(P1,4) + toCat(P2,4)
A2' = sub(cat, join(
        {a_(4,0,0)=>1, a_(3,1,0)=>1},
        for g in drop(gens A, {0,1}) list g=>0
        ))
```

Case $r = 1$ We give the defining equations of the linear span $\langle F_{\mathcal{C}}(A_1)\rangle \simeq \mathbb{P}^{14}$, made of matrices in $(\mathbb{P}\mathbb{S}^6)^\vee$ whose entries in the first row and column are all zero. We identify this linear space with a suitable space of 5×5 symmetric matrices. Then, we compute the equations for $F_{\mathcal{C}}(A_1)$ as the reciprocal variety of the space of 5×5 submatrices in $\mathbb{P}\mathrm{Cat}(2,3)$, obtained by erasing the first row and the first column. Equations are obtained by saturating the ideal of pull-back (cut out by 3 quartics, and the 6 linear equations), with the determinant polynomial.

```
PP14 = ideal(for i from 1 to 6 list y_(1,i))
sym' = submatrix'(sym,{0},{0})
quartics = {
    det submatrix'(sym',{0},{3}) - det submatrix'(sym',{1},{2}),
    det submatrix'(sym',{0},{4}) - det submatrix'(sym',{1},{3}),
    det submatrix'(sym',{2},{4}) - det submatrix'(sym',{3},{3})
    };
Q = trim ideal(quartics) + PP14
d5 = ideal(det sym')
FA1 = Q:d5 --takes 4h ca.
```

We verify that the cubics defining $F_{\mathcal{C}}(A_1)$ are the cubic Pfaffians of a 7×7 skew matrix S_1.

```
cubics = ideal(for i from 6 to 12 list FA1_i)
S1 = matrix{
    {0, y_(6,6), y_(5,6), y_(4,6), y_(3,6), y_(2,6), 0},
    {0, 0, y_(5,5)-y_(4,6), y_(4,5), y_(3,5)-y_(2,6), y_(2,5), -y_(3,6)},
    {0, 0, 0, y_(4,4), y_(3,4)-y_(2,5), y_(2,4), -y_(3,5)},
    {0, 0, 0, 0, -y_(2,4), 0, -y_(3,4)},
    {0, 0, 0, 0, 0, y_(2,2), -y_(3,3)},
    {0, 0, 0, 0, 0, 0, -y_(2,3)},
    {0, 0, 0, 0, 0, 0, 0}
    }
S1 = S1 - transpose S1
pfaffians(6,S1) == cubics --true
```

Case $r = 2$, secant Analogously, we give the defining equations of the linear span $\langle F_{\mathcal{C}}(A_2)\rangle \simeq \mathbb{P}^9$, made of matrices of $(\mathbb{P}\mathbb{S}^6)^\vee$ whose entries in the first and last row and in the first and last column are all zero. Then, we compute the defining equations of $F_{\mathcal{C}}(A_2)$ by saturating the pull-back ideal of the reciprocal variety of a suitable subspace of 4×4 symmetric matrices.

```
PP9 = ideal(flatten for i from 1 to 6 list {y_(1,i), y_(i,6)})
sym' = submatrix'(sym,{0,5},{0,5})
cubic = {det submatrix'(sym',{0},{3}) - det submatrix'(sym',{1},{2})};
C = ideal(cubic) + PP9
d4 = ideal(det sym')
FA2 = C:d4
```

We verify that the cubic defining $F_{\mathcal{C}}(A_2)$ is the cubic Pfaffian of a 6×6 skew matrix S_2.

```
S2 = matrix{
    {0, 0, y_(2,2), y_(2,3), y_(2,4), y_(2,5)},
    {0, 0, y_(2,3), y_(3,3), y_(3,4), y_(3,5)},
    {0, 0, 0, y_(3,4)-y_(2,5), y_(4,4), y_(4,5)},
    {0, 0, 0, 0, y_(4,5), y_(5,5)},
    {0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0}
    }
S2 = S2 - transpose S2
pfaffians(6,S2) == ideal(FA2_11) --true
```

Case $r = 2$, tangent As in the previous case, but now the linear span $\langle F_{\mathcal{C}}(A_2')\rangle \simeq \mathbb{P}^9$ is made of matrices of $(\mathbb{PS}^6)^\vee$ whose entries in the first two rows and first two columns are all zero.

```
PP9 = ideal(flatten(
    for i from 1 to 2 list(for j from i to 6 list y_(i,j))
    ))
sym' = submatrix'(sym,{0,1},{0,1})
cubic = {det submatrix'(sym',{1},{3}) - det submatrix'(sym',{2},{2})};
C = ideal(cubic) + PP9
d4 = ideal(det sym')
FA2' = C:d4
```

We verify that the cubic defining $F_{\mathcal{C}}(A_2')$ is the cubic Pfaffian of a 6×6 skew matrix S_2'.

```
S2' = matrix{
    {0, 0, y_(5,6), y_(6,6), y_(3,6), y_(4,6)},
    {0, 0, y_(3,5), y_(3,6), y_(3,3), y_(3,4)},
```

```
    {0, 0, 0, y_(4,6)-y_(5,5), y_(3,4), y_(4,4)},
    {0, 0, 0, 0, y_(3,5), y_(4,5)},
    {0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0}
    }
S2' = S2' - transpose S2'
pfaffians(6,S2') == ideal(FA2'_11) --true
```

A.1.4 Rank loci in the orthogonal space

This is the auxiliary code for Remark 2.3.7. First, we show that the general rank of $\mathbb{P}\mathrm{Cat}(2,3)^{\perp}$ is 6. Then, we give defining equations for the rank loci, showing that for $r = 1, 2$ we have the emptyset, for $r = 3$ a Veronese surface $\nu_2(\mathbb{P}^2)$ and for $r = 4, 5$ the secant variety $\sigma_2(\nu_2(\mathbb{P}^2))$.

```
ortho = orthogonal(cat,sym)
rank sub(sym, Y/ortho)

use Y
rk1locus = ortho + minors(2,sym);
rk2locus = ortho + minors(3,sym);
rk3locus = ortho + minors(4,sym);
rk4locus = ortho + minors(5,sym);
rk5locus = ortho + ideal(det sym);

dim rk1locus-1, dim rk2locus-1

mat = matrix{
    {y_(2,2), y_(2,3), y_(3,4)},
    {y_(2,3), y_(3,3), y_(3,5)},
    {y_(3,4), y_(3,5), y_(5,5)}
    }
surface = minors(2,mat)
secant = ideal det(mat)

radical rk3locus == surface + ortho --true
radical rk4locus == secant + ortho --true
radical rk5locus == secant + ortho --true
```

A.1.5 Intersection with the orthogonal space

This is the auxiliary code for Proposition 2.3.8. We show that $\mathbb{P}\mathrm{Cat}(2,3)^{\perp}$ intersects $F_{\mathcal{C}}(A_2)$ in the emptyset, while it intersect $F_{\mathcal{C}}(A_1)$ and $F_{\mathcal{C}}(A_2')$ in a point (the same one).

```
dim (FA1 + ortho)-1
dim (FA2 + ortho)-1
dim (FA2' + ortho)-1
```

A.1.6 Tangent cone at rank-1 points

This is the auxiliary code for the revisited proof of Theorem 2.4.1. We prove that the tangent cone to $\mathbb{P}\mathrm{Cat}(2,3)^{-1}$ at a rank-1 point spans the entire space $(\mathbb{P}\mathbb{S}^6)^\vee$. Specifically, we iterate 10 times the following procedure.

(1) We fix a random 5-tuple of points $P_1,\ldots,P_5 \in \nu_2(\mathbb{P}^1)$ (to make sure they are in general position and they span a proper 5-secant space).

(2) We compute their image $\nu_4(\nu_2(P_i))$, for $i = 1,\ldots,5$. These are points lying on the image of the same conic.

(3) We pick a random rank-5 point A on the 5-secant space. Then, we pick 15 random points of rank 6 in $\mathbb{P}\mathrm{Cat}(2,3)$. Joining them with A, we obtain 15 lines through A. The images of these lines are quintic curves passing through $B = \mathrm{Adj}_6(A)$.

(4) We compute the tangent directions to these curves at B and store them in a list.

(5) We check that our random choice was a good choice (the image of A is constant).

(6) We iterate the process, and in the end we find a list of directions which spans $(\mathbb{P}\mathbb{S}^6)^\vee$.

```
Z = QQ[gens A, gens Y, t];
AY = Y[gens A];
cat = sub(cat, AY);
sym = sub(sym, AY);

nSteps = 1
tangentsList = {};
rk1Pts = {};

for j from 1 to nSteps do (
    use AY;
    pts := for i from 1 to 5 list veronese({random QQ,random QQ},2);
    for i to 5 do (
        rk5 := sum(for P in pts list (random QQ) * toCat(P,4));
        nL := 15;
        rk6Cats := for i from 1 to nL list (
```

```
            sub(cat, for g in gens AY list g=>random QQ)
            );
        joiningLines := for cat in rk6Cats list (
            t*sub(cat,Z) + sub(rk5,Z)
            );
        imageLines := for L in joiningLines list adjugate L;
        imgRk5 := sub(adjugate rk5,Z);
        rk1Pts = rk1Pts | {symmetricEntries(imgRk5,6)};
        if rank matrix rk1Pts == 1 then print (
            "Until step "|j|","|i|": all rank-5 points"|
            "are mapped to the same rank-1 point."
            )
        else (
            print (
                "At step "|j|","|i|": a different rank-1 point was found."|
                "Process interrupted."
                );
            break
            );
        b := true;
        for curve in imageLines do (
            b = b and (mutableMatrix sub(curve,t=>0) == mutableMatrix imgRk5)
            );
        if b == true then print (
            "The curves contain the same rank-1 point"
            )
        else (
            print (
                "Found a curve not containing the same rank-1 point. "|
                "Pocess interrupted."
                );
            break
            );
        tgDirections := for curve in imageLines list(
            symmetricEntries(sub(diff(curve, t), t=>0), 6)
            );
        tangentsList = tangentsList | tgDirections;
        );
    )

rank matrix tangentsList
```

A.2 Julia codes

In this section, we need many technical functions, so we introduce them at different steps. Every function depends only on those defined before it.

A.2.1 ML-degree of the linear covariance model

The following code is used to compute the ML-degree of the linear concentration model represented by $\mathrm{Cat}(2,3)$ (cfr. Theorem 2.1.3). First, we import the required packages.

```
using LinearCovarianceModels
import HomotopyContinuation
const HC = HomotopyContinuation
```

Then, we define functions to create linear covariance models of catalecticant matrices.

```
function nvectors_sumingupm(n, m)
    ntuples = Iterators.product(ntuple(_ -> 0:m, n)...)
    return filter(I -> sum(I) == m, collect(ntuples))
end

function Catalecticantindices(r, s)
    rowindeces = collect.(nvectors_sumingupm(s, r))
    m = binomial(r + s - 1, s - 1)
    M = reduce(hcat, fill(rowindeces, m))
    return M + permutedims(M)
end

function Catalecticantmatrix(r, s)
    return [HC.Variable(:t, I...) for I in Catalecticantindices(r, s)]
end

Catalecticant = LCModel(Catalecticantmatrix)
```

When applied to the case of catalecticants of ternary quartics, we obtain:

```
S = Catalecticant(2, 3)
ml_degree_witness(S; dual = true)
```

A.2.2 Suite of functions for computing the degree

We load the required packages and set HomotopyContinuation to not compile, since the computations are small.

```
using HomotopyContinuation, LinearAlgebra
set_default_compile(:none)
```

Similarly to Section A.1, we define functions giving the symmetric matrix corresponding to the vector of its upper triangular part, and viceversa.

```
sym_to_vec(L,m) = [L[i, j] for i = 1:m for j = i:m]
function vec_to_sym(l::AbstractVector{T},m::Int64) where T
    k = 0
    L = Matrix{T}(undef, m, m)
    for i in 1:m, j in i:m
        k += 1
        L[i, j] = L[j, i] = l[k]
    end
    L
end
```

The following function indexes the entries of a space of catalecticant matrices.

```
function Catalecticant_dims(r, s)
    m = binomial(r + s - 1, s - 1)
    N = binomial(2r + s - 1, 2r)
    M = binomial(m + 1, 2)
    return m, N, M
end
```

This can be used to compute the defining equations of any catalecticant space.

```
function CatalecticantIndices(r,s,m)
    tf = [sum(I) == r for I in Iterators.product(ntuple(i->0:r, s)...)]
    indx = collect.(collect(Iterators.product(ntuple(i->0:r, s)...))[tf])
    M = hcat([indx for _ in 1:m]...)
    return M+permutedims(M)
end

function CatalecticantSpace(M::AbstractMatrix,A::AbstractMatrix,m::Integer)
    eq = Vector{typeof(A[1,1]+A[1,1])}(undef,0)
    count = Vector{typeof(M[1,1])}(undef,0)
    for i in 1:m
        for j in i:m
            for i2 in (i+1):m
                for j2 in i2:m
                    if M[i,j] == M[i2,j2]
                       if ([i,j] in count)==false
                         push!(eq, A[j,i]-A[j2,i2]);
                         push!(count,[i,j]);
```

```
                    end
                end
            end
        end
    end
return eq
end
```

Finally, a function for computing the adjugate of a square matrix:

```
function adjugate(M::AbstractMatrix{T}) where T
    nr, nc = size(M)
    out = similar(M)
    rows = BitArray(ones(Int16,nr))
    cols = BitArray(ones(Int16,nc))
    for r in 1:nr
        for c in r:nc
            rows[r] = 0
            cols[c] = 0
            out[c, r] = (-1)^(c+r)*det(M[rows,cols])
            rows[r] = 1
            cols[c] = 1
        end
    end
    return out
end
```

A.2.3 Degree of the reciprocal variety

We now give the code for computing the degree of the reciprocal variety of the catalecticant space of ternary quartics (cfr. Theorem 2.1.3). We consider the catalecticant space $\mathbb{P}\mathcal{C} := \mathbb{P}\mathrm{Cat}(2,3) \simeq \mathbb{P}^N$ as a linear subspace of $\mathbb{P}\mathbb{S}^m \simeq \mathbb{P}^M$.

```
r = 2;
s = 3;
m, N, M = Catalecticant_dims(r, s)
```

We give the defining equations of $\mathbb{P}\mathcal{C}$

```
@var x[0:2r, 0:2r, 0:2r], y[1:M]
cat = CatalecticantIndices(r,s,m)
```

```
adj = adjugate(Expression.(vec_to_sym(y,m)));
eq = CatalecticantSpace(cat,adj,m);
```

We set up the system and compute a point on the reciprocal variety $\mathbb{P}\mathcal{C}^{-1}$.

```
F = System(eq; variables = y);
x0 = randn(ComplexF64, N)
@var x[0:2r, 0:2r, 0:2r]
X = map(ijk -> x[(ijk .+ 1)...], cat)
y0 = sym_to_vec(inv(X(variables(X) => x0)),m)
```

We build an affine space through this point and perform monodromy.

```
V0 = let
   A0 = randn(ComplexF64, N, M)
   b0 = A0 * y0
   LinearSubspace(A0, b0)
end
mres = monodromy_solve(
   F,
   y0,
   V0,
   parameter_sampler=_ -> LinearSubspace(
       randn(ComplexF64, N, M),
       randn(ComplexF64, N),
   ),
)
```

We find 85 solution for this system. These solutions are then certified.

```
delta = randn(ComplexF64, N);
V1 = translate(V0, delta);
V2 = translate(V0, -delta);

r1 = solve(F, solutions(mres), start_subspace = V0, target_subspace = V1);
r2 = solve(F, solutions(mres), start_subspace = V0, target_subspace = V2);
sigma = svdvals(hcat(sum.(solutions.([r1, mres, r1]))...))
trace = sigma[3] / sigma[1]
```

A.2.4 Number of cubics and quartics

We introduce here some functions for generating random catalecticant matrices.

```
using HomotopyContinuation, LinearAlgebra, Combinatorics

function Catalecticant_indices(r, s)
    M = reduce(hcat, [collect(
        multiexponents(s, r)) for _ in 1:length(multiexponents(s, r))])
    return M .+ permutedims(M)
end

function Catalecticant_generic(r, s)
    return [Variable(:x, I...) for I in Catalecticant_indices(r, s)]
end

function Catalecticant_randn(::Type{T}, r, s) where T
    catmat = Catalecticant_generic(r, s)
    return randnvalue(T, catmat, unique(catmat))
end

Catalecticant_randn(r, s) = Catalecticant_randn(ComplexF64, r, s)
```

We study the kernel of

$$\begin{array}{rccc} \phi\colon & \mathbb{P}^n \simeq H^0(\mathbb{PS}^m, \mathcal{O}_{\mathbb{PS}^m}(d)) & \to & H^0(\mathbb{P}\mathcal{C}^{-1}, \mathcal{O}_{\mathbb{P}\mathcal{C}^{-1}}(d)) \\ & f & \mapsto & f_{|\mathbb{P}\mathcal{C}^{-1}}. \end{array}$$

We do the case $d = 3$ ($d = 4$ is analogous). First, we pick n general points on $\mathbb{P}\mathcal{C}^{-1}$. These points are generated as inverses of randomly generated catalecticant matrices. We check that they are in general position

```
d = 3;
n3 = length(multiexponents(M, d))
points = [inv(Catalecticant_randn(r, s)) for _ in 1:n3];
all(==(m), rank.(points))
```

We evaluate the monomials in $\mathcal{O}_{\mathbb{PS}^m}(d)$ at these points and compute the number of cubics in the kernel of ϕ.

```
mplanes = mplanes1(M, d, sym_to_vec.(points));
n3 - rank(mplanes)
```

A.2.5 Rank of the Jacobian

This is the auxiliary code for the numerical proof of Theorem 2.4.1. We build the (numerical) system of the 27 cubic polynomials in the defining ideal of $\mathbb{P}\mathcal{C}^{-1}$.

```
mcubics = nullspace(mplanes);
@var y[1:M];
pcubics = transpose(mcubics) * forms_through_point(M, d)(y);
variables(pcubics)
F = System(pcubics; variables = y);
```

We evaluate the Jacobian matrix of the 27 cubics at a rank-1 symmmetric point (every such point belongs to $\mathbb{P}\mathcal{C}^{-1}$), and check that the evaluation is identically zero.

```
v = randn(ComplexF64, m);
S = v * transpose(v);
all(x -> norm(x) < 1e-10, F(sym_to_vec(S)))
J = jacobian(F, sym_to_vec(S)); maximum(svdvals(J))
```

Now we evaluate the Jacobian at a rank-2 symmetric point (every such point belongs to $\mathbb{P}\mathcal{C}^{-1}$) and compute the rank.

```
S = zeros(m, m);
S[1,1] = S[2,2] = 1;
all(x -> norm(x) < 1e-10, F(sym_to_vec(S)))
J = jacobian(F, sym_to_vec(S)); rank(J, atol=1e-11)
```

Finally, we consider system JF of equations defining the variety of points of $(\mathbb{P}\mathbb{S}^6)^\vee$ at which the Jacobian is constantly zero.

```
JF = System(vec(jacobian(F)); variables = y);
```

We pick a rank-1 point in $\mathbb{P}\mathcal{C}^{-1}$ and build an affine space through it.

```
v = randn(ComplexF64, m);
S = v * transpose(v);
y = sym_to_vec(S);
codim = 6;
V = let
    A = randn(ComplexF64, codim, M)
    b = A * y
    LinearSubspace(A, b)
end;
```

The degree of JF is then computed with monodromy.

```
mres = monodromy_solve(
    JF,
    y,
    V,
    parameter_sampler=_ -> LinearSubspace(
        randn(ComplexF64, codim, M),
        randn(ComplexF64, codim),
    ),
)
```

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