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Instanton Floer Homology and Binary Polyhedral Spaces

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For better or worse, among the many things I have learned during this period as a PhD fellow the most outstanding is not of a mathematical nature. The great Stoic philosopher Epictetus said

> "*To accuse others for one's own misfortunes is a sign of want of education. To accuse oneself shows that one's education has begun. To accuse neither oneself nor others shows that one's education is complete*". *(Epictetus, Enchiridion, 5. Translation: P.E. Matheson, 1916)*

Gard Olav Helle
Oslo, February 2022

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Chapter 1

Introduction

This thesis is concerned with the topic of equivariant instanton Floer homology; especially in relation with a family of 3-manifolds known as the binary polyhedral spaces. It is the product of research conducted during my time as a PhD fellow supervised by Kim Frøyshov at the Department of Mathematics at the University of Oslo. It consists of the following two papers:

Paper I) G.O. Helle. “Equivariant Instanton Floer Homology and Calculations for the Binary Polyhedral Spaces”. 2021.

Paper II) G.O. Helle. “Singular Quiver Varieties over Extended Dynkin Quivers”. 2021.

The first paper develops and modifies a part of the algebraic component of the construction of the equivariant instanton Floer homology (EIFH) groups of Miller Eismeier [Eis19] and provides explicit calculations for the binary polyhedral spaces. The second paper gives a classification of the singularities in a family of quiver varieties intimately linked with the binary polyhedral spaces. As an application, a number of hyper-Kähler bordisms between various members of the latter family of 3-manifolds are constructed.

This introductory chapter has two main goals. The first is to present and provide context for the key concepts involved in the two papers, while the second is to explain how they are related. The exposition is fairly informal with a focus on the main ideas involved. Nevertheless, some amount of detail is needed for the purpose of discussing the relevant mathematical objects.

The first section is devoted to introducing instanton Floer homology. I start on a basic level introducing the instanton equation and discuss some of the analytical tools needed to make sense of the solution space as a topological space and under ideal circumstances a smooth finite-dimensional manifold. While this topic is not visibly present in the actual papers, it is included here to at least give a taste of this important aspect of the theory. After presenting the basic idea of instanton Floer homology, the equivariant generalization of [Eis19] is introduced. Here I include a simplified version of the package of geometric data involved and explain in which form the corresponding algebraic invariants are delivered. In the final part of the section I introduce the algebraic procedure used to transform the geometric data into algebraic invariants. An integral part of the first paper is concerned with precisely this subject.

The second section is devoted to explaining the general context in which the second paper is written and how this relates to the content of the first paper. I start by introducing the binary polyhedral groups and spaces and discuss some of the key ideas needed to perform the EIFH calculations contained in the first paper. The two initial parts of the section are spent introducing the McKay correspondence and the process of hyper-Kähler reduction. These are the two basic concepts needed to form the bridge between instantons, binary polyhedral spaces and quiver varieties. The final part of the section is devoted to elaborating further on these relations, which form the key connection between the two papers. This also includes a brief discussion of some ideas for future work.

1.1 Background on Instanton Floer Theory

In this section we introduce and discuss some of the key ideas underlying instanton Floer homology and, more importantly, its equivariant generalization. In the final part we give a brief exposition of the algebra relevant in the first paper of this thesis.

1.1.1 The Instanton Equation

Let M be a smooth 4-manifold equipped with a Riemannian metric and an orientation. The Riemannian metric induces an inner product in $\Lambda^p T^*M$ for each $0 \leq p \leq 4$. The Hodge star operator $*\colon \Lambda^p T^*M \to \Lambda^{4-p}T^*M$ is for each p defined pointwise by the formula $\alpha \wedge *\beta = (\alpha, \beta)\,\mathrm{vol}_q$ for $\alpha, \beta \in (\Lambda^p T^*M)_q$, where $\mathrm{vol} \in \Omega^4(M)$ is the Riemannian volume element and $(\cdot,\cdot)$ is the induced inner product. It satisfies $* \circ * = (-1)^p\,\mathrm{id}\colon \Lambda^p T^*M \to \Lambda^p T^*M$, so for $p = 2$ it holds true that $*^2 = \mathrm{id}$. This relation gives rise to a splitting

$$\Lambda^2 T^*M = \Lambda^+ T^*M \oplus \Lambda^- T^*M \tag{1.1}$$

into the ± 1-eigenbundles of $*$. This natural bundle decomposition of the second exterior power of the cotangent bundle is special to dimension four and gives rise to the instanton equation, which we will introduce shortly.[1]

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and let $\pi\colon P \to M$ be a principal G-bundle. Given a connection $A \in \Omega^1(P, \mathfrak{g})$, its curvature

$$F_A = dA + \frac{1}{2} A \wedge_{ad} A \in \Omega^2(P, \mathfrak{g})$$

descends to a bundle valued 2-form $F_A \in \Omega^2(M, \mathfrak{g}_P)$, where $\wedge_{ad}$ combines the wedge product with the Lie bracket $\mathrm{ad} = [\cdot,\cdot]$ and $\mathfrak{g}_P = P \times_G \mathfrak{g}$ is the bundle associated with the adjoint action on $\mathfrak{g}$. The bundle splitting in (1.1) induces a splitting

$$\Omega^2(M, \mathfrak{g}_P) = \Omega^+(M, \mathfrak{g}_P) \oplus \Omega^-(M, \mathfrak{g}_P)$$

and we may accordingly write $F_A = F_A^+ + F_A^-$. The instanton or anti-self-dual (ASD) equation is then simply $F_A^+ = 0$, or equivalently $*F_A = -F_A$.

[1] There is an analogous splitting of $\Lambda^{2n}T^*M$ when $\dim M = 4n$.

Example 1.1.1. Let $M = \mathbb{R}^4$ with the constant metric and standard orientation. Let $P = \mathbb{R}^4 \times G$ be the trivial bundle. Then a connection can be represented by a 1-form $a = \sum_{i=1}^4 a_i dx^i \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$, where the $a_i\colon \mathbb{R}^4 \to \mathfrak{g}$ are smooth functions. From the formula $F = da + \frac{1}{2} a \wedge_{ad} a$ one finds that the curvature $F = \sum_{i<j} F_{ij} dx^i dx^j$ has components

$$F_{ij} = \frac{\partial a_j}{\partial x^i} - \frac{\partial a_i}{\partial x^j} + [a_i, a_j].$$

In this case the bundles $\Lambda^{\pm} T^* \mathbb{R}^4$ have global frames given by

$$(dx^1 dx^2 \pm dx^3 dx^4, dx^1 dx^3 \pm dx^4 dx^2, dx^1 dx^4 \pm dx^2 dx^3),$$

respectively, so the equation $F^+ = 0$ takes the form $F_{12} + F_{34} = F_{13} + F_{42} = F_{14} + F_{23} = 0$.

The instanton equation, or rather the splitting in (1.1), is intimately linked with the quaternion algebra $\mathbb{H}$. This is the real 4-dimensional algebra with standard basis $1, i, j, k$ and multiplication rules specified by the equations

$$i^2 = j^2 = k^2 = ijk = -1.$$

We define a real inner product in $\mathbb{H}$ by declaring $(1, i, j, k)$ to be an orthonormal basis. There is also an involution $q \mapsto q^*$ defined on the standard basis by $1, i, j, k \mapsto 1, -i, -j, -k$. The group of unit quaternions $\mathrm{Sp}(1) := S^3 \subset \mathbb{H}$ is then a simply connected compact Lie group whose Lie algebra is naturally identified with

$$\mathfrak{sp}(1) = \operatorname{Im} \mathbb{H} := \operatorname{Span}_{\mathbb{R}} \{i, j, k\}.$$

Let $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ act on $\mathbb{H}$ by the rule $(a, b) \cdot x = axb^{-1}$. This action preserves the inner product on $\mathbb{H}$ and therefore defines a homomorphism $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$. This map turns out to be surjective with kernel $\{\pm(1, 1)\}$ and therefore induces an isomorphism of Lie algebras

$$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \cong \mathfrak{so}(4). \tag{1.2}$$

The adjoint action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ factors through $\mathrm{SO}(4)$, so this is an isomorphism of $\mathrm{SO}(4)$-representations. In view of the fact that $\Lambda^2(\mathbb{R}^4)^* \cong \mathfrak{so}(4)$ as $\mathrm{SO}(4)$-representations, the isomorphism in (1.2) is the structure group incarnation of the bundle splitting in (1.1).

One of the most basic, yet non-trivial, examples of a solution to the instanton equation is most easily expressed using quaternions. A detailed exposition of the following example may be found in [Ati79].

Example 1.1.2. Regard $\mathbb{R}^4 = \mathbb{H}$ and consider the trivial $\mathrm{Sp}(1)$-bundle over $\mathbb{H}$. Let $x\colon \mathbb{H} \to \mathbb{H}$ denote the identity map. Then

$$a := \operatorname{Im} \left(\frac{x^* dx}{1 + |x|^2} \right) \in \Omega^1(\mathbb{H}, \operatorname{Im} \mathbb{H}) \tag{1.3}$$

defines a connection form, where $\mathrm{Im}\colon \mathbb{H} \to \mathrm{Im}\,\mathbb{H}$ denotes the orthogonal projection. The curvature is given by the formula

$$F = \frac{dx^* \wedge dx}{(1+|x|^2)^2} \in \Omega^2(\mathbb{H}, \mathrm{Im}\,\mathbb{H}),$$

where the wedge product is combined with multiplication in $\mathbb{H}$ in the numerator. One may check by explicit calculation that this curvature form is anti-self-dual.

On a more conceptual level, if we regard $\mathbb{H} \subset \mathbb{H}P^1$ as a standard coordinate patch in the quaternionic projective line, the connection form in (1.3) corresponds to the canonical connection of the quaternionic line bundle $\gamma^\perp \to \mathbb{H}P^1$. This is the orthogonal complement of the tautological line bundle $\gamma \to \mathbb{H}P^1$ regarded as a subbundle of the trivial bundle with fiber $\mathbb{H}^2$.

The basic idea in instanton gauge theory is to study the space of all solutions of the instanton equation modulo a natural notion of symmetry, namely, gauge equivalence. We first give a naive outline and then elaborate on some of the technicalities involved in the next section. Let M be a compact, oriented, Riemannian 4-manifold and let $\pi\colon P \to M$ be a principal G-bundle. Let $\mathcal{A}_P$ denote the space of all connections in P. This is an affine space over $\Omega^1(M, \mathfrak{g}_P)$, i.e., if we fix a base connection $A_0 \in \mathcal{A}_P$ then $\mathcal{A}_P = \{A_0 + a : a \in \Omega^1(M, \mathfrak{g}_P)\}$, where a is regarded as an element of $\Omega^1(P, \mathfrak{g})$.[2]The gauge group $\mathcal{G}_P$ of P is the group of bundle automorphisms $f\colon P \to P$ covering the identity map in M. There is a group action

$$\mathcal{A}_P \times \mathcal{G}_P \to \mathcal{A}_P$$

given by pullback; $(A, f) \mapsto f^*A$. The orbit space $\mathcal{B}_P := \mathcal{A}_P/\mathcal{G}_P$ is called the configuration space. The gauge group also acts linearly on the vector space $\Omega^p(M, \mathfrak{g}_P)$ for each p. The rule that assigns the curvature F_A to a connection A then takes the form of a $\mathcal{G}_P$-equivariant map $F\colon \mathcal{A}_P \to \Omega^2(M, \mathfrak{g}_P)$. The projection $p^+\colon \Omega^2(M, \mathfrak{g}_P) \to \Omega^+(M, \mathfrak{g}_P)$ is also equivariant, so the same holds true for the composite

$$F^+ := p^+ \circ F\colon \mathcal{A}_P \to \Omega^+(M, \mathfrak{g}_P). \tag{1.4}$$

The moduli space of instantons or ASD connections is then

$$\mathcal{M}_{ASD}(P) := (F^+)^{-1}(0)/\mathcal{G}_P = \{A \in \mathcal{A}_P : F_A^+ = 0\}/\mathcal{G}_P \subset \mathcal{B}_P.$$

These moduli spaces are the basic objects of study in instanton gauge theory. In the next section we will outline how these spaces are topologized and under which conditions one can expect them to carry the structure of a smooth finite-dimensional manifold in the simplest case of a compact base manifold. Further important aspects of the theory are the compactness properties of the moduli spaces and whether they are orientable. Establishing orientability and appropriate compactness properties of certain instanton moduli spaces over a cylinder $\mathbb{R} \times Y$ for a closed 3-manifold Y are of vital importance in order to construct instanton Floer homology.

1.1.2 Some Analytical Aspects

Let M be a closed Riemannian 4-manifold and let $\pi\colon P \to M$ be a principal G bundle for some compact Lie group G. For our later purposes we may take $G = \mathrm{SO}(3)$ or $G = \mathrm{SU}(2)$. The first step needed to make sense of $\mathcal{M}_{ASD}(P)$ as a topological space is to provide every space in sight with the structure of a smooth infinite-dimensional manifold. The standard and most convenient way of doing this is to use Sobolev completions. Let $E \to M$ be a vector bundle equipped with a fiber metric and a connection. The metrics and connections in TM and E induce a metric and connection in each bundle $T^k(T^*M) \otimes E = T^*M \otimes \cdots \otimes T^*M \otimes E$. For an integer $k \geq 0$, the L^2_k norm on the space $\Gamma(E)$ of smooth sections is defined by

$$||s||_{L^2_k} = \left(\sum_{j=0}^{k} \int_M |\nabla^{(j)} s|^2 \,\mathrm{vol} \right)^{1/2},$$

where $\nabla^{(j)}$ denotes the iterated covariant derivative and vol is the Riemannian volume form. The completion of $\Gamma(E)$ with respect to the norm $|| \cdot ||_{L^2_k}$ is a Hilbert space denoted by $L^2_k(E)$ and is called the Sobolev space of L^2_k-sections. We should note that the norm $|| \cdot ||_{L^2_k}$ depends on our choices of connections and metric, but the resulting completion is as a topological vector space independent of all of these choices since the base manifold M is compact. The spaces $L^2_k(E)$ for various $k \geq 0$ satisfy a number of useful properties (see for instance [FU91, p. 92]). Here we will just note that there is a sequence of compact, continuous inclusions

$$L^2(E) = L^2_0(E) \supset L^2_1(E) \supset \cdots \supset L^2_k(E) \supset L^2_{k+1}(E) \supset \cdots,$$

whose inverse limit recovers $\Gamma(E)$ in the C^∞-topology. This is a consequence of the Sobolev embedding theorem, which in this 4-dimensional situation states that there is a compact inclusion $L^2_k(E) \hookrightarrow C^l(E)$ provided $k - 4/2 > l$, where $C^l(E)$ is the space of l times differentiable sections in the C^l-topology.

In our situation of interest we may fix a smooth base connection $A_0 \in \mathcal{A}_P$ and define $\mathcal{A}^2_{P,k} = A_0 + L^2_k(T^*M \otimes \mathfrak{g}_P)$ for each $0 \leq k < \infty$. Similarly, $\Omega^+(M, \mathfrak{g}_P)^2_k = L^2_k(\Lambda^+ T^*M \otimes \mathfrak{g}_P)$. Provided there is a Sobolev embedding $L^2_k \hookrightarrow C^0$, i.e., $k \geq 3$, we may also define a Sobolev completion $\mathcal{G}^2_{P,k}$ of the gauge group. For any fixed integer $k \geq 2$, it then holds true that

(i) $\mathcal{G}^2_{P,k+1}$ is a Hilbert Lie group, $\mathcal{A}^2_{P,k}$ is an affine Hilbert space and $\Omega^+(M, \mathfrak{g}_P)^2_{k-1}$ is a Hilbert space,

[2] There is a natural bijection between $\Omega^p(M, \mathfrak{g}_P)$ and the basic forms in $\Omega^p(P, \mathfrak{g})$, that is, the G-invariant forms α for which $\alpha(v_1, \cdots, v_p) = 0$ whenever v_i is tangent to the fiber for some $1 \leq i \leq p$.

(ii) there is an extended gauge action $\mathcal{A}^2_{P,k} \times \mathcal{G}^2_{P,k+1} \to \mathcal{A}^2_{P,k}$ which is smooth and proper, and

(iii) the map F^+ of (1.4) extends to a smooth map

$$F^+ \colon \mathcal{A}^2_{P,k} \to \Omega^+(M, \mathfrak{g}_P)^2_{k-1}.$$

Therefore, for each integer $k \geq 2$ we get a moduli space $\mathcal{M}_{ASD}(P,k)$ equipped with a well-defined topology. Due to the elliptic nature of the ASD equation it turns out that this topological space is independent of the integer k (see [DK90, Prop. 4.2.16]).

What the above Sobolev completions achieve is to express the problem in the category of Banach (or Hilbert) manifolds.[3]In fact, a number of results familiar from finite-dimensional manifold theory extend to this infinite-dimensional setting (see [Lan85]). In particular, the inverse and implicit function theorems apply with minor modifications. To understand the local structure around $[A] \in \mathcal{M}_{ASD}(P,k)$ for $A \in \mathcal{A}^2_{P,k}$ consider the sequence $\mathcal{G}^2_{P,k+1} \to \mathcal{A}^2_{P,k} \to \Omega^+(M, \mathfrak{g}_P)^2_{k-1}$ where the first map is the orbit map $g \mapsto g^*A$ and the second is F^+. Since A is ASD the composite is constant and the derivatives at $1 \in \mathcal{G}^2_{P,k+1}$ and $A \in \mathcal{A}^2_{P,k}$ form a three term complex

$$\Omega^0(M, \mathfrak{g}_P)^2_{k+1} \xrightarrow{d_A} \Omega^1(M, \mathfrak{g}_P)^2_k \xrightarrow{d_A^+} \Omega^+(M, \mathfrak{g}_P)^2_{k-1}, \tag{1.5}$$

where d_A is the exterior derivative twisted by A and $d_A^+ = p^+ \circ d_A$. Here we have used that the Lie algebra of $\mathcal{G}_P$ is $\Omega^0(M, \mathfrak{g}_P)$ and that $T_A\mathcal{A}^2_{P,k} \cong \Omega^1(M, \mathfrak{g}_P)^2_{P,k}$. The above sequence is an elliptic complex, that is, the composition is zero and the sequence is exact on the level of symbols. This implies that the cohomology groups, written H^i_A for $i = 0, 1, 2$, are finite-dimensional and independent of the integer k. The group $H^0_A = \mathrm{Ker}(d_A)$ is precisely the Lie algebra of the stabilizer of A in the gauge group. Hence, $H^0_A = 0$ implies that the stabilizer is zero-dimensional. While, if $H^2_A = 0$, then $d(F^+)_A = d^+_A$ is surjective, which implies that $(F^+)^{-1}(0)$ carries the structure of a manifold in a neighborhood of A. At least for the structure group $G = \mathrm{SU}(n)$, one can conclude that if $H^0_A = H^2_A = 0$ then $\mathcal{M}_{ASD}(P)$ carries the structure of a smooth manifold of dimension $H^1_A < \infty$ in a neighborhood of $[A]$.[4]

Using the formal adjoint construction, the two key conditions for local smoothness can be neatly expressed in terms of the single elliptic operator

$$D_A := (d^*_A, d^+_A) \colon \Omega^1(M, \mathfrak{g}_P) \to \Omega^0(M, \mathfrak{g}_P) \oplus \Omega^+(M, \mathfrak{g}_P).$$

[3]A (smooth) Banach manifold is simply a Hausdorff topological space X equipped with a maximal atlas $\{\phi_i \colon U_i \to X : i \in I\}$ of local charts, where U_i is an open subset of some Banach space E_i for each $i \in I$ and the transition functions are smooth.

[4]The center $Z(G)$ can be realized naturally as a subgroup of $\mathcal{G}_P$. The action of the gauge group on $\mathcal{A}_P$ factors through $\mathcal{G}_P/Z(G)$. For $G = \mathrm{SU}(n)$ it holds true that the stabilizer of a connection A in $\mathcal{G}$ is discrete, i.e. $H^0_A = 0$, if and only if it coincides with $Z(G)$.

Explicitly, $\mathrm{Coker}(D_A) \cong H^0_A \oplus H^2_A$ and $\mathrm{Ker}\, D_A \cong H^1_A$. This operator therefore plays an important part in the theory. Its index $\mathrm{Ind}(D_A) = \dim \mathrm{Ker}(D_A) - \dim \mathrm{Coker}(D_A)$ can be expressed using the Atiyah-Singer index theorem.

Example 1.1.3. An $\mathrm{SU}(2) = \mathrm{Sp}(1)$-bundle E over a closed oriented 4-manifold M is classified up to isomorphism by the second Chern number $c_2(E)[M] \in \mathbb{Z}$. In general, if M is connected, the stabilizer $\mathcal{G}_{P,A}$ of a connection $A \in \mathcal{A}_P$ can be identified with the centralizer of the holonomy group $\mathrm{Hol}_A(x_0) \subset \mathrm{Aut}(E_{x_0}) \cong \mathrm{SU}(2)$ over a base point $x_0 \in M$. By direct calculation one may show that the possible stabilizers are $\mathrm{SU}(2)$, $U(1)$ or $\{\pm 1\}$. In particular, the only 0-dimensional stabilizer is the center $\{\pm 1\}$. Consequently, if D_A is surjective for each ASD connection $A \in \mathcal{A}_P$, the moduli space $\mathcal{M}_{ASD}(E)$ carries the structure of a smooth manifold of dimension

$$\mathrm{Ind}(D_A) = 8c_2(E)[M] - \frac{3}{2}(\chi(M) + \sigma(M))$$

where $\chi(M)$ is the Euler characteristic of M and $\sigma(M)$ is the signature ([DK90, p. 137]).

We end this brief analytical tour with a few remarks concerning what happens in the case D_A fails to be surjective. As we have seen, there are two obstructions involved, one concerning the failure of the group action of $\mathcal{G}_P$ to be free (more precisely $\mathcal{G}_P/Z(G)$) and one concerning the failure of 0 to be a regular value for the function F^+. The second issue can typically be solved by introducing a perturbation of the function, or changing the Riemannian metric on the base manifold M. The first issue cannot be removed in this fashion though. Either one can just be fine with the fact that $\mathcal{M}_{ASD}(P)$ is a singular space or, as we will see later, one can work with framed connections yielding a smooth moduli space $\widetilde{\mathcal{M}}_{ASD}(P)$ with a residual action of $G/Z(G)$ and try to analyze this space using equivariant techniques.

For a more complete account of the material presented here we refer to [DK90], [FU91] and [Pal68].

1.1.3 Instanton Floer Theory

The basic idea of instanton Floer homology is to do Morse theory in an infinite-dimensional setting, where the Chern-Simons functional plays the role of a Morse function. To introduce this functional and see where the instanton equation enters into the picture, let us for simplicity consider a trivial $\mathrm{SU}(2)$-bundle $P = Y \times \mathrm{SU}(2) \to Y$ over a closed oriented Riemannian 3-manifold Y. In this setting there is a canonical identification $\mathcal{A}_P \cong \Omega^1(Y, \mathfrak{su}(2)$, where $\mathfrak{su}(2) \subset M_2(\mathbb{C})$ denotes the Lie algebra of skew-adjoint matrices. In terms of this identification the Chern-Simons functional $\mathrm{cs}\colon \mathcal{A}_P \to \mathbb{R}$ is given by the explicit formula

$$\mathrm{cs}(A) = \frac{1}{8\pi^2}\int_Y \mathrm{trace}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \quad \text{for} \quad A \in \Omega^1(Y, \mathfrak{su}(2))$$

where the wedge product is defined using matrix multiplication in $\mathfrak{su}(2) \subset M_2(\mathbb{C})$ (see [Don02, p. 18]). This function is gauge invariant up to an integer and therefore descends to a function $\mathrm{cs}\colon \mathcal{B}_P \to \mathbb{R}/\mathbb{Z} = S^1$. It is this function one seeks to do Morse theory with in instanton Floer homology. Briefly, this means that one tries to construct a chain complex generated by the critical points of cs and whose differential counts negative gradient flow lines between critical points.

The derivative of cs is up to a constant given by $d(\mathrm{cs})_A(a) = \int_Y \mathrm{trace}(F_A \wedge a)$. By using the L^2 inner product in $T_A\mathcal{A}_P \cong \Omega^1(Y, \mathfrak{su}(2))$, $(a,b)_{L^2} = \int_Y (a,b)\,\mathrm{vol} = \int_Y \mathrm{trace}(a \wedge *_3 b)$, [5]the corresponding gradient vector field is given by

$$V\colon A \in \mathcal{A}_P \mapsto *_3 F_A \in \Omega^1(Y, \mathfrak{su}(2)).$$

In particular, the critical set of cs is precisely the set

$$\mathcal{C} := \{[A] \in \mathcal{B}_P : F_A = 0\}$$

of gauge equivalence classes of flat connections. The negative gradient flow equation for cs is therefore given by

$$\frac{dA_t}{dt} = - *_3 F_{A_t} \tag{1.6}$$

for a path of connections $t \in \mathbb{R} \mapsto A_t \in \mathcal{A}_P$. Such a path also defines a connection $\mathbf{A}$ in the trivial bundle over the cylinder $\mathbb{R} \times Y$. In terms of this relation the equation (1.6) takes precisely the form $F^+_{\mathbf{A}} = 0$. Therefore, at least heuristically, the spaces of negative gradient flow lines correspond to (translation reduced) instanton moduli spaces over the cylinder $\mathbb{R} \times Y$.

In [Flo88] Andreas Floer introduced the above idea and succeeded in constructing a $\mathbb{Z}/8$ graded abelian group $I_*(Y)$, later coined instanton Floer homology, associated with a homology 3-sphere Y, that is, $H_*(Y;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$. The reason for the $\mathbb{Z}/8$ grading is that one can only make sense of an index function $i\colon \mathcal{C} \to \mathbb{Z}/8$. Setting this theory up in detail requires quite a lot of delicate analysis (see [Don02]), and there are obstructions to making the construction go through for a general closed, oriented 3-manifold. Nevertheless, the construction can be generalized to apply for a class of manifolds including all rational homology 3-spheres using equivariant techniques.

1.1.4 Equivariant Instanton Floer Homology

Let $\pi\colon P \to Y$ be a principal $G = \mathrm{SU}(2)$-bundle over a closed oriented 3-manifold. The action of the gauge group $\mathcal{G}_P$ on the space of connections $\mathcal{A}_P$ is typically not free, so the configuration space $\mathcal{B}_P$ will generally be a singular space. One way of circumventing this issue is to choose a base point $b \in Y$ and define the space of framed connections $\widetilde{\mathcal{A}}_P := \mathcal{A}_P \times P_b$, where $P_b = \pi^{-1}(b)$ is the fiber over b. Each gauge transformation $f\colon P \to P$ restricts to an automorphism $f_b\colon P_b \to P_b$. The gauge group therefore also acts (say from the left) on $\widetilde{\mathcal{A}}_P$ by

[5]This L^2-inner product is negative definite. If we replace it with the positive definite inner product by introducing a sign, the instanton equation becomes the gradient flow equation rather than the negative gradient flow equation.

$f \cdot (A, u) = ((f^{-1})^* A, f_b(u))$. In this case the action is free, and we obtain a framed configuration space

$$\widetilde{\mathcal{B}}_P \coloneqq \widetilde{\mathcal{A}}_P / \mathcal{G}_P.$$

The right SU(2)-action on the fiber P_b descends to an $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm 1\}$ action on $\widetilde{\mathcal{B}}_P$. In appropriate Sobolev completions $\widetilde{\mathcal{B}}_P$ becomes a smooth Hilbert manifold equipped with a smooth SO(3)-action.

The equivariant instanton Floer homology of [Eis19] considered in this thesis is obtained by applying techniques from equivariant Morse theory or Morse-Bott theory (see [AB95] or [BH10] for the finite-dimensional perspective) to the natural SO(3)-invariant extension of the Chern-Simons functional to $\mathrm{cs}\colon \widetilde{\mathcal{B}}_P \to \mathbb{R}/\mathbb{Z} = S^1$. [6]In this introductory text we will only give a brief outline of what one should expect in this situation and refer to the original resource mentioned above and the outline given in the first paper of this thesis for further details.

For simplicity we consider a situation where Y is a rational homology 3-sphere, i.e., $H_*(Y;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$, and $\pi\colon P \to Y$ is the trivial SU(2)-bundle. Since $\mathrm{cs}\colon \widetilde{\mathcal{B}}_P \to \mathbb{R}/\mathbb{Z}$ is SO(3)-equivariant, the critical set admits a partition into SO(3)-orbits. The set $\mathcal{C}$ of critical orbits is in natural correspondence with the set of gauge equivalence classes of flat connections in P. An orbit $\alpha \in \mathcal{C}$ is of the form

$$\mathrm{SO}(3), \quad \mathrm{SO}(3)/\,\mathrm{SO}(2) \cong S^2 \quad \text{or} \quad \mathrm{SO}(3)/\,\mathrm{SO}(3) = *$$

and we say that α is irreducible, reducible or fully reducible, respectively. To ensure that $\mathcal{C}$ consists of finitely many orbits and that the moduli spaces, we will soon introduce, are cut out transversely, one generally has to introduce a perturbation of cs. In this outline we will for simplicity just assume that the above two conditions are satisfied.

Let $\alpha, \beta \in \mathcal{C}$ be a pair of distinct critical orbits. For each relative homotopy class $z \in \pi_1(\widetilde{\mathcal{B}}_P, \alpha, \beta)$ there is then a moduli space $\widetilde{\mathcal{M}}_z(\alpha, \beta)$ of framed instantons approaching α as $t \to -\infty$ and β as $t \to \infty$ whose corresponding path in $\widetilde{\mathcal{B}}_P$ belongs to the homotopy class z. This is a (possibly empty) finite-dimensional smooth SO(3)-manifold equipped with SO(3)-equivariant end-point maps as in the diagram

$$\alpha \xleftarrow{\;e_-\;} \widetilde{\mathcal{M}}_z(\alpha, \beta) \xrightarrow{\;e_+\;} \beta.$$

There is a number $\mathrm{gr}_z(\alpha, \beta) \in \mathbb{Z}$ such that if $\mathrm{gr}_z(\alpha, \beta) \leq 0$ the moduli space is empty and otherwise

$$\mathrm{gr}_z(\alpha, \beta) = \widetilde{\mathcal{M}}_z(\alpha, \beta) - \dim \alpha.$$

This number is independent of the homotopy class z modulo 8, and it therefore descends to a function $\mathrm{gr}\colon \mathcal{C} \times \mathcal{C} \to \mathbb{Z}/8$ called the relative grading. It satisfies the property $\mathrm{gr}(\alpha, \beta) + \mathrm{gr}(\beta, \gamma) = \mathrm{gr}(\alpha, \gamma)$ for all $\alpha, \beta, \gamma \in \mathcal{C}$. In our case of a

[6]The equivariant instanton Floer homology of [Eis19] is defined more generally for a weakly admissible SO(3)-bundle over a closed oriented 3-manifold. An SO(3)-bundle $E \to Y$ is weakly admissible if either Y is a rational homology sphere or if the second Stiefel Whitney class $w_2(E) \in H^2(Y;\mathbb{Z}/2)$ does not admit a torsion lift in $H^2(Y;\mathbb{Z})$.

trivial SU(2)-bundle we obtain an absolute grading $j\colon \mathcal{C} \to \mathbb{Z}/8$ by taking the trivial connection θ as a reference point, that is, $j(\alpha) := \mathrm{gr}(\alpha, \theta)$.

The moduli spaces $\widetilde{\mathcal{M}}_z(\alpha, \beta)$ carry an action of $\mathbb{R}$ by translations. Provided $\alpha \neq \beta$ and $\mathrm{gr}_z(\alpha, \beta) \leq 10$ the resulting quotient admits a compactification $\widetilde{\mathcal{M}}_z(\alpha, \beta)/\mathbb{R} \subset \overline{\mathcal{M}}_z(\alpha, \beta)$ into a topological manifold with corners and smooth structure on each stratum. The SO(3)-action and the equivariant end-point maps are defined on this compactification as well. We now have all the pieces needed to define the framed Floer complex $(\widetilde{CI}(Y, P), \partial)$. There is a choice of coefficient ring involved, but we will omit this from the notation. We set

$$\widetilde{CI}(Y, P)_{s,t} = \bigoplus_{j(\alpha) \equiv s} C_*^{gm}(\alpha) \quad \text{and} \quad \widetilde{CI}(Y, P)_n = \bigoplus_{s+t=n} \widetilde{CI}(Y, P)_{s,t}, \tag{1.7}$$

where C_*^{gm} is a geometric homology functor defined on smooth manifolds. For the moment, we will just say that $C_*^{gm}(\alpha)$ is a chain complex supported in degrees $0 \leq i \leq \dim(\alpha) + 1$, whose homology recovers the usual singular homology of α. For the specifics concerning this type of geometric homology see [Eis19, Section 6.1]. The differential $\partial\colon \widetilde{CI}(Y, P)_n \to \widetilde{CI}(Y, P)_{n-1}$ has components

$$\partial^r\colon \widetilde{CI}(Y, P)_{s,t} \to \widetilde{CI}(Y, P)_{s-r,t+r-1} \quad \text{for} \quad 0 \leq r \leq 5.$$

The component ∂^0 is defined to be the sum over the internal differentials $C_t^{gm}(\alpha) \to C_{t-1}^{gm}(\alpha)$ for $\alpha \in \mathcal{C}$ with $j(\alpha) \equiv s$. The other components, ∂^r for $1 \leq r \leq 5$, are defined via fiber products using the diagrams

$$\alpha \xleftarrow{e_-} \overline{\mathcal{M}}_z(\alpha, \beta) \xrightarrow{e_+} \beta, \tag{1.8}$$

where z is the unique homotopy class for which $-2 \leq \mathrm{gr}_z(\alpha, \beta) \leq 5$. To make this precise one has to delve into the inner workings of the geometric homology functor C_*^{gm}, so instead we will give a motivational analogy.

Suppose we are given a diagram of closed, oriented smooth manifolds

$$X \xleftarrow{p} M \xrightarrow{q} Y \ .$$

Then there is a map $p_!\colon H_*(X) \to H_{*+l}(M)$, where $l = \dim M - \dim X$. It is defined to be the composite $\mathrm{PD}_M \circ p^* \circ \mathrm{PD}_X^{-1}$, where PD_M, PD_X are Poincaré duality isomorphisms and p^* is the pullback map in cohomology. The above diagram therefore gives rise to the map $f = q_* \circ p_!\colon H_*(X) \to H_{*+l}(Y)$. In geometric terms, suppose that $\alpha \in H_i(X)$ can be represented by a closed oriented submanifold $\iota\colon Z \hookrightarrow X$, i.e., $\iota_*([Z]) = \alpha$ where $[Z]$ is the fundamental class of Z. Then, if p is transverse to Z, $p_!(\alpha) \in H_{i+l}(M)$ is represented by $p^{-1}(Z) \hookrightarrow M$, so that $f(\alpha)$ is represented by $p^{-1}(Z) \hookrightarrow M \to Y$. Now $p^{-1}(Z)$ is nothing but the fiber product $Z \times_X M \hookrightarrow M$, so alternatively $f(\alpha)$ is represented by the composite $Z \times_X M \to M \to Y$. What the geometric homology functor C_*^{gm} achieves is to represent maps such as f on the chain level. The diagram (1.8) gives rise to a fiber product map $f_{\beta\alpha}\colon C_t^{gm}(\alpha) \to C_{t+\mathrm{gr}_z(\alpha,\beta)-1}^{gm}(\beta)$ and these maps are the components of ∂^r for $1 \leq r \leq 5$.

The complex $\widetilde{CI}(Y,P)$ carries the structure of a right $C_*^{gm}(\mathrm{SO}(3))$ module and a filtration by index $\{F_p\widetilde{CI}(Y,P)\}_{p\in\mathbb{Z}}$. In terms of the bigrading given in (1.7) it is given by

$$F_p\widetilde{CI}(Y,P)_n = \bigoplus_{\substack{s+t=n\\ s\le p}} \widetilde{CI}(Y,P)_{s,t}$$

for $p, n \in \mathbb{Z}$. This is an increasing filtration by $C_*^{gm}(\mathrm{SO}(3))$ submodules and is called the index filtration. Moreover, since the index function j takes values in $\mathbb{Z}/8$ there is a periodicity isomorphism $\widetilde{CI}(Y,P)_n \cong \widetilde{CI}(Y,P)_{n+8}$ for all $n \in \mathbb{Z}$ compatible with all the additional structure mentioned above. The equivariant Floer groups are extracted from the framed Floer complex $\widetilde{CI}(Y,P)$, the action of $C_*^{gm}(\mathrm{SO}(3))$ and the index filtration by purely algebraic means, and come in the form of three groups $I^+(Y,P)$, $I^-(Y,P)$ and $I^\infty(Y,P)$. They are called the positive, negative and Tate equivariant instanton Floer homology of the pair (Y,P). These groups carry module structure over $H^{-*}(B\,\mathrm{SO}(3))$ and fit into an exact triangle

$$\begin{array}{ccc} I^+(Y,P) & \xrightarrow{\;[+3]\;} & I^-(Y,P) \\ & {\scriptstyle [-4]}\nwarrow \qquad \swarrow{\scriptstyle [0]} & \\ & I^\infty(Y,P) & \end{array} \qquad , \tag{1.9}$$

where the numbers indicates the degree of the respective maps.

An integral part of the first paper contained in this thesis is concerned with the homological algebra needed for the construction of the above groups from the framed Floer complex $\widetilde{CI}(Y,P)$. We therefore give a short introduction into the ideas needed for this next.

1.1.5 The Algebra of EIFH

Fix a principal ideal domain R, which serve as the ground ring for all modules, algebras and their graded and differential graded (DG) analogues. Let A be a differential graded algebra and M a right A-module. The situation to have in mind is of course $A = C_*^{gm}(\mathrm{SO}(3); R)$ and $M = \widetilde{CI}(Y,P)$. In this case M also carries the index filtration and we will discuss the effect of this later.

In [Eis19, Appendix A] the author constructs a triple of functors C_A^+, C_A^- and C_A^∞ from the category of right A-modules to the category DG R-modules. In fact, $C_A^-(R)$ carries the structure of a differential graded algebra and the three functors take values in the category of left $C_A^-(R)$-modules. Before we consider the construction of these let us look at a more basic example of a similar construction.

Example 1.1.4. Let G be a finite group and let A be a G-module, i.e., a module over the integral group ring $\mathbb{Z}[G]$. Then the homology and cohomology of G with coefficients in A are defined by

$$H_*(G;A) \coloneqq \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, A) \quad \text{and} \quad H^*(G;A) \coloneqq \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, A)$$

where $\mathbb{Z}$ is regarded as a trivial G-module. These groups can, at least in theory, be calculated using the (normalized) bar resolution $B_*(G)$, a functorial free resolution of $\mathbb{Z}$. By definition $B_n(G)$ is the free $\mathbb{Z}[G]$-module on the set of symbols $[g_1|g_2|\cdots|g_n]$ for $g_i \in G - \{1\}$, $1 \le i \le n$. In particular, $B_0(G)$ is the free module on the empty symbol $[]$. The differential ∂ is defined by sending a generator $[g_1|\cdots|g_n]$ to

$$g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}]$$

where a term with $g_ig_{i+1} = 1$ is interpreted to be 0. Thus for a G-module A one has $H_*(G;A) = H(B_*(G) \otimes_G A)$ and $H^*(G;A) = H(\mathrm{Hom}_G(B_*(G), A))$.

In addition, to the homology and cohomology one has the Tate (co)homology of G with coefficients in A. Define $N = \sum_{g\in G} g \in \mathbb{Z}[G]$ to be the norm element. Write $C_G^+(A) = A \otimes B_*(G)$ and $C_G^-(A) = \mathrm{Hom}_G(B_*(G), A)$ for the chain complexes calculating the group (co)homology of G with coefficients in A. Then $C_G^+(A)$ is concentrated in non-negative degrees while $C_G^-(A)$ is concentrated in non-positive degrees, that is, we omit raising indices and also regard the latter as a homological chain complex. Moreover, since $B_0(G) \cong \mathbb{Z}[G]$ we may identify $C_G^+(A)_0 \cong A$ and $C_G^-(A)_0 \cong A$. Then multiplication by the norm element $N \in \mathbb{Z}[G]$

$$N\cdot\colon C_G^+(A)_0 \cong A \to A \cong C_G^-(A)_0$$

can be shown to define a chain map. Let $C_G^\infty(A)$ be the mapping cone complex of the above norm map. The homology of this complex can be called the Tate homology of G with coefficients in A. In this situation it is more usual to raise indices and write $\hat{H}^*(G;A) = H(C_G^\infty(A))_{-*}$. In this basic situation it is not necessary to introduce a mapping cone. Indeed, one can define $\hat{H}^*(G;A)$ directly in terms of $H_*(G;A)$, $H^*(G;A)$ and the induced norm map $N\colon A_G \to A^G$ between coinvariants and invariants. See for instance [Wei94, Chapter 6] for background and additional information concerning this example.

Let as earlier A be a DG algebra and assume that A is equipped with an augmentation $\epsilon\colon A \to R$. Given a right A-module M and a left A-module N there is a complex $B(M, A, N)$ called the bar construction on the triple (M, A, N) (see [GM74, Appendix A]). It is a derived version of the tensor product $M \otimes_A N$ in an appropriate sense (see [BMR14]). The functors $C_A^\pm$ are then defined for a right A-module M by

$$C_A^+(M) \coloneqq B(M, A, R) \quad \text{and} \quad C_A^-(M) \coloneqq \mathrm{Hom}_A(B(R, A, A), M),$$

where we note that $B(R, A, A)$ inherits a right A-module structure from the right hand factor. The construction of C_A^∞ is more involved, and we will not give the definition here. However, we will mention that it is constructed as the cone of an appropriate norm map as in the above example. This eventually gives rise to the exact triangle of (1.9).

Let $\bullet$ denote any element of $\{+,-,\infty\}$. Provided the algebra A is degreewise free over R, the functor $C_A^\bullet$ is exact. Therefore, if M is equipped with an increasing filtration

$$\cdots \subset F_{p-1}M \subset F_pM \subset F_{p+1}M \subset \cdots \subset M$$

there is an induced filtration $F_pC_A^\bullet(M) \coloneqq C_A^\bullet(F_pM)$. This filtration gives rise to a spectral sequence $\{E^r, d^r\}_{r\geq 1}$ with

$$E_{s,t}^1 = H_A^\bullet(F_sM/F_{s-1}M)_{s+t}.$$

However, this spectral sequence may fail to converge to $H_A^\bullet(M)$ in any sensible way. To ensure that the above spectral sequence satisfy a structural property called conditional convergence (see [Boa99]), a completion procedure is introduced.[7]In the first paper of this thesis we have taken a different approach to this completion procedure than the one initially given in [Eis19]. In any case, there is a completed version $\hat{C}_A^\bullet$ of $C_A^\bullet$. The equivariant instanton Floer group $I^\bullet(Y,P)$ is then defined to be the homology of $\hat{C}_A^\bullet(M)$ for $A = C_*^{gm}(\mathrm{SO}(3);R)$ and $M = \widetilde{CI}(Y,P)$.

1.2 Binary Polyhedral Groups and Quivers

A central theme in this thesis are the binary polyhedral groups, that is, the finite subgroups of SU(2). The classification of these up to conjugacy is classical and is presented nicely in [Wol11]. They consist of the cyclic groups C_l, $l \geq 1$, the binary dihedral groups D_n^*, $n \geq 2$, the binary tetrahedral group T^*, the binary octahedral group O^* and the binary icosahedral group I^*. The binary polyehdral spaces are the orbit spaces $Y_\Gamma \coloneqq S^3/\Gamma$, where $\Gamma \subset \mathrm{SU}(2)$ acts in the standard way on $S^3 \subset \mathbb{C}^2$. These are closed, orientable smooth 3-manifolds.

One of the main purposes of the first paper is to calculate the equivariant instanton Floer homology associated with the trivial SU(2)-bundle over Y_Γ. The key input needed for the calculations is complete control over the set $\mathcal{C}$ of gauge equivalence classes of flat SU(2)-connections, and some geometric information concerning the low-dimensional moduli spaces of instantons $\overline{\mathcal{M}}_z(\alpha,\beta)$ over $\mathbb{R} \times Y_\Gamma$ approaching $\alpha, \beta \in \mathcal{C}$ along the ends. The gauge equivalence classes of flat SU(2)-connections over Y_Γ can be identified with the isomorphism classes of SU(2)-representations of $\pi_1(Y_\Gamma) \cong \Gamma$. The complex representation theory of the binary polyhedral groups is well-known and from this it is mostly a matter of bookkeeping to determine the set $\mathcal{C}$ in all cases.

The necessary information concerning the instanton moduli space $\overline{\mathcal{M}}(\alpha,\beta)$ is essentially contained in [Aus95]. In this paper Austin also considers calculations of EIFH groups (restricted to real coefficients). One of the key ideas is that instantons over the cylinder $\mathbb{R} \times Y_\Gamma$ can in a natural way be identified with equivariant instantons over the four sphere S^4. Moreover, the latter problem

[7] Conditional convergence is actually not a property of the spectral sequence, but rather the underlying exact couple.

can be tackled using an equivariant version of the ADHM correspondence. The classical ADHM correspondence gives a description of various instanton moduli spaces over S^4 in terms of (finite-dimensional) linear algebraic data for a number of structure groups (see [Ati79] and [DK90, Section 3.3]). In both cases the linear algebraic descriptions can be placed within the general framework of quiver varieties in the sense of [Nak94], at least for the structure groups $\mathrm{SU}(n)$.

The purpose of this section is give some background on quivers and quiver varieties and their relation to the binary polyhdedral groups and spaces. This will serve to place the second paper of this thesis in a more general context as well as explaining the relations to the first paper.

1.2.1 The McKay Correspondence

In [McK80] it was observed that the finite subgroups of SU(2) could be put in a natural correspondence with the simply laced Dynkin diagrams of type ADE, that is, A_n, D_n, E_6, E_7 or E_8, or more precisely their extended analogues $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. The construction can be explained swiftly as follows. Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup and let Q be the canonical 2-dimensional representation associated with the inclusion $\Gamma \subset \mathrm{SU}(2)$. Furthermore, let $R_0, R_1, \cdots, R_n$ be a complete set of representatives for the isomorphism classes of irreducible complex representations of Γ, where we take R_0 to be the trivial representation. We may then define an unoriented graph $\overline{\Delta}_\Gamma$ by taking $\{0, 1, 2, \cdots, n\}$ as the set of vertices and a single edge connecting i and j if and only if

$$\dim_{\mathbb{C}}(\mathrm{Hom}_\Gamma(R_i \otimes Q, R_j)) = 1.$$

This graph will then be an extended Dynkin diagram and McKay's observation was that the rule sending (the conjugacy class of) $\Gamma \subset \mathrm{SU}(2)$ to $\overline{\Delta}_\Gamma$ sets up a bijection between the binary polyehdral groups and the extended Dynkin diagrams of type $\widetilde{ADE}$. The graph Δ_Γ obtained from $\overline{\Delta}_\Gamma$ by deleting the vertex 0 recovers the corresponding Dynkin diagram of type ADE.

There are also many other incarnations of the McKay correspondence (see for instance [Cra00]). From an algebro geometric perspective one may consider the quotient singularities $\mathbb{C}^2/\Gamma$ for finite $\Gamma \subset \mathrm{SU}(2)$. These are known as the Kleinian singularities. There is then an essentially unique minimal resolution $\pi\colon \widetilde{\mathbb{C}^2/\Gamma} \to \mathbb{C}^2/\Gamma$. The exceptional set $\pi^{-1}(0)$ is a union of 2-spheres ($\mathbb{C}P^1$) whose intersection matrix is precisely the Cartan matrix associated with the Dynkin diagram Δ_Γ.

1.2.2 Hyper-Kähler Reduction

Both the ADHM description of (equivariant) instantons over S^4 and quiver varieties are expressed in terms of a hyper-Kähler quotients. We therefore give a brief introduction to this concept before we proceed. A hyper-Kähler manifold [Hit+87] is a Riemannian manifold (M, g) equipped with a quaternionic structure on its tangent bundle such that multiplication $q\cdot\colon TM \to TM$ by unit quaternions $q \in \mathbb{H}$ is orthogonal and covariantly constant with respect to the Levi-Civita connection. If we specify a unit quaternion $s \in \operatorname{Im}\mathbb{H}$, the corresponding multiplication endomorphism $S \colon TM \to TM$ defines a complex structure on M. In fact, the triple (M, g, S) is a Kähler manifold (see for instance [Voi07]) with Kähler form $\omega_S \in \Omega^2(M, \mathbb{R})$ given by $\omega_S(v, w) = g(Sv, w)$ for $v, w \in T_pM$ and $p \in M$. In particular, we have the three standard complex structures $I, J, K \colon TM \to TM$ given by multiplication by $i, j, k \in \mathbb{H}$, respectively. A hyper-Kähler manifold is typically specified by the tuple (M, g, I, J, K).

Let (M, g, I, J, K) be a hyper-Kähler manifold and suppose that G is a compact Lie group acting on M preserving all structure in sight, i.e., for each $a \in G$ and $p \in M$ the linear map $da_p \colon T_pM \to T_{ap}M$ is orthogonal and commutes with I, J and K. A hyper-Kähler moment map for the action is defined to be a smooth function $\mu = (\mu_I, \mu_J, \mu_K) \to \mathbb{R}^3 \otimes \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of $\mathfrak{g} = \operatorname{Lie}(G)$, such that

(a) $\mu(g \cdot p) = \operatorname{Ad}^*_g \mu(p)$ for all $g \in G$, $p \in M$, and

(b) for each $S \in \{I, J, K\}$ one has $d\mu_S(v)(\xi) = \omega_S(v, V^\xi)$ for all $v \in TM$ and $\xi \in \mathfrak{g}$, where V^ξ is the vector field on M induced by ξ.

In other words, the three components μ_I, μ_J and μ_K are moment maps in the symplectic sense [Can01] for the symplectic forms ω_I, ω_J and ω_K, respectively. Let $Z = \{\alpha \in \mathfrak{g}^* : \operatorname{Ad}^*_g \alpha = \alpha \ \forall\ g \in G\}$. Then for each $\zeta \in \mathbb{R}^3 \otimes Z$ the fiber $\mu^{-1}(\zeta)$ is preserved by G and one may consider the quotient $\mu^{-1}(\zeta)/G$. The passage from (M, G, μ) to $\mu^{-1}(\zeta)/G$ for $\zeta \in \mathbb{R}^3 \otimes Z$ is called hyper-Kähler reduction and the spaces $\mu^{-1}(\xi)/G$ are called hyper-Kähler quotients. Generally these spaces will be singular, but if if G acts freely on $\mu^{-1}(\xi)$, then this is a smooth submanifold and the quotient $\mu^{-1}(\xi)/G$ is a smooth manifold equipped with a natural hyper-Kähler structure (see [Hit+87] or [GN92]).

The most basic instance of this procedure occurs when M is a quaternionic representation of a compact Lie group G. In that case we may equip M with an inner product g compatible with the quaternionic structure and the action of G, and thereby regard M as a flat hyper-Kähler manifold. In this situation the unique hyper-Kähler moment map vanishing at $0 \in M$ has the simple form

$$\mu(x)(\xi) = (g(i\xi x, x), g(j\xi x, x), g(k\xi x, x)) \ \text{ for } \ x \in M,\ \xi \in \mathfrak{g}.$$

The resulting hyper-Käher quotients may be very interesting. Indeed, as we will see in the next section, this procedure produces the ALE spaces. Moreover, as mentioned earlier, the ADHM correspondence and its equivariant modification describe various instanton moduli spaces in this way. The fact that this procedure enters in the description of instantons is partially justified by the following example (see [Sal13, Remark 6.5] and [GN92] for a non-compact example).

Example 1.2.1. Let (M, g, I, J, K) be a closed 4-dimensional hyper-Kähler 4-manifold oriented such that $(v, Iv, Jv, Kv) \in T_pM$ is a positive basis for any nonzero $v \in T_pM$. Then the three Kähler forms $\omega_I, \omega_J, \omega_K$ form a global frame for the bundle Λ^+T^*M. Let $\pi \colon P \to M$ be a principal G-bundle. Then the space of connections $\mathcal{A}_P$ inherits a hyper-Kähler structure. The Riemannian metric is the L^2-metric and the three hyper-Kähler endomorphisms I', J', K' are defined on $T_A\mathcal{A}_P = \Omega^1(M, \mathfrak{g}_P)$ by $I'\alpha = -\alpha \circ I$ and similarly for J' and K'. The action of the gauge group $\mathcal{G}_P$ preserve all of this structure and $F^+ \colon \mathcal{A}_P \to \Omega^+(M, \mathfrak{g}_P)$ can be regarded as the corresponding moment map. To elaborate, given $A \in \mathcal{A}_P$ we may write $F_A^+ = \xi_I\omega_I + \xi_J\omega_J + \xi_K\omega_K$ for unique $\xi_I, \xi_J, \xi_K \in \Omega^0(M, \mathfrak{g}_P)$. The assignment $A \mapsto (\xi_I, \xi_J, \xi_K)$ are then the three components of the moment map when $\mathrm{Lie}(\mathcal{G}_P) = \Omega^0(M, \mathfrak{g}_P)$ is identified with its dual using an L^2-inner product. Consequently, the instanton moduli space $\mathcal{M}_{ASD}(P) = (F^+)^{-1}(0)/\mathcal{G}_P$ is an example of a hyper-Kähler quotient in this setting.

1.2.3 ALE Spaces and Kronheimer's Construction

In [Kro89] Kronheimer used the technique of hyper-Kähler reduction to construct the so-called ALE (asymptotically locally Euclidean) spaces. Basically, given a finite subgroup $\Gamma \subset \mathrm{SU}(2)$ he considered the complex vector space $M = \mathrm{Hom}_\Gamma(Q \otimes R, R)$, where $R = \mathbb{C}[\Gamma]$ is the regular representation and Q is the canonical representation as defined earlier. We assume that Q and R are equipped with Γ-invariant Hermitian inner products. The complex vector space M admits a quaternionic structure preserved by the natural action of the Lie group $F = U(R)^\Gamma/U(1)$, that is, the group of unitary Γ-equivariant transformations of R divided out by the scalar subgroup. There is an associated hyper-Kähler moment map

$$\mu \colon M \to \mathbb{R}^3 \otimes \mathfrak{f}$$

where $\mathfrak{f} = \mathrm{Lie}(F)$ and he studied the hyper-Kähler quotients

$$X_\xi := \mu^{-1}(\xi)/F \quad \text{for} \quad \xi \in \mathbb{R}^3 \otimes Z(\mathfrak{f}). \tag{1.10}$$

Here, $Z(\mathfrak{f})$ denotes the center of the Lie algebra.

Using the McKay correspondence Kronheimer showed that $Z(f)$ could be identified with a Cartan subalgebra $\mathfrak{h}$ of the simple Lie algebra associated with the Dynkin graph Δ_Γ. The corresponding root system Φ can therefore be regarded as a finite subset of the dual space, i.e., $\Phi \subset \mathfrak{h}^*$. For each $\theta \in \Phi$ there is a root wall

$$D_\theta := \{\zeta \in \mathfrak{h} : \theta(\zeta) = 0\}.$$

One of the main results of [Kro89] is then that if $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \otimes \mathfrak{h}$, then X_ξ is a non-singular 4-dimensional hyper-Kähler manifold provided ξ_i, $i = 1, 2, 3$, avoids all the root walls D_θ for $\theta \in \Phi$. Moreover, in that case X_ξ is diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$ and the metric is ALE. The latter condition can be explained as follows. There is a resolution map $\pi \colon X_\xi \to \mathbb{C}^2/\Gamma$ and the restriction $\pi \colon (X_\xi - \pi^{-1}(0)) \to (\mathbb{C}^2 - \{0\})/\Gamma$ is a diffeomorphism. This can be regarded as a coordinate system at infinity. The ALE condition then asserts that if we push the hyper-Kähler metric forward along the above restriction and pull back to $\mathbb{C}^2 - \{0\}$, then it approximates the Euclidean metric up to order 4 as the parameter tends to infinity.

In the second paper of this thesis we study the singular members of the family $\{X_\xi : \xi \in \mathbb{R}^3 \otimes \mathfrak{h}\}$ for the various finite subgroups $\Gamma \subset \mathrm{SU}(2)$. For the purpose of this introduction let us call them singular ALE spaces. As stated in Kronheimer's paper mentioned above,[8]the singularities in X_ξ are discrete and locally modeled on $\mathbb{C}^2/\Gamma'$ for other finite subgroups $\Gamma' \subset \mathrm{SU}(2)$. Therefore, by deleting the singularities we expected to obtain various hyper-Kähler 4-manifolds with a number of ends modeled on

$$(\mathbb{C}^2 - \{0\})/\Gamma' \cong (0,\infty) \times S^3/\Gamma'.$$

The basic motivation for this was the most degenerate case.

Example 1.2.2. By [Kro89, Corollary 3.2] it holds true that $X_0 = \mathbb{C}^2/\Gamma$, so by deleting the singularity one recovers the cylinder

$$(0,\infty) \times S^3/\Gamma \cong \mathbb{R} \times S^3/\Gamma.$$

Alternatively, we can regard these spaces as hyper-Kähler bordisms between various binary polyhedral spaces. The results of our second paper show that there is a fairly simple classification of the singularities in the spaces X_ξ in terms of the parameter $\xi \in \mathbb{R}^3 \otimes \mathfrak{h}$ provided we require $\xi_1 = 0$. Furthermore, given a pair of finite subgroups $\Gamma_1, \Gamma_2 \subset \mathrm{SU}(2)$ we show that this procedure can be used to construct a hyper-Kähler bordism between S^3/Γ_1 and S^3/Γ_2 if and only if the Dynkin graph Δ_{Γ_1} can be realized as a subgraph of the Dynkin graph Δ_{Γ_2}, or conversely. While we find the question of classifying the singularities in these spaces interesting by itself, we also regard it as an opportunity for future investigations. We will elaborate further on this in the following section.

1.2.4 Quiver Varieties and Instantons on ALE spaces

In this final part we explain our initial motivation for studying the singular ALE spaces, the basic relations between the two papers in this thesis and discuss some ideas for future work that provide additional context.

In the paper [KN90] the authors develop a generalized ADHM correspondence describing moduli spaces of instantons on unitary bundles over the ALE spaces described in the previous section. Later in [Nak94] Nakajima introduced a family of spaces he called quiver varieties and reformulated the above correspondence in the language of quivers. A quiver (Q, I, s, t) is a finite directed graph, where I is the set of vertices, Q is set of edges and $s, t\colon Q \to I$ are the source and target maps. Usually we let I and s, t be implicit and denote the quiver simply by Q. A (extended) Dynkin quiver is a quiver whose underlying unoriented graph is a (extended) Dynkin diagram.

For a fixed quiver (Q, I, s, t) Nakajima defined for each pair of dimension vectors $v, w \in \mathbb{Z}^I_{\geq 0}$ a family of quiver varieties $\mathcal{M}_\xi(v, w)$ depending on a

[8]This is [Kro89, Lemma 3.3]. As noted in the introduction of the second paper, I am not convinced that the proof given of this lemma is valid. Nevertheless, the result remains true, but we have reached the same conclusion using different tools.

parameter $\xi \in \mathbb{R}^3 \otimes \mathbb{R}^I$.[9] We will not give the construction here, but note that it is another instance of (linear) hyper-Kähler reduction. In general this space may be singular, but there is an open subset $\mathcal{M}_\xi^{\mathrm{reg}}(v,w)$ carrying the structure of a smooth hyper-Kähler manifold. Its dimension can be determined from the underlying graph Q and the two dimension vectors v, w. If Q is an extended Dynkin quiver whose underlying graph is associated with $\Gamma \subset \mathrm{SU}(2)$ under the McKay correspondence, then there is a distinguished dimension vector $\delta \in \mathbb{Z}^I$, the so-called minimal positive imaginary root. The quiver variety $\mathcal{M}_\xi(\delta, 0)$ is precisely the space X_ξ in (1.10) associated with Γ. In other words, the ALE spaces can also be described as quiver varieties over extended Dynkin graphs.

A brief and rather imprecise statement of the generalized ADHM correspondence of [KN90] can then be stated as follows. If $X_\xi = \mathcal{M}_\xi(\delta, 0)$ is non-singular, then the quiver varieties $\mathcal{M}_{-\xi}^{reg}(v,w)$ for various $v, w \in \mathbb{Z}_{\geq 0}^I$ describe instanton moduli spaces on unitary bundles over X_ξ.[10] The equivariant ADHM correspondence as laid out in [Aus95] can be regarded as a degenerate case of this. Indeed, the quiver varieties $\mathcal{M}_0^{\mathrm{reg}}(v,w)$ describe instanton moduli spaces over $\mathcal{M}_0^{\mathrm{reg}}(\delta, 0) \cong \mathbb{R} \times S^3/\Gamma$ where $\Gamma \subset \mathrm{SU}(2)$ is the finite subgroup associated with the underlying graph of Q under the McKay correspondence (see example 1.2.2). This is precisely the relation used to determine the geometric data needed for the calculations of the equivariant instanton Floer groups.

In this final paragraph we describe our initial motivation for pursuing the research contained in the second paper in relation to the first paper of this thesis. First, a bordism equipped with an appropriate bundle between a pair of rational homology spheres should induce a map between the corresponding equivariant instanton Floer groups (see [Eis19, Section 6] for the current state of affairs). These maps depend on various instanton moduli spaces on the bundle over the bordism. Our motivation for studying the singular ALE spaces was precisely for the purpose of constructing bordisms between distinct binary polyhedral spaces, in order to possibly study such induced maps in equivariant instanton Floer homology. We find the bordisms constructed in this way to be ideal candidates for this purpose due to the generalized ADHM correspondence mentioned above. We therefore ask: Can the generalized ADHM correspondence of [KN90] be extended to also apply for the singular ALE spaces? The equivariant ADHM correspondence gives a positive answer to this question in the most degenerate case, and this is precisely the input needed for the EIFH calculations of the first paper. Now, if the answer to the above question is true, it might be possible to also study induced maps in equivariant instanton Floer homology. The EIFH calculations for binary polyhedral spaces are contained in the first paper of this thesis and the construction of the appropriate bordisms is contained in the second. The above conjectural leap would therefore provide an absolute bridge between the two papers contained in this thesis.

[9] In our second paper we are only concerned with quiver varieties of the form $\mathcal{M}_\xi(v,0)$ and write this as $\mathcal{M}_\xi(Q,v)$ instead, incorporating the quiver into the notation.

[10] The ALE spaces $\mathcal{M}_\xi(\delta,0) = X_\xi$ are non-compact, so the moduli spaces depend in addition on a choice of flat structure at infinity. The two dimension vectors v, w determine this flat structure in addition to the topological type of the bundle.

Chapter 2

Summary of Papers

In the following we give summaries of the two papers contained in this thesis. Most of the background material may be found in the previous chapter.

2.1 Equivariant Instanton Floer Homology and Calculations for the Binary Polyhedral Spaces

This paper is concerned with the equivariant instanton Floer homology introduced in [Eis19] and it has two basic goals. The first is to rework some of the homological algebra needed to extract the equivariant Floer groups

$$I^+(Y,E;R), \quad I^-(Y,E;R) \quad \text{and } I^\infty(Y,E;R) \tag{2.1}$$

from the framed Floer complex $\widetilde{CI}(Y,E;R)$ associated with an (weakly admissible) SO(3)-bundle $E \to Y$ over a closed oriented 3-manifold and a PID of coefficients R. The second goal is to give explicit calculations of the groups in (2.1) for the binary polyhedral spaces equipped with the trivial bundle. Our calculations are restricted to the case where $2 \in R$ is invertible in the ring of coefficients. The universal case to have in mind is $R = \mathbb{Z}[\frac{1}{2}]$, the integers localized at 2.

Before we proceed to the results proved in this paper we need to explain the implications of the condition $\frac{1}{2} \in R$. In this situation it is shown in [Eis19, Section 7.2] that one may replace the pair $(C_*^{gm}(\mathrm{SO}(3);R), \widetilde{CI}(Y,E;R))$ by a much simpler pair $(H_*(\mathrm{SO}(3),R), DCI(Y,E;R))$. As $2 \in R$ is invertible, it follows that $H_*(\mathrm{SO}(3);R) = \Lambda_R[u]$ is an exterior algebra on a single generator u in degree 3. Furthermore, the complex $DCI(Y,E;R)$, called the Donaldson model, is essentially obtained from $\widetilde{CI}(Y,E)$ by passing to homology along the columns in (1.7), that is,

$$DCI(Y,E;R)_{s,t} = \bigoplus_{j(\alpha)\equiv s} H_t(\alpha;R) \text{ and } \mathrm{DCI}(Y,E;R)_n = \bigoplus_{s+t=n} DCI(Y,E;R)_{s,t}$$

where the direct sum runs over the set $\mathcal{C}$ of critical orbits and $j\colon \mathcal{C} \to \mathbb{Z}/8$ is the index function. If we write $\mathcal{C} = \mathcal{C}^{irr} \cup \mathcal{C}^{red} \cup \mathcal{C}^{f.red}$ for the decomposition of the critical orbits into irreducibles, reducibles and fully reducibles we have

$$H_*(\alpha,R) = \begin{cases} R \oplus R[3] & \text{if } \alpha \in \mathcal{C}^{irr} \\ R \oplus R[2] & \text{if } \alpha \in \mathcal{C}^{red} \\ R & \text{if } \alpha \in \mathcal{C}^{f.red}, \end{cases}$$

where the shift operation is for a graded module D defined by $D[p]_{p+n} = D_n$. This shows that $DCI(Y,E;R)$ is degreewise free and finitely generated over R. Moreover, the differential has a concrete description.

With these preliminaries in mind the main body of the paper is divided into three main parts:

(i) We determine the complexes $\mathrm{DCI}(\overline{S^3/\Gamma}, E; R)$ for each finite subgroup $\Gamma \subset \mathrm{SU}(2)$, where the orientation is the opposite of the standard one induced from the covering $S^3 \to S^3/\Gamma$.

(ii) We review some of the homological algebra of [Eis19, Appendix A] and then work out a modified completion procedure that we use to define the equivariant Floer groups in (2.1).

(iii) We calculate the groups I^+, I^- and I^∞ for each binary polyhedral space $\overline{S^3/\Gamma}$ and also give the calculations for the standard orientation S^3/Γ in slightly less detail.

We will go through the main results contained in each of these starting with the second part (ii).

Let R be a fixed ground ring, which is required to be a PID. We will omit the reference to R in the notation throughout. The framed Floer complex $\widetilde{CI}(Y, E)$ is a module over the DG algebra $C_*^{gm}(\mathrm{SO}(3))$ and carries the index filtration. The DG algebra $C_*^{gm}(\mathrm{SO}(3))$ carries an augmentation and is in addition degreewise free over R. To extract equivariant groups in such a situation we consider a slightly more general algebraic context. Let A be DG algebra equipped with an augmentation $\epsilon\colon A \to R$. In [Eis19, Appendix A] the author defines four functors C_A^+, C_A^-, $C_A^{+,tw}$ and C_A^∞ from the category of right A-modules to the category of left $C_A^-(R)$-modules using the bar construction. We let $\bullet \in \{+, -, (+, tw), \infty\}$ denote a generic element. The homology of $C_A^\bullet(M)$ is denoted by $H_A^\bullet(M)$. According to [Eis19, Appendix A.5], if A satisfies a Poincaré duality hypothesis of degree d, valid for $C_*^{gm}(\mathrm{SO}(3))$ with $d = 3$, there is an isomorphism $H_A^+(M) \cong H_A^{+,tw}(M)$ of degree d for each right A-module M. We give a different proof of this assertion in the special case $A = \Lambda_R[u]$. This case is sufficient for the applications to equivariant instanton Floer homology when $\frac{1}{2} \in R$.

The functor $C_A^\bullet$ is exact and preserve quasi-isomorphisms provided A is degreewise free. Moreover, if $f\colon A \to B$ is a quasi-isomorphisms of degreewise R-free DG-algebras and M is a right B-module, there is a natural isomorphism $H_A^\bullet(f^{-1}M) \cong H_B^\bullet(M)$, where $f^{-1}M$ denotes the A-module obtained by restriction along f. In [Eis19] the author also defines what he calls a completed version $\hat{C}_A^\bullet$ of $C_A^\bullet$. These completed functors are used to define the equivariant Floer groups with $A = C_*^{gm}(\mathrm{SO}(3))$ and $M = \widetilde{CI}(Y, E)$. Due to the exactness of $C_A^\bullet$ it preserves filtrations of the argument, i.e., if M is filtered by $\{F_pM\}_p$ then $C_A^\bullet(M)$ is filtered by $\{F_pC_A^\bullet(M) \coloneqq C_A^\bullet(F_pM)\}_p$. The basic motivation for introducing the completed functors is to force the resulting index spectral sequences $\{E^r, d^r\}_{r\geq 1}$ with

$$E^1_{s,t} = H_A^\bullet(F_sM/F_{s-1}M)_{s+t}$$

to converge conditionally, in the sense of [Boa99, Def. 5.10], to its natural target. The fact that the completions of [Eis19] achieve this purpose is not justified. I believe that his type of completion works in the cases $\bullet \in \{+,-\}$, but seems to fail for $\bullet \in \{(+,tw),\infty\}$. The purpose of the algebraic part of this paper is to give a different, and arguably more natural, completion procedure that fixes this possible issue. The basic idea is to first promote $C_A^\bullet$ to a functor between the corresponding filtered categories and then compose with a full completion functor. Explicitly, given a right A-module M equipped with a filtration $\{F_pM\}_{p\in\mathbb{Z}}$ we define

$$\begin{aligned} \hat{C}_A^\bullet(M) &\coloneqq \lim_q \operatorname{colim}_p C_A^\bullet(F_pM/F_qM) \quad \text{filtered by} \\ F_p\hat{C}_A^\bullet(M) &\coloneqq \lim_q C_A^\bullet(F_pM/F_qM). \end{aligned}$$

The homology of $\hat{C}_A^\bullet(M)$ is denoted by $\hat{H}_A^\bullet(M)$. This functor take values in the category of filtered left $C_A^-(R)$-modules. This approach has the advantage that the existence of a conditional convergent spectral sequence $\{E^r,d^r\}_{r\geq 1}$ with

$$E_{s,t}^1 = H_A^\bullet(F_sM/F_{s-1}M) \implies \hat{H}_A^\bullet(M) \tag{2.2}$$

is essentially an immediate consequence of the definition. Our main results concerning this construction are summarized in the following theorem. By a filtered quasi-isomorphism of filtered A-modules we mean an A-linear chain map $f\colon M\to N$ such that the induced map $\overline{f}_p\colon F_pM/F_{p-1}M \to F_pN/F_{p-1}N$ is a quasi-isomorphism for each $p\in\mathbb{Z}$.

Theorem 2.1.1. *Let* $\bullet \in \{+,-,(+,tw),\infty\}$.

(a) *If $f\colon A\to B$ is a quasi-isomorphism of degreewise R-free DG algebras, then there is a natural isomorphism $\hat{H}_A^\bullet(f^{-1}M)\cong \hat{H}_B^\bullet(M)$ for each filtered B-module M.*

(b) *If $g\colon M\to N$ is a filtered quasi-isomorphism of filtered A-modules and A is degreewise free over R, then there is an induced filtered quasi-isomorphism $\hat{g}^\bullet\colon \hat{C}_A^\bullet(M)\to \hat{C}_A^\bullet(N)$. In particular, $H(\hat{g}^\bullet)\colon \hat{H}_A^\bullet(M)\cong \hat{H}_A^\bullet(M)$.*

(c) *If $A=\Lambda_R[u]$ with $|u|=3$, then for each filtered A-module M there is a natural degree 3 isomorphism $\hat{H}_A^+(M)\cong \hat{H}_A^{+,tw}(M)$ of $H_A^-(R)\cong R[U]$-modules, where $|U|=-4$. Moreover, there is a natural long exact sequence of $R[U]$-modules*

$$\hat{H}_A^+(M)_n \longrightarrow \hat{H}_A^-(M)_{n+3} \longrightarrow \hat{H}_A^\infty(M)_{n+3} \longrightarrow \hat{H}_{n-1}^+(M)$$

We define $I^\bullet(Y,E)\coloneqq \hat{H}_A^\bullet(M)$ for $A=C^{gm}(\mathrm{SO}(3))$ and $M=\widetilde{CI}(Y,E)$ and we verify that this definition coincides with that of [Eis19] for $\bullet\in\{+,-\}$. The two parts (a) and (b) of the above theorem are precisely the input needed to show that the equivariant groups $I^\bullet(Y,E)$ can be calculated using $A=\Lambda_R[u]$

and $M = \mathrm{DCI}(Y, E)$. The final part (c) establishes the exact triangle relating $I^+(Y, E)$, $I^-(Y, E)$ and $I^\infty(Y, E)$ when $\frac{1}{2} \in R$.

In addition to this we treat the case $A = \Lambda_R[u]$ in some detail. The main novel result is the construction of a simplified version of the Tate complex $C_A^\infty(M)$. In the case $M = DCI(Y, E)$ we also extend this model to the completed version $\hat{C}_A^\infty(M)$ and this leads to a new proof of the fact that the action of $U \in R[U] = H_A^-(R)$ is an isomorphism in Tate homology; $U\colon I^\infty(Y, E) \cong I^\infty(Y, E)$. This concludes the summary of the main algebraic results proved in this paper.

We now turn to the calculational part of the paper. For each finite subgroup $\Gamma \subset \mathrm{SU}(2)$ let $Y_\Gamma := S^3/\Gamma$ equipped with the orientation making the covering map $S^3 \to S^3/\Gamma$ orientation preserving and write $\overline{Y}_\Gamma$ for the same manifold equipped with the opposite orientation. The main body of the paper focuses on the calculations for $\overline{Y}_\Gamma$, while the necessary modifications in the case of Y_Γ are laid out in the final section of the paper.

The first step is to determine the Donaldson models $DCI(\overline{Y}_\Gamma)$. The bundle in question will always be the trivial one, so we omit it from the notation. The key tools needed for this is mainly contained in [Aus95]. The result can be described as follows. For each finite subgroup $\Gamma \subset \mathrm{SU}(2)$ there is a graph $\mathcal{S}_\Gamma$, whose vertices correspond to the set $\mathcal{C}$ of critical orbits, or equivalently the set of isomorphism classes of $\mathrm{SU}(2) = \mathrm{Sp}(1)$ representations of $\pi_1(\overline{Y}_\Gamma) = \Gamma$. This graph is a tree in all cases. The indexing function $j\colon \mathcal{C} \to \mathbb{Z}/8$ is then given by $j(\alpha) \equiv 4p(\alpha) \pmod 8$, where $p(\alpha)$ is the number of edges in the minimal path connecting α to the trivial representation θ (this is the main reason for having a preference for $\overline{Y}_\Gamma$ over Y_Γ). The grading determines $DCI(\overline{Y}_\Gamma)$ as a graded $\Lambda_R[u]$-module. In particular, it implies that $DCI(\overline{Y}_\Gamma)_{s,*} = 0$ unless $4|s$. The only non-trivial component of the differential is therefore $\partial^4\colon DCI(\overline{Y}_\Gamma)_{4s,0} \to DCI(\overline{Y}_\Gamma)_{4(s-1),3}$ for $s \in \mathbb{Z}$, which in turn has components $\partial^4_{\beta\alpha}\colon H_0(\alpha) \cong R \to R \cong H_3(\beta)$ for β irreducible. The latter map is given by multiplication by an integer $n_{\beta\alpha}$, which can only be nonzero if α and β are adjacent in $\mathcal{S}_\Gamma$ and in that case there is a simple procedure to find it (up to a sign ambiguity). We attach the integers $n_{\alpha\beta}$ to the edges of the graphs $\mathcal{S}_\Gamma$, so that the resulting labeled graph $\mathcal{S}_\Gamma$ contains all the information needed to construct $\mathrm{DCI}(\overline{Y}_\Gamma)$. Our main result lay out the labeled graph $\mathcal{S}_\Gamma$.

In the final part of the paper we calculate $I^+(\overline{Y}_\Gamma)$, $I^-(\overline{Y}_\Gamma)$ and $I^\infty(\overline{Y}_\Gamma)$ for each finite subgroup $\Gamma \subset \mathrm{SU}(2)$. One of the main tools in the calculations are the index spectral sequences. This is simply the spectral sequences of (2.2). For $A = \Lambda_R[u]$, $M = \mathrm{DCI}(\overline{Y}_\Gamma)$ and $\bullet \in \{+, -, \infty\}$ it takes the form

$$E^1_{s,t} = \bigoplus_{j(\alpha) \equiv s} H_A^\bullet(\alpha)_t \implies I^\bullet(\overline{Y}_\Gamma)_{s+t},$$

where $H_A^\bullet(\alpha) := H_A^\bullet(H_*(\alpha))$ is explicitly available for all types of orbits $\alpha \in \mathcal{C}$. These are spectral sequences of $H_A^-(R) \cong R[U]$-modules and they carry a periodicity isomorphism $E^r_{s,t} \cong E^r_{s+8,t}$ for all s, t, r compatible with the differentials, the $R[U]$-module structure and the target $I^\bullet(\overline{Y}_\Gamma)$. We will not give the complete formulas for all the calculations here, but rather give two sample calculations to indicate what kind of results one should expect. We note that all the equivariant groups repeats with period 8 just as the underlying complexes $\widetilde{CI}(Y, E)$ and $\mathrm{DCI}(Y, E)$.

Example 2.1.2. (Positive EIFH) For $\Gamma = T^*$ the set of critical orbits takes the form $\mathcal{C} = \{\theta, \alpha, \lambda\}$, where θ is the trivial representation, α is irreducible and λ is reducible. The grading is given by $j(\theta) = j(\lambda) = 0$ and $j(\alpha) = 4$. We introduce a variable V_θ of degree 4, a generator g_α of degree 0 and a generator W_λ of degree 2. Set

$$M = R[V_\theta] \oplus R[W_\lambda] \oplus R\{g_\alpha\}[-4]$$

regarded as a graded $R[U]$-module, where the action is determined by $U \cdot V_\theta^i = V_\theta^{i-1}$ for $i \geq 1$, $U \cdot Z_\lambda^i = Z_\lambda^{i-2}$ for $i \geq 2$, $U \cdot g_\alpha = U \cdot Z_\lambda^1 = 0$, $U \cdot Z_\lambda^0 = 3g_\alpha$ and $U \cdot V_\theta^0 = g_\alpha$. The polynomial algebra notation is only used to effectively express the graded modules, it should not be interpreted to mean that there is any algebra structure on M or the summands.

To express the equivariant Floer group we use the following construction. Given an $R[U]$-module X we define $X^{\Pi,8} = \prod_{s\in\mathbb{Z}} X[8s]$ with the induced $R[U]$-module structure. In particular note that, $X_n^{\Pi,8} = \prod_{s\in\mathbb{Z}} X_{n+8s}$ for each $n \in \mathbb{Z}$. Then our result is that $I^+(\overline{Y}_{T^*}) = M^{\Pi,8}$. It is not difficult to determine the individual groups $I^+(\overline{Y}_{T^*})_n$ from this. For instance

$$I^+(\overline{Y}_{T^*})_0 = \left(\prod_{i\geq 0} R\{V_\theta^{2i}\}\right) \oplus \left(\prod_{i\geq 0} R\{Z_\lambda^{4i}\}\right).$$

This is typical for the positive Floer groups; the fully reducible contributes a pair of towers $\prod_{n\geq 0} R$ in degree 0 and 4, while the reducibles contribute 4 towers in degree 0, 2, 4 and 6.

Example 2.1.3. (Negative EIFH) For $\Gamma = O^*$ the set of critical orbits takes the form $\mathcal{C} = \{\theta, \alpha, \beta, \eta\}$, where θ and η are fully reducible and α and β are irreducible. The grading is given by $j(\theta) = j(\beta) = 0$ and $j(\alpha) = j(\eta) = 4$. Introduce variables U_θ and U_η of degree -4 and let $N = R[U_\theta] \oplus R[U_\eta][4]$ with $R[U]$-module structure determined by $U \cdot U_\rho^i = U_\rho^{i+1}$ for $\rho = \theta, \eta$. Let M be the free $R[U]$-submodule of N generated by U_θ^1 and U_η^1. Then $I^-(\overline{Y}_\Gamma) = N^{\oplus,8} = \bigoplus_{s\in\mathbb{Z}} N[8s]$. Explicitly,

$$I^-(\overline{Y}_{O^*})_0 = \left(\bigoplus_{i\geq 1} R\{U_\theta^{2i}\}\right) \oplus \left(\bigoplus_{i\geq 0} R\{U_\eta^{2i+1}\}\right)$$

and similarly, $I^-(\overline{Y}_{O^*})_4$ is generated by $\{U_\theta^{2i+1} : i \geq 0\} \cup \{U_\eta^{2i} : i \geq 1\}$, while $I^-(\overline{Y}_{O^*})_n$ vanishes for n not divisible by 4.

Let us briefly explain why the irreducibles α and β are not present in the above formula. The irreducibles give rise to a pair of generators h_α and h_β in the relevant index spectral sequence. The above calculation is the result of the fact that there are differentials mapping U_η^0 and U_η^0 to h_α and h_β, so these four generators are killed off and does not contribute to the homology.

2.2 Singular Quiver Varieties over Extended Dynkin Quivers

In this paper we study the singular members of a family of quiver varieties associated with extended Dynkin quivers (the singular ALE spaces). Given a quiver Q with vertex set I there is a quiver variety $\mathcal{M}_\xi(Q,v)$ for each choice of dimension vector $v \in \mathbb{Z}^I$ and parameter $\xi \in \mathbb{R}^3 \otimes \mathbb{R}^I$. Let us briefly recall the construction. Form the doubled quiver $\overline{Q}$ obtained from Q by adjoining a reverse arrow $\overline{h}\colon j \to i$ for each arrow $h\colon i \to j$ in Q. For each dimension vector $v = (v_i)_{i\in I} \in \mathbb{Z}^I_{\geq 0}$ the complex vector space

$$\operatorname{Rep}(\overline{Q}, v) \coloneqq \bigoplus_{(h\colon i\to j)\in\overline{Q}} \operatorname{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j})$$

carries a quaternionic structure preserved by the natural action of the compact Lie group $G_v \coloneqq \prod_{i\in I} U(v_i)$ and there is an associated hyper-Kähler moment map

$$\mu\colon \operatorname{Rep}(\overline{Q}, v) \to \mathbb{R}^3 \otimes \mathfrak{g}_v,$$

where $\mathfrak{g}_v \coloneqq \operatorname{Lie}(G_v)$. There is a natural map from $\mathbb{R}^I$ onto the center $Z(\mathfrak{g}_v)$ used to regard an element $\xi \in \mathbb{R}^3 \otimes \mathbb{R}^I$ as an element of $\mathbb{R}^3 \otimes Z(\mathfrak{g}_v) \subset \mathbb{R}^3 \otimes \mathfrak{g}_v$. With this in mind, for each $\xi \in \mathbb{R}^3 \otimes \mathbb{R}^I$ the quiver variety associated with Q, v and ξ is defined to be the hyper-Kähler quotient

$$\mathcal{M}_\xi(Q,v) \coloneqq \mu^{-1}(\xi)/G_v. \tag{2.3}$$

There is a decomposition $\mathcal{M}_\xi(Q,v) = \mathcal{M}^{\text{reg}}_\xi(Q,v) \cup \mathcal{M}^{\text{sing}}_\xi(Q,v)$. The regular set $\mathcal{M}^{\text{reg}}_\xi(Q,v)$ is open and carries the structure of a smooth hyper-Kähler manifold and the singular set $\mathcal{M}^{\text{sing}}_\xi(Q,v)$ is its closed complement. If the regular set is non-empty, its dimension is given by $4 - 2(v,v)_Q$ where $(\cdot,\cdot)_Q\colon \mathbb{R}^I \times \mathbb{R}^I$ is the symmetric bilinear form given by

$$(\alpha,\beta)_Q = 2\sum_{i\in I} \alpha_i\beta_i - \sum_{(h\colon i\to j)\in\overline{Q}} \alpha_i\beta_j.$$

It only depends on the unoriented graph underlying Q. The spaces defined in (2.3) are a special case of the quiver varieties defined in [Nak94].

An extended Dynkin quiver is a quiver Q whose underlying unoriented graph is an extended Dynkin diagram of type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. If Q is an extended Dynkin quiver with vertex set I, there is a distinguished dimension vector $\delta \in \mathbb{Z}^I_{\geq 0}$ called the minimal positive imaginary root. In this paper we are concerned with the singular members of the family of quiver varieties $\mathcal{M}_\xi(Q,\delta)$ for Q an extended Dynkin quiver. These spaces were first introduced and studied in [Kro89] in a slightly different form. In fact, the non-singular members are precisely the ALE spaces.

There are two main results of this paper. The first results gives a complete classification of the singularities in $\mathcal{M}_\xi(Q,\delta)$ for parameters $\xi = (\xi_1,\xi_2,\xi_3) \in$

$\mathbb{R}^3 \otimes \mathbb{R}^I$ satisfying $\xi_1 = 0$. The basic reason for this restriction is that in this case this hyper-Kähler quotient can also be described as a complex symplectic quotient in the category of complex analytic spaces, and this allows us to employ representation theoretic techniques to determine the singularities. The classification is described in terms of a root space decomposition. Briefly, by deleting a suitable vertex from Q one recovers the underlying non-extended Dynkin diagram. This diagram determines a root system $\Phi \subset \mathbb{Z}^J \subset \mathbb{R}^J$, where J is I with the one vertex deleted. If $\mathcal{M}_\xi(Q, \delta)$ is non-empty, the parameter ξ can be identified with a parameter $\tau = (0, \tau_2, \tau_3) \in \mathbb{R}^J$. Then our first main result sets up a bijection between $\mathcal{M}_\xi^{\mathrm{sing}}(Q, \delta)$ and the components in the root space decomposition

$$\Phi \cap \tau^\perp = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_r,$$

where $\tau^\perp = \{\zeta \in \mathbb{R}^J : \zeta \cdot \tau_1 = \zeta \cdot \tau_2 = 0\}$. Furthermore, we show that a neighborhood of the singularity corresponding to Φ_j is homeomorphic to a neighborhood of 0 in $\mathbb{C}^2/\Gamma_j$, where $\Gamma_j \subset \mathrm{SU}(2)$ is the finite subgroup associated with the ADE root system Φ_j under the McKay correspondence.

Our second main result uses this to construct a number of hyper-Kähler 4-manifolds with ends modeled on $(0, \infty) \times S^3/\Gamma$ for various finite subgroups $\Gamma \subset \mathrm{SU}(2)$. Without stating the full result we can mention the following corollary. Let $\Gamma_1, \Gamma_2 \subset \mathrm{SU}(2)$ be a pair of finite subgroups with corresponding (non-extended) Dynkin graphs Δ_{Γ_1} and Δ_{Γ_2}. Then, if Δ_{Γ_1} can be realized as a subgraph of Δ_{Γ_2} or conversely, there is an extended Dynkin quiver Q with vertex set I, a parameter $\xi \in \mathbb{R}^3 \otimes \mathbb{R}^I$ such that $X = \mathcal{M}_\xi^{\mathrm{reg}}(Q, \delta)$ satisfies the following properties.

(i) X is a smooth, connected hyper-Kähler 4-manifold.

(ii) There are disjoint open subsets $U_1, U_2 \subset X$ and diffeomorphisms

$$\phi_i : U_i \cong (0, \infty) \times S^3/\Gamma_i \quad \text{for} \quad i = 1, 2.$$

(iii) The complement $Y = X - (U_1 \cup U_2)$ is a compact 4-manifold with two boundary components S^3/Γ_1 and S^3/Γ_2.

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Papers

Paper I

Equivariant Instanton Floer Homology and Calculations for the Binary Polyhedral Spaces

Gard Olav Helle

I

Abstract

We calculate the equivariant instanton Floer homology of the binary polyhedral spaces with coefficients in a PID R away from characteristic 2. Along the way we modify a part of the algebraic construction needed to define the equivariant instanton Floer groups.

I.1 Introduction

Instanton Floer homology has appeared in many forms in the literature. In this paper we will be concerned with a fairly recent version: the equivariant instanton Floer homology in the sense of Miller Eismeier [Eis19]. A pair (Y, E) consisting of a closed oriented 3-manifold Y and an SO(3)-bundle $E \to Y$ is said to be weakly admissible if either Y is a rational homology sphere, $H_*(Y;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$, or the second Stiefel-Whitney class $w_2(E)$ admits no torsion lift in $H^2(Y;\mathbb{Z})$. Associated with such a pair and a commutative ring of coefficients R (required to be a principal ideal domain), there are four (relatively) $\mathbb{Z}/8$ graded groups

$$I^+(Y,E;R),\ \ I^-(Y,E;R),\ \ I^\infty(Y,E;R)\ \text{ and }\ \widetilde{I}(Y,E;R), \tag{I.1}$$

that together constitute the equivariant instanton Floer homology of the pair (Y, E). The first three groups carry module structure over $H^{-*}(\mathrm{BSO}(3);R)$ and fit into an exact triangle

$$\begin{array}{ccc} I^+(Y,E;R) & \xrightarrow{\quad [3] \quad} & I^-(Y,E;R) \\ & {\scriptstyle [-4]}\nwarrow \qquad \swarrow{\scriptstyle [0]} & \\ & I^\infty(Y,E;R) & \end{array}\ , \tag{I.2}$$

where the numbers denote the degrees of the maps, while the final group is a module over $H_*(\mathrm{SO}(3);R)$. The construction of the equivariant instanton Floer groups is naturally divided in two steps. The first step consists of applying Morse-theoretic techniques to the Chern-Simons functional, defined on a framed configuration space of connections in the bundle, to construct a type of Morse-Bott complex $\widetilde{CI}(Y,E;R)$ equipped with an action of $C_*(\mathrm{SO}(3);R)$. The second step, purely algebraic in nature, extracts the equivariant groups from this complex using the bar construction from algebraic topology.

There are two main purposes of this paper. The first is to rework some of the algebra needed to set up the equivariant Floer groups. Below we will elaborate further on why we have found this to be necessary. The second is to calculate the equivariant Floer groups associated with the trivial SO(3)-bundle over the binary polyhedral spaces, that is, the orbit spaces $Y_\Gamma := S^3/\Gamma$ for finite subgroups $\Gamma \subset \mathrm{SU}(2)$. For reasons that will become apparent later, our calculations will be restricted to the case where 2 is invertible in the ring of coefficients R. In fact, we will focus primarily on the calculations for $\overline{Y}_\Gamma$, the manifold Y_Γ equipped with the opposite orientation, and only briefly treat the calculations for Y_Γ.

To be able to explain our results we will first elaborate further on the brief explanation of the equivariant instanton Floer groups given above. To simplify the exposition we will assume that Y is a rational homology sphere and that $E \to Y$ is the trivial SO(3)-bundle, both of which are satisfied in our situation of binary polyhedral spaces. In this case one may just as well replace E by the trivial SU(2)-bundle, so we will make this assumption as well. Let $\mathcal{A}_E$ denote the space of connections in E and let $\mathcal{G}_E$ be the group of gauge transformations, that is, the group of bundle automorphisms $u\colon E \to E$. The gauge group acts by pullback on the space of connections, but this action is not free. Indeed, the stabilizer of a connection $A \in \mathcal{A}_E$ may be identified, through evaluation at a point $b \in Y$, with the centralizer of the holonomy group $\mathrm{Hol}_A(b) \subset \mathrm{Aut}(E_b) \cong \mathrm{SU}(2)$. To remedy this one may fix a base point $b \in Y$ and define the space of framed connections by

$$\widetilde{\mathcal{A}}_E := \mathcal{A}_E \times E_b.$$

Here the gauge group acts diagonally, through evaluation in the second component. This action is free and in suitable Sobolev completions (see [Eis19, Section 2.2]) the configuration space

$$\widetilde{\mathcal{B}}_E := \widetilde{\mathcal{A}}_E/\mathcal{G}_E$$

is a smooth Hilbert manifold. The right SU(2)-action on the fiber descends to a right $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm 1\}$-action on the configuration space.

The Chern-Simons functional $\mathrm{cs}\colon \widetilde{\mathcal{B}}_E \to \mathbb{R}/\mathbb{Z} \cong S^1$ may be defined for $[A,u] \in \widetilde{\mathcal{B}}_E$ represented by $(A,u) \in \widetilde{\mathcal{A}}_E$ by

$$\mathrm{cs}([A,u]) = \frac{1}{8\pi^2}\int_Y \mathrm{Tr}(a \wedge da + \frac{2}{3} a \wedge a \wedge a) \in \mathbb{R}/\mathbb{Z} \tag{I.3}$$

where $a \in \Omega^1(Y, \mathfrak{su}(2))$ is a connection form representing A in a global trivialization of the bundle (see [Don02, p. 18]). The fact that this is independent of the gauge equivalence class of A, or equivalently the choice of global trivialization of the bundle, is shown in [Fre02, Proposition 1.27]. Clearly, cs is independent of the framing coordinate and is therefore SO(3)-invariant. The equivariant framed Floer complex $\widetilde{CI}(Y,E)$ is constructed as a type of Morse-Bott complex for the functional $\mathrm{cs}\colon \widetilde{\mathcal{B}}_E \to S^1$. This means that the complex is, in a certain sense, generated by the set of critical orbits of cs and the differentials are defined via fiber products with the spaces of negative gradient flow lines between the various critical orbits. First, the set of critical points of cs regarded as a map $\mathcal{A}_E \to \mathbb{R}/\mathbb{Z}$ is precisely the set of flat connections in E, so that $\mathcal{C}$, the set of critical orbits in $\widetilde{\mathcal{B}}_E$, consists precisely of the SO(3)-orbits of the gauge equivalence classes of the flat connections. Furthermore, the negative gradient flow equation for cs is, for a path of connections $A_t \in \mathcal{A}_E$, given by, disregarding the normalizing constant,

$$\frac{dA_t}{dt} = - * F_{A_t}$$

where F_{A_t} is the curvature of the connection A_t and $* : \Omega^2(Y, \mathrm{su}(E)) \to \Omega^1(Y, \mathrm{su}(E))$ is the Hodge star operator. Now, a path of connections A_t may in a natural way be identified with a connection $\mathbf{A}$ in the bundle $\mathbb{R} \times E \to \mathbb{R} \times Y$ over the cylinder. In terms of this relation the negative gradient flow equation takes the form of the familiar anti-self-dual or instanton equation

$$F^+_{\mathbf{A}} = 0.$$

In general, as in finite dimensional Morse theory, one is typically forced to introduce a perturbation of the functional to ensure that the critical points are non-degenerate and that the spaces of flow lines are cut out transversally. A flat connection α, regarded as a critical point of cs, is non-degenerate if the twisted cohomology group $H^1_A(Y; \mathrm{su}(E))$ vanishes. Given a pair of non-degenerate flat connections α and β and a relative homotopy class $z \in \pi_1(\widetilde{\mathcal{B}}_E, \alpha, \beta)$ one may form a configuration space $\widetilde{\mathcal{B}}_{z,\mathbb{R}\times E}(\alpha, \beta)$ of connections framed at $(0, b) \in \mathbb{R} \times Y$ approaching α at $-\infty$ and β at $+\infty$, whose corresponding path in $\widetilde{\mathcal{B}}_E$ belongs to the homotopy class z, modulo gauge transformations that approach α-harmonic and β-harmonic gauge transformations at $\pm\infty$. This space also carries a residual action of SO(3) and there are equivariant end-point maps

$$\alpha \xleftarrow{e_+} \widetilde{\mathcal{B}}_{z,\mathbb{R}\times E}(\alpha, \beta) \xrightarrow{e_-} \beta \tag{I.4}$$

defined by parallel transport of the framing to $\pm\infty$ along $\mathbb{R} \times \{b\}$. The moduli space of framed instantons $\widetilde{\mathcal{M}}_z(\alpha, \beta) = \{[\mathbf{A}, u] \in \widetilde{\mathcal{B}}_{\mathbb{R}\times Y, z}(\alpha, \beta) : F^+_{\mathbf{A}} = 0\}$ is then cut out transversally provided the framed analogue of the anti-self-dual operator

$$D_{\mathbf{A}} = (d^*_{\mathbf{A}}, d^+_{\mathbf{A}}) \colon \Omega^1(\mathbb{R} \times Y, \mathfrak{su}(E)) \to \Omega^0(\mathbb{R} \times Y, \mathfrak{su}(E)) \oplus \Omega^+(\mathbb{R} \times Y, \mathfrak{su}(E))$$

is (in appropriate Sobolev completions) surjective for all $[\mathbf{A}, u] \in \widetilde{\mathcal{M}}_z(\alpha, \beta)$. For the trivial SU(2)-bundle over the binary polyhedral spaces all of the flat connections are non-degenerate and the moduli spaces are cut out transversally, so there is no need to introduce a perturbation (see [Aus90, Section 4.5]).

In [Eis19] it is shown, more generally, that there exist generic perturbations that achieve the above (equivariant) transversality conditions. We will assume that such a perturbation has been chosen, leaving it out from the notation. There is then a finite set $\mathcal{C}$ of critical orbits. Given $\alpha \in \mathcal{C}$ the possible stabilizers in SO(3) are $\{1\}$, SO(2) or SO(3), so that

$$\alpha \cong \mathrm{SO}(3), \quad \alpha \cong \mathrm{SO}(3)/\,\mathrm{SO}(2) \cong S^2 \quad \text{or} \quad \alpha \cong *. \tag{I.5}$$

We say that α is irreducible, reducible or fully reducible respectively. Given a pair of critical orbits $\alpha, \beta \in \mathcal{C}$ there is then for each homotopy class $z \in \pi_1(\widetilde{\mathcal{B}}_E, \alpha, \beta)$ a finite-dimensional moduli space $\widetilde{\mathcal{M}}_z(\alpha, \beta)$ equipped with an SO(3)-action and equivariant end-point maps. There is a natural $\mathbb{R}$-action by translations on the moduli space. Provided $\alpha \neq \beta$ this action is free and the quotient is denoted by $\widetilde{\mathcal{M}}^0_z(\alpha, \beta)$. The situation is summarized in the following diagram,

$$\begin{array}{ccccc} & & \widetilde{\mathcal{M}}_z(\alpha,\beta) & & \\ & \swarrow^{e_-} & \downarrow{\scriptstyle /\mathbb{R}} & \searrow^{e_+} & \\ \alpha & \xleftarrow[e_-]{} & \widetilde{\mathcal{M}}^0_z(\alpha,\beta) & \xrightarrow[e_+]{} & \beta, \end{array}$$

where the upper part is obtained by restriction from diagram (I.4). A relative grading, depending on the relative homotopy class z, may then be defined by

$$\mathrm{gr}_z(\alpha, \beta) = \dim \widetilde{\mathcal{M}}_z(\alpha, \beta) - \dim \alpha \in \mathbb{Z}.$$

This function satisfies a few key properties.

(i) For $\alpha, \beta, \gamma \in \mathcal{C}$ and relative homotopy classes z, w connecting α to β and β to γ respectively one has

$$\mathrm{gr}_{z*w}(\alpha, \gamma) = \mathrm{gr}_z(\alpha, \beta) + \mathrm{gr}_w(\beta, \gamma).$$

(ii) For $\alpha, \beta \in \mathcal{C}$ and relative homotopy classes z, z' connecting them one has

$$\mathrm{gr}_z(\alpha, \beta) - \mathrm{gr}_{z'}(\alpha, \beta) \in 8\mathbb{Z}.$$

Moreover, for any integer $n \equiv \mathrm{gr}_z(\alpha, \beta) \pmod 8$ there exists a homotopy class w such that $\mathrm{gr}_w(\alpha, \beta) = n$.

The relative grading $\mathrm{gr}\colon \mathcal{C} \times \mathcal{C} \to \mathbb{Z}/8$ is then defined by $\mathrm{gr}(\alpha, \beta) \equiv \mathrm{gr}_z(\alpha, \beta) \pmod 8$ for any choice of z. In our case of a trivial bundle, we may lift this to an absolute grading $j\colon \mathcal{C} \to \mathbb{Z}/8$ by taking the trivial product connection θ as a reference point, that is, $j(\alpha) := \mathrm{gr}(\alpha, \theta)$.

As a graded module, the framed Floer complex $\widetilde{CI}(Y, E; R)$ may now be defined to be the totalization of the bigraded module,

$$\widetilde{CI}(Y, E; R)_{s,t} = \bigoplus_{j(\alpha) \equiv s} C^{gm}_t(\alpha; R)$$

for all $s, t \in \mathbb{Z}$, where the direct sum is taken over all $\alpha \in \mathcal{C}$ with $j(\alpha) \equiv s$ mod 8. Here, for a smooth manifold X, $C^{gm}_*(X; R)$ is a chain complex of free

R-modules generated by (equivalence classes of) maps $\sigma\colon P \to X$ from a certain set of stratified topological spaces with quite a lot of additional structure (see [Eis19, Section 6]). Importantly, this version of geometric homology supports a notion of fiber products on the chain level, and the homology $H_*^{gm}(X;R)$ coincides with the usual singular homology of X. The differential in $\widetilde{CI}(Y,E)$ is then defined on a generator $c = [\sigma\colon P \to \alpha] \in C_t^{gm}(\alpha) \subset \widetilde{CI}(Y,E)_{s,t}$ by taking the sum of the internal differential $\partial^{gm} c \in C_{t-1}^{gm}(\alpha)$ and various fiber products with (compactified) moduli spaces

$$[P \times_\alpha \overline{\mathcal{M}}_z(\alpha,\beta) \to \beta] \in C^{gm}_{t+\mathrm{gr}_z(\alpha,\beta)-1}(\beta).$$

We will state the key results that make all of this precise in section 2.

From this point we omit the reference to the ring of coefficients in the notation. The framed Floer complex $\widetilde{CI}(Y,E)$ carries two important pieces of additional structure. First, $A := C_*^{gm}(\mathrm{SO}(3))$ is a differential graded algebra and the action on $C_*^{gm}(\alpha)$ for $\alpha \in \mathcal{C}$ gives $\widetilde{CI}(Y,E)$ the structure of a right differential graded A-module. Second, $\widetilde{CI}(Y,E)$ is filtered by index:

$$F_p\widetilde{CI}(Y,E)_n = \bigoplus_{j(\alpha)\equiv s,\ s\leq p} C^{gm}_{n-s}(\alpha)$$

for $p \in \mathbb{Z}$. This is a filtration by differential graded A-submodules and is naturally referred to as the index filtration.

When 2 is invertible in the ground ring, Miller Eismeier shows in [Eis19, Section 7] that it is possible to replace the pair $(\widetilde{CI}(Y,E), C_*^{gm}(\mathrm{SO}(3))$ with a simpler pair $(DCI(Y,E), H_*(\mathrm{SO}(3))$. The complex $DCI(Y,E)$, called the Donaldson model, is defined as a graded module to be the totalization of the bigraded module

$$DCI(Y,E)_{s,t} = \bigoplus_{j(\alpha)\equiv s} H_t(\alpha)$$

and the differential has a concrete description. Furthermore, as $1/2 \in R$, $H_*(\mathrm{SO}(3)) \cong \Lambda_R[u]$ is an exterior algebra on a single generator u in degree 3. Using the results of [Aus95], we are able to describe these complexes explicitly for all the binary polyhedral spaces. This is the key input needed for our calculations.

The fourth group, $\widetilde{I}(Y,E)$, in (I.1) is simply defined to be the homology of $\widetilde{CI}(Y,E)$. To explain the other three groups, we have to delve into some differential graded algebra. Let A be an augmented differential graded (DG) algebra. In [Eis19, Appendix A] four functors $C_A^+, C_A^{+,tw}, C_A^-, C_A^\infty$ from the category of right DG A-modules (degree 0 maps assumed) to the category of left DG $C_A^-(R)$-modules are constructed. The basic tool employed is the two sided bar construction $B(M,A,N)$ associated with the algebra A, a right A-module M and a left A-module N (see Definition I.5.1). Then

$$C_A^+(M) := B(M,A,R) \quad\text{and}\quad C_A^-(M) := \mathrm{Hom}_A(B(R,A,A),M). \tag{I.6}$$

Furthermore, $C_A^{+,tw}(M) := B(M,A,D_A)$ where D_A is a dualizing object associated with A (see Definition I.5.11). There is a natural transformation

$N\colon C_A^{+,tw} \to C_A^-$, called the norm map, and the Tate functor C_A^∞ is defined to be the mapping cone of N. It is this that eventually gives rise to the exact triangle connecting the three Floer groups in (I.2). For a right A-module M the homology of $C_A^+(M)$ is denoted by $H_A^+(M)$ and similarly for the other functors.

Provided the ground ring R is a principal ideal domain and A is degreewise free over R, these functors are all exact and preserve quasi-isomorphisms (isomorphisms upon passage to homology). Aside from the role played in the construction of C_A^∞ the functor $C_A^{+,tw}$ is mostly inessential. Indeed, provided A satisfies a Poincaré duality hypothesis of degree $d \in \mathbb{Z}$, valid in the situations of interest with $d = 3$, there is a degree d isomorphism $H_A^+(M) \cong H_A^{+,tw}(M)$ for all right A-modules M.

To define the equivariant Floer groups in (I.1), Miller Eismeier also constructs what he calls completed versions, $\hat{C}_A^\bullet$ for $\bullet \in \{+, -, (+, tw), \infty\}$, of the above functors. The equivariant Floer groups $I^\bullet(Y, E)$ are then defined by applying $\hat{C}_A^\bullet$ to $\widetilde{CI}(Y, E)$, with $A = C_*^{gm}(\mathrm{SO}(3))$, and then passing to homology. To explain these completed versions it is important to note that the bar construction $B(M, A, N)$ is obtained as the totalization of a double complex $(B_{*,*}(M, A, N), \partial', \partial'')$. It therefore carries two filtrations, one by the first simplicial degree, and one by the second internal degree. In [Eis19, Definition A.6] the completed bar construction $\hat{B}(M, A, N)$ is defined to be the completion of $B(M, A, N)$ with respect to the filtration by internal degree, and the finitely supported cobar construction $c\hat{B}(N, A, M) \subset \mathrm{Hom}_A(B(N, A, A), M)$ is defined to be the submodule of functionals $B(N, A, A) \to M$ vanishing on $B_{p,q}(N, A, A)$ for all sufficiently large p. The completed versions $\hat{C}_A^+$ and $\hat{C}_A^-$ are then defined by replacing $B(M, A, R)$ with $\hat{B}(M, A, R)$ and $\mathrm{Hom}_A(B(R, A, A), M)$ with $c\hat{B}(R, A, M)$ in (I.6), while $\hat{C}_A^{+,tw}(M) := \hat{B}(M, A, \hat{D}_A)$ where $\hat{D}_A$ is a completed version of the dualizing object. Finally, the norm map extends to a map $\hat{N} : \hat{C}_A^{+,tw}(M) \to \hat{C}_A^-(M)$ and $\hat{C}_A^\infty(M)$ is defined to be the cone of this extended map.

Let $M = \widetilde{CI}(Y, E)$ be equipped with the index filtration and let $A = C_*^{gm}(\mathrm{SO}(3)$. The intention of the above completions is to make sure that for $\bullet \in \{+, -, (+, tw), \infty\}$ the induced filtration $F_p\hat{C}_A^\bullet(M) := \hat{C}_A^\bullet(F_pM)$ is well-behaved; by which we mean exhaustive and complete Hausdorff (see [Boa99, Definition 2.1]). This is to ensure that the associated spectral sequence converges conditionally, in the sense of [Boa99], to the correct target. However, the fact that this induced filtration is well-behaved in the above sense is not justified in [Eis19]. In fact, we find this difficult to believe in the case of $\hat{C}_A^{+,tw}$ and $\hat{C}_A^\infty$. To remedy this potential issue we suggest a different approach to the completed functors. Rather than defining completed functors on the whole category of A-modules, we simply promote the existing functors $C_A^\bullet$ to functors between the corresponding filtered categories and then compose with a full completion functor to ensure that the resulting filtrations are well-behaved. To be precise, given a right DG A-module M equipped with an increasing filtration by DG A-submodules $\{F_pM\}_p$ we define

$$\hat{C}_A^\bullet(M) := \lim_q \operatorname{colim}_p C_A^\bullet(F_pM/F_qM) \quad \text{filtered by}$$

$$F_p\hat{C}^\bullet_A(M) \coloneqq \lim_{q<p} C^\bullet_A(F_pM/F_qM) \quad \text{for} \quad p \in \mathbb{Z}$$

for each $\bullet \in \{+, -, (+, tw), \infty\}$. This approach has the advantage that essentially all the relevant properties of the functors $C^\bullet_A$ extend in a straightforward way to the functors $\hat{C}^\bullet_A$.

For $A = C^{gm}_*(\mathrm{SO}(3))$ and $M = \widetilde{CI}(Y, E)$ we define filtered complexes of $C^-_A(R)$-modules

$$CI^\bullet(Y, E) = \hat{C}^\bullet_A(M) \quad \text{for} \quad \bullet \in \{+, -, (+, tw), \infty\}$$

and define $I^\bullet(Y, E)$ to be its homology. When $\frac{1}{2} \in R$ we also define $DCI^\bullet(Y, E)$ as above with $A = \Lambda_R[u]$ and $M = DCI(Y, E)$. Our main results concerning this construction are simply that all of the desired formal properties of the groups mentioned at the beginning of the section are valid. This is established in section 5. Furthermore, by the nature of our definitions we are with certainty able to establish the existence of the index spectral sequences

$$E^1_{s,t} = \bigoplus_{j(\alpha) \equiv s} H^\bullet_{\mathrm{SO}(3)}(\alpha)_t \implies I^\bullet_{s+t}(Y, E)$$

of $H^{-*}(B\,\mathrm{SO}(3))$-modules converging conditionally to the desired target. Here, $H^\bullet_{\mathrm{SO}(3)}(\alpha) \coloneqq H^\bullet_A(C^{gm}_*(\alpha))$. All of these spectral sequences carry a periodicity isomorphism $E^r_{s,t} \cong E^r_{s+8,t}$ for all $s, t \in \mathbb{Z}$, compatible with the module structure, the differentials and the target.

We should note that this modification of the completion procedure has no essential consequences for any of the results in the main body of [Eis19] as far as we can see. Moreover, we explain, see Remark I.5.46, that our construction coincides with the original one for $I^\pm(Y, E)$.

For $A = \Lambda_R[u]$ with $|u| = 3$ it holds true that $C^-_A(R) = H^-_A(R) = R[U]$, a polynomial algebra on a single generator U of degree -4. In this situation we give concrete models for $C^+_A(M)$, $C^-_A(M)$ and $C^\infty_A(M)$. In the first two cases these are easy to extract and are also essentially (up to signs) contained in [Eis19], but the chain level model for C^∞_A seems to be new. For $M = \widetilde{CI}(Y, E)$ or $M = DCI(Y, E)$ equipped with the index filtration we are able extend these models to the completed functors $\hat{C}^\pm_A(M)$ and $\hat{C}^\infty_A(M)$. In all cases these models express the relevant complex as a suitable totalization of a double complex $(D_{*,*}, \partial', \partial'')$ concentrated in the right half plane in the $+$ case, the left half-plane in the $-$ case and the whole plane in the ∞ case. Each nonzero column $(D_{s,*}, \partial'')$ is a shifted copy of M and the horizontal differential ∂' is given by the action of $u \in \Lambda_R[u]$ up to a sign. Moreover, the action of $U \in R[U]$ is easy to express in these models. This leads to a new proof of the fact that $U\colon I^\infty(Y, E) \to I^\infty(Y, E)$ is an isomorphism ($\frac{1}{2} \in R$ assumed) established in [Eis19]. We find these models very convenient to use at various places in our calculations.

We now move over to the calculational content of this paper. The finite subgroups of SU(2) are naturally divided into two infinite families consisting of cyclic groups C_m and binary dihedral groups D_n^*, $n \geq 2$, as well as three exceptional ones: the binary tetrahedral group T^*, the binary octahedral group O^* and the binary icosahedral group I^*. The quotients S^3/C_m are lens spaces and S^3/I^* is the famous Poincaré homology sphere. Calculations in these cases are contained in [Eis19, Section 7] for one of the two possible orientations.

Calculations of equivariant instanton Floer groups for binary polyhedral spaces have been considered before. Austin and Braam introduced a type of equivariant instanton Floer groups in [AB96]. Their theory, although similar to the one considered here, relies on using differential forms to express the equivariant complexes and is therefore restricted to using real coefficients. Austin calculated these groups for binary polyhedral spaces in [Aus95]. Most of the necessary background material needed to set up the Donaldson model $DCI(Y, E)$ is therefore contained in this paper. The key idea is to exploit a natural bijection between instantons over the cylinder $\mathbb{R} \times S^3/\Gamma$ and Γ-invariant instantons on Γ-equivariant SU(2)-bundles over $S^4 = (\mathbb{R} \times S^3) \cup \{\pm\infty\}$ with the suspended Γ-action, and then use an equivariant version of the well-known ADHM correspondence to determine the relevant moduli spaces. We review and expand on several of the necessary results in section 4. In particular, to state the equivariant ADHM-correspondence clearly we have included a classification of Γ-equivariant SU(2)-bundles over S^4, whose proof is given in Appendix B.

The conclusion of the above is that we are able to describe the complex $DCI(\overline{Y}_\Gamma)$ for all finite subgroups $\Gamma \subset \mathrm{SU}(2)$ explicitly. The description can be summarized as follows. Given a finite subgroup $\Gamma \subset \mathrm{SU}(2)$, we construct a graph $\mathcal{S}_\Gamma$ whose vertices correspond to the flat connections in the trivial SU(2)-bundle over $Y_\Gamma = S^3/\Gamma$. In all cases this graph is a tree. Attached to an edge connecting two vertices α and β there is a symbol encoding the action of the differential between these. If α and β are both irreducible the symbol is $(n_{\alpha\beta}|n_{\beta\alpha})$ and if only, say α, is irreducible the symbol is $n_{\alpha\beta}$, where the $n_{\alpha\beta}$ are certain integers. Then the labeled graph $\mathcal{S}_\Gamma$ contains all the information needed to set up $DCI(\overline{Y}_\Gamma)$. The grading is determined by $j(\alpha) = 4p(\alpha)$ where $p(\alpha)$ is the length of a minimal path connecting α to the trivial connection θ in $\mathcal{S}_\Gamma$. In particular, $DCI(\overline{Y}_\Gamma)_{s,t} = 0$ unless $4|s$. This has the implication that the only possibly nontrivial part of the differential is $\partial : DCI_{4s,0} \to DCI_{4(s-1),3}$. This differential has components $\partial_{\alpha\beta} : H_0(\beta) \to H_3(\alpha)$, which is nonzero precisely when α and β are adjacent in $\mathcal{S}_\Gamma$ and α is irreducible. In that case, the differential is specified by $\partial_{\alpha\beta}(b_\beta) = n_{\alpha\beta} t_\alpha$ where $b_\beta \in H_0(\beta)$ and $t_\alpha \in H_3(\alpha)$ are generators. The labeled graphs $\mathcal{S}_\Gamma$ for all the finite subgroups $\Gamma \subset \mathrm{SU}(2)$ are contained in Proposition I.4.25 and examples of the resulting Donaldson complexes are given in Example I.4.26.

In the final section of the paper we explicitly calculate $I^+(\overline{Y}_\Gamma)$, $I^-(\overline{Y}_\Gamma)$ and $I^\infty(\overline{Y}_\Gamma)$ for each finite subgroup $\Gamma \subset \mathrm{SU}(2)$ and observe that the exact triangle relating them splits into a short exact sequence. The calculations are contained in Theorem I.6.7, Theorem I.6.13, Theorem I.6.14 and Theorem I.6.16. The main tools in the calculations are the explicit complexes $DCI(\overline{Y}_\Gamma)$, the index spectral sequences of Theorem I.5.58 and the explicit models for DCI^+, DCI^- mentioned earlier. Interestingly, we obtain essentially the same results as in

[Aus95]. Indeed, the fact that $\frac{1}{2} \in R$ ensures that exactly the same type of degeneration pattern seen in Austin's calculations occur for us as well.

In the final part of Appendix B we also include a discussion of the Chern-Simons invariant of the flat connections in the trivial SU(2)-bundle over the various binary polyhedral spaces. The main result we wish to bring out is that this numerical invariant can easily be calculated with the help of a few results established in section 4. Methods for these calculations are certainly known, see for instance [KK90] or [Auc94]. However, we also show, possibly more interestingly, that this invariant can in a natural way be identified with the second Chern class of the holonomy representation associated with the flat connection.

I.1.1 Organization of Paper

We start in section 2 by stating the key theorem of [Eis19] and some additional details on the geometric homology functor C_*^{gm} needed to rigorously define the framed Floer complex $\widetilde{CI}(Y, E)$. Moreover, we also introduce the Donaldson model $DCI(Y, E)$ under a simplifying assumption valid in the case of binary polyhedral spaces. Finally, we state the precise result that ensures that $DCI(Y, E)$ also may be used to calculate the various flavours of equivariant instanton Floer homology.

In section 3 we recall the classification of finite subgroups of SU(2) and specify our conventions concerning the binary polyhedral spaces. We also recall some simple representation theory and the McKay correspondence. This is needed to efficiently arrange the set of flat connections in the trivial SU(2)-bundle over the binary polyhedral spaces. At the end of the section we define the graphs $\mathcal{S}_\Gamma$.

In section 4 we review the key results from [Aus95] that explicitly determine the low-dimensional instanton moduli spaces needed to set up the Donaldson model for the binary polyhedral spaces. Here we have, for the purpose of clarity and precision, expanded on Austin's rather terse exposition in a number of places. In the final part of the section we give an explicit description of $DCI(\overline{Y}_\Gamma)$ for each finite subgroup $\Gamma \subset \text{SU}(2)$.

In section 5 we move into the world of differential graded algebra. First, we review the construction of the functors $C_A^\bullet$; their invariance and functorial properties for $\bullet \in \{+, -, (+, tw), \infty\}$ mainly following [Eis19, Appendix A.2, A.3, A.5]. Detailed calculations are included in the case $A = \Lambda_R[u]$ for $|u| = 3$, but we note that all of these calculations easily extend to the case of $|u|$ of arbitrary odd degree. After this we explain how to extend the functors to the category of filtered modules and establish all the necessary functoriality and invariance results in detail. We then define the complexes $CI^\bullet(Y, E)$ for $\bullet \in \{+, -, (+, tw), \infty\}$ calculating the various flavors of equivariant Floer homology, and similarly for $DCI^\bullet(Y, E)$. In the final part of the section we recall some theory on spectral sequences and their convergence properties needed for our later calculations, and then go on to set up the index spectral sequences.

In the final section we calculate the various flavors of equivariant instanton Floer homology for the trivial SU(2)-bundle over the binary polyhedral spaces

$\overline{Y}_\Gamma$. In the last part of the section we explain how DCI(Y_Γ) can be recovered from DCI($\overline{Y}_\Gamma$) and then give the calculations for Y_Γ, omitting some details.

In Appendix A we have included various facts concerning the finite subgroups $\Gamma \subset \mathrm{SU}(2)$. This include explicit realizations within Sp(1), information on their complex representation theory, character tables for T^*, O^* and I^*, extended Dynkin diagrams and complete lists over the 1-dimensional quaternionic representations. This is needed to determine all the flat connections in the trivial SU(2)-bundle over $\overline{Y}_\Gamma$ and for the indexing and differentials in the complexes $DCI(\overline{Y}_\Gamma)$.

In Appendix B we have included proofs of two results used in section 4: the classification of Γ-equivariant SU(2)-bundles over S^4 and an index calculation. As a byproduct, we obtain a simple procedure for the computation of the Chern-Simons invariants for the various flat connections over the binary polyhedral spaces. We also discuss a relation between these invariants and the group cohomology of the binary polyhedral groups.

I.2 Equivariant Instanton Floer Homology

In this section we will expand on, and make more rigorous, the definition of the framed instanton Floer complex $\widetilde{CI}(Y, E)$. We will assume, as in the introduction, that Y is a rational homology sphere and that $E \to Y$ is the trivial SU(2)-bundle as this is sufficient for our purpose. For the more general case of weakly admissible bundles we refer to [Eis19]. We will assume as in the introduction that a perturbation of the Chern-Simons functional has been fixed such that every critical orbit is non-degenerate and all the relevant moduli spaces are cut out transversally. As already noted, there is no need to introduce a perturbation in the case of binary polyhedral spaces (see [Aus95, Section 4.5]). The set of critical orbits will always be denoted by $\mathcal{C}$.

I.2.1 The Framed Instanton Floer Complex

Recall that we have fixed a basepoint $b \in Y$ and that $\widetilde{\mathcal{B}}_E$ is a configuration space of framed connections in E modulo gauge equivalence. Moreover, for each pair $\alpha, \beta \in \mathcal{C}$ and a relative homotopy class $z \in \pi_1(\widetilde{\mathcal{B}}_E, \alpha, \beta)$ there is a configuration space $\widetilde{\mathcal{B}}_{z,\mathbb{R}\times E}(\alpha, \beta)$ of connections in $\mathbb{R} \times E \to \mathbb{R} \times Y$ framed at $(0, b)$ that approaches α and β respectively as t tends to $\pm\infty$, modulo gauge. There is a residual SO(3)-action and equivariant end-point maps $(e_-, e_+)\colon \widetilde{\mathcal{B}}_{z,\mathbb{R}\times E}(\alpha, \beta) \to \alpha \times \beta$. In reality, these spaces are completed with respect to certain weighted Sobolev norms (see [Eis19, Section 2.2] for the specifics). We may now state the main theorem required to set up the framed instanton Floer complex. Here we have extracted the relevant part of [Eis19, Theorem 6.8].

Theorem I.2.1. *Let $E \to Y$ be the trivial* SU(2)*-bundle over Y and let $\mathcal{C}$ be the finite set of critical orbits of the (perturbed) Chern-Simons functional.*

(a). *For each pair* $\alpha, \beta \in \mathcal{C}$ *and relative homotopy class* z *between them in* $\widetilde{\mathcal{B}}_E$ *there is a number* $\mathrm{gr}_z(\alpha, \beta) \in \mathbb{Z}$. *If* $\gamma \in \mathcal{C}$, w *is a relative homotopy class from* β *to* γ *and* $z * w$ *denotes the concatenation of paths then*

$$\mathrm{gr}_{z*w}(\alpha, \gamma) = \mathrm{gr}_z(\alpha, \beta) + \mathrm{gr}_w(\beta, \gamma).$$

Moreover, if z' *is a different homotopy class from* α *to* β *it holds true that* $\mathrm{gr}_z(\alpha, \beta) - \mathrm{gr}_{z'}(\alpha, \beta) \in 8\mathbb{Z}$. *Hence,* $\mathrm{gr}(\alpha, \beta) \in \mathbb{Z}/8$ *is well defined.*

(b). *For each pair* $\alpha, \beta \in \mathcal{C}$ *and relative homotopy class* $z \in \pi_1(\widetilde{\mathcal{B}}_E, \alpha, \beta)$ *there is a moduli space of framed instantons* $\widetilde{\mathcal{M}}_z(\alpha, \beta)$ *approaching* α *as* $t \to -\infty$ *and* β *as* $t \to \infty$. *This is a smooth right* $\mathrm{SO}(3)$*-manifold of dimension* $\mathrm{gr}_z(\alpha, \beta) + \dim(\alpha)$.

(c). *There is a smooth* $\mathbb{R}$*-action by translations and parallel transport of the framing on* $\widetilde{\mathcal{M}}_z(\alpha, \beta)$ *with smooth quotient denoted by* $\widetilde{\mathcal{M}}^0_z(\alpha, \beta)$.

(d). *There are smooth* $\mathrm{SO}(3)$*-equivariant end-point maps as in the diagram*

$$\begin{array}{ccccc} & & \widetilde{\mathcal{M}}_z(\alpha, \beta) & & \\ & \swarrow^{e_-} & \downarrow /\mathbb{R} & \searrow^{e_+} & \\ \alpha & \xleftarrow{e_-} & \widetilde{\mathcal{M}}^0_z(\alpha, \beta) & \xrightarrow{e_+} & \beta \end{array}$$

obtained by parallel transport of the framing towards $\pm\infty$. *Here we identify* $\alpha \cong E_b/\Gamma_\alpha$ *and* $\beta \cong E_b/\Gamma_\beta$, *where* $\Gamma_\alpha, \Gamma_\beta \subset \mathrm{Aut}(E_b)$ *correspond to the stabilizers of* α *and* β.

(e). *For each* $\alpha \in \mathcal{C}$ *there is a two-element set* $\Lambda(\alpha)$ *corresponding to a choice of orientation. For a pair* α, β *any choice of orientations in* $\Lambda(\alpha)$ *and* $\Lambda(\beta)$ *induces an orientation in the fiber of* $e_- : \widetilde{\mathcal{M}}^0_z(\alpha, \beta) \to \alpha$. *If either of the choices are changed the resulting fiber orientation is reversed.*

(f). *If* $\mathrm{gr}_z(\alpha, \beta) - \dim(\alpha) \leq 10$ *there is a compactification*

$$\widetilde{\mathcal{M}}^0_z(\alpha, \beta) \subset \overline{\mathcal{M}}_z(\alpha, \beta)$$

into a topological $\mathrm{SO}(3)$*-manifold with corners and smooth structure on each stratum. The endpoint maps extend over the compactification. Furthermore, the* $\mathrm{SO}(3)$ *action is free.*

(g). *Given any choice of elements from* $\Gamma(\alpha)$, $\Gamma(\beta)$ *and* $\Gamma(\gamma)$ *there is a decomposition respecting fiber orientations*

$$(-1)^{\dim \alpha} \partial \overline{\mathcal{M}}_z(\alpha, \beta) = \coprod_{w_1, w_2 \colon w_1 * w_2 = z} \overline{\mathcal{M}}_{w_1}(\alpha, \gamma) \times_\gamma \overline{\mathcal{M}}_{w_2}(\gamma, \beta).$$

The function $\mathrm{gr}\colon \mathcal{C}\times\mathcal{C}\to\mathbb{Z}/8$ is the relative grading on the set of critical orbits. Since the bundle $E\to Y$ is trivial we may define an absolute grading $j\colon \mathcal{C}\to\mathbb{Z}/8$ by setting $j(\alpha) := \mathrm{gr}(\alpha,\theta)$, where θ denotes the trivial product connection. The relative grading may be recovered from j:

$$j(\alpha)-j(\beta)=\mathrm{gr}(\alpha,\theta)-\mathrm{gr}(\beta,\theta)=\mathrm{gr}(\alpha,\beta)+\mathrm{gr}(\beta,\theta)-\mathrm{gr}(\beta,\theta)=\mathrm{gr}(\alpha,\beta).$$

To construct the framed Floer complex Miller Eismeier uses a type of geometric homology. For each smooth manifold X he constructs a functorial chain complex $C_*^{gm}(X)$ whose generators are equivalences classes of maps from a set of spaces called strong δ-chains. For the precise definition see [Eis19, Def.:6.1]. This class of spaces contains all topological manifolds with corners and smooth structure on each stratum, so in particular contains the compactified instanton moduli spaces in part (f) of the above theorem.

Proposition I.2.2. *Given a commutative ring of coefficients R there is a functor $C_*^{gm}(-;R)$ from the category of smooth manifolds to the category of homological chain complexes of R-modules. For a smooth manifold X of dimension d the following holds true.*

(i) *There is a natural isomorphism $H_*^{sing}(X;R)\cong H_*^{gm}(X;R)$.*

(ii) *$C_n^{gm}(X;R)$ is a free R-module for each $n\in\mathbb{Z}$.*

(iii) *$C_n^{gm}(X;R)=0$ for all $n<0$ and $n>d+1$.*

There is one additional property of C_*^{gm} that is needed for the definition of the framed instanton Floer complex. First, for a pair of smooth manifolds X and Y there is a cross product $C_i^{gm}(X;R)\times C_j^{gm}(Y;R)\to C_{i+j}^{gm}(X\times Y)$. This ensures that $C_*^{gm}(\mathrm{SO}(3);R)$ obtains the structure of a differential graded algebra and for any right $\mathrm{SO}(3)$-manifold X, $C_*^{gm}(X;R)$ obtains the structure of a right differential graded $C_*^{gm}(\mathrm{SO}(3);R)$-module.

The framed Floer complex $\widetilde{CI}(Y,E)$ will be defined to be the chain complex associated with a multicomplex, whose definition we first recall.

Definition I.2.3. A (homological) multicomplex is a pair $(M_{*,*},\{\partial^r\}_{r\geq 0})$ consisting of a bigraded module $M_{*,*}$ and differentials $\partial^r\colon M\to M$ of bidegree $(-r,r-1)$ for $r\geq 0$ such that

(i) $\sum_{i+j=k}\partial^i\circ\partial^j=0 : M_{s,t}\to M_{s-k,t-k+2}$ for all $k\geq 0$, $s,t\in\mathbb{Z}$, and

(ii) for each $a\in M_{s,t}$ there exists $r_0\geq 0$ such that $\partial^r(a)=0$ for all $r\geq r_0$.

The associated chain complex (M_*,∂) is defined by $M_n=\bigoplus_{s+t=n}M_{s,t}$ for all $n\in\mathbb{Z}$ and $\partial=\sum_{r\geq 0}\partial^r$.

For a commutative ring of coefficients R define the bigraded R-module $\widetilde{CI}(Y,E;R)_{*,*}$ by

$$\widetilde{CI}(Y,E;R)_{s,t}=\bigoplus_{j(\alpha)\equiv s}C_t^{gm}(\alpha;R)$$

for all $s, t \in \mathbb{Z}$. Observe that since the grading function j takes values in $\mathbb{Z}/8$ there is a periodicity isomorphism $\widetilde{CI}(Y, E; R)_{s,t} \cong \widetilde{CI}(Y, E; R)_{s+8,t}$ for all $s, t \in \mathbb{Z}$. We make a preliminary definition before we define the differentials $\{\partial^r\}_{r \geq 0}$. For each pair $\alpha, \beta \in \mathcal{C}$ fix the unique relative homotopy class z from α to β satisfying $-2 \leq \mathrm{gr}_z(\alpha, \beta) \leq 5$. Then according to part (f) of the above theorem the moduli space $\widetilde{\mathcal{M}}^0_z(\alpha, \beta)$ admits a compactification $\overline{\mathcal{M}}_z(\alpha, \beta)$ and we have a diagram

$$\alpha \xleftarrow{\;e_-\;} \overline{\mathcal{M}}_z(\alpha, \beta) \xrightarrow{\;e_+\;} \beta\ .$$

By [Eis19, Lemma 6.6] there is an induced fiber product map

$$f_{\alpha\beta} : C^{gm}_*(\alpha; R) \to C^{gm}_{*+\mathrm{gr}_z(\alpha,\beta)-1}(\beta; R)$$

defined as follows. Given a basic chain $\sigma \colon P \to \alpha$ with $\dim P = t$ representing an element $[\sigma \colon P \to \alpha] \in C^{gm}_t(\alpha; R)$ we form the fiber product $P \times_\alpha \overline{\mathcal{M}}_z(\alpha, \beta)$ as in the commutative diagram

$$\begin{array}{ccccc} P & \longleftarrow & P \times_\alpha \overline{\mathcal{M}}_z(\alpha, \beta) & & \\ \big\downarrow{\scriptstyle \sigma} & & \big\downarrow{\scriptstyle \tau} & \searrow^{e_+ \circ \tau} & \\ \alpha & \xleftarrow{\;e_-\;} & \overline{\mathcal{M}}_z(\alpha, \beta) & \xrightarrow{\;e_+\;} & \beta, \end{array}$$

and define

$$f_{\alpha\beta}([\sigma : P \to \alpha]) = [e_+ \circ \tau \colon P \times_\alpha \overline{\mathcal{M}}_z(\alpha, \beta) \to \beta] \in C^{gm}_{t+\mathrm{gr}_z(\alpha,\beta)-1}(\beta; R).$$

In case $\mathrm{gr}_z(\alpha, \beta) \leq 0$ the compactified moduli space $\overline{\mathcal{M}}_z(\alpha, \beta)$ is empty and the map $f_{\alpha\beta}$ vanishes. Otherwise, we note that the dimension formula

$$\dim \overline{\mathcal{M}}_z(\alpha, \beta) = \mathrm{gr}_z(\alpha, \beta) + \dim \alpha - 1 \tag{I.7}$$

ensures that the chain $e_+ \circ \tau \colon P \times_\alpha \overline{\mathcal{M}}_z(\alpha, \beta) \to \beta$ has dimension $\dim P + \mathrm{gr}_z(\alpha, \beta) - 1$ as asserted. Finally, we note that the maps $f_{\alpha\beta}$ commute with the action of $C^{gm}_*(\mathrm{SO}(3); R)$.

Definition I.2.4. The framed instanton Floer complex $\widetilde{CI}(Y, E; R)$ with coefficients in R is defined to be the chain complex associated with the multicomplex $(\widetilde{CI}(Y, E; R)_{*,*}, \{\partial^r\}_{r\geq 0})$ where

$$\widetilde{CI}(Y, E; R)_{s,t} = \bigoplus_{\alpha \in \mathcal{C}\,:\, j(\alpha) \equiv s} C^{gm}_t(\alpha; R)$$

and the differentials $\partial^r : \widetilde{CI}(Y, E; R)_{s,t} \to \widetilde{CI}(Y, E; R)_{s-r,t+r-1}$ for $r \geq 0$ are defined as follows:

(i) ∂^0 is the sum of the internal differentials $\partial^{gm} \colon C^{gm}_*(\alpha; R) \to C^{gm}_{*-1}(\alpha; R)$.

(ii) For $1 \le r \le 5$ the differential ∂^r is the sum of the maps $f_{\alpha\beta} : C_*^{gm}(\alpha; R) \to C^{gm}_{*+\mathrm{gr}_z(\alpha,\beta)-1}(\beta; R)$ for pairs α, β with $\mathrm{gr}_z(\alpha, \beta) = r$.

(iii) $\partial^r = 0$ for $r > 5$.

The homology of $\widetilde{CI}(Y, E; R)$ is denoted by $\widetilde{I}(Y, E)$ and is called the framed instanton Floer homology of the pair (Y, E). The $C_*^{gm}(\mathrm{SO}(3); R)$-module structure of $C_*^{gm}(\alpha; R)$ for $\alpha \in \mathcal{C}$ gives $\widetilde{CI}(Y, E; R)$ the structure of a right differential graded $C_*^{gm}(\mathrm{SO}(3); R)$-module.

The filtration by $C_*^{gm}(\mathrm{SO}(3))$-modules given degreewise by

$$F_p\widetilde{CI}(Y, E; R)_n := \bigoplus_{s \le p} \widetilde{CI}(Y, E; R)_{s,n-s}$$

is called the index filtration.

Remark I.2.5. In the above we have omitted discussing orientations. To get the complex $\widetilde{CI}(Y, E)$ in the above form one has to fix an orientation $\mathfrak{o}_\alpha \in \Lambda(\alpha)$ for each critical orbit $\alpha \in \mathcal{C}$. The verification that $\partial^2 = 0$ and that this complex has the structure of a right differential graded $C_*^{gm}(\mathrm{SO}(3))$-module are contained in [Eis19, Lemma 6.11].

From this point we will omit the ring of coefficients from the notation. The complex given in the above definition is the unrolled framed instanton Floer complex. Instead of having a $\mathbb{Z}/8$-graded complex, we have a $\mathbb{Z}$-graded complex equipped with an evident periodicity isomorphism

$$\widetilde{CI}(Y, E)_n \cong \widetilde{CI}(Y, E)_{n+8}$$

for all $n \in \mathbb{Z}$, compatible with the $C_*^{gm}(\mathrm{SO}(3))$-action and the differential. This is more convenient when we apply the algebraic machinery of section 5 to extract the equivariant Floer groups. It should be noted that the above periodicity isomorphisms interact with the index filtration in the following way

$$F_p\widetilde{CI}(Y, E)_n \cong F_{p+8}\widetilde{CI}(Y, E)_{n+8}$$

for all $p, n \in \mathbb{Z}$.

I.2.2 The Donaldson Model

In the previous definition we introduced the framed Floer complex $\widetilde{CI}(Y, E)$ as a differential graded module over the differential graded algebra $C_*^{gm}(\mathrm{SO}(3))$. Since we are still working on the chain level, the complex and action are difficult to handle. It is shown in [Eis19, Section 7.2] that provided $2 \in R$ is invertible, one may replace the pair $(C_*^{gm}(\mathrm{SO}(3)), \widetilde{CI}(Y, E))$ with a simpler pair $(H_*(\mathrm{SO}(3)), DCI(Y, E))$. The complex $DCI_*(Y, E)$ is called the Donaldson model and is associated with a multicomplex whose bigraded module is given by

$$DCI(Y, E)_{s,t} = \bigoplus_{\alpha \in \mathcal{C}\, :\, j(\alpha) \equiv s} H_t(\alpha).$$

Note that $H_*(\mathrm{SO}(3); R) = \Lambda_R[u]$ is an exterior algebra on a single generator u of degree 3 as $\frac{1}{2} \in R$. For $\alpha \in \mathcal{C}$ we have by (I.5)

$$H_*(\alpha) \cong \begin{cases} R \oplus R[3] & \alpha \text{ irreducible} \\ R \oplus R[2] & \alpha \text{ reducible} \\ R & \alpha \text{ fully reducible} \end{cases} \tag{I.8}$$

Here our convention regarding the shift operation is that if C is a graded module then $C[n]_{n+i} = C_i$ for all $i, n \in \mathbb{Z}$. The above isomorphisms are made into identifications by fixing a base point and an orientation for each orbit $\alpha \in \mathcal{C}$. The $\Lambda_R[u] = H_*(\mathrm{SO}(3))$-action is uniquely determined by the map $\cdot u : DCI_{*,*} \to DCI_{*,*+3}$, which has components $\mathrm{id}\colon R = H_0(\alpha) \to H_3(\alpha) = R$ for α irreducible and vanishes otherwise.

The complete description of the differentials in this complex is given in [Eis19, p.:175-176]. In the following we will only concern ourselves with the components relevant in the situation for binary polyhedral spaces. We therefore make the following assumption (see Lemma I.4.17)

(i) For each pair $\alpha, \beta \in \mathcal{C}$ we have $\mathrm{gr}(\alpha, \beta) \equiv 0 \pmod 4$.

This implies that $DCI(Y, E)_{s,t}$ can only be nontrivial if $4|s$ and $0 \leq t \leq 3$. The only possibly nontrivial differential is therefore ∂^4 of bidegree $(-4, 3)$, whose components are $R = H_0(\alpha) \to H_3(\beta) = R$ for $\alpha, \beta \in \mathcal{C}$ with $j(\alpha) - j(\beta) = 4$ and β irreducible.

Recall that for $\alpha, \beta \in \mathcal{C}$ we defined z to be the unique homotopy class between α and β for which $-2 \leq \mathrm{gr}_z(\alpha, \beta) \leq 5$. Combining this with the above assumption and the boundary formula in part (g) of Theorem I.2.1 we may conclude that $\mathcal{M}^0_z(\alpha, \beta) = \overline{\mathcal{M}}_z(\alpha, \beta)$ is compact without boundary. If $\mathrm{gr}_z(\alpha, \beta) = 0$ the moduli space is empty and if $\mathrm{gr}_z(\alpha, \beta) = 4$ the dimension is $\dim \alpha + 3$ (provided it is nonempty). For simplicity of notation we write $\overline{\mathcal{M}}_{\alpha,\beta} = \overline{\mathcal{M}}_z(\alpha, \beta)$ with this choice of homotopy class z implicit.

To define the differential ∂^4 we have to make a little technical detour. For each orbit $\alpha \in \mathcal{C}$ let $\widetilde{\alpha} \to \alpha$ denote its universal cover. For irreducible orbits α we have $\widetilde{\alpha} = \mathrm{SU}(2)$, while for reducible or fully reducible orbits $\widetilde{\alpha} = \alpha$. These coverings carry $\mathrm{SU}(2)$-actions such that the projections $\tilde{\alpha} \to \alpha$ are equivariant along $\mathrm{SU}(2) \to \mathrm{SO}(3)$. For each pair $\alpha, \beta \in \mathcal{C}$ define $\overline{\mathcal{M}}^{cov}_{\alpha,\beta}$ to be the pullback as in the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}^{cov}_{\alpha,\beta} & \xrightarrow{(\widetilde{e}_-, \widetilde{e}_+)} & \tilde{\alpha} \times \tilde{\beta} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{\alpha,\beta} & \xrightarrow{(e_-, e_+)} & \alpha \times \beta. \end{array}$$

Thus $\overline{\mathcal{M}}^{cov}_{\alpha,\beta}$ is a 1, 2 or 4-sheeted covering of $\overline{\mathcal{M}}_{\alpha,\beta}$ depending on whether α and/or β are irreducible. The free $\mathrm{SO}(3)$-action on $\overline{\mathcal{M}}_{\alpha,\beta}$ lifts to a free $\mathrm{SU}(2)$-action on $\overline{\mathcal{M}}^{cov}_{\alpha,\beta}$ such that $(\widetilde{e}_-, \widetilde{e}_+)$ is equivariant. Define

$$\mathcal{M}_{\alpha,\beta} := \overline{\mathcal{M}}^{cov}_{\alpha,\beta} / \mathrm{SU}(2) \quad \text{and} \quad X_{\alpha,\beta} := (\tilde{\alpha} \times \tilde{\beta}) / \mathrm{SU}(2)$$

and let $e\colon \mathcal{M}_{\alpha,\beta} \to X_{\alpha,\beta}$ be the map induced by $(\widetilde{e}_-, \widetilde{e}_+)$.

The component $\partial^4 : R = H_0(\alpha) \to H_3(\beta) = R$, for β irreducible, is then defined to be multiplication by the degree of the map

$$e : \mathcal{M}_{\alpha,\beta} \to X_{\alpha,\beta}.$$

From the dimension formula we have $\dim \mathcal{M}_{\alpha,\beta} = \dim \alpha = \dim X_{\alpha,\beta}$, so as these spaces are compact oriented manifolds this makes sense. Note that $X_{\alpha,\beta} \cong \widetilde{\alpha}$ as β is irreducible.

For later purposes it will be important to relate the degree of the map $e\colon M_{\alpha,\beta} \to X_{\alpha,\beta}$ to the degree of a different map determined by Austin in [Aus95]. Recall that if X and Y are closed, oriented manifolds of dimension n and Y is connected then the degree of a map $f\colon X \to Y$ is defined by the relation $f_*([X]) = \deg(f)[Y]$ where $f_*\colon H_n(X) \to H_n(Y)$ is the induced map and $[X]$, $[Y]$ denote the fundamental classes. If X and Y in addition carry smooth structures and f is a smooth map one may also calculate the degree as

$$\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sgn}(df_x : T_xX \to T_yY),$$

for a regular value y, where $\operatorname{sgn}(df_x\colon T_xX \to T_yY)$ is ± 1 depending on whether this map preserves or reverses orientation, respectively. The proof of the following proposition is a simple exercise in differential topology, so we leave it out.

Proposition I.2.6. *In the statements below assume that X, Y and Z are smooth, closed, oriented manifolds and that Y and Z are connected.*

(a) *Let G be a compact Lie group acting freely on the manifolds X and Y of the same dimension. Then if $f\colon X \to Y$ is a smooth G-equivariant map it holds true that*

$$\deg(f\colon X \to Y) = \deg(f/G\colon X/G \to Y/G).$$

(b) *Let $f\colon X \to Y$ be a smooth map between manifolds of equal dimension and let $p : \widetilde{Y} \to Y$ be a finite, connected covering space with the induced orientation. Let $q : \widetilde{X} = f^*\widetilde{Y} \to X$ be the pull-back covering and let $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ be the induced map. Then*

$$\deg(f\colon X \to Y) = \deg(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}).$$

(c) *Let $(f, g)\colon X \to Y \times Z$ be a smooth map and assume that $\dim X = \dim Y + \dim Z$. If $y \in Y$ is a regular value for f, so that $f^{-1}(y) \subset X$ is a closed, oriented submanifold of dimension $\dim f^{-1}(y) = \dim Z$, then*

$$\deg((f, g)\colon X \to Y \times Z) = \deg(g|_{f^{-1}(y)}\colon f^{-1}(y) \to Z).$$

Lemma I.2.7. *Let $\alpha, \beta \in \mathcal{C}$ be a pair with β irreducible and $\mathrm{gr}(\alpha, \beta) = 4$. Consider the diagram*

$$\alpha \xleftarrow{e_-} \overline{\mathcal{M}}_z(\alpha, \beta) \xrightarrow{e_+} \beta,$$

where z is the relative homotopy class with $\mathrm{gr}_z(\alpha, \beta) = 4$. Then if $j(\alpha) \equiv s \pmod 8$, the component $H_0(\alpha) \to H_3(\beta)$ of $\partial^4 : DCI_{s,0} \to DCI_{s-4,3}$ is given by multiplication by the degree of the map

$$e_+|_{e_-^{-1}(*)} : e_-^{-1}(*) \to \beta$$

where $* \in \alpha$ is a point.

Proof. Write $\overline{\mathcal{M}}_{\alpha,\beta} = \overline{\mathcal{M}}_z(\alpha, \beta)$ as earlier. By part (c) of the above proposition we have

$$\deg(e_+|_{e_-^{-1}(*)} \colon e_-^{-1}(*) \to \beta) = \deg((e_-, e_+) \colon \overline{\mathcal{M}}_{\alpha,\beta} \to \alpha \times \beta),$$

and by part (b)

$$\deg((e_-, e_+) \colon \overline{\mathcal{M}}_{\alpha,\beta} \to (\alpha \times \beta) = \deg((\widetilde{e}_-, \widetilde{e}_+) \colon \overline{\mathcal{M}}^{cov}_{\alpha,\beta} \to \widetilde{\alpha} \times \widetilde{\beta}).$$

Finally by part (a) the latter integer coincides with the degree of

$$e \colon \overline{\mathcal{M}}^{cov}_{\alpha,\beta} / \mathrm{SU}(2) = \mathcal{M}_{\alpha,\beta} \to X_{\alpha,\beta} = (\tilde{\alpha} \times \tilde{\beta}) / \mathrm{SU}(2),$$

which by definition is the component of ∂^4 from $R = H_0(\alpha) \to H_3(\beta) = R$. ∎

Definition I.2.8. Assume that $\mathrm{gr}(\alpha, \beta) \equiv 0 \mod 4$ for all $\alpha, \beta \in \mathcal{C}$. Then the Donaldson model $DCI(Y, E)$ for the Floer complex is the chain complex associated with the multicomplex

$$(DCI(Y, E)_{*,*}, \{\partial^r\}_{r \geq 0}) \text{ where } DCI(Y, E)_{s,t} = \bigoplus_{\alpha \in \mathcal{C} \,:\, j(\alpha) \equiv s} H_t(\alpha),$$

∂^4 is given as in Lemma I.2.7 and $\partial^r = 0$ for $r \neq 4$.

The complex $DCI(Y, E)$ enjoys the same periodicity as $\widetilde{CI}(Y, E)$. Furthermore, it also carries a natural filtration

$$F_p DCI(Y, E)_n := \bigoplus_{s \leq p} DCI(Y, E)_{s, n-s}$$

by $H_*(\mathrm{SO}(3)) \cong \Lambda_R[u]$-submodules. This filtration is also referred to as the index filtration.

Finally, we state the result that allows the replacement of the pair $(C_*^{gm}(\mathrm{SO}(3)), \widetilde{CI}(Y, E))$ by the pair $(\Lambda_R[u], DCI(Y, E))$.

Proposition I.2.9. *[Eis19, Corollary 7.6] Assume that $2 \in R$ is invertible. There is a quasi-isomorphism $i : \Lambda_R[u] \to C_*^{gm}(\mathrm{SO}(3); R)$ of differential graded algebras and there is a zigzag of $\Lambda_R[u]$-equivariant quasi-isomorphisms*

$$DCI(Y,E) \xrightarrow{\ f\ } X \xleftarrow{\ g\ } \widetilde{CI}(Y,E)\ .$$

Furthermore, X also carries a filtration, f and g are filtration preserving and the induced maps

$$\frac{F_p DCI(Y,E)}{F_{p-1} DCI(Y,E)} \longrightarrow \frac{F_p X}{F_{p-1} X} \longleftarrow \frac{F_p \widetilde{CI}(Y,E)}{F_{p-1} \widetilde{CI}(Y,E)}$$

are quasi-isomorphisms for all p.

Remark I.2.10. The statement in the cited result is slightly weaker. The fact that the intermediate objects carry a type of index filtration and that all the quasi-isomorphisms preserve this filtration, inducing quasi-isomorphisms on the minimal filtration quotients, is obtained by a close inspection of the quite involved proof.

I.3 Binary Polyhedral Spaces and Flat Connections

The purpose of this section is to fix our conventions concerning the binary polyhedral spaces and to recall the basic representation theory needed to effectively classify the gauge equivalence classes of flat SU(2)-connections. We also recall the McKay correspondence between the finite subgroups of SU(2) and the simply laced extended Dynkin diagrams, that is, the diagrams of type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. This surprising relation is vital for our work in the next section and is essential in the definition of the graphs $\mathcal{S}_\Gamma$, given in the end of the section, mentioned in the introduction.

To avoid making this rather basic section too long we have moved a number of facts concerning the finite subgroups of SU(2) into Appendix *A*. This includes their complex representation theory, the corresponding McKay graphs used in the construction of $\mathcal{S}_\Gamma$ and complete lists of the flat SU(2)-connections over the binary polyhedral spaces.

I.3.1 Binary Polyhedral Spaces

We briefly recall the classification of finite subgroups of SU(2). Let C_l denote the cyclic group of order l for $l \geq 1$, let D_k denote the dihedral group of order $2k$ for $k \geq 2$, and let T, O and I denote the subgroups of SO(3) that leave a regular tetrahedron, octahedron and icosahedron, respectively, in $\mathbb{R}^3$ invariant. T is called the tetrahedral group, O the octahedral group and I the icosahedral group. Define D_k^*, T^*, O^* and I^* in SU(2) by pulling back D_k, T, O and I along the standard double covering homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Every cyclic group C_l may also be realized as a subgroup of SU(2) by embedding them in a copy of $U(1) \subset \mathrm{SU}(2)$. We have thus constructed two infinite families C_l, D_k^* of subgroups in SU(2), as well as the three exceptional ones T^*, O^* and I^*. It is a classical fact that up to conjugacy these exhaust all the finite subgroups of SU(2). A proof may be found in [Wol11, p. 83].

Theorem I.3.1. *Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup. Then Γ is isomorphic to precisely one of the groups C_l for $l \geq 1$, D_k^* for $k \geq 2$, T^*, O^* or I^*. Moreover, if two finite subgroups $\Gamma, \Gamma' \subset \mathrm{SU}(2)$ are isomorphic, then they are conjugate in* SU(2).

Let $\mathbb{C}^2$ carry the canonical complex orientation and orient $S^3 \subset \mathbb{C}^2$ by the outward pointing normal first convention; that is, an ordered basis $(v_1, v_2, v_3) \in T_x S^3 \subset T_x\mathbb{C}^2 \cong \mathbb{C}^2$ is positive if and only if (x, v_1, v_2, v_3) is a positive basis for $\mathbb{C}^2$. The standard left action of SU(2) on $\mathbb{C}^2$ restricts to a transitive and free action by orientation preserving isometries on the unit sphere $S^3 \subset \mathbb{C}^2$.

Definition I.3.2. For any finite subgroup $\Gamma \subset \mathrm{SU}(2)$ let Γ act on $S^3 \subset \mathbb{C}^2$ by restricting the standard free action of SU(2). Define Y_Γ to be the quotient manifold S^3/Γ equipped with the Riemannian metric and orientation induced from the standard round metric and orientation of S^3. Define $\overline{Y}_\Gamma$ to be the same Riemannian manifold equipped with the opposite orientation. The spaces Y_Γ and $\overline{Y}_\Gamma$ are called binary polyhedral spaces.

Note that by the above theorem Y_Γ is determined up to SU(2)-equivariant isometry by the isomorphism class of Γ.

Lemma I.3.3. *For $\Gamma \subset \mathrm{SU}(2)$ we have*

$$H_i(Y_\Gamma;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0,3 \\ \Gamma^{ab} & \text{for } i = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H^i(Y_\Gamma;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0,3 \\ \Gamma^{ab} & \text{for } i = 2 \\ 0 & \text{otherwise,} \end{cases}$$

where $\Gamma^{ab} = \Gamma/[\Gamma,\Gamma]$ is the abelianization of Γ. In particular, $H_*(Y_\Gamma;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$, so the Y_Γ are rational homology 3-spheres.

Proof. As Y_Γ is a connected, closed and orientable the (co)homology is concentrated in degrees $0 \leq i \leq 3$ and $H_i(Y_\Gamma) \cong H^i(Y_\Gamma) \cong \mathbb{Z}$ for $i = 0, 3$. By construction, S^3 is a universal cover of Y_Γ so that $\pi_1(Y_\Gamma) \cong \Gamma$ and hence $H_1(Y_\Gamma) \cong \Gamma^{ab}$. Since this group is finite it follows by the universal coefficient theorem that $H^1(Y_\Gamma) = 0$. By Poincaré duality we conclude that $H_2(Y_\Gamma) = 0$ and $H^2(Y_\Gamma) \cong \Gamma^{ab}$. The final statement follows from the universal coefficient theorem as $H_1(Y_\Gamma;\mathbb{Q}) \cong \mathbb{Q} \otimes \Gamma^{ab} = 0$. ■

I.3.2 A Few Results from Representation Theory

The framed Floer complex $\widetilde{CI}(Y, E)$ associated with the trivial SU(2)-bundle $E \to Y_\Gamma$ is generated, in the sense of Definition I.2.4, by the gauge equivalence classes of the flat connections in E. By a well-known result (see for instance

[Tau11] for a proof) these gauge equivalence classes are in natural bijection with the equivalence classes of representations

$$\Gamma = \pi_1(Y_\Gamma) \to \mathrm{SU}(2)$$

or in other words, as $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$, the isomorphism classes of 1-dimensional quaternionic representations of Γ. It is convenient to express this set in terms of the complex representation theory of Γ, since in this setting we have the calculational power of character theory at our disposal. The following three results may be extracted from [BD85, p. II.6].

Lemma I.3.4. *Let G be a finite group. The set $\mathrm{Irr}(G, \mathbb{C})$ of isomorphism classes of irreducible complex representations admits a decomposition into disjoint subsets*

$$\mathrm{Irr}(G, \mathbb{C}) = \mathrm{Irr}(G, \mathbb{C})_\mathbb{R} \cup \mathrm{Irr}(G, \mathbb{C})_\mathbb{C} \cup \mathrm{Irr}(G, \mathbb{C})_\mathbb{H}.$$

The set $\mathrm{Irr}(G, \mathbb{C})_\mathbb{C}$ consists of the classes $[V]$ where $V \ncong V^*$, while $\mathrm{Irr}(G, \mathbb{C})_\mathbb{R}$ and $\mathrm{Irr}(G, \mathbb{C})_\mathbb{H}$ consists of the classes $[W]$ admitting a conjugate linear equivariant map $s : W \to W$ satisfying $s^2 = 1$ or $s^2 = -1$, respectively.

An irreducible representation V is said to be of real, complex or quaternionic type according to whether $[V] \in \mathrm{Irr}(G, \mathbb{C})_K$ for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively. There is a simple test for determining the type of an irreducible complex representation.

Lemma I.3.5. *Let V be an irreducible complex representation of the finite group G with character χ_V. Then*

$$\frac{1}{|G|} \sum_{g \in G} \chi_V(g^2) = \begin{cases} 1 & \text{if } [V] \in \mathrm{Irr}(G, \mathbb{C})_\mathbb{R} \\ 0 & \text{if } [V] \in \mathrm{Irr}(G, \mathbb{C})_\mathbb{C} \\ -1 & \text{if } [V] \in \mathrm{Irr}(G, \mathbb{C})_\mathbb{H} \end{cases}$$

Finally we recall how the set $\mathrm{Irr}(G, \mathbb{H})$ of isomorphism classes of irreducible quaternionic representations may be recovered from $\mathrm{Irr}(G, \mathbb{C})$. Let $\mathrm{Rep}_K(G)$ denote the category of finite dimensional G-representations over $K = \mathbb{C}, \mathbb{H}$. There is a restriction functor $r\colon \mathrm{Rep}_\mathbb{H}(G) \to \mathrm{Rep}_\mathbb{C}(G)$ given by pullback along the inclusion $\mathbb{C} \hookrightarrow \mathbb{H}$ and an extension functor $e\colon \mathrm{Rep}_\mathbb{C}(G) \to \mathrm{Rep}_\mathbb{H}(G)$ given by $e(V) = \mathbb{H} \otimes_\mathbb{C} V$. Then we have the following result.

Proposition I.3.6. *There is a decomposition into disjoint subsets*

$$\mathrm{Irr}(G, \mathbb{H}) = \mathrm{Irr}(G, \mathbb{H})_\mathbb{R} \cup \mathrm{Irr}(G, \mathbb{H})_\mathbb{C} \cup \mathrm{Irr}(G, \mathbb{H})_\mathbb{H}.$$

Furthermore, the following maps are bijections

$$\begin{aligned} r &: \mathrm{Irr}(G, \mathbb{H})_\mathbb{H} \to \mathrm{Irr}(G, \mathbb{C})_\mathbb{H} \\ e &: \frac{1}{2}\mathrm{Irr}(G, \mathbb{C})_\mathbb{C} \to \mathrm{Irr}(G, \mathbb{H})_\mathbb{C} \\ e &: \mathrm{Irr}(G, \mathbb{C})_\mathbb{R} \to \mathrm{Irr}(G, \mathbb{H})_\mathbb{R}, \end{aligned}$$

where $\frac{1}{2}\mathrm{Irr}(G, \mathbb{C})_\mathbb{C}$ denotes the set of unordered pairs $\{[V], [V^*]\}$ for $[V] \in \mathrm{Irr}(G, \mathbb{C})_\mathbb{C}$.

Let $\mathrm{Irr}^n(G,\mathbb{C})_K$ denote the set of irreducible G-representations of dimension n of type $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Corollary I.3.7. *The set of isomorphism classes of* 1*-dimensional quaternionic representations of the finite group* G *are in bijection with*

$$\mathrm{Irr}^2(G,\mathbb{C})_{\mathbb{H}} \cup \frac{1}{2}\,\mathrm{Irr}^1(G,\mathbb{C})_{\mathbb{C}} \cup \mathrm{Irr}^1(G,\mathbb{C})_{\mathbb{R}}.$$

In particular, if $G = \Gamma$ *is a finite subgroup of* $\mathrm{SU}(2)$ *then the above set is in bijection with the set of gauge equivalence classes of flat connections in the trivial* $\mathrm{SU}(2)$*-bundle over* Y_Γ*.*

Note that a flat connection is irreducible, reducible or fully reducible if and only if the associated representation is of quaternionic, complex or real type, respectively.

I.3.3 Arrangement of the Flat Connections

The McKay correspondence [McK80] sets up a bijection between the isomorphism classes of the nontrivial finite subgroups of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ and the simply-laced extended Dynkin diagrams. The extended Dynkin graph is constructed from the group Γ as follows. Let $R_0, R_1, \cdots, R_n$ be a complete set of representatives for the elements of $\mathrm{Irr}(\Gamma,\mathbb{C})$, where we take R_0 to be the trivial representation. Let Q denote the two dimensional representation associated with the inclusion $\Gamma \subset \mathrm{SU}(2)$ and define a matrix $A = (a_{ij})_{ij}$ by

$$Q \otimes R_i = \bigoplus_{j=0}^{n} a_{ij} R_j \quad \text{for} \quad 0 \le i \le n.$$

One may show that A is symmetric with $a_{ii} = 0$ and $a_{ij} \in \{0,1\}$. Define an unoriented graph $\overline{\Delta}_\Gamma$ by taking $I = \{0, 1, \cdots, n\}$ as the set of vertices and an edge connecting i to j if and only if $a_{ij} = a_{ji} = 1$. This graph will then be an extended Dynkin diagram of type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. Furthermore, if we let Δ_Γ be the graph obtained from $\overline{\Delta}_\Gamma$ by deleting the vertex 0 corresponding to the trivial representation we obtain the underlying Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8. The precise correspondence is given in the following table where $l, k \ge 2$.

Γ	C_l	D_k^*	T^*	O^*	I^*
Δ_Γ	A_{l-1}	D_{k+2}	E_6	E_7	E_8

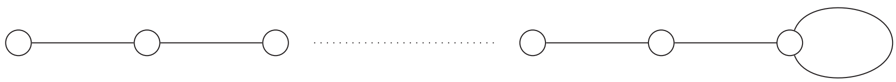
The set of vertices, say $\{1, 2, \cdots, n\}$, in a Dynkin diagram corresponds to a set of simple roots $\{r_i : 1 \le i \le n\}$ in the associated root system. There is then a unique maximal positive root $r_{\max} = \sum_{i=1}^n d_i r_i$ where $d_i \in \mathbb{N}$ are positive integers. The final fact we wish to bring out is that these integers are determined by $d_i = \dim_{\mathbb{C}} R_i$.

By mapping an irreducible representation R_i to its dual representation R_i^* we obtain an involution $\iota\colon \{0,1,\cdots,n\} \to \{0,1,\cdots,n\}$, i.e., $R_{\iota(i)} \cong R_i^*$. This map extends to a graph involution of $\overline{\Delta}_\Gamma$ as

$$a_{\iota(i)\iota(j)} = \dim_{\mathbb{C}} \mathrm{Hom}_\Gamma(Q \otimes R_i^*, R_j^*) = \dim_{\mathbb{C}} \mathrm{Hom}_\Gamma(Q \otimes R_j, R_i) = a_{ji} = a_{ij}$$

for all $0 \leq i, j \leq n$. In view of Lemma I.3.4 we see that the fixed points of ι correspond to the vertices of quaternionic or real type, while the nontrivial orbits are pairs $\{i,j\}$ for which $i \neq j$ and R_i, $R_j \cong R_i^*$ are of complex type. The set of vertices in the quotient graph $\overline{\Delta}_\Gamma/(\iota)$ may therefore by Proposition I.3.6 be identified with $\mathrm{Irr}(\Gamma, \mathbb{H})$. In particular, we may identify our set $\mathcal{C}$ of flat connections with a subset of the vertices in this quotient graph. The following may be observed from the quotient graphs given in Appendix A.

Lemma I.3.8. *For each finite subgroup* $\Gamma \subset \mathrm{SU}(2)$ *the quotient graph* $\overline{\Delta}_\Gamma/(\iota)$ *is a connected tree, except for* $\Gamma = C_l$ *with* l *odd in which case it has the form*



From the above lemma it follows that for each pair of vertices there is a unique minimal edge path connecting them.

Definition I.3.9. [Aus95, p. 297] Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup. Define a graph $\mathcal{S}_\Gamma$ by letting the vertices be the set of 1-dimensional quaternionic representations of Γ and an (unoriented) edge connecting a pair of distinct vertices α and β if and only if the minimal edge path connecting α to β in $\overline{\Delta}_\Gamma/(\iota)$ does not pass through a vertex corresponding to a 1-dimensional quaternionic representation different from α and β.

Two vertices in $\mathcal{S}_\Gamma$ are said to be adjacent if there is an edge connecting them.

At the end of the next section we will attach symbols to the edges of the graph $\mathcal{S}_\Gamma$ such that the resulting labeled graph contains all the necessary information needed to set up the complex $DCI(\overline{Y}_\Gamma)$. All of the graphs $\mathcal{S}_\Gamma$ will be given in Proposition I.4.25.

I.4 Determination of Moduli Spaces and Differentials

In this section we review the key results that allow us to determine the Donaldson models $DCI(\overline{Y}_\Gamma)$, for the finite subgroups $\Gamma \subset \mathrm{SU}(2)$, explicitly. The main result of [Aus95] necessary for the calculations, originally due to Kronheimer, is the following.

Theorem I.4.1. *Let* $\Gamma \subset \mathrm{SU}(2)$ *be a finite subgroup and let* α, β *be flat connections in the trivial* $\mathrm{SU}(2)$*-bundle over* $\overline{Y}_\Gamma$*. Let* z *be the unique homotopy class from*

α to β with $-2 \leq \mathrm{gr}_z(\alpha,\beta) \leq 5$. Then if α and β are adjacent in $\mathcal{S}_\Gamma$ (see Def. I.3.9) then in the diagram

$$\alpha \xleftarrow{e_-} \overline{\mathcal{M}}_z(\alpha,\beta) \xrightarrow{e_+} \beta$$

there is an identification $e_-^{-1}(*) \cong Y_{\Gamma'}$ for some other finite subgroup $\Gamma' \subset \mathrm{SU}(2)$. If α and β are not adjacent in $\mathcal{S}_\Gamma$, then $\overline{\mathcal{M}}_z(\alpha,\beta)$ is empty.

There is also fairly simple graphical procedure to determine the subgroup Γ' from the positions of α and β in the extended Dynkin diagram $\overline{\Delta}_\Gamma$ associated with Γ under the McKay correspondence (see Proposition I.4.12).

The main idea of the proof is to translate the problem of determining instantons over $\mathbb{R}\times Y_\Gamma$ with L^2 curvature modulo gauge transformations into the problem of determining Γ-invariant instantons in Γ-equivariant SU(2)-bundles over S^4 modulo Γ-invariant gauge transformations, and then to solve the latter problem with the use of the equivariant ADHM correspondence.

We begin by setting up the first correspondence. Recall that the instantons on the trivial SU(2)-bundle over the cylinder $\mathbb{R}\times Y_\Gamma$ with L^2-curvature are naturally partitioned into subsets by three pieces of data: the limiting flat connections $\alpha_\pm \in \mathcal{C} \cong \mathrm{Irr}^1(\Gamma,\mathbb{H})$ and the relative homotopy class z traced out by the connection. For our purpose it is more convenient to replace this latter invariant by the relative Chern number (see [Don02, p. 48])

$$\hat{c}_2(A) := \int_{\mathbb{R}\times Y_\Gamma} \mathrm{tr}\, F_A^2 = \int_{\mathbb{R}\times Y_\Gamma} |F_A|^2 \in \mathbb{R}.$$

Thus to any instanton A with L^2-curvature over $\mathbb{R}\times Y_\Gamma$ we may associate a unique triple

$$(\alpha_+,\alpha_-,\hat{c}_2(A)) \in \mathrm{Irr}^1(\Gamma,\mathbb{H}) \times \mathrm{Irr}^1(\Gamma,\mathbb{H}) \times \mathbb{R}.$$

Let Γ act on $\mathbb{R}\times S^3$ by $\gamma\cdot(t,x)\mapsto(t,\gamma x)$ such that $q=(1\times p)\colon \mathbb{R}\times S^3 \to \mathbb{R}\times Y_\Gamma$ is an orientation preserving covering map. Regard $S^4 \subset \mathbb{C}^2\oplus\mathbb{R}$ such that the suspended action $\Gamma\times S^4 \to S^4$ takes the form $\gamma\cdot(x,t) = (\gamma x, t)$. By composing the conformal equivalence $\mathbb{R}\times S^3 \cong \mathbb{C}^2 - \{0\}$, $(t,x)\mapsto e^t x$, with the inverse of stereographic projection from the north pole we obtain an equivariant conformal equivalence $\tau\colon \mathbb{R}\times S^3 \cong S^4 - \{N,S\}$, where $N,S = (0,\pm 1)\in S^4$ are the poles. Explicitly,

$$\tau(t,x) = (\cosh(t)x, \tanh(t)).$$

Let S^4 carry the orientation that makes this map orientation preserving, that is, the opposite of the standard orientation.

Recall that a Γ-equivariant vector-bundle over a Γ-space X is a vector bundle $E\to X$ with a Γ-action on the total space making the projection equivariant and such that $\gamma: E_x \to E_{\gamma x}$ is linear for each $x\in X$ and $\gamma\in\Gamma$. If E carries an SU(2)-structure, that is, a Hermitian metric and a fixed unitary trivialization of $\Lambda^2_{\mathbb{C}}E$, then we call E a Γ-equivariant SU(2)-bundle provided the action preserves this additional structure. In that case, there is an induced action of Γ on the space of connections $\mathcal{A}_E$. A connection is said to be invariant if it is fixed under this action. The subspace of invariant connections is denoted by $\mathcal{A}_E^\Gamma$. The action of the full gauge group does not preserve this subspace, but the subgroup of equivariant gauge transformations $\mathcal{G}_E^\Gamma \subset \mathcal{G}_E$ does. Therefore, the natural configuration space in this equivariant setting is $\mathcal{B}_{E,\Gamma} := \mathcal{A}_E^\Gamma / \mathcal{G}_E^\Gamma$. We may also include a framing coordinate to obtain the framed configuration space $\widetilde{\mathcal{B}}_{E,\Gamma}$. All of the above may, of course, be spelled out in terms of principal bundles as well.

Let $F \to \mathbb{R} \times Y_\Gamma$ be the trivial $\mathrm{SU}(2)$-bundle. Then $E := q^*F \to \mathbb{R} \times S^3$ obtains the structure of a Γ-equivariant vector bundle in a natural way. Moreover, since q is a finite covering map, pullback induces a bijection $\mathcal{A}_F \cong \mathcal{A}_E^\Gamma$ that matches the subsets of anti-self-dual connections with curvature in L^2. Let B be an ASD connection in F with curvature in L^2 and let $A = q^*(B)$ be the pullback connection in E. According to Uhlenbeck's removable singularities theorem there exists a bundle $\widetilde{E} \to S^4$ with connection $\widetilde{A}$ and a bundle map $\rho : E \to \widetilde{E}$ covering $\tau \colon \mathbb{R} \times S^3 \to S^4$ such that $\rho^*(\widetilde{A}) = A$. The following diagram summarizes the situation

$$\begin{array}{ccccc} (F,B) & \longleftarrow & (E,A) & \xrightarrow{\rho} & (\widetilde{E},\widetilde{A}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}\times Y_\Gamma & \xleftarrow{q} & \mathbb{R}\times S^3 & \xrightarrow{\tau} & S^4. \end{array} \tag{I.9}$$

According to Austin [Aus95] the Γ-action extends over $\widetilde{E}$ making $\widetilde{A}$ invariant and the flat limiting connections of the connection B are encoded in the isomorphism classes of the representations $\widetilde{E}_N$ and $\widetilde{E}_S$ over the fixed points. Furthermore, the following calculation shows that $c_2(\widetilde{E})$ is uniquely determined by $\hat{c}_2(B)$

$$c_2(\widetilde{E})[S^4] = \int_{S^4} \mathrm{tr}(F_{\widetilde{A}}^2) = \int_{S^4} |F_{\widetilde{A}}|^2 = \int_{\mathbb{R}\times S^3} |F_A|^2 \tag{I.10}$$

$$= |\Gamma| \int_{\mathbb{R}\times Y_\Gamma} |F_B|^2 = |\Gamma|\hat{c}_2(B). \tag{I.11}$$

Here we have used the fact that the Chern-Weil integrand $\mathrm{tr}(F_{\widetilde{A}}^2)$ coincides pointwise with the norm $|F_{\widetilde{A}}|^2$ for ASD connections, the conformal invariance of the integral of 2-forms in dimension four and that $\mathbb{R} \times S^3 \to \mathbb{R} \times Y_\Gamma$ is a finite Riemannian covering map with fibers of cardinality $|\Gamma|$.

The following theorem classifies the Γ-equivariant $\mathrm{SU}(2)$-bundles over S^4. The proof is given in Appendix B (see Theorem I.B.6).

Theorem I.4.2. *A Γ-equivariant $\mathrm{SU}(2)$-bundle $\widetilde{E} \to S^4$ is determined up to isomorphism by the ordered triple $([\widetilde{E}_N], [\widetilde{E}_S], k)$, where $k = c_2(\tilde{E})[S^4]$ is the second Chern number of the underlying $\mathrm{SU}(2)$-bundle and $[\widetilde{E}_N], [\widetilde{E}_S]$ denote the isomorphism classes of the Γ-representations over the fixed points. Moreover, for each pair $\alpha, \beta \in \mathrm{Irr}^1(\Gamma, \mathbb{H})$ there is a constant $c \in \mathbb{Z}$ such that for each*

$k \equiv c \pmod{|\Gamma|}$ *there exists a* Γ*-equivariant* $\mathrm{SU}(2)$*-bundle with invariants* (α, β, k).

Let $\widetilde{E} \to S^4$ be a Γ-equivariant $\mathrm{SU}(2)$-bundle with invariants (α_+, α_-, k). Let $E = \tau^*(E) \to \mathbb{R} \times S^3$ and $F = E/\Gamma \to \mathbb{R} \times Y_\Gamma$. Then we may reverse the above procedure to produce a map from the invariant instantons in $\widetilde{E}$ to the instantons over $\mathbb{R} \times Y_\Gamma$ with limiting flat connection $\alpha_\pm$ and relative Chern number $k/|\Gamma|$. More is true, the Γ-equivariant gauge transformations in $\widetilde{E}$ correspond to gauge transformations in F that approach $\alpha_\pm$-harmonic gauge transformations on the ends. Let $b_0 \in Y_\Gamma$ denote a basepoint and let $\tilde{b} \in S^3$ be a lift. Let $\widetilde{\mathcal{M}}_{\Gamma,\infty}(\widetilde{E}) \subset \widetilde{\mathcal{B}}_{\widetilde{E},\Gamma}$ be the moduli space of invariant instantons in $\widetilde{E}$ framed at $(0, \widetilde{b})$. Then the above discussion yields a bijective correspondence

$$\widetilde{\mathcal{M}}_\Gamma(\widetilde{E}) \cong \widetilde{\mathcal{M}}_z(\alpha_-, \alpha_+),$$

where z is the relative homotopy class corresponding to the relative Chern number $k/|\Gamma|$ and the latter space is framed at $(0, b) \in \mathbb{R} \times Y_\Gamma$. Recall that the end-point maps $e_\pm : \widetilde{\mathcal{M}}_z(\alpha_-, \alpha_+) \to \alpha_\pm$ was defined by parallel transport of the framing to $\pm\infty$. Under the above bijection this corresponds to parallel transport of the framing along the great circle through the basepoint $(\tilde{b}, 0) \in S^4$ and the poles $N, S = (0, \pm 1)$. The equivariant ADHM correspondence, which we turn to next, describes the instanton moduli space with fixed framing at N and we then see that this corresponds to the fiber $e_+^{-1}(*)$. If we change the orientation in Y_Γ we may precompose with the map $(t, x) \mapsto (-t, x)$ in $\mathbb{R} \times Y_\Gamma$ to see that the ADHM correspondence will describe the fiber $e_-^{-1}(*)$ instead.

I.4.1 The ADHM Description of Instantons

We first recall the non-equivariant ADHM classification of $\mathrm{SU}(2)$-instantons over S^4 [DK90, Sec. 3.3]. This construction describes the various moduli spaces of instantons over S^4 in terms of linear algebraic data. In the following we regard $\mathbb{C}^2 \cup \{\infty\} = S^4$ where the orientation is determined by the complex orientation on $\mathbb{C}^2$. Recall that an $\mathrm{SU}(2)$-bundle over S^4, or any other closed, oriented 4-manifold, is determined up to isomorphim by the second Chern number $k = c_2(E)[S^4] \in \mathbb{Z}$. The bundle can only support an ASD connection if $k \geq 0$ and in case $k = 0$ every ASD connection is flat.

For a fixed integer $k \geq 1$ let $\mathscr{H}$ be a Hermitian vector space of dimension k and let E_∞ be a 2-dimensional complex vector space with symmetry group $\mathrm{SU}(2)$. Define the complex vector space

$$M := \mathrm{Hom}(\mathbb{C}^2 \otimes \mathscr{H}, \mathscr{H}) \oplus \mathrm{Hom}(\mathscr{H}, E_\infty) \oplus \mathrm{Hom}(E_\infty, \mathscr{H}).$$

By identifying $\mathbb{C}^2 \otimes \mathscr{H} = \mathscr{H} \oplus \mathscr{H}$ we may write an element $m \in M$ in terms of its components $m = (\tau_1, \tau_2, \pi, \sigma)$ where $\tau_1, \tau_2 \in \mathrm{End}(\mathscr{H})$, $\pi \colon \mathscr{H} \to E_\infty$ and $\sigma \colon E_\infty \to \mathscr{H}$. Let $U(\mathscr{H})$, the group of unitary automorphisms of $\mathscr{H}$, act on M by

$$g \cdot (\tau_1, \tau_2, \pi, \sigma) = (g\tau_1 g^{-1}, g\tau_2 g^{-1}, \pi g^{-1}, g\sigma).$$

Set $\mathfrak{u}(\mathscr{H}) := \mathrm{Lie}(U(\mathscr{H}))$ and define $\mu = (\mu_{\mathbb{C}}, \mu_{\mathbb{R}}) : M \to \mathrm{End}(\mathscr{H}) \oplus \mathfrak{u}(\mathscr{H})$ by

$$\begin{aligned} \mu_{\mathbb{C}}(\tau_1, \tau_2, \pi, \sigma) =& [\tau_1, \tau_2] + \sigma\pi \\ \mu_{\mathbb{R}}(\tau_1, \tau_2, \pi, \sigma) =& [\tau_1, \tau_1^*] + [\tau_2, \tau_2^*] + \sigma\sigma^* - \pi^*\pi \end{aligned} \tag{I.12}$$

These maps are equivariant when $U(\mathscr{H})$ acts by conjugation on $\mathrm{End}(\mathscr{H})$ and $\mathfrak{u}(\mathscr{H})$.

Given $(\tau_1, \tau_2, \pi, \sigma) \in M$ we may for each $z = (z_1, z_2) \in \mathbb{C}^2$ form the sequence

$$\mathscr{H} \xrightarrow{A_z} \mathbb{C}^2 \otimes \mathscr{H} \oplus E_\infty \xrightarrow{B_z} \mathscr{H} \tag{I.13}$$

where

$$A_z = \begin{pmatrix} \tau_1 - z_1 \\ \tau_2 - z_2 \\ \pi \end{pmatrix} \quad \text{and} \quad B_z = \begin{pmatrix} -(\tau_2 - z_2) & (\tau_1 - z_1) & \sigma \end{pmatrix}.$$

The condition $\mu_{\mathbb{C}}(\tau_1, \tau_2, \pi, \sigma) = 0$ is equivalent to $B_z \circ A_z = 0$ for all $z \in \mathbb{C}^2$. Define $U_{reg} \subset M$ to be the open set consisting of the $(\tau_1, \tau_2, \pi, \sigma)$ for which A_z is injective and B_z is surjective for all $z \in \mathbb{C}^2$. From the sequence (I.13) we may form a bundle homomorphism R between the trivial bundles over $\mathbb{C}^2$ with fibers $\mathbb{C}^2 \otimes \mathscr{H}$ and $\mathbb{C}^2 \otimes \mathscr{H} \oplus E_\infty$, respectively, by $R_z = (A_z, B_z^*)$. This map extends naturally to a bundle homomorphism

$$\begin{array}{ccc} \gamma \oplus \gamma \oplus \cdots \oplus \gamma & \xrightarrow{\quad R \quad} & \underline{\mathbb{C}^2 \otimes \mathscr{H} \oplus E_\infty} \\ & \searrow \quad \swarrow & \\ & \mathbb{C}^2 \cup \{\infty\} = S^4, & \end{array}$$

where γ is the SU(2)-bundle with $c_2(\gamma)[S^4] = -1$ and we take k copies of γ on the left hand side. This fact is most easily understood using quaternions and the model $\mathbb{H}P^1 \cong S^4$ (see [Ati79]). If $m = (\tau_1, \tau_2, \pi, \sigma) \in U_{reg} \cap \mu_{\mathbb{C}}^{-1}(0)$ it follows that R is everywhere injective and we obtain an SU(2)-bundle

$$E := \mathrm{Im}(R)^\perp \cong \mathrm{Coker}(R)$$

with $c_2(E)[S^4] = k$, and the fiber over the point at ∞ is the space E_∞ we started with. This bundle carries a natural connection A obtained by orthogonal projection from the product connection in the trivial bundle. This connection is ASD provided $\mu_{\mathbb{R}}(m) = 0$ as well.

The above construction associates a pair (E, A) consisting of an SU(2)-bundle $E \to S^4$ with $c_2(E) = k$ and an ASD connection A in E to each element $m \in U_{reg} \cap \mu^{-1}(0)$. If m is replaced by $g \cdot m$ for $g \in U(\mathscr{H})$ the resulting pair (E', A') is equivalent to (E, A) by a bundle isomorphism $\rho^g : E \to E'$ fixing the fiber at ∞. The content of the ADHM classification is that this construction produces all instantons in the SU(2)-bundle over S^4 with Chern number k.

Theorem I.4.3. *(The ADHM-Correspondence) The assignment* $(\tau_1, \tau_2, \pi, \sigma) \mapsto (E, A)$ *descends to an equivalence*

$$(U_{reg} \cap \mu^{-1}(0))/U(\mathscr{H}) \cong \mathcal{M}_\infty(k)$$

where $\mathcal{M}_\infty(k)$ *denotes the moduli space of gauge equivalences classes of instantons framed at* ∞ *in the bundle of Chern number* k *over* $S^4 = \mathbb{C}^2 \cup \{\infty\}$.

For the proof of the theorem and additional details we refer to [DK90, Section 3.3]. A part of the proof involves giving an inverse procedure to the construction sketched above. We will need one fact from this inverse procedure. Given an SU(2)-bundle $E \to S^4$ with $c_2(E)[S^4] = k$ and ASD connection A, the space $\mathscr{H}$ is recovered by

$$\mathrm{Ker}[D_A^*\colon \Gamma(S^- \otimes E) \to \Gamma(S^+ \otimes E)], \tag{I.14}$$

where $S^\pm$ are the complex spinor bundles associated with the unique spin structure on S^4 and D_A^* is the formal adjoint of the Dirac operator twisted by the connection A.

Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup. Then according to [Aus95] the ADHM correspondence is natural with respect to the standard linear action $\Gamma \times \mathbb{C}^2 \to \mathbb{C}^2$ and the corresponding suspended action on $\mathbb{C}^2 \cup \{\infty\} = S^4$. This means that if $E \to S^4$ is a Γ-equivariant bundle with a Γ-invariant ASD connection A, then the spaces E_∞, $\mathscr{H}$ are complex Γ-representations and $m = (\tau_1, \tau_2, \pi, \sigma) \in U_{reg} \cap \mu^{-1}(0)$ associated with (E, A) belongs to the subspace M^Γ of invariants. Conversely, if $\mathscr{H}$ and E_∞ are Γ-representations and $m \in U_{reg} \cap \mu^{-1}(0) \cap M^\Gamma$ then the associated bundle E naturally obtains the structure of a Γ-equivariant bundle such that the connection A is invariant.

As we have seen the Γ-equivariant SU(2)-bundles over S^4 are determined up to isomorphism by the triple of invariants (α, β, k) as in Theorem I.4.2. We need to know which representation $\mathscr{H}$ that produces invariant instantons in the bundle with data (α, β, k). The relevant formula is obtained by an application of the equivariant index theorem to the twisted Dirac operator in (I.14). Let $R(\Gamma)$ denote the complex representation ring of Γ and let $\epsilon : R(\Gamma) \to \mathbb{Z}$ denote the augmentation. Furthermore, let as before Q denote the 2-dimensional representation associated with the inclusion $\Gamma \subset \mathrm{SU}(2)$.

Lemma I.4.4. *Let* $E \to S^4$ *be a* Γ*-equivariant* SU(2)*-bundle with invariants* $([E_N], [E_S], c_2(E)[S^4]) = (\alpha, \beta, k)$ *and set*

$$\mathscr{H} = \mathrm{Ind}[D_A^*\colon \Gamma(S^- \otimes E) \to \Gamma(S^+ \otimes E)] \in R(\Gamma)$$

where A *is a* Γ*-invariant connection in* E. *Then we have the following relations in the representation ring* $R(\Gamma)$*:*

$$(2 - Q)\mathscr{H} = \alpha - \beta$$
$$\epsilon(\mathscr{H}) = k.$$

These formulas are given in [Aus95, Sec. 4.3]. We have included a proof in Appendix I.B.11. The following lemma ensures that the above two equations determine the virtual representation $\mathscr{H}$ uniquely.

Lemma I.4.5. *Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup and let Q be the canonical 2-dimensional representation of Γ. Then the kernel of the map $R(\Gamma) \to R(\Gamma)$ given by $V \mapsto (2-Q)V$ is given by $\{nR : n \in \mathbb{Z}\}$ where $R \cong \mathbb{C}[\Gamma]$ denotes the regular representation.*

Proof. Assume that $(2-Q)V = 0$ in $R(\Gamma)$. In terms of the associated characters this means that $(2-\chi_Q(g))\chi_V(g) = 0$ for all $g \in \Gamma$. Since Q is a faithful representation of Γ it follows that $\chi_Q(g) \neq \chi_Q(1) = 2$ for all $g \neq 1$, from which we deduce that $\chi_V(g) = 0$ for all $g \neq 1$. There exist representations U, W of Γ such that $V = U - W$ in $R(\Gamma)$. Therefore,

$$\chi_V(1)/|\Gamma| = (\chi_V, 1_\Gamma) = (\chi_U, 1_\Gamma) - (\chi_W, 1_\Gamma) = \dim_{\mathbb{C}} U^\Gamma - \dim_{\mathbb{C}} W^\Gamma \in \mathbb{Z},$$

where 1_Γ is the character of the trivial representation and $(\cdot,\cdot)$ is the Hermitian inner product on characters. As the character of the regular representation R is given by $\chi_R(1) = |\Gamma|$ and $\chi_R(g) = 0$ for $g \neq 1$, we conclude that for $m = \chi_V(1)/|\Gamma|$ we have $\chi_V = m\chi_R$, and hence $V = mR$ in $R(\Gamma)$. ■

Recall that $R(\Gamma)$ is, as an abelian group, freely generated by the isomorphism classes of the irreducible representations of Γ. Therefore, if we label the irreducible representations by $R_0, \cdots, R_m$, then every element in $R(\Gamma)$ may be expressed uniquely as a linear combination $\sum_{i=0}^m n_i[R_i]$ for $n_i \in \mathbb{Z}$. Such a sum is called a virtual representation and in the case all $n_i \geq 0$ we call it an actual representation since it determines the representation $\bigoplus_i n_i R_i$ up to isomorphism.

We are now in a position to state the equivariant ADHM correspondence. This is a more precise version of [Aus95, Lemma 4.3].

Theorem I.4.6. *(The Equivariant ADHM Correspondence) Let E be a Γ-equivariant $\mathrm{SU}(2)$-bundle over S^4 with $([E_N],[E_S],c_2(E)[S^4]) = (\alpha,\beta,k)$. Let $\mathscr{H} \in R(\Gamma)$ be the unique element satisfying $(2-Q)\mathscr{H} = \alpha - \beta$ and $\epsilon(\mathscr{H}) = k$. Then if $\mathscr{H}$ is an actual representations the moduli space $\widetilde{\mathcal{M}}_{\Gamma,\infty}(E)$ of Γ-invariant instantons modulo Γ-equivariant gauge transformations framed at $\infty = N$ is given by*

$$(U_{reg} \cap \mu^{-1}(0) \cap M^\Gamma)/U(\mathscr{H})^\Gamma,$$

where $U(\mathscr{H})^\Gamma$ is the group of unitary Γ-equivariant automorphisms of $\mathscr{H}$ and

$$M^\Gamma = \mathrm{Hom}_\Gamma(Q \otimes \mathscr{H}, \mathscr{H}) \oplus \mathrm{Hom}_\Gamma(\mathscr{H}, E_\infty) \oplus \mathrm{Hom}_\Gamma(E_\infty, \mathscr{H}).$$

If $\mathscr{H}$ is not an actual representation, the moduli space is empty.

Note that in the above statement $E_\infty = E_N$.

I.4.2 Determination of the Low-Dimensional Moduli Spaces

The ADHM equations (I.12) are a particularly simple example of the hyper-Kähler moment map equations associated with an action of a Lie group on a hyper-Kähler manifold (see [Hit+87]). We briefly review the set up in the linear case relevant for our purpose.

Let V be a (left) quaternionic vector space equipped with a compatible real inner product $g\colon V \times V \to \mathbb{R}$, i.e., multiplication by unit quaternions are orthogonal. Let $\mathrm{Sp}(V)$ denote the group of $\mathbb{H}$-linear orthogonal automorphisms of V. Define an $\operatorname{Im}\mathbb{H} \coloneqq \operatorname{Span}_{\mathbb{R}}\{i, j, k\}$ valued 2-form $\omega\colon V \times V \to \operatorname{Im}\mathbb{H}$ by the formula

$$\omega(v, w) = ig(iv, w) + jg(jv, w) + kg(kv, w).$$

Note that the unique symplectic inner product $s : V \times V \to \mathbb{H}$ determined by g is given by $s = g - \omega$.

Suppose that G is a compact Lie group acting on V through a homomorphism $\rho : G \to \mathrm{Sp}(V)$. The Lie algebra $\mathfrak{g}$ of G acts on V through the derivative $d\rho_1 : \mathfrak{g} \to \mathrm{sp}(V) = \mathrm{Lie}(\mathrm{Sp}(V))$. A hyper-Kähler moment map for the action $G \times V \to V$ is a G-equivariant map $\mu\colon V \to \operatorname{Hom}(\mathfrak{g}, \operatorname{Im}\mathbb{H})$, where G acts on the target through the adjoint representation, satisfying

$$d\mu_x(v) = [\xi \mapsto \omega(\xi \cdot x, v)] \in \operatorname{Hom}(\mathfrak{g}, \operatorname{Im}\mathbb{H})$$

for all $x \in V$ and $v \in V \cong T_xV$. A moment map is clearly unique up to a constant and one may check that the unique moment map vanishing at $0 \in V$ is given by

$$\mu(x) = [\xi \in \mathfrak{g} \mapsto \frac{1}{2}\omega(\xi \cdot x, x) \in \operatorname{Im}\mathbb{H}]. \tag{I.15}$$

Definition I.4.7. For a pair $(V, \rho\colon G \to \mathrm{Sp}(V, g))$ as above we define the hyper-Kähler quotient of V by G to be the space

$$\mu^{-1}(0)/G.$$

In general neither $\mu^{-1}(0)$ nor $\mu^{-1}(0)/G$ will be manifolds due to singularities. However, if $U \subset V$ is an open G-invariant subset on which G acts freely, then $\mu^{-1}(0) \cap U$ and the quotient by G will be manifolds. Moreover, in this case the quotient inherits the structure of a hyper-Kähler manifold.

Example I.4.8. (The ADHM equations) Let $\mathscr{H}$ and E_∞ be a pair of Hermitian vector space and define

$$V = \operatorname{Hom}(\mathbb{C}^2 \otimes \mathscr{H}, \mathscr{H}) \oplus \operatorname{Hom}(\mathscr{H}, E_\infty) \oplus \operatorname{Hom}(E_\infty, \mathscr{H})$$

equipped with the Hermitian metric induced from $\mathscr{H}$ and E_∞. Using the standard basis of $\mathbb{C}^2$ an element of V is written $(\tau_1, \tau_2, \pi, \sigma)$. Let $G = U(\mathscr{H})$ act on V by $g \cdot (\tau_1, \tau_2, \pi, \sigma) = (g\tau_1 g^{-1}, g\tau_2 g^{-1}, \pi g^{-1}, g\sigma)$. Define a quaternionic structure map $J\colon V \to V$ by

$$J(\tau_1, \tau_2, \pi, \sigma) = (-\tau_2^*, \tau_1^*, -\sigma^*, \pi^*).$$

Then the action of G commutes with J and therefore defines a homomorphism $\rho\colon G \to \mathrm{Sp}(V)$. If we use the trace inner product in $\mathfrak{u}(\mathscr{H})$ and the decomposition $\mathrm{Im}(\mathbb{H}) \cong \mathbb{R} \oplus \mathbb{C}$ to identify

$$\mathrm{Hom}(\mathfrak{u}(\mathscr{H}), \mathrm{Im}\,\mathbb{H}) \cong \mathfrak{u}(\mathscr{H}) \otimes \mathrm{Im}(\mathbb{H}) \cong \mathfrak{u}(\mathscr{H}) \oplus \mathfrak{u}(\mathscr{H}) \otimes \mathbb{C} \cong \mathfrak{u}(\mathscr{H}) \oplus \mathrm{End}_{\mathbb{C}}(\mathscr{H}),$$

the resulting moment map $\mu\colon V \to \mathfrak{u}(\mathscr{H}) \oplus \mathrm{End}_{\mathbb{C}}(\mathscr{H})$ vanishing at 0 is given by the ADHM equations in (I.12).

Kronheimer [Kro89] used this construction in the following situation. Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup and let Q and R denote the canonical representation and the regular representation, respectively. Assume that Q and R are equipped with Γ-invariant Hermitian metrics and fix a Γ-invariant quaternionic structure $j : Q \to Q$ compatible with the metric. Let

$$P = Q \otimes \mathrm{End}(R) \cong \mathrm{Hom}(Q \otimes R, R)$$

equipped with the induced Hermitian metric. By combining the quaternionic structure on Q with the real structure $f \mapsto f^*$ on $\mathrm{End}(R)$ one obtains a Γ-equivariant quaternionic structure $J\colon P \to P$ compatible with the Hermitian metric. As this structure is preserved by the action of Γ, the subspace $K = P^\Gamma$ carries the same structure.

The group $U(R)$ acts on P by conjugation in the second factor. This action preserves the quaternionic structure and the Hermitian metric. The subgroup $U(R)^\Gamma$ of Γ-equivariant unitary automorphisms preserves the subspace K and therefore defines a homomorphism $U(R)^\Gamma \to \mathrm{Sp}(K)$. Here, the subgroup $U(1)$ of scalars is contained in the kernel so the homomorphism descends to a homomorphism $F := U(R)^\Gamma/U(1) \to \mathrm{Sp}(K)$. Let $\mu : K \to \mathrm{Hom}(\mathfrak{f}, \mathrm{Im}\,\mathbb{H})$ be the unique hyper-Kähler moment vanishing at $0 \in K$. In this situation Kronheimer identified the hyper-Kähler quotient.

Lemma I.4.9. *[Kro89] There is a homeomorphism $\mu^{-1}(0)/F \cong \mathbb{C}^2/\Gamma$ that restricts to an isometry*

$$\frac{\mu^{-1}(0) - \{0\}}{F} \cong \frac{\mathbb{C}^2 - \{0\}}{\Gamma}.$$

Remark I.4.10. A consequence of this is that F acts freely on $\mu^{-1}(0) - \{0\}$.

The space K and the group F admit simple descriptions in terms of the McKay graph $\overline{\Delta}_\Gamma$ associated with Γ. Let $R_0, \cdots, R_m$ be a complete set of representatives for $\mathrm{Irr}(\Gamma, \mathbb{C})$, where we take $R_0 = \mathbb{C}$ to be the trivial representation. The regular representation $R = \mathbb{C}[\Gamma]$ decomposes as

$$R \cong \bigoplus_{i=0}^{m} n_i R_i,$$

where $n_i = \dim_{\mathbb{C}} R_i$. Recall from subsection I.3.3 that a matrix A was defined by $Q \otimes R_i = \bigoplus_j a_{ij} R_j$, or equivalently $a_{ij} = \dim_{\mathbb{C}} \mathrm{Hom}_\Gamma(Q \otimes R_i, R_j)$. Furthermore, the McKay graph $\overline{\Delta}_\Gamma$ was defined by taking the irreducible representations as vertices and an (unoriented) edge connecting R_i to R_j precisely when $a_{ij} = a_{ji} = 1$. Using this

$$\begin{aligned} K = \mathrm{Hom}_\Gamma(Q \otimes R, R) &= \bigoplus_{i,j} \mathrm{Hom}_\mathbb{C}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) \otimes \mathrm{Hom}_\Gamma(Q \otimes R_i, R_j) \\ &= \bigoplus_{i \to j \text{ in } \overline{\Delta}_\Gamma} \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) \end{aligned}$$

where each edge is repeated twice, once with each orientation, and

$$F = U(R)^\Gamma / U(1) = U\left(\bigoplus_i n_i R_i\right)^\Gamma / U(1) = \left(\prod_i U(n_i)\right) / U(1).$$

This has a simple graphical interpretation on the Dynkin graph $\overline{\Delta}_\Gamma$ corresponding to Γ. Attach the vector space $\mathbb{C}^{n_i}$ to the vertex corresponding to R_i. Then an element of K is an assignment of a linear map $f_{ji} : \mathbb{C}^{n_i} \to \mathbb{C}^{n_j}$ whenever $a_{ij} = 1$. An element $g = (g_i)_i \in \prod_i U(n_i)$ acts on $f = (f_{ji})_{i,j} \in K$ by $(g \cdot f)_{ji} = g_j f_{ji} g_i^{-1}$.

Example I.4.11. For $\Gamma = O^*$ an element of $K = \mathrm{Hom}_{O^*}(Q \otimes R, R)$ is given by assigning a linear map to each arrow in the following diagram

$$\begin{array}{ccccccccccccc} & & & & & & \mathbb{C}^2 & & & & & & \\ & & & & & & \uparrow\downarrow & & & & & & \\ \mathbb{C} & \rightleftarrows & \mathbb{C}^2 & \rightleftarrows & \mathbb{C}^3 & \rightleftarrows & \mathbb{C}^4 & \rightleftarrows & \mathbb{C}^3 & \rightleftarrows & \mathbb{C}^2 & \rightleftarrows & \mathbb{C} \end{array}$$

The next result is the key step in relating Kronheimer's hyper-Kähler quotient to the space occuring in the equivariant ADHM correspondence. Recall that an SU(2)-representation α of Γ is either

(1) of quaternionic type and irreducible as a complex representation,

(2) of complex type and splits $\alpha = \lambda \oplus \lambda^*$ with $\lambda \ncong \lambda^*$ or

(3) of real type and splits $\alpha = 2\eta$ for some η with $\eta \cong \eta^*$.

By the vertex or vertices corresponding to α in $\overline{\Delta}_\Gamma$ we will mean the vertex corresponding to α in case (1), the vertices corresponding to λ and λ^* in case (2) and the vertex corresponding to η in case (3). The following is a more refined version of [Aus95, Lemma 4.6]. Austin gives no explicit proof, but states that one may verify it by a case by case analysis.

Proposition I.4.12. *Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup and let α and β be* SU(2)*-representations that are adjacent in the graph $\mathcal{S}_\Gamma$ (Definition I.3.9). Then the solution $\mathscr{H}$ of the equation $(2 - Q)\mathscr{H} = \alpha - \beta$ in $R(\Gamma)$ with minimal $\epsilon(\mathscr{H}) = k > 0$ is given in the following way.*

(i) *Delete the vertex or vertices corresponding to* β *in* $\overline{\Delta}_\Gamma$. *The resulting graph has one or two components. Let* $\Delta_{\mathscr{H}}$ *be the component containing* α.

(ii) *Up to isomorphism there is a unique nontrivial finite subgroup* $\Gamma' \subset \mathrm{SU}(2)$ *and an isomorphism of graphs* $\phi\colon \Delta_{\Gamma'} \to \Delta_{\mathscr{H}}$.

(iii) *Let* $S_0, S_1, \cdots, S_r$ *be the irreducible representations of* Γ', *where* S_0 *is the trivial representation. Then*

$$\mathscr{H} := \bigoplus_{j=1}^{r} (\dim_{\mathbb{C}} S_j) R_{\phi(j)}.$$

Furthermore, if $Q' = m_1 S_{i_1} \oplus m_2 S_{i_2}$ *is the isotypical decomposition of the canonical representation of* Γ', *then* $\alpha = m_1 R_{\phi(i_1)} \oplus m_2 R_{\phi(i_2)}$ *is the isotypical decomposition of* α. *In particular,* ϕ *takes the vertex or vertices associated with* Q' *to the vertex or vertices corresponding to* α.

Remark I.4.13. The isotypical decomposition of the canonical representation Q of a finite subgroup $\Gamma \subset \mathrm{SU}(2)$ is $Q = Q$ for Γ non-cyclic and $Q = \rho \oplus \rho^*$ for some 1-dimensional representation ρ for Γ cyclic.

Corollary I.4.14. *Let* $\Gamma \subset \mathrm{SU}(2)$ *be a finite subgroup and let* $\alpha \neq \beta$ *be* $\mathrm{SU}(2)$-*representations that are not adjacent in* $\mathcal{S}_\Gamma$. *Let* $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_s = \beta$ *be the vertices of the minimal edge path connecting* α *to* β *in* $\mathcal{S}_\Gamma$ *and let* $\mathscr{H}_i \in R(\Gamma)$ *be the minimal positive solution of the equation* $(2-Q)\mathscr{H}_i = \alpha_i - \alpha_{i+1}$ *for* $0 \leq i \leq s-1$. *Then* $\mathscr{H} := \bigoplus_{i=0}^{s-1} \mathscr{H}_i$ *is the minimal positive solution* $(2-Q)\mathscr{H} = \alpha - \beta$ *in* $R(\Gamma)$.

Proof. The calculation

$$(2-Q)\mathscr{H} = \sum_{i=0}^{s-1} (2-Q)\mathscr{H}_i = \sum_{i=0}^{s-1} \alpha_i - \alpha_{i+1} = \alpha_0 - \alpha_s = \alpha - \beta$$

shows that $\mathscr{H}$ solves the relevant equation. By Lemma I.4.5 every other solution is given by $\mathscr{H} + nR$ for some $n \in \mathbb{Z}$, where R is the regular representation. To see that $\mathscr{H}$ is the minimal positive solution, it therefore suffices to show that $\mathscr{H} - R$ is not positive, i.e., not an actual representation. Let $\Delta_i = \Delta_{\mathscr{H}_i}$ be the subgraph of $\overline{\Delta}_\Gamma$ described in the above proposition. By the construction of these graphs and Lemma I.3.8 we conclude that $\Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_{s-1}$. This implies that the representation $\mathscr{H}$ is only supported on the vertices of Δ_{s-1}, which by construction does not contain the vertices corresponding to $\beta = \alpha_s$. Hence, in the isotypical decomposition $\mathscr{H} = \bigoplus_{i=0}^{n} k_i R_i$ there exists some i with $k_i = 0$. We therefore conclude that $\mathscr{H} - R$ is not an actual representation. ∎

Let Γ, α, β, $\mathscr{H}$, Γ' and $\phi\colon \Delta_{\Gamma'} \to \Delta_{\mathscr{H}}$ be defined as in Proposition I.4.12 and set

$$N = M^\Gamma = \mathrm{Hom}_\Gamma(\mathscr{H} \otimes Q, \mathscr{H}) \oplus \mathrm{Hom}_\Gamma(\mathscr{H}, E_\infty) \oplus \mathrm{Hom}_\Gamma(E_\infty, \mathscr{H}),$$

as in Theorem I.4.6. This space is endowed with a quaternionic structure and compatible metric preserved by the action of $G := U(\mathscr{H})^\Gamma$ as in Example I.4.8. Note that $E_\infty = \alpha$. Let R' and Q' denote the regular and canonical representation of Γ' respectively and set

$$K = \mathrm{Hom}_{\Gamma'}(Q' \otimes R', R') \quad \text{and} \quad F = U(R')^{\Gamma'}/U(1)$$

as in Kronheimer's construction. The following is a more precise version of [Aus95, Lemma 4.7].

Proposition I.4.15. *In the above situation there is an isometric isomorphism $f : N \to K$ of quaternionic vector spaces and an isomorphism of groups $\tau : G \to F$ such that $f(g \cdot n) = \tau(g) f(n)$ for all $n \in N$ and $g \in G$.*

Proof. Let S be the regular representation of Γ' and write $S = S_0 \oplus S'$, where S_0 denotes the trivial representation. Then

$$K = \mathrm{Hom}_{\Gamma'}(Q' \otimes S, S) = \mathrm{Hom}_{\Gamma'}(Q' \otimes S', S') \oplus \mathrm{Hom}_{\Gamma'}(Q', S') \oplus \mathrm{Hom}_{\Gamma'}(S', Q')$$

since $\mathrm{Hom}_{\Gamma'}(Q', S_0) = 0$ and $\mathrm{Hom}_{\Gamma'}(Q' \otimes S', S_0) \cong \mathrm{Hom}_{\Gamma'}(S', Q')$ as Q' is self-dual. We will now match these components with the components of N. Let $n_i = \dim_{\mathbb{C}} S_i$. Recall that the graph $\Delta_{\Gamma'}$ is the graph obtained from $\overline{\Delta}_{\Gamma'}$ by deleting the vertex corresponding to S_0. Using part (iii) of the above proposition we can write $\mathscr{H} = \oplus_{i=1}^r n_i R_{\phi(i)}$ where $\phi : \Delta_{\Gamma'} \to \Delta_{\mathscr{H}}$ is the isomorphism of graphs. Then

$$\begin{aligned} \mathrm{Hom}_\Gamma(Q \otimes \mathscr{H}, \mathscr{H}) &= \bigoplus_{i \to j \text{ in } \Delta_{\Gamma'}} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) \otimes \mathrm{Hom}_\Gamma(Q \otimes R_{\phi(i)}, R_{\phi(j)}) \\ &\cong \bigoplus_{i \to j \text{ in } \Delta_{\Gamma'}} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) \otimes \mathrm{Hom}_{\Gamma'}(Q' \otimes S_i, S_j) = \mathrm{Hom}_\Gamma(Q' \otimes S', S'). \end{aligned} \tag{I.16}$$

To make this an $\mathbb{H}$-linear isometric isomorphism we specify the isomorphisms

$$\mathrm{Hom}_\Gamma(Q \otimes R_{\phi(i)}, R_{\phi(j)}) \cong \mathbb{C} \cong \mathrm{Hom}_{\Gamma'}(Q' \otimes S_i, S_j) \tag{I.17}$$

for i adjacent to j in $\Delta_{\Gamma'}$ as follows. Choose an arbitrary orientation on the edges in $\Delta_{\Gamma'}$. Then for each positive arrow $i \to j$ in $\Delta_{\Gamma'}$ we pick basis vectors $u_{ij} \in \mathrm{Hom}_\Gamma(Q \otimes R_{\phi(i)}, R_{\phi(j)})$ and $v_{ij} \in \mathrm{Hom}_{\Gamma'}(Q' \otimes S_i, S_j)$ of unit length. Then $u_{ji} = J u_{ij} \in \mathrm{Hom}_\Gamma(Q \otimes R_{\phi(j)}, R_{\phi(i)})$ and $v_{ji} = J v_{ij} \in \mathrm{Hom}_{\Gamma'}(Q' \otimes S_j, S_i)$ are also unit basis vectors. We then define the isomorphisms in (I.17) by sending u_{ij} to v_{ij} for each $i \to j$ in $\Delta_{\Gamma'}$.

Next, by the final part of the above proposition we know that if $Q' = m_1 S_{i_1} \oplus m_2 S_{i_2}$ is the isotypical decomposition, then $\alpha = E_\infty = m_1 R_{\phi(i_1)} \oplus m_2 R_{\phi(i_2)}$ is the isotypical decomposition of α. We may therefore identify

$$\mathrm{Hom}_{\Gamma'}(S', Q') = \bigoplus_{j=1}^{2} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{n_{i_j}}, \mathbb{C}^{m_j}) \otimes \mathrm{Hom}_{\Gamma'}(S_{i_j}, S_{i_j}) \tag{I.18}$$

$$\cong \bigoplus_{j=1}^{2} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{n_{i_j}}, \mathbb{C}^{m_j}) \otimes \mathrm{Hom}_\Gamma(R_{\phi(i_j)}, R_{\phi(i_j)}) \cong \mathrm{Hom}_\Gamma(\mathscr{H}, E_\infty)$$

and similarly $\mathrm{Hom}_{\Gamma'}(Q', S') \cong \mathrm{Hom}_\Gamma(E_\infty, \mathscr{H})$. Once again one should match unit vectors carefully to ensure that the direct sum of these isomorphisms is an $\mathbb{H}$-linear isometry. The direct sum of these isomorphisms gives the required $\mathbb{H}$-linear isometry $N \cong K$.

The (orthogonal) decomposition $\mathscr{H} = \bigoplus_{i=1}^{r} n_i R_{\phi(i)}$ induces an isomorphism $G = U(\mathscr{H})^\Gamma \cong \prod_{i=1}^{r} U(n_i)$ and this group only acts on the matrix components in the decompositions (I.16) and (I.18). In the latter case it only acts on factors coming from $\mathscr{H}$. Moreover,

$$F \cong \left(\prod_{i=0}^{r} U(n_i)\right) / U(1) \cong \prod_{i=1}^{r} U(n_i)$$

as $n_0 = 1$, and the action can again be seen to be on the matrix component in (I.16) and (I.18), and in the latter case only on the factors coming from S'. Therefore, the isomorphism

$$F \cong \prod_{i=1}^{r} U(n_i) \cong G$$

does the trick and the proof is complete. ∎

Proposition I.4.16. *Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup and let α and β be $\mathrm{SU}(2)$-representations adjacent in the graph $\mathcal{S}_\Gamma$. Let $E \to S^4$ be the Γ-equivariant $\mathrm{SU}(2)$-bundle with $[E_N] = \alpha$, $[E_S] = \beta$ and minimal $k = c_2(E)[S^4] > 0$. Then the moduli space $\widetilde{\mathcal{M}}_{\Gamma,\infty}(E)$ of Γ-invariant instantons in E framed at $N = \infty$ is given by*

$$(\mathbb{C}^2 - \{0\})/\Gamma' \cong \mathbb{R} \times S^3/\Gamma'$$

where the finite subgroup $\Gamma' \subset \mathrm{SU}(2)$ is determined as in Proposition I.4.12.

Proof. We continue to use the notation of the two preceding propositions. According to the equivariant ADHM correspondence the moduli space $\widetilde{\mathcal{M}}_{\Gamma,\infty}(E)$ is given by

$$\frac{\mu^{-1}(0) \cap U_{reg} \cap M^\Gamma}{U(\mathscr{H})^\Gamma}, \tag{I.19}$$

where in the above notation $N = M^\Gamma$ and $G = U(\mathscr{H})^\Gamma$. Let $\mu_N \colon N \to \mathfrak{g} \otimes \mathrm{Im}\,\mathbb{H}$ and $\mu_K \colon K \to \mathfrak{f} \otimes \mathrm{Im}\,\mathbb{H}$ be the unique hyper-Kähler moment maps vanishing at 0 associated with the action of G on N and F on K. Due to the uniqueness of the moment map it follows that the isomorphisms $f \colon N \to K$ and $\psi \colon G \to F$ of the previous proposition induce a homeomorphism $\mu_N^{-1}(0) \cong \mu_K^{-1}(0)$ equivariant along the map ψ. The group F acts freely on $\mu_K^{-1}(0) - \{0\}$ (see Remark I.4.10), so by Lemma I.4.9 we obtain isometries of hyper-Kähler 4-manifolds

$$\frac{\mu_N^{-1}(0) - \{0\}}{G} \cong \frac{\mu_K^{-1}(0) - \{0\}}{F} \cong \frac{\mathbb{C}^2 - \{0\}}{\Gamma'}.$$

To complete the proof we have to show that the left hand side coincides with the space in (I.19). This will be achieved by proving that

(1) μ_N is the restriction of μ to $N = M^\Gamma \subset M$, so that $\mu_N^{-1}(0) = \mu^{-1}(0) \cap N$ and

(2) $U_{reg} \cap \mu_N^{-1}(0) = \mu_N^{-1}(0) - \{0\}$.

We may decompose an element $m \in M^\Gamma$ as

$$m = (\tau_1, \tau_2, \pi, \sigma) \in \mathrm{Hom}_\Gamma(Q \otimes \mathscr{H}, \mathscr{H}) \oplus \mathrm{Hom}_\Gamma(\mathscr{H}, E_\infty) \oplus \mathrm{Hom}_\Gamma(E_\infty, \mathscr{H}),$$

where we regard $Q = \mathbb{C}^2$, as earlier. Here π and σ are Γ-equivariant, while τ_1, τ_2 satisfy the properties (see [Kro89, Eq. 2.2])

$$\begin{matrix} \gamma^{-1}\tau_1\gamma = a\tau_1 + b\tau_2 \\ \gamma^{-1}\tau_2\gamma = -\bar{b}\tau_1 + \bar{a}\tau_2 \end{matrix} \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \Gamma \subset \mathrm{SU}(2), \tag{I.20}$$

where on the left hand side γ is regarded as an isometry $\mathscr{H} \to \mathscr{H}$. Using the above description and the moment map equations (I.12)

$$\begin{aligned} \mu_{\mathbb{C}}(\tau_1, \tau_2, \pi, \sigma) =& [\tau_1, \tau_2] + \sigma\pi \\ \mu_{\mathbb{R}}(\tau_1, \tau_2, \pi, \sigma) =& [\tau_1, \tau_1^*] + [\tau_2, \tau_2^*] + \sigma\sigma^* - \pi^*\pi \end{aligned}$$

one easily verifies that $\mu_{\mathbb{R}}$ and $\mu_{\mathbb{C}}$ map $N = M^\Gamma$ into $\mathfrak{u}(\mathscr{H})^\Gamma$ and $\mathrm{End}_\Gamma(\mathscr{H})$, respectively. By the uniqueness of the moment map it follows that μ_N is the restriction of μ to $N = M^\Gamma$. This proves the first point.

Before we proceed to the second point recall that $U_{reg} \subset M$ was defined to be the set of $m = (\tau_1, \tau_2, \pi, \sigma)$ for which

$$\begin{aligned} A_z &= (\tau_1 - z_1, \tau_2 - z_2, \pi)^t \colon \mathscr{H} \oplus \mathscr{H} \oplus E_\infty \to \mathscr{H} \\ B_z &= (-(\tau_2 - z_2), \tau_1 - z_1, \sigma) \colon \mathscr{H} \oplus \mathscr{H} \oplus E_\infty \to \mathscr{H} \end{aligned}$$

were injective and surjective, respectively, for all $z = (z_1, z_2) \in \mathbb{C}^2$ (see (I.13)). Observe that the equation $\mu_{\mathbb{R}}(m) = 0$ may be rewritten as

$$\tau_1\tau_1^* + \tau_2\tau_2^* + \sigma\sigma^* = \tau_1^*\tau_1 + \tau_2^*\tau_2 + \pi^*\pi,$$

or equivalently $B_0 B_0^* = A_0^* A_0$. From this it follows that

$$\mathrm{Ker}(A_0) = \mathrm{Ker}(A_0^* A_0) = \mathrm{Ker}(B_0 B_0^*) = \mathrm{Ker}(B_0^*).$$

Hence, A_0 is injective if and only if B_0 is surjective. In view of the easily verified fact that for any $(z_1, z_2) \in \mathbb{C}^2$ we have $\mu(\tau_1 - z_1, \tau_2 - z_2, \pi, \sigma) = \mu(\tau_1, \tau_2, \pi, \sigma)$, the above argument applies to show that if $m \in \mu^{-1}(0)$, then for any $z \in \mathbb{C}^2$, A_z is injective if and only if B_z is surjective.

We may now verify the second claim as follows. As $U_{reg} \cap \mu_N^{-1}(0) \subset \mu_N^{-1}(0) - \{0\}$, it suffices to show that if $m = (\tau_1, \tau_2, \pi, \sigma) \in \mu_N^{-1}(0) - U_{reg}$ then $m = 0$. To this end assume that $m \in \mu_N^{-1}(0) - U_{reg}$. In view of the above considerations this means that there exists $z = (z_1, z_2) \in \mathbb{C}^2$ and $h \neq 0 \in \mathscr{H}$ such that $A_z(h) = 0$. Then h belongs to $\operatorname{Ker} \pi$ and is a common eigenvector for τ_1, τ_2 with eigenvalues z_1, z_2 respectively. Using the equations in (I.20) we deduce that $\gamma \cdot h$ is a common eigenvector for τ_1, τ_2 with eigenvalues

$$\begin{pmatrix} z_1^\gamma \\ z_2^\gamma \end{pmatrix} := \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \Gamma \subset \mathrm{SU}(2).$$

Let $L := \operatorname{Span}_{\mathbb{C}}\{\gamma h : \gamma \in \Gamma\}$. This is a nontrivial Γ-invariant subspace of $\mathscr{H}$ satisfying $\tau_i(L) \subset L$ for $i = 1, 2$ and $\pi(L) = 0$. Moreover, since $\gamma \cdot h \in \operatorname{Ker}(A_{(z_1^\gamma, z_2^\gamma)}) = \operatorname{Ker}(B^*_{(z_1^\gamma, z_2^\gamma)})$, it follows that $\tau_i^*(\gamma h) = \overline{z_i^\gamma} h$ for $i = 1, 2$ and $\sigma^*(\gamma h) = 0$. Therefore, $\tau_i^*(L) \subset L$ for $i = 1, 2$ and $\sigma^*(L) = 0$ as well. Hence, $\tau_i(L^\perp) \subset L^\perp$ for $i = 1, 2$ and $\sigma(E_\infty) \subset L^\perp$.

To summarize, we have proved that there is an orthogonal splitting of Γ-modules $\mathscr{H} = L \oplus L^\perp$ such that $\tau_i = \tau_i' \oplus \tau_i''$ preserves the splitting for $i = 1, 2,$, $\pi(L) = 0$ and $\operatorname{Im}(\sigma) \subset L^\perp$. Pick $\lambda \neq 1 \in U(1)$ and define $g \in U(\mathscr{H})^\Gamma$ by setting $g = \lambda \operatorname{id}$ on L and $g = \operatorname{id}$ on $L^\perp$. Then $(g\tau_1 g^{-1}, g\tau_2 g^{-1}, \pi g^{-1}, g\sigma) = (\tau_1, \tau_2, \pi, \sigma)$, so as $g \neq 1$ and $U(\mathscr{H})^\Gamma$ acts freely on $\mu_N^{-1}(0) - \{0\}$ we conclude that $m = 0$ as desired. ∎

Proof of Proposition I.4.1. Given an adjacent pair $\alpha, \beta \in \mathcal{C}$ let $E \to S^4$ be the unique Γ-equivariant $\mathrm{SU}(2)$-bundle with $[E_N] = \alpha$ and $[E_S] = \beta$ and minimal $c_2(E)[S^4] > 0$. Then by the above proposition the moduli space $\widetilde{\mathcal{M}}_{\Gamma,\infty}(E)$ of Γ-invariant instantons framed at $\infty = N$ may be identified with $\mathbb{R} \times Y_{\Gamma'}$ for some other finite subgroup $\Gamma' \subset \mathrm{SU}(2)$. Using the correspondence between invariant instantons in $E \to S^4$ and instantons over $\mathbb{R} \times \overline{Y}_\Gamma$, taking the framing and orientation reversal into account, one obtains

$$\mathbb{R} \times Y_{\Gamma'} \cong e_-^{-1}(*) \subset \widetilde{\mathcal{M}}_z(\alpha, \beta),$$

where $e_- : \widetilde{\mathcal{M}}_z(\alpha, \beta) \to \alpha$ is the end-point map and z is the homotopy class from α to β corresponding to the relative Chern number $\hat{c}_2 = c_2(E)/|\Gamma|$. According to [Aus95] the natural translation action on the left hand side corresponds to the translation action on the right hand side. Hence,

$$Y_{\Gamma'} \cong e_-^{-1}(*) \subset \widetilde{\mathcal{M}}^0_z(\alpha, \beta).$$

Since the end-point map is $\mathrm{SO}(3)$-equivariant it follows that $\widetilde{\mathcal{M}}^0_z(\alpha, \beta) = e_-^{-1}(*) \cdot \mathrm{SO}(3)$ is compact. Therefore, $\widetilde{\mathcal{M}}^0_z(\alpha, \beta) = \overline{\mathcal{M}}_z(\alpha, \beta)$, and as $\mathrm{gr}_z(\alpha, \beta) = \dim(e_-^{-1}(*)) + 1 = 4$, we conclude that z is indeed the unique homotopy class with $-2 \leq \mathrm{gr}_z(\alpha, \beta) \leq 5$.

To complete the proof we have to verify that $\overline{\mathcal{M}}_z(\alpha, \beta)$ is empty when α and β are not adjacent in $\mathcal{S}_\Gamma$. Let $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_s = \beta$ be the vertices of the minimal edge path connecting α to β in $\mathcal{S}_\Gamma$. Let z_i denote the unique homotopy class connecting α_i to α_{i+1} with $\mathrm{gr}_{z_i}(\alpha_i, \alpha_{i+1}) = 4$ for each $0 \leq i < s$. Let $\mathscr{H}_i$ be the minimal positive solution of $(2 - Q)\mathscr{H}_i = \alpha_i - \alpha_{i+1}$ for $0 \leq i < s$. By

Corollary I.4.14 the minimal positive solution of $(2-Q)\mathscr{H} = \alpha - \beta$ is given by $\mathscr{H} = \bigoplus_{i=0}^{s-1} \mathscr{H}_i$. Put $k = \epsilon(\mathscr{H})$ and let $E^l \to S^4$ denote the Γ-equivariant SU(2)-bundle with invariants $([E^l_N], [E^l_S], c_2(E^l)[S^4]) = (\alpha, \beta, k + |\Gamma| l)$ for $l \in \mathbb{Z}$. The representation $\mathscr{H}_l$ used in the description of the moduli space $\widetilde{\mathcal{M}}_{\Gamma,\infty}(E^l)$ in the equivariant ADHM correspondence is then given by $\mathscr{H}_l = \mathscr{H} + lR$, where R is the regular representation. The dimension of the moduli space increases linearly with l so, as $\mathscr{H}_0 = \mathscr{H}$ is the minimal positive solution of $(2-Q)\mathscr{H} = \alpha - \beta$, we conclude that $\widetilde{\mathcal{M}}_{\Gamma,\infty}(E^l)$ is empty for $l < 0$. This space corresponds to the fiber $e_-^{-1}(*) \subset \widetilde{\mathcal{M}}_w(\alpha, \beta)$ where $w = z_0 * z_1 * \cdots * z_{s-1}$. By Theorem I.2.1 part (a) we have

$$\mathrm{gr}_w(\alpha, \beta) = \sum_{i=0}^{s-1} \mathrm{gr}_{z_i}(\alpha_i, \alpha_{i+1}) = 4s > 4,$$

since by assumption $s > 1$. Consequently, the fiber $e_-^{-1}(*) \subset \widetilde{\mathcal{M}}_z(\alpha, \beta)$ where z is the unique homotopy class with $\mathrm{gr}_z(\alpha, \beta) = 0, 4$, corresponds to $\widetilde{\mathcal{M}}_{\Gamma,\infty}(E^l)$ for some $l < 0$ and must therefore be empty. This implies that the whole moduli space must be empty and the proof is complete. ■

I.4.3 The Structure of $DCI(\overline{Y}_\Gamma)$

In this section we determine the complexes $DCI(\overline{Y}_\Gamma)$ explicitly. Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup and let $\mathcal{C}$ denote the set of gauge equivalence classes of flat connections in the trivial SU(2)-bundle over $\overline{Y}_\Gamma$. In this section we make the convention that the relative and absolute gradings

$$\mathrm{gr}\colon \mathcal{C} \times \mathcal{C} \to \mathbb{Z}/8 \quad \text{and} \quad j\colon \mathcal{C} \to \mathbb{Z}/8,$$

introduced after Theorem I.2.1, are always defined with respect to the orientation $\overline{Y}_\Gamma$ of S^3/Γ. Furthermore, the vertices of the graph $\mathcal{S}_\Gamma$, introduced in Definition I.3.9, i.e., the set of isomorphism classes of SU(2)-representations of Γ, will be identified freely with the set $\mathcal{C}$.

Lemma I.4.17. *For any pair $\alpha, \beta \in \mathcal{C}$ that are adjacent in $\mathcal{S}_\Gamma$ it holds true that* $\mathrm{gr}(\alpha, \beta) = 4$. *For each $\alpha \in \mathcal{C}$ let $p(\alpha)$ denote the number of edges in the minimal edge path connecting the trivial representation θ to α. Then $j(\alpha) \equiv 4p(\alpha) \pmod 8$ for all $\alpha \in \mathcal{C}$.*

Proof. Suppose that α and β are adjacent in $\mathcal{S}_\Gamma$ and let z be the unique homotopy class with $-2 \leq \mathrm{gr}_z(\alpha, \beta) \leq 5$. Then according to Theorem I.4.1 the fiber of $e_-\colon \overline{\mathcal{M}}_z(\alpha, \beta) \to \alpha$ has dimension 3. Using the dimension formula (I.7) we deduce that

$$\mathrm{gr}(\alpha, \beta) = \dim \overline{\mathcal{M}}_z(\alpha, \beta) - \dim(\alpha) + 1 = \dim e_-^{-1}(*) + 1 = 4$$

proving the first assertion. For the second let $\alpha \in \mathcal{C}$ be arbitrary and let $\theta = \alpha_0, \alpha_1, \cdots, \alpha_s = \alpha$ denote the vertices in the minimal edge path connecting

θ to α in $\mathcal{S}_\Gamma$. Then $p(\alpha) = s$ and by the additivity of the grading

$$j(\alpha) = \mathrm{gr}(\alpha, \theta) = \sum_{i=0}^{s-1} \mathrm{gr}(\alpha_{i+1}, \alpha_i) = 4s \in 4\mathbb{Z}/8.$$

Consequently, $j(\alpha) \equiv 4p(\alpha) \pmod 8$ as required. ■

To determine the differentials in $DCI(\overline{Y}_\Gamma)$ we need the following result, which is contained in the proof of [Aus95, Lemma 5.1].

Lemma I.4.18. *Let $\alpha, \beta \in \mathcal{C}$ be adjacent in $\mathcal{S}_\Gamma$ with β irreducible, let $\mathscr{H} \in R(\Gamma)$ be the solution of $(2-Q)\mathscr{H} = \alpha - \beta$ given by Proposition I.4.12 and let $\Gamma' \subset \mathrm{SU}(2)$ be the corresponding finite subgroup. Let z be the unique homotopy class with $\mathrm{gr}_z(\alpha, \beta) = 4$. Then the degree of the map*

$$e_-^{-1}(*) = Y_{\Gamma'} \to \beta = \mathrm{SO}(3)$$

is up to a sign given by $2 \dim_{\mathbb{C}} \mathscr{H} / |\Gamma'|$.

Remark I.4.19. We are not aware of a procedure to pin down the orientation of $Y_{\Gamma'}$ in the above statement. For this reason we are only able to determine the degree up to a sign. As we will see, this inaccuracy will not affect our calculations in any relevant way.

For $i = 0, 1$ let $\mathcal{C}^i = \{\alpha \in \mathcal{C} : j(\alpha) \equiv 4i \mod 8\}$ such that $\mathcal{C} = \mathcal{C}^0 \cup \mathcal{C}^i$ is a disjoint union. Let us fix generators $b_\alpha \in H_0(\alpha)$ for all $\alpha \in \mathcal{C}$, and $t_\alpha \in H_2(\alpha)$ for all reducible α and $t_\alpha := b_\alpha u \in H_3(\alpha)$ for all irreducible α.

Definition I.4.20. Let $\mathcal{C}^{irr} \subset \mathcal{C}$ be the subset of irreducibles. For each adjacent pair $\alpha, \beta \in \mathcal{C}$ with $\beta \in \mathcal{C}^{irr}$ define $n_{\beta\alpha}$ to be the integer $2 \dim_C \mathscr{H} / |\Gamma'|$ where $\mathscr{H} \in R(\Gamma)$ is the minimal positive solution of the equation $(2 - Q)\mathscr{H} = \alpha - \beta$. If α and β are not adjacent in $\mathcal{S}_\Gamma$ we define $n_{\beta\alpha} = 0$.

Theorem I.4.21. *Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup. Then the multicomplex $(DCI(\overline{Y}_\Gamma)_{*,*}, \{\partial^r\}_{r \geq 0})$ is given by*

$$DCI_{8s,*} = \bigoplus_{\alpha \in \mathcal{C}^0} H_*(\alpha) \text{ and } DCI_{8s+4,*} = \bigoplus_{\beta \in \mathcal{C}^1} H_*(\beta)$$

for all $s \in \mathbb{Z}$ and $DCI_{s,*} = 0$ otherwise. The differentials are given by $\partial^r = 0$ for $r \neq 4$ and $\partial^4 \colon DCI_{4s,0} \to DCI_{4(s-1),3}$ is determined on generators by

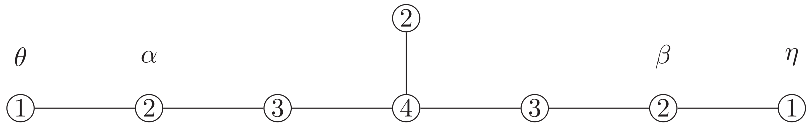
$$\partial^4(b_\beta) = \sum_{\alpha \in \mathcal{C}^{irr}} n_{\alpha\beta} t_\alpha$$

where the integers $n_{\alpha\beta}$ are defined above.

Proof. The description of $DCI(Y, E)$ as a graded module is a simple consequence of the definition and Lemma I.4.17. By the final part of Theorem I.4.1 the moduli space $\overline{\mathcal{M}}_z(\alpha, \beta)$, where z is the homotopy class with $-2 \leq \mathrm{gr}_z(\alpha, \beta) \leq 5$, is empty unless α and β are adjacent in $\mathcal{S}_\Gamma$. The statement concerning the differentials therefore follows from Lemma I.2.7 and Lemma I.4.18. ■

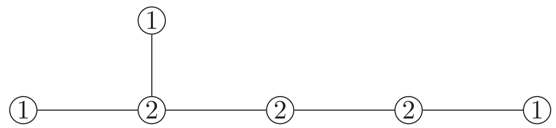
We will end this section by giving a few applications of Proposition I.4.12 and show how we can easily extract both the grading and the integers $n_{\alpha\beta}$ provided we know where the one dimensional quaternionic representations are placed in the graph $\overline{\Delta}_\Gamma$. All the necessary information is contained in Appendix A.

Example I.4.22. ($\Gamma = O^*$) There are four 1-dimensional quaternionic representations; two of real type θ, η and two of quaternionic type α, β. The graph $\overline{\Delta}_{O^*}$ is shown below.



The integers are the dimensions of the corresponding irreducible complex representations and we have labeled the vertices corresponding to the 1-dimensional quaternionic representation. Here there are no representations of complex type, in particular $\overline{\Delta}_\Gamma = \overline{\Delta}_\Gamma/(\iota)$. One may immediately conclude that the indexing is given by $j(\theta) = j(\beta) = 0$ and $j(\alpha) = j(\eta) = 4$. To determine the integer $n_{\beta\eta}$ we need to solve the equation $(2-Q)\mathscr{H} = \eta - \beta$ in $R(O^*)$. According to the recipe given in Proposition I.4.12 we first delete the vertex corresponding to β and let $\Delta_{\mathscr{H}}$ be the component containing η; that is, $\Delta_{\mathscr{H}}$ is just the single vertex corresponding to η. We then recognize this graph as Δ_{C_2}, from which we deduce that $\Gamma' = C_2$ and $\mathscr{H} = \frac{1}{2}\eta$. Therefore, $n_{\beta\eta} = 2\dim_{\mathbb{C}} \mathscr{H}/|\Gamma'| = 2/2 = 1$. The situation with θ and α is completely symmetric so $n_{\alpha\theta} = 1$ as well.

The case of $n_{\beta\alpha}$ is more interesting. If we delete α, the component $\Delta_{\mathscr{H}}$ containing β is given by



We recognize this as the Dynkin graph D_6 which corresponds to the group D_4^* of order 16. We find the integral weights of $\mathscr{H}$, as given in the above graph, by knowledge of the dimensions of the irreducible representations of D_4^* (see Appendix A.2). From this we calculate

$$\dim_{\mathbb{C}} \mathscr{H} = 3 \cdot 1 + 4 \cdot 2 + 2 \cdot 1 + 3 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 = 24$$

and hence $n_{\beta\alpha} = 2 \cdot 24/16 = 3$. By symmetry $n_{\alpha\beta} = 3$ as well.

The calculation of $n_{\alpha\theta}$ and $n_{\beta\eta}$ in the above example generalizes.

Lemma I.4.23. *Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup and let Q be the canonical representation. Suppose ρ is a 1-dimensional complex representation of real type and that $\rho \otimes Q$ is irreducible so that $\eta = 2\rho$ and $\beta = \rho \otimes Q$ correspond to a pair of distinct vertices in $\mathcal{S}_\Gamma$. Then $n_{\beta\eta} = 1$.*

Proof. We trivially have $(2-Q)\rho = 2\rho - Q \otimes \rho = \eta - \beta$ so that $\mathscr{H} = \rho$. This means that $\Delta_{\mathscr{H}}$ consists of a single vertex and that the associated subgroup is $\Gamma' = C_2$. It follows that $n_{\beta\eta} = 2\dim_{\mathbb{C}} \mathscr{H}/|\Gamma'| = 2/2 = 1$. ∎

We also consider the most involved case $\Gamma = D_n^*$. The 1-dimensional quaternionic representations and the graph $\overline{\Delta}_{D_n^*}$ are given in Appendix $A.2$. For n even we label the quaternionic representations by

$$\theta, \eta_1, \eta_2, \eta_3, \alpha_1, \alpha_2, \cdots, \alpha_{n/2}$$

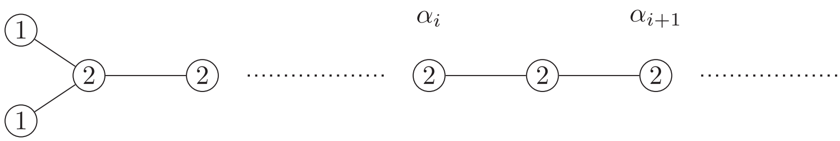
and for n odd by

$$\theta, \eta, \alpha_1, \alpha_2, \cdots, \alpha_{(n-1)/2}, \lambda.$$

Here θ, η and the η_i are fully reducible, the α_i are irreducible and λ is reducible.

Lemma I.4.24. *In the above situation we have $n_{\alpha_i,\alpha_{i+1}} = n_{\alpha_{i+1},\alpha_i} = 2$ for all i.*

Proof. Due to the symmetry in the graph $\overline{\Delta}_{D_n^*}$ it suffices to show that $n_{\alpha_{i+1},\alpha_i} = 2$ for all i. We therefore have to solve $(2-Q)\mathscr{H} = \alpha_i - \alpha_{i+1}$ for $\mathscr{H}$. The relevant portion of the graph $\overline{\Delta}_{D_n^*}$ is shown below



Note that α_i is the $2i-1$'th vertex labeled with 2 from left to right. Following the description in Proposition I.4.12 we delete the vertex corresponding to α_{i+1} and let $\Delta_{\mathscr{H}}$ be the component containing α_i. This graph has a total of $2(i+1)$ vertices so we recognize $\Delta_{\mathscr{H}} = D_{2(i+1)} = \Delta_{D_{2i}^*}$. Thus $\Gamma' = D_{2i}^*$ of order $8i$. The weights of $\mathscr{H}$ are therefore given by attaching the integer 2 to each internal vertex (of degree ≥ 2) and 1 to the rest. This yields

$$n_{\alpha_{i+1}\alpha_i} = 2\dim_{\mathbb{C}} \mathscr{H}/|D_{2i}^*| = 2(1+1+4(2i-1)+2)/8i = 2$$

as required. ∎

We note that for cyclic groups there are no irreducible SU(2)-representations, from which it follows that $DCI(\overline{Y}_{C_m})$ has trivial differential for each m. There are now only a finite number of $n_{\alpha\beta}$ left to calculate. We state the calculations of these below without proof and trust that the interested reader will verify these for themselves.

For a finite subgroup $\Gamma \subset \mathrm{SU}(2)$ we attach labels to the edges in the graph $\mathcal{S}_\Gamma$ to express all the information needed to set up $DCI(\overline{Y}_\Gamma)$. If α and β are adjacent and both irreducible we attach the symbol $(n_{\alpha\beta}|n_{\beta\alpha})$ to the edge connecting α to β such that $n_{\alpha\beta}$ is closest to α. If α and β are adjacent and only α is irreducible we attach $n_{\alpha\beta}$ to the edge.

Proposition I.4.25. *The graphs $\mathcal{S}_\Gamma$ with labels for the finite subgroups $\Gamma \subset \mathrm{SU}(2)$ are given below.*

$$\mathcal{S}_{I^*}: \quad \theta \overset{}{\underset{1}{—}} \alpha \underset{(3|4)}{—} \beta$$

$$\mathcal{S}_{O^*}: \quad \theta \underset{1}{—} \alpha \underset{(3|3)}{—} \beta \underset{1}{—} \eta$$

$$\mathcal{S}_{T^*}: \quad \theta \underset{1}{—} \alpha \underset{3}{—} \lambda$$

$$\mathcal{S}_{C_{2m}}: \quad \theta — \lambda_1 — \lambda_2 \cdots\cdots \lambda_{m-1} — \eta$$

$$\mathcal{S}_{C_{2m+1}}: \quad \theta — \lambda_1 — \lambda_2 \cdots\cdots \lambda_{m-1} — \lambda_m$$

$$\mathcal{S}_{D^*_{2m}}: \quad \begin{matrix} \theta \overset{1}{\searrow} \\ \eta_1 \underset{1}{\nearrow} \end{matrix} \alpha_1 \underset{(2|2)}{—} \alpha_2 \cdots\cdots \alpha_{m-1} \underset{(2|2)}{—} \alpha_m \begin{matrix} \overset{1}{\nearrow} \eta_2 \\ \underset{1}{\searrow} \eta_3 \end{matrix}$$

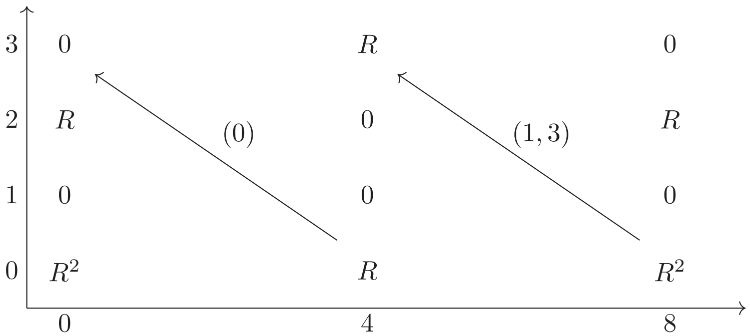
$$\mathcal{S}_{D^*_{2m+1}}: \quad \begin{matrix} \theta \overset{1}{\searrow} \\ \eta \underset{1}{\nearrow} \end{matrix} \alpha_1 \underset{(2|2)}{—} \alpha_2 \cdots\cdots \alpha_{m-1} \underset{(2|2)}{—} \alpha_m \underset{2}{—} \lambda$$

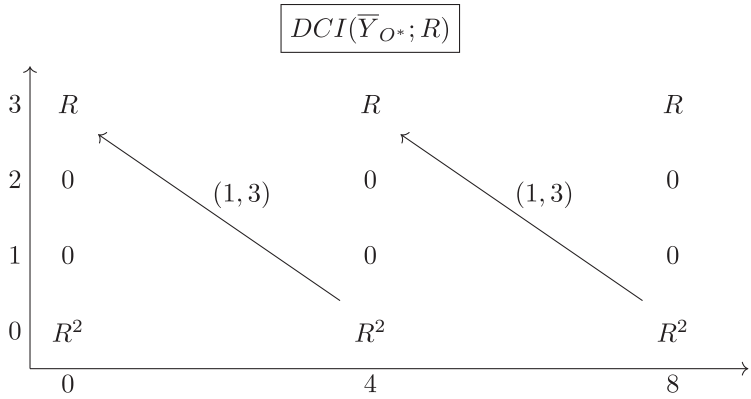
In all cases θ denotes the trivial representation. The letters α and β are reserved for representation of quaternionic type, λ for representations of complex type and η for representations of real type.

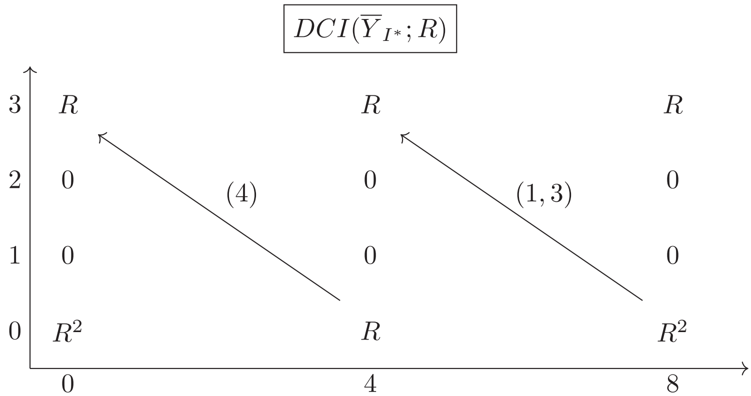
The notation for the 1-dimensional quaternionic representations is compatible with the notation in Appendix A.

Example I.4.26. The multicomplexes $(DCI(\overline{Y}_\Gamma)_{*,*}, \partial^4)$ for $\Gamma = T^*, O^*, I^*$ are shown below for (s,t) with $0 \le s \le 8$ and $0 \le t \le 3$.

$DCI(\overline{Y}_{T^*}; R)$







I.5 The Algebraic Construction of Equivariant Floer Homology

The equivariant instanton Floer homology associated with the trivial SU(2)-bundle over a closed oriented 3-manifold Y comes in three flavors: $I^+(Y)$, $I^-(Y)$, $I^\infty(Y)$, the positive, negative and Tate homology, respectively. These are all constructed algebraically from the complex $\widetilde{CI}(Y)$, the associated action of $C_*^{gm}(\mathrm{SO}(3); R)$ and the index filtration $\{F_p\widetilde{CI}(Y)\}_{p\in\mathbb{Z}}$.

The construction proceeds as follows. First, following [Eis19, Appendix A], we review the construction of four functors C_A^+, C_A^-, $C^{(+,tw)}$ and C_A^∞ from the category of right modules over a differential graded R-algebra A to the category of differential graded R-modules. In fact, $C_A^-(R)$ will carry the structure of a differential graded algebra and the three functors take values in the subcategory of left $C_A^-(R)$-modules. Under the assumptions that the ground ring R is a principal ideal domain and the algebra A is degreewise free over R, the four functors will be exact and preserve quasi-isomorphisms.

Next, we promote the functors to the category of filtered right A-modules. If M is a right A-module equipped with an increasing filtration

$$\cdots \subset F_pM \subset F_{p+1}M \subset \cdots \subset M$$

of A-submodules, the exactness of the functors ensure that $C_A^\bullet(M)$ is naturally filtered by $F_pC_A^\bullet(M) := C_A^\bullet(F_pM)$ for each$\bullet \in \{+,-,(+,tw),\infty\}$. However, this filtration may not be very well behaved. To remedy this issue we pass to the so-called full completion, thereby obtaining four new functors $\hat{C}_A^\bullet$, $\bullet \in \{(+,-,(+,tw),\infty\}$ from the category of filtered right A-modules to the category of filtered left $C_A^-(R)$-modules; to be specific,

$$\hat{C}_A^\bullet(M) = \lim_q \operatorname{colim}_p C_A^\bullet(F_pM/F_qM).$$

For $A = C_*^{gm}(\mathrm{SO}(3); R)$ the groups $I^\bullet(Y)$ for $\bullet \in \{+,-,\infty\}$ are obtained by applying $\hat{C}_A^\bullet$ to $M = \widetilde{CI}(Y)$ equipped with the index filtration and then passing to homology. These groups will then be modules over $H(C_A^-(R))$, which, provided $\frac{1}{2} \in R$, is isomorphic to the polynomial algebra $R[U]$ with a single generator U of degree -4. The mod 8 periodicity of M will be preserved so that $I_n^\bullet \cong I_{n+8}^\bullet$ for all $n \in \mathbb{Z}$.

The above constructions are also functorial in a suitable sense depending on $\bullet \in \{+,-,\infty\}$, in the differential graded algebra A. Moreover, this functoriality will preserve quasi-isomorphisms in the situation of interest; that is, we may equally well calculate $I^\bullet(Y)$ using $A = \Lambda_R[u]$ and $M = DCI(Y)$. When we proceed to the calculations for binary polyhedral spaces in the next section, we will indeed work in this context.

I.5.1 Conventions

Throughout R will be a fixed principal ideal domain. A module or algebra will always mean an R-module or R-algebra, and the same applies in the graded or differential graded setting. We will work with homological gradings and differentials ∂ of degree -1. We follow the conventions on graded and differential graded (DG) modules given in [Mac95, Chapter VI]. We briefly recall the most central concepts. If M and N are DG modules then $M \otimes N$ and $\mathrm{Hom}(M,N)$ are the DG modules given by

$$(M \otimes N)_n = \bigoplus_{p+q=n} M_n \otimes N_q \quad \text{and} \quad \mathrm{Hom}(M,N)_n = \prod_{q-p=n} \mathrm{Hom}(M_p, N_q)$$

with differentials $\partial_{M\otimes N} = \partial_M \otimes 1 + 1 \otimes \partial_N$ and $\partial_{\mathrm{Hom}(M,N)} = \mathrm{Hom}(1, \partial_N) - \mathrm{Hom}(\partial_M, 1)$. This means that $\partial_{M\otimes N}(a \otimes b) = \partial_M a \otimes b + (-1)^{|a|} a \otimes \partial_N b$ and $\partial_{\mathrm{Hom}(M,N)}(f) = \partial_N \circ f - (-1)^{|f|} f \circ \partial_M$, where we write $|x|$ for the degree of an element in a graded module. There is a natural interchange isomorphism $\tau : M \otimes N \cong N \otimes M$ given by $m \otimes n \mapsto (-1)^{|m||n|} n \otimes m$.

For each integer $p \in \mathbb{Z}$ let $R[p]$ denote the complex with a single copy of R concentrated in degree p. For a DG module M the shifted complex $M[p]$ is defined to be $R[p] \otimes M$. Therefore, $M[p]_{n+p} = M_n$ for all $n \in \mathbb{Z}$ with differential $\partial_{M[p]} = (-1)^p \partial_M$. It is also possible to define $M[p] = M \otimes R[p]$, but this will result in different sign conventions.

A differential graded algebra is a graded algebra A equipped with a differential satisfying the Leibniz rule $\partial_A(ab) = \partial_A(a)b + (-1)^{|a|} a \partial_A b$ for all $a, b \in A$. We will always assume that A is unital, i.e., there is a unit $1 \in A_0$ for the multiplication. A left or right A-module is a graded left or right A-module equipped with a differential satisfying a corresponding Leibniz rule. This ensures that $H(M)$ is a graded $H(A)$-module. If M is a left or right A-module then so is $M[p]$. For $a \in A$ and $x \in M[p]_n = M_{n-p}$ the induced product is given by $a \cdot x = (-1)^{|a|p} ax$ in the left module context and by $x \cdot a = xa$ in the right module context. This is a consequence of our left shifting (suspension) convention.

We also mention that there is a natural adjunction isomorphism of DG modules

$$\operatorname{Hom}(M \otimes N, P) \cong \operatorname{Hom}(M, \operatorname{Hom}(N, P))$$

given by $(f\colon M \otimes N \to P) \mapsto (g\colon M \to \operatorname{Hom}(N, P))$ where $f(m \otimes n) = g(m)(n)$. This isomorphism admits various natural generalizations to the case where M, N and P carry additional module structures over various DG algebras, see [Mac95, Chapter VI.8]. All the signs occurring in the upcoming theory can be deduced from these conventions: the interchange morphism, the natural adjunction and the left shift convention.

Finally, a homomorphism $f\colon M \to N$ of DG modules will mean a degree 0 chain map unless otherwise stated. We say that f is a quasi-isomorphism if the induced map in homology, $H(f)\colon H(M) \to H(N)$, is an isomorphism.

I.5.2 The Bar Construction and the Functors $C_A^{\pm}$

Let A be a unital DG algebra and assume that A is equipped with an augmentation $\epsilon\colon A \to R$, that is, ϵ is a map of DG algebras where R is regarded as a DG algebra concentrated in degree 0. The augmentation gives R the structure of an A-bimodule. We define $\bar{A} = A/R \cdot 1$ and note that there is a canonical identification of $\bar{A}$ with the augmentation ideal $\operatorname{Ker}(\epsilon)$. Given a right A-module M and a left A-module N define the bigraded module $B_{*,*}(M, A, N)$ by

$$B_{p,q}(M, A, N) = (M \otimes \bar{A}^{\otimes p} \otimes N)_q.$$

We adopt the standard notation and write

$$m[a_1|a_2|\cdots|a_p]n \coloneqq m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_p \otimes n \in B_{p,q}(M, A, N).$$

Here, p is the simplicial degree, $q = |m| + (\sum_{i=1}^p |a_i|) + |n|$ is the internal degree and $p + q$ is the total degree. Let $\partial^s\colon B_{p,q}(M, A, N) \to B_{p-1,q}(M, A, N)$ and

$\partial^i\colon B_{p,q}(M,A,N) \to B_{p,q-1}(M,A,N)$ act on $x = m[a_1|\cdots|a_p]n$ by

$$\begin{aligned}\partial^s(x) &= ma_1[a_2|\cdots|a_p]n + \sum_{i=1}^{p-1}(-1)^i m[a_1|\cdots|a_ia_{i+1}|\cdots|a_p]n \\ &\quad +(-1)^p m[a_1|\cdots|a_{p-1}]a_pn \\ \partial^i(x) &= (-1)^p(\partial_M m[a_1|\cdots|a_p]n + \sum_{i=1}^{p}(-1)^{\epsilon_{i-1}} m[a_1|\cdots|\partial a_i|\cdots|a_p]n \\ &\quad + (-1)^{\epsilon_p} m[a_1|\cdots|a_p]\partial_N n),\end{aligned}$$

where $\epsilon_i = |m| + |a_1| + \cdots + |a_i|$. Note that ∂^i is the usual tensor product differential in $M \otimes \bar{A}^{\otimes p} \otimes N$ twisted by a sign depending on the simplicial degree. We call ∂^s the simplicial differential and ∂^i the internal differential. It holds true that $(\partial^i)^2 = 0$, $(\partial^s)^2 = 0$ and $\partial^i\partial^s + \partial^s\partial^i = 0$. This ensures that the triple $(B_{*,*}(M,A,N), \partial^s, \partial^i)$ is a double complex.

Definition I.5.1. [GM74, Appendix A] The bar construction $B(M,A,N)$ is the total complex associated with the double complex $(B_{*,*}(M,A,N), \partial^s, \partial^i)$ introduced above, that is,

$$B_k(M,A,N) = \bigoplus_{p+q=n} B_{p,q}(M,A,N) \quad \text{and} \quad \partial = \partial^s + \partial^i.$$

The bar construction is functorial in the triple (M,A,N). For a triple $(f,g,h)\colon (M,A,N) \to (M',A',N')$ where g is a map of augmented DG algebras and f,h are chain maps linear along g, there is an induced map $B(f,g,h)\colon B(M,A,N) \to B(M',A',N')$ of DG modules. There is also a natural augmentation $B(M,A,N) \to M \otimes_A N$ given by $m[a_1|\cdots|a_p]n \mapsto 0$ for $p > 0$ and $m[]n \mapsto m \otimes n$.

Remark I.5.2. The DG module $B(M,A,A)$ inherits a right A-module structure from the last factor. The augmentation $B(M,A,A) \to M$ is a map of right A-modules and should be regarded as an analogue of a projective resolution of M. Moreover, $B(M,A,N) \cong B(M,A,A) \otimes_A N$, so that $B(M,A,N)$ is a derived tensor product $M \otimes_A^L N$ in a certain sense. These statements are made precise in the language of model categories in [BMR14, Section 10.2].

Following [Eis19] we introduce the notation

$$BA \coloneqq B(R,A,R) \quad \text{and} \quad EA \coloneqq B(R,A,A).$$

Furthermore, we write $[a_1|a_2|\cdots|a_p] \coloneqq 1_R[a_1|a_2|\cdots|a_p]1_R \in BA$ and similar conventions apply for $B(R,A,N)$ and $B(M,A,R)$.

Lemma I.5.3. *For any right A-module M and left A-module N there is a natural map of DG modules $\psi_{M,N}\colon B(M,A,N) \to B(M,A,R) \otimes B(R,A,N)$ given by*

$$m[a_1|\cdots|a_p]n \mapsto \sum_{i=0}^{p}(-1)^{(p-i)\epsilon_i} m[a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_p]n,$$

where $\epsilon_i = |m| + |a_1| + \cdots + |a_i|$*. The map* $\psi_{R,R}\colon BA \to BA \otimes BA$ *along with the augmentation* $BA \to R$ *give* BA *the structure of a DG coalgebra. The map* $\psi_{M,R}$ *gives* $B(M, A, R)$ *the structure of a right DG* BA*-comodule and the map* $\psi_{R,N}$ *gives* $B(R, A, N)$ *the structure of a left DG* BA*-comodule.*

Definition I.5.4. [Eis19, Definition A.5] Let A be an augmented DG algebra and let M be a right A-module. Define the positive and negative A-chains of M to be

$$C_A^+(M) \coloneqq B(M, A, R) \quad \text{and} \quad C_A^-(M) \coloneqq \mathrm{Hom}_A(EA, M).$$

The homology of $C_A^\pm(M)$ is denoted by $H_A^\pm(M)$ and is called the positive and negative A-homology of M, respectively.

The above constructions are functorial in the following sense. A homomorphism $f\colon A \to B$ of augmented DG algebras induces maps

$$f_*\colon C_A^+(f^{-1}M) \to C_B^+(M) \ \text{ and } \ f^*\colon C_B^-(M) \to C_A^-(f^{-1}M)$$

where M is a B-module and $f^{-1}M$ is the A-module obtained from M by restriction along f. Furthermore, a homomorphism $g\colon M \to N$ of A-modules induces maps

$$g^+\colon C_A^+(M) \to C_A^+(N) \ \text{ and } \ g^-\colon C_A^-(M) \to C_A^-(N).$$

The following invariance result is proven in [Eis19, Theorem A.1, A.2]; it relies on the standing assumption that R is a PID.

Proposition I.5.5. *Let* $f\colon A' \to A$ *be a homomorphism of augmented DG algebras and let* $g\colon M \to M'$ *be a homomorphism of* A*-modules.*

(i) *If* A *and* A' *are degreewise free and* f *is a quasi-isomorphism, then* f_* *and* f^* *are quasi-isomorphisms.*

(ii) *If* A *is degreewise free and* g *is a quasi-isomorphism, then* g^+ *and* g^- *are quasi-isomorphisms.*

The DG modules $C_A^\pm(M)$ carry additional structure. First, we have the following sequence of isomorphisms of DG modules

$$C_A^-(R) = \mathrm{Hom}_A(EA, \epsilon^{-1}(R)) \cong \mathrm{Hom}_R(EA \otimes_A R, R) \cong \mathrm{Hom}(BA, R),$$

which identifies $C_A^-(R)$ with the R-dual of BA. The DG coalgebra structure of BA dualizes to give $C_A^-(R)$ the structure of a DG algebra. Next, for any right A-module M the DG modules $C_A^\pm(M)$ obtain left $C_A^-(R)$-module structures in the following way. Write $BA^\vee = \mathrm{Hom}_R(BA, R) = C_A^-(R)$, then the structure map $C_A^-(R) \otimes C_A^+(M) \to C_A^+(M)$ is given by the following composition

$$\begin{aligned} BA^\vee \otimes C_A^+(M) &\xrightarrow{1\otimes\psi_M} BA^\vee \otimes C_A^+(M) \otimes BA \\ &\xrightarrow{\tau\otimes 1} C_A^+(M) \otimes BA^\vee \otimes BA \xrightarrow{1\otimes \mathrm{ev}} C_A^+(M) \end{aligned} \tag{I.21}$$

where $\psi_M = \psi_{M,R}$ is the comodule structure map of $C_A^+(M) = B(M, A, R)$ in Lemma I.5.3, τ is the interchange isomorphism $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$ and $\mathrm{ev}\colon BA^\vee \otimes BA \to R$ is the evaluation map. In the other case the structure map $C_A^-(R) \otimes C_A^-(M) \to C_A^-(M)$ is given by the following composition

$$BA^\vee \otimes \mathrm{Hom}_A(EA, M) \xrightarrow{\zeta} \mathrm{Hom}_A(BA \otimes EA, M) \xrightarrow{\psi_{EA}^*} \mathrm{Hom}_A(EA, M) \tag{I.22}$$

where $\zeta(f \otimes g)(a \otimes b) = (-1)^{|a||g|} f(a) \cdot g(b)$ and $\psi_{EA}^* = \mathrm{Hom}_A(\psi_{R,A}, 1)$ is the dual of the comodule structure map $\psi_{R,A}\colon EA \to BA \otimes EA$.

The properties of the functors $C_A^\pm$ that will be important for us are summarized in the following theorem. Here the morphisms in the category of right or left modules over a DG algebra A are taken to be the A-linear chain maps of degree 0.

Theorem I.5.6. *Let A be a degreewise R-free augmented DG algebra. Then the assignments*

$$M \mapsto C_A^\pm(M) \quad \text{and} \quad (g\colon M \to M') \mapsto (g^\pm\colon C_A^\pm(M) \to C_A^\pm(M')$$

define functors $C_A^\pm$ from the category of right A-modules to the category of left $C_A^-(R)$-modules. Moreover, these functors preserve quasi-isomorphisms and short exact sequences.

We will now treat the case $A = \Lambda_R[u]$ with $|u| = 3$ in detail. These calculations will be important for later purposes and will serve as an excellent example of how the above constructions work. We should note that all of these calculations easily generalize to the case where the degree of u is any odd number.

We will write $R[U]$ for the graded polynomial algebra on a single generator U of even non-zero degree. This is also a coalgebra with counit given by the identity in degree 0 and diagonal determined by

$$U^p \mapsto \sum_{i=0}^{p} U^i \otimes U^{p-i}$$

where $U^0 = 1$. This structure is compatible with the algebra structure and turns $R[U]$ into a Hopf algebra, but this will not be important for us.

Lemma I.5.7. *Let $A = \Lambda_R[u]$ with $|u| = 3$. Then there is an isomorphism of DG coalgebras $BA \cong R[V]$ where $|V| = 4$, and an isomorphism of DG algebras $C_A^-(R) = BA^\vee \cong R[U]$ where $|U| = -4$. In particular, these complexes have trivial differentials so that $H_A^+(R) = R[V]$ and $H_A^-(R) = R[U]$. The first isomorphism is given by*

$$V^p \mapsto (-1)^{p(p+1)/2} [u|\cdots|u] \quad (p \text{ times})$$

for $p \geq 0$. The second is obtained by dualization of the first where U^p is determined by the evaluation $U^p(V^p) = 1$ for all $p \geq 0$. Furthermore, the left $BA^\vee$-module structure of BA is given by $U^s \cdot V^p = V^{p-s}$, where we interpret $V^i = 0$ for $i < 0$.

Proof. We have $A = \Lambda_R[u]$ so that $A_0 = R$, $A_3 = Ru$ and $A_n = 0$ otherwise. The augmentation is necessarily given by the identity in degree 0. Hence, $\overline{A} = Ru$ concentrated in degree 3. From the definition of $B(R, A, R)$ we find that $BA_{p,3p} = R \cdot Z^p$ for $p \geq 0$, where $Z^p = [u|\cdots|u]$ with u repeated p times, and $B_{p,q} = 0$ otherwise. It follows immediately that the differential vanishes identically. Using the formula for the coalgebra structure map in Lemma I.5.3 we find that

$$\psi_{R,R}(Z^p) = \sum_{i=0}^{p} (-1)^{(p-i)3i} Z^i \otimes Z^{p-i}.$$

Define $V^p = (-1)^{p(p+1)/2} Z^p$ for $p \geq 0$. Then using the identity

$$\frac{p(p+1)}{2} = (p-i)i + \frac{i(i+1)}{2} + \frac{(p-i)(p-i+1)}{2}$$

we deduce that the above formula transforms into $\psi_{R,R}(V^p) = \sum_{i=0}^{p} V^i \otimes V^{p-i}$. Hence, $BA \cong R[V]$ as DG coalgebras with trivial differentials.

Define $U^p \in BA^\vee$ to be the dual of V^p, i.e., $U^p \colon BA \to R$ has degree $-4p$ and acts by $U^p(V^p) = 1$ and vanishes otherwise. The differential is still of course trivial. The product $U^s \cdot U^t$ has degree $-4(s+t)$ and must therefore be an R-multiple of U^{s+t}. The calculation

$$(U^s \cdot U^t)(V^{s+t}) = (U^s \otimes U^t)\left(\sum_{i=0}^{s+t} V^i \otimes V^{s+t-i}\right) = U^s(V^s)U^t(V^t) = 1$$

shows that $U^s \cdot U^t = U^{s+t}$. Hence, $C_A^-(R) \cong BA^\vee \cong R[U]$ as DG algebras with trivial differentials. Finally, we need to check that the left module structure $BA^\vee \otimes BA \to BA$ is given by $U^s \cdot V^p = V^{p-s}$. Using the definition we find

$$U^s \cdot V^p = \sum_{i=0}^{p} (-1)^{|U^s||V^i|} V^i \cdot U^s(V^{p-i}) = \begin{cases} V^{p-s} & \text{if } p \geq s \\ 0 & \text{otherwise.} \end{cases}$$

as required. ∎

We will also need explicit calculations of $H_A^{\pm}(M)$ when $M = H_*(\alpha; R)$ for $\alpha = \mathrm{SO}(3)/\mathrm{SO}(2)$ or $\alpha = \mathrm{SO}(3)$, under the assumption $\frac{1}{2} \in R$. In other words, $M = R \oplus R[2]$ with the trivial A-module structure or $M = A = \Lambda_R[u]$ (see (I.8)). For this purpose we give concrete models for the complexes $C_A^{\pm}(M)$. Note that a right $\Lambda_R[u]$-module M is simply a DG module equipped with a degree 3 map $u : M \to M$ satisfying $u^2 = 0$ and $u\partial_M = \partial_M u$ (because A acts from the right).

Proposition I.5.8. *Let M be a right $A = \Lambda_R[u]$-module. Define bigraded modules $D^{\pm}_{*,*}$ by*

$$D^+_{p,q} = \begin{cases} M_{q-3p} & p \geq 0 \\ 0 & p < 0 \end{cases} \quad \text{and} \quad D^-_{p,q} = \begin{cases} M_{q-3p} & p \leq 0 \\ 0 & p > 0 \end{cases}.$$

Define $U\colon D^{\pm}_{p,q} \to D^{\pm}_{p-1,q-3}$ *to be the identity* $M_{q-3p} \to M_{q-3p}$ *when* $p \geq 1$ *and* $p \leq 0$*, respectively, and otherwise to be zero. Define*

$$\partial' = (-1)^{p+q+1}u\colon D^{\pm}_{p,q} = M_{q-3p} \to M_{q-3p+3} = D^{\pm}_{p-1,q}$$
$$\partial'' = \partial_M\colon D^{\pm}_{p,q} = M_{q-3p} \to M_{q-3p-1} = D^{\pm}_{p,q-1},$$

in the range it makes sense. Then $(D^+_{*,*}, \partial', \partial'')$ *and* $(D^-_{*,*}, \partial', \partial'')$ *are double complexes and there are natural isomorphisms of DG* $R[U]$*-modules*

$$\mathrm{Tot}^{\oplus}(D^+_{*,*}, \partial', \partial'') \cong C^+_A(M) \quad \textit{and} \quad \mathrm{Tot}^{\Pi}(D^-_{*,*}, \partial', \partial'') \cong C^-_A(M).$$

Proof. We have $\bar{A} = R \cdot u$ with u in degree 3. Therefore, $B_{p,q} = (M \otimes \bar{A}^{\otimes p})_q \cong M_{q-3p} \otimes \bar{A}^{\otimes p}_{3p} \cong M_{q-3p}$. The isomorphism is given by $m \mapsto m[u|u|\cdots|u]$ with u repeated p times. For simplicity of notation write this as $m \otimes Z^p$. Then, using the formulas for the differentials, we find $\partial^s(m \otimes Z^p) = mu \otimes Z^{p-1}$ and $\partial^i(m \otimes Z^p) = (-1)^p \partial_M m \otimes Z^p$. The action of U is given by first applying the comodule structure map to $m \otimes Z^p$, and then evaluating U on the right factors in the resulting sum (U has even degree so the interchange introduces no sign). We have

$$\psi_{M,R}(m \otimes Z^p) = \sum_{i=0}^{p} (-1)^{(p-i)(|m|+3i)} (m \otimes Z^i) \otimes Z^{p-i}$$

so that $U \cdot (m \otimes Z^p) = (-1)^{|m|+p-1}(m \otimes Z^{p-1}) \cdot U(Z^1) = (-1)^{|m|+p} m \otimes Z^{p-1}$ as U was defined to be the dual of $V^1 = -Z^1$. It is now easy to check that when we introduce the sign change $m \in D^+_{p,q} = M_{q-3p} \mapsto (-1)^{p(p+1)/2+p|m|} m \otimes Z^p$ the U action and the differentials are given as in the proposition.

In the second case, $EA = B(R, A, A)$ has generators over R of the form $Z^p = [u|u|\cdots|u]$ and $Z^p \otimes u = [u|u|\cdots|u]u$ in degrees $4p$ and $4p+3$ respectively. It is convenient as earlier to introduce the sign twist $V^p = (-1)^{p(p+1)/2} Z^p$, since then the differential is given by $\partial(V^p) = V^{p-1} \otimes u$ and $\partial(V^p \otimes u) = 0$ and the comodule structure map $\psi_{R,A}\colon EA \to BA \otimes EA$ is given by

$$\psi_{R,A}(V^p) = \sum_{i=0}^{p} V^i \otimes V^{p-i}.$$

An element $f \in \mathrm{Hom}_A(EA, M)_n$ is by A-linearity uniquely determined by the elements $f(V^p) \in M_{n+4p}$ for $p \geq 0$. The differential is given by

$$(\partial f)(V^p) = \partial_M(f(V^p)) - (-1)^{|f|} f(\partial V^p) = \partial_M(f(V^p)) - (-1)^{|f|} f(V^{p-1})u$$

and the action of U is given by

$$(U \cdot f)(V^p) = (U \otimes f)(\psi_{R,A}(V^p)) = (U \otimes f)(V \otimes V^{p-1}) = f(V^{p-1}).$$

It is now easy to check that the complex $C^-_A(M) = \mathrm{Hom}_A(EA, M)$ can be described as the product totalization of the double complex given in the proposition. ■

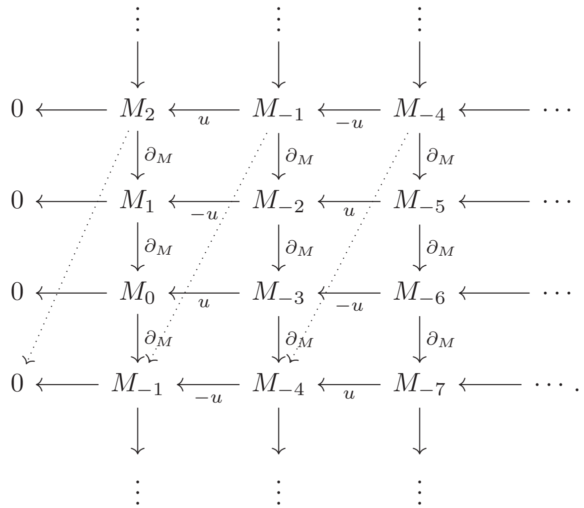
For later reference we record the following facts from the above proof. The isomorphism $\mathrm{Tot}^{\oplus}(D^+, \partial', \partial'') \cong C_A^+(M)$ is given by sending an element $m \in M_{q-3p} = D^+_{p,q}$ to

$$(-1)^{p(q-p)+p(p+1)/2} m \, \overbrace{[u|u|\cdots|u]}^{p} \in B(M, A, R)_{p,q} \tag{I.23}$$

and the isomorphism $C_A^-(M) \cong \mathrm{Tot}^{\Pi}(D^-, \partial', \partial'')$ is given by sending an element $f \in \mathrm{Hom}_A(EA, M)_n$ to

$$((-1)^{p(p+1)/2} f(\overbrace{[u|u|\cdots|u]}^{p}))_{p\geq 0} \in \prod_{p\geq 0} M_{n+4p} = \prod_{p\geq 0} D^-_{-p,n+p}. \tag{I.24}$$

It should be noted that there are other choices of signs possible. We have chosen the signs that ensure that the action of U corresponds to the identity map. A part of the double complex $(D^+_{*,*}, \partial', \partial'')$ corresponding to a right A-module M is shown below



The dotted arrows represent the action of U. An element x of $C_A^+(M)_n$ can be represented by a sequence $(m_{n-4p})_{p\geq 0}$, where almost all terms vanish. This element x is a cycle if and only if $\partial_M m_{n-4p} = (-1)^n m_{n-4(p-1)} u$ for each $p \geq 0$. Each column is a shifted copy of the original complex M. The double complex $(D^-_{*,*}, \partial', \partial'')$ is obtained by shifting the whole diagram from the right half plane over to the left half plane. A more concise representation of this double complex is given in the following diagram

$$\cdots \longleftarrow M[-9[\xleftarrow{\pm u} M[-6] \xleftarrow{\mp u} M[-3] \xleftarrow{\pm u} M \longleftarrow 0.$$

Note that in this case the internal differential in $M[-3n]$, $n \geq 0$, is not twisted by a sign, and that the $\pm u$ notation means that the sign depends on the internal degree.

With these concrete models in hand it is easy to calculate the following.

Corollary I.5.9. *We have* $H_A^+(A) \cong R$*,* $H_A^-(A) \cong R[3]$*, with necessarily trivial* $R[U]$*-module structure, and* $H_A^+(R \oplus R[2]) \cong R[W]$ *with* $|W| = 2$*,* $H_A^-(R \oplus R[2]) = R[Z][2]$ *with* $|Z| = -2$*. The* $R[U]$*-module structures are determined by* $U \cdot W^p = W^{p-2}$ *and* $U \cdot Z^p = Z^{p+2}$*.*

Remark I.5.10. The notation $R[W]$ and $R[Z][2]$ in the above result is only used to effectively express the underlying graded modules and the action of $R[U]$, it should not be interpreted to mean that $H_A^{\pm}(R \oplus R[2])$ carry any (co)algebra structure.

I.5.3 Tate Homology

We will now proceed to the construction of $C_A^{\infty}(M)$.

Definition I.5.11. For an augmented DG algebra A define the dualizing object by

$$D_A \coloneqq C_A^-(A) = \mathrm{Hom}_A(EA, A).$$

The dualizing object is a (left-left) $A - C_A^-(R)$-bimodule. The left $C_A^-(R)$-module structure is the one already given in (I.22), while the left A-module structure is obtained from the second factor in $D_A = \mathrm{Hom}_A(EA, A)$, that is, $(a \cdot f)(x) := af(x)$ for all $a \in A$, $f \in \mathrm{Hom}_A(EA, A)$ and $x \in EA$. This justifies the following definition.

Definition I.5.12. For a right A-module M define the twisted positive A-chains of M to be the DG module

$$C_A^{+,tw}(M) \coloneqq B(M, A, D_A).$$

The homology of this complex is denoted by $H_A^{+,tw}(M)$ and is called the positive twisted A-homology of M.

The complex $C_A^{+,tw}(M)$ carries a left $C_A^-(R)$-module structure induced from the left $C_A^-(R)$-module structure of D_A.

Definition I.5.13. The norm map $N_M \colon C_A^{+,tw}(M) \to C_A^-(M)$ is for each right A-module M defined to be the composite

$$B(M, A, D_A) \xrightarrow{\ \epsilon\ } M \otimes_A D_A \xrightarrow{\ \xi\ } \mathrm{Hom}_A(EA, M),$$

where ϵ is the augmentation and ξ is the adjoint of

$$M \otimes_A (D_A \otimes EA) \xrightarrow{1\otimes \mathrm{ev}} M \otimes_A A = M.$$

Explicitly, for $m[a_1|\cdots|a_p]v \in B(M, A, D_A)$ and $x \in EA$,

$$N_M(m[a_1|\cdots|a_p]v)(x) = \begin{cases} m \cdot v(x) & \text{for } p = 0 \\ 0 & \text{for } p > 0. \end{cases} \tag{I.25}$$

Lemma I.5.14. *The norm map* $N_M\colon C_A^{+,tw}(M) \to C_A^-(M)$ *is* $C_A^-(R)$*-linear and natural in* M*.*

Proof. The naturality in M is clear from the definition. To verify the $C_A^-(R)$-linearity let $f \in C_A^-(R)$ and $w = m[a_1|\cdots|a_p]u \in C_A^{+,tw}(M) = B(M, A, D_A)$. Then $f \cdot w = (-1)^\eta m[a_1|\cdots|a_p](f \cdot w)$ for $\eta = p + |m| + |a_1| + \cdots + |a_p|$ and therefore $N_M(f \cdot w) = 0 = f \cdot N_M(w)$ provided $p > 0$. We may therefore assume that $p = 0$ such that $w = m[]u$. Let $v = N_M(w)$ and note that $|v| = |m| + |u|$. Let $x \in EA$ and write $\psi_{R,A}(x) = \sum_i y_i \otimes z_i \in BA \otimes EA$ where $\psi_{R,A}$ is the map of Lemma I.5.3. Then by definition $(f \cdot v)(x) = (f \otimes v)(\psi_{R,A}(x))$, which equals

$$\sum_i (-1)^{|v||y_i|} f(y_i) v(z_i) = \sum_i (-1)^{|m||y_i| + |u||y_i|} m \cdot f(y_i) u(z_i). \tag{I.26}$$

On the other hand, $N_M(f \cdot m[]u) = (-1)^{|f||m|} N_M(m[](f \cdot u))$, and by (I.25) this coincides with

$$(-1)^{|f||m|} \sum_i m \cdot (f \otimes u)(y_i \otimes z_i) = \sum_i (-1)^{|f||m| + |u||y_i|} m \cdot f(y_i) u(z_i).$$

The final expression is equal to the final expression in (I.26) because $f\colon BA \to R$ so that $f(y_i)$ can only be nonzero if $|f| = -|y_i|$. Hence, $f \cdot N_M(w) = N_M(f \cdot w)$ and the proof is complete. ∎

The Tate complex $C_A^\infty(M)$ will be defined to be the mapping cone complex of the norm map. At a later stage we will also have to consider the cone of a map of nonzero degree, so we fix our conventions in the following definition.

Definition I.5.15. Let B be a DG algebra and suppose that $f\colon M \to N$ is a degree d chain map of left B-modules. This means that

(a) $f(M_n) \subset N_{n+d}$ for all $n \in \mathbb{Z}$,

(b) $f(bm) = (-1)^{|b|d} b f(m)$ for all $b \in B$ and $m \in M$ and

(c) $f \circ \partial_M = (-1)^d \partial_N \circ f$.

The left B-module Cone(f) is defined by $\mathrm{Cone}(f)_k = N_k \oplus M_{k-d-1}$ for all $k \in \mathbb{Z}$, with differential

$$\partial_{\mathrm{Cone}(f)} \cdot \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} \partial_N & (-1)^d f \\ 0 & (-1)^{d+1}\partial_M \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} \partial_N n + (-1)^d f m \\ (-1)^{d+1} \partial_M m \end{pmatrix}$$

and B-module structure $b \cdot (n, m) = (bn, (-1)^{|b|(d+1)} bm)$.

Remark I.5.16. If $B = R$ is concentrated in degree 0 and f has degree 0, then this definition agrees with [Mac95, p. 46]. The above definition is what we get when we regard f as a degree 0 map $f\colon M[d] \to N$ and take into account the sign twist in the B-action upon shifting M.

The cone complex sits in a natural short exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{Cone}(f) \longrightarrow M[d+1] \longrightarrow 0$$

of B-modules, whose connecting homomorphism $\delta\colon H(M)_n \to H_{n+d}(N)$ recovers $H(f)$ up to a sign. In particular, there is a natural exact triangle in homology.

Definition I.5.17. Let M be a right A-module. Then the Tate complex of M is defined to be the $C_A^-(R)$-module

$$C_A^\infty(M) \coloneqq \mathrm{Cone}(N_M\colon C_A^{+,tw}(M) \to C_A^-(M)).$$

The homology of this complex is denoted by $H_A^\infty(M)$ and is called the Tate homology of M.

Given a map $g\colon M \to M'$ of a right A-modules, there are induced maps

$$g^{+,tw}\colon C_A^{+,tw}(M) \to C_A^{+,tw}(M') \quad \text{and} \quad g^\infty\colon C_A^\infty(M) \to C_A^\infty(M')$$

of $C_A^-(R)$-modules. The key properties of the functors $C_A^{+,tw}$ and C_A^∞ are summarized in the following theorem.

Theorem I.5.18. *Let A be a degreewise R-free augmented DG algebra. Then the assignments*

$$(M \mapsto C_A^\bullet(M)) \ \ \textit{and} \ \ (g\colon M \to M') \mapsto (g^\bullet\colon C_A^\bullet(M) \to C_A^\bullet(M'))$$

for $\bullet \in \{(+,tw),\infty\}$, define functors $C_A^{+,tw}$, C_A^∞ from the category of right A-modules to the category of left $C_A^-(R)$-modules. They are both exact and preserve quasi-isomorphisms. Moreover, there is a natural exact triangle of $H_A^-(R)$-modules

$$\begin{array}{ccc} H_A^{+,tw}(M) & \xrightarrow{H(N_M)} & H_A^-(M) \\ & {\scriptstyle [-1]}\nwarrow \quad \swarrow & \\ & H_A^\infty(M). & \end{array}$$

Suppose $f\colon A \to B$ is a quasi-isomorphism of augmented DG algebras. For a given right B-module M there is no induced map between $C_A^\bullet(f^{-1}M)$ and $C_B^\bullet(M)$ for $\bullet \in \{(+,tw),\infty\}$. Rather one has to introduce an intermediate object to compare them.

Definition I.5.19. Let $f\colon A \to B$ be a homomorphism of augmented DG algebras. Define the relative dualizing object to be the (left-left) $C_A^-(R) - B$-bimodule

$$D_f \coloneqq C_A^-(f^{-1}B) = \mathrm{Hom}_A(EA, f^{-1}B),$$

where $f^{-1}B$ is regarded as a (right-left) $A - B$-bimodule.

Given a right B-module M define the relative positive twisted complex to be $C_f^{+,tw}(M) \coloneqq B(M, B, D_f)$ equipped with the left $C_A^-(R)$-module structure induced from D_f.

The relative positive twisted complex is a functor from the category of right B-modules to the category of left $C_A^-(R)$-modules. Given a map $g\colon M \to N$ of right B-modules the induced map is given by $B(g,1,1)\colon B(M,B,D_f) \to B(N,B,D_f)$.

Lemma I.5.20. *In the situation of the above definition, if A and B are degreewise free, the relative functor $C_f^{+,tw}$ preserves quasi-isomorphisms and short exact sequences.*

Proof. The assumptions that A and B are degreewise free along with the fact that R is PID ensure that the relative dualizing object D_f is degreewise flat. This is verified as in [Eis19, Lemma A.10]. It then follows from the invariance theorem [Eis19, Theorem A.1] that $C_f^{+,tw} = B(-,B,D_f)$ is an exact functor preserving quasi-isomorphisms. ■

Proposition I.5.21. *Let $f\colon A \to B$ be a homomorphism of augmented DG algebras and let M be a right B-module. There is a relative norm map $N_M^f\colon C_f^{+,tw}(M) \to C_A^-(f^{-1}M)$ fitting into the following commutative diagram, natural in M,*

$$\begin{array}{ccccc}
C_B^{+,tw}(M) & \xrightarrow{\beta_M} & C_f^{+,tw}(M) & \xleftarrow{\alpha_M} & C_A^{+,tw}(f^{-1}M) \\
\big\downarrow N_M^B & & \big\downarrow N_M^f & & \big\downarrow N_M^A \\
C_B^-(M) & \xrightarrow{f^*} & C_A^-(f^{-1}M) & \xleftarrow{=} & C_A^-(f^{-1}M).
\end{array}$$

All of the maps in the diagram are $C_A^-(R)$-linear, $C_B^-(R)$-linear or linear along $f^*: C_B^-(R) \to C_A^-(R)$, as appropriate.

Moreover, if f is a quasi-isomorphism and A and B are degreewise R-free, then all the horizontal maps in the diagram are quasi-isomorphisms.

Proof. Define $g\colon D_B \to D_f$ and $h\colon D_A \to D_f$ to be the maps

$$g = f^*\colon C_B^-(B) \to C_A^-(f^{-1}B) \quad \text{and} \quad h = f^-: C_A^-(A) \to C_A^-(f^{-1}B).$$

Note that D_A is a $C_A^-(R) - A$-bimodule, D_f is a $C_A^-(R) - B$-bimodule and D_B is a $C_B^-(R) - B$-bimodule. Moreover, observe that g is a homomorphism along $(f^*,1)\colon (C_B^-(R),B) \to (C_A^-(R),B)$ and that h is a homomorphism along $(1,f)\colon (C_A^-(R),A) \to (C_A^-(R),B)$. We may therefore define the upper horizontal maps in the diagram by

$$\begin{aligned}
\beta_M &\coloneqq B(1,1,g)\colon B(M,B,D_B) \to B(M,B,D_f) \\
\alpha_M &\coloneqq B(1,f,h)\colon B(f^{-1}M,A,D_A) \to B(M,B,D_f).
\end{aligned}$$

It is clear by construction that these maps are natural in M. Furthermore, by the above observations β_M is linear along $f^*\colon C_B^-(R) \to C_A^-(R)$ and α_M is $C_A^-(R)$-linear.

We define the relative norm map $N_M^f\colon B(M,B,D_f) \to C_A^-(f^{-1}M)$ to be the composition of the augmentation $B(M,B,D_f) \to M \otimes_B D_f$ and the map $\kappa\colon M \otimes_B D_f \to C_A^-(M)$ which is given as the adjoint of

$$M \otimes_B (\mathrm{Hom}_A(EA, f^{-1}B) \otimes EA) \xrightarrow{1\otimes \mathrm{ev}} M \otimes_B f^{-1}B \cong f^{-1}M.$$

Then N_M^B is $C_B^-(R)$-linear while N_M^f and N_M^A are $C_A^-(R)$-linear. Consider the diagram

$$\begin{array}{ccccc}
B(M,B,D_B) & \xrightarrow{B(1,1,g)} & B(M,B,D_f) & \xleftarrow{B(1,f,h)} & B(f^{-1}M,A,D_A) \\
\downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\
M \otimes_B D_B & \xrightarrow{1\otimes_B g} & M \otimes_B D_f & \xleftarrow{1\otimes_f h} & f^{-1}M \otimes_A D_A \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \xi} \\
\mathrm{Hom}_B(EB,B) & \xrightarrow{Ef^*} & \mathrm{Hom}_A(EA, f^{-1}M) & \xleftarrow{=} & \mathrm{Hom}_A(EA, f^{-1}M),
\end{array}$$

where ϵ denotes the augmentations and ξ are the natural maps in the definition of the norm map (see I.5.13). In particular, the outer compositions $\xi \circ \epsilon$ are the respective norm maps and by definition $\kappa \circ \epsilon = N_M^f$. We wish to verify that the diagram commutes. The two upper squares commute by the naturality of the augmentation in the bar construction. To verify that the lower squares commute, consider the diagram obtained by passing to the adjoints of the vertical maps ξ, κ and ξ in the bottom half of the diagram. The resulting diagram is then seen to commute using the fact that each of the maps ξ, κ and ξ was defined as the adjoint of $1 \otimes \mathrm{ev}$ for a suitable evaluation. By omitting the middle row of the above diagram, we obtain the commutative diagram in the statement of the proposition.

Finally, we need to verify that all the horizontal maps in the diagram are quasi-isomorphisms provided that $f\colon A \to B$ is a quasi-isomorphism of degreewise R-free algebras. We already know that this is true for $f^*\colon C_B^-(M) \to C_A^-(f^{-1}M)$ by Proposition I.5.5. For the upper horizontal maps, note first that $g = f^*$ and $h = f^-$ are quasi-isomorphisms by the same result. Then, since D_A, D_B and D_f are degreewise flat, one concludes by [Eis19, Theorem A.1] that $\beta_M = B(1,1,g)$ and $\alpha_M = B(1,f,g)$ are quasi-isomorphisms as well. This completes the proof. ∎

From this result one quickly obtains the required invariance result. Nevertheless, we will need the more refined statement of the above proposition in the next section.

Corollary I.5.22. *Let $f\colon A \to B$ be a quasi-isomorphism of degreewise R-free augmented DG algebras. Then for every right B-module M there are natural isomorphisms of $H_A^-(R) \cong H_B^-(R)$ modules*

$$H_A^{+,tw}(f^{-1}M) \cong H_B^{+,tw}(M) \quad \text{and} \quad H_A^{\infty}(f^{-1}M) \cong H_B^{\infty}(M).$$

Moreover, these isomorphisms are compatible with the exact triangles of Theorem I.5.18.

A DG algebra A is said to satisfy Poincaré duality of degree $d \in \mathbb{Z}$ provided there is a weak equivalence $A \simeq \operatorname{Hom}(A, R[d])$ of $A-A$-bimodules, where the right and left actions on the target are induced by the left action on A and the trivial left action on $R[d]$, respectively. Here, weak equivalence means that there exist a finite number $A = X_0, X_1, \cdots, X_r = \operatorname{Hom}(A, R[d])$ of A-bimodules and for each $0 \le i < r$ there is a quasi-isomorphism $X_i \to X_{i+1}$ or $X_{i+1} \to X_i$ of A-bimodules. According to [Eis19, Theorem A.19] there is under this hypothesis, an isomorphism $H_A^{+,tw}(M) \cong H_A^+(M)[n]$ of $H_A^-(R)$-modules. We will give a different proof of this result for $A = \Lambda_R[u]$ with $|u| = 3$. This case is sufficient for the application to equivariant instanton Floer homology when $\frac{1}{2} \in R$.

We will now consider these constructions for $A = \Lambda_R[u]$ with $|u| = 3$ as we did for $C_A^\pm$. The Tate complex $C_A^\infty(M)$, even in this simple case, is big and difficult to handle explicitly. To remedy this we will construct a much simpler and very explicit complex that also computes the Tate homology $H_A^\infty(M)$. Indeed, we will show that if we take one of the double complexes $(D^\pm_{*,*}, \partial', \partial'')$ of Proposition I.5.8 and extend them in the natural way to the whole plane, then a suitable totalization of this complex will do the trick.

Lemma I.5.23. *Let $A = \Lambda_R[u]$ with $|u| = 3$. Then there is an isomorphism of left A-modules $A \otimes R[\alpha] \cong D_A$ where $|\alpha| = -4$. Under this isomorphism the differential on $A \otimes R[\alpha]$ is given by $\partial(1 \otimes \alpha^p) = -u \otimes \alpha^{p+1}$ and $\partial(u \otimes \alpha^p) = 0$ for $p \ge 0$, and the action of $R[U] = C_A^-(R)$ is given by $U \cdot (a \otimes \alpha^p) = a \otimes \alpha^{p+1}$ for $a \in A$ and $p \ge 0$.*

Furthermore, the map $\rho\colon R \to A \otimes R[\alpha]$ of degree 3 given by $1 \mapsto u \otimes \alpha^0$ is an A-linear quasi-isomorphism.

Proof. Using the explicit model of Theorem I.5.8 we see that

$$(D_A)_n = C_A^-(A)_n \cong \begin{cases} A_0 & \text{for } n = 4m, m \le 0 \\ A_3 & \text{for } n = 4m+3, m \le 0 \\ 0 & \text{otherwise} \end{cases}$$

and a quick check shows that these identifications are compatible with the left A-module structure. The required isomorphism is therefore obtained by letting $1 \otimes \alpha^p$ and $u \otimes \alpha^p$ correspond to the generators $1 \in A_0 \cong C_A^-(A)_{-4p}$ and $u \in A_3 = C_A^-(A)_{-4p+3}$, respectively, for $p \ge 0$. The same result shows that the differential and $R[U]$-action are determined by the formulas $\partial(u \otimes \alpha^p) = 0$, $\partial(1 \otimes \alpha^p) = -u \otimes \alpha^{p+1}$ and $U \cdot (a \otimes \alpha^p) = a \otimes \alpha^{p+1}$ for $a \in A$ and $p \ge 0$.

From this description we find that $\partial\colon (D_A)_{-4p} \to (D_A)_{-4(p+1)-3}$ is an isomorphism for each $p \ge 0$. Therefore, the map $\rho\colon R \to A \otimes R[\alpha]$ of degree 3 given by $1 \mapsto u \otimes \alpha^0$ is an A-linear quasi-isomorphism. ■

In the following result we establish the isomorphism $H_A^{+,tw}(M) \cong H_A^+(M)[3]$ of $R[U]$-modules for each right A-module M. It also contains the key ingredients needed to establish the simplified model for $C_A^\infty(M)$.

Lemma I.5.24. *Let $A = \Lambda_R[u]$ with $|u| = 3$ and let M be a right A-module. Then there is a quasi-isomorphism $f_M\colon C_A^+(M) \to C_A^{+,tw}(M)$ of degree 3 and a chain homotopy $s_M\colon C_A^+(M) \to C_A^{+,tw}(M)$ of degree 0 such that*

$$f_M \circ U - U \circ f_M = \partial \circ s_M - s_M \circ \partial.$$

In particular, $H(f_M)\colon H_A^+(M) \to H_A^{+,tw}(M)$ is a degree 3 isomorphism of $R[U]$-modules. Furthermore, f_M and s_M are natural in the module M, and in terms of the explicit models $C_A^+(M) \cong \mathrm{Tot}^{\oplus}(D^+_{*,*}, \partial', \partial'')$ and $C_A^-(M) \cong \mathrm{Tot}^{\Pi}(D^-_{*,*}, \partial', \partial'')$ of Proposition I.5.8 the compositions $N_M \circ f_M$ and $N_M \circ s_M$ are given in degree n by

$$\bigoplus_{p\geq 0} M_{n-4p} \xrightarrow{\ \pi\ } M_n \xrightarrow{(-1)^n u} M_{n+3} \xrightarrow{\ \iota\ } \prod_{q\geq 0} M_{n+3+4q}$$

$$\bigoplus_{p\geq 0} M_{n-4p} \xrightarrow{\ \pi\ } M_n \xrightarrow{\ \iota\ } \prod_{q\geq 0} M_{n+4q},$$

respectively.

Proof. Recall that $C_A^+(M) = B(M, A, R)$ and $C_A^{+,tw}(M) = B(M, A, D_A)$. It is convenient to introduce the notation

$$\begin{aligned} m \otimes V^p &:= (-1)^{p(p+1)/2} m[u|u|\cdots|u] \in B(M, A, R) \\ m \otimes V^p \otimes \psi &:= (-1)^{p(p+1)/2} m[u|u|\cdots|u]\psi \in B(M, A, D_A) \end{aligned}$$

for $m \in M$ and $\psi \in D_A$, where u is repeated p times in both formulas. In terms of this notation the differentials and actions of $U \in R[U]$ of $B(M, A, R)$ and $B(M, A, D_A)$ are given by the formulas

$$\begin{aligned} U(m \otimes V^p) &= (-1)^{|m|} m \otimes V^{p-1} \\ U(m \otimes V^p \otimes \psi) &= m \otimes V^p \otimes (U\psi) \\ \partial(m \otimes V^p) &= (-1)^p(\partial_M m + mu \otimes V^{p-1}) \\ \partial(m \otimes V^p \otimes \psi) &= (-1)^p(\partial_M m \otimes V^p + mu \otimes V^{p-1}) \otimes \psi \\ &\quad + m \otimes (V^{p-1} \otimes u\psi + (-1)^{|m|} V^p \otimes \partial_{D_A}\psi). \end{aligned}$$

By the previous lemma there is an isomorphism $D_A \cong A \otimes R[\alpha]$ with $|\alpha| = -4$. To simplify the notation write $a\alpha^p := a \otimes \alpha^p$ for $a \in A$. There is also an A-linear quasi-isomorphism $\rho : R \to D_A$ of degree 3 given by $\rho(1) = u\alpha^0$. This map induces a quasi-isomorphism

$$f_M := B(1, 1, \rho)\colon B(M, A, R) \to B(M, A, D_A)$$

of degree 3, which in terms of the above notation is given by $f_M(m \otimes V^p) = (-1)^{|m|} m \otimes V^p \otimes u\alpha^0$ (the sign is forced upon us by the fact that we regard ρ as a map of degree 3). Using the above formulas we find that

$$(f_M \circ U - U \circ f_M)(m \otimes V^p) = m \otimes V^{p-1} \otimes u\alpha^0 - (-1)^{|m|} m \otimes V^p \otimes u\alpha^1. \quad \text{(I.27)}$$

Define $s_M\colon B(M,A,R) \to B(M,A,D_A)$ of degree 0 by $s_M(m \otimes V^p) = m \otimes V^p \otimes \alpha^0$. A straightforward calculation using the above formulas and the formula for the differential in $D_A \cong A \otimes R[\alpha]$ given in the previous lemma shows that

$$\begin{aligned}\partial s_M(m \otimes V^p) &= (-1)^p(\partial_M m \otimes V^p \otimes \alpha^0 + mu \otimes V^{p-1} \otimes \alpha^0)\\ &\quad + m \otimes V^{p-1} \otimes u\alpha^0 - (-1)^{|m|} m \otimes V^p \otimes u\alpha^1\\ s_M\partial(m \otimes V^p) &= (-1)^p(\partial_M m \otimes V^p \otimes \alpha^0 + mu \otimes V^{p-1} \otimes \alpha^0).\end{aligned}$$

Combining this with (I.27) we obtain $f_M \circ U - U \circ f_M = \partial s_M - s_M \partial$ as required.

Finally, we need to consider the compositions of f_M and s_M with the norm map $N_M\colon C_A^{+,tw}(M) \to C_A^-(M)$ in terms of the models given in Proposition I.5.8. By (I.23) $m \in M_{q-3p} = D^+_{p,q}$ corresponds to $(-1)^{|m|p} m \otimes V^p$, which is mapped by f_M to $(-1)^{|m|(p+1)} m \otimes V^p \otimes u\alpha^0$. The norm map kills off all such elements with $p > 0$ and sends $m \otimes V^0 \otimes u\alpha^0 \mapsto mu \cdot \alpha^0$. This is the functional that sends V^0 to $mu \in M_{q+3}$ and vanishes on V^q for $q > 0$. By (I.24) the identification $C_A^-(M)_n \cong \prod_{q\geq 0} M_{n+4q}$ is given by $\psi \mapsto (\psi(V^q))_{q\geq 0}$. We conclude that $N_M \circ f_M$ is given by the composition in the statement of the lemma. The verification for $N_M \circ s_M$ is analogous. ■

We need two simple but slightly technical results before we can reach our goal.

Lemma I.5.25. *Let X and Y be DG $R[U]$-modules and suppose that $f\colon X \to Y$ is a degree d map of $R[U]$-modules up to homotopy; that is, f is a chain map of degree d and there is a chain homotopy $s\colon X \to Y$ such that $fU - Uf = \partial_Y s + (-1)^d s\partial_X$. Let $\mathrm{Cone}(f)$ be given as in Definition I.5.15, except that we redefine the $R[U]$-module structure by setting*

$$U \cdot \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} U & (-1)^{|s|}s \\ 0 & U \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} Uy - (-1)^d sx \\ Ux \end{pmatrix}$$

for all $(y,x) \in Y_n \oplus X_{n-d-1} = \mathrm{Cone}(f)_n$. Then $\mathrm{Cone}(f)$ is a DG $R[U]$-module and the maps in the natural short exact sequence

$$0 \longrightarrow Y \longrightarrow \mathrm{Cone}(f) \longrightarrow X[d+1] \longrightarrow 0$$

are (strict) maps of $R[U]$-modules.

Proof. It suffices to check that $\partial_{\mathrm{Cone}(f)}U = U\partial_{\mathrm{Cone}(f)}$. This follows easily from the relations $\partial_X U = U\partial_X$, $\partial_Y U = U\partial_Y$ and $fU - Uf = \partial_Y s + (-1)^d s\partial_X$. Next, the maps in the natural exact sequence are given degreewise by the inclusion $Y_n \to Y_n \oplus X_{n-d-1}$ and the projection $Y_n \oplus X_{n-d-1} \to X_{n-d-1}$ and are seen to commute with the action of U. ■

Lemma I.5.26. *Let $f\colon X \to Y$ be a degree 3 map of DG $R[U]$-modules up to homotopy, and let $g\colon Y \to Z$ be a degree 0 map of DG $R[U]$-modules. Then*

$g \circ f : X \to Z$ is a degree 3 map of $R[U]$-modules up to homotopy and there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \longrightarrow & \mathrm{Cone}(g\circ f) & \longrightarrow & X[4] & \longrightarrow & 0 \\
 & & \big\downarrow{=} & & \big\downarrow{\scriptstyle \mathrm{Cone}(f,1)} & & \big\downarrow{\scriptstyle f[1]} & & \\
0 & \longrightarrow & Z & \longrightarrow & \mathrm{Cone}(g) & \longrightarrow & Y[1] & \longrightarrow & 0
\end{array}$$

with short exact rows. If $\mathrm{Cone}(g \circ f)$ is given the DG $R[U]$-module structure of the previous lemma, then the maps $\mathrm{Cone}(f, 1)$ and $f[1]$ are maps of DG $R[U]$-modules up to homotopy and all the other maps in the diagram are strict maps of DG $R[U]$-modules.

Proof. By assumption there is a chain homotopy $s \colon X \to Y$ of degree 0 such that $fU - Uf = \partial s - s\partial$. Then for $h \coloneqq g \circ f$ and $t \coloneqq g \circ s$ we have $hU - Uh = \partial t - t\partial$ as well. This shows that h is a map of $R[U]$-modules up to homotopy. By functoriality of the cone construction we obtain a commutative diagram of DG modules with short exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \longrightarrow & \mathrm{Cone}(g\circ f) & \longrightarrow & X[4] & \longrightarrow & 0 \\
 & & \big\downarrow{=} & & \big\downarrow{\scriptstyle \mathrm{Cone}(f,1)} & & \big\downarrow{\scriptstyle f[1]} & & \\
0 & \longrightarrow & Z & \longrightarrow & \mathrm{Cone}(g) & \longrightarrow & Y[1] & \longrightarrow & 0.
\end{array}$$

Here, $\mathrm{Cone}(f, 1) \colon \mathrm{Cone}(g \circ f)_n = Z_n \oplus X_{n-4} \to Z_n \oplus Y_{n-1} = \mathrm{Cone}(g)_n$ is given by $(z, x) \mapsto (z, -fx)$, and $f[1] \colon X[4]_n = X_{n-4} \to Y_{n-1} = Y[1]_n$ is given by $x \mapsto -fx$. Equip $\mathrm{Cone}(g \circ f)$ with the adjusted $R[U]$-module structure given in the previous lemma. Then all the horizontal maps in the diagram are strict maps of $R[U]$-modules. The right hand vertical map is a map of $R[U]$-modules up to homotopy, so the only thing we have to verify is that $\mathrm{Cone}(f, 1)$ is a map of $R[U]$-modules up to homotopy.

We claim that $v \colon \mathrm{Cone}(h) \to \mathrm{Cone}(g)$ given by $v(z, x) = (0, sx)$ is the required homotopy. Indeed, using matrix notation for the various maps,

$$\begin{aligned}
C(f,1)U - UC(f,1) &= \begin{pmatrix} 1 & 0 \\ 0 & -f \end{pmatrix}\begin{pmatrix} U & t \\ 0 & U \end{pmatrix} - \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -f \end{pmatrix} \\
&= \begin{pmatrix} 0 & t \\ 0 & Uf - fU \end{pmatrix} \\
\partial_{\mathrm{Cone}(g)} v + v\partial_{\mathrm{Cone}(h)} &= \begin{pmatrix} \partial_Z & g \\ 0 & -\partial_Y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} \partial_Z & -h \\ 0 & \partial_X \end{pmatrix} \\
&= \begin{pmatrix} 0 & g \circ s \\ 0 & s\partial_X - \partial_Y s \end{pmatrix},
\end{aligned}$$

where $C(f, 1)$ is shorthand for $\mathrm{Cone}(f, 1)$. Therefore, as $t = g \circ s$ and $\partial_Y s - s\partial_X s = fU - Uf$, we deduce that $\mathrm{Cone}(f, 1)$ is $R[U]$-linear up to homotopy. ∎

Given a sequence of R-modules $\{D_s\}_{s\in\mathbb{Z}}$ we write $\prod_{s\to-\infty} D_s$ for the submodule of $\prod_s D_s$ consisting of the sequences $(x_s)_{s\in\mathbb{Z}}$ for which there exists s_0 such that $x_s = 0$ for all $s \geq s_0$. Similarly, $\prod_{s\to\infty} D_s$ is the submodule of $(x_s)_{s\in\mathbb{Z}}$ with $x_s = 0$ for all sufficiently small s.

Definition I.5.27. Given a double complex $E = (D_{*,*}, \partial', \partial'')$, define $\operatorname{Tot}^{\Pi,-\infty}(E)$ and $\operatorname{Tot}^{\Pi,+\infty}(E)$ to be the complexes that in degree n are given by

$$\prod_{s\to-\infty} D_{s,n-s} \quad\text{and}\quad \prod_{s\to\infty} D_{s,n-s}$$

respectively, with differential $\partial = \partial' + \partial''$.

Theorem I.5.28. *Let $A = \Lambda_R[u]$ with $|u| = 3$ and let M be a right DG A-module. Define $D^\infty_{p,q} = M_{q-3p}$ for all $p, q \in \mathbb{Z}$. Set*

$$\begin{aligned}\partial' = (-1)^{p+q+1}u\colon D^\infty_{p,q} = M_{q-3p} \to M_{q-3p+3} = D^\infty_{p-1,q}\\ \partial'' = \partial_M\colon D^\infty_{p,q} = M_{q-3p} \to M_{q-3p-1} = D^\infty_{p,q-1}\end{aligned}$$

and let $U\colon D^\infty_{p,q} = M_{q-3p} \to M_{q-3p} = D^\infty_{p-1,q-3}$ be the identity for all p, q. Then there is a quasi-isomorphism, natural in M,

$$\operatorname{Tot}^{\Pi,-\infty}(D_{*,*}, \partial', \partial'') \to C^\infty_A(M),$$

which is a homomorphism of $R[U]$-modules up to homotopy.

Proof. By Lemma I.5.24 there is a quasi-isomorphism $f = f_M\colon C^+_A(M) \to C^{+,tw}_A(M)$ and a homotopy $s = s_M\colon C^+_A(M) \to C^{+,tw}_A(M)$ such that $fU - Uf = \partial s - s\partial$. We apply Lemma I.5.26 for $f\colon C^+_A(M) \to C^{+,tw}_A(M)$ and the norm map $N_M\colon C^{+,tw}_A(M) \to C^-_A(M)$ to obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^-_A(M) & \longrightarrow & \operatorname{Cone}(N_M\circ f) & \longrightarrow & C^+_A(M)[4] & \longrightarrow & 0\\ & & \big\downarrow{=} & & \big\downarrow{\operatorname{Cone}(f,1)} & & \big\downarrow{f[1]} & & \\ 0 & \longrightarrow & C^-_A(M) & \longrightarrow & C^\infty_A(M) & \longrightarrow & C^{+,tw}_A(M)[1] & \longrightarrow & 0,\end{array} \tag{I.28}$$

where we recall that $C^\infty_A(M) = \operatorname{Cone}(N_M)$. By the same result $\operatorname{Cone}(f,1)$ is a map of $R[U]$-modules up to homotopy. As both the outer vertical maps are quasi-isomorphisms, it follows by passage to the long exact sequences in homology and the 5-lemma that $\operatorname{Cone}(f,1)$ is a quasi-isomorphism. Thus, $\operatorname{Cone}(f,1)$ is the required map. This quasi-isomorphism is natural in M, since both f and N_M are natural transformations in M and the cone construction is functorial.

The remaining task is to establish an isomorphism $\operatorname{Cone}(N_M\circ f) \cong \operatorname{Tot}^{\Pi,-\infty}(D^\infty_{*,*}, \partial', \partial'')$. Using the identifications of Proposition I.5.8 we obtain

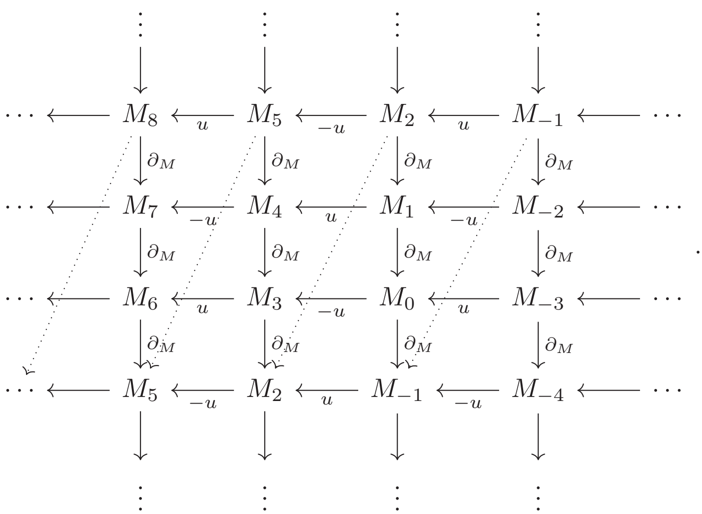
$$\operatorname{Cone}(N_M\circ f)_n = C^-_A(M)_n \oplus C^+_A(M)_{n-4} \cong \left(\prod_{q\geq 0} M_{n+4q}\right) \oplus \left(\bigoplus_{p\geq 0} M_{n-4(p+1)}\right)$$

for each $n \in \mathbb{Z}$. Moreover, by Lemma I.5.24 $h \coloneqq N_M \circ f$ and $t \coloneqq N_M \circ s$ are given on $(x_p)_{p\geq 0} \in C_A^+(M)_n$ by $h((x_p)_p) = ((-1)^{|x_0|}x_0 u, 0, \cdots)$ and $t((x_p)_{p\geq 0}) = (x_0, 0, 0, \cdots)$. Therefore, using the formulas

$$\partial_{\mathrm{Cone}(h)} = \begin{pmatrix} \partial_{C_A^-(M)} & -h \\ 0 & \partial_{C_A^+(M)} \end{pmatrix} \text{ and } U = \begin{pmatrix} U & t \\ 0 & U \end{pmatrix},$$

as well as the description of the differentials and the U-action in terms of the models for $C_A^{\pm}(M)$, we deduce that $\mathrm{Cone}(h) \cong \mathrm{Tot}^{\Pi,-\infty}(D^{\infty}_{*,*}, \partial', \partial'')$. ■

A part of the double complex $(D^{\infty}_{*,*}, \partial', \partial'')$ is shown below.



This is indeed the natural concatenation of the two double complexes $D^+_{*,*}$ and $D^-_{*,*}$.

Corollary I.5.29. *Let M be a right $A = \Lambda_R[u]$-module. Then $U \colon H_A^{\infty}(M) \to H_A^{\infty}(M)$ is an isomorphism.*

Proof. $U \colon \mathrm{Tot}^{\Pi,-\infty}(D^{\infty}_{*,*}, \partial', \partial'') \to \mathrm{Tot}^{\Pi,-\infty}(D^{\infty}_{*,*}, \partial', \partial'')$ is an isomorphism. ■

As in the cases $C_A^{\pm}$ we will need calculations for $M = R$, $M = R \oplus R[2]$ and $M = \Lambda_R[u]$. This is easily achieved with the help of the above explicit model.

Corollary I.5.30. *We have*

$$H_A^{\infty}(A) = 0, \quad H_A^{\infty}(R) = R[T, T^{-1}] \quad \textit{and} \quad H_A^{\infty}(R \oplus R[2]) = R[S, S^{-1}],$$

where $|T| = -4$, $|S| = -2$ and the $R[U]$-action is given by $U \cdot T^i = T^{i+1}$ and $U \cdot S^i = S^{i+2}$ for all $i \in \mathbb{Z}$.

We note once again that our notation should not be interpreted to mean that there is any product structure in $H_A^{\infty}(R \oplus R[2])$ or $H_A^{\infty}(R)$.

I.5.4 The Equivariant Instanton Floer Groups

In this section we will give the definition of the various flavors of equivariant instanton Floer homology. Before we can get to this we have to recall a few definitions concerning filtered DG modules. Let M be a DG module equipped with an increasing filtration

$$\cdots \subset F_{p-1}M \subset F_pM \subset F_{p+1}M \subset \cdots \subset M$$

by DG submodules. The minimal filtration quotients $F_pM/F_{p-1}M$ enter many times in the following discussion, so we introduce the simpler notation $\overline{F}_pM := F_pM/F_{p-1}M$. The filtration gives rise to a homological spectral sequence with

$$E^1_{p,q} = H_{p+q}(\overline{F}_pM) \implies H_{p+q}(M) \tag{I.29}$$

(we will elaborate on this in the next section). However, without any additional information, this spectral sequence may fail to converge in any sense to the desired target $H(M)$.

The following properties of a filtration are highly desirable in order to achieve any reasonable convergence properties of the above spectral sequence. The filtration $\{F_pM\}_p$ of M is said to be [Boa99, Definition 2.1]

(i) exhaustive if $\operatorname{colim}_p F_pM = \bigcup_p F_pM = M$,

(ii) Hausdorff if $\lim_p F_pM = \bigcap_p F_pM = 0$ and

(iii) complete if $\operatorname{Rlim}_p F_pM = 0$.

Here Rlim is the first (and only nonzero) right derived functor of lim (see [Boa99, Section 1] for additional details). It is important to note that these limits are calculated degreewise. In particular, if the filtration is degreewise bounded below, that is, for each degree n there exists $p_0(n)$ such that $F_pM_n = 0$ for all $p \leq p_0$, then the filtration is automatically complete Hausdorff. Similarly, if the filtration is degreewise bounded above then the filtration is automatically exhaustive.

Definition I.5.31. Let M be a DG module equipped with an increasing filtration $\{F_pM\}_p$ by DG submodules. Then the full completion of $(M, \{F_pM\}_p)$ is defined to be the complex

$$\hat{M} := \lim_q \operatorname{colim}_p F_pM/F_qM$$

equipped with the filtration $F_p\hat{M} := \lim_q F_pM/F_qM$.

Remark I.5.32. The standard completion of a filtered module M is typically defined by $\lim_q M/F_qM$ (see for instance [Boa99, p. 2.7]) and is naturally filtered by $\{\lim_q F_pM/F_qM\}_p$. We have therefore chosen to use the terminology full completion to distinguish it from this notion. The lemma below demonstrates that the two notions coincide if the filtration of M is exhaustive.

A map $f\colon M \to M'$ between filtered DG modules is filtered if $f(F_pM) \subset F_pM'$ for all $p \in \mathbb{Z}$. There is therefore a category of filtered DG modules and filtered maps. A sequence $M' \to M \to M''$ of filtered DG modules and filtered maps is short exact if $F_pM' \to F_pM \to F_pM''$ is short exact for each $p \in \mathbb{Z}$. The relevant properties of the full completion are summarized in the following lemma.

Lemma I.5.33. *Let M be a filtered DG module. Then the following holds true.*

(i) *The filtration $F_p\hat{M} = \lim_{q<p} F_pM/F_qM$ of $\hat{M}$ is exhaustive and complete Hausdorff.*

(ii) *For each $q < p$ there is a natural isomorphism $F_pM/F_qM \to F_p\hat{M}/F_q\hat{M}$.*

(iii) *If the filtration of M is exhaustive, then $\hat{M} = \lim_q M/F_qM$.*

(iv) *If the filtration of M is complete Hausdorff, then $\hat{M} = \operatorname{colim}_p F_pM = \bigcup_p F_pM$.*

The full completion $M \mapsto \hat{M}$ defines a functor from the category of filtered modules to itself, which preserves short exact sequences.

Proof. Let M be a filtered DG module and set $N = \operatorname{colim}_p F_pM = \bigcup_p F_pM$ filtered by $F_pN = F_pM$. Then the full completion of M coincides with the standard completion of N, i.e., $\hat{M} = \lim_q N/F_qN$. Therefore part (i) and (ii) follow from the corresponding properties of the standard completion (see [Boa99, Proposition 2.8]). If the filtration of M is exhaustive, then $\operatorname{colim}_p F_pM/F_qM = M/F_qM$ for each q and part (iii) easily follows. For part (iv)

$$\hat{M} = \lim_q \operatorname{colim}_p F_pM/F_qM = \lim_q N/F_qN = N = \operatorname{colim}_p F_pM.$$

It is clear that $M \mapsto \hat{M}$ is functorial with respect to filtered maps. The exactness of $\hat{M}$ follows from the exactness of sequential colimits and exactness of the standard completion in the category of R-modules (see for instance [AM69, Corollary 10.3]). ∎

Remark I.5.34. If B is a DG algebra and M is a left or right B-module equipped with a filtration by B-submodules the definition of the full completion and the above lemma apply without change. The reason for this is that quotients, limits and colimits are formed in the category of graded modules, with the B-action and differentials carried along.

Recall from the previous section that for a degreewise R-free augmented DG algebra A, four functors $C_A^\bullet$, $\bullet \in \{+, -, (+, tw), \infty\}$, from the category of right A-modules to the category of left $C_A^-(R)$-modules were defined. By Theorem I.5.6 and Theorem I.5.18 these are all exact and preserve quasi-isomorphisms. Therefore, if M is a right A-module equipped with a filtration F_pM by A-submodules, $C_A^\bullet(M)$ carries a natural filtration $F_pC_A^\bullet(M) := C_A^\bullet(F_pM)$ by $C_A^-(R)$-submodules. Now, even if the initial filtration of M is exhaustive and complete Hausdorff, the resulting filtration of $C_A^\bullet(M)$ may fail to satisfy these properties. This motivates the following definition.

Definition I.5.35. Let A be a degreewise R-free augmented DG algebra. For $\bullet \in \{+, -, (+, tw), \infty\}$ define a functor $\hat{C}_A^\bullet$ from the category of filtered right A-modules to the category of filtered left $C_A^-(R)$-modules by

$$\begin{aligned} \hat{C}_A^\bullet(M) &:= \lim_q \operatorname{colim}_p C_A^\bullet(F_pM/F_qM) && \text{filtered by} \\ F_p\hat{C}_A^\bullet(M) &:= \lim_{q<p} C_A^\bullet(F_pM/F_qM) && \text{for all } p \in \mathbb{Z}. \end{aligned}$$

In other words, $\hat{C}_A^\bullet$ is the composition of the natural extension of $C_A^\bullet$ to filtered A-modules followed by the full completion functor. We denote the homology of $\hat{C}_A^\bullet(M)$ by $\hat{H}_A^\bullet(M)$.

The various flavors of equivariant instanton Floer homology are obtained by applying this construction to the framed Floer complex.

Definition I.5.36. Let $A = C_*^{gm}(\mathrm{SO}(3))$ and let $M = \widetilde{CI}(Y, E)$ be the framed Floer complex associated with an SO(3)-bundle E over a closed oriented three-manifold Y, equipped with the index filtration. For $\bullet \in \{+, -, (+, tw), \infty\}$ define

$$CI^\bullet(Y, E) := \hat{C}_A^\bullet(M) \quad \text{and} \quad I^\bullet(Y, E) := H(CI^\bullet(Y, E)).$$

The groups $I^\bullet(Y, E)$, for $\bullet \in \{+, -, (+, tw), \infty\}$, are called the positive, negative, positive twisted and Tate equivariant instanton Floer homologies of (Y, E), respectively. These homology groups are modules over $H_A^-(R)$.

For $A = \Lambda_R[u]$ and $M = DCI(Y, E)$ we similarly define the complexes

$$DCI^\bullet(Y, E) = \hat{C}_A^\bullet(M).$$

Remark I.5.37. For $A = C_*(\mathrm{SO}(3); R)$ and $M = R$ it holds true that $H_A^-(R) \cong H^{-*}(\mathrm{BSO}(3); R)$ (see [Eis19, Lemma A.21]).

The norm map $N_M \colon C_A^{+,tw}(M) \to C_A^-(M)$ is a natural in M and therefore extends to a natural map $\hat{N}_M \colon \hat{C}_A^{+,tw}(M) \to \hat{C}_A^-(M)$ for any filtered module M. The following simple lemma ensures that $\hat{C}_A^\infty(M)$ is naturally isomorphic to $\mathrm{Cone}(\hat{N}_M)$.

Lemma I.5.38. *Let A be a degreewise R-free augmented DG algebra and let M be a filtered right A-module. Then there is a natural (filtered) isomorphism of $C_A^-(R)$-modules*

$$\mathrm{Cone}(\hat{N}_M) \cong \hat{C}_A^\infty(M).$$

In particular, we have the following exact triangle of $H_A^-(R)$-modules.

$$\begin{array}{ccc} \hat{H}_A^{+,tw}(M) & \xrightarrow{\quad H(\hat{N}_M) \quad} & \hat{H}_A^-(M) \\ & {\scriptstyle [-1]} \nwarrow \qquad \swarrow {\scriptstyle [0]} & \\ & \hat{H}_A^\infty(M) & \end{array}$$

Proof. For $q < p$ and n there is a natural identification

$$\frac{F_pC_A^\infty(M)}{F_qC_A^\infty(M)}_n \cong C_A^\infty\left(\frac{F_pM}{F_qM}\right)_n = C_A^-\left(\frac{F_pM}{F_qM}\right)_n \oplus C_A^{+,tw}\left(\frac{F_pM}{F_qM}\right)_{n-1}.$$

In terms of the right hand side the differential is given by

$$\begin{pmatrix} \partial^-_{p,q} & N_{p,q} \\ 0 & -\partial^{+,tw}_{p,q} \end{pmatrix},$$

where $\partial^-_{p,q}$ and $\partial^{+,tw}_{p,q}$ denote the differential in the first and second summand respectively and $N_{p,q}$ denotes the evident norm map. Applying $\lim_q \operatorname{colim}_p$ to this identification, taking into account that this is calculated degreewise with differentials and A-module structure carried along, we obtain $\hat{C}_A^\infty(M)_n \cong \hat{C}_A^-(M)_n \oplus \hat{C}_A^{+,tw}(M)_{n-1}$ in each degree n. The differential on the right hand side is given by

$$\begin{pmatrix} \lim_q \operatorname{colim}_p \partial^-_{p,q} & \lim_q \operatorname{colim}_p N_{p,q} \\ 0 & \lim_q \operatorname{colim}_p \partial^{+,tw}_{p,q} \end{pmatrix}.$$

Therefore, the right hand side is precisely $\operatorname{Cone}(\hat{N}_M)$, since by definition $\hat{N}_M = \lim_q \operatorname{colim}_p N_{p,q}$. ∎

To verify that $DCI^\bullet(Y,E)$ also calculates $I^\bullet(Y,E)$ when $\frac{1}{2} \in R$ we will establish the necessary invariance results for the functors $\hat{C}_A^\bullet$. The appropriate tool is the following lemma, which is a simple consequence of the Eilenberg-Moore comparison theorem of [Wei94, Theorem 5.5.11]. For a filtered map $f\colon M \to N$ between filtered DG modules we use the notation $\overline{f}_p\colon \overline{F}_pM \to \overline{F}_pN$ for the induced maps between the the minimal filtration quotients.

Lemma I.5.39. *Let M and N be DG modules equipped with exhaustive and complete Hausdorff filtrations. Then if $f\colon M \to N$ is a filtered map such that the induced map $\overline{f}_p\colon \overline{F}_pM \to \overline{F}_pN$ is a quasi-isomorphism for each $p \in \mathbb{Z}$, then f is a quasi-isomorphism.*

Proposition I.5.40. *Let A be a degreewise R-free augmented DG algebra. Suppose $f\colon M \to N$ is a filtered map of filtered right A-modules such that $\overline{f}_p\colon \overline{F}_pM \to \overline{F}_pN$ is a quasi-isomorphism for each $p \in \mathbb{Z}$. Then for $\bullet \in \{+,-,(+,tw),\infty\}$ the induced map*

$$\hat{f}^\bullet\colon \hat{C}_A^\bullet(M) \to \hat{C}_A^\bullet(N)$$

is a (filtered) quasi-isomorphism of $C_A^-(R)$-modules. Furthermore, upon passage to homology, these isomorphisms are compatible with the exact triangle of Lemma I.5.38.

Proof. Let $\bullet \in \{+,-,(+,tw),\infty\}$ and note that the functors $C_A^\bullet$ and $\hat{C}_A^\bullet$ are covariant in M. For simplicity of notation let $g = \hat{f}^\bullet\colon \hat{C}_A^\bullet(M) \to \hat{C}_A^\bullet(N)$ denote the induced filtered map of $C_A^-(R)$-modules. Then by Lemma I.5.33 part (ii) and the exactness of $C_A^\bullet$, there is for each p a commutative diagram

$$\begin{array}{ccc} C_A^\bullet(\overline{F}_pM) & \xrightarrow{(\overline{f}_p)^\bullet} & C_A^\bullet(\overline{F}_pN) \\ \downarrow\cong & & \downarrow\cong \\ \overline{F}_p\hat{C}_A^\bullet(M) & \xrightarrow{\overline{g}_p} & \overline{F}_p\hat{C}_A^\bullet(N). \end{array}$$

By assumption $\overline{f}_p$ is a quasi-isomorphism, so since the functor $C_A^\bullet$ preserves quasi-isomorphisms it follows that $\overline{f}_p^\bullet$ is a quasi-isomorphism. As the vertical maps are isomorphisms, we conclude that $\overline{g}_p$ is a quasi-isomorphism. By Lemma I.5.33 part (i) the filtrations of $\hat{C}_A^\bullet(M)$ and $\hat{C}_A^\bullet(N)$ are exhaustive and complete Hausdorff. We conclude that $g = \hat{f}^\bullet$ is a quasi-isomorphism using the comparison result; Lemma I.5.39. The lemma applies equally well to the filtered subcomplexes. Consequently, the restrictions $g_p\colon F_p\hat{C}_A^\bullet(M) \to F_p\hat{C}^\bullet(N)$ are also quasi-isomorphisms for all p. This shows that g is in fact a filtered quasi-isomorphism. ■

Next we have to establish a similar invariance result for variations in the DG algebra.

Proposition I.5.41. *Let $f\colon A \to B$ be a quasi-isomorphism of degreewise R-free augmented algebras and let M be a filtered B-module. Then there is for each $\bullet \in \{+, -, (+, tw), \infty\}$ a natural isomorphism of $H_A^-(R) \cong H_B^-(R)$-modules*

$$\hat{H}_A^\bullet(f^{-1}M) \cong \hat{H}_B^\bullet(M),$$

compatible with the exact triangles of Lemma I.5.38.

Proof. We first consider the case $\bullet = +$. For each pair $q < p$ there is by Proposition I.5.5 a natural quasi-isomorphism

$$f_*\colon C_A^+(F_pf^{-1}M/F_qf^{-1}M) \to C_B^+(F_pM/F_qM)$$

linear along $f^*\colon C_B^-(R) \to C_A^-(R)$. By applying $\lim_q \operatorname{colim}_p$ we obtain a filtered map $g = \hat{f}_*\colon \hat{C}_A^+(f^{-1}M) \to \hat{C}_B^+(M)$ linear along $f^* : C_B^-(R) \to C_A^-(R)$. By construction, the following diagram commutes for each p.

$$\begin{array}{ccc} C_A^+(\overline{F}_pf^{-1}M) & \xrightarrow{\overline{(f_*)}_p} & C_B^+(\overline{F}_pM) \\ \downarrow\cong & & \downarrow\cong \\ \overline{F}_p\hat{C}_A^+(f^{-1}M) & \xrightarrow{\overline{g}_p} & \overline{F}_p\hat{C}_B^+(M) \end{array}$$

As the upper horizontal map is a quasi-isomorphism, so is the lower horizontal map and we conclude that $g = \hat{f}_*$ is a filtered quasi-isomorphism by Lemma I.5.39. For $\bullet = -$ there is for each pair $q < p$ a natural quasi-isomorphism

$$f^*\colon C_B^-(F_pM/F_qM) \to C_A^-(F_pf^{-1}M/F_qf^{-1}M).$$

By passing to the colimit and limit we obtain a filtered map $\hat{f}^*\colon \hat{C}^-_B(M) \to \hat{C}^-_A(f^{-1}M)$, linear along $f^*\colon C^-_B(R) \to C^-_A(R)$. The same argument as above shows that this is a filtered quasi-isomorphism.

For the two remaining cases recall that there is a relative functor $C^{+,tw}_f$, see Definition I.5.19, from the category of right B-modules to the category of left $C^-_A(R)$-modules. By Lemma I.5.20 this functor is exact and preserves quasi-isomorphism, so we may promote it to a functor $\hat{C}^{+,tw}_f$ between the corresponding filtered categories as we did with the others. By passing to the colimit and then the limit in the commutative diagram of Proposition I.5.21 applied to F_pM/F_qM for all $q < p$, we obtain a commutative diagram

$$\begin{array}{ccccc} \hat{C}^{+,tw}_B(M) & \xrightarrow{\hat{\beta}_M} & \hat{C}^{+,tw}_f(M) & \xleftarrow{\hat{\alpha}_M} & \hat{C}^{+,tw}_A(f^{-1}M) \\ \downarrow{\scriptstyle \hat{N}^B_M} & & \downarrow{\scriptstyle \hat{N}^f_M} & & \downarrow{\scriptstyle \hat{N}^A_M} \\ \hat{C}^-_B(M) & \xrightarrow{\hat{f}^*} & \hat{C}^-_A(f^{-1}M) & \xleftarrow{=} & \hat{C}^-_A(f^{-1}M) \end{array}$$

of filtered DG modules over $C^-_A(R)$ or $C^-_B(R)$ as appropriate. Furthermore, all the maps are filtered and $C^-_A(R)$-linear, $C^-_B(R)$-linear or linear along $f^*\colon C^-_B(R) \to C^-_A(R)$ as appropriate. By the same proposition, as $f\colon A \to B$ is a quasi-isomorphism of degreewise free DG algebras, all the horizontal arrows in the diagram applied to F_pM/F_qM are quasi-isomorphisms. This implies, by the now familiar argument, that all the horizontal arrows in the above diagram are filtered quasi-isomorphisms. In particular, there is a natural isomorphism $\hat{H}^{+,tw}_A(f^{-1}M) \cong \hat{H}^{+,tw}_B(M)$. Moreover, we obtain a zigzag of quasi-isomorphisms between the cones of the vertical maps

$$\mathrm{Cone}(\hat{N}^B_M) \longrightarrow \mathrm{Cone}(\hat{N}^f_M) \longleftarrow \mathrm{Cone}(\hat{N}^A_{f^{-1}M})\ .$$

In view of Lemma I.5.38, we obtain an isomorphism $\hat{H}^\infty_A(f^{-1}M) \cong \hat{H}^\infty_B(M)$ linear along $H^-_A(R) \cong H^-_B(R)$ as well. The above diagram also ensures that these isomorphisms are compatible with the corresponding exact triangles. ■

The key application of these invariance results is the following.

Corollary I.5.42. *Suppose that $\frac{1}{2} \in R$. Then for each $\bullet \in \{+, -, (+, tw), \infty\}$ there is an isomorphism $H(DCI^\bullet(Y, E)) \cong I^\bullet(Y, E)$ of $H^-_A(R) \cong R[U]$-modules. These isomorphisms are compatible with the corresponding exact triangles.*

Proof. As $\frac{1}{2} \in R$ there are, according to Proposition I.2.9, a quasi-isomorphism $i\colon \Lambda_R[u] \to C^{gm}_*(\mathrm{SO}(3))$ of DG algebras and a zigzag

$$\widetilde{CI}(Y, E) \xrightarrow{f} X \xleftarrow{g} DCI(Y, E)$$

of filtered objects and filtered $\Lambda_R[u]$-homomorphisms such that $\overline{f}_p$ and $\overline{g}_p$ are quasi-isomorphism for each $p \in \mathbb{Z}$. The required result now follows from Propositions I.5.40 and I.5.41. ■

For $A = \Lambda_R[u]$ we can extend a number of the results from the previous section.

Proposition I.5.43. *Let $A = \Lambda_R[u]$ and let M be a filtered A-module. Then there is a filtered quasi-isomorphism $\hat{f}_M : \hat{C}^+_A(M) \to \hat{C}^{+,tw}_A(M)$ of degree 3 and a (filtered) homotopy $\hat{s}_M : \hat{C}^+_A(M) \to \hat{C}^{+,tw}_A(M)$ such that $\hat{f}_M U - U\hat{f}_M = \partial\hat{s}_M - \hat{s}_M\partial$ for $U \in R[U] \cong C^-_A(R)$. Moreover, there is a (filtered) quasi-isomorphism*

$$\mathrm{Cone}(\hat{N}_M \circ \hat{f}) \to \mathrm{Cone}(\hat{N}_M) \cong \hat{C}^\infty_A(M),$$

which is $R[U]$-linear up to homotopy, where the source is given the $R[U]$-module structure determined by the homotopy $\hat{t}_M = \hat{N}_M \circ \hat{s}_M$ as in Lemma I.5.25. Furthermore, the resulting isomorphism in homology is compatible with the associated exact triangles.

Proof. Recall from Lemma I.5.24 that there is for each right A-module N a quasi-isomorphism $f_N : C^+_A(N) \to C^{+,tw}_A(N)$ of degree 3 and a homotopy $s_N : C^+_A(N) \to C^{+,tw}_A(N)$ such that $f_N U - U f_N = \partial s_N - s_N\partial$. Let $f_{p,q}$ and $s_{p,q}$ denote the maps obtained for $N = F_pM/F_qM$ for $q < p$. By naturality we may pass to the colimit over p and then to the limit over q to obtain maps $\hat{f}_M : \hat{C}^+_A(M) \to \hat{C}^{+,tw}_A(M)$ and $\hat{s}_M : \hat{C}^+_A(M) \to \hat{C}^{+,tw}_A(M)$. As $f_{p,q}U - Uf_{p,q} = s_{p,q}\partial - \partial s_{p,q}$ is valid for each pair $q < p$, it follows that $\hat{f}_M U - U\hat{f}_M = \partial\hat{s}_M - \hat{s}_M\partial$. The fact that $\hat{f}_M$ is a quasi-isomorphism is proved as before using Lemma I.5.39.

The final statement follows by applying $\lim_q \operatorname{colim}_p$ to the diagram (I.28) in the proof of Theorem I.5.28 for F_pM/F_qM and another application of Lemma I.5.39. It is clear from this construction that one obtains an isomorphism between the corresponding exact triangles in homology. ∎

Combining the above proposition with Corollary I.5.42 we obtain the following.

Corollary I.5.44. *Suppose that $\frac{1}{2} \in R$. Then there is a degree 3 isomorphism $I^+(Y,E) \cong I^{+,tw}(Y,E)$ of $R[U]$-modules and there is an exact triangle of $R[U]$-modules*

$$\begin{array}{ccc} I^+(Y,E) & \xrightarrow{\quad [3] \quad} & I^-(Y,E) \\ {\scriptstyle [-4]}\nwarrow & & \swarrow{\scriptstyle [0]} \\ & I^\infty(Y,E) & \end{array},$$

where the numbers specify the degrees of the maps.

We will now restrict ourselves to the cases of interest, namely,

$$(A, M) = (C^{gm}_*(\mathrm{SO}(3)), \widetilde{CI}(Y,E)) \quad \text{or} \quad (A, M) = (\Lambda_R[u], DCI(Y,E)),$$

where M is equipped with the index filtration. Recall from Definition I.2.8 and Definition I.2.4 that M was obtained from a multicomplex $(M_{*,*}, \{\partial^r\}_{r=0}^5)$, and that the index filtration was defined to be the column filtration, i.e.,

$F_pM_n = \bigoplus_{s\le p} M_{s,n-s}$. In both cases it holds true that $M_{s,t} = 0$ for all $t > 4$ and $t < 0$ (for $M = DCI(Y,E)$ this is true for $t \ge 4$). This implies that the filtration is degreewise finite and therefore exhaustive and complete Hausdorff. Explicitly,

$$F_pM_n = \begin{cases} M_n & \text{for all } p \ge n \\ 0 & \text{for all } p \le n-5 \end{cases} \tag{I.30}$$

The following lemma indicates why it is desirable to introduce the full completions of the various functors.

Lemma I.5.45. *The filtration of $C_A^+(M)$, $F_pC_A^+(M) = C_A^+(F_pM)$, is degreewise bounded above, exhaustive and Hausdorff. The filtration of $C_A^-(M)$ is degreewise bounded below and complete Hausdorff. If $M \neq 0$ the filtration of $C_A^+(M)$ fails to be complete and the filtration of $C_A^-(M)$ fails to be exhaustive. Therefore,*

$$\hat{C}_A^+(M) = \lim_q C_A^+(M/F_qM) \quad \text{and} \quad \hat{C}_A^-(M) = \operatorname{colim}_p C_A^-(F_pM).$$

Proof. We will use the underlying bigrading of the multicomplex $M_{*,*}$. As a graded module (not DG) we have $C_A^+(M) = B(M,A,R) \cong M \otimes BA$. Therefore, in degree n

$$C_A^+(M)_n = \bigoplus_{u+r=n} M_u \otimes BA_r = \bigoplus_{s+t+r=n} M_{s,t} \otimes BA_r, \tag{I.31}$$

and $F_pC_A^+(M)$ is obtained by imposing the condition $s \le p$ in the above right hand direct sum. First, notice that A and hence BA are supported in nonnegative degrees. Then, as $M_{s,t} = 0$ for $t < 0$, it follows $F_pC_A^+(M)_n = C_A^+(M)_n$ for $p \ge n$. This shows that the filtration is degreewise bounded above and hence exhaustive. It also follows from the above description that the filtration is Hausdorff. On the other hand

$$\lim_p C_A^+(M/F_pM)_n \cong \lim_p \bigoplus_{\substack{s+t+r=n \\ s>p}} M_{s,t} \otimes BA_r \cong \prod_{s+t+r=n} M_{s,t} \otimes BA_r. \tag{I.32}$$

This does not coincide with the direct sum in equation (I.31) if $M \neq 0$, due to the periodicity $M_{s,t} \cong M_{s+8,t}$ for all s,t. Thus, $C_A^+(M)$ fails to be complete if $M \neq 0$.

Next, there is an isomorphism $C_A^-(M) = \operatorname{Hom}_A(EA, M) \cong \operatorname{Hom}(BA, M)$ of graded modules. In a fixed degree n, we find

$$C_A^-(M)_n = \operatorname{Hom}(BA, M)_n = \prod_{s+t-r=n} \operatorname{Hom}(BA_r, M_{s,t}) \tag{I.33}$$

and as above $F_pC_A^-(M)$ is obtained by imposing the condition $s \le p$ in the right hand product. For the factor $\operatorname{Hom}(BA_r, M_{s,t})$ to be nonzero it is necessary that $r \ge 0$ and $0 \le t \le 4$, and therefore $s = n + r - t \ge n - 4$. Hence, $F_pC_A^-(M)_n = 0$ for $p < n-4$ showing that the filtration is degreewise bounded below, and thus complete Hausdorff. Moreover, the above also implies that the right hand product in (I.33) subject to the condition $s \leq p$ is finite and therefore coincides with the direct sum. Consequently, $\operatorname{colim}_p F_p C_A^-(M)$ is given by

$$\operatorname{colim}_p \bigoplus_{\substack{s+t-r=n \\ s\leq p}} \operatorname{Hom}(BA_r, M_{s,t}) = \bigoplus_{s+t-r=n} \operatorname{Hom}(BA_r, M_{s,t}), \tag{I.34}$$

which does not coincide with the product in (I.33) as long as $M \neq 0$. This shows that the filtration fails to be exhaustive if $M \neq 0$.

The description of $\hat{C}_A^+(M)$ and $\hat{C}_A^-(M)$ given in the statement is now a consequence of Lemma I.5.33 part (iii) and (iv). ■

Remark I.5.46. Miller [Eis19, Appendix A] defines completed versions of $C_A^\pm$ using what he names the completed bar construction and the finitely supported cobar construction. He defines the completed bar construction, $\hat{B}(M, A, N)$, to be the completion of $B(M, A, N)$ with respect to the filtration by internal degree, while the finitely supported cobar construction $c\hat{B}(N, A, M) \subset \operatorname{Hom}_A(B(N, A, A), M)$ is the subcomplex consisting of those functionals that vanish on $B_{p,q}(N, A, A)$ for all sufficiently large p. The above lemma, in particular equations (I.32) and (I.34), verifies that our $\hat{C}_A^+(M)$ and $\hat{C}_A^-(M)$ coincide with Miller's $\hat{B}(M, A, R)$ and $c\hat{B}(R, A, M)$ for $M = \widetilde{CI}(Y, E)$ and $M = DCI(Y, E)$ with A as appropriate.

The final part of this section is devoted to extending the concrete models of Proposition I.5.8 and Theorem I.5.28 to the complex $DCI^\bullet(Y, E)$ for $\bullet \in \{+, -, \infty\}$. We will therefore assume that $\frac{1}{2} \in R$ throughout.

Theorem I.5.47. *Let $A = \Lambda_R[u]$ and let $(D^\pm_{*,*}, \partial', \partial'')$ be the double complexes associated with $M = DCI(Y, E)$ in Proposition I.5.8. Then there are isomorphisms of DG $R[U]$-modules*

$$DCI^+(Y, E) \cong \operatorname{Tot}^\Pi(D^+_{*,*}, \partial', \partial'') \quad \text{and} \quad DCI^-(Y, E) \cong \operatorname{Tot}^\oplus(D^-_{*,*}, \partial', \partial'').$$

In particular, there are identifications

$$DCI^+(Y)_n \cong \prod_{p\geq 0} M_{n-4p} \quad \text{and} \quad DCI^-(Y)_n \cong \bigoplus_{p\geq 0} M_{n+4p}$$

for each $n \in \mathbb{Z}$. An element $x \in DCI^\pm(Y)_n$ can therefore be expressed by a sequence $x = (m_{n+4p})_p$ where we require $p \leq 0$ in the $+$ case and $p \geq 0$ and almost all $m_{n+4p} = 0$ in the $-$ case. The differential is in both cases given by

$$\partial x = (\partial_M m_{n+4p} - (-1)^n m_{n+4(p-1)} u)_p$$

and the action of $R[U]$ is determined by $U \cdot x = (m_{n+4(p+1)})_p$.

Proof. The isomorphism $C_A^+(M) \cong \operatorname{Tot}^\oplus(D^+_{*,*}, \partial', \partial'')$ of Proposition I.5.8 yields in degree n

$$C_A^+(M)_n \cong \bigoplus_{s\geq 0} M_{n-4s} = \bigoplus_{s\geq 0, t} M_{t, n-4s-t}$$

and $F_pC_A^+(M)$ is obtained by imposing the condition $t \leq p$ in the right hand sum. Now the calculation of the limit $\lim_p C_A^+(M)_n/F_pC_A^+(M)_n$ goes through as in equation (I.32) of the previous proof to show that $\hat{C}_A^+(M)_n \cong \prod_{s\geq 0} M_{n-4s}$.

The other case is analogous and the calculation of the relevant colimit proceeds as in equation (I.34) of the previous proof. ∎

Note that the only difference between the above theorem and Proposition I.5.8 is that we have interchanged the type of totalization we apply to the double complexes.

We also obtain the following explicit model computing $I^\infty(Y,E)$.

Theorem I.5.48. *Let $A = \Lambda_R[u]$, let $(D^\pm_{*,*}, \partial', \partial'')$ and $(D^\infty_{*,*}, \partial', \partial'')$ be the double complexes associated with $M = DCI(Y,E)$ in the above theorem and Theorem I.5.28, respectively. Let $\nu, \psi\colon \operatorname{Tot}^\Pi(D^+, \partial', \partial'') \to \operatorname{Tot}^\oplus(D^-, \partial', \partial'')$ of degree 3 and 0 respectively, be given degreewise by the compositions*

$$\prod_{s\geq 0} M_{n-4s} \xrightarrow{\pi} M_n \xrightarrow{(-1)^n u} M_{n+3} \xrightarrow{\iota} \bigoplus_{t\geq 0} M_{n+3+4t}$$

$$\prod_{s\geq 0} M_{n-4s} \xrightarrow{\pi} M_n \xrightarrow{\iota} \bigoplus_{t\geq 0} M_{n+4t},$$

where π and ι denote the projection and inclusion, respectively. Then ν is a chain map satisfying $\nu U - U\nu = \partial\psi - \psi\partial$. Define $\operatorname{Cone}(\nu)$ with the $R[U]$-module structure adjusted by the homotopy ψ as in Lemma I.5.25. Then there is an isomorphism $R[U]$-modules $\operatorname{Tot}^{\Pi,\infty}(D^\infty_{*,*}, \partial', \partial'') \cong \operatorname{Cone}(\nu)$ and a filtered quasi-isomorphism between this complex and $DCI^\infty(Y,E)$, which is $R[U]$-linear up to homotopy. Moreover, this quasi-isomorphism is compatible with the corresponding exact triangles in homology.

Proof. By Proposition I.5.43 there is a filtered quasi-isomorphism $\hat{f}_M \colon \hat{C}_A^+(M) \to \hat{C}_A^{+,tw}(M)$ of degree 3 and a filtered homotopy $\hat{s}_M : \hat{C}_A^+(M) \to \hat{C}_A^{+,tw}(M)$ such that $\hat{f}_M U - U\hat{f}_M = \partial\hat{s}_M - \hat{s}_M\partial$. Furthermore, there is an induced filtered quasi-isomorphism $\operatorname{Cone}(\hat{N}_M \circ \hat{f}_M) \to \hat{C}_A^\infty(M)$, which is $R[U]$-linear up to homotopy when the cone is given the adjusted $R[U]$-module structure determined by the homotopy $\hat{t}_M = \hat{N}_M \circ \hat{s}_M$. In terms of the explicit models of the above theorem $\hat{N}_M \circ \hat{f}_M$ and $\hat{N}_M \circ \hat{s}_M$ are precisely ν and ψ as defined in the statement. The verification of this is done just as in Lemma I.5.24. The fact that $\operatorname{Cone}(\nu) \cong \operatorname{Tot}^{\Pi,\infty}(D^\infty_{*,*}, \partial', \partial'')$ is proved just as in Theorem I.5.28. ∎

Remark I.5.49. The proofs of Theorem I.5.47 and Theorem I.5.48 go through without any essential change with $M = \widetilde{CI}(Y,E)$, regarded as a $\Lambda_R[u]$-module, in place of $M = DCI(Y,E)$.

We may extract the following noteworthy consequences, showing in particular that the framed homology groups $\widetilde{I}(Y,E) := H(DCI(Y,E)) \cong H(\widetilde{CI}(Y,E))$ may be recovered from the $R[U]$-action on $I^\pm(Y,E)$ up to extension.

Corollary I.5.50. *There are long exact sequences*

$$\tilde{I}_n(Y,E) \longrightarrow I^+(Y,E)_n \xrightarrow{U} I^+(Y,E)_{n-4} \longrightarrow \tilde{I}(Y,E)_{n-1}$$

$$\tilde{I}(Y,E)_{n-3} \longrightarrow I^-(Y,E) \xrightarrow{U} I^-(Y,E)_{n-4} \longrightarrow \tilde{I}(Y,E)_{n-4}.$$

Moreover, $U : I^\infty(Y,E) \to I^\infty(Y,E)$ is an isomorphism.

Proof. Let $M = DCI(Y,E)$ and $A = \Lambda_R[u]$. In terms of the explicit models for $DCI^\pm(Y,E) = \hat{C}^\pm_A(M)$ the action of $U \in R[U]$ are in degree n given by the natural shifts

$$U\colon \prod_{s\geq 0} M_{n-4s} \to \prod_{s\geq 0} M_{n-4-4s} \quad \text{and} \quad U\colon \bigoplus_{s\geq 0} M_{n+4s} \to \bigoplus_{s\geq 0} M_{n-4+4s},$$

respectively. Explicitly, $(x_s)_{s\geq 0} \mapsto (x_{s+1})_{s\geq 0}$ in the first case and $(y_s)_{s\geq 0} \mapsto (0, y_0, y_1, \cdots)$ in the second. From this we deduce that the first map is surjective with kernel $D^+_{0,*} \cong M$ and the second is injective with cokernel isomorphic to $D^-_{0,*} \cong M$. There are therefore short exact sequences of chain complexes

$$0 \longrightarrow M \longrightarrow D^+_* \xrightarrow{U} D^+_*[4] \longrightarrow 0$$

$$0 \longrightarrow D^-_*[-4] \xrightarrow{U} D^-_* \longrightarrow M \longrightarrow 0$$

giving rise to the stated long exact sequences in homology. The final statement is a consequence of the fact that the shift

$$U\colon D^\infty_n = \prod_{s\to\infty} M_{n-4s} \to \prod_{s\to\infty} M_{n-4-4s} = D^\infty_{n-4}$$

is an isomorphism in each degree n and therefore also an isomorphism upon passage to homology. ∎

Remark I.5.51. The fact that $U\colon I^\infty(Y,E) \to I^\infty(Y,E)$ is an isomorphism when $\frac{1}{2} \in R$ was first proved using a spectral sequence argument in [Eis19].

I.5.5 The Index Spectral Sequences

For $\bullet \in \{+,-,(+,tw),\infty\}$ the complexes $CI^\bullet(Y,E)$ and $DCI^\bullet(Y,E)$ introduced in Definition I.5.36 come equipped with filtrations and therefore give rise to spectral sequences. These will be called index spectral sequences as the filtrations are induced by the index filtrations of $\widetilde{CI}(Y,E)$ and $DCI(Y,E)$. The purpose of this section is to recall some theory on spectral sequences and then to establish the basic properties of the index spectral sequences. We will mainly follow Boardman's paper [Boa99], but we will stick to homological notation. All of this will be very useful for our calculations in the next section.

Recall that an (unrolled) exact couple (A, E, i, j, k) consists of two sequences of graded modules $(A_s)_s$ and $(E_s)_s$, indexed over $s \in \mathbb{Z}$, and homomorphisms $i = (i_s : A_{s-1} \to A_s)_s$, $j = (j_s : A_s \to E_s)_s$ and $k = (k_s : E_s \to A_{s-1})_s$ such that the sequence

$$A_{s-1} \xrightarrow{i_s} A_s \xrightarrow{j_s} E_s \xrightarrow{k_s} A_{s-1} \xrightarrow{i_s} A_s$$

is exact for each s. We will generally omit the lower index on the maps, letting the source and target specify the map in question. In our situation the degrees of (i, j, k) will be $(0, 0, -1)$. For each $r \geq 1$ and s set

$$\begin{aligned} Z_s^r &:= k^{-1}\,\mathrm{Im}[i^{(r-1)} : A_{s-r} \to A_{s-1}] \\ B_s^r &:= j\,\mathrm{Ker}[i^{(r-1)} : A_s \to A_{s+r-1}] \\ E_s^r &:= Z_s^r / B_s^r, \end{aligned}$$

where $i^{(r)}$ denotes the r-fold composition $i \circ \cdots \circ i$ for $r \geq 1$ and $i^{(0)}$ is the identity. Furthermore, put $Z_s^\infty := \bigcap_r Z_s^r$, $B_s^\infty := \bigcup_r B_s^r$ and $E_s^\infty := Z_s^\infty / B_s^\infty$. For the later convergence theory it is also important to introduce the derived limit $RE_s^\infty := \mathrm{Rlim}_s\, Z_s^r$. There are differentials $d^r \colon E_s^r \to E_{s-r}^r$ for each s and $r \geq 1$, defined in the following way. Let $x \in Z_s^r$ represent the class $[x] \in E_s^r$. Choose $y \in A_{s-r}$ such that $i^{(r-1)}(y) = k(x)$. Then $d^r([x]) = [j(y)] \in E_{s-r}^r$. The situation is illustrated in the following diagram

$$\begin{array}{ccccc} & y \in A_{s-r} & \xrightarrow{\;i^{(r-1)}\;} & k(x) \in A_{s-1} & \\ \swarrow^{j} & & & & \nwarrow^{k} \\ j(y) \in E_{s-r} & & & & x \in E_s. \end{array} \tag{I.35}$$

One may then show that d^r is well-defined, that $d^r \circ d^r = 0$ and that

$$E_s^{r+1} \cong \mathrm{Ker}(d^r : E_s^r \to E_{s-r}^r) / \mathrm{Im}(d^r : E_{s+r}^r \to E_s^r)$$

for each $r \geq 1$ and s. In other words, $(E^r, d^r)_{r \geq 1}$ is a spectral sequence. We call it the spectral sequence associated with the exact couple (A, E, i, j, k). Every exact couple we will meet is covered by the following example (see [Boa99, Section 9]).

Example I.5.52. Let C be a DG module equipped with an increasing filtration $\{F_s C\}_s$ by DG submodules. For each $s \in \mathbb{Z}$ let $A_s = H(F_s C)$ and $E_s = H(F_s C / F_{s-1} C)$. Define $i \colon A_{s-1} \to A_s$, $j \colon A_s \to E_s$ and $k \colon E_s \to A_{s-1}$ to be the maps induced by the short exact sequence

$$0 \longrightarrow F_{s-1}C \longrightarrow F_s C \longrightarrow F_s C / F_{s-1} C \longrightarrow 0$$

upon passage to homology. In particular, k is the connecting homomorphism of degree -1. Then (A, E, i, j, k) is an exact couple. The spectral sequence associated with this exact couple is the spectral sequence mentioned in (I.29). If B is a DG algebra and C is a B-module filtered by B-submodules, then A_s, E_s carry the structure of graded $H(B)$-modules, and the maps i, j, k are $H(B)$-linear. Moreover, this structure carries in a natural way over to the whole spectral sequence.

Returning to the situation of an exact couple (A, E, i, j, k) put

$$A_\infty := \operatorname{colim}_s A_s, \quad A_{-\infty} := \lim_s A_s \quad \text{and} \quad RA_{-\infty} := \operatorname{Rlim}_s A_s. \tag{I.36}$$

For us the relevant target for the spectral sequence is the colimit A_∞, filtered by $F_sA_\infty := \operatorname{Im}(A_s \to A_\infty)$. For each $s \in \mathbb{Z}$ there is a natural inclusion $\phi_s \colon F_sA_\infty/F_{s-1}A_\infty \to E_s^\infty$ given by the composition

$$F_sA_\infty/F_{s-1}A_\infty \xrightarrow{\cong} \operatorname{Im} j/B_s^\infty \longrightarrow Z_s^\infty/B_s^\infty = E_s^\infty. \tag{I.37}$$

Here, the first isomorphism is given by sending $[x] \in F_sA_\infty/F_{s-1}A_\infty$, represented by $x \in F_sA_\infty$, to $[j(y)] \in \operatorname{Im} j/B_s^\infty$, where $y \in A_s$ is a lift of x along the surjection $A_s \to F_sA_\infty$, and the second map is induced by the inclusion $\operatorname{Im} j = \operatorname{Ker} k \subset Z_s^\infty$ (see [Boa99, Lemma 5.6] and its proof).

Definition I.5.53. The spectral sequence $(E^r, d^r)_{r\geq 1}$ associated with the exact couple (A, E, i, j, k) converges strongly to the colimit A_∞ if the filtration $F_sA_\infty = \operatorname{Im}(A_s \to A_\infty)$ of A_∞ is exhaustive and complete Hausdorff and the map $\phi_s \colon F_sA_\infty/F_{s-1}A_\infty \to E_s^\infty$ is an isomorphisms for each $s \in \mathbb{Z}$.

Strong convergence is the ideal form of convergence. In theory, if we can can calculate E^∞ we would know the subquotients $F_sA_\infty/F_{s-1}A_\infty$ and provided we can solve the extension problems to determine F_sA_∞/F_tA_∞ for each pair $s > t$, we can recover the target group as

$$A_\infty \cong \lim_s \operatorname{colim}_t F_tA_\infty/F_sA_\infty.$$

Definition I.5.54. [Boa99, Definition 5.10] Let (A, E, i, j, k) be an exact couple with associated spectral sequence $(E^r, d^r)_{r\geq 1}$. If $RA_{-\infty} = 0$ and $A_{-\infty} = 0$ (see (I.36)), then the spectral sequence is said to converge conditionally to the colimit A_∞

In our situation of a filtered DG module and the corresponding exact couple described in Example I.5.52 we have the following result.

Theorem I.5.55. *[Boa99, Theorem 9.2] Let C be a DG module equipped with an increasing filtration F_sC by DG submodules. Let (A_s, E_s, i, j, k) be the associated exact couple with corresponding spectral sequence $(E^r, d^r)_{r\geq 1}$. If the filtration of C is exhaustive and complete Hausdorff, then the spectral sequence $(E^r, d^r)_{r\geq 1}$ converges conditionally to the colimit $A_\infty = H(C)$.*

In the presence of conditional convergence there are two remaining obstructions to achieving strong convergence; namely, the vanishing of the group RE^∞ and another group W. We refer to [Boa99, Lemma 8.5] for the general definition of W, but we note that if the exact couple is constructed from a filtered complex, then W is isomorphic to the kernel of the canonical interchange map (see [HR19])

$$\kappa\colon \operatorname{colim}_p \lim_q H(F_pC/F_qC) \to \lim_q \operatorname{colim}_p H(F_pC/F_qC).$$

We then have the following convergence theorem.

Theorem I.5.56. *[Boa99, Theorem 8.2] Let (A, E, i, j, k) be an exact couple and assume that the associated spectral sequence $(E^r, d^r)_{r\geq 1}$ converges conditionally to the colimit A_∞. Then if $W = 0$ and $RE^\infty = 0$ the spectral converges strongly to the colimit A_∞.*

The following result gives a few standard criteria for the vanishing of RE^∞ and W in the situation of a filtered complex.

Proposition I.5.57. *Let C be a filtered complex with associated exact couple (A, E, i, j, k) and spectral sequence $(E^r, d^r)_{r\geq 1}$.*

(i) *If the filtration of C is degreewise bounded below, then $RE^\infty = 0$ and $W = 0$.*

(ii) *If the filtration of C is degreewise bounded above then $W = 0$.*

(iii) *If the spectral sequence degenerates at some finite stage; that is, $E^{r_0} = E^{r_0+1} = \cdots = E^\infty$ for some finite $r_0 \geq 1$, then $RE^\infty = 0$ and $W = 0$.*

We may now apply this theory to the filtered complexes $CI^\bullet(Y, E)$ for $\bullet \in \{+, -, (+, tw), \infty\}$. This is a good point to explain that the mod 8 periodicity of $\widetilde{CI}(Y, E)$ carries over to the complexes $CI^\bullet(Y, E)$. Let $M = \widetilde{CI}(Y, E)$ and let $A = C_*^{gm}(\mathrm{SO}(3))$ or $A = \Lambda_R[u]$. Then the periodicity may be expressed as an isomorphism $M \cong M[8]$ of A-modules. Moreover, this isomorphism is compatible with the filtration in the sense that $F_{p+8}M \cong (F_pM)[8]$ for all $p \in \mathbb{Z}$. There is therefore an induced isomorphism

$$\begin{aligned} CI^\bullet(Y, E) &= \lim_q \operatorname{colim}_p C_A^\bullet(F_{p+8}M/F_{q+8}M) \\ &\cong \lim_q \operatorname{colim}_p C_A^\bullet(F_pM/F_qM)[8] = CI^\bullet(Y, E)[8]. \end{aligned}$$

This isomorphism is seen to be filtered in the same sense: $F_pCI^\bullet(Y, E)_n \cong F_{p+8}CI^\bullet(Y, E)_{n+8}$ for all p and n.

Theorem I.5.58. *Let $A = C_*^{gm}(\mathrm{SO}(3))$. There is a conditionally convergent spectral sequence of $H_A^-(R)$-modules*

$$E^1_{s,t} = \bigoplus_{j(\alpha)\equiv s} H^\bullet_{\mathrm{SO}(3)}(\alpha)_t \implies I^\bullet(Y, E)_{s+t}$$

for each $\bullet \in \{+, -, (+, tw), \infty\}$, where

$$H^\bullet_{\mathrm{SO}(3)}(\alpha) := H^\bullet_A(C_*^{gm}(\alpha))$$

for each critical orbit $\alpha \in \mathcal{C}$. Moreover, the spectral sequences are periodic in the sense that there are isomorphisms $E^r_{s,t} \cong E^r_{s+8,t}$ for all s, t, r. These isomorphisms commute with the differentials and the $H^-_A(R)$-action and they are compatible with the target in the sense that

$$\begin{array}{ccc} \overline{F}_s I^\bullet_{s+t} & \xrightarrow{\cong} & \overline{F}_{s+8} I^\bullet_{s+t+8} \\ \downarrow & & \downarrow \\ E^\infty_{s,t} & \xrightarrow{\cong} & E^\infty_{s+8,t} \end{array}$$

commutes for all s, t.

The spectral sequence for $I^-(Y,E)$ converges strongly, while the spectral sequence for $I^+(Y,E)$ converges strongly provided $RE^\infty = 0$.

Proof. Let $M = \widetilde{CI}(Y,E)$ equipped with the index filtration. For each $\bullet \in \{+, -, (+, tw), \infty\}$ the complex $CI^\bullet(Y,E) = \hat{C}^\bullet_A(M)$ carries by construction an exhaustive and complete Hausdorff filtration with minimal filtration quotients

$$\overline{F}_p \hat{C}^\bullet_A(M) \cong C^\bullet_A(\overline{F}_p M).$$

There is therefore by Theorem I.5.55 a conditionally convergent spectral sequence

$$E^1_{s,t} = H^\bullet_A(\overline{F}_s M)_{s+t} \implies \hat{H}^\bullet_A(M)_{s+t} = I^\bullet(Y,E)_{s+t}.$$

From the definition of $\widetilde{CI}(Y,E)$ (Definition I.2.4) we may identify

$$\overline{F}_s M_{s+t} = \bigoplus_{j(\alpha) \equiv s} C^{gm}_t(\alpha)$$

so that $E^1_{s,t} \cong \bigoplus_{j(\alpha)\equiv s} H_{\mathrm{SO}(3)}(\alpha)_t$ in the notation of the statement.

Next, the periodicity isomorphisms $F_p \hat{C}^\bullet_A(M)_n \cong F_{p+8} \hat{C}^\bullet_A(M)_{n+8}$, compatible with the differentials and the $C^-_A(R)$-structure, induce a morphism of the associated exact couple. That is, there are isomorphisms $A_s \cong A_{s+8}$, $E_s \cong E_{s+8}$ each s, compatible with the structure maps. It is then a straightforward exercise to verify that this gives rise to the periodicity in the spectral sequence, compatible with the target, as stated.

Finally, for the convergence we know by Lemma I.5.45 that the filtration of $\hat{C}^+_A(M)$ is degreewise bounded above and that the filtration of $\hat{C}^-_A(M)$ is degreewise bounded below. This implies by Proposition I.5.57 and Theorem I.5.56 that the spectral sequence associated with $I^-(Y,E)$ converges strongly and that the spectral sequence associated with $I^+(Y,E)$ converges strongly provided $RE^\infty = 0$. ∎

Provided $\frac{1}{2} \in R$, these spectral sequences can equally well be constructed from $M = DCI(Y,E)$ equipped with the index filtration and $A = \Lambda_R[u]$. This is what we will assume when we calculate these groups for binary polyhedral spaces in the next section. Furthermore, in this setting the calculation of the groups

$$H^{\bullet}_{\mathrm{SO}(3)}(C^{gm}_*(\alpha)) \cong H^{\bullet}_{\Lambda_R[u]}(H_*(\alpha))$$

for each $\bullet \in \{+, -, (+, tw), \infty\}$, are contained in Lemma I.5.7, Corollary I.5.9 and Corollary I.5.30. From these calculations it becomes clear that the spectral sequences for I^+ and I^- are contained in the upper and lower half-planes respectively, while the spectral sequence for I^∞ is a whole-plane spectral sequence.

I.6 Calculations for Binary Polyhedral Spaces

We will from this point require that $2 \in R$ is invertible. The universal example to have in mind is $R = \mathbb{Z}[\frac{1}{2}]$. Our main aim is to give explicit calculations of the $H^-_{\Lambda_R[u]}(R) = R[U]$-modules

$$I^+(\overline{Y}_\Gamma, E), \quad I^-(\overline{Y}_\Gamma, E) \quad \text{and} \quad I^\infty(\overline{Y}_\Gamma, E)$$

for each finite subgroup $\Gamma \subset \mathrm{SU}(2)$, where $E \to \overline{Y}_\Gamma$ is the trivial SU(2)-bundle. To simplify the notation slightly we will omit the reference to the bundle in the notation from this point. In the final part we also include the calculations of $I^\bullet(Y_\Gamma)$ for $\bullet \in \{+, -, \infty\}$, omitting some details.

The two results Theorem I.4.21 and Proposition I.4.25 give a concrete description of the Donaldson model $\mathrm{DCI}(\overline{Y}_\Gamma)$ in all cases. The key tools needed in the calculations are the index spectral sequence in combination with the explicit models for $DCI^\pm$ of Theorem I.5.8. In the I^+ case, we will see that the index spectral sequence immediately degenerates, and we are left with solving an extension problem in the category of $R[U]$-modules. In the I^- case there are a number of differentials in the index spectral sequence, but we will in each case verify that the spectral sequence degenerates on some finite page E^{r_0}. The extension problems in this case are neatly solved by observing that E^∞_s is a free $R[U]$-module for all s.

In [Eis19, Corollary 8.7] Miller calculates $I^\infty(Y)$ for all rational homology spheres. This, of course, includes $\overline{Y}_\Gamma$, so our calculation of this group should only be regarded as a verification of his result. In addition, we observe that the norm map vanishes in homology so that the exact triangle relating the three groups splits up into short exact sequences.

Throughout this section we let $A = \Lambda_R[u]$ with $|u| = 3$. Given a finite subgroup $\Gamma \subset \mathrm{SU}(2)$ we will always write $\mathcal{C}$ for the set of critical orbits, or equivalently, the set of 1-dimensional quaternionic representations of Γ. Recall that for each $\alpha \in \mathcal{C}$ we have fixed generators $b_\alpha \in H_0(\alpha)$, $t_\alpha = b_\alpha \cdot u \in H_3(\alpha)$ if α is irreducible and $t_\alpha \in H_2(\alpha)$ if α is reducible.

I.6.1 The Case I^+

In this section we will calculate $I^+(\overline{Y}_\Gamma)$ as an $R[U]$-module for all finite subgroups $\Gamma \subset \mathrm{SU}(2)$. Even though our arguments do not explicitly rely on the index spectral sequence, we note that the motivation for the upcoming arguments is the following observation.

Lemma I.6.1. *Let* $M = DCI(\overline{Y}_\Gamma)$ *for a finite subgroup* $\Gamma \subset \mathrm{SU}(2)$*. Then the index spectral sequence*

$$E^1_{s,t} = \bigoplus_{j(\alpha)\equiv s} H^+_A(\alpha)_t \implies I^+(\overline{Y}_\Gamma)_{s+t}$$

degenerates at the E^1 *page, that is, all differentials are trivial and* $E^1 = E^\infty$*.*

Proof. From the calculations in Lemma I.5.7 and Corollary I.5.9 we see that $H^+_A(\alpha)$ vanishes in odd degrees for all types of orbits α. Then, as $j(\alpha) \equiv 0 \pmod 4$ for all $\alpha \in \mathcal{C}$, it follows that $E^1_{s,t} = 0$ if s or t is odd. Since the differential d^r has bidegree $(-r, r-1)$ we conclude that $d^r = 0$ for all $r \geq 1$ as required. ■

From this one may determine $I^+(\overline{Y}_\Gamma)$ as an R-module. However, the structure as an $R[U]$-module is more subtle and this is the reason we spell out our argument more directly in terms of the filtration of the chain complex $DCI^+(\overline{Y}_\Gamma)$.

Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup and let $M = DCI(\overline{Y}_\Gamma)$. Let $\mathcal{C}^{irr} \subset \mathcal{C}$ denote the subset of free critical orbits or equivalently irreducible flat connections. Define $M^{irr} \subset M$ to be the submodule generated by $\mathcal{C}^{irr}$, i.e.,

$$M^{irr}_{s,t} = \bigoplus_{\substack{\alpha\in\mathcal{C}^{irr}\\ j(\alpha)\equiv s}} H_t(\alpha) \quad \text{and} \quad M^{irr}_n = \bigoplus_{s+t=n} M^{irr}_{s,t}.$$

The following lemma is the key tool in resolving our extension problems in the category of $R[U]$-modules.

Lemma I.6.2. *There is an R-linear map* $\psi\colon M \to M$ *of degree* -4 *satisfying*

(i) $\psi(x) \cdot u = (-1)^{|x|}\partial_M x$ *for all* $x \in M$*,*

(ii) $\mathrm{Im}(\psi) \subset M^{irr}$*,*

(iii) $\psi(\mathrm{Im}(u)) = 0$ *and*

(iv) $\psi(\mathrm{Im}\,\partial_M) = 0$.

Proof. By Theorem I.4.21 the only possibly nontrivial differentials in M are $\partial_M\colon M_{4s} \to M_{4s-1}$ for $s \in \mathbb{Z}$. The same result also shows that $u\colon M^{irr}_{4s-4} \to M^{irr}_{4s-1} = M_{4s-1}$ is an isomorphism for each s. We may therefore define $\psi\colon M_{4s} \to M^{irr}_{4s-4} \subset M_{4s-4}$ by $u^{-1} \circ \partial_M$ and set $\psi = 0\colon M_t \to M_{t-4}$ in all other degrees. The relation $\psi(x) \cdot u = (-1)^{|x|}\partial_M x$ is then satisfied for all $x \in M$ and by construction $\mathrm{Im}(\psi) \subset M^{irr}$. For (iii) and (iv) it is then sufficient to note that $\mathrm{Im}(u)$ and $\mathrm{Im}(\partial_M) \subset M^{irr}$ are supported in the degrees M_{4s-1} for $s \in \mathbb{Z}$. ■

Remark I.6.3. It is not difficult to show that the conditions (i) and (ii) determine the map ψ uniquely, but we will not use this explicitly.

Recall from Corollary I.5.47 that for $M = DCI(\overline{Y}_\Gamma)$ there is an identification $DCI^+(\overline{Y}_\Gamma)_n \cong \prod_{p\geq 0} M_{n-4p}$ for each n. Moreover, the differential is given by

$$\partial(m_{n-4p})_{p\geq 0} = (\partial_M m_{n-4p} - (-1)^n m_{n-4(p+1)} u)_{p\geq 0}.$$

The point of the map ψ should now be clear; given $m \in M_n$ the element $(\psi^p(x))_{p\geq 0} \in \prod_{p\geq 0} M_{n-4p}$ gives an explicit extension of m to a cycle in $DCI^+(Y)$.

Using the description of the differentials in Proposition I.4.21 we note that ψ is given on a generator $b_\alpha \in H_0(\alpha) \subset M_{4p}$ by

$$\psi(b_\alpha) = u^{-1}\left(\sum_{\beta\in\mathcal{C}^{irr}} n_{\beta\alpha} t_\beta\right) = \sum_{\beta\in\mathcal{C}^{irr}} n_{\beta\alpha} b_\beta \tag{I.38}$$

(the integers $n_{\beta\alpha}$ were defined in Definition I.4.20).

Recall from Lemma I.5.45 that

$$DCI^+(\overline{Y}_\Gamma) = \hat{C}_A^+(M) = \lim_p C_A^+(M/F_pM),$$

filtered by $F_p\hat{C}_A^+(M) = \hat{C}_A^+(F_pM) = \lim_{q<p} C_A^+(F_pM/F_qM)$. The concrete model given in Corollary I.5.47 applies equally well to these subcomplexes. Recall that we use the notation $\overline{F}_pM = F_pM/F_{p-1}M$.

Lemma I.6.4. *Let $M = DCI(\overline{Y}_\Gamma)$ be equipped with the index filtration. Then $F_{4p}M = F_{4p+t}M$ for all $p \in \mathbb{Z}$ and $0 \leq t \leq 3$. Moreover, there are identifications for $s \in \mathbb{Z}$ and $0 \leq t \leq 3$*

$$C_A^+(\overline{F}_{4p}M)_{4s+t} \cong \begin{cases} M_{4p+t} & \text{for } s \geq p \\ 0 & \text{for } s < p \end{cases} \tag{I.39}$$

$$\hat{C}_A^+(F_{4p}M)_{4s+t} \cong \begin{cases} \prod_{q\geq 0} M_{4(p-q)+t} & \text{for } s \geq p \\ \prod_{q\geq 0} M_{4(s-q)+t} & \text{for } s < p. \end{cases}$$

Define $\zeta\colon C_A^+(\overline{F}_{4p}M) \to \hat{C}_A^+(F_{4p}M)$ for $x \in M_{4p+t} \cong C^+(\overline{F}_{4p}M)_{4s+t}$, $s \geq p$, by

$$\zeta(x) = (\psi^q(x))_{q\geq 0} \in \prod_{q\geq 0} M_{4(p-q)+t} = \hat{C}_A^+(F_{4p}M),$$

where ψ is the map of Lemma I.6.2, and otherwise to be zero. Then ζ is a chain map splitting the exact sequence of DG R-modules (i.e., $\pi \circ \zeta = 1$)

$$0 \longrightarrow \hat{C}_A^+(F_{4p-1}M) \xrightarrow{\ i\ } \hat{C}_A^+(F_{4p}M) \underset{\curvearrowleft\,\zeta}{\xrightarrow{\ \pi\ }} C_A^+(\overline{F}_{4p}M) \longrightarrow 0.$$

Proof. The fact that $F_{4p+t}M = F_{4p}M$ for $0 \leq t \leq 3$ follows from the structure theorem I.4.21. From the same result one concludes that $F_{4p}M_n = M_n$ for $n \leq 4p+3$ and $F_{4p}M_n = 0$ for $n \geq 4(p+1)$. Therefore, $\overline{F}_{4p}M_n$ coincides with M_n for $4p \leq n \leq 4p+3$ and vanishes otherwise. Using this, the given degreewise formulas for $C_A^+(\overline{F}_{4p}M)$ and $\hat{C}_A^+(F_{4p}M)$ follow from Proposition I.5.8 and Corollary I.5.47, respectively.

Next we have to verify that ζ is a chain map. The differential in $C_A^+(\overline{F}_{4p}M)$ is given by $u\colon M_{4p} \to M_{4p+3}$ in degree $4s$ for $s > p$ and is otherwise 0. The differential in $\hat{C}_A^+(F_{4p}M)$ is given in degree $4s+t$ by

$$\partial((m_{4(s-q)+t})_{q\geq 0}) = (\partial_M m_{4(s-q)+t} - (-1)^t m_{4(s-q-1)+t}u)_{q\geq 0},$$

where if $s > p$ it is assumed that $m_{4(s-q)+t} = 0$ for $q \leq s-p$. In view of this formula and the fact that $\psi(x) \cdot u = (-1)^{|x|}\partial_M x$ for all $x \in M$ by Lemma I.6.2, we see that for $x \in M_{4p+t} = C_A^+(\overline{F}_{4p}M)_{4s+t}$, $s \geq p$,

$$\partial \circ \zeta(x) = \partial(\psi^q x)_{q\geq 0} = \begin{cases} 0 & \text{if } (s,t) = (p,0) \text{ or } 1 \leq t \leq 3 \\ (x \cdot u, 0, 0, \cdots) & \text{if } s > p, t = 0 \end{cases}$$

In the first case we also have $\partial x = 0$ and hence $\zeta \circ \partial(x) = 0$, while in the second case we have $\partial x = x \cdot u$ so that $\zeta \circ \partial(x) = (\psi^q(x \cdot u))_{q\geq 0} = (x \cdot u, 0, 0, \cdots)$ as ψ vanishes on $\operatorname{Im} u$ by Lemma I.6.2. In all other degrees ζ vanishes, so we may conclude from the above that ζ is a chain map.

Finally, the projection $\pi\colon \hat{C}_A^+(F_{4p}M) \to C_A^+(\overline{F}_{4p}M)$ is in terms of the formulas of (I.39) given by the projection $\prod_{q\geq 0} M_{4(p-q)+t} \to M_{4p+t}$ in degree $4s+t$ for $s \geq p$ and otherwise 0. From this description it is clear that $\pi \circ \zeta = 1$ and the proof is complete. ■

We need one final lemma describing the interaction between the map ζ and the action of $R[U]$.

Lemma I.6.5. *In the situation of the above lemma the map* $\zeta\colon C_A^+(\overline{F}_{4p}M) \to \hat{C}_A^+(F_{4p}M)$ *satisfies* $\zeta(Ux) = U\zeta(x)$ *for each* x *with* $|x| \neq 4p$. *Moreover, in degree* $4p$ *we have the following commutative diagram*

$$\begin{array}{ccccc} C_A^+(\overline{F}_{4p}M)_{4p} & \xrightarrow{\zeta} & \hat{C}_A^+(F_{4p}M)_{4p} & \xrightarrow{U} & \hat{C}_A^+(F_{4p}M)_{4(p-1)} \\ \downarrow{\scriptstyle\psi} & & & & \downarrow{\scriptstyle =} \\ C_A^+(\overline{F}_{4(p-1)}M)_{4(p-1)} & & \xrightarrow{\qquad\zeta\qquad} & & \hat{C}_A^+(F_{4(p-1)}M)_{4(p-1)}. \end{array}$$

Proof. In degree n for $n < 4p$ the map ζ vanishes, so the first assertion is clear in this case. In degree $4s+t$ with $s > p$ and $0 \leq t \leq 3$ we have, using the formulas of Lemma I.6.4,

$$C_A^+(\overline{F}_{4p}M)_{4s+t} = M_{4p+t} \quad \text{and} \quad \hat{C}_A^+(F_{4p}M)_{4s+t} = \prod_{q\geq 0} M_{4(p-q)+t}.$$

It then follows from Proposition I.5.8 and Corollary I.5.47 that the U action is given by the identity in these degrees. Therefore, $U \circ \zeta = \zeta \circ U$ in this case as well.

For $x \in M_{4p+t} = C_A^+(\overline{F}_{4p}M)_{4p+t}$, where $0 \leq t \leq 3$, we have $Ux = 0$, while

$$U(\zeta(x)) = U((\psi^q x)_{q\geq 0}) = (\psi^{q+1}x)_{q\geq 0} = (\psi^q(\psi x))_{q\geq 0} = \zeta(\psi(x)),$$

where we regard $\psi x \in M_{4(p-1)+t} = C_A^+(\overline{F}_{4(p-1)}M)_{4(p-1)+t}$. This shows that the given diagram commutes in degree $4p+t$ for $0 \leq t \leq 3$. However, in the proof of Lemma I.6.2 it was shown that $\psi\colon M_{4p+t} \to M_{4(p-1)+t}$ vanishes for $1 \leq t \leq 3$. Therefore $\zeta \circ U = U \circ \zeta$ in degree $4p+t$ for $1 \leq t \leq 3$ as well. This completes the proof. ∎

To describe $I^+(\overline{Y}_\Gamma)$ and later $I^-(\overline{Y}_\Gamma)$, it is convenient to introduce the following definition.

Definition I.6.6. For an $R[U]$-module X define the mod 8 periodic $R[U]$-modules $X^{\Pi,8}$ and $X^{\oplus,8}$ degreewise by

$$X_n^{\Pi,8} = \prod_{s\in\mathbb{Z}} X_{n+8s} \quad \text{and} \quad X_n^{\oplus,8} = \bigoplus_{s\in\mathbb{Z}} X_{n+8s}.$$

The maps $U\colon X_n^{\Pi,8} \to X_{n-4}^{\Pi,8}$ and $U\colon X_n^{\oplus,8} \to X_{n-4}^{\oplus,8}$ are defined to be the product and direct sum over the maps $U : X_{n+8s} \to X_{n-4+8s}$ for $s \in \mathbb{Z}$, respectively.

Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup. Given $\alpha \in \mathcal{C}$ we will use the shorthand notation $H_A^+(\alpha) := H_A^+(H_*(\alpha))$. Write $\mathcal{C} = \mathcal{C}^{irr} \cup \mathcal{C}^{red} \cup \mathcal{C}^{f.red}$ for the decomposition of the critical orbits into irreducible, reducibles and fully reducibles. For each $\eta \in \mathcal{C}^{f.red}$, $\lambda \in \mathcal{C}^{red}$ and $\alpha \in \mathcal{C}^{irr}$ introduce variables V_η, W_λ and a generator g_α so that

$$H_A^+(\eta) = R[V_\eta], \quad H_A^+(\lambda) = R[W_\lambda] \quad \text{and} \quad H_A^+(\alpha) = R \cdot g_\alpha,$$

where $|V_\eta| = 4$, $|W_\lambda| = 2$ and $|g_\alpha| = 0$. This is justified by the calculations of Lemma I.5.7 and Corollary I.5.9. In the following we regard the relative grading $j : \mathcal{C} \to \mathbb{Z}/8$ as taking values in $\{0, 4\}$.

Theorem I.6.7. *The positive equivariant instanton Floer homology $I^+(\overline{Y}_\Gamma)$ associated with the trivial $\mathrm{SU}(2)$-bundle over $\overline{Y}_\Gamma$ is given by*

$$\left[\left(\bigoplus_{\alpha\in\mathcal{C}^{irr}} R\cdot g_\alpha[j(\alpha)]\right) \oplus \left(\bigoplus_{\lambda\in\mathcal{C}^{red}} R[W_\lambda][j(\lambda)]\right) \oplus \left(\bigoplus_{\eta\in\mathcal{C}^{f.red}} R[V_\eta][j(\eta)]\right)\right]^{\Pi,8}.$$

The $R[U]$-module structure is determined by

$$U \cdot V_\eta^p = \begin{cases} V_\eta^{p-1} & \text{for } p > 0 \\ \sum_{\rho\in C^{irr}} n_{\rho\eta} g_\rho & \text{for } p = 0 \end{cases}$$

$$U \cdot W_\lambda^p = \begin{cases} W_\lambda^{p-2} & \text{for } p > 0 \\ \sum_{\rho\in C^{irr}} n_{\rho\lambda} g_\rho & \text{for } p = 0 \end{cases}$$

$$U \cdot g_\alpha = \sum_{\rho\in C^{irr}} n_{\rho\alpha} g_\rho,$$

where we interpret $W_\beta^{-1} = 0$ *and the integers* $n_{\rho_1\rho_2}$ *are given in Proposition I.4.25 for each pair* $(\rho_1, \rho_2) \in \mathcal{C}$.

Proof. Let as usual $M = DCI(\overline{Y}_\Gamma)$ be equipped with the index filtration. By Lemma I.6.4 we have $F_{4p}M = F_{4p+t}M$ for $0 \leq t \leq 3$, and the map $\zeta\colon C_A^+(\overline{F}_{4p}M) \to \hat{C}_A^+(F_{4p}M)$ along with the inclusion $i\colon \hat{C}_A^+(F_{4(p-1)}M) \to \hat{C}_A^+(F_{4p}M)$ define a chain isomorphism

$$i + \zeta : \hat{C}_A^+(F_{4(p-1)}M) \oplus C_A^+(\overline{F}_{4p}M) \cong \hat{C}_A^+(F_{4p}M)$$

of *DG R*-modules for each p. By induction we obtain isomorphisms

$$\hat{C}_A^+(F_{4p}M) \cong \hat{C}_A^+(F_{4(p-r-1)}M) \oplus \left(\bigoplus_{q=0}^{r} C_A^+(\overline{F}_{4(p-q)}M)\right)$$

for each p and $r \geq 1$. Since the filtration $F_p\hat{C}_A^+(M) = \hat{C}_A^+(F_pM)$ is degreewise bounded above and complete Hausdorff, it follows by first passing to the limit over r and then to the colimit over p that we obtain an isomorphism of DG R-modules

$$\hat{C}_A^+(M) \cong \prod_{p\in\mathbb{Z}} C_A^+(\overline{F}_{4p}M).$$

Hence, as products commutes with homology,

$$I^+(\overline{Y}_\Gamma) = \hat{H}_A^+(M) \cong \prod_{p\in\mathbb{Z}} H_A^+(\overline{F}_{4p}M). \tag{I.40}$$

Write $\mathcal{C} = \mathcal{C}^0 \cup \mathcal{C}^1$ where $\mathcal{C}^i = \{\alpha \in \mathcal{C} : j(\alpha) \equiv 4i \pmod 8\}$ for $i = 0, 1$. Then

$$H_A^+(\overline{F}_{4p}M) = \begin{cases} \bigoplus_{\alpha\in\mathcal{C}^0} H_A^+(\alpha)[4p] & \text{if } p \equiv 0 \pmod 2 \\ \bigoplus_{\alpha\in\mathcal{C}^1} H_A^+(\alpha)[4p] & \text{if } p \equiv 1 \pmod 2, \end{cases}$$

and we may simplify the expression in (I.40)

$$\prod_{p\in\mathbb{Z}} H_A^+(\overline{F}_{4p}M) = \prod_{p\in\mathbb{Z}} \bigoplus_{\alpha\in\mathcal{C}} H_A^+(\alpha)[8p + j(\alpha)] = \left(\bigoplus_{\alpha\in\mathcal{C}} H_A^+(\alpha)[j(\alpha)]\right)^{\Pi,8}. \tag{I.41}$$

By replacing each $H_A^+(\alpha)$ with $R[V_\alpha]$, $R[W_\alpha]$ or $R \cdot g_\alpha$ according to whether α is fully reducible, reducible or irreducible we obtain the additive statement of the theorem.

To finish the proof we need to determine the action of U. Returning to the notation of equation (I.40), for a fixed degree n, we have $I^+(\overline{Y}_\Gamma)_n = \hat{H}_A^+(F_{4p}M)_n$ for each p with $4p > n$. This is a consequence of the formula (I.39) given in Lemma I.6.4. Fix the minimal p with $4p > n$ and consider the commutative diagram

$$\begin{array}{ccccc} I^+(\overline{Y}_\Gamma)_n & \xleftarrow{\cong} & \hat{H}_A^+(F_{4p}M)_n & \xleftarrow{\phi} & \prod_{q:4q\leq n} H_A^+(\overline{F}_{4q}M)_n \\ \downarrow U & & \downarrow U & & \downarrow U' \\ I^+(\overline{Y}_\Gamma)_{n-4} & \xleftarrow{\cong} & \hat{H}_A^+(F_{4p}M)_{n-4} & \xleftarrow{\phi} & \prod_{q:4q\leq n} H_A^+(\overline{F}_{4(q-1)}M)_{n-4}. \end{array}$$

Here, ϕ is the isomorphism obtained by taking the limit, as $j \to \infty$, over the maps

$$([x_q])_{j \leq q \leq n/4} \mapsto \sum_{j \leq q \leq n/4} [\zeta(x_q)],$$

where $x_q \in C_A^+(\overline{F}_{4q}M)$ represents $[x_q] \in H_A^+(\overline{F}_{4q}M)$ and U' is the map that forces the right rectangle to commute. Our task is to determine U'. By Lemma I.6.5 we have for $y \in H_A^+(\overline{F}_{4q}M)_n$ that $H(\zeta) \circ U(y) = U \circ H(\zeta)(y)$ provided $n > 4q$, while if $n = 4q$, and $y = [x]$ then $U \circ H(\zeta)(y) = [\psi x]$. It follows that $U' = \prod U$ if n is not divisible by 4. Otherwise, if $4|n$ so that $n = 4(p-1)$, then $U' = \tau + \prod U$, where $\tau = ([x_q])_{q \leq p-1} = [\psi(x_{p-1})]$. Therefore, in the description

$$I^+(\overline{Y}_\Gamma) = \prod_{p \in \mathbb{Z}} \bigoplus_{\alpha \in \mathcal{C}} H_A^+(\alpha)[8p + j(\alpha)]$$

of (I.41), the action of U is given by taking the product over the internal $R[U]$-module structure of each factor $H_A^+(\alpha)$ and adding the correction term τ. For each p, τ only affects the terms $H_A^+(\alpha)[8p + j(\alpha)]_{8p+j(\alpha)} = H_A^+(\alpha)_0$. In terms of our generators this correspond to V_α^0 if α is fully reducible, W_α^0 if α is reducible and g_α if α is irreducible. Now, we may require that these generators correspond to $b_\alpha \in H_0(\alpha) \subset M_{4q} \cong C_A^+(\overline{F}_{4q}M) = H_A^+(\overline{F}_{4q}M)_{4q}$ under the identifications of Lemma I.6.4. Therefore, using the explicit formula (I.38) for ψ, we obtain

$$U \cdot V_\eta^0 = \sum_{\rho \in \mathcal{C}^{irr}} n_{\rho\eta} g_\rho$$

and similarly for the other types of generators. This completes the proof. ■

Example I.6.8. For $Y = \overline{Y}_{O^*}$ there are two irreducibles α, β, the trivial connection θ and a fully reducible η. The grading is given by $j(\theta) = j(\beta) = 0$ and $j(\alpha) = j(\eta) = 4$. Furthermore, $n_{\alpha\beta} = n_{\beta\alpha} = 3$ and $n_{\alpha\theta} = n_{\beta\eta} = 1$. This is obtained from Proposition I.4.25. The above theorem gives

$$I^+(Y) = \left(R[V_\theta] \oplus R \cdot g_\beta \oplus R[V_\eta][4] \oplus R \cdot g_\alpha[4]\right)^{\Pi,8}.$$

This means that

$$I^+(Y)_n = \begin{cases} R\{g_\beta\} \oplus \prod_{i \geq 0} R\{V_\theta^{2i}, V_\eta^{2i-1}\} & n \equiv 0 \pmod 8 \\ R\{g_\alpha\} \oplus \prod_{i \geq 0} R\{V_\theta^{2i+1}, V_\eta^{2i}\} & n \equiv 4 \pmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

For $x = (r_{2i} V_\eta^{2i})_{i \geq 0} \in I^+(Y)_4$, $r_{2i} \in R$, we have $U \cdot x = r_0 g_\alpha + (r_{2i} V_\eta^{2i-1})_{i \geq 1} \in I^+(Y)_0$. For $x = r g_\alpha$, $r \in R$, we have $U \cdot x = 3 r g_\beta$.

I.6.2 The Case I^-

In this section we will calculate $I^-(\overline{Y}_\Gamma)$ for all finite subgroups $\Gamma \subset \mathrm{SU}(2)$. In contrast to the calculations in the previous section, there will be a number of nontrivial differentials in the index spectral sequence. Nevertheless, the spectral sequence will still stabilize after a finite number of steps in each case.

Lemma I.6.9. *Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup. Then the only possibly nontrivial differentials in the index spectral sequence*

$$E^1_{s,t} = \bigoplus_{j(\alpha)\equiv s} H^-_A(\alpha)_t \implies I^-(\overline{Y}_\Gamma)_{s+t}$$

are $d^{4r}\colon E^{4r}_{4(s+r),-4(r-1)} \to E^{4r}_{4s,3}$ for s and $r \geq 1$.

Proof. As $j(\alpha) \equiv 0 \mod 4$ for each $\alpha \in \mathcal{C}$, we have $E^1_{s,*} = 0$ unless $4|s$. By Lemma I.5.7 and Corollary I.5.9 the $R[U]$-module $H^-_A(\alpha)$ is given by $R[U]$ with $|U| = -4$, $R[Z][2]$ with $|Z| = -2$ or $R[3]$ depending on whether α is fully reducible, reducible or irreducible, respectively. This means that $E^1_{4s,t} = 0$ for $t \geq 4$ and odd $t \leq 2$. Since the differential d^r has bidegree $(-r, r-1)$ we deduce that it can only be nonzero if it lands in the bidegree $(4s, 3)$ for some s. The only such differentials that also begin in a possibly nonzero group are $d^{4r}\colon E^{4r}_{4(s-r),-4(r-1)} \to E^{4r}_{4s,3}$. ∎

Our first task will be to give an explicit formula for the nontrivial differentials. Fix a finite subgroup $\Gamma \subset \mathrm{SU}(2)$ and let $M = DCI(\overline{Y}_\Gamma)$ be equipped with the index filtration. Recall that by Lemma I.5.45

$$DCI^-(\overline{Y}_\Gamma) = \operatorname{colim}_p C^-_A(F_pM) = \bigcup_p C^-_A(F_pM).$$

In the following we will use the fact from Lemma I.6.4 that $F_{4p+t}M = F_{4p}M$ for all $p \in \mathbb{Z}$ and $0 \leq t \leq 3$ without further mention.

Lemma I.6.10. *For all $p, s \in \mathbb{Z}$ and $0 \leq t \leq 3$ we have*

$$C^-_A(F_{4p}M)_{4s+t} \cong \begin{cases} \bigoplus_{0\leq q\leq p-s} M_{4(s+q)+t} & \text{if } s \leq p \\ 0 & \text{if } s > p \end{cases}$$

$$C^-_A(\overline{F}_{4p}M)_{4s+t} \cong \begin{cases} M_{4p+t} & \text{if } s \leq p \\ 0 & \text{if } s > p \end{cases}$$

The differential in $C^-_A(\overline{F}_{4p}M)$ is given by $u\colon M_{4p} \to M_{4p+3}$ in degree $4s$ with $s \leq p$, and vanishes otherwise. The differential in $C^-_A(F_{4p}M)$ is given by

$$\partial((x_q)_{0\leq q\leq p-s}) = (\partial_M x_q - (-1)^t x_{q-1}u)_{0\leq q\leq p-s+1}$$

for $(x_q)_{0\leq q\leq p-s} \in \bigoplus_{0\leq q\leq p-s} M_{4(s+q)+t} = C^-_A(F_{4p}M)_{4s+t}$ where $s \leq p$, and we interpret $x_{-1} = 0 = x_{p-s+1}$.

Proof. This is a simple consequence of Proposition I.5.8 using the fact that $F_{4p}M_n = M_n$ for $n \leq 4p+3$ and $F_{4p}M_n = 0$ otherwise. ∎

For each $\eta \in \mathcal{C}^{f.red}$, $\lambda \in \mathcal{C}^{red}$ and $\alpha \in \mathcal{C}^{irr}$ we introduce variables U_η, Z_λ and a generator h_α. Then, according to Lemma I.5.7 and Corollary I.5.9,

$$H^-_A(\eta) = R[U_\eta], \quad H^-_A(\lambda) = R[Z_\lambda][2] \quad \text{and} \quad H^-_A(\alpha) = R \cdot h_\alpha$$

where $|U_\eta| = -4$, $|Z_\lambda| = -2$ and $|h_\alpha| = 3$.

Lemma I.6.11. *Let $\{E^r, d^r\}_{r\geq 1}$ be the index spectral sequence with*

$$E^1_{s,t} = \bigoplus_{j(\alpha)\equiv s} H^-_A(\alpha)_t \implies I^-_{s+t}(\overline{Y}_\Gamma).$$

Then, for each $r \geq 1$ and s, $E^{4r}_{4s,-4(r-1)} = E^1_{4s,-4(r-1)}$ is a free R-module with generators

$$\{U^{r-1}_\eta : \eta \in \mathcal{C}^{f.red} \ni j(\eta) \equiv 4s\} \cup \{Z^{2r-1}_\lambda : \lambda \in \mathcal{C}^{red} \ni j(\lambda) \equiv 4s\}$$

and $E^1_{4s,3}$ is a free R-module with generators $\{h_\alpha : \alpha \in \mathcal{C}^{irr} \ni j(\alpha) \equiv 4s\}$. Let $p_r : E^1_{4s,3} \to E^{4r}_{4s,3}$ denote the natural surjection. Then in terms of these generators the differential $d^r \colon E^{4r}_{4(s+r),-4(r-1)} \to E^{4r}_{4s,3}$ is determined by

$$d^{4r}(U^{r-1}_\eta) = p_r\left(\sum_{(\alpha_1,\cdots,\alpha_r)} n_{\alpha_1\eta}\left(\prod_{i=1}^{r-1} n_{\alpha_{i+1}\alpha_i}\right)\cdot h_{\alpha_r}\right)$$
$$d^{4r}(W^{2r-1}_\lambda) = p_r\left(\sum_{(\alpha_1,\cdots,\alpha_r)} n_{\alpha_1\lambda}\left(\prod_{i=1}^{r-1} n_{\alpha_{i+1}\alpha_i}\right)\cdot h_{\alpha_r}\right),$$

where the sums are taken over tuples $(\alpha_1,\cdots,\alpha_r) \in (\mathcal{C}^{irr})^r$ for which $(\eta,\alpha_1,\cdots,\alpha_r)$, respectively $(\lambda,\alpha_1,\cdots,\alpha_r)$, form a path in the graph $\mathcal{S}_\Gamma$ (see Definition I.3.9).

Proof. Before we introduce the explicit generators we will work out a formula for the differentials in terms of the complex M. By Lemma I.6.9 we have $E^{4r}_{4(s+r),-4(r-1)} = E^1_{4(s+r),-4(r-1)} = H^-_A(\overline{F}_{4(s+r)}M)_{4(s+1)}$ and by Lemma I.6.10 the latter group may be identified with $\mathrm{Ker}(u) \subset M_{4(s+r)}$. Recall from diagram (I.35) that the differential d^{4r} is defined by first applying the connecting homomorphism $\delta\colon H^-_A(\overline{F}_{4(s+r)}M) \to H^-_A(F_{4(s+r)-1}M)$, then lifting along the map $H^-_A(F_{4s}M) \to H^-_A(F_{4(s+r)-1}M)$ and then pushing down along $H^-_A(F_{4s}M) \to H^-_A(\overline{F}_{4s}M)$. On the chain level we have the following diagram

$$\begin{array}{ccccc}
C^-_A(\overline{F}_{4(s+r)}M)_{4(s+1)} & \overset{\delta}{\dashrightarrow} & C^-_A(F_{4(s+r)-1}M)_{4s+3} & \overset{\iota}{\longleftarrow} & C^-_A(F_{4s}M)_{4s+3} \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
M_{4(s+r)} & \overset{\delta}{\dashrightarrow} & \bigoplus_{q=0}^{r-1} M_{4(s+q)+3} & \overset{\iota}{\longleftarrow} & M_{4s+3}.
\end{array}$$

The arrows correspond to the connecting homomorphism δ are dashed as they are not well-defined before passing to homology. Given $x \in \mathrm{Ker}\, u \subset M_{4(s+r)}$ the element $\delta(x)$ is represented on the chain level by $(0,0,\cdots,0,\partial_M x)$. As ι is the inclusion of the first summand, this element does not lift along ι. Let $\psi : M \to M$ be the map of Lemma I.6.2 and let

$$y := (\psi^{r-q}x)^{r-1}_{q=1} \in \bigoplus_{q=1}^{r-1} M_{4(s+q)} \cong C^-_A(F_{4(s+r-1)}M)_{4(s+1)}.$$

Then $\partial(y) = (\partial_M \psi^{r-1}x, 0, 0, \cdots, 0, -\partial_M x)$, which implies that $\iota[\partial_M \psi^{r-1}x] = [(0, \cdots, 0, \partial_M x)]$ in homology. By Lemma I.6.10 we have the following sequence of isomorphisms

$$M_{4s+3} \cong C_A^-(F_{4s}M)_{4s+3} \cong C_A^-(\overline{F}_{4s}M)_{4s+3} \cong H_A^-(\overline{F}_{4s}M)_{4s+3} \cong E^1_{4s,3}.$$

Moreover, according to Lemma I.6.9, there are no outgoing differentials from $E^r_{4s,3}$ for $r \geq 1$. We conclude that $d^{4r}(x)$ is given by the image of $\partial_M \psi^{r-1}x$ along the surjection $E^1_{4s,3} \to E^{4r}_{4s,3}$ for each $x \in E^{4r}_{4(s+r),-4(r-1)} \subset M_{4(s+r)}$. In other words, we have the following commutative diagram

$$\begin{array}{ccccc} E^{4r}_{4(s+r),-4(r-1)} & \xrightarrow{=} & E^1_{4(s+r),-4(r-1)} & \xrightarrow{\cong} & \operatorname{Ker}(u) \subset M_{4(s+r)} \\ \downarrow{\scriptstyle d^{4r}} & & & & \downarrow{\scriptstyle \partial_M \circ \psi^{(r-1)}} \\ E^{4r}_{4s,3} & \xleftarrow{p_r} & E^1_{4s,3} & \xrightarrow{\cong} & M_{4s+3}. \end{array} \tag{I.42}$$

Now, $\operatorname{Ker} u \subset M_{4(s+r)}$ is freely generated by $b_\eta \in H_0(\eta)$, $b_\lambda \in H_0(\lambda)$ for $\eta \in \mathcal{C}^{f.red}$ and $\lambda \in \mathcal{C}^{red}$ with $j(\eta) \equiv j(\lambda) \equiv 4(s+r)$. It is important to note that under the top horizontal identification in (I.42) we may, and will, require that these correspond to the generators U_η^{r-1} and W_λ^{2r-1} in $E^1_{4(s+r),-4(r-1)}$. Similarly, M_{4s+3} is freely generated by $t_\alpha = b_\alpha \cdot u \in H_3(\alpha)$ for $\alpha \in \mathcal{C}^{irr}$ with $j(\alpha) \equiv 4s$, and under the lower horizontal isomorphism we require that these correspond to the generators $h_\alpha \in E^1_{4s,3}$. Recall from equation (I.38) that

$$\psi(b_\eta) = u^{-1}\partial_M(b_\eta) = \sum_{\alpha \in \mathcal{C}^{irr}} u^{-1}(n_{\alpha\eta}t_\alpha) = \sum_{\alpha \in \mathcal{C}^{irr}} n_{\alpha\eta}b_\alpha.$$

and similarly for $\psi(b_\lambda)$. Taking into account the fact that $n_{\alpha\beta} = 0$ whenever α and β are not adjacent in the graph $\mathcal{S}_\Gamma$, we obtain

$$\partial_M \psi^{r-1}(b_\eta) = \sum_{(\alpha_1, \cdots, \alpha_r)} (n_{\alpha_1\eta}n_{\alpha_2\alpha_2} \cdots n_{\alpha_r\alpha_{r-1}}) \cdot t_{\alpha_r},$$

where the sum is taken over $(\alpha_1, \cdots, \alpha_r) \in (\mathcal{C}^{irr})^r$ for which $(\eta, \alpha_1, \cdots, \alpha_r)$ forms an edge path in S_Γ. An analogous formula holds for b_λ, so in view of diagram (I.42) the proof is complete. ■

By combining the above lemma with Theorem I.4.25 we obtain complete control over all the differentials in the spectral sequences. The following result takes care of all the extension problems we will meet.

Lemma I.6.12. *Let* $\Gamma \subset \mathrm{SU}(2)$ *be a finite subgroup and let* $(E^r, d^r)_{r \geq 1}$ *be the index spectral sequence with*

$$E^1_{s,t} = \bigoplus_{j(\alpha) \equiv s} H_A^-(\alpha)_t \implies I^-(\overline{Y}_\Gamma)_{s+t}.$$

Assume that E_s^∞ *is a free* $R[U]$*-module for* $s = 0, 4$. *Then there is an isomorphism of* $R[U]$*-modules*

$$I^-(\overline{Y}_\Gamma) \cong (E_0^\infty \oplus E_4^\infty)^{\oplus,8}.$$

Proof. Recall that the index spectral sequence is periodic in the sense that $E^r_{s,t} \cong E^r_{s+8,t}$, or equivalently $E^r_s[8] \cong E^r_{s+8}$, for all $s, t \in \mathbb{Z}$ and $r \in \mathbb{N} \cup \{\infty\}$. Therefore, the assumption that E^∞_0 and E^∞_4 are free $R[U]$-modules implies that E^∞_s is a free $R[U]$-module for each $s \in \mathbb{Z}$, as $E^1_s = 0$ for all s not divisible by 4.

Write $I = I^-(\overline{Y}_\Gamma)$. By Theorem I.5.58 $(E^r, d^r)_{r\geq 1}$ is a spectral sequence of $R[U]$-modules that converges strongly to I. In other words, I carries an exhaustive and complete Hausdorff filtration $\{F_s I\}_s$ of $R[U]$-submodules and there is an isomorphism $E^\infty_s \cong \overline{F}_s I$ of $R[U]$-modules for each $s \in \mathbb{Z}$. In view of the fact that E^∞_s is free over $R[U]$, the short exact sequence

$$0 \longrightarrow F_{s-1} I \xrightarrow{\iota_s} F_s I \xrightarrow{\pi_s} \overline{F}_s I \cong E^\infty_s \longrightarrow 0$$

splits in the category of $R[U]$-modules for each $s \in \mathbb{Z}$. Hence, $F_{s-1} I \oplus \overline{F}_s I \cong F_s I$ and inductively

$$F_{s-1} I \oplus \left(\bigoplus_{t=0}^{r} \overline{F}_{s+t} I \right) \cong F_{s+r} I$$

for $s \in \mathbb{Z}$ and $r \geq 1$. These are all isomorphisms of $R[U]$-modules. By Lemma I.6.10 we have $E^1_{s,t} = 0$ for all s and $t \geq 4$. Therefore, for fixed $n \in \mathbb{Z}$, $(E^1_s)_n = E^1_{s,n-s} = 0$ for all $s \leq n-4 =: s_0$. Hence, $(\overline{F}_s I)_n = 0$ for all $s \leq s_0$. Since the filtration of I is complete Hausdorff it follows that $(F_s I)_n = 0$ for all $s \leq s_0$. Consequently, using the above isomorphisms, we obtain

$$I_n = \mathrm{colim}_{s \geq s_0} (F_s I)_n \cong \mathrm{colim}_{s\geq s_0} \bigoplus_{i=s_0}^{s} (\overline{F}_s I)_n \cong \bigoplus_s (E^\infty_s)_n$$

for each $n \in \mathbb{Z}$. These isomorphisms piece together to an isomorphism $I \cong \bigoplus_s E^\infty_s$ of $R[U]$-modules. Finally, we may exploit the periodicity of E^∞ and the fact that $E^\infty_s = 0$ unless $4|s$ to simplify:

$$I \cong \bigoplus_s E^\infty_s \cong \bigoplus_s (E^\infty_{8s} \oplus E^\infty_{8s+4}) \cong \bigoplus_s (E^\infty_0[8s] \oplus E^\infty_4[8s]) = (E^\infty_0 \oplus E^\infty_4)^{\oplus,8}$$

and the proof is complete. ■

It is not possible to give a uniform result for the calculation of $I^-(\overline{Y}_\Gamma)$ for all the subgroups $\Gamma \subset \mathrm{SU}(2)$ simultaneously. We will therefore treat the simplest cases $\Gamma = I^*, O^*, T^*, C_m$ first and treat the more involved case of binary dihedral groups afterwards.

We have the following table over the flat connections extracted from Appendix *A*. The notation is compatible with Proposition I.4.25.

Γ	$\mathcal{C}^{f.red}$	$\mathcal{C}^{irr}$	$\mathcal{C}^{red}$
C_{2m}	θ, η		$\lambda_1, \cdots, \lambda_{m-1}$
C_{2m+1}	θ		$\lambda_1, \cdots, \lambda_m$
I^*	θ	α, β	
O^*	θ, η	α, β	
T^*	θ	α	λ

Theorem I.6.13. *The negative equivariant instanton Floer homology associated with the trivial* $\mathrm{SU}(2)$*-bundle over* $\overline{Y}_\Gamma$ *for* $\Gamma = C_{2m}, C_{2m+1}$ *is given by*

$$I^-(\overline{Y}_{C_{2m}}) = \left((R[U_\theta] \oplus R[U_\eta][4m] \oplus \bigoplus_{i=1}^{m-1} R[Z_{\lambda_i}][4i+2] \right)^{\oplus,8}$$

$$I^-(\overline{Y}_{C_{2m+1}}) = \left(R[U_\theta] \oplus \bigoplus_{i=1}^{m} R[Z_{\lambda_i}][4i+2] \right)^{\oplus,8},$$

while for $\Gamma = I^*, O^*, T^*$ *we have* $I^-(\overline{Y}_\Gamma) = (X_\Gamma)^{\oplus,8}$ *where* X_Γ *is the* $R[U]$*-submodule of* P_Γ *generated by* G_Γ *specified in the following table.*

Γ	P_Γ	G_Γ
I^*	$R[U_\theta]$	$\{U_\theta^2\}$
O^*	$R[U_\theta] \oplus R[U_\eta][4]$	$\{U_\theta^1, U_\eta^1\}$
T^*	$R[U_\theta] \oplus R[Z_\lambda][2]$	$\{U_\theta^1, Z_\lambda^0, 3U_\theta^0 - Z_\lambda^1\}$

Proof. For $\Gamma = I^*, O^*, T^*, C_n$ we will write $\mathcal{C} = \mathcal{C}_\Gamma$ for the set of flat connections and $(E^r, d^r)_{r\geq 1}$ will denote the corresponding index spectral sequence with

$$E^1_{s,t} = \bigoplus_{j(\alpha)\equiv s} H_A^-(\alpha)_t \implies I^-(\overline{Y}_\Gamma)_{s+t}.$$

In each case we will calculate E^∞ and observe that E_0^∞ and E_4^∞ are free $R[U]$-modules. By Lemma I.6.12 this is enough to conclude that $I^-(\overline{Y}_\Gamma) = (E_0^\infty \oplus E_4^\infty)^{\oplus,8}$. In the following we will make consistent use of Lemma I.6.9, Theorem I.4.25 and Lemma I.6.11 that together determine all the differentials explicitly.

$\mathbf{C_n}$. As $\mathcal{C}^{irr} = \emptyset$ there are no nontrivial differentials in the spectral sequence so that $E^1 = E^\infty$. For $n = 2m$ we have $\mathcal{C} = \{\theta, \eta, \lambda_1, \cdots, \lambda_{m-1}\}$ with θ, η fully reducible and the λ_i reducible. The grading is given by $j(\lambda_i) \equiv 4i$ and $j(\eta) \equiv 4m$. This gives

$$E_0^\infty \oplus E_4^\infty = R[U_\theta] \oplus R[U_\eta][j(\eta)] \oplus \left(\bigoplus_{i=1}^{m-1} R[Z_{\lambda_i}][j(\lambda_i)+2] \right).$$

For $n = 2m+1$ we have $\mathcal{C} = \{\theta, \lambda_1, \cdots, \lambda_m\}$ with θ fully reducible and the λ_i reducible. The grading is given by $j(\lambda_i) \equiv 4i$. Hence,

$$E_0^\infty \oplus E_4^\infty = R[U_\theta] \oplus \left(\bigoplus_{i=1}^{m} R[Z_{\lambda_i}][j(\lambda_i)+2] \right).$$

In both cases these are free $R[U]$-modules ($R[Z_\lambda]$ is freely generated by $\{Z_\lambda^0, Z_\lambda^1\}$). The stated results are obtained by applying $X \mapsto X^{\oplus,8}$ to the above formulas for $E_0^\infty \oplus E_4^\infty$.

I*. In this case we have $\mathcal{C} = \{\theta, \alpha, \beta\}$ with α and β irreducible. The grading is given by $j(\theta) = j(\beta) = 0$ and $j(\alpha) = 4$. The first nontrivial differential in the index spectral sequence is $d^4\colon E^4_{8s,0} = R\{U^0_\theta\} \to R\{h_\alpha\} = E^4_{8s-4,3}$ and is given by $d^4(U^0_\theta) = n_{\alpha\eta}h_\alpha = h_\alpha$. Therefore, $E^\infty_{8s+4,3} = 0 = E^\infty_{8s,0}$. The next differential is $d^8\colon E^8_{8s,-4} = R\{U^1_\theta\} \to R\{h_\beta\} = E^8_{8(s-1),3}$ and is given by $d^8(U^1_\theta) = n_{\beta\alpha}n_{\alpha\eta}h_\beta = 4h_\beta$. As $2 \in R$ is invertible, this is an isomorphism and we conclude that $E^\infty_{8s,3} = 0$ and $E^\infty_{8s,-4} = 0$. There are no more nontrivial differentials in the spectral sequence. Therefore, $E^\infty_4 = 0$ and $E^\infty_0 \subset R[U_\theta]$ is the free $R[U]$-submodule generated by U^2_θ as required.

O*. Here we have $\mathcal{C} = \{\theta, \eta, \alpha, \beta\}$ with θ, η fully reducible and α, β irreducible. The grading is $j(\theta) = j(\beta) = 0$ and $j(\alpha) = j(\eta) = 4$. The first nontrivial differentials are

$$d^4\colon E^4_{8s+4,0} = R\{U^0_\eta\} \to R\{h_\beta\} = E^4_{8s,3}$$
$$d^4\colon E^4_{8s,0} = R\{U^0_\theta\} \to R\{h_\alpha\} = E^4_{8s-4,3}.$$

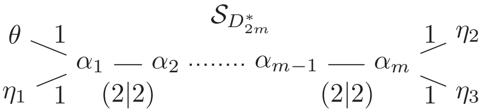
These are both isomorphisms as $n_{\beta\eta} = 1 = n_{\alpha\theta}$. This implies that $E^\infty_{8s,3} = E^5_{8s,3} = 0$ and $E^\infty_{8s-4,3} = E^5_{8s-4,3} = 0$. There can therefore be no more nontrivial differentials. Hence, $E^\infty_0 \subset R[U_\theta]$ and $E^\infty_4 \subset R[U_\eta][4]$ are the free $R[U]$-submodules generated by U^1_θ and U^1_η, respectively.

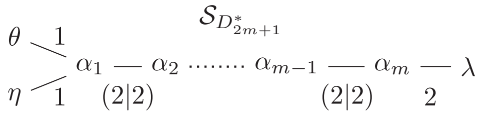
T*. In this case $\mathcal{C} = \{\theta, \alpha, \lambda\}$ with θ fully reducible, α irreducible and λ reducible. The grading is $j(\theta) = j(\lambda) = 0$ and $j(\alpha) = 4$. The first nontrivial differential is $d^4\colon E^4_{8s,0} = R\{Z^1_\lambda, U^0_\theta\} \to R\{h_\alpha\} = E^4_{8s-4,3}$ and is determined by $d^4(U^0_\theta) = n_{\alpha\theta}h_\alpha = h_\alpha$ and $d^4(Z^1_\lambda) = n_{\alpha\lambda}h_\alpha = 3h_\alpha$. This map is surjective with kernel $R\{3U^0_\theta - Z^1_\lambda\}$. Therefore, $E^\infty_{8s-4,3} = E^4_{8s-4,3} = 0$ and there are no more nontrivial differentials. We find $E^\infty_4 = 0$, $(E^\infty_0)_0 = R\{3U^0_\theta - Z^1_\lambda\}$ and $(E^\infty_0)_n = (R[U_\theta] \oplus R[Z_\lambda][2])_n$ in all other degrees n. It now suffices to observe that E^∞_0 is indeed the free submodule of $R[U_\theta] \oplus R[Z_\lambda][2]$ generated by $\{Z^0_\lambda, 3U^0_\theta - Z^1_\lambda, U^1_\theta\}$. This completes the final case and the proof. ■

We will now consider the binary dihedral groups D^*_m. It is necessary to partition the calculations into cases depending on the residue of m mod 4. We have the following table over the flat connections

	$\mathcal{C}^{f.red}$	$\mathcal{C}^{irr}$	$\mathcal{C}^{red}$
D^*_{4n}	$\theta, \eta_1, \eta_2, \eta_3$	$\alpha_1, \cdots, \alpha_{2n}$	
D^*_{4n+1}	θ, η	$\alpha_1, \cdots, \alpha_{2n}$	λ
D^*_{4n+2}	$\theta, \eta_1, \eta_2, \eta_3$	$\alpha_1, \cdots, \alpha_{2n+1}$	
D^*_{4n+3}	θ, η	$\alpha_1, \cdots, \alpha_{2n+1}$	λ

in agreement with Appendix A and Proposition I.4.25. In all cases the grading of the irreducibles are given by $j(\alpha_i) \equiv 4i \mod 8$. For D^*_{4n+2} and D^*_{4n+3} all the fully reducibles and reducibles satisfy $j(\rho) = 0$. For D^*_{4n} we have $j(\theta) = j(\eta_1) = 0$, $j(\eta_2) = j(\eta_3) = 4$, while for D^*_{4n+1} we have $j(\theta) = j(\eta) = 0$ and $j(\lambda) = 4$. For the convenience of the reader we include the relevant diagrams for $S_{D^*_m}$ from Proposition I.4.25.





Theorem I.6.14. *The negative instanton Floer homology associated with the trivial* $\mathrm{SU}(2)$*-bundle over* $\overline{Y}_\Gamma$ *for* $\Gamma = D^*_m$ *is given by* $I^-(\overline{Y}_{D^*_m}) = (X_m)^{\oplus,8}$*, where* X_m *is the* $R[U]$*-submodule of* P_m *generated by* G_m *specified in the following tables.*

m	P_m
$4n$	$R[U_\theta] \oplus R[U_{\eta_1}] \oplus R[U_{\eta_2}][4] \oplus R[U_{\eta_3}][4]$
$4n+1$	$R[U_\theta] \oplus R[U_\eta] \oplus R[Z_\lambda][6]$
$4n+2$	$R[U_\theta] \oplus R[U_{\eta_1}] \oplus R[U_{\eta_2}] \oplus R[U_{\eta_3}]$
$4n+3$	$R[U_\theta] \oplus R[U_\eta] \oplus R[Z_\lambda][2]$

m	G_m
$4n$	$\{U^0_\theta - U^0_{\eta_1}, U^0_{\eta_2} - U^0_{\eta_3}, U^n_\theta, U^n_{\eta_2}\}$
$4n+1$	$\{U^0_\theta - U^0_\eta, Z^0_\lambda, U^n_\theta, Z^{2n+1}_\lambda\}$
$4n+2$	$\{U^0_\theta - U^0_{\eta_1}, U^0_{\eta_2} - U^0_{\eta_3}, U^n_\theta - U^n_{\eta_2}, U^{n+1}_\theta\}$
$4n+3$	$\{U^0_\theta - U^0_\eta, Z^0_\lambda, 2U^n_\theta - Z^{2n+1}_\lambda, U^{n+1}_\theta\}$

Proof. Observe first that the modules P_m in the above table coincides with the $R[U]$-submodule of $E^1_0 \oplus E^1_4$ in the index spectral sequence generated by the reducibles and fully reducibles. Furthermore, the submodule generated by G_m is free in each case. Therefore, following the same procedure as in the proof of Theorem I.6.13, it will be sufficient to show that $E^\infty_0 \oplus E^\infty_4$ is the $R[U]$-submodule of P_m generated by G_m in each case.

For $m = 4n, 4n+1$ we have

$$E^1_{8s,3} = R\{h_{\alpha_2}, h_{\alpha_4}, \cdots, h_{\alpha_{2n}}\} \text{ and } E^1_{8s+4,3} = R\{h_{\alpha_1}, h_{\alpha_3}, \cdots, h_{\alpha_{2n-1}}\}$$

for each $s \in \mathbb{Z}$. For $m = 4n+2, 4n+3$ we have the same formulas except that one generator $h_{\alpha_{2n+1}}$ is adjoined to the latter group. To simplify the notation put $h_i := h_{\alpha_i}$ for $1 \le i \le 2n+1$. For each $r \ge 1$ let $p_r \colon E^1_{4s,3} = E^4_{4s,3} \to E^{4r}_{4s,3}$ denote the natural surjection.

Consider first the cases $m = 4n, 4n+1$. To unify the notation slightly we write $(U_1, U_2, U_3, U_4) = (U_\theta, U_{\eta_1}, U_{\eta_2}, U_{\eta_3})$ for $m = 4n$ and $(U_1, U_2, Z) = (U_\theta, U_\eta, Z_\lambda)$

for $m = 4n+1$. We claim that for $1 \le r \le n$ the differentials

$$f^r := d^{4r} : E^{4r}_{8s,-4(r-1)} \to E^{4r}_{8s-4r,3}$$
$$g^r := d^{4r} : E^{4r}_{8s+4,-4(r-1)} \to E^{4r}_{8s-4(r-1),3}$$

are given by the formulas

$$f^r(U_1^{r-1}) = f^r(U_2^{r-1}) = 2^{r-1} p_r(h_r)$$
$$g^r(U_3^{r-1}) = g^r(U_4^{r-1}) = 2^{r-1} p_r(h_{2n-r+1})$$
$$g^r(Z^{2r-1}) = 2^r p_r(h_{2n-r+1}).$$

We verify this by induction on r. Let $1 \le r \le n-1$ and assume that the statement is true for all i with $1 \le i < r$. This implies in particular that $p_r(h_i) = 0$ for $1 \le i < r$ and $2n - r + 1 < i \le 2n$. By Lemma I.6.11 we have

$$f^r(U_1^{r-1}) = p_r \left(\sum_{(\beta_1,\cdots,\beta_r)} (n_{\beta_1\theta} n_{\beta_2\beta_1} \cdots n_{\beta_r\beta_{r-1}}) \cdot h_{\beta_r} \right),$$

where the sum runs over all $(\beta_1, \cdots, \beta_r) \in (\mathcal{C}^{irr})^r$ for which $(\theta, \beta_1, \cdots \beta_r)$ forms an edge path in $\mathcal{S}_\Gamma$. As such a path has length r, it must, in view of the graphs $\mathcal{S}_{D^*_m}$ shown above, terminate at some vertex α_i with $i \le r$. By the inductive hypothesis $p_r(h_{\alpha_i}) = 0$ for $i < r$, so the only nonzero term in the formula corresponds to the path $(\alpha_1, \alpha_2, \cdots, \alpha_r)$. As $n_{\alpha_{i+1}\alpha_i} = 2$ for each i and $n_{\alpha_1\theta} = 1$ we conclude that $f^r(U_1^{r-1}) = 2^{r-1} p_r(h_r)$. The other formulas follow by essentially identical arguments, we only note that the additional factor 2 picked up in the last formula follows from the fact that $n_{\alpha_{2n}\lambda} = 2$. This completes the inductive step and the claim is verified.

The above formulas for the differentials imply, as $2 \in R$ is invertible, that the generators h_i, $1 \le i \le 2n$, of $E^1_{8s,3}$ and $E^1_{8s+4,3}$ are killed off two by two until we reach $E^{4n+1}_{8s,3} = 0$, $E^{4n+1}_{8s+4,3} = 0$. Therefore, by Lemma I.6.9, $E^{4n+1} = E^\infty$. Moreover, we find $E^\infty_{8s,-4(r-1)} = \mathrm{Ker}(f^r) = R\{(U_1^{r-1} - U_2^{r-1})\}$ for $1 \le r \le n$, and similarly

$$E^\infty_{8s+4,-4(r-1)} = \begin{cases} R\{(U_3^{r-1} - U_4^{r-1})\} & \text{for } m = 4n \\ 0 & \text{for } m = 4n+1 \end{cases}$$

for $1 \le r \le n$. In all other degrees we have $E^1_{s,t} = E^\infty_{s,t}$. We therefore see that E^∞_0 is freely generated as an $R[U]$-module by $\{(U_1^0 - U_2^0), U_1^n\}$ for both $m = 4n$ and $m = 4n+1$, while E^0_4 is freely generated by $\{(U_3^0 - U_4^0), U_3^n\}$ for $m = 4n$ and by $\{Z^0_\lambda, Z^{2n+1}_\lambda\}$ for $m = 4n+1$. We have thus completed the cases $m = 4n, 4n+1$.

Consider now the cases $m = 4n+2, 4n+3$. As earlier we write $(U_1, U_2, U_3, U_4) = (U_\theta, U_{\eta_1}, U_{\eta_2}, U_{\eta_3})$ for $m = 4n+2$ and $(U_1, U_2, Z) = (U_\theta, U_\eta, Z_\lambda)$. Here $U_i^{r-1}, Z^{2r-1} \in E^1_{8s,-4(r-1)}$ for each i and $r \ge 1$. For $1 \le r \le n+1$ the differentials

$$d^{4r} : E^{4r}_{8s,-4(r-1)} \to E^{4r}_{8s-4r,3}$$

are given by the formulas

$$\begin{aligned} d^{4r}(U_1^{r-1}) &= d^{4r}(U_2^{r-1}) = 2^{r-1}p_r(h_r) \\ d^{4r}(U_3^{r-1}) &= d^{4r}(U_4^{r-1}) = 2^{r-1}p_r(h_{2n+2-r}) \\ d^{4r}(Z^{2r-1}) &= 2^r p_r(h_{2n+2-r}). \end{aligned}$$

This may be verified by induction in exactly the same way as in the cases $n = 4m, 4m+1$. We conclude that $E^{4n+5}_{8s,3} = 0 = E^{4n+5}_{8s+4,3}$ and hence $E^{4n+5} = E^{\infty}$. In addition, for $0 \le r \le n-1$ we obtain

$$E^{\infty}_{8s,-4r} = \mathrm{Ker}(d^{4(r+1)}) = \begin{cases} R\{(U_1^r - U_2^r), (U_3^r - U_4^r)\} & \text{if } m = 4n+2 \\ R\{(U_1^r - U_2^r)\} & \text{if } m = 4n+3 \end{cases}$$

Furthermore, for $m = 4n+2$ we have $d^{4(n+1)}(U_i^n) = 2^n p_{n+1}(h_{n+1})$ for each $1 \le i \le 4$, and for $m = 4n+3$ we have

$$2d^{4(n+1)}(U_i^n) = d^{4(n+1)}(Z^{2n+1}) = 2^{n+1}p_{n+1}(h_{2n+1})$$

for $i = 1, 2$. From this we deduce that

$$E^{\infty}_{8s,-4n} = \begin{cases} R\{(U_1^n - U_2^n), (U_3^n - U_4^n), (U_1^n - U_3^n)\} & \text{if } m = 4n+2 \\ R\{(U_1^n - U_2^n), (2U_1^{n+1} - Z^{2n+1})\} & \text{if } m = 4n+3 \end{cases}$$

In all other degrees we have $E^1_{s,t} = E^{\infty}_{s,t}$. For $m = 4n+2$ we conclude that E^{∞}_0 is freely generated by $\{U_1 - U_2, U_3 - U_4, U_1^n - U_3^n, U_1^{n+1}\}$ and for $m = 4n+3$ we conclude that E^{∞}_0 is freely generated by $((U_1^0 - U_2^0), Z^0_\lambda, 2U_1^n - Z^{2n+1}, U_1^{n+1}\}$. In both cases $E^{\infty}_4 = 0$. This completes the final two cases and hence the proof. ■

I.6.3 The Case I^{∞}

This is the simplest calculation. Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup and write as usual $M = DCI(\overline{Y}_\Gamma)$ and $\mathcal{C}$ for the set of flat connections. Introduce a variable T_η for each $\eta \in \mathcal{C}^{f.red}$ and a variable S_λ for each $\lambda \in \mathcal{C}^{red}$, where $|T_\eta| = -4$, $|S_\lambda| = -2$. Then according to Corollary I.5.30 we have $H^{\infty}_A(\alpha) = 0$ for α irreducible, $H^{\infty}_A(\lambda) = R[S_\lambda, S_\lambda^{-1}]$ for λ reducible and $H^{\infty}_A(\eta) = R[T_\eta, T_\eta^{-1}]$ for η fully reducible. The $R[U]$-module structure is given by $U \cdot T^i_\eta = T^{i+1}_\eta$ and $U \cdot S^i_\lambda = S^{i-2}_\lambda$ for each $i \in \mathbb{Z}$.

Definition I.6.15. For an $R[U]$-module X we define the mod 8 periodic $R[U]$-module $X^{\Pi_\infty,8}$ degreewise by

$$X^{\Pi_\infty,8}_n = \prod_{s\to\infty} X_{n+8s}$$

and define $U\colon X^{\Pi_\infty,8}_n \to X^{\Pi_\infty,8}_{n-4}$ to be the product over $U\colon X_{n+8s} \to X_{n-4+8s}$ for $s \in \mathbb{Z}$.

Theorem I.6.16. *The Tate equivariant instanton Floer homology associated with the trivial* $\mathrm{SU}(2)$*-bundle over* $\overline{Y}_\Gamma$ *is given by*

$$I^\infty(\overline{Y}_\Gamma) = \left[\left(\bigoplus_{\eta\in\mathcal{C}^{f.red}} R[T_\eta, T_\eta^{-1}][j(\eta)]\right) \oplus \left(\bigoplus_{\lambda\in\mathcal{C}^{red}} R[S_\lambda, S_\lambda^{-1}][j(\lambda)]\right)\right]^{\Pi_\infty,8}.$$

Proof. Let $(E^r, d^r)_{r\geq 1}$ be the index spectral sequence with

$$E^1_{s,t} = \bigoplus_{j(\alpha)\equiv s} H^\infty_A(\alpha)_t \implies I^\infty(\overline{Y}_\Gamma)_{s+t}.$$

From the fact that $j(\alpha) \equiv 0 \pmod 4$ for each $\alpha \in \mathcal{C}$ and that $H^\infty_A(\alpha)$ is concentrated in even degrees for all types of orbits, we conclude that $E^1_{s,t} = 0$ for all (s,t) with s odd or t odd. It follows that there are no nontrivial differentials in the spectral sequence so that $E^1 = E^\infty$. This implies by Proposition I.5.57 part (iii) and Theorem I.5.55 that the spectral sequence converges strongly.

Write $I = I^\infty(\overline{Y}_\Gamma)$. To resolve the extension problems observe first that E^1_{4s} is a (shifted) direct sum of $R[U]$-modules $R[T_\eta, T_\eta^{-1}]$ and $R[S_\lambda, S_\lambda^{-1}]$. These contain free $R[U]$-submodules $R[T_\eta]$ and $R[S_\lambda]$ (freely generated by $\{S^0_\lambda, S^1_\lambda\}$), respectively, and the whole module is the localization of these free submodules in the multiplicatively closed subset $\{U^i\}_{i\geq 0} \subset R[U]$. By Corollary I.5.50 $U\colon I \to I$ is an isomorphism, and the same is valid for the $R[U]$-submodule $F_s I$ for each $s \in \mathbb{Z}$. We may therefore for each s construct a splitting of the short exact sequence

$$0 \longrightarrow F_{s-1}I \longrightarrow F_s I \underset{\xi_s}{\overset{\curvearrowleft}{\longrightarrow}} \overline{F}_s I \cong E^\infty_s \longrightarrow 0$$

by defining ξ to be a section on the free submodule and then extending to the whole module using the universal property of localization. By induction we obtain an isomorphism

$$F_s I / F_t I \cong \bigoplus_{q=t+1}^{s} E^\infty_q$$

of $R[U]$-modules for each pair $s > t$. Since the filtration of I is exhaustive and complete Hausdorff we obtain, exploiting the periodicity,

$$I \cong \lim_t \operatorname{colim}_s F_s I/F_t I \cong \prod_{s\to-\infty} E^\infty_s \cong \prod_{s\to-\infty} (E^\infty_0[-8s] \oplus E^\infty_4[-8s])$$

and the final term is precisely $(E^\infty_0 \oplus E^\infty_4)^{\Pi_\infty,8} = (E^1_0 \oplus E^1_4)^{\Pi_\infty,8}$. The proof is completed by observing that

$$E^1_0 \oplus E^1_4 = \left(\bigoplus_{\eta\in\mathcal{C}^{f.red}} R[T_\eta, T_\eta^{-1}][j(\eta)]\right) \oplus \left(\bigoplus_{\lambda\in\mathcal{C}^{red}} R[S_\lambda, S_\lambda^{-1}][j(\lambda)]\right).$$

∎

Proposition I.6.17. *For each finite subgroup $\Gamma \subset \mathrm{SU}(2)$ the homology norm map*

$$H(N)\colon I^+(\overline{Y}_\Gamma)_n \to I^-(\overline{Y}_\Gamma)_{n+3}$$

vanishes. In particular, the exact triangle of $R[U]$-modules of Corollary I.5.44 splits into a short exact sequence

$$0 \longrightarrow I^-(\overline{Y}_\Gamma) \longrightarrow I^\infty(\overline{Y}_\Gamma) \longrightarrow I^+(\overline{Y}_\Gamma)[4] \longrightarrow 0\ .$$

Proof. The calculations of Theorem I.6.7, Theorem I.6.13 and Theorem I.6.14 show that $I^+(\overline{Y}_\Gamma)$ and $I^-(\overline{Y}_\Gamma)$ are concentrated in even degrees. Therefore, as the norm map has degree 3, it must vanish. ∎

I.6.4 Calculations for Y_Γ

In this section we explain, omitting some details, the necessary modifications needed to calculate the equivariant instanton Floer groups for Y_Γ, that is, S^3/Γ equipped with the standard orientation inherited from S^3. The tools needed to handle this are contained in [Eis19, Theorem 7.10]. First, the given result states that there is an isomorphism $I^\infty(Y_\Gamma) \cong I^\infty(\overline{Y}_\Gamma)$. Second, in the proof it is verified that $DCI(Y_\Gamma) \cong DCI(\overline{Y}_\Gamma)^\vee$ and from this it is not hard to express $DCI(Y_\Gamma)$ as the totalization of a multicomplex.

Lemma I.6.18. *Let $i\colon \mathcal{C} \to \mathbb{Z}/8$ denote the grading for Y_Γ and let $j\colon \mathcal{C} \to \mathbb{Z}/8$ denote the grading for $\overline{Y}_\Gamma$. Then*

$$i(\alpha) = \begin{cases} j(\alpha) - 3 & \text{if } \alpha \in \mathcal{C}^{irr} \\ j(\alpha) - 2 & \text{if } \alpha \in \mathcal{C}^{red} \\ j(\alpha) & \text{if } \alpha \in \mathcal{C}^{f.red} \end{cases}$$

Moreover, $DCI(Y_\Gamma) = \mathrm{Tot}^\oplus(DCI(Y_\Gamma)_{*,*}, \{\partial^r\}_{r=1}^4)$ where

$$DCI(Y_\Gamma)_{s,t} = \bigoplus_{i(\alpha)\equiv s} H_t(\alpha),$$

$\partial^2 = 0$ and $\partial^1, \partial^3, \partial^4$ are given for $b_\alpha \in H_0(\alpha) \subset DCI(Y_\Gamma)_{4s+1,0}$, $\alpha \in \mathcal{C}^{irr}$, by

$$\partial^1(b_\alpha) = \sum_{\eta \in \mathcal{C}^{f.red}} n_{\alpha\eta} b_\eta$$
$$\partial^3(b_\alpha) = \sum_{\lambda \in \mathcal{C}^{red}} n_{\alpha\lambda} t_\lambda$$
$$\partial^4(b_\alpha) = \sum_{\beta \in \mathcal{C}^{irr}} n_{\alpha\beta} t_\beta .$$

and vanish otherwise.

The patterns seen in the calculations of $I^\pm(\overline{Y}_\Gamma)$ are reversed for $I^\pm(Y_\Gamma)$. Compare the following with Lemma I.6.1 and Lemma I.6.9.

Lemma I.6.19. *Let $(E^r, d^r)_{r\geq 1}$ be the index spectral sequence with*

$$E^1_{s,t} = \bigoplus_{i(\alpha)\equiv s} H^-_A(\alpha)_t \implies I^-(Y_\Gamma)_{s+t}.$$

Then the spectral sequence immediately degenerates, i.e., $E^1 = E^\infty$.

Let $(E^r, d^r)_{r\geq 1}$ be the index spectral sequence with

$$E^1_{s,t} = \bigoplus_{i(\alpha)\equiv s} H^+_A(\alpha)_t \implies I^+(Y_\Gamma)_{s+t}.$$

Then the only possibly nontrivial differentials are $d^{2r+1}\colon E^{2r+1}_{4s+1,0} \to E^{2r+1}_{4s-2r,2r}$ for $r \geq 0$ and $s \in \mathbb{Z}$.

Proof. In the first case recall that $H^-_A(\alpha) = R\cdot h_\alpha$ with $|h_\alpha| = 3$ for $\alpha \in \mathcal{C}^{irr}$, $H^-_A(\lambda) = R[Z_\lambda][2]$, $|Z_\lambda| = -2$ for $\lambda \in \mathcal{C}^{red}$ and $H^-_A(\eta) = R[U_\eta]$, $|U_\eta| = -4$ for $\eta \in \mathcal{C}^{f.red}$. By the above lemma $i(\rho) \equiv 0 \pmod 2$ for each $\rho \in \mathcal{C}^{ref} \cup \mathcal{C}^{f.red}$ and $i(\alpha) \equiv 1 \pmod 4$ for each $\alpha \in \mathcal{C}^{irr}$. Therefore $E^1_{s,t}$ can only be nonzero if the total degree $s+t$ is even. As each differential d^r reduces the total degree by one, they must all vanish and the first assertion follows.

In the second case, $H^+_A(\eta) = R[V_\eta]$, $|V_\eta| = 4$, $H^+_A(\lambda) = R[W_\lambda]$, $|W_\lambda| = 2$ and $H^+_A(\alpha) = R\cdot g_\alpha$, $|g_\alpha| = 0$, where $\eta \in \mathcal{C}^{f.red}$, $\lambda \in \mathcal{C}^{red}$ and $\alpha \in \mathcal{C}^{irr}$. Due to the indexing mentioned above the only possibly nontrivial module $E^1_{s,t}$ with $s+t$ odd are $E^1_{4s+1,0}$. This module is freely generated by g_α for $\alpha \in \mathcal{C}^{irr}$ satisfying $i(\alpha) \equiv 4s+1 \pmod 8$. Therefore, a nontrivial differential must start in $E^r_{4s+1,0}$ for some r and s. As $E^r_{s,t} = 0$ for each $t > 0$ when s is odd, we conclude that the only possibly nontrivial differentials are $d^{2r+1}\colon E^{2r+1}_{4s+1,0} \to E^{2r+1}_{4s-2r,2r}$. ∎

For the calculation of $I^-(Y_\Gamma)$ one is therefore left with solving an extension problem. In this case the extension problem is slightly simpler. We only state the conclusion. Note that in the statement below we are using the grading function $j\colon \mathcal{C} \to \mathbb{Z}/8$ for $\overline{Y}_\Gamma$, regarded as taking values in $\{0,4\}$.

Theorem I.6.20. *The negative equivariant instanton Floer homology $I^-(Y_\Gamma)$ associated with the trivial* SU(2)*-bundle over Y_Γ is given by*

$$\left[\left(\bigoplus_{\alpha\in\mathcal{C}^{irr}} R\cdot h'_\alpha[j(\alpha)]\right) \oplus \left(\bigoplus_{\lambda\in\mathcal{C}^{red}} R[Z_\lambda][j(\lambda)]\right) \oplus \left(\bigoplus_{\eta\in\mathcal{C}^{f.red}} R[U_\eta][j(\eta)]\right)\right]^{\oplus,8},$$

where $|h'_\alpha| = 0$, $|Z_\lambda| = -2$ and $|U_\eta| = -4$. The $R[U]$-module structure is determined by $U\cdot U^p_\eta = U^{p+1}_\eta$, $U\cdot Z^p_\lambda = Z^{p+2}_\lambda$ for $p \geq 0$ and

$$U\cdot h'_\alpha = \left(\sum_{\beta\in\mathcal{C}^{irr}} n_{\alpha\beta} h'_\beta\right) + \left(\sum_{\lambda\in\mathcal{C}^{red}} n_{\alpha\lambda} Z^0_\lambda\right) + \left(\sum_{\eta\in\mathcal{C}^{f.red}} n_{\alpha\eta} V^0_\eta\right).$$

To calculate $I^+(Y_\Gamma)$ one may proceed as in Lemma I.6.11 to determine the differentials in the spectral sequence explicitly and then resolve the resulting extension problem. However, one can avoid this rather technical endeavor by using the following result. If M is a DG $R[U]$-module we write $M^\vee = \mathrm{Hom}(M, R)$ for the dual complex. Our convention is that $(\partial_{M^\vee} f)(x) = (-1)^{|f|+1} f(\partial_M x)$ and $(U \cdot f)(x) = f(Ux)$ for $f \in M^\vee = \mathrm{Hom}(M_n, R)$. The content of the following lemma, which applies more generally, is stated in the proof of [Eis19, Theorem 7.10].

Lemma I.6.21. *There is an isomorphism of DG $R[U]$-modules*

$$DCI^+(Y_\Gamma) \cong DCI^-(\overline{Y}_\Gamma)^\vee.$$

Proof. Let $M = DCI(\overline{Y}_\Gamma)$ and recall that $M^\vee \cong DCI(Y_\Gamma)$. Using the explicit models for $DCI^\pm$ given in Theorem I.5.47 we find

$$\begin{aligned} DCI^-(\overline{Y}_\Gamma)^\vee_n = \mathrm{Hom}\left(\bigoplus_{p\geq 0} M_{-n+4p}, R\right) &\cong \prod_{p\geq 0} \mathrm{Hom}(M_{-n+4p}, R) \\ &= \prod_{p\geq 0} M^\vee_{n-4p} \cong DCI^+(Y_\Gamma)_n. \end{aligned}$$

It is a straightforward calculation to check that these isomorphisms piece together to give an isomorphism of DG $R[U]$-modules taking into account that the $\Lambda_R[u]$-module structure on $M^\vee$ is given by $(f \cdot u)(x) = (-1)^{|u||x|} f(xu) = (-1)^{|x|} f(xu)$. ■

Theorem I.6.22. *There is an isomorphisms of $R[U]$-modules*

$$I^+(Y_\Gamma) \cong I^-(\overline{Y}_\Gamma)^\vee$$

for each finite subgroup $\Gamma \subset \mathrm{SU}(2)$.

Proof. Since R is a principal ideal domain and the complex $DCI^-(\overline{Y}_\Gamma)$ is degreewise free over R (use the explicit model and the fact that $DCI(\overline{Y}_\Gamma)$ is degreewise free), there is a universal coefficent theorem. In view of the above lemma, this takes the form of a short exact sequence

$$\mathrm{Ext}^1_R(I^-_{n-1}(\overline{Y}_\Gamma), R) \longrightarrow I^+_{-n}(Y_\Gamma) \longrightarrow \mathrm{Hom}(I^-(\overline{Y}_\Gamma)_n, R)$$

for each $n \in \mathbb{Z}$. By Theorem I.6.13 and Theorem I.6.14 we know that $I^-(\overline{Y}_\Gamma)$ is degreewise free over R. Consequently, the first term in the above sequence vanishes and we are lift with an isomorphism $I^+(Y_\Gamma) \cong (I^-(\overline{Y})_\Gamma)^\vee$ of $R[U]$-modules as required. ■

Note that the calculations of Theorem I.6.20 and Theorem I.6.7 are compatible with the duality $I^+(\overline{Y}_\Gamma) \cong (I^-(Y_\Gamma))^\vee$ as well.

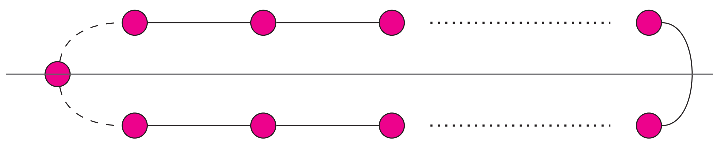
Appendix I.A Character Tables, Dynkin Graphs and Flat Connections

This appendix contains various useful facts concerning the binary polyhedral groups. Of particular importance are their complex representation theory, the corresponding McKay graphs and complete lists of the 1-dimensional quaternionic representations.

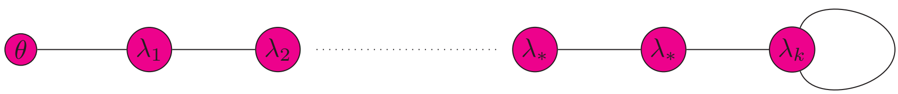
I.A.1 Cyclic Groups

The complex representation theory of cyclic groups are of course very well known, so we only include a brief summary for completeness. Let $C_l = \langle g : g^l = 1\rangle$ be the cyclic group of order l. The irreducible complex representations are $\rho_k\colon C_l \to U(1)$ for $0 \leq k < l$, where $\rho_k(g) = e^{\frac{2\pi i k}{l}}$. If we interpret the indices modulo l we have $\rho_k \otimes \rho_{k'} \cong \rho_{k+k'}$ and $\rho_k^* \cong \rho_{-k}$. We embed $C_l \subset \mathrm{SU}(2)$ through the representation $\rho_1 \oplus \rho_{-1}$.

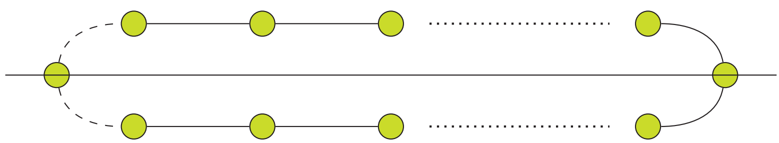
Suppose that l is odd. Then all of the ρ_k, $1 \leq k < l$, are of complex type, so the 1-dimensional quaternionic representations are given by $\theta = 2\rho_0$ and $\lambda_k = \rho_k \oplus \rho_{-k}$ for $1 \leq k < l/2$. The McKay graph $\overline{\Delta}_{C_l} \cong \widetilde{A}_{l-1}$ takes the form



where the line indicates the involution symmetry. The quotient graph $\overline{\Delta}_{C_l}/(\iota)$ is given by



Suppose that l is even. Then $\rho_{l/2}$ is of real type, so the flat connections are given by $\theta = 2\rho_0$, $\eta = 2\rho_{k/2}$ and $\lambda_k = \rho_k \oplus \rho_{-k}$ for $1 \leq k < l/2$. The McKay graph takes the form



with quotient graph

I.A.2 Binary Dihedral Groups

A presentation for the binary dihedral group D_n^*, $n \geq 2$, is given by

$$D_n^* = \langle a, x \mid a^n = x^2, axa = x \rangle.$$

Note that $a^{2n} = a^n x^2 = xa^{-n}x = xx^{-2}x = 1$ follows from the two relations. Each element may be expressed uniquely in the form $a^k x^\epsilon$ for $0 \leq k < 2n$ and $0 \leq \epsilon \leq 1$. The group may be realized within $\mathrm{Sp}(1) \subset \mathbb{H}$ by taking $a = e^{\pi i/n}$ and $x = j$. In this realization $a^n = x^2 = -1$ so this element is central.

Lemma I.A.1. *We have* $[D_n^*, D_n^*] = \langle a^2 \rangle$ *so that*

$$D_n^*/[D_n^*, D_n^*] \cong \begin{cases} \mathbb{Z}/4 & \text{if } n \equiv 1 \pmod 2 \\ \mathbb{Z}/2 \times \mathbb{Z}/2 & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

Proof. Using the fact that $xa^k = a^{-k}x$ one sees that every commutator is of the form a^k. Moreover, $axa^{-1}x^{-1} = a^2$ from which it follows that $\langle a^2 \rangle \subset [D_n^*, D_n^*]$. As $\langle a^2 \rangle$ is normal with quotient of order 4 and thus abelian, we must have equality. If n is odd x has order 4 modulo $\langle a^2 \rangle$, so the quotient is cyclic If n is even every element has order 2 mod $\langle a^2 \rangle$, so the quotient is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. ■

Lemma I.A.2. *The conjugacy classes of* D_n^* *are given by* $\{1\}, \{-1\}$, $\{a^j, a^{-j}\}$ *for* $1 \leq j < n$,

$$\{x, a^2x, \cdots, xa^{2n-2}x\} \quad \text{and} \quad \{ax, a^3x, \cdots, a^{2n-1}x\}.$$

Proof. The elements $1, -1$ are central so $\{1\}$, $\{-1\}$ are conjugacy classes. From the calculation $(a^s x)a^j(a^s x)^{-1} = a^{-j}$, valid for any s and j, it follows that $\{a^j, a^{-j}\}$, for $1 \leq j < n$, are conjugacy classes. Next, $a^j x a^{-j} = a^{2j}x$ from which we deduce that that $\{x, a^2x, \cdots, a^{2n-2}x\}$ is a conjugacy class. Finally, from the relation $a^j(ax)a^{-j} = a^{2j+1}x$ we deduce that $\{ax, a^3x, \cdots, a^{2n-1}x\}$ is at least a portion of a conjugacy class. However, since we have already exhausted every element of the group, we conclude that this must in fact be the whole conjugacy class and that we have found all the conjugacy classes. ■

Proposition I.A.3. *The irreducible representations of* D_n^* *consist of the four* 1*-dimensional representations associated with the four distinct homomorphisms* $D_n^*/[D_n^*, D_n^*] \to \{\pm 1, \pm i\}$ *and the* 2*-dimensional representations* $\tau_k \colon D_n^* \to \mathrm{GL}_2(\mathbb{C})$, *for* $1 \leq k < n$, *given by*

$$\tau_k(a) = \begin{pmatrix} \xi^k & 0 \\ 0 & \xi^{-k} \end{pmatrix} \quad \text{and} \quad \tau_k(x) = \begin{pmatrix} 0 & (-1)^k \\ 1 & 0 \end{pmatrix}$$

where $\xi = e^{\pi i/n}$.

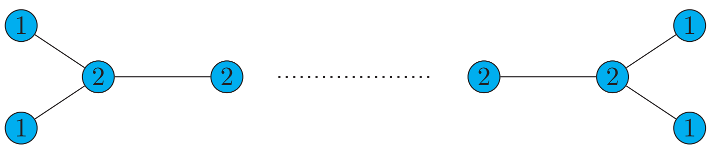
Proof. The statement about the 1-dimensional representations is clear. Let χ_k denote the character of τ_k. Then $\chi_k(a^j) = 2\cos(\pi jk/n)$, $\chi_k(1) = 2$, $\chi_k(-1) = 2(-1)^k$ and χ_k vanishes on the remaining conjugacy classes. Hence,

$$||\chi_k||^2 = \frac{1}{4n}(8 + 2\sum_{j=1}^{n-1} 4\cos^2(\pi kj/n))) = \frac{1}{4n}(8 + 4\sum_{j=1}^{n-1}(\cos(2\pi jk/n) + 1))$$
$$= \frac{1}{4n}(8 + 4(n-2)) = 1,$$

which shows that each τ_k is irreducible. The representations τ_k are inequivalent for $1 \leq k < n$ because the values $\chi_k(a) = 2\cos(\pi k/n)$ are distinct for $1 \leq k < n$. The sum of the squares of the dimensions of the irreducible representations found is $4 + (n-1)4 = 4n = |D_n^*|$, so we conclude that we have found all of them. ■

The representation corresponding to the inclusion $D_n^* \subset \mathrm{Sp}(1) = \mathrm{SU}(2)$ is τ_1. We denote the four representations corresponding to the homomorphisms $(D_n^*)^{ab} \to \{\pm 1, \pm i\}$ by ρ_j for $0 \leq j \leq 3$, where we take ρ_0 to be the trivial representation and fix ρ_1 by the requirement $\tau_1 \otimes \tau_1 = \tau_2 \oplus \rho_1 \oplus \rho_0$. Then ρ_0, ρ_1 are of real type, while if n is odd $\rho_2 \cong \rho_3^*, \rho_3$ are of complex type and if n is even ρ_2 and ρ_3 are of real type.

The McKay graph $\overline{\Delta}_{D_n^*} \cong \widetilde{D}_{n+2}$ takes the form



where we have included the dimensions of the representations in the graph. For D_n^* there are $n-1$ vertices in the middle corresponding to $\tau_1, \cdots, \tau_{n-1}$ from left to right. On the left hand side we have ρ_0, ρ_1 and on the right hand side we have ρ_2, ρ_3.

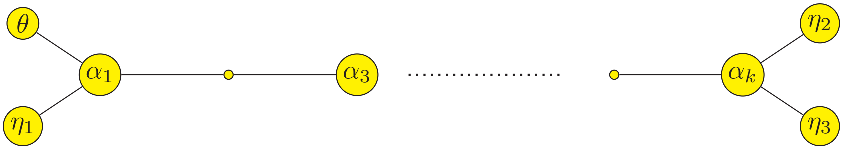
Lemma I.A.4. *The representation τ_k is quaternionic for k odd and real for k even.*

Proof. We use the criterion of Lemma I.3.5. Since $\sum_{j=0}^{2n-1} \chi_k(a^{2j}) = 0$ and $(xa^j)^2 = x^2 = -1$ for each j, we obtain $\sum_{g \in D_n^*} \chi_k(g^2) = 4n(-1)^k$ and the result follows. ■

Let $\theta = 2\rho_0$ and $\alpha_k = \tau_{2k-1}$ for $1 \leq k \leq n/2$. For n even the 1-dimensional quaternionic representations are given by

$$\theta, \eta_1 = 2\rho_1, \eta_2 = 2\rho_2, \eta_3 = 2\rho_3, \alpha_1, \cdots, \alpha_{n/2}$$

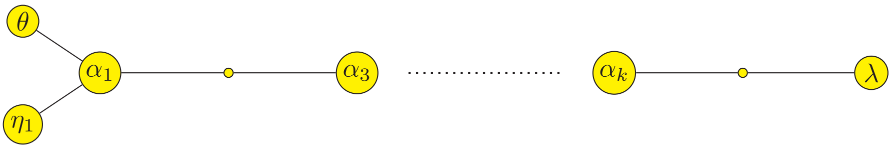
and the quotient graph $\overline{\Delta}_{D_n^*}/(\iota)$ is given by



For n odd the 1-dimensional quaternionic representations are given by

$$\theta, \eta = 2\rho_1, \alpha_1, \cdots, \alpha_{(n-1)/2}, \lambda = \rho_2 \oplus \rho_3$$

and the corresponding quotient graph is given by



I.A.3 The Binary Tetrahedral Group

Let $Q = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathrm{Sp}(1)$ be the quaternion group (note that $Q \cong D_2^*$). Then T^* may be realized in $\mathrm{Sp}(1) \subset \mathbb{H}$ as

$$T^* = Q \cup \{\frac{1}{2}(\epsilon_0 1 + \epsilon_1 i + \epsilon_2 j + \epsilon_3 k) : \epsilon_r = \pm 1,\ 0 \le r \le 3\}.$$

We have $[T^*, T^*] = Q$ so as $|T^*| = 24$, it follows that $(T^*)^{ab} = T^*/[T^*, T^*] \cong \mathbb{Z}/3$.

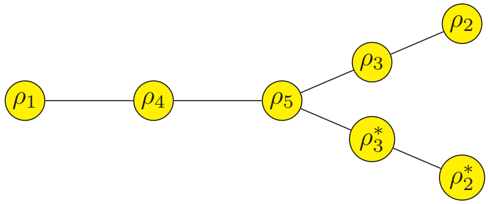
The character table of T^* is given below. Here $\xi = e^{2\pi i/3}$ is a primite third root of unity.

	1	2	$3a$	$3b$	4	$6a$	$6b$	type
ρ_1	1	1	1	1	1	1	1	$\mathbb{R}$
ρ_2	1	1	ξ	ξ^2	1	ξ	ξ^2	$\mathbb{C}$
ρ_2^*	1	1	ξ^2	ξ	1	ξ^2	ξ	$\mathbb{C}$
ρ_3	2	-2	$-\xi$	$-\xi^2$	0	ξ	ξ^2	$\mathbb{C}$
ρ_3^*	2	-2	$-\xi^2$	$-\xi$	0	ξ^2	ξ	$\mathbb{C}$
ρ_4	2	-2	-1	-1	0	1	1	$\mathbb{H}$
ρ_5	3	3	0	0	-1	0	0	$\mathbb{R}$

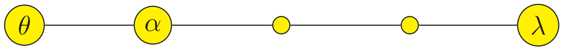
The type of the representation in the right hand column is calculated using Lemma I.3.5 and the following table. The first row gives a representative for each conjugacy class and the second row gives the conjugacy class in which the square of this element belongs. Here $x = (1 + i + j + k)/2 \in T^*$.

	1	2	$3a$	$3b$	4	$6a$	$6b$
rep	1	-1	$-x$	$-x^*$	i	x	x^*
sm	1	1	$3b$	$3a$	2	$3b$	$3a$

The canonical representation given by $T^* \hookrightarrow \mathrm{SU}(2)$ corresponds to the irreducible character ρ_4. From this one may calculate the McKay graph $\overline{\Delta}_{T^*} \cong \widetilde{E}_6$ to be



From the type calculation given in the character table and Proposition I.3.6, we see that the 1-dimensional quaternionic representations are $\theta = 2\rho_1$, $\alpha = \rho_4$ and $\lambda = \rho_2 \oplus \rho_2^*$. The quotient graph $\overline{\Delta}_{T^*}/(\iota)$ then takes the form



I.A.4 The Binary Octahedral Group

Let $Q \subset T^* \subset \mathrm{Sp}(1)$ be as in the previous section. Then O^* may be realized as

$$O^* = T^* \cup \{(u+v)/\sqrt{2} : u, v \in Q\ ,\ u \neq \pm v\} \subset \mathrm{Sp}(1).$$

We have $[O^*, O^*] = T^*$, so as $|O^*| = 48$ it follows that $(O^*)^{ab} = O^*/[O^*, O^*] \cong \mathbb{Z}/2$.

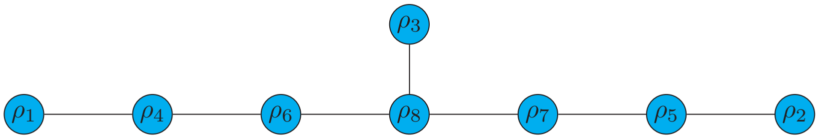
The character table of O^* is given below

	1	2	3	$4a$	$4b$	6	$8a$	$8b$	type
ρ_1	1	1	1	1	1	1	1	1	$\mathbb{R}$
ρ_2	1	1	1	1	-1	1	-1	-1	$\mathbb{R}$
ρ_3	2	2	-1	2	0	-1	0	0	$\mathbb{R}$
ρ_4	2	-2	-1	0	0	1	$\sqrt{2}$	$-\sqrt{2}$	$\mathbb{H}$
ρ_5	2	-2	-1	0	0	1	$-\sqrt{2}$	$\sqrt{2}$	$\mathbb{H}$
ρ_6	3	3	0	-1	-1	0	1	1	$\mathbb{R}$
ρ_7	3	3	0	-1	1	0	-1	-1	$\mathbb{R}$
ρ_8	4	-4	1	0	0	-1	0	0	$\mathbb{H}$

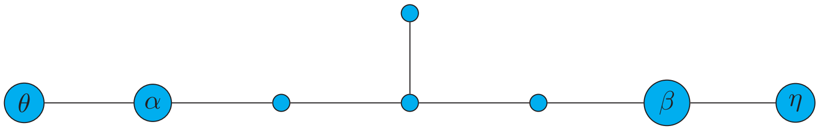
Let $x = (1+i+j+k)/2$, as in the previous section, let $y = (1+i)/\sqrt{2}$ and let $z = (i+j)/\sqrt{2}$. The following table contains the information needed for the type calculation given in the above right hand column.

	1	2	3	$4a$	$4b$	6	$8a$	$8b$
rep	1	-1	$-x$	i	z	x	y	$-y$
sm	1	1	3	2	2	3	6	6

The canonical representation given by the inclusion $O^* \hookrightarrow \mathrm{SU}(2)$ corresponds to ρ_4. From this one obtains the McKay graph $\overline{\Delta}_{O^*} \cong \widetilde{E}_7$.



Using the by now standard method one finds the 1-dimensional quaternionic representation to be $\theta = 2\rho_1$, $\alpha = \rho_4$, $\beta = \rho_5$ and $\eta = 2\rho_2$. In this case there are no irreducible representations of complex type, so the quotient graph is simply



I.A.5 The Binary Icosahedral Group

Let $\phi = (1+\sqrt{5})/2 = 2\cos(\pi/5)$ be the golden ratio (thus $\phi^2 = \phi + 1$) and let S be the set of quaternions obtained by even coordinate permutations of

$$(\epsilon_0 i + \epsilon_1 \phi^{-1} j + \epsilon_2 \phi k)/2$$

where $\epsilon_r \in \{\pm 1\}$ for $0 \leq r \leq 2$. Note that there are $12 = |A_4|$ even permutations and $8 = 2^3$ sign combinations so $|S| = 12 \cdot 8 = 96$. The binary icosahedral group may be realized in Sp(1) as $I^* = T^* \cup S$. It is well-known that I^* is a perfect group, that is, $[I^*, I^*] = I^*$ or equivalently $(I^*)^{ab} = 0$. The binary polyhedral space $Y_{I^*} = S^3/I^*$ is also called the Poincaré sphere and is an integral homology sphere.

The character table of I^* is given below.

	1	2	3	4	5a	5b	6	10a	10b	type
ρ_1	1	1	1	1	1	1	1	1	1	$\mathbb{R}$
ρ_2	2	-2	-1	0	ϕ^{-1}	$-\phi$	1	ϕ	$-\phi^{-1}$	$\mathbb{H}$
ρ_3	2	-2	-1	0	$-\phi$	ϕ^{-1}	1	$-\phi^{-1}$	ϕ	$\mathbb{H}$
ρ_4	3	3	0	-1	$-\phi^{-1}$	ϕ	0	ϕ	$-\phi^{-1}$	$\mathbb{R}$
ρ_5	3	3	0	-1	ϕ	$-\phi^{-1}$	0	$-\phi^{-1}$	ϕ	$\mathbb{R}$
ρ_6	4	4	1	0	-1	-1	1	-1	-1	$\mathbb{R}$
ρ_7	4	-4	1	0	-1	-1	-1	1	1	$\mathbb{H}$
ρ_8	5	5	-1	1	0	0	-1	0	0	$\mathbb{R}$
ρ_9	6	-6	0	0	1	1	0	-1	-1	$\mathbb{H}$

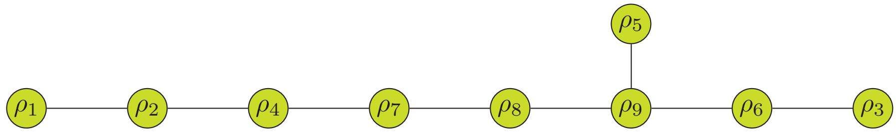
Put

$$u = (\phi \cdot 1 + \phi^{-1} \cdot i + j)/2 \in S$$

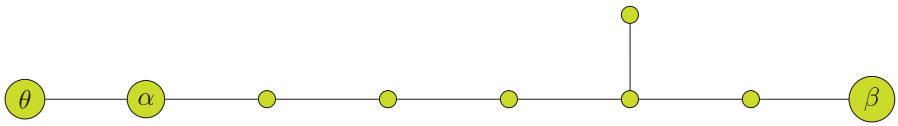
and let $x \in T^*$ be defined as earlier. We then have the following table for the type calculation in the above right hand column.

	1	2	3	4	$5a$	$5b$	6	$10a$	$10b$
rep	1	-1	$-x$	i	u^2	$-u$	x	u	$-u^2$
sm	1	1	3	2	$5b$	$5a$	3	$5a$	$5b$

The character corresponding to the canonical representation $I^* \hookrightarrow \mathrm{Sp}(1) = \mathrm{SU}(2)$ is ρ_2. From this one may show that the McKay graph $\overline{\Delta}_{I^*} \cong \widetilde{E}_8$ is given by



The 1-dimensional quaternionic representations are $\theta = 2\rho_1$, $\alpha = \rho_2$ and $\beta = \rho_3$. Once again there are no irreducible representations of complex type, so the quotient graph is simply



Appendix I.B Equivariant SU(2)-Bundles over S^4 and the Chern-Simons Invariant

In this appendix we give a proof of the classification of Γ-equivariant SU(2)-bundles over S^4 and apply the equivariant index theorem to the twisted Dirac operators $D_A \colon \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$ to obtain a proof of the equations in Lemma I.4.4. As a byproduct of this we obtain a simple way to calculate the Chern-Simons invariants of the flat connections in the trivial SU(2)-bundle over the binary polyhedral spaces. In the final part we also show that this invariant can in a natural way be related to the algebraic Chern class of the holonomy representation associated with the flat connection.

I.B.1 Conventions on Quaternions

It will be convenient to employ quaternions in the the upcoming theory, so in this section we fix our conventions. We will regard $\mathbb{H}$ as the 4-dimensional real algebra with standard basis $(1, i, j, k)$ and multiplication rules

$$i^2 = j^2 = k^2 = -1 \text{ and } ij = -ji = k.$$

The involution determined by $1, i, j, k \mapsto 1, -i, -j, -k$ is written as $q \mapsto q^*$. Our convention is that a quaternionic vector space is a left $\mathbb{H}$-module and that the standard quaternionic structure on $\mathbb{H}^n$ is given by left multiplication. We write

$M_n(F)$ for the algebra of $n \times n$ matrices with entries in $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$. The standard identification

$$M_n(\mathbb{H}) \cong \mathrm{End}_{\mathbb{H}}(\mathbb{H}^n)$$

is then given by $A \mapsto (x \mapsto xA^*)$, that is, we multiply the conjugate transpose of A with x, regarded as a row vector, from the right. To avoid confusion we will always write $q \cdot x := xq^*$ for the left action of $q \in \mathrm{Sp}(1)$ on $x \in \mathbb{H}$ and always omit the dot when we mean actual multiplication.

We define the standard inclusion by $\mathbb{C} = \mathrm{Span}_{\mathbb{R}}(1, i) \subset \mathbb{H}$ and the standard identification $\mathbb{C}^2 \cong \mathbb{H}$ by $(z, w) \mapsto z + wj$. Note that the complex orientation on $\mathbb{C}^2$ corresponds to the standard orientation on $\mathbb{H} \cong \mathbb{R}^4$ in which $(1, i, j, k)$ is a positive basis.

With these conventions in place we fix the isomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ by requiring the following diagram to commute

$$\begin{array}{ccccc}
\mathrm{Sp}(1) & \xrightarrow{\subset} & \mathbb{H} & \xrightarrow{\cong} & \mathrm{End}_{\mathbb{H}}(\mathbb{H}) \\
\big\downarrow{\scriptstyle\cong} & & \big\downarrow & & \big\downarrow{\scriptstyle\subset} \\
\mathrm{SU}(2) & \xrightarrow{\subset} & M_2(\mathbb{C}) & \xrightarrow{\cong} & \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2).
\end{array}$$

Explicitly,

$$q = z + wj \in \mathrm{Sp}(1) \mapsto \begin{pmatrix} z^* & w^* \\ -w & z \end{pmatrix}.$$

These conventions ensure that the standard left action $\mathrm{SU}(2) \times \mathbb{C}^2 \to \mathbb{C}^2$ corresponds to the above defined left action $\mathrm{Sp}(1) \times \mathbb{H} \to \mathbb{H}$. We define the orientation of $\mathrm{Sp}(1) = \mathrm{SU}(2)$ by requiring that $(i, j, k) \in \mathrm{Im}\,\mathbb{H} = \mathfrak{sp}(1)$ is a positive basis. The orbit map $\mathrm{Sp}(1) \to S^3 \subset \mathbb{H}$ given by $q \mapsto q \cdot 1 = 1q^* = q^*$ is orientation reversing. Since our work in this appendix is very sensitive to orientations, we will for this reason distinguish between the group $\mathrm{Sp}(1) = \mathrm{SU}(2)$ and the unit sphere in the representation space $S^3 \subset \mathbb{H} = \mathbb{C}^2$.

I.B.2 Classification of Γ-equivariant $\mathrm{SU}(2)$-bundles over S^4

Let $\Gamma \subset \mathrm{Sp}(1) = \mathrm{SU}(2)$ be a finite subgroup. Recall that Γ acts on S^3 by restriction of the linear action on $\mathbb{H} = \mathbb{C}^2$. By regarding $S^4 \subset \mathbb{H} \oplus \mathbb{R}$ the suspended action $\Gamma \times S^4 \to S^4$ takes the form $\gamma \cdot (x, t) = (\gamma \cdot x, 1)$. The fixed points of this action are $N := (0, 1)$ and $S := (0, -1)$.

Let $\mathrm{Rep}^1(\Gamma, \mathbb{H})$ denote the set of isomorphism classes of 1-dimensional quaternionic representations of Γ and let $\mathrm{Vec}^1_\Gamma(S^4, \mathbb{H})$ denote the set of isomorphism classes of Γ-equivariant quaternionic line bundles over S^4. If V is a 1-dimensional quaternionic representation of Γ we write $[V]$ for its isomorphism class in $\mathrm{Rep}^1(\Gamma, \mathbb{H})$. The same applies to Γ-equivariant bundles and $\mathrm{Vec}^1_\Gamma(S^4, \mathbb{H})$. Define

$$\phi\colon\ \mathrm{Vec}^1_\Gamma(S^4, \mathbb{H}) \to \mathrm{Rep}^1(\Gamma, \mathbb{H}) \times \mathrm{Rep}^1(\Gamma, \mathbb{H}) \times \mathbb{Z}$$

by $[E] \mapsto ([E_N], [E_S], c_2(E)[S^4])$. Our aim is to show that this map is injective and identify the image. We will need the following two well-known results for our work.

Lemma I.B.1. *Let Y be a closed 3-manifold. Then every $\mathrm{Sp}(1) = \mathrm{SU}(2)$-bundle over Y is trivial.*

Proof. Every closed 3-manifold admits the structure of a finite CW complex with cells of dimension ≤ 3. We have $B\,\mathrm{Sp}(1) = \mathbb{H}P^\infty$ and this space has a CW structure with a single cell in each dimension $n = 4k$, $k \geq 0$. By cellular approximation every map $f\colon Y \to \mathbb{H}P^\infty$ is homotopic to a map with image in the 3-skeleton $(\mathbb{H}P^\infty)^3 = *$. This implies that every $\mathrm{Sp}(1)$-bundle must be trivial. ■

Lemma I.B.2. *Let Y be a closed oriented 3-manifold and let $f, g\colon Y \to \mathrm{Sp}(1)$ be a pair of continuous maps. Define $h\colon Y \to \mathrm{Sp}(1)$ by $h(y) = f(y)g(y)$ for $y \in Y$. Then*

$$\deg(h) = \deg(f) + \deg(g).$$

Proof. The map h is given by the following composition

$$Y \xrightarrow{\Delta} Y \times Y \xrightarrow{f \times g} \mathrm{Sp}(1) \times \mathrm{Sp}(1) \xrightarrow{\mu} \mathrm{Sp}(1),$$

where Δ is the diagonal map and μ is the multiplication map. Let $\pi_i\colon \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{Sp}(1)$ for $i = 1, 2$ be the two projections and let $\iota_i\colon \mathrm{Sp}(1) \to \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ for $i = 1, 2$ the two inclusions given by $\iota_1(a) = (a, 1)$ and $\iota_2(a) = (1, a)$ for $a \in \mathrm{Sp}(1)$. Then $\pi_i \circ \iota_i = \mathrm{id}$ for $i = 1, 2$ and $\pi_i \circ \iota_j$ is constant for $i \neq j$. Clearly, $H_3(\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \cong \mathbb{Z} \oplus \mathbb{Z}$ and from this we deduce that

$$q \coloneqq ((\pi_1)_*, (\pi_2)_*)\colon H_3(\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \to H_3(\mathrm{Sp}(1)) \oplus H_3(\mathrm{Sp}(1))$$

is an isomorphism with inverse $(\iota_1)_* + (\iota_2)_*$. Furthermore,

$$\mu_* \circ ((\iota_1)_* + (\iota_2)_*) = (\mu \circ \iota_1)_* + (\mu \circ \iota_2)_* = \mathrm{id}_* + \mathrm{id}_*\,.$$

These considerations imply that the following diagram commutes

$$\begin{array}{ccccc} H_3(Y) & \xrightarrow{(f,g)_* \circ \Delta_*} & H_3(\mathrm{Sp}(1) \times \mathrm{Sp}(1)) & \xrightarrow{\mu_*} & H_3(\mathrm{Sp}(1)) \\ & \searrow^{(f_*, g_*)} & \downarrow \cong & \nearrow^{+} & \\ & & H_3(\mathrm{Sp}(1)) \oplus H_3(\mathrm{Sp}(1)) & & \end{array}.$$

Since the upper composition is $h_*\colon H_3(Y) \to H_3(\mathrm{Sp}(1))$ by definition, it follows that

$$\deg(h)[\mathrm{Sp}(1)] = h_*([Y]) = f_*([Y]) + g_*([Y]) = (\deg(f) + \deg(g))[\mathrm{Sp}(1)],$$

where $[Y] \in H_3(Y)$ and $[\mathrm{Sp}(1)] \in H_3(\mathrm{Sp}(1))$ are the fundamental classes. Hence, $\deg(h) = \deg(f) + \deg(g)$ as required. ■

Let $\alpha \in \mathrm{Rep}^1(\Gamma, \mathbb{H})$ and choose a homomorphism $\rho_\alpha \colon \Gamma \to \mathrm{Sp}(1)$ representing α. Write $\mathrm{Sp}(1)^\alpha$ for the group $\mathrm{Sp}(1)$ equipped with the Γ-action given by $\gamma \cdot q = \rho_\alpha(\gamma) q$. If X and Y are Γ-manifolds we write $[X, Y]_\Gamma$ for the set of Γ-equivariant homotopy classes of equivariant maps.

Lemma I.B.3. *There exist Γ-equivariant maps $S^3 \to \mathrm{Sp}(1)^\alpha$. Moreover, for any pair g, g' of such maps it holds true that $\deg g \equiv \deg g' \pmod{|\Gamma|}$.*

Proof. Since Γ acts freely on S^3 the set of equivariant maps $S^3 \to \mathrm{Sp}(1)^\alpha$ is in natural bijection with the set of sections of the associated principal $\mathrm{Sp}(1)$-bundle

$$S^3 \times_\Gamma \mathrm{Sp}(1)^\alpha \to S^3/\Gamma.$$

According to Lemma I.B.1 this bundle must be trivial, and hence admits global sections. This proves the first assertion.

For the second assertion assume that $f, g \colon S^3 \to \mathrm{Sp}(1)^\alpha$ is a pair of equivariant maps. We may then form the map $h \colon S^3 \to \mathrm{Sp}(1)$ given by $h(x) = f(x)^* g(x)$. This map is equivariant when Γ acts trivially on $\mathrm{Sp}(1)$ and therefore descends to a map $\bar{h} \colon S^3/\Gamma \to \mathrm{Sp}(1)$. Since the covering map $S^3 \to S^3/\Gamma$ has degree $|\Gamma|$, we deduce that $\deg h = |\Gamma| \deg(\bar{h}) \equiv 0 \pmod{|\Gamma|}$. Finally, by Lemma I.B.2 and the fact that $\deg f^* = -\deg f$ we obtain $\deg h = \deg(g) - \deg f$ and the proof is complete. ■

Let $\alpha, \beta \in \mathrm{Rep}^1(\Gamma, \mathbb{H})$ have representatives $\rho_\alpha, \rho_\beta \colon \Gamma \to \mathrm{Sp}(1)$. Let $\mathbb{H}(\alpha)$ and $\mathbb{H}(\beta)$ denote $\mathbb{H}$ equipped with the Γ-action determined by ρ_α and ρ_β, respectively. Furthermore, write $\mathrm{Sp}(1)^{(\alpha;\beta)}$ for the group $\mathrm{Sp}(1)$ equipped with the Γ-action $\gamma \cdot q = \rho_\alpha(\gamma) q \rho_\beta(\gamma)^*$.

Proposition I.B.4. *The set of isomorphism classes of Γ-equivariant $\mathrm{Sp}(1)$-bundles $E \to S^4$ with $[E_N] = \alpha$ and $[E_S] = \beta$ is in natural bijection with*

$$[S^3, \mathrm{Sp}(1)^{(\alpha;\beta)}]_\Gamma.$$

Proof. This is well-known in the non-equivariant setting, so we will only include the details necessary to adapt the usual proof to the equivariant setting.

Let E be a Γ-equivariant $\mathrm{Sp}(1)$-bundle over S^4 (regarded as a vector bundle) with $[E_N] = \alpha$ and $[E_S] = \beta$. Let $U_N = S^4 - \{S\}$ and $U_S = S^4 - \{N\}$. Let A be a Γ-invariant $\mathrm{Sp}(1)$-connection in E. Then, using parallel transport along radial geodesic from S and N, one obtains equivariant trivializations $U_S \times E_S \cong E|_{U_S}$ and $U_N \times E_N \cong E|_{U_N}$. By making suitable choices of unit basis vectors in E_S and E_N, we may identify $E_S \cong \mathbb{H}(\beta)$ and $E_N \cong \mathbb{H}(\alpha)$. The transition function between these trivializations is an equivariant map $U_N \cap U_S \to \mathrm{Hom}_{\mathbb{H}}(E_S, E_N)$ taking values in the subset of isometries. This subset may be identified with $\mathrm{Sp}(1)^{(\alpha;\beta)}$ through

$$\mathrm{Sp}(1)^{(\alpha;\beta)} \hookrightarrow \mathrm{Hom}_{\mathbb{H}}(\mathbb{H}(\beta), \mathbb{H}(\alpha)) \cong \mathrm{Hom}_{\mathbb{H}}(E_S, E_N),$$

where we recall that by convention the first map is given by $q \mapsto r_{q^*}$. Let $t \colon S^3 \to S^3(\alpha, \beta)$ denote the restriction of this map to the middle sphere. One may verify that the Γ-equivariant homotopy class of this map is independent of the trivializations chosen. We have therefore constructed one direction of the equivalence.

The inverse may be defined as follows. Let $f\colon S^3 \to \mathrm{Sp}(1)^{(\alpha;\beta)}$ represent a homotopy class in $[S^3, \mathrm{Sp}(1)^{(\alpha;\beta)}]_\Gamma$. Let $p\colon U_N \cap U_S \to S^3$ be the equivariant projection onto the middle sphere. Using f we may therefore construct an equivariant bundle E by gluing $U_S \times \mathbb{H}(\beta)$ and $U_N \times \mathbb{H}(\alpha)$ along $\psi\colon U_N \cap U_S \times \mathbb{H}(\beta) \to U_S \cap U_N \times \mathbb{H}(\alpha)$ given by $\psi(x, v) = (x, f(p(x)) \cdot v) = (x, vf(p(x))^*)$. The equivariance of f and p ensure that the actions of Γ on $U_N \times \mathbb{H}(\alpha)$ and $U_S \times \mathbb{H}(\beta)$ match over the intersection $U_N \cap U_S$. The resulting bundle is therefore a Γ-equivariant quaternionic line bundle. One may verify that the isomorphism class of this bundle is independent of the representative f chosen using the equivariant bundle homotopy theorem of [Seg68, Prop. 1.3].

The verification of the fact that these two constructions are mutual inverses proceeds just as in the non-equivariant case. ■

Recall that we regard $S^4 \subset \mathbb{H} \oplus \mathbb{R}$. Let $u\colon S^4 - \{N\} \cong \mathbb{H}$ be stereographic projection from the north pole. We give $\mathbb{H}$ the standard orientation and orient S^4 by requiring u to be orientation preserving. Moreover, we orient the middle sphere $S^3 \subset S^4$ by requiring that the restriction $u|_{S^3} : S^3 \cong S^3 \subset \mathbb{H}$ preserves orientation.

Proposition I.B.5. *Let* $\alpha, \beta \in \mathrm{Rep}^1(\Gamma, \mathbb{H})$ *have representatives* $\rho_\alpha, \rho_\beta\colon \Gamma \to \mathrm{Sp}(1)$*. Then the natural map*

$$[S^3, \mathrm{Sp}(1)^{(\alpha;\beta)}]_\Gamma \to [S^3, \mathrm{Sp}(1)^{(\alpha;\beta)}]$$

is injective. In other words, two equivariant maps $f, g\colon S^3 \to \mathrm{Sp}(1)^{(\alpha;\beta)}$ *are homotopic if and only if they are equivariantly homotopic. Moreover, if we identify* $[S^3, \mathrm{Sp}(1)^{(\alpha;\beta)}] \cong \mathbb{Z}$ *through* $[f] \mapsto \deg(f)$*, there is a constant* $c \in \mathbb{Z}$ *so that the image is* $\{n \in \mathbb{Z} : n \equiv c \mod |\Gamma|\}$*.*

Proof. By Lemma I.B.3 we can find equivariant maps $g\colon S^3 \to \mathrm{Sp}(1)^\alpha$ and $h\colon S^3 \to \mathrm{Sp}(1)^\beta$. Given an equivariant map $f\colon S^3 \to \mathrm{Sp}(1)^{(\alpha;\beta)}$ define $\tau(f) = \tau_{g,h}(f)\colon S^3 \to \mathrm{Sp}(1)^\theta$ to be the composition

$$S^3 \xrightarrow{(h,f,g)} \mathrm{Sp}(1)^\alpha \times \mathrm{Sp}(1)^{(\alpha;\beta)} \times \mathrm{Sp}(1)^\beta \xrightarrow{\mu} \mathrm{Sp}(1)^\theta,$$

where $\mathrm{Sp}(1)^\theta$ denotes $\mathrm{Sp}(1)$ equipped with the trivial Γ-action and μ is the equivariant map $(x, y, z) \mapsto x^*yz$. Observe that if $f'\colon S^3 \to \mathrm{Sp}(1)^{(\alpha;\beta)}$ is another equivariant map, then f and f' are equivariantly homotopic if and only if $\tau(f)$ and $\tau(f')$ are equivariantly homotopic. Indeed, if $f_t\colon S^3 \to \mathrm{Sp}(1)^{(\alpha;\beta)}$ is an equivariant homotopy between f and f', then $\tau(f_t)$ is an equivariant homotopy between $\tau(f)$ and $\tau(f')$ and the same argument applies for the converse using $\tau_{g^{-1},h^{-1}}$ in place of $\tau = \tau_{g,h}$.

Since the Γ-action on $\mathrm{Sp}(1)^\theta$ is trivial, we obtain a bijection

$$[S^3, \mathrm{Sp}(1)^\theta]_\Gamma \cong [S^3/\Gamma, \mathrm{Sp}(1)] \cong \mathbb{Z},$$

where we give S^3/Γ the orientation induced from S^3 and the second isomorphism is given by $[u] \mapsto \deg(u)$. Given an equivariant map $f\colon S^3 \to \mathrm{Sp}(1)^{(\alpha;\beta)}$ let $\overline{\tau(f)}\colon S^3/\Gamma \to \mathrm{Sp}(1)$ denote the map induced by $\tau(f)$. The above work then amounts to the fact that the map $[S^3, \mathrm{Sp}(1)^{(\alpha;\beta)}]_\Gamma \to \mathbb{Z}$ given by $f \mapsto \deg(\overline{\tau(f)})$ is a bijection. To complete the proof we will relate this degree to the degree of f. First, since $S^3 \to S^3/\Gamma$ has degree $|\Gamma|$, we deduce that $|\Gamma| \deg(\overline{\tau(f)}) = \deg \tau(f)$. Using Lemma I.B.2 and the definition $\tau(f) = \mu \circ (g, f, h)$ one finds that

$$\deg(\tau(f)) = \deg h - \deg g + \deg(f).$$

Hence, $\deg(f) = (\deg g - \deg h) + |\Gamma| \deg(\overline{\tau(f)})$. We therefore have the following commutative diagram

$$\begin{array}{ccc} [S^3, \mathrm{Sp}(1)^{(\alpha;\beta)}]_\Gamma & \longrightarrow & [S^3, \mathrm{Sp}(1)^{(\alpha;\beta)}] \\ \big\downarrow \cong & & \big\downarrow \cong \\ \mathbb{Z} & \xrightarrow{\quad m \quad} & \mathbb{Z}, \end{array} \tag{I.43}$$

where the vertical maps are given by $[f] \mapsto \deg(\overline{\tau(f)})$ and $[f] \mapsto \deg f$, from left to right, and $m(n) = |\Gamma| n + (\deg g - \deg h)$. Since m is injective we conclude that the upper horizontal map is injective. The final assertion follows from the fact that $\deg g - \deg h \pmod{|\Gamma|}$ is independent of g and h by Lemma I.B.3. ■

Theorem I.B.6. *Let $\Gamma \subset \mathrm{Sp}(1)$ be a finite subgroup acting on $S^4 \subset \mathbb{H} \oplus R$ by $\gamma \cdot (x, t) = (\gamma \cdot x, t) = (x\gamma^*, t)$. Then the map*

$$\phi\colon \mathrm{Vec}^1_\Gamma(S^4, \mathbb{H}) \to \mathrm{Rep}^1(\Gamma, \mathbb{H}) \times \mathrm{Rep}^1(\Gamma, \mathbb{H}) \times \mathbb{Z}$$

given by $[E] \mapsto ([E_N], [E_S], c_2(E)[S^4])$ is injective. Furthermore, for each pair $\alpha, \beta \in \mathrm{Rep}^1(\Gamma, \mathbb{H})$ there is a constant $c \in \mathbb{Z}$ such that $(\alpha, \beta, k) \in \mathrm{Im}\,\phi$ if and only if $k \equiv c \pmod{|\Gamma|}$.

Proof. The follows from Proposition I.B.4 and Proposition I.B.5 in view of the fact that $c_2(E)[S^4]$ coincides with the degree of the transition function $t\colon S^3 \to \mathrm{Sp}(1) \subset \mathrm{Hom}_{\mathbb{H}}(E_S, E_N)$. ■

The proof of Proposition I.B.5 gives a formula for the constant c that will be useful later.

Corollary I.B.7. *The constant c in the above theorem may be taken to be $c = \deg g - \deg h$ for any choices of equivariant maps $g\colon S^3 \to \mathrm{Sp}(1)^\alpha$ and $h\colon S^3 \to \mathrm{Sp}(1)^\beta$.*

I.B.3 Application of the Equivariant Index Theorem

Let $E \to S^4$ be a Γ-equivariant SU(2) bundle and let S^+ and S^- be the complex spinor bundles associated with the unique spin structure on S^4. It turns out that this spin structure is naturally Γ-equivariant. This means that the spinor bundles carry the structure of Γ-equivariant bundles and that the Clifford multiplication map $TS^4 \to \mathrm{Hom}_{\mathbb{C}}(S^+, S^-)$ is a map of equivariant bundles. Therefore, if A is a Γ-invariant connection in E the twisted Dirac operator

$$D_A\colon \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$$

is a Γ-equivariant elliptic operator. The non-equivariant index of such an operator takes values in the K-theory of a point $K(*) = \mathbb{Z}$. In the Γ-equivariant setting it takes values in the Γ-equivariant K-theory of a point, namely, $K_\Gamma(*) = R(\Gamma)$; the complex representation ring of Γ. It is this index we will calculate in this section.

For this purpose it is more convenient to use the model $\mathbb{H}P^1$ for S^4. This is the identification space $\mathbb{H}^2 - \{0\}/\sim$ where $(x, y) \sim (qx, qy)$ for $q \in \mathbb{H} - \{0\}$. The equivalence class of (x, y) in $\mathbb{H}P^1$ is denoted by $[x : y]$. There are two canonical charts

$$\mathbb{H} \cong U = \{[x : y] \in \mathbb{H}P^1 : x \neq 0\} \text{ and } \mathbb{H} \cong V = \{[x : y] \in \mathbb{H}P^1 : y \neq 0\}$$

given by $z \mapsto [1 : z]$ and $w \mapsto [w : 1]$, respectively. The transition function $\mathbb{H} - \{0\} \to \mathbb{H} - \{0\}$ (in either direction) is given by $q \mapsto q^{-1}$. This map is orientation preserving so we define the orientation of $\mathbb{H}P^1$ by requiring that both of the canonical charts are positive, where $\mathbb{H} = \mathbb{C}^2$ has the standard orientation.

The tautological quaternionic line bundle $\gamma \to \mathbb{H}P^1$ is defined to be

$$\gamma := \{([x : y], (z, w)) \in \mathbb{H}P^1 \times \mathbb{H}^2 : \exists q \in \mathbb{H} - \{0\} \ni (z, w) = (qx, qy)\}.$$

Let the trivial bundle $\underline{\mathbb{H}}^2 = \mathbb{H}P^1 \times \mathbb{H}^2$ carry the standard symplectic inner product, i.e., $(z_1, z_2) \cdot (w_1, w_2) = z_1 w_1^* + z_2 w_2^*$. Then $\widetilde{\gamma} := \gamma^\perp$ is another quaternionic line bundle. We give γ and $\tilde{\gamma}$ the connections obtained from the product connection in $\underline{\mathbb{H}}^2$ by orthogonal projection and equip them with the symplectic inner products inherited from the trivial bundle $\underline{\mathbb{H}}^2$.

The standard action $\mathrm{Sp}(2) \times \mathbb{H}^2 \to \mathbb{H}^2$ descends to a transitive action $\mathrm{Sp}(2) \times \mathbb{H}P^1 \to \mathbb{H}P^1$. The diagonal action on $\mathbb{H}P^1 \times \mathbb{H}^2$ preserves the subbundles $\gamma, \tilde{\gamma} \subset \underline{\mathbb{H}}^2$ and therefore give these the structure of Sp(2)-equivariant bundles over $\mathbb{H}P^1$. The connections are invariant under this action.

The unique spin structure on $\mathbb{H}P^1$ may now be realized explicitly with $S^+ = \gamma$, $S^- = \widetilde{\gamma}$ and Clifford multiplication $\chi\colon T\mathbb{H}P^1 \cong \mathrm{Hom}_{\mathbb{H}}(\gamma, \widetilde{\gamma})$ defined in the following way. Let $\iota\colon \gamma \to \underline{\mathbb{H}}^2$ denote the inclusion and $\pi\colon \underline{\mathbb{H}}^2 \to \widetilde{\gamma}$ the projection. Given $p \in \mathbb{H}P^1$, $v \in T_p\mathbb{H}P^1$ and $w \in \gamma_p$ choose a local smooth section s of γ with $s(p) = w$. Then $\chi(v)(w) = \pi(d(\iota(s)))_p$, where d is the product connection in $\underline{\mathbb{H}}^2$. In suitable trivializations of γ and $\tilde{\gamma}$ over $\mathbb{H} \cong U \subset \mathbb{H}P^1$, the map $\chi_q\colon T_q\mathbb{H} = \mathbb{H} \to \mathrm{Hom}_{\mathbb{H}}(\mathbb{H}, \mathbb{H})$ for $q \in \mathbb{H}$ takes the form $v \mapsto r_v\colon \mathbb{H} \to \mathbb{H}$, where $r_v(w) = wv$ is right multiplication. Finally, we note that this spin structure is Sp(2)-equivariant.

Remark I.B.8. With our orientation conventions it holds true that $c_2(\gamma)[\mathbb{H}P^1] = -1$. This can be verified, as suggested in [FU91, p. 85], as follows. First check that the transition function restricted to $S^3 \subset \mathbb{H} \cong U = \mathbb{H}P^1 - \{[0 : 1]\}$

is a diffeomorphism and hence has degree ± 1. Then explicitly calculate the connection and curvature forms in the chart $\mathbb{H} \cong U$ and conclude that the connection of γ is self-dual. From these two pieces of information the assertion follows. As $\gamma \oplus \widetilde{\gamma}$ is trivial it follows that $c_2(\tilde{\gamma})[\mathbb{H}P^1] = 1$ as well.

Let $\Gamma \subset \mathrm{Sp}(1)$ be a finite subgroup. We define a Γ-action on $\mathbb{H}P^1$ by restricting the $\mathrm{Sp}(2)$-action along $\Gamma \subset \mathrm{Sp}(1) \subset \mathrm{Sp}(2)$, where the latter inclusion is specified by

$$q \in \mathrm{Sp}(1) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}. \tag{I.44}$$

From the above discussion it follows that the spinor bundles $S^{\pm}$ obtain Γ-equivariant structures and that the spinor connections, are Γ-invariant. By convention the action of $\mathrm{Sp}(2)$ on $\mathbb{H}P^1$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : y] = [x : y] \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = [xa^* + yb^* : xc^* + yd^*].$$

From this we see that in the chart $\mathbb{H} \cong U \subset \mathbb{H}P^1$, $z \mapsto [1 : z]$, this action corresponds to the standard linear action $\Gamma \times \mathbb{H} \to \mathbb{H}$ given by $\gamma \cdot x = x\gamma^*$.

Remark I.B.9. By regarding $S^4 \subset \mathbb{H} \oplus \mathbb{R}$ as before the map $\psi \colon \mathbb{H}P^1 \to S^4$ given by

$$\psi([x : y]) = \frac{1}{||x||^2 + ||y||^2}(2x^*y, ||y||^2 - ||x||^2)$$

is a Γ-equivariant, orientation preserving diffeomorphism, when S^4 is given the suspended action and orientation described earlier (note that this is the opposite of the standard orientation on S^4).

Let G be a compact Lie group, X a G-manifold and $P \colon \Gamma(V) \to \Gamma(W)$ an elliptic G-operator, that is, V and W are G-equivariant complex vector bundles and P is equivariant with respect the induced actions. Then the G-index of P is defined by

$$\mathrm{Ind}_G(P) = [\mathrm{Ker}\, P] - [\mathrm{Coker}(P)] \in R(G).$$

The character of this virtual representation is denote by $\mathrm{Ind}_g(P) = \mathrm{tr}(g|_{\mathrm{Ker}(P)}) - \mathrm{tr}(g|_{\mathrm{Coker}(P)}) \in \mathbb{C}$ for each $g \in G$. The fixed-point formula due to Atiyah, Segal and Singer expresses the fact that $\mathrm{Ind}_g(P)$ only depends on information above the fixed point set $F_g \coloneqq \{x \in X : gx = x\}$. In the case where F_g is a finite set the formula (see [LM89, Theorem 14.3]) simplifies to

$$\mathrm{Ind}_g P = \sum_{x \in F_g} \frac{\mathrm{ch}_g(V_x) - \mathrm{ch}_g(W_x)}{\mathrm{ch}_g(\lambda_{-1}(T_x X)_{\mathbb{C}})}. \tag{I.45}$$

Here, $\mathrm{ch}_g(V_x) = \mathrm{tr}(g \colon V_x \to V_x)$, $(T_x X)_{\mathbb{C}}$ is the complexification of $T_x X$ and λ_{-1} is defined for a G representation U by

$$\lambda_{-1}(U) = \sum_{k=0}^{\dim_{\mathbb{C}} U} (-1)^k \Lambda^k_{\mathbb{C}}(U) \in R(G).$$

Let $\Gamma \subset \mathrm{SU}(2) = \mathrm{Sp}(1)$ be a finite subgroup acting on $\mathbb{H}P^1$ in the way described above. This action is free away from the two fixed points $S = [1:0]$ and $N = [0:1]$. Let $E \to \mathbb{H}P^1 \cong S^4$ be a Γ-equivariant $\mathrm{Sp}(1)$-bundle with $[E_S] = \alpha$, $[E_N] = \beta$ and $c_2(E)[\mathbb{H}P^1] = k$. Let A be a Γ-invariant connection in E. Then the twisted Dirac operator $D_A \colon \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$ is an elliptic Γ-operator as described above. For our purpose the formal adjoint $D_A^* \colon \Gamma(S^- \otimes E) \to \Gamma(S^+ \otimes E)$ is more relevant.

Remark I.B.10. Here we regard $S^\pm$ and E as complex vector bundles by restriction along $\mathbb{C} \subset \mathbb{H}$. In particular, the tensor products $S^\pm \otimes E = S^\pm \otimes_{\mathbb{C}} E$ are formed over the complex numbers.

Proposition I.B.11. *Let $E \to \mathbb{H}P^1 \cong S^4$ be a Γ-equivariant $\mathrm{SU}(2)$-bundle with $[E_N], [E_S] = \alpha, \beta \in R(\Gamma)$ and $c_2(E)[\mathbb{H}P^1] = k \in \mathbb{Z}$. Then $\mathscr{H} := \mathrm{Ind}_\Gamma(D_A^*) \in R(\Gamma)$ satisfies*

$$(2 - Q)\mathscr{H} = \alpha - \beta,$$

where Q is the 2-dimensional complex representation associated with the inclusion $\Gamma \subset \mathrm{SU}(2)$. Furthermore, if $\epsilon : R(\Gamma) \to \mathbb{Z}$ denotes the augmentation, then $\epsilon(\mathscr{H}) = k$.

Proof. To apply the formula (I.45) we have to determine the action of Γ on the fibers of $S^\pm \otimes E$ over the fixed points $S = [1:0]$ and $N = [0:1]$ as well as the action on the tangent spaces $T_N\mathbb{H}P^1, T_S\mathbb{H}P^1$. From the definition of $S^+ = \gamma$ and $S^- = \widetilde{\gamma}$ we observe that in terms of the inclusions $\gamma, \widetilde{\gamma} \subset \underline{\mathbb{H}}^2$

$$\begin{array}{ll} \gamma_S = \mathbb{H} \oplus 0 & \gamma_N = 0 \oplus \mathbb{H} \\ \widetilde{\gamma}_S = 0 \oplus \mathbb{H} & \widetilde{\gamma}_N = \mathbb{H} \oplus 0 \end{array}.$$

From the description of the action in (I.44) it follows that $\gamma_S \cong \widetilde{\gamma}_N \cong \mathbb{C}^2$, the trivial representation, and $\gamma_N \cong \widetilde{\gamma}_S \cong Q$.

Next we need to investigate the action on the tangent spaces. We have already seen that in the chart $\mathbb{H} \cong U \subset \mathbb{H}P^1$ the action corresponds to the standard linear action $\gamma \cdot x = x\gamma^*$. Here $0 \in \mathbb{H}$ corresponds to $S \in \mathbb{H}P^1$. It follows that $T_S\mathbb{H}P^1 \cong Q_{\mathbb{R}}$, that is, the canonical representation regarded as a real representation. In the other chart $\mathbb{H} \cong V \subset \mathbb{H}P^1$ given by $z \mapsto [z:1]$ one may verify that the action is given by $\gamma \cdot z = \gamma z$. The differentiated action on $\mathbb{H} \cong T_0\mathbb{H}$ is given by the same formula. The conjugation map $* \colon \mathbb{H} \to \mathbb{H}$ is real linear and satisfies $(\gamma x)^* = x^*\gamma^*$. Hence, $T\mathbb{H}P^1_N \cong Q_{\mathbb{R}}$ as well. We need to calculate the term $\lambda_{-1}((Q_{\mathbb{R}})_{\mathbb{C}})$. First, note that $(Q_{\mathbb{R}})_{\mathbb{C}} \cong Q \oplus Q^* \cong 2Q$, since Q is an $\mathrm{SU}(2)$-representation and therefore self-dual. Thus,

$$\lambda_{-1}((Q_{\mathbb{R}})_{\mathbb{C}}) = \lambda_{-1}(2Q) = (\lambda_{-1}Q)^2 = (\mathbb{C} - Q + \Lambda^2 Q)^2 = (2 - Q)^2,$$

where we have used that $\lambda_{-1}(V \oplus W) = \lambda_{-1}(V)\lambda_{-1}(W)$ and the fact that $\Lambda^2 Q \cong \mathbb{C}$, the trivial representation.

Given an element $V \in R(\Gamma)$ we let $\chi_V : \Gamma \to \mathbb{C}$ denote the associated character. For any $g \neq 1 \in \Gamma$ the index formula (I.45) now gives $\chi_{\mathscr{H}}(g)$ as

$$\left(\frac{2\chi_\alpha(g) - \chi_Q(g)\chi_\alpha(g)}{(2 - \chi_Q(g))^2}\right) + \left(\frac{\chi_Q(g)\chi_\beta(g) - 2\chi_\beta(g)}{(2 - \chi_Q(g))^2}\right) = \frac{\chi_\alpha(g) - \chi_\beta(g)}{2 - \chi_Q(g)},$$

where the first term corresponds to the fixed point $S = [1 : 0]$ and the second term to $N = [0 : 1]$. This yields the equality of characters $(2-\chi_Q)\chi_{\mathscr{H}} = \chi_\alpha - \chi_\beta$, since by the above it holds for all $g \neq 0$, while for $g = 1$ it is trivially satisfied as $\chi_Q(1) = \chi_\alpha(1) = \chi_\beta(1) = 2$. We therefore obtain the relation $(2-Q)\mathscr{H} = \alpha - \beta$ in $R(\Gamma)$. The final assertion follows from the regular index theorem

$$\epsilon(\mathscr{H}) = \mathrm{Ind}(D_A^*) = c_2(E)[S^4].$$

■

I.B.4 The Chern-Simons Invariant

Let Y be a closed oriented 3-manifold and let $E \to Y$ be the necessarily trivial $\mathrm{Sp}(1) = \mathrm{SU}(2)$-bundle over Y. The Chern-Simons invariant of a flat connection A in E is the value $\mathrm{cs}(A) \in \mathbb{R}/\mathbb{Z}$, where cs is the Chern-Simons functional defined in equation (I.3). In this section we will explain that the theory of section (4) lead to a simple procedure for the calculation of this invariant when $Y = Y_\Gamma$ for some finite subgroup $\Gamma \subset \mathrm{SU}(2)$.

Equip $\mathfrak{sp}(1) = \mathrm{Im}\,\mathbb{H}$ with the standard invariant inner product for which i, j, k is an orthonormal basis and write $g(x, y) = -(x, y)$ for its negative. The identification $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$ matches g with the symmetric bilinear form $(x, y) \mapsto \frac{1}{2}\,\mathrm{tr}(xy)$. In this section we will prefer to work with the group $\mathrm{Sp}(1)$ so we make the following definition (compare equation (I.3)).

Definition I.B.12. Let X be a manifold. For any $a \in \Omega^1(X, \mathfrak{sp}(1))$ define the Chern-Simons form $\zeta(a) \in \Omega^3(X, \mathbb{R})$ by

$$\zeta(a) = \frac{1}{4\pi^2}[a \wedge_g da + \frac{1}{3} a \wedge_g (a \wedge_{ad} a)].$$

Here $\wedge_g$ and $\wedge_{ad}$ denote the combination of the wedge product and the bilinear maps g and $\mathrm{ad} = [\cdot, \cdot]$, respectively. Note that $f^*(\zeta(a)) = \zeta(f^*(a))$ for any smooth map $f : X \to Y$ and $a \in \Omega^1(Y, \mathfrak{sp}(1))$.

Let Y be a closed oriented 3-manifold and let $E \to Y$ be an $\mathrm{Sp}(1)$-bundle. Fix a global trivialization $E \cong Y \times \mathrm{Sp}(1)$ so that the space of connections $\mathcal{A}_E$ is identified with $\Omega^1(Y, \mathfrak{sp}(1))$. Then for $a \in \Omega^1(Y, \mathfrak{sp}(1))$ we have $\mathrm{cs}(a) = \int_Y \zeta(a)$ mod $\mathbb{Z}$. Observe that if a is flat, that is, $da + \frac{1}{2} a \wedge_{ad} a = 0$, then

$$\zeta(a) = \frac{-1}{24\pi^2} a \wedge_g (a \wedge_{ad} a). \tag{I.46}$$

Lemma I.B.13. *Let $\theta \in \Omega^1(\mathrm{Sp}(1), \mathfrak{sp}(1))$ be the left Maurer-Cartan form, that is, $\theta_g(v) = dl_{g^{-1}}(v)$ for all $g \in \mathrm{Sp}(1)$ and $v \in T_g\,\mathrm{Sp}(1)$. Then*

$$\int_{\mathrm{Sp}(1)} \zeta(\theta) = 1,$$

when $\mathrm{Sp}(1)$ is oriented by requiring that (i, j, k) is a positive basis for $T_1\,\mathrm{Sp}(1) = \mathfrak{sp}(1) = \mathrm{Im}\,\mathbb{H}$.

Proof. As θ is left invariant and $l_g^*\zeta(\theta) = \zeta(l_g^*(\theta)) = \zeta(\theta)$ for each $g \in \mathrm{Sp}(1)$, it follows that $\zeta(\theta) \in \Omega^3(\mathrm{Sp}(1), \mathbb{R})$ is left invariant. Every left invariant form of top dimension in a compact, connected Lie group is necessarily right invariant as well. Therefore, if we regard $\mathrm{Sp}(1) = S^3 \subset \mathbb{H} = \mathbb{R}^4$, we see that $\zeta(\theta)$ is invariant under the action $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times S^3 \to S^3$ given by $(a, b) \cdot x = axb^{-1}$. This action factors through the double covering $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$ where $\mathrm{SO}(4)$ acts in the standard way on $S^3 \subset \mathbb{R}^4$. We conclude that $\zeta(\theta)$ is $\mathrm{SO}(4)$-invariant.

The Riemannian volume form of S^n equipped with the standard round metric is the restriction of

$$\omega_n = \sum_{i=1}^{n+1} (-1)^{i+1} x_i dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n \in \Omega^n(\mathbb{R}^{n+1}, \mathbb{R}),$$

where the hat denotes omission. This form is invariant under the standard transitive action of $\mathrm{SO}(n+1)$. Specializing to the case $n = 3$, it follows that there is a constant $c \in \mathbb{R}$ such such that $\zeta(\theta) = c\omega_3$.

We determine the constant c by evaluating at $1 = (1, 0, 0, 0) \in \mathbb{H} = \mathbb{R}^4$. By the formula above we have $(\omega_3)_1 = dx_2 \wedge dx_3 \wedge dx_4$, so that $c = \zeta(\theta)(i, j, k)$. As $\theta_1 : \mathfrak{sp}(1) \to \mathfrak{sp}(1)$ is the identity, we have for $a, b, c \in \mathfrak{sp}(1)$ that

$$\theta \wedge_g (\theta \wedge_{ad} \theta)(a, b, c) = g(a, 2[b, c]) - g(b, 2[a, c]) + g(c, 2[a, b]) = 6g(a, [b, c]),$$

since g is symmetric and $g(b, [a, c]) = g([b, a], c) = -g([a, b], c) = -g(a, [b, c])$. Inserting $(a, b, c) = (i, j, k)$ we find

$$\theta \wedge_g (\theta \wedge_{ad} \theta)(i, j, k) = 6g(i, [j, k]) = 6g(i, 2i) = -12(i, i) = -12.$$

Consequently, using formula (I.46) we obtain $\zeta(\theta)_1(i, j, k) = \frac{1}{2\pi^2}$ and therefore $\zeta(\theta) = \frac{1}{2\pi^2}\omega_3$. The volume of S^3 is given by $\mathrm{Vol}(S^3) = \int_{S^3} \omega = 2\pi^2$, so it follows that $\int_{\mathrm{Sp}(1)} \zeta(\theta) = 1$ as required. ∎

In the following we give $Y_\Gamma = S^3/\Gamma$ the orientation induced from the standard orientation on $S^3 \subset \mathbb{H}$.

Lemma I.B.14. *Let $\alpha \in \mathrm{Rep}^1(\Gamma, \mathbb{H})$ correspond to a flat connection in the $\mathrm{Sp}(1)$-bundle over Y_Γ. Choose a representative $\rho_\alpha : \Gamma \to \mathrm{Sp}(1)$ for α and let $\mathrm{Sp}(1)^\alpha$ denote the group $\mathrm{Sp}(1)$ equipped with the Γ-action $\gamma \cdot q = \rho_\alpha(\gamma)q$. Then for any Γ-equivariant map $g : S^3 \to \mathrm{Sp}(1)^\alpha$ it holds true that*

$$\mathrm{cs}(\alpha) = \deg(g)/|\Gamma| \mod \mathbb{Z}.$$

Proof. Choose a representative $a \in \Omega^1(Y_\Gamma, \mathfrak{sp}(1))$ for the flat connection in a global trivialization of the bundle. Then by definition

$$\mathrm{cs}(\alpha) = \int_{Y_\Gamma} \zeta(a) \mod \mathbb{Z}.$$

Let $p : S^3 \to Y_\Gamma$ be the quotient map. Since Y_Γ carries the orientation induced from S^3 this map has degree $|\Gamma|$. Let $b = p^*(a) \in \Omega^1(S^3, \mathfrak{sp}(1))$. Since a and b

are flat and S^3 is simply connected, there exists a smooth map $g\colon S^3 \to \mathrm{Sp}(1)$ such that $b = g^*\theta$, where $\theta \in \Omega^1(\mathrm{Sp}(1), \mathfrak{sp}(1))$ is the left Maurer-Cartan form. Moreover, the map g is unique up to left translation in the sense that if $h\colon S^3 \to \mathrm{Sp}(1)$ is another map with $h^*\theta = b$, then $h = l_u \circ g$ for some $u \in \mathrm{Sp}(1)$, where $l_u(v) = uv$ denotes left translation in $\mathrm{Sp}(1)$. For each $\gamma \in \Gamma$ we have

$$\gamma^*(b) = \gamma^* p^*(a) = (p \circ \gamma)^* a = p^*(a) = b.$$

Then as $(g \circ \gamma)^*(\theta) = \gamma^* b = b$ we deduce from the uniqueness of g that there is a unique $\rho(\gamma) \in \mathrm{Sp}(1)$ such that $g \circ \gamma = l_{\rho(\gamma)} \circ g$. One may easily verify that $\rho\colon \Gamma \to \mathrm{Sp}(1)$ is a homomorphism, in fact, it encodes the holonomy representation of the flat connection $a \in \Omega^1(Y_\Gamma, \mathfrak{sp}(1))$ we started with. If we replace $g\colon S^3 \to \mathrm{Sp}(1)$ with $l_u \circ g$, then $\rho : \Gamma \to \mathrm{Sp}(1)$ is replaced by $c_u \circ \rho$, where c_u is conjugation by u in $\mathrm{Sp}(1)$. Therefore, for a suitable choice of $u \in \mathrm{Sp}(1)$ we can arrange that $\rho = \rho_\alpha$. From the defining formula $g \circ \gamma = l_{\rho_\alpha(\gamma)} \circ g$ it follows that $g\colon S^3 \to \mathrm{Sp}(1)^\alpha$ is equivariant.

Our setup is summarized in the following diagram

$$\begin{array}{ccccc} \theta & \xrightarrow{\;g^*\;} & b & \xleftarrow{\;p^*\;} & a \\ & & & & \\ \mathrm{Sp}(1) & \xleftarrow{\;g\;} & S^3 & \xrightarrow{\;p\;} & Y_\Gamma. \end{array}$$

We may now calculate

$$\int_{Y_\Gamma} \zeta(a) = \frac{1}{|\Gamma|} \int_{S^3} p^*(\zeta(a)) = \frac{1}{|\Gamma|} \int_{S^3} g^*(\zeta(\theta)) = \frac{\deg(g)}{|\Gamma|} \int_{\mathrm{Sp}(1)} \zeta(\theta).$$

By the previous lemma $\int_{\mathrm{Sp}(1)} \zeta(\theta) = 1$, so by reducing this modulo integers we obtain $\mathrm{cs}(\alpha) = \mathrm{cs}(a) = \frac{\deg g}{|\Gamma|} \mod \mathbb{Z}$. To complete the proof we must show that $\deg(h)/|\Gamma| \mod \mathbb{Z}$ is independent of the equivariant map $h\colon S^3 \to \mathrm{Sp}(1)^\alpha$. This is a consequence of Lemma I.B.3, which shows that $\deg h \mod |\Gamma|$ is independent of h. ■

Example I.B.15. Let $\Gamma \subset \mathrm{Sp}(1) = \mathrm{SU}(2)$ be any finite subgroup. Let $\alpha = Q$ denote the the canonical representation associated with the inclusion above. Then the map $g\colon S^3 \to \mathrm{Sp}(1)^\alpha$ given by $g(x) = x^*$ is equivariant. Indeed,

$$g(\gamma \cdot x) = g(x\gamma^*) = (x\gamma^*)^* = \gamma x^* = \gamma \cdot g(x)$$

for each $\gamma \in \Gamma$ and $x \in S^3$. Since $S^3 \subset \mathbb{H}$ and $\mathrm{Sp}(1) \subset \mathbb{H}$ both carry the standard orientation for which $(i, j, k) \in T_1 S^3 = T_1 \mathrm{Sp}(1)$ is a positive basis, it follows that $\deg g = -1$ and therefore

$$\mathrm{cs}(\alpha) = \frac{-1}{|\Gamma|} \mod \mathbb{Z}.$$

If we reverse the orientation of Y_Γ then the value of cs changes by a sign.

Proposition I.B.16. *Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup and let α, β be a pair of $\mathrm{SU}(2)$-representations of $\Gamma = \pi_1(Y_\Gamma)$ corresponding to a pair of flat connections in the $\mathrm{SU}(2)$-bundle over Y_Γ. Let $\mathscr{H}$ be a solution of the equation*

$$(2 - Q)\mathscr{H} = \alpha - \beta$$

in $R(\Gamma)$. Then $\mathrm{cs}(\alpha) - \mathrm{cs}(\beta) = \epsilon(\mathscr{H})/|\Gamma| \in \mathbb{R}/\mathbb{Z}$.

Proof. Let $E \to S^4$ be a Γ-equivariant $\mathrm{SU}(2)$-bundle with $[E_N] = \alpha$ and $[E_S] = \beta$. Then by Proposition I.B.11 the element $\mathscr{H} := \mathrm{Ind}(D_A^*) \in R(\Gamma)$, where A is any Γ-invariant connection in E, satisfies $(2 - Q)\mathscr{H} = \alpha - \beta$ and $\epsilon(\mathscr{H}) = c_2(E)[S^4]$. Moreover, by the classification theorem I.B.6 and Corollary I.B.7 the integer $c_2(E)[S^4]$ satisfies a congruence

$$c_2(E)[S^4] \equiv \deg g - \deg h \mod |\Gamma|,$$

where $g\colon S^3 \to \mathrm{Sp}(1)^\alpha$ and $h\colon S^3 \to \mathrm{Sp}(1)^\beta$ are any choices of equivariant maps. From Lemma I.B.14 we obtain

$$\mathrm{cs}(\alpha) - \mathrm{cs}(\beta) \equiv (\deg g - \deg h)/|\Gamma| \equiv c_2(E)[S^4]/|\Gamma| \equiv \epsilon(\mathscr{H})/|\Gamma| \mod \mathbb{Z}.$$

This shows that the result is valid for a specific solution $\mathscr{H}$. However, by Lemma I.4.5 any other solution is given by $\mathscr{H} + mR$, where $m \in \mathbb{Z}$ and $R = \mathbb{C}[\Gamma]$ is the regular representation. Since $\dim_{\mathbb{C}} R = |\Gamma|$ we conclude that $\epsilon(\mathscr{H})/|\Gamma|$ is independent of the solution $\mathscr{H}$ modulo $\mathbb{Z}$ and the proof is complete. ■

This theorem reduces the calculation of Chern-Simons invariants to solving an equation in $R(\Gamma)$. Employing the graphical solution procedure of Proposition I.4.12 one may easily solve for all the flat connections recursively starting at the trivial connection. We will give an example in the next section.

I.B.5 Connections to Group Cohomology

Given a finite group G its group cohomology may be defined by $H^*(G;\mathbb{Z}) := H^*(BG;\mathbb{Z})$, where BG is its classifying space (see for instance [AM04, Chapter II]). A representation $\rho\colon G \to U(n)$ gives rise to characteristic classes $c_i(\rho) \in H^{2i}(\Gamma;\mathbb{Z})$ by pulling back the universal Chern classes $c_i \in H^{2i}(BU(n);\mathbb{Z})$ along the induced map $B\rho\colon B\Gamma \to BU(n)$. The purpose of this section is to show that for a finite group $\Gamma \subset \mathrm{SU}(2)$ there is a natural way to identify the Chern-Simons invariant $\mathrm{cs}(\alpha) \in \mathbb{R}/\mathbb{Z}$ with the second Chern class $c_2(\rho_\alpha) \in H^4(\Gamma;\mathbb{Z})$, where $\rho_\alpha : \Gamma \to \mathrm{SU}(2)$ is the holonomy representation associated with α.

The following calculation is certainly known. We include a quick proof for completeness.

Lemma I.B.17. *Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup. Then $H^0(\Gamma;\mathbb{Z}) = \mathbb{Z}$, while for $i > 0$*

$$H^i(\Gamma;\mathbb{Z}) \cong \begin{cases} \Gamma^{ab} & \text{for } i \equiv 2 \pmod 4 \\ \mathbb{Z}/|\Gamma| & \text{for } i \equiv 0 \pmod 4 \\ 0 & \text{for } i \equiv 1, 3 \pmod 4 \end{cases}$$

Moreover, for any generator $e \in H^4(\Gamma;\mathbb{Z})$ the cup product $e\cup\colon H^i(\Gamma;\mathbb{Z}) \cong H^{i+4}(\Gamma;\mathbb{Z})$ is an isomorphism for each $i > 0$.

Proof. Let Γ act freely on S^3 in the standard way. The action induces a fibration $S^3 \to S^3/\Gamma \to B\Gamma$ (see for instance [AM04, Lemma 6.2]). The associated Gysin sequence breaks up into the following pieces of information:

(1) There are isomorphisms $H^i(\Gamma) \cong H^i(S^3/\Gamma)$ for $0 \leq i \leq 2$.

(2) There is an exact sequence

$$0 \longrightarrow H^3(\Gamma) \longrightarrow H^3(S^3/\Gamma) \xrightarrow{p^*} H^3(S^3) \longrightarrow H^4(\Gamma) \longrightarrow 0,$$

where $p : S^3 \to S^3/\Gamma$ is the quotient map.

(3) The map $e\cup\colon H^i(\Gamma) \to H^{i+4}(\Gamma)$ is surjective for $i = 0$ and an isomorphism for $i \geq 1$, where $e \in H^4(\Gamma)$ is the Euler class of the fibration.

By Lemma I.3.3 we have $H^1(S^3/\Gamma) = 0$ and $H^2(S^3/\Gamma) = \Gamma^{ab}$ so that $H^1(\Gamma) = 0$ and $H^2(\Gamma) \cong \Gamma^{ab}$ by the first point. Since $p\colon S^3 \to S^3/\Gamma$ is a covering map of degree $|\Gamma|$, it follows from the second point that $H^3(\Gamma) = 0$ and $H^4(\Gamma) \cong \mathbb{Z}/|\Gamma|$. The rest of the statement now follows from the third point. ■

The above lemma does not give complete information about the product structure in $H^*(\Gamma)$. The underlying Serre spectral sequence giving rise to the Gysin sequence simply does not contain information about the cup product $H^2(\Gamma;\mathbb{Z}) \times H^2(\Gamma;\mathbb{Z}) \to H^4(\Gamma;\mathbb{Z})$. It is of course known that $H^*(\mathbb{Z}/m,\mathbb{Z}) \cong \mathbb{Z}[\beta]/(m\beta)$ with $|\beta| = 2$ for cyclic groups [Wei94, Exercise 6.7.4]. For the binary icosahedral group I^* one has $(I^*)^{ab} = 0$, so the cup product is necessarily trivial. However, for the remaining groups more refined techniques seem to be required.

Definition I.B.18. Define $e_\Gamma \in H^4(\Gamma;\mathbb{Z})$ to be $c_2(\iota)$ where $\iota\colon \Gamma \subset \mathrm{SU}(2)$ denotes the inclusion.

Proposition I.B.19. *Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup. Then $e_\Gamma \in H^4(\Gamma;\mathbb{Z})$ is a generator. Define $\tau\colon H^4(\Gamma;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ to be the group homomorphism given by $\tau(ke_\Gamma) = k/|\Gamma| \mod \mathbb{Z}$. Let α be a flat connection in the trivial $\mathrm{SU}(2)$-bundle over Y_Γ with associated representation $\rho_\alpha\colon \Gamma \to \mathrm{SU}(2)$. Then*

$$\tau(c_2(\rho_\alpha)) = -\operatorname{cs}(\alpha).$$

Proof. According to Lemma I.B.3 there exists an equivariant map $g\colon S^3 \to \mathrm{Sp}(1) \cong \mathrm{SU}(2)$ in the sense that $g(\gamma \cdot q) = \rho_\alpha(\gamma) \cdot g(q)$ for each $\gamma \in \Gamma$ and $q \in S^3$. Composing this map with the orientation reversing orbit map $\mathrm{SU}(2) \to S^3$ given by $x \mapsto x \cdot 1$, we obtain a map $h : S^3 \to S^3$, which is equivariant along $\rho_\alpha : \Gamma \to \mathrm{SU}(2)$. Here $\mathrm{SU}(2)$ acts on $S^3 \subset \mathbb{C}^2$ in the standard way. Note that

$\deg h = -\deg g$. There is an induced morphism of fibrations

$$\begin{array}{ccccc} S^3 & \longrightarrow & E\Gamma \times_\Gamma S^3 & \longrightarrow & B\Gamma \\ \downarrow h & & \downarrow & & \downarrow B\rho_\alpha \\ S^3 & \longrightarrow & E\,\mathrm{SU}(2) \times_{\mathrm{SU}(2)} S^3 & \longrightarrow & B\,\mathrm{SU}(2), \end{array}$$

where $E\Gamma \to B\Gamma$ and $E\,\mathrm{SU}(2) \to B\,\mathrm{SU}(2)$ are the universal bundles. As both Γ and $\mathrm{SU}(2)$ act freely on S^3 there are homotopy equivalences

$$E\Gamma \times_\Gamma S^3 \simeq S^3/\Gamma \quad \text{and} \quad E\,\mathrm{SU}(2) \times_{\mathrm{SU}(2)} S^3 \simeq S^3/\,\mathrm{SU}(2) = *.$$

The above morphism of fibrations induces a morphism of the associated (cohomology) Serre spectral sequences. As both fibrations have S^3 as fiber the only nontrivial differentials are $d_4\colon E_4^{p,3} \to E_4^{p+4,0}$ for $p \geq 0$. In particular, we obtain a commutative diagram

$$\begin{array}{ccc} H^3(S^3;\mathbb{Z}) = E_4^{0,3} & \xrightarrow{d_4} & E_4^{4,0} = H^4(B\,\mathrm{SU}(2);\mathbb{Z}) \\ \downarrow h^* & & \downarrow B\rho_\alpha^* \\ H^3(S^3;\mathbb{Z}) = E_4^{0,3} & \xrightarrow{d_4} & E_4^{4,0} = H^4(B\Gamma;\mathbb{Z}) \end{array}$$

The upper differential must be an isomorphism since the spectral sequence converges to $H^*(*) = \mathbb{Z}$. Let $u \in H^3(S^3;\mathbb{Z})$ be the unique generator with $d_4(u) = c_2 \in H^4(B\,\mathrm{SU}(2);\mathbb{Z})$. Then from the commutativity of the diagram we obtain

$$c_2(\alpha) = (B\rho_\alpha)^*(c_2) = d_4(h^*u) = (\deg h)d_4(u) = (\deg h)e_\Gamma.$$

For the final equality, consider the same commutative diagram with ρ_α replaced by the inclusion $\iota\colon \Gamma \subset \mathrm{SU}(2)$. Then we can take $h\colon S^3 \to S^3$ to be the identity, and it follows from the same calculation as above that $e_\Gamma := B\iota^*(c_2) = d^4(u)$. This also shows that e_Γ is a generator for $H^4(B\Gamma;\mathbb{Z})$ as the lower differential must be surjective because $H^4(S^3/\Gamma;\mathbb{Z}) = 0$.

To conclude, recall from Lemma I.B.14 that for the equivariant map $g\colon S^3 \to \mathrm{SU}(2)$ we started with, it holds true that $\mathrm{cs}(\alpha) = \deg(g)/|\Gamma| \mod \mathbb{Z}$. Hence,

$$\tau(c_2(\alpha)) = \tau(\deg(h)e_\Gamma) = -\tau(\deg(g)e_\Gamma) = -\deg(g)/|\Gamma| \mod \mathbb{Z}$$

and the proof is complete. ■

Example I.B.20. Let $\Gamma = T^*$. There are three flat connections θ, α and λ. Here θ corresponds to the trivial representation, α corresponds to the canonical representation Q, while λ is reducible and corresponds to a representation $\rho \oplus \rho^*$ where $\rho\colon T^* \to U(1)$ is one of the 1-dimensional representations (see the character table in A.3). By example I.B.15 we know that $\mathrm{cs}(\alpha) = -1/|T^*| = -1/24$, and, of course, $\mathrm{cs}(\theta) = 0$. To determine $\mathrm{cs}(\lambda)$ we solve the equation $(2-Q)\mathscr{H} = \alpha - \lambda$ in $R(\Gamma)$ using the technique of Proposition I.4.12. This yields a solution $\mathscr{H}$ with $\epsilon(\mathscr{H}) = \dim_{\mathbb{C}} \mathscr{H} = 15$. Then by Proposition I.B.16 we obtain

$$\mathrm{cs}(\alpha) - \mathrm{cs}(\lambda) = \epsilon(\mathscr{H})/|T^*| = 15/24 \mod \mathbb{Z}.$$

Hence, $\mathrm{cs}(\lambda) = -2/3 = 1/3 \mod \mathbb{Z}$.

This calculation can be used to determine the cup product $H^2(T^*;\mathbb{Z}) \times H^2(T^*;\mathbb{Z}) \to H^4(T^*;\mathbb{Z})$ using the above proposition. As $c_1\colon \mathrm{Hom}(T^*, U(1)) \to H^2(T^*;\mathbb{Z})$ is an isomorphism, we can take $c_1(\rho) \in H^2(T^*;\mathbb{Z}) \cong \mathbb{Z}/3$ as a generator. Then

$$-c_1(\rho)^2 = c_1(\rho) \cup c_1(\rho^*) = c_2(\rho \oplus \rho^*) = c_2(\lambda) = 8e_\Gamma.$$

In particular, the cup product is an isomorphism from $\mathbb{Z}/3 \otimes \mathbb{Z}/3 \cong \mathbb{Z}/3$ onto the unique cyclic subgroup $8\mathbb{Z}/(24)$ of order 3.

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Paper II

Singular Quiver Varieties over Extended Dynkin Quivers

Gard Olav Helle

Abstract

We classify the singularities in certain unframed Nakajima quiver varieties associated with extended Dynkin quivers with a small restriction on the parameter and use this to construct a number of hyper-Kähler bordisms between binary polyhedral spaces.

Contents

II.1 Introduction

In [Nak94] Nakajima introduced a family of spaces he called quiver varieties. A quiver is simply a finite directed graph (Q, I) where I is the set of vertices and Q is the set of edges. We typically denote the quiver by Q. Given a dimension vector $v \in \mathbb{Z}^I_{\geq 0}$, we form the vector space

$$\operatorname{Rep}(Q, v) \coloneqq \bigoplus_{(h\colon i \to j) \in Q} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}),$$

which carries a natural linear action of the compact Lie group $G_v \coloneqq \prod_{i \in I} U(v_i)$. The doubled quiver $\overline{Q}$ is obtained from Q by adjoining an opposite edge $\overline{h}\colon j \to i$

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for each edge $h : i \to j$ in Q. In this situation one may give the complex vector space $\mathrm{Rep}(\overline{Q}, v)$ a natural quaternionic structure preserved by the action of G_v. There is an associated hyper-Kähler moment map $\mu\colon \mathrm{Rep}(\overline{Q}, v) \to \mathbb{R}^3 \otimes \mathfrak{g}_v$, where $\mathfrak{g}_v = \mathrm{Lie}(G_v)$. The quiver varieties associated with Q and v are then defined to be the hyper-Kähler quotients

$$\mathcal{M}_\xi(Q, v) := \mu^{-1}(\xi)/G_v$$

for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \otimes \mathbb{R}^I$. Here, ξ is regarded as an element of $\mathbb{R}^3 \otimes \mathfrak{g}_v$ using a canonical linear map from $\mathbb{R}^I$ onto the center of the Lie algebra. Given $w \in \mathbb{Z}^I$ let $D_w = \{\zeta \in \mathbb{R}^I : \zeta \cdot w = \sum_i \zeta_i w_i = 0\} \subset \mathbb{R}^I$. It is then necessary that $\xi \in \mathbb{R}^3 \otimes D_v$ for $\mathcal{M}_\xi(Q, v)$ to be nonempty, however, for almost all such parameters the quiver variety $\mathcal{M}_\xi(Q, v)$ carries the structure of a smooth hyper-Kähler manifold. More generally, there is a decomposition

$$\mathcal{M}_\xi(Q, v) = \mathcal{M}_\xi^{\mathrm{reg}}(Q, v) \cup \mathcal{M}_\xi^{\mathrm{sing}}(Q, v),$$

where the regular set $\mathcal{M}_\xi^{\mathrm{reg}}(Q, v)$ is open and carries the structure of a smooth hyper-Kähler manifold, while the singular set $\mathcal{M}_\xi^{\mathrm{sing}}(Q, v)$ is its closed complement.

An extended Dynkin quiver Q is a quiver whose underlying unoriented graph is an extended Dynkin diagram of type $\widetilde{ADE}$, that is, type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. In this situation there is a distinguished dimension vector $\delta \in \mathbb{Z}^I_{\geq 0}$; the minimal positive imaginary root in the associated root system. The purpose of this paper is to study the singular members of the family of quiver varieties $\mathcal{M}_\xi(Q, \delta)$ when Q is an extended Dynkin quiver. This family of spaces, whose non-singular members are the so-called ALE spaces, was first constructed and studied by Kronheimer [Kro89] in a slightly different form. The fact that Kronheimer's construction can be expressed in the above form is explained in [Nak94, p. 372-373].

The McKay correspondence [McK80] sets up a bijection between the isomorphism classes of finite subgroups $\Gamma \subset \mathrm{SU}(2)$ and the extended Dynkin diagrams of type $\widetilde{ADE}$. Kronheimer exploited this correspondence to show that the (non-empty) non-singular members of the family $\mathcal{M}_\xi(Q, \delta)$ for $\xi \in \mathbb{R}^3 \otimes D_\delta$ are smooth 4-dimensional hyper-Kähler manifolds diffeomorphic to the minimal resolution of the quotient singularity $\mathbb{C}^2/\Gamma$ where $\Gamma \subset \mathrm{SU}(2)$ is the finite subgroup associated with the underlying graph of Q under the McKay correspondence.

To state our first main result let Q be an extended Dynkin quiver with vertex set I and minimal positive imaginary root $\delta \in \mathbb{Z}^I$. By deleting any vertex $i \in I$ with $\delta_i = 1$ from Q one recovers the associated (non-extended) Dynkin graph of type ADE. Identify the set of vertices with $\{0, 1, \cdots, n\}$ for some $n \in \mathbb{N}$ such that $\delta_0 = 1$. One may then realize the root system associated with the underlying Dynkin graph as a subset $\Phi \subset \mathbb{Z}^n \subset \mathbb{R}^n$ with the coordinate vectors as a set of simple roots. Furthermore, there is a natural way to identify $\mathbb{R}^n \cong D_\delta \subset \mathbb{R}^{n+1}$ thereby identifying the set of parameters $\mathbb{R}^3 \otimes D_\delta \cong \mathbb{R}^3 \otimes \mathbb{R}^n$. With this in mind, our first main result can be stated as follows.

Theorem II.1.1. *Assume that* $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \otimes \mathbb{R}^n$ *satisfies* $\xi_1 = 0$. *Then if* $\Phi \cap \xi^\perp = \{\alpha \in \Phi : \alpha \cdot \xi_2 = \alpha \cdot \xi_3 = 0\}$ *is nonempty, it is a root system in the subspace it spans and admits a decomposition into root systems of type ADE:*

$$\Phi \cap \xi^\perp = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_r. \tag{II.1}$$

Furthermore, there is a natural bijection $\rho \colon \mathcal{M}_\xi^{\mathrm{sing}}(Q, \delta) \cong \{\Phi_1, \Phi_2, \cdots, \Phi_r\}$ *and the local structure around the singularities can be described as follows. Let* $x \in \mathcal{M}_\xi^{\mathrm{sing}}(Q, \delta)$ *and let* $\Gamma_x \subset \mathrm{SU}(2)$ *be the finite group associated with* $\rho(x)$ *under the McKay correspondence. Then there is an open neighborhood* $x \in U_x \subset \mathcal{M}_\xi(Q, \delta)$ *and a homeomorphism* $\phi_x \colon U_x \to B_r(0)/\Gamma_x$, *where* $B_r(0) \subset \mathbb{C}^2$ *is the open ball of radius* r, *that restricts to a diffeomorphim*

$$\mathcal{M}_\xi^{reg} \supset (U_x - \{x\}) \cong (B_r(0) - \{0\})/\Gamma_x.$$

The fact that $\mathcal{M}_\xi(Q, \delta)$ is non-singular when ξ avoids all the root walls D_θ for $\theta \in \Phi$ is the content of [Kro89, Corollary 2.10].

We give a brief outline of the proof of Theorem II.1.1 and, in particular, explain why we make the restriction $\xi_1 = 0$. The action of the compact group G_v on $\mathrm{Rep}(\overline{Q}, v)$ extends to a linear action of the complexification $G_\delta^c = \prod_{i=0}^n \mathrm{GL}(\delta_i, \mathbb{C})$. Moreover, the hyper-Kähler moment map splits

$$\mu = (\mu_\mathbb{R}, \mu_\mathbb{C}) \colon \mathrm{Rep}(\overline{Q}, \delta) \to \mathbb{R}^3 \otimes \mathfrak{g}_\delta \cong \mathfrak{g}_\delta \oplus \mathfrak{g}_\delta^c,$$

where $\mathfrak{g}_\delta^c = \mathrm{Lie}(G_\delta^c)$, and the second component is a moment for the action of G_δ^c with respect to a complex symplectic form on $\mathrm{Rep}(\overline{Q}, \delta)$. In the situation where the parameter $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \otimes \mathbb{R}^n$ satisfies $\xi_1 = 0$, there is a homeomorphism between the hyper-Kähler quotient $\mathcal{M}_\xi(Q, \delta)$ and the GIT quotient $\mu_\mathbb{C}^{-1}((\xi_2, \xi_3))//G_\delta^c$. The elements of the latter quotient have a representation theoretic interpretation. Indeed, if we write $\lambda = \xi_2 + i\xi_3 \in \mathbb{C}^I$, the points of $\mu_\mathbb{C}^{-1}(\lambda)//G_\delta^c$ are in natural bijection with the isomorphism classes of semi-simple modules of dimension δ over the deformed preprojective algebra $\Pi^\lambda = \Pi^\lambda(Q)$ introduced in [CH98]. Under these bijections the singularities in $\mathcal{M}_\xi(Q, \delta)$ correspond precisely to the non-simple, semi-simple modules. Using a result of Crawley-Boevey [Cra01] on the existence and uniqueness of simple Π^λ-modules, we are able to set up a bijection between the latter set and the root systems in the statement of the theorem.

To establish the homeomorphisms $\phi_x \colon U_x \to B_r(0)/\Gamma_x$ we employ a result of [May18] that essentially reduces the statement to the determination of the complex symplectic slice (see Definition II.7.1) at a point $\widetilde{x} \in \mu^{-1}(0, \lambda)$ above x. We should note that a result along these lines is given in [Kro89, Lemma 3.3], however, the proof given there seems to contain a gap that we have been unable to close. For this reason we have chosen to rely on the above mentioned result instead.

The finite subgroups $\Gamma \subset \mathrm{SU}(2)$ are called the binary polyhedral groups. By restricting the canonical action of Γ to the three-sphere $S^3 \subset \mathbb{C}^2$ we obtain the binary polyhedral spaces S^3/Γ. In Proposition II.8.3 we determine what kind of root space decomposition

$$\Phi \cap \tau = \Phi_1 \cup \cdots \Phi_r$$

one can obtain by varying the parameter τ. Combining this with the above theorem we obtain the following constructive procedure for hyper-Kähler manifolds with a number of ends modeled on $(0, \infty) \times S^3/\Gamma$ for finite subgroups $\Gamma \subset \mathrm{SU}(2)$. In the following statement we say that a subgraph H of G is a full subgraph if every edge in G connecting a pair of vertices in H belongs to H.

Theorem II.1.2. *Let $\Gamma_0, \Gamma_1, \cdots, \Gamma_r \subset \mathrm{SU}(2)$ be finite subgroups and let $K_0, K_1, \cdots, K_r$ denote the corresponding (non-extended) Dynkin graphs. Let Q be an extended Dynkin quiver with vertex set I, whose underlying unoriented graph is the extended version of K_0. Then if $K_1 \sqcup K_2 \sqcup \cdots \sqcup K_r$ can be realized as a full subgraph of K_0, there exists a parameter $\xi \in \mathbb{R}^3 \otimes \mathbb{R}^I$ such that $X = \mathcal{M}_\xi^{\mathrm{reg}}(Q, \delta)$ satisfies the following properties.*

(1) *X is a connected hyper-Kähler manifold of dimension 4.*

(2) *There are disjoint open subsets $U_0, U_1, \cdots, U_r \subset X$ and for each $0 \le i \le r$ a diffeomorphism*
$$\phi_i \colon U_i \to (0, \infty) \times S^3/\Gamma_i.$$

(3) *The complement $Y = X - \bigcup_{i=0}^r U_i$ is a compact 4-manifold with boundary components S^3/Γ_i for $0 \le i \le r$.*

Note that the diffeomorphism ϕ_i, $0 \le i \le r$, will generally not preserve the hyper-Kähler structure. We do not expect the hyper-Kähler metric in $\mathcal{M}_\xi^{\mathrm{reg}}(Q, \delta)$ to be complete.

The paper is organized as follows. In section (2) we give the basic definitions concerning hyper-Kähler manifolds and hyper-Kähler reduction. In section (3) we introduce quivers and quiver varieties and state the key results that will be needed concerning these. In section (4) we recall the basic elements of the complex representation theory of quivers. Afterwards, we give the definition of the deformed preprojective algebras $\Pi^\lambda(Q)$ and spell out the relation between the quiver variety $\mathcal{M}_{(0,\lambda)}(Q, v)$ and the isomorphism classes of semi-simple $\Pi^\lambda(Q)$-modules. Finally, we recall the key result of [Cra01] that eventually allow us to classify the singularities in $\mathcal{M}_{(0,\lambda)}(Q, v)$. In section (5) we give the construction of the extended Dynkin diagrams from the underlying Dynkin diagram and review the necessary root space theory of the associated root systems.

Our original work starts in section (6) where we establish the bijection between the singularities in the (extended Dynkin) quiver varieties and the components in the corresponding root space decomposition as in (II.1). In section (7) we establish the local models around the singularities using a result of [May18] and give the proof of Theorem II.1.1. In the final section we determine the possible configurations of singularities in the various quiver varieties and complete the proof Theorem II.1.2.

II.2 Hyper-Kähler Reduction

A hyper-Kähler manifold is a tuple (M, g, I, J, K) consisting of a smooth manifold M, a Riemannian metric g and three almost complex structure maps $I, J, K \colon TM \to TM$ subject to the following conditions:

(a) I, J and K are orthogonal with respect to g,

(b) $IJK = -1_{TM}$ and

(c) $\nabla^g I = \nabla^g J = \nabla^g K = 0$, where ∇^g is the Levi-Civita connection.

In particular, for each $S \in \{I, J, K\}$ the triple (M, g, S) is a Kähler manifold with Kähler form ω_S given by $(\omega_S)_p(v, w) = g_p(Sv, w)$ for each $p \in M$ and $v, w \in T_pM$.

Following the terminology of [May18] a tri-Hamiltonian hyper-Kähler manifold is a triple (M, K, μ) consisting of a hyper-Kähler manifold M, a compact Lie group K acting on M preserving the hyper-Kähler structure and a hyper-Kähler moment map $\mu = (\mu_I, \mu_J, \mu_K) \colon M \to \mathbb{R}^3 \otimes \mathfrak{k}^*$, where $\mathfrak{k}$ is the Lie algebra of K. Note that by definition μ is a hyper-Kähler moment map if and only if the components μ_I, μ_J, μ_K are moment maps for the corresponding symplectic forms $\omega_I, \omega_J, \omega_K$, respectively, in the sense familiar from symplectic geometry (see for instance [Can01]).

The group K acts on $\mathfrak{k}^*$ through the coadjoint action and we denote the set of fixed points by $(\mathfrak{k}^*)^K$. For each $\xi \in \mathbb{R}^3 \otimes (\mathfrak{k}^*)^K$ the fiber $\mu^{-1}(\xi)$ is K-invariant and the quotient space $\mu^{-1}(\xi)/K$ is called a hyper-Kähler quotient.

Theorem II.2.1. *[Hit+87] Let (M, K, μ) be a tri-Hamiltonian hyper-Kähler manifold and let $\xi \in \mathbb{R}^3 \otimes (\mathfrak{k}^*)^K$. If K acts freely on $\mu^{-1}(\xi)$, then the following holds true.*

(a) *ξ is a regular value for μ so that $\mu^{-1}(\xi)$ is a smooth submanifold of M.*

(b) *The quotient $\mu^{-1}(\xi)/K$ is a smooth manifold of dimension* $\dim M - 4 \dim K$ *and the projection $\pi \colon \mu^{-1}(\xi) \to \mu^{-1}(\xi)/K$ is a principal K-bundle.*

(c) *There is a unique hyper-Kähler structure on $\mu^{-1}(\xi)/K$ with Kähler forms $\omega'_I, \omega'_J, \omega'_K$ such that $\pi^*(\omega'_S) = \omega_S|_{\mu^{-1}(\xi)}$ for each $S \in \{I, J, K\}$.*

The passage from (M, K, μ) to $\mu^{-1}(\xi)/K$ for $\xi \in \mathbb{R}^3 \otimes (\mathfrak{k}^*)^K$ is called hyper-Kähler reduction. Even if the action of K on $\mu^{-1}(\xi)$ fails to be free, the hyper-Kähler quotient $X := \mu^{-1}(\xi)/K$ admits a decomposition into smooth hyper-Kähler manifolds of various dimensions (see [May18, Theorem 1.1]). For our purpose it will be sufficient to note that if $U \subset M$ denotes the open (possibly empty) set consisting of the free K-orbits, then $\mu|_U : U \to \mathbb{R}^3 \otimes \mathfrak{k}^*$ is a moment map for the action of K on U, and therefore $(\mu^{-1}(\xi) \cap U)/K =: X^{\mathrm{reg}} \subset X$ carries the structure of a smooth hyper-Kähler manifold by the above theorem. The open subset X^{reg} is called the regular set and its closed complement $X^{\mathrm{sing}} := X - X^{reg}$ is called the singular set.

We will only be interested in a very simple instance of the above procedure. Let V be a quaternionic vector space equipped with a compatible real inner product $g\colon V \times V \to \mathbb{R}$, that is, V is a real vector space equipped with three orthogonal endomorphisms $I, J, K\colon V \to V$ satisfying the relations of the quaternion algebra:

$$I^2 = J^2 = K^2 = IJK = -1_V.$$

Using the standard identification $T_pV \cong V$ for each $p \in V$, we may regard (V, g, I, J, K) as a (flat) hyper-Kähler manifold. Let K be a compact Lie group acting linearly on V preserving (g, I, J, K). In this situation the unique hyper-Kähler moment map vanishing at $0 \in V$, $\mu = (\mu_I, \mu_J, \mu_K)\colon V \to \mathbb{R}^3 \otimes \mathfrak{k}^*$, is given by

$$\mu_I(x)(\xi) = \frac{1}{2}\omega_I(\xi \cdot x, x) = \frac{1}{2}g(\xi \cdot Ix, x)$$

for $x \in V$, $\xi \in \mathfrak{k}$ and similarly for μ_J and μ_K. We call the triple (V, K, μ) a linear tri-Hamiltonian hyper-Kähler manifold.

II.3 Quiver Varieties

A quiver is a finite directed graph (Q, I, s, t), where I is the set of vertices, Q is the set of edges and $s, t\colon Q \to I$ are the source and target maps. Given an edge $h \in Q$ with $s(h) = i \in I$ and $t(h) = j \in I$ we write $h\colon i \to j$. We will abuse notation slightly and refer to the quiver simply as Q or (Q, I) letting s and t be implicit. The purpose of this section is to fix our notation, define the quiver varieties of interest and state a few results needed for our later work. We will later restrict our attention to the quivers specified in the following definition.

Definition II.3.1. An extended Dynkin quiver is a quiver Q whose underlying unoriented graph is an extended Dynkin diagram of type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. Similarly, a Dynkin quiver is a quiver whose underlying unoriented graph is a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8.

Let (Q, I) be a quiver. For each $v = (v_i)_{i \in I} \in \mathbb{Z}^I_{\geq 0}$, called a dimension vector, define

$$\begin{aligned}
\operatorname{Rep}(Q, v) &:= \bigoplus_{h \in Q} \operatorname{Hom}(\mathbb{C}^{v_{s(h)}}, \mathbb{C}^{v_{t(h)}}) \\
G_v &:= \prod_{i \in I} U(v_i) \\
G_v^c &:= \prod_{i \in I} \operatorname{GL}(v_i, \mathbb{C}),
\end{aligned}$$

where $U(v_i) \subset \operatorname{GL}(v_i, \mathbb{C})$ denotes the group of unitary matrices for each $i \in I$. There is an evident inclusion $G_v \subset G_v^c$ witnessing the fact that G_v^c is the complexification of G_v. The Lie algebras $\mathfrak{g}_v := \operatorname{Lie}(G_v)$ and $\mathfrak{g}_v^c := \operatorname{Lie}(G_v^c)$ are given by

$$\mathfrak{g}_v = \bigoplus_{i\in I} \mathfrak{u}(v_i) \quad \text{and} \quad \mathfrak{g}_v^c = \bigoplus_{i\in I} \operatorname{End}(\mathbb{C}^{v_i}).$$

The group G_v^c acts linearly on $\operatorname{Rep}(Q,v)$ by the formula

$$g\cdot x = (g_{t(h)}x_h g_{s(h)}^{-1})_{h\in Q} \text{ for } g=(g_i)_{i\in I}\in G_v^c \text{ and } x=(x_h)_{h\in Q}\in \operatorname{Rep}(Q,v).$$

The subgroup G_v acts by restriction along the inclusion $G_v \subset G_v^c$. The space $\operatorname{Rep}(Q,v)$ carries a Hermitian inner product preserved by the action of G_v. Explicitly,

$$(x,y) = \sum_{h\in Q} \operatorname{tr}(x_h y_h^*),$$

where tr is the trace and y_h^* is the adjoint of y_h with respect to the standard Hermitian inner product on $\mathbb{C}^{v_i}$ for $i\in I$.

Definition II.3.2. Let Q be a quiver. The opposite quiver Q^{op} is defined by taking the same set of vertices and reverse the orientation of each edge. For an edge $h\in Q$ the opposite edge is denoted by $\overline{h}\in Q^{op}$. The doubled quiver $\overline{Q}$ is defined by taking the same set of vertices and let the set of edges be $Q\cup Q^{op}$. The orientation map $\epsilon\colon \overline{Q}\to\{\pm1\}$ is defined by $\epsilon(h)=+1$ if $h\in Q$ and $\epsilon(h)=-1$ if $h\in Q^{op}$.

We extend the bijection $Q\to Q^{op}$, $h\mapsto \overline{h}$, to an involution of $\overline{Q}$ by setting $\overline{h_2}=h_1$ if and only if $\overline{h_1}=h_2$ for $h_1\in Q$ and $h_2\in Q^{op}$.

Given a quiver Q with vertex set I and a dimension vector $v\in\mathbb{Z}^I_{\geq 0}$, there is a natural decomposition

$$\operatorname{Rep}(\overline{Q},v) = \operatorname{Rep}(Q,v)\oplus \operatorname{Rep}(Q^{op},v).$$

This gives rise to a quaternionic structure $J\colon \operatorname{Rep}(\overline{Q},v)\to\operatorname{Rep}(\overline{Q},v)$. In terms of the above decomposition, J is given by $J(x,y)=(-y^*,x^*)$, where $(x^*)_h := (x_{\overline{h}})^*$ and similarly for y. The action of G_v commutes with this quaternionic structure and we may therefore regard $\operatorname{Rep}(\overline{Q},v)$ as a quaternionic representation of the compact group G_v. The components of the unique hyper-Kähler moment map $\mu=(\mu_{\mathbb{R}},\mu_{\mathbb{C}}) := \operatorname{Rep}(\overline{Q},v)\to\mathfrak{g}_v\oplus\mathfrak{g}_v^c$ vanishing at zero, where the Lie algebras are identified with their duals using the trace pairing, have the explicit forms [Nak94, p. 370]

$$\mu_{\mathbb{R}}(x) = \frac{\sqrt{-1}}{2}\left(\sum_{h\in t^{-1}(i)} x_h x_h^* - x_{\overline{h}}^* x_{\overline{h}}\right)_{i\in I}$$

$$\mu_{\mathbb{C}}(x) = \left(\sum_{h\in t^{-1}(i)} \epsilon(h) x_h x_{\overline{h}}\right)_{i\in I}. \tag{II.2}$$

In the terminology of the previous section $(\operatorname{Rep}(\overline{Q},v),G_v,\mu)$ is a linear tri-Hamiltonian hyper-Kähler manifold.

Under the identifications of $\mathfrak{g}_v$ and $\mathfrak{g}_v^c$ with their dual spaces, the subspaces fixed under the coadjoint action are identified with the centers $Z(\mathfrak{g}_v)$ and $Z(\mathfrak{g}_v^c)$. There are natural maps $\mathbb{R}^I \to Z(\mathfrak{g}_v)$ and $\mathbb{C}^I \to Z(\mathfrak{g}_v^c)$ given by

$$(\xi_i)_{i\in I} \in \mathbb{R}^I \mapsto (\sqrt{-1}\xi_i \operatorname{Id}_{\mathbb{C}^{v_i}})_{i\in I} \in \bigoplus_{i\in I} Z(\mathfrak{u}(v_i))$$
$$(\lambda_i)_{i\in I} \in \mathbb{C}^I \mapsto (\lambda_i \operatorname{Id}_{\mathbb{C}^{v_i}})_{i\in I} \in \bigoplus_{i\in I} Z(\operatorname{End}(\mathbb{C}^{v_i}).$$

If $v_i \neq 0$ for each $i \in I$, then both of these are isomorphisms. Otherwise, they restrict to isomorphisms from $\mathbb{R}^{\operatorname{supp} v}$ and $\mathbb{C}^{\operatorname{supp} v}$, respectively, where $\operatorname{supp} v = \{i \in I : v_i \neq 0\}$. For any dimension vector $v \in \mathbb{Z}^I_{\geq 0}$ we will tacitly regard elements $\xi \in \mathbb{R}^I$ and $\lambda \in \mathbb{C}^I$ as elements of $Z(\mathfrak{g}_v)$ and $Z(\mathfrak{g}_v^c)$, respectively, using the above maps.

Definition II.3.3. Let Q be a quiver with vertex set I. For any dimension vector $v \in \mathbb{Z}^I_{\geq 0}$ and parameter $\xi = (\xi_{\mathbb{R}}, \xi_{\mathbb{C}}) \in \mathbb{R}^I \oplus \mathbb{C}^I$ define

$$\mathcal{M}_\xi(Q, v) := \mu^{-1}(\xi)/G_v.$$

These hyper-Kähler quotients are called (unframed) quiver varieties.

Remark II.3.4. In [Nak94] Nakajima defines what one may call framed quiver varieties $\mathcal{M}_\xi(v, w)$ associated with a quiver Q with vertex set I and two dimension vectors $v, w \in \mathbb{Z}^I$. The above defined spaces $\mathcal{M}_\xi(Q, v)$ correspond to his $\mathcal{M}_\xi(v, 0)$. According to [Cra01, p. 261] the spaces $\mathcal{M}_\xi(v, w)$ can be expressed as $\mathcal{M}_{\xi'}(\overline{Q}_1, v')$, where Q_1 is a quiver obtained from Q by adjoining a single vertex and a number of arrows depending on w. There is therefore no loss in generality in only considering these (unframed) quivers.

The subgroup T of scalars, i.e., $U(1) \cong T \subset G_v$, acts trivially on $\operatorname{Rep}(\overline{Q}, v)$ so the action factors through $G_v \to G_v/T =: G_v'$. As explained in the previous section we obtain a decomposition

$$\mathcal{M}_\xi(Q, v) = \mathcal{M}_\xi^{\mathrm{reg}}(Q, v) \cup \mathcal{M}_\xi^{\mathrm{sing}}(Q, v),$$

where the regular set $\mathcal{M}_\xi^{\mathrm{reg}}(Q, v)$ is the image of the free G_v'-orbits in $\mu^{-1}(\xi)$ or equivalently the points $x \in \mu^{-1}(\xi)$ with stabilizer T in G_v. The regular set is open in $\mathcal{M}_\xi(Q, v)$ and carries the structure of a smooth hyper-Kähler manifold. The singular set $\mathcal{M}_\xi^{\mathrm{sing}}(Q, v)$ is the closed complement of the regular set.

The fact that the action of G_v factors through G_v' has another important implication, namely, that the moment map $\mu\colon \operatorname{Rep}(\overline{Q}, v) \to \mathbb{R}^3 \otimes \mathfrak{g}_v$ takes values in the subspace $\mathfrak{g}_{v,0} \subset \mathfrak{g}_v$ corresponding to $(\mathfrak{g}_v')^* = \operatorname{Lie}(G_v')^*$ under the isomorphism $\mathfrak{g}_v^* \cong \mathfrak{g}_v$. This subspace consists precisely of the $(a_i)_{i\in I} \in \mathfrak{g}_v$ satisfying $\sum_{i\in I} \operatorname{tr}(a_i) = 0$. A parameter $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \otimes \mathbb{R}^I$ corresponds to an element satisfying this condition precisely when

$$v \cdot \xi_k = \sum_{i\in I} \operatorname{tr}((\xi_k)_i \operatorname{Id}_{\mathbb{C}^{v_i}}) = 0 \quad \text{for} \quad k = 1, 2, 3,$$

where $\cdot$ denotes the usual scalar product. For each $\theta \in \mathbb{Z}^I$ define

$$D_\theta = \{u \in \mathbb{R}^I : u \cdot \theta = 0\} \subset \mathbb{R}^I.$$

The above then amounts to the fact that $\mu^{-1}(\xi) = \emptyset$ whenever $\xi_k \notin D_v$ for some $1 \le k \le 3$. However, for most parameters $\xi \in \mathbb{R}^3 \otimes D_v$ the space $\mathcal{M}_\xi(Q, v)$ will be a smooth hyper-Kähler manifold. To state the relevant result we have to recall the definition of the symmetric bilinear form associated with a quiver (see for instance [Cra01, Section 2]).

Definition II.3.5. Let Q be a quiver with vertex set I. The symmetric bilinear form $(\cdot,\cdot)\colon \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$ associated with the quiver is defined by

$$(v, w) := 2\sum_{i\in I} v_i w_i - \sum_{h\in\overline{Q}} v_{s(h)} w_{t(h)} \quad \text{for} \quad v, w \in \mathbb{Z}^I.$$

If we identify the set of vertices $I \cong \{1, 2, \cdots, n\}$ for some $n \in \mathbb{N}$ and let $A = (a_{ij})$ be the adjacency matrix of the unoriented graph underlying Q, i.e., $a_{ij} = a_{ji}$ is the number of edges connecting i and j, then $(v, w) = 2v \cdot w - v \cdot Aw$. Alternatively, $(v, w) = v \cdot Cw$ where $C = 2\,\mathrm{id} - A$. The symmetric bilinear form therefore only depends on the underlying unoriented graph. If Q is a (extended) Dynkin quiver, then C is the Cartan matrix associated with the corresponding (extended) Dynkin diagram.

The following theorem is [Nak94, Theorem 2.8] adapted to the unframed setting. Let $\mathbb{Z}^I$ be partially ordered by $v \le w$ if and only if $v_i \le w_i$ for each $i \in I$.

Theorem II.3.6. *Let Q be a quiver with vertex set I. Given a dimension vector $v \in \mathbb{Z}^I_{\ge 0}$ define*

$$R_+(v) = \{\theta \in \mathbb{Z}^n : 0 < \theta < v \ \text{ and } \ (\theta, \theta) \le 2\}.$$

Then if

$$\xi \in \mathbb{R}^3 \otimes D_v - \left(\bigcup_{\theta \in R_+(v)} \mathbb{R}^3 \otimes (D_v \cap D_\theta) \right),$$

the group G'_v acts freely on $\mu^{-1}(\xi) \subset \mathrm{Rep}(\overline{Q}, v)$ and the quiver variety $\mathcal{M}_\xi(Q, v)$ is a (possibly empty) smooth hyper-Kähler manifold of dimension $4 - 2(v, v)$.

Let (Q, I) be a quiver and fix a dimension vector $v \in \mathbb{Z}^I_{\ge 0}$. The complex Lie group G^c_v acts on $\mathrm{Rep}(\overline{Q}, v)$ preserving the complex symplectic form $\omega_\mathbb{C}$ given by the formula

$$\omega_\mathbb{C}(x, y) = \sum_{h\in\overline{Q}} \epsilon(h)\,\mathrm{tr}(x_h y_{\overline{h}}) \quad \text{for} \quad x, y \in \mathrm{Rep}(\overline{Q}, v). \tag{II.3}$$

The corresponding moment map is precisely the component $\mu_\mathbb{C}\colon \mathrm{Rep}(\overline{Q}, v) \to \mathfrak{g}^c_v$ in (II.2). From the given formula it is clear that $\mu_\mathbb{C}$ is algebraic and therefore

$\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})$ carries the structure of an affine variety for each $\xi_{\mathbb{C}} \in \mathbb{C}^I$. The action of the reductive group G_v^c is algebraic so there is a complex analytic quotient $\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) \to \mu^{-1}(\xi_{\mathbb{C}})//G_v^c$. This is the analytification of the affine GIT quotient

$$\operatorname{Spec}\mathbb{C}[\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})] \to \operatorname{Spec}(\mathbb{C}[\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})]^{G_v^c}).$$

We will need a few standard facts concerning this construction (see for instance [Dol03, Chapter 6] for the algebraic side of the story and [May18, Section 2.4.1] and the references contained therein for the analytical perpective).

Lemma II.3.7. *As a topological space $\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})//G_v^c$ is homeomorphic to the quotient space $\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})/\sim$ where $x \sim y$ if and only if $\overline{G_v^c \cdot x} \cap \overline{G_v^c \cdot y} \neq \emptyset$. Let $q\colon \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) \to \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})//G_v^c$ denote the quotient map. Then each fiber $q^{-1}(x)$ contains a unique closed orbit $G_v^c \cdot \tilde{x}$, and if $y \in q^{-1}(x)$ then $G_v^c \cdot x \subset \overline{G_v^c \cdot y}$.*

In this setting we have the following result comparing the analytic quotient and the hyper-Kähler quotient.

Theorem II.3.8. *[Nak94, Theorem 3.1] Let Q be a quiver with vertex set I and let $v \in \mathbb{Z}^I$ be a dimension vector. Then for each $\xi_{\mathbb{C}} \in \mathbb{C}^I$ the inclusion $\mu^{-1}(0, \xi_{\mathbb{C}}) = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) \hookrightarrow \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})$ descends to a homeomorphism*

$$\mathcal{M}_{(0,\xi_{\mathbb{C}})}(Q, v) = \left(\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})\right)/G_v \cong \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})//G_v^c.$$

Moreover, each closed orbit $G_v^c \cdot x \subset \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})$ intersects $\mu_{\mathbb{R}}^{-1}(0)$ in a unique G_v-orbit.

Remark II.3.9. The final statement is not explicitly stated in [Nak94], but seems to be well-known. See for instance [Sja95, Proposition 2.2].

The above result implies that $\mathcal{M}_{(0,\xi_{\mathbb{C}})}(Q, v)$ carries the structure of a complex analytic space. We will have use for one final result. Let $v \in \mathbb{Z}_{\geq 0}^I$ be a fixed dimension vector and let $\xi_{\mathbb{C}} \in \mathbb{C}^I$ such that $\operatorname{Re}\xi_{\mathbb{C}}, \operatorname{Im}\xi_{\mathbb{C}} \in D_v$. Choose $\xi_{\mathbb{R}} \in D_v - \bigcup_{\theta \in R_+(v)} D_\theta$ and set $\xi = (0, \xi_{\mathbb{C}})$ and $\widetilde{\xi} = (\xi_{\mathbb{R}}, \xi_{\mathbb{C}})$. The space $\mathcal{M}_{\widetilde{\xi}}(Q, v)$ is a smooth hyper-Kähler manifold by Theorem II.3.6. The inclusion

$$\mu^{-1}(\widetilde{\xi}) = \mu_{\mathbb{R}}^{-1}(\xi_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) \hookrightarrow \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})$$

induces a map $\pi\colon \mathcal{M}_{\widetilde{\xi}}(Q, v) \to \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})//G_v^c \cong \mathcal{M}_\xi(Q, v)$. In the following result we regard $\mathcal{M}_{\widetilde{\xi}}(Q, v)$ as a complex manifold by fixing the complex structure induced by the (standard) complex vector space structure of $\operatorname{Rep}(\overline{Q}, v)$.

Theorem II.3.10. *[Nak94, Theorem 4.1] The map π is holomorphic. Moreover, if $\mathcal{M}_\xi^{reg}(Q, v)$ is nonempty, π is a resolution of singularities. That is,*

(1) *$\pi\colon \mathcal{M}_{\widetilde{\xi}}(Q, v) \to \mathcal{M}_\xi(Q, v)$ is proper,*

(2) *π induces an isomorphism $\pi^{-1}(\mathcal{M}_\xi^{\mathrm{reg}}(Q, v)) \cong \mathcal{M}_\xi^{\mathrm{reg}}(Q, v)$ and*

(3) *$\pi^{-1}(\mathcal{M}_\xi^{\mathrm{reg}}(Q, v))$ is a dense subset of $\mathcal{M}_{\widetilde{\xi}}(Q, v)$.*

II.4 Representations of Quivers

We briefly recall a few basic notions concerning the representation theory of quivers. An excellent reference for this material is [Bri12]. Afterwards we give the definition of the deformed preprojective algebras $\Pi^\lambda = \Pi^\lambda(Q)$ of [CH98] and spell out the correspondence between $\mathcal{M}_{(0,\lambda)}(Q,v)$ and the isomorphism classes of semi-simple Π^λ-modules. Finally, we recall the construction of the root system associated with a quiver and state the key result of [Cra01] relevant for our purpose.

A (complex) representation of a quiver Q is a pair (V, f) where $V = (V_i)_{i\in I}$ is a family of complex vector spaces and $f = (f_h : V_{s(h)} \to V_{t(h)})_{h\in Q}$ is a family of linear maps. We will only be concerned with finite dimensional representations, i.e., V_i is finite dimensional for each $i \in I$. The dimension of a representation (V, f) is $\dim V \coloneqq (\dim(V_i))_{i\in I} \in \mathbb{Z}^I_{\geq 0}$. A homomorphism $u\colon (V, f) \to (W, g)$ of representations is a collection of linear maps $u_i\colon V_i \to W_i$ for $i \in I$ such that $f_h u_{s(h)} = u_{t(h)} g_h$ for each $h \in Q$. We therefore have a category of complex representations of Q. This category is equivalent with the category of left modules over the quiver algebra $\mathbb{C}Q$: the complex algebra generated by $\{e_i : i \in I\}$ and $\{h : h \in Q\}$ subject to the relations

$$e_i e_j = \delta_{ij} e_i, \quad e_i h = \delta_{it(h)} h \quad \text{and} \quad h e_j = \delta_{s(h)j} h$$

for all $i, j \in I$ and $h \in Q$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. The $\{e_i\}_{i\in I}$ is a complete set of mutually orthogonal idempotents, in particular $1_{\mathbb{C}Q} = \sum_{i\in I} e_i$.

We briefly recall the equivalence between representations of Q and left $\mathbb{C}Q$-modules. Let (V, f) be a representation of Q and put $X = \oplus_{i\in I} V_i$. For each $i \in I$ let $\iota_i\colon V_i \to X$ and $\pi_i\colon X \to V_i$ denote the inclusion and projection, respectively. Define $\rho\colon \mathbb{C}Q \to \mathrm{End}_{\mathbb{C}}(X)$ by $\rho(e_i) = \iota_i \circ \pi_i$ for each $i \in I$ and $\rho(h) = \iota_{t(h)} \circ f_h \circ \pi_{s(h)}$ for each $h \in Q$. One may then verify that ρ is a well-defined homomorphism of $\mathbb{C}$-algebras and therefore endows X with a $\mathbb{C}Q$-module structure. One may recover (V, f) from (X, ρ) by setting $V_i = e_i X$ for $i \in I$ and $f_h = \pi_{t(h)} \circ \rho(h) \circ \iota_{s(h)}$ for $h \in Q$. With this in mind, we will pass freely between the notion of a Q representation and a $\mathbb{C}Q$-module.

A $\mathbb{C}Q$-module X of dimension $v \in \mathbb{Z}^I$ defines a unique G^c_v-orbit $\mathcal{O}_X \subset \mathrm{Rep}(Q, v)$. A representative x for the orbit is obtained by choosing a basis for $V_i = e_i X$, thereby identifying $V_i \cong \mathbb{C}^{v_i}$, for each $i \in I$ and then letting $x_h\colon \mathbb{C}^{v_{s(h)}} \to \mathbb{C}^{v_{t(h)}}$ be the corresponding linear maps. The correspondence $X \mapsto \mathcal{O}_X$ sets up a bijection between the isomorphism classes of $\mathbb{C}Q$-modules of dimension v and the set of G^c_v-orbits in $\mathrm{Rep}(Q, v)$. Given a parameter $\lambda \in \mathbb{C}^I$ the G^c_v-orbits in $\mu_{\mathbb{C}}^{-1}(\lambda) \subset \mathrm{Rep}(\overline{Q}, v)$ have a representation theoretic interpretation as well.

Definition II.4.1. [CH98, p. 611] Let Q be a quiver with vertex set I. The deformed preprojective algebra $\Pi^\lambda = \Pi^\lambda(Q)$ of weight $\lambda \in \mathbb{C}^I$ is defined to be the quotient of the quiver algebra $\mathbb{C}\overline{Q}$ by the two sided ideal generated by

$$c = \sum_{i \in I} \lambda_i e_i - \sum_{h \in Q} [h, \overline{h}].$$

Observe that there is a decomposition $c = \sum_i c_i$ where

$$c_i = e_i \left(\lambda_i 1_{\mathbb{C}\overline{Q}} - \sum_{h \in t^{-1}(i)} \epsilon(h) h \overline{h} \right).$$

In view of the formula (II.2) for $\mu_{\mathbb{C}}$, it is not hard to see that the G_v^c-orbit of a $\mathbb{C}\overline{Q}$-module X is contained in $\mu^{-1}(\lambda)$ precisely when X descends to a Π^λ-module along the projection $\mathbb{C}\overline{Q} \to \Pi^\lambda$. Therefore, the G_v^c-orbits in $\mu_{\mathbb{C}}^{-1}(\lambda) \subset \mathrm{Rep}(\overline{Q}, v)$ are in natural bijection with the isomorphism classes of Π^λ-modules of dimension v.

We have the following result describing the closed G_v^c-orbits in $\mathrm{Rep}(Q, v)$ (see for instance [Bri12, Section 2] for a proof). Note that a G_v^c-orbit is closed in the Zariski topology if and only if it is closed in the analytic topology.

Proposition II.4.2. *Let Q be a quiver with vertex set I and let X be a finite dimensional $\mathbb{C}Q$-module of dimension $v \in \mathbb{Z}^I_{\geq 0}$. Let $\mathcal{O}_X$ denote the orbit corresponding to the isomorphism class of X in $\mathrm{Rep}(Q, v)$. Then $\mathcal{O}_X$ is closed if and only if X is semi-simple. Moreover, let*

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$$

be a composition series for X, i.e., each quotient X_k/X_{k-1}, $1 \leq k \leq n$, is a simple module, and let $X_{ss} = \bigoplus_{i=1}^n X_i/X_{i-1}$ be the semi-simplification of X. Then $\mathcal{O}_{X_{ss}}$ is the unique closed orbit contained in the closure of $\mathcal{O}_X$.

Let $\mathcal{SS}(\Pi^\lambda, v)$ denote the set of isomorphism classes of semi-simple Π^λ-modules of dimension v. For a semi-simple Π^λ-module X we let $[X]$ denote its isomorphism class in $\mathcal{SS}(\Pi^\lambda, v)$.

Proposition II.4.3. *Let Q be a quiver with vertex set I and let Π^λ be the associated deformed preprojective algebra of weight $\lambda \in \mathbb{C}^I$. Then for each dimension vector $v \in \mathbb{Z}^I$, the map*

$$\rho \colon \mathcal{M}_{(0,\lambda)}(Q, v) \to \mathcal{SS}(\Pi^\lambda, v),$$

that assigns to a point $x \in \mathcal{M}_{(0,\lambda)}(Q, v)$ the isomorphism class of the Π^λ-module corresponding to any point $\tilde{x} \in \mu^{-1}(0, \lambda)$ in the fiber over x, is a well-defined bijection.

Moreover, if $\rho(x) = [X]$ and $X = \bigoplus_{j=1}^k n_j X_j$ with the X_j simple and $n_j \in \mathbb{N}$, then for any point $\tilde{x} \in \mu^{-1}(0, \lambda)$ above x there are isomorphisms

$$(G_v)_{\tilde{x}} \cong \prod_{j=1}^k U(n_j) \quad \textit{and } (G_v^c)_{\tilde{x}} \cong \prod_{j=1}^k \mathrm{GL}(n_j, \mathbb{C}).$$

In particular, $x \in \mathcal{M}^{\mathrm{reg}}_{(0,\lambda)}(Q, v)$ if and only if X is simple.

Proof. We divide the proof into four steps. The first sentence in each step is a claim that we then go on to verify.

Step 1: The rule $[X] \mapsto \mathcal{O}_X \subset \mu_{\mathbb{C}}^{-1}(\lambda)$ defines a bijection between $\mathcal{SS}(\Pi^\lambda, v)$ and the set of closed G_v^c-orbits in $\mu_{\mathbb{C}}^{-1}(\lambda)$. We have seen that the given rule sets up a bijection between the set of isomorphism classes of Π^λ-modules of dimension v and the G_v^c-orbits contained in $\mu_{\mathbb{C}}^{-1}(\lambda)$. Since a Π^λ-module X is semi-simple if and only if it is semi-simple as a $\mathbb{C}\overline{Q}$-module, Proposition II.4.2 ensures that this bijection restricts to a bijection between the isomorphism classes of the semi-simple Π^λ-modules and the closed G_v^c-orbits in $\mu_{\mathbb{C}}^{-1}(\lambda)$.

Step 2: The rule $(G_v \cdot x) \mapsto (G_v^c \cdot x)$ for $x \in \mu^{-1}(0,\lambda)$ defines a bijection between the G_v-orbits in $\mu^{-1}(0,\lambda)$ and the closed G_v^c-orbits in $\mu_{\mathbb{C}}^{-1}(\lambda)$. For any dimension vector $v \in \mathbb{Z}^I$ we have a commutative diagram

$$\begin{array}{ccc} \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\lambda) & \xrightarrow{\quad i \quad} & \mu_{\mathbb{C}}^{-1}(\lambda) \\ \Big\downarrow p & & \Big\downarrow q \\ \mathcal{M}_{(0,\lambda)}(Q,v) & \xrightarrow{\quad j \quad} & \mu_{\mathbb{C}}^{-1}(\lambda)//G_v^c, \end{array}$$

where p and q are the quotient maps, i is the inclusion and j is the induced map between the quotients. According to Theorem II.3.8 the map j is a homeomorphism and in particular a bijection. Therefore, the only thing we need to prove is that for each $x \in \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\lambda)$ the orbit $G_v^c \cdot x \subset \mu_{\mathbb{C}}^{-1}(\lambda)$ is closed. By Lemma II.3.7 there is a unique closed orbit $G_v^c \cdot y \subset q^{-1}q(i(x))$. Moreover, by the second statement in Theorem II.3.8 we may assume that $y = i(z)$ for some $z \in \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\lambda)$. Then as $jp(x) = qi(x) = qi(z) = jp(z)$ and j is injective we conclude that $p(x) = p(z)$ and hence $G_v \cdot x = G_v \cdot z$. This implies that $G_v^c \cdot x = G_v^c \cdot z$ and as the latter orbit is closed by construction the claim has been verified.

Step 3: The map $\rho\colon \mathcal{M}_{(0,\lambda)} \to \mathcal{SS}(\Pi^\lambda, v)$ is a well-defined bijection. Let $p\colon \mu^{-1}(0,\lambda) \to \mathcal{M}_{(0,\lambda)}(Q,v)$ denote the quotient map as in the above diagram. The map sending $x \in \mathcal{M}_{(0,\lambda)}(Q,v)$ to the G_v-orbit $p^{-1}(x) \subset \mu^{-1}(0,\lambda)$ is clearly a bijection. The map ρ sending a point $x \in \mathcal{M}_{(0,\lambda)}(Q,v)$ to the isomorphism class of the Π^λ-module associated with any choice of $\tilde{x} \in p^{-1}(x)$ is then precisely the composition of the bijection $x \mapsto p^{-1}(x) = G_v \cdot \tilde{x}$, the bijection of step 2 and the inverse of the bijection of step 1. It is then clear that ρ is a well-defined bijection.

Step 4: If $\rho(x) = [X]$ and $X = \sum_{j=1}^k n_j X_j$ is a decomposition of X into simple modules, then for any $\tilde{x} \in p^{-1}(x)$ it holds true that

$$(G_v)_{\tilde{x}} \cong \prod_{j=1}^k \quad \text{and} \quad (G_v^c)_{\tilde{x}} \cong \prod_{j=1}^k \mathrm{GL}(n_j, \mathbb{C}).$$

Let $y \in \mu_{\mathbb{C}}^{-1}(\lambda) \subset \mathrm{Rep}(\overline{Q}, v)$ and denote the corresponding Π^λ-module by Y. It is then easy to see that the stabilizer $(G_v^c)_y$ coincides with the module theoretic automorphism group $\mathrm{Aut}_{\Pi^\lambda}(Y)$. If Y is semi-simple and $Y = \oplus_{j=1}^k n_j Y_j$ is a decomposition into simple modules, it follows by Schur's Lemma that

$$\mathrm{Aut}_{\Pi^\lambda}(Y) \cong \prod_{j=1}^{k} \mathrm{GL}(n_j, \mathbb{C}).$$

Let $x \in \mathcal{M}_{(0,\lambda)}$, let $\tilde{x} \in \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\lambda)$ be a point above x and let $X = \sum_{j=1}^{k} n_j X_k$ be the corresponding semi-simple Π^λ-module decomposed into simple summands. From the above considerations we may deduce that there is an isomorphism $(G_v^c)_{\tilde{x}} \cong \prod_{j=1}^{k} \mathrm{GL}(n_j, \mathbb{C})$. For any point $y \in \mu_{\mathbb{R}}^{-1}(0)$ it holds true that the inclusion of stabilizers $\iota\colon (G_v)_y \hookrightarrow (G_v^c)_y$ induces an isomorphism between the complexification of $(G_v)_y$ and $(G_v^c)_y$ (see [Sja95, Proposition 1.6]). Applying this in the situation above we deduce that $\prod_{j=1}^{k} \mathrm{GL}(n_j, \mathbb{C})$ is isomorphic to the complexification of $(G_v)_{\tilde{x}}$. In particular, $(G_v)_{\tilde{x}}$ is isomorphic to a maximal compact subgroup of $\prod_{j=1}^{k} \mathrm{Gl}_{n_j}(\mathbb{C})$ and as all such subgroups are conjugate we deduce that there is an isomorphism

$$(G_v)_{\tilde{x}} \cong \prod_{j=1}^{k} U(n_j).$$

This completes the final step and hence the proof. ∎

In [Cra01] Crawley-Boevey gives a strong result on the existence and uniqueness of simple Π^λ-modules. To state the result we need to recall the construction of the root system associated with a quiver. Here we follow [Cra01, Section 2].

Let Q be a quiver with vertex set I and let $(\cdot,\cdot)\colon \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$ be the associated symmetric bilinear form of Definition II.3.5. Let $\{\epsilon_i \in \mathbb{Z}^I : i \in I\}$ denote the standard basis of $\mathbb{Z}^I$, that is, $(\epsilon_i)_j = \delta_{ij}$ for $i, j \in I$. To simplify the exposition slightly we will assume that Q contains no edge loops, i.e., there is no $h \in Q$ with $s(h) = t(h)$. This is valid in the case of (extended) Dynkin quivers. Note that this condition implies that $(\epsilon_i, \epsilon_i) = 2$ for each $i \in I$.

For each $i \in I$ there is a reflection $s_i\colon \mathbb{Z}^I \to \mathbb{Z}^i$ defined by $s_i(v) = v - (v, \epsilon_i)\epsilon_i$. These reflections generate a finite subgroup $W \subset \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^I)$ called the Weyl group. The action of the Weyl group on $\mathbb{Z}^I$ preserves the symmetric bilinear form associated with the quiver. The support of $\alpha \in \mathbb{Z}^I$ is the full subquiver of Q with vertex set $\{i \in I : \alpha_i \neq 0\}$. The fundamental domain $F \subset \mathbb{Z}^I_{\geq 0} - \{0\}$ is then defined to be the set of $\alpha \in \mathbb{Z}^I_{\geq 0}$ with connected support satisfying $(\alpha, \epsilon_i) \leq 0$ for each $i \in I$. The root system associated with the quiver Q is defined to be $\Phi := \Phi^{\mathrm{re}} \cup \Phi^{\mathrm{im}} \subset \mathbb{Z}^I$ where

$$\Phi^{\mathrm{re}} = \bigcup_{i \in I} W \cdot \epsilon_i \quad \text{and} \quad \Phi^{\mathrm{im}} = W \cdot (F \cup -F).$$

The elements of Φ^{re} are called real roots and the elements of Φ^{im} are called imaginary roots. One may show that there is a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots, where a root α is positive (respectively negative) if $\alpha \in \mathbb{Z}^I_{\geq 0}$ (respectively $\alpha \in \mathbb{Z}^I_{\leq 0}$). We record the following elementary fact.

Lemma II.4.4. *For each $\alpha \in \Phi^{\mathrm{re}}$ it holds true that $(\alpha, \alpha) = 2$. For each $\beta \in \Phi^{\mathrm{im}}$ it holds true that $(\beta, \beta) \leq 0$.*

Proof. As already noted $(\epsilon_i, \epsilon_i) = 2$ for each $i \in I$. The first assertion now follows from the fact that each $\alpha \in \Phi^{\mathrm{re}}$ may be expressed in the form $w \cdot \epsilon_i$ for some $w \in W$ and $i \in I$. For the second assertion we may assume without loss of generality that $\beta \in F$. Writing $\beta = \sum_{i \in I} b_i \epsilon_i$ with $b_i \geq 0$ we find

$$(\beta, \beta) = \sum_{i \in I} b_i (\beta, \epsilon_i) \leq 0$$

since by definition $(\beta, \epsilon_i) \leq 0$ for each $i \in I$. ∎

We may now state the key result on the existence and uniqueness of simple Π^λ-modules. In the following result the function $p \colon \mathbb{Z}^I \to \mathbb{Z}$ is defined by the formula $p(\alpha) = 1 - \frac{1}{2}(\alpha, \alpha)$.

Theorem II.4.5. *[Cra01, Theorem 1.2] Let Q be a quiver with vertex set I. Let Π^λ be the associated deformed preprojective algebra of weight $\lambda \in \mathbb{Z}^I$. Then for each $\alpha \in \mathbb{Z}^I_{\geq 0}$ the following is equivalent*

(i) *There exists a simple Π^λ-module of dimension α.*

(ii) *α is a positive root with $\lambda \cdot \alpha = 0$ and for every decomposition $\alpha = \sum_t \beta^{(t)}$ into positive roots satisfying $\lambda \cdot \beta^{(t)} = 0$ one has*

$$p(\alpha) > \sum_t p(\beta^{(t)}).$$

In that situation $\mu_{\mathbb{C}}^{-1}(\lambda) \subset \mathrm{Rep}(\overline{Q}, \alpha)$ is a reduced and irreducible complete intersection of dimension $\alpha \cdot \alpha - 1 + 2p(\alpha)$ and the general element is a simple representation.

II.5 Extended Dynkin Quivers and their Root Systems

In our later work it will be important to have a firm grip on the relation between the Dynkin diagrams and root systems of type ADE and their extended counterparts of type $\widetilde{ADE}$. In this section we briefly review the necessary root space theory, establish our notation and prove two basic lemmas needed to effectively apply Theorem II.4.5.

Let K be a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8, for short type ADE. Fix an identification of the set of vertices with $\{1, 2, \cdots, n\}$ for some $n \in \mathbb{N}$. The Cartan matrix $C = (c_{ij})_{ij} \in M_n(\mathbb{Z})$ of K is then defined by $c_{ij} = 2\delta_{ij} - a_{ij}$, where $a_{ij} = a_{ji} = 1$ precisely when there is an edge connecting i to j in K and otherwise 0. The associated root system $\Phi \subset \mathbb{Z}^n$ is then constructed just as in the previous section using the pairing $(v, w)_C \coloneqq v \cdot Cw$. Note that this pairing is positive definite so in particular $\Phi = \Phi^{\mathrm{re}}$. In particular, the coordinate vectors $\{\epsilon_i : 1 \leq i \leq n\}$ is a set of simple roots for Φ. There is a unique maximal root $d \in \Phi^+ \subset \mathbb{Z}^n$ with respect to the partial ordering $\leq$ on $\mathbb{Z}^n$ (see [Hum78, Section 10.4]). The extended Dynkin diagram $\widetilde{K}$ is constructed from K by adjoining a single vertex 0 and one edge connecting 0 to i if $(d, \epsilon_i)_C = 1$ for each $1 \leq i \leq n$. The extended Cartan matrix $\tilde{C}$ is constructed from $\widetilde{K}$ in the same way C was constructed from K. Explicitly, if we identify $\mathbb{Z}^{n+1} = \mathbb{Z}\epsilon_0 \oplus \mathbb{Z}^n$,

$$\widetilde{C} = \begin{pmatrix} 2 & -d^tC \\ -Cd & C \end{pmatrix}.$$

The associated root system $\widetilde{\Phi} \subset \mathbb{Z}^{n+1}$ is then constructed using the pairing $(v, w)_{\widetilde{C}} := v^t\widetilde{C}w$. We have the following useful description of the real roots in Φ and $\widetilde{\Phi}$ (see [Kac90, Proposition 5.10])

$$\Phi = \{\alpha \in \mathbb{Z}^n : (\alpha, \alpha)_C = 2\} \quad \text{and} \quad \widetilde{\Phi}^{re} = \{\beta \in \mathbb{Z}^{n+1} : (\beta, \beta)_{\widetilde{C}} = 2\}. \tag{II.4}$$

To understand the imaginary roots in $\widetilde{\Phi}$ define a linear map $\psi\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ by $\psi(\epsilon_0) = -d$ and $\psi(\epsilon_i) = \epsilon_i$ for $1 \leq i \leq n$. Then, using the above explicit description of $\widetilde{C}$, one obtains the following identity

$$(v, w)_{\widetilde{C}} = (\psi(v), \psi(w))_C.$$

As the latter pairing is positive definite one deduces that $(\cdot, \cdot)_{\widetilde{C}}$ is positive semi-definite. It follows by Lemma II.4.4 that the set of imaginary roots must coincide with the nonzero elements of $\mathrm{Ker}(\psi)$, that is,

$$\widetilde{\Phi}^{\mathrm{im}} = \{r\delta : r \in \mathbb{Z} - \{0\}\}$$

where $\delta = (1, d)^t \in \mathbb{Z}\epsilon_0 \oplus \mathbb{Z}^n = \mathbb{Z}^{n+1}$ is the minimal positive imaginary root. We will need two lemmas concerning these root systems.

Lemma II.5.1. *Define $\Sigma := \{\beta \in \widetilde{\Phi} : 0 < \beta < \delta\}$. Then the map $\psi\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ restricts to a bijection $\psi\colon \Sigma \to \Phi$ with inverse given by*

$$\psi^{-1}(\alpha) = \begin{cases} (0, \alpha) & \text{if } \alpha \in \Phi^+ \\ (1, d + \alpha) & \text{if } \alpha \in \Phi^- \end{cases}$$

with respect to the decomposition $\mathbb{Z}^{n+1} = \mathbb{Z}\epsilon_0 \oplus \mathbb{Z}^n$. Furthermore, the adjoint $\psi^*\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$, determined by $\psi(\theta) \cdot \tau = \theta \cdot \psi^*(\tau)$ for $\theta \in \mathbb{Z}^{n+1}$ and $\tau \in \mathbb{R}^n$, is given by $\psi^*(\tau) = (-d \cdot \tau, \tau)$ and corestricts to an isomorphism $\mathbb{R}^n \cong \delta^\perp \subset \mathbb{R}^{n+1}$.

Proof. Note first $\Sigma \subset \widetilde{\Phi}^{\mathrm{re}}$ since δ is the minimal positive imaginary root. As $(\alpha, \beta)_{\widetilde{C}} = (\psi(\alpha), \psi(\beta))_C$ for all $\alpha, \beta \in \mathbb{Z}^{n+1}$, it follows from the description of the real roots in (II.4) that $\psi(\Sigma) \subset \Phi$. The same result shows that the map $\kappa\colon \Phi \to \Sigma$ given by $\kappa(\alpha) = (0, \alpha)$ if $\alpha \in \Phi^+$ and $\kappa(\beta) = (1, d + \beta)$ if $\beta \in \Phi^-$ is well-defined. Using the definition of ψ one easily verifies that $\psi\kappa = \mathrm{id}_\Phi$ and $\kappa\psi = \mathrm{id}_\Sigma$. Hence, ψ is a bijection with inverse $\psi^{-1} = \kappa$.

For the second part note that ψ extends uniquely to a linear map $\psi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ and hence has an adjoint $\psi^* : \mathbb{R}^n \to \mathbb{R}^{n+1}$ uniquely determined by the formula given in the statement. For each $1 \leq i \leq n$ we find $\psi^*(\tau)_i = \psi^*(\tau) \cdot \epsilon_i = \tau \cdot \psi(\epsilon_i) = \tau_i$, while $\psi^*(\tau_0) = \tau \cdot \psi(\epsilon_0) = -\tau \cdot d$. Thus $\psi^*(\tau) = (-d \cdot \tau, \tau)$. Finally, since $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is surjective, it follows that ψ^* corestricts to an isomorphism onto $\operatorname{Ker}(\psi)^\perp = \delta^\perp$. ∎

In the following lemma we regard $\Phi \subset \mathbb{Z}^n \subset \mathbb{R}^n$ as above, and we write $(\cdot, \cdot)\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ for the Cartan pairing.

Lemma II.5.2. *For $\tau \in \mathbb{C}^n$ define $\tau^\perp := \operatorname{Span}_{\mathbb{R}}(\operatorname{Re}\tau, \operatorname{Im}\tau)^\perp \subset \mathbb{R}^n$ with respect to the standard scalar product on $\mathbb{R}^n$. Then if $\tau^\perp \cap \Phi$ is non-empty, it is a root system in the subspace it spans and decomposes into a disjoint union of root systems of type ADE*

$$\tau^\perp \cap \Phi = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_r.$$

Furthermore, Φ_j admits a unique base contained in Φ^+ for each $1 \leq j \leq r$.

Proof. Write $\Phi_\tau = \tau^\perp \cap \Phi$. The fact that $\tau^\perp \cap \Phi$ is a root system in the subspace it spans follows from [Hum78, Exercise III.9.7]. To see that Φ_τ admits a base contained in $\Phi^+ \subset \mathbb{Z}^n_{\geq 0}$ we mimic the proof for the existence of bases in a root system in [Hum78, p. 48]. We may write $\Phi_\tau = \Phi_\tau^+ \cup \Phi_\tau^-$ where $\Phi_\tau^\pm = \tau^\perp \cap \Phi^\pm$. As $\Phi_\tau^- = -\Phi_\tau^+$, it follows that Φ_τ is nonempty if and only if Φ_τ^+ is nonempty. We may therefore define $S \subset \Phi_\tau^+$ to be the subset of $\alpha \in \Phi_\tau^+$ that admits no decomposition $\alpha = \beta + \gamma$ for $\beta, \gamma \in \Phi_\tau^+$. This set is nonempty since any $\alpha = \sum_i a_i \epsilon_i \in \Phi_\tau^+$, $a_i \geq 0$, with $\sum_i a_i$ minimal must belong to S. For any pair $\alpha \neq \beta \in S$ we have have $(\alpha, \beta) \leq 0$. Indeed, if $(\alpha, \beta) = 1$, then either $\alpha - \beta$ or $\beta - \alpha$ will belong to Φ_τ^+ contradicting either $\alpha \in S$ or $\beta \in S$. To see that the set S is linearly independent, suppose that $\sum_{s \in S} a_s s = 0$. Put $S_1 = \{s \in S : a_s > 0\}$, $S_2 = S - S_1$ and write $u = \sum_{s \in S_1} a_s s = \sum_{t \in S_2} b_t t$ where $a_s > 0$ and $b_t = -a_t \geq 0$. Then

$$(u, u) = \sum_{s,t} a_s b_t (s, t) \leq 0,$$

which is only possible if $u = 0$. Hence, as each $s \in S$ is nonzero and has non-negative coefficients with respect to the standard basis ϵ_i, $1 \leq i \leq n$, it follows that $a_s = 0$ for all $s \in S$ as required. It is clear that every root $\alpha \in \Phi_\tau^+$ can be written as a positive integral linear combination of the elements of S and we have thus verified that S is a base for Φ_τ. At this point we may decompose $S = S_1 \cup S_2 \cup \cdots \cup S_r$ into pairwise orthogonal sets in such a way that each S_i is indecomposable, i.e., admits no further decomposition into pairwise orthogonal sets. This yields a corresponding decomposition into irreducible root systems (see [Hum78, Section 10.4]) $\Phi_\tau = \Phi_1 \cup \Phi_2 \cdots \cup \Phi_r$, where S_i is a base for Φ_i for each $1 \leq i \leq r$. As each Φ_j is contained in Φ, all the roots have the same length and this implies that Φ_j must be of type ADE for each j. ∎

The graphs K and $\widetilde{K}$ are transformed into quivers by giving the edges arbitrary orientations. As already mentioned the corresponding symmetric bilinear forms and root systems are independent of the choice of orientations. In particular, if Q is an extended Dynkin quiver we may identify the set of vertices with $\{0, 1, \cdots, n\}$ for some $n \in \mathbb{N}$ and assume that we have root systems $\widetilde{\Phi} \subset \mathbb{Z}^{n+1}$, $\Phi \subset \mathbb{Z}^n$ such that the minimal positive imaginary root δ takes the form $(1, d) \in \mathbb{Z}\epsilon_0 \oplus \mathbb{Z}^n$, where $d \in \Phi$ is the maximal positive root. Furthermore, by Lemma II.5.1 we have the map $\psi\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ relating them and the adjoint $\psi^* : \mathbb{R}^n \to \mathbb{R}^{n+1}$ that allows us to identify $\mathbb{R}^n \cong \delta^\perp$. We will work under these assumptions whenever convenient in the rest of the paper.

II.6 Classification of Singularities

Let Q be an extended Dynkin quiver with vertex set I and minimal imaginary root $\delta \in \mathbb{Z}^I$. In this section we will give a description of the singular set in the quiver variety $\mathcal{M}_{(0,\lambda)}(Q, \delta)$ for $\lambda \in \mathbb{C}^I$. According to Proposition II.4.3 the singular set $\mathcal{M}^{\text{sing}}_{(0,\lambda)}(Q, \delta)$ is in natural bijection with the isomorphism classes of semi-simple, non-simple Π^λ-modules of dimension δ, so it suffices to determine the latter set.

For this purpose let $\widetilde{\Phi}$ denote the root system associated with Q and let $\Sigma = \{\alpha \in \widetilde{\Phi} : 0 < \alpha < \delta\}$ as in Lemma II.5.1. For $\lambda \in \mathbb{C}^I$ define $\Sigma_\lambda = \{\alpha \in \Sigma : \alpha \cdot \lambda = 0\}$ and let this set be partially ordered by $\alpha \prec \beta$ if and only if $\beta - \alpha = \sum_t \gamma^{(t)}$ for some $\gamma^{(t)} \in \Sigma_\lambda$. Finally, let $\Sigma_\lambda^{min} \subset \Sigma_\lambda$ denote the subset of minimal elements with respect to this partial ordering.

Lemma II.6.1. *There exists a simple Π^λ-module of dimension δ if and only if $\delta \cdot \lambda = 0$. Moreover, there exists a simple Π^λ-module of dimension α satisfying $0 < \alpha < \delta$ if and only if $\alpha \in \Sigma_\lambda^{min}$ and in that case the simple module is unique up to isomorphism.*

Proof. According to Theorem II.4.5 there exists a simple Π^λ-module of dimension $\alpha \in \mathbb{Z}^I_{\geq 0}$ if and only if α is a root satisfying $\alpha \cdot \lambda = 0$ and for every decomposition $\alpha = \sum_t \beta^{(t)}$ into positive roots satisfying $\beta^{(t)} \cdot \lambda = 0$, it holds true that $p(\alpha) > \sum_t p(\beta^{(t)})$, where we recall that $p(\alpha) = 1 - \frac{1}{2}(\alpha, \alpha)$. In our case of an extended Dynkin quiver we have $p(\delta) = 1 - \frac{1}{2}(\delta, \delta) = 1$ and $p(\alpha) = 1 - \frac{1}{2}(\alpha, \alpha) = 0$ for every real root $\alpha \in \widetilde{\Phi}^{re}$. In any decomposition $\delta = \sum_t \beta^{(t)}$ into positive roots with at least two summands the roots $\beta^{(t)}$ must be real because δ is the minimal positive imaginary root. Therefore, the condition $p(\delta) = 1 > 0 = \sum_t p(\beta^{(t)}$ is trivially satisfied. We conclude that there exists a simple Π^λ-module of dimension δ if and only if $\delta \cdot \lambda = 0$.

If α satisfies $0 < \alpha < \delta$, there exists a simple Π^λ-module of dimension α if and only if $\alpha \in \Sigma_\lambda$ and for every decomposition $\alpha = \sum_t \beta^{(t)}$ with $\beta^{(t)} \in \Sigma_\lambda$ it holds true that $p(\alpha) > \sum_t p(\beta^{(t)})$. This inequality is never satisfied since both sides reduce to zero. Consequently, the above condition can only be satisfied if α does not admit such a decomposition at all and this is equivalent to $\alpha \in \Sigma_\lambda^{min}$.

The fact that the simple Π^λ-module is unique up to isomorphism in this case follows from the final part of Theorem II.4.5 as explained in [Cra01, p. 260]. ■

Before we proceed we record the following consequence.

Lemma II.6.2. *Let Q be an extended Dynkin quiver with vertex set I and minimal imaginary root δ. Let $\lambda \in \mathbb{C}^I$ satisfy $\lambda \cdot \delta = 0$. Then the quiver variety $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ is connected and $\mathcal{M}^{\text{reg}}_{(0,\lambda)}(Q,\delta)$ is nonempty.*

Proof. By the above lemma there exists a simple Π^λ-module of dimension δ in this situation. By Proposition II.4.3 this implies that $\mathcal{M}^{\text{reg}}_{(0,\lambda)}(Q,\delta)$ is nonempty. Furthermore, by the final part of Theorem II.4.5 the variety $\mu_{\mathbb{C}}^{-1}(\lambda)$ is irreducible in the Zariski topology. It is therefore connected in the analytic topology and it follows that the quotient $\mathcal{M}_{(0,\lambda)}(Q,\delta) \cong \mu_{\mathbb{C}}^{-1}(\lambda)//G^c_\delta$ is connected as well. ■

In the following theorem we make the assumptions on the extended Dynkin quiver Q as explained in the end of the previous section.

Theorem II.6.3. *Let Q be an extended Dynkin quiver with vertex set $\{0,1,\cdots,n\}$ and let Π^λ be the associated deformed preprojective algebra of weight $\lambda \in \mathbb{C}^{n+1}$ satisfying $\lambda \cdot \delta = 0$. Let $\Phi \subset \mathbb{Z}^n$ be the root system of type ADE associated with Q. Write $\lambda = (\lambda_1, \tau)$ where $\lambda_1 \in \mathbb{C}$ and $\tau \in \mathbb{C}^n$ and let*

$$\tau^\perp \cap \Phi = \Phi_1 \cup \cdots \cup \Phi_r$$

be a decomposition into (irreducible) subsystems of type ADE as in Lemma II.5.2. Then there is a bijection between $\{\Phi_1,\cdots,\Phi_r\}$ and the isomorphism classes of semi-simple, non-simple Π^λ-modules of dimension δ.

Proof. Let $\widetilde{\Phi} \subset \mathbb{Z}^{n+1}$ be the root system associated with Q. Let $\Sigma_\lambda^{min} \subset \Sigma_\lambda \subset \Sigma \subset \widetilde{\Phi}$ be defined as in the beginning of the section. The content of Lemma II.6.1 is then that there exists a simple Π^λ-module of dimension α, $0 < \alpha < \delta$ if and only if $\alpha \in \Sigma_\lambda^{min}$ and in that case the module is unique up to isomorphism. This implies that a semi-simple, non-simple Π^λ-module $X = \sum_{t=0}^k n_t X_t$ of dimension δ is uniquely determined up to isomorphism by the roots $\gamma_t := \dim X_t \in \Sigma_\lambda^{min}$ and the multiplicities $n_t \in \mathbb{N}$. We therefore have a bijective correspondence between the isomorphism classes of semi-simple, non-simple Π^λ-modules of dimension δ and sets $\{(n_t,\gamma_t)\}_{t=0}^k$ for which $n_t \in \mathbb{N}$, $\gamma_t \in \Sigma_\lambda^{min}$ for each t, $\delta = \sum_t n_t\gamma_t$ and either $k \geq 1$ or $n_0 > 1$.

Our task is to relate the collection of such sets with the root systems in the decomposition

$$\tau^\perp \cap \Phi = \Phi_1 \cup \cdots \cup \Phi_r$$

given in the statement of the theorem. Suppose that $\{(n_t,\gamma_t)\}_{t=0}^k$ is such a set. As $\delta = (1,d) \in \mathbb{Z} \oplus \mathbb{Z}^n$, where $d \in \Phi$ is the maximal root, the condition $\delta = \sum_t n_t\gamma_t$ implies that there is a distinguished root γ_t with nonzero first component and thus necessarily $n_t = 1$. After possibly rearranging the roots we may take this root to be γ_0. By Lemma II.5.1 there are unique positive roots $\beta, \alpha_t \in \Phi^+$, $1 \leq t \leq k$, such that $\gamma_0 = \psi^{-1}(-\beta) = (1, d-\beta)$ and $\gamma_t = \psi^{-1}(\alpha_t) = (0, \alpha_t)$ for $1 \leq t \leq k$. Moreover, since $\lambda \cdot \delta = 0$, there is a unique $\tau \in \mathbb{C}^n$ such that $\lambda = (-d \cdot \tau, \tau) = \psi^*(\tau)$. The relation $\theta \cdot \lambda = \psi(\theta) \cdot \tau$ for each $\theta \in \mathbb{Z}^{n+1}$ ensures that the bijection $\psi\colon \Sigma \cong \Phi$ restricts to a bijection $\Sigma_\lambda \cong \Phi \cap \tau^\perp$. In particular, $\beta, \alpha_1, \cdots, \alpha_k \in \Phi \cap \tau^\perp$. Moreover, the minimality of $\gamma_t = (0, \alpha_t)$, $1 \leq t \leq k$, translates to the fact that each α_t is minimal among the roots in $\Phi^+ \cap \tau^\perp$, while the minimality of $\gamma_0 = (1, d-\beta)$ translates to the fact that $\beta \in \Phi^+ \cap \tau^\perp$ is maximal. This means that β must be the unique maximal positive root in precisely one of the systems Φ_j occurring in the decomposition of $\Phi \cap \tau^\perp$. Furthermore, since the equality $\delta = \sum_t n_t \gamma_t$ is equivalent to the equality $\beta = \sum_t n_t \alpha_t$, we also deduce that $\{\alpha_t : 1 \leq t \leq k\}$ must be the unique positive base in the same system.

This procedure is clearly reversible. Given a system Φ_j let α_t, $1 \leq t \leq k$ be the unique positive base and let $\beta = \sum_t n_t \alpha_t$ be the maximal root. We may then define $\gamma_0 = (1, d-\beta) \in \Sigma_\lambda^{min}$, $n_0 = 1$ and $\gamma_t = (0, \alpha_t) \in \Sigma_\lambda^{min}$ for $1 \leq t \leq k$. It then follows from our previous arguments that the set $\{(n_t, \gamma_t\}_{t=1}^k$ satisfies the required conditions: $n_t \in \mathbb{N}$, $\gamma_t \in \Sigma_\lambda^{min}$ for all t and $\sum_t n_t \gamma_t = \delta$. This completes the proof of the theorem. ■

II.7 Local Structure and the Proof of Theorem II.1.1

The combination of Proposition II.4.3 and Theorem II.6.3 give full control over the singularities in $\mathcal{M}_{(0,\lambda)}(Q, \delta)$ for an extended Dynkin quiver Q. In this section we establish the final results needed to complete the proof of Theorem II.1.1.

Let Q be a quiver with vertex set I and let $\lambda \in \mathbb{C}^I$ be a parameter. Given a point $x \in \mu^{-1}(\lambda)$ consider the sequence

$$G_v^c \xrightarrow{b_x} \operatorname{Rep}(\overline{Q}, \delta) \xrightarrow{\mu_{\mathbb{C}}} \mathfrak{g}_v^c,$$

where $b_x(g) = g \cdot x$ is the orbit map at x. As $\mu_{\mathbb{C}}$ is G_v^c-equivariant and $\lambda \in \mathbb{C}^I$ is identified with an element of $Z(\mathfrak{g}_v^c)$, the composition $\mu_{\mathbb{C}} \circ b_x$ is the constant map at λ. Hence, by differentiating this sequence at $1 \in G_v^c$ we obtain a three term complex

$$0 \longrightarrow \mathfrak{g}_v^c \xrightarrow{\sigma_x} \operatorname{Rep}(\overline{Q}, v) \xrightarrow{\nu_x} \mathfrak{g}_v^c \longrightarrow 0 \tag{II.5}$$

where $\sigma_x = d(b_x)_1$ and $\nu_x = d(\mu_{\mathbb{C}})_x$. If $x \in \mu_{\mathbb{R}}^{-1}(0)$ such that the orbit $G_v^c \cdot x$ is closed and hence an embedded complex submanifold of $\operatorname{Rep}(\overline{Q}, v)$, one may identify $\operatorname{Im}(\sigma_x) = T_x(G_v^c \cdot x)$. By general properties of the moment map it holds true that $\operatorname{Ker}(\nu_x) = \operatorname{Im}(\sigma_x)^{\omega_{\mathbb{C}}}$, where the upper case $\omega_{\mathbb{C}}$ denotes the complex symplectic complement. In particular, the space $T_x(G_v^c \cdot x)$ is isotropic with respect to $\omega_{\mathbb{C}}$. Moreover, the stabilizer $H \coloneqq (G_v^c)_x$ acts linearly on all the spaces involved and the maps σ_x and ν_x are H-equivariant. Therefore, $T_x(G_v^c \cdot x)^{\omega_{\mathbb{C}}} / T_x(G_v^c \cdot x) = \operatorname{Ker}(\nu_x) / \operatorname{Im}(\sigma_x)$ obtains a complex symplectic form preserved by the induced action of H.

Definition II.7.1. Let $x \in \mu^{-1}(0,\lambda)$. Then the complex symplectic slice at x is the complex symplectic $(G_v^c)_x$-representation

$$T_x(G_v^c \cdot x)^{\omega_\mathbb{C}}/T_x(G_v^c \cdot x) = \mathrm{Ker}(\nu_x)/\,\mathrm{Im}(\sigma_x).$$

The following result is a consequence of [May18, Theorem 1.4(iv)]. Here we regard $\mathcal{M}_{(0,\lambda)}(Q,v)$ as a complex analytic space using Theorem II.3.8.

Lemma II.7.2. *Let Q be a quiver with vertex set I, let $v \in \mathbb{Z}^I$ be a dimension vector and let $\lambda \in \mathbb{C}^I$ be a parameter. Let $y \in \mathcal{M}_{(0,\lambda)}(Q,v)$ and let $x \in \mu^{-1}(0,\lambda) \subset \mathrm{Rep}(\overline{Q},\delta)$ be a point above y. Set*

$$H \coloneqq (G_v^{\mathbb{C}})_x \quad \textit{and} \quad W \coloneqq T_p(G_v^c \cdot x)^{\omega_\mathbb{C}}/T_x(G_v^c \cdot x)$$

Let $\mu_W : W \to \mathfrak{h}^*$ be the unique complex symplectic moment map vanishing at 0, where $\mathfrak{h} = \mathrm{Lie}(H)$. Then a neighborhood of $y \in \mathcal{M}_{(0,\lambda)}(Q,v)$ is biholomorphic with a neighborhood of 0 in (the analytification of) the GIT quotient $\mu_W^{-1}(0)//H$.

In view of this result our task is to determine the complex symplectic slices at the points above the singular points in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$. It will be useful to introduce the following notation.

Definition II.7.3. Let Q be a quiver with vertex set I. For a pair of dimension vectors $v, w \in \mathbb{Z}^I_{\geq 0}$ define

$$\mathrm{Hom}(v,w) \coloneqq \bigoplus_{i\in I} \mathrm{Hom}(V_i,W_i) \quad \text{and} \quad \mathrm{Rep}(Q;v,w) \coloneqq \bigoplus_{h\in\overline{Q}} \mathrm{Hom}(V_{s(h)},W_{t(h)}),$$

where $V_i = \mathbb{C}^{v_i}$ and $W_i = \mathbb{C}^{w_i}$ for each $i \in I$.

Note that $\mathrm{Rep}(Q;v,v) = \mathrm{Rep}(Q,v)$ and that $\mathrm{End}(v) \coloneqq \mathrm{Hom}(v,v) = \mathfrak{g}_v^c$. The complex in (II.5) also has a relative analogue. Let $v, w \in \mathbb{Z}^I$ be a pair of dimension vectors and let $x \in \mathrm{Rep}(\overline{Q},v)$ and $y \in \mathrm{Rep}(\overline{Q},w)$ satisfy $\mu_\mathbb{C}(x) = \mu_\mathbb{C}(y) = \lambda$ for some $\lambda \in \mathbb{C}^I$. Define $C_Q(x,y)$ to be the sequence given by

$$0 \longrightarrow \mathrm{Hom}(v,w) \xrightarrow{\sigma_{x,y}} \mathrm{Rep}(Q;v,w) \xrightarrow{\nu_{x,y}} \mathrm{Hom}(v,w) \longrightarrow 0$$

where

$$\begin{aligned} \sigma_{x,y}((u_i)_{i\in I}) &= \ (u_{t(h)}x_h - y_h u_{s(h)})_{h\in\overline{Q}} \\ \nu_{x,y}((v_h)_{h\in\overline{Q}}) &= \left(\sum_{h\in t^{-1}(i)} \epsilon(h)(u_h x_{\overline{h}} + y_h u_{\overline{h}})\right)_{i\in I}. \end{aligned}$$

Note that $C_Q(x,x)$ is the complex of (II.5).

Lemma II.7.4. *Let X and Y denote the Π^λ-modules corresponding to $x \in \mathrm{Rep}(\overline{Q},v)$ and $y \in \mathrm{Rep}(\overline{Q},w)$. Then $C_Q(x,y)$ is a chain complex, i.e., $\nu_{x,y} \circ \sigma_{x,y} = 0$ and if we denote the cohomology groups from left to right by $H^i_Q(x,y)$ for $0 \leq i \leq 2$ we have*

(1) $H^0_Q(x,y) \cong \mathrm{Hom}_{\Pi^\lambda}(X,Y)$,

(2) $H^2_Q(x,y) \cong \mathrm{Hom}_{\Pi^\lambda}(Y,X)^*$,

(3) $\dim_{\mathbb{C}} H^1_Q(x,y) = \dim_{\mathbb{C}} H^0_Q(x,y) + \dim_{\mathbb{C}} H^2_Q(x,y) - (v,w)$.

Proof. To simplify the notation we will write $V_i = \mathbb{C}^{v_i}$ and $W_i = \mathbb{C}^{w_i}$ for $i \in I$. Let $u = (u_i : V_i \to W_i)_{i\in I} \in \mathrm{Hom}(v,w)$. Then using the definitions of $\sigma_{x,y}$ and $\nu_{x,y}$ we see that $\nu_{x,y} \circ \sigma_{x,y}(u)$ equals

$$= \left(\sum_{h\in t^{-1}(i)} \epsilon(h)(u_{t(h)}x_h x_{\overline{h}} - y_h u_{s(h)} x_{\overline{h}} + y_h u_{t(\overline{h})} x_{\overline{h}} - y_h y_{\overline{h}} u_{s(\overline{h})}) \right)_{i\in I}$$
$$= \left(\sum_{h\in t^{-1}(h)} u_i(\epsilon(h)x_h x_{\overline{h}}) - (\epsilon(h) y_h y_{\overline{h}})u_i \right)_{i\in I} = (u_i\lambda_i - \lambda_i u_i)_{i\in I} = 0.$$

Here we have used that $s(\overline{h}) = t(h)$, $t(\overline{h}) = s(h)$ and that $\mu_{\mathbb{C}}(x) = \mu_{\mathbb{C}}(y) = \lambda$. This shows that $C_Q(x,y)$ is a chain complex.

Recall that Π^λ was defined to be a quotient of the quiver algebra $\mathbb{C}\overline{Q}$. Therefore, we may also regard X and Y as $\mathbb{C}\overline{Q}$-modules and clearly $\mathrm{Hom}_{\mathbb{C}\overline{Q}}(X,Y) = \mathrm{Hom}_{\Pi^\lambda}(X,Y)$. From the definition of a homomorphism of representations it is clear that $\mathrm{Hom}_{\mathbb{C}\overline{Q}}(X,Y) = \mathrm{Ker}(\sigma_{x,y}) = H^0_Q(x,y)$ proving part (1).

For the second part we use an idea from the proof of [Cra00, Lemma 3.1] (this lemma and its proof implies our result for $\lambda = 0$). Let $\phi\colon \mathrm{Hom}(w,v) \to \mathrm{Hom}(v,w)^*$ be the isomorphism given by $\phi(u)(v) = \sum_{i\in I} \mathrm{tr}(u_i v_i)$ and let $\psi\colon \mathrm{Rep}(Q;w,v) \to \mathrm{Rep}(Q;v,w)^*$ be the isomorphism given by $\psi(f)(g) = \sum_{h\in\overline{Q}} \epsilon(h)\,\mathrm{tr}(f_h g_{\overline{h}})$. Then a rather tedious calculation shows that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}(w,v) & \xrightarrow{\sigma_{y,x}} & \mathrm{Rep}(Q;w,v) \\ \downarrow \phi & & \downarrow \psi \\ \mathrm{Hom}(v,w)^* & \xrightarrow{(\nu_{x,y})^*} & \mathrm{Rep}(Q;v,w)^*. \end{array}$$

Since both the vertical maps are isomorphism we conclude that

$$\mathrm{Coker}(\nu_{x,y})^* \cong \mathrm{Ker}((\nu_{x,y})^*) \cong \mathrm{Ker}(\sigma_{y,x}) = \mathrm{Hom}_{\Pi^\lambda}(Y,X),$$

where the final equality follows from the first part. Hence, $H^2_Q(x,y) \cong \mathrm{Hom}_{\Pi^\lambda}(Y,X)^*$.

For the final part observe that

$$(v,w) = 2\sum_{i\in I} v_i w_i - \sum_{h\in\overline{Q}} v_{s(h)} w_{t(h)} = 2\dim_{\mathbb{C}} \mathrm{Hom}(v,w) - \dim_{\mathbb{C}} \mathrm{Rep}(Q;v,w)$$

is the Euler characteristic of the complex $C_Q(v,w)$. Since the Euler characteristic is preserved upon passage to cohomology, we obtain $(v,w) = \dim_{\mathbb{C}} H^0_Q(v,w) - \dim_{\mathbb{C}} H^1_Q(v,w) + \dim_{\mathbb{C}} H^2_Q(v,w)$ and this is equivalent to the formula stated in part (3). ■

Remark II.7.5. It is in fact also true that $H^1_Q(x,y) \cong \mathrm{Ext}^1_{\Pi^\lambda}(X,Y)$. We give a sketch of the proof. By [Bri12, Corollary 1.4.2] it holds true that $\mathrm{Coker}(\sigma_{x,y}) = \mathrm{Ext}^1_{\mathbb{C}\overline{Q}}(X,Y)$. Moreover, there is an explicit way to relate this group to the set of isomorphism classes of extensions $0 \to Y \to Z \to X \to 0$. Given an element $[z] \in \mathrm{Ext}^1_{\mathbb{C}\overline{Q}}(X,Y)$ represented by $z = (z_u \colon V_{s(h)} \to W_{t(h)})$ one may construct the extension Z by setting $e_i Z = U_i = V_i \oplus W_i$ for each $i \in I$ and letting $z_h \colon U_{s(h)} \to U_{t(h)}$ for $h \in \overline{Q}$ be given by the matrix

$$z_h = \begin{pmatrix} x_h & 0 \\ z_h & y_h \end{pmatrix}.$$

The exact sequence $0 \to Y \to Z \to X \to 0$ is given componentwise by the canonical exact sequence $0 \to W_i \to V_i \oplus W_i \to V_i \to 0$. This is then an extension of Π^λ-modules if and only if $\mu_{\mathbb{C}}(Z) = \lambda$. It is then a matter of calculation to check that this is the case if and only if $z \in \mathrm{Ker}(\nu_{x,y})$.

Let Q be an extended Dynkin quiver with vertex set identified with $\{0, 1, \cdots, n\}$ and minimal imaginary root $\delta \in \mathbb{Z}^{n+1}$. Let $\lambda = (\lambda_1, \tau) \in \mathbb{C} \oplus \mathbb{C}^n = \mathbb{C}^{n+1}$ satisfy $\delta \cdot \lambda = 0$. Denote the root systems by $\tilde{\Phi} \subset \mathbb{Z}^{n+1}$ and $\Phi \subset \mathbb{Z}^n$ as usual. By Proposition II.4.3 and Theorem II.6.3 the singular points in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ are in bijection with the components in the root space decomposition

$$\Phi \cap \tau^\perp = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_r.$$

Write $\mathcal{M}^{\mathrm{sing}}_{(0,\lambda)}(Q,\delta) = \{y_1, y_2, \cdots, y_r\}$ where y_i corresponds to Φ_i for each $1 \le i \le r$.

Proposition II.7.6. *In the above situation fix i, $1 \le i \le r$, and let $x \in \mu^{-1}(0,\lambda) \subset \mathrm{Rep}(Q,\delta)$ be a point above y_i. Let Q' be the extended Dynkin quiver associated with the root system Φ_i and let δ' denote its minimal imaginary root. Then there is an isomorphism $(G^c_\delta)_x \cong G^c_{\delta'}$ and there is a complex symplectic isomorphism*

$$T_x(G^c_\delta \cdot x)^{\omega_{\mathbb{C}}} / T_x(G^c_\delta \cdot x) \cong \mathrm{Rep}(\overline{Q'}, \delta')$$

equivariant along the above isomorphism of groups.

Proof. First note that the complex symplectic slice at x is precisely the cohomology group $H^1_Q(x,x)$. We will determine this complex symplectic space as an $H \coloneqq (G^c_\delta)_x$ representation. Let $X = \oplus_{t=0}^k n_t Z_t$ denote the semi-simple Π^λ-module corresponding to x decomposed into simple summands. Then, according to Proposition II.4.3, we have $H = \prod_{t=0}^k \mathrm{GL}(n_t, \mathbb{C})$. Recall from the proof of Theorem II.6.3 that if we write $\gamma_t = \dim Z_t \in \mathbb{Z}^{n+1}$ for $0 \le t \le k$, then after possibly rearranging the indices we have $\gamma_0 = (1, d - \beta) \in \mathbb{Z}^{n+1}$ and

$\gamma_t = (0, \alpha_t) \in \mathbb{Z}^{n+1}$, $1 \leq t \leq k$, where $\alpha_1, \alpha_2, \cdots, \alpha_t \in \Phi_i \subset \Phi \cap \tau^\perp$ is a base and $\beta = \sum_{t=1}^n n_t \alpha_t$ is the maximal root.

Let $z_j \in \mu_{\mathbb{C}}^{-1}(\lambda) \subset \mathrm{Rep}(Q, \gamma_j)$ be the point corresponding to Z_j. Then the complex $C_Q(x,x)$ decomposes according to the decomposition $X = \sum_{t=0}^k n_t Z_t$, namely,

$$C_Q(x,x) \cong \bigoplus_{t,s} \mathrm{Hom}(\mathbb{C}^{n_s}, \mathbb{C}^{n_t}) \otimes C_Q(z_s, z_t).$$

The stabilizer $H = \prod_{t=0}^k \mathrm{GL}(n_t, \mathbb{C})$ only acts on the first factors, i.e.,

$$(u_j)_j \cdot (f_{t,s} \otimes B_{t,s})_{t,s} = (u_t f_{t,s} u_s^{-1} \otimes B_{t,s})_{t,s}$$

for $(u_j)_j \in H$ and $f_{t,s} \otimes B_{t,s} \in \mathrm{Hom}(\mathbb{C}^{n_s}, \mathbb{C}^{n_t}) \otimes C_Q(z_t, z_s)$. Passing to cohomology we obtain

$$H^1_Q(x,x) \cong \bigoplus_{s,t} \mathrm{Hom}(\mathbb{C}^{n_s}, \mathbb{C}^{n_t}) \otimes H^1_Q(z_s, z_t) \tag{II.6}$$

and the action of H is the same as described above. By Lemma II.7.4 part (3) and the fact that each Z_t is a simple module we find

$$\begin{aligned} \dim_{\mathbb{C}} H^1_Q(z_s, z_t) &= \dim_C \mathrm{Hom}_{\Pi^\lambda}(Z_s, Z_t) + \dim_{\mathbb{C}} \mathrm{Hom}_{\Pi^\lambda}(Z_t, Z_s)^* - (\gamma_s, \gamma_t) \\ &= 2\delta_{st} - (\gamma_s, \gamma_t). \end{aligned} \tag{II.7}$$

Let $\widetilde{K}$ be the extended Dynkin graph associated with the root system Φ_i. Specifically, the vertex set is $I = \{0, 1, \cdots, k\}$ corresponding to the roots $\alpha_0 = -\beta, \alpha_1, \cdots, \alpha_t$ and a single edge connecting s to t if and only if $(\alpha_s, \alpha_t) = -1$. As $(\gamma_s, \gamma_t) = (\alpha_s, \alpha_t)$ for all s, t, we conclude by the dimension formula (II.7) that $H^1_Q(z_s, z_t) \cong \mathbb{C}$ precisely when $s \neq t$ and s and t are adjacent in $\widetilde{K}$ and $H^1_Q(z_s, z_t) = 0$ otherwise. The expression in (II.6) then takes the form

$$H^1_Q(x,x) \cong \bigoplus_{s \to t \text{ in } \widetilde{K}} \mathrm{Hom}(\mathbb{C}^{n_s}, \mathbb{C}^{n_t}),$$

where each edge is repeated twice once with each orientation. If the identifications $H^1_Q(z_s, z_t) \cong \mathbb{C}$ for s and t adjacent in $\widetilde{K}$ are chosen appropriately, the induced symplectic form is given by

$$\omega((f_{s,t})_{s,t}, (g_{s,t})_{s,t})) = \sum_{s<t} \epsilon(s,t)(\mathrm{tr}(f_{s,t} g_{t,s}) - \mathrm{tr}(f_{t,s} g_{s,t})).$$

form some $\epsilon(s,t) = \pm 1$. If $s < t$ and there is an edge connecting s to t, we specify the orientation of the edge by $s \to t$ if $\epsilon(s,t) = 1$ and $t \to s$ if $\epsilon(t,s) = -1$. This gives rise to an extended Dynkin quiver Q' with minimal imaginary root $\delta' = (n_0 = 1, n_1, \cdots, n_k)$. It is now clear from the above work that $H^1_Q(x,x) \cong \mathrm{Rep}(\overline{Q'}, \delta')$ as complex symplectic $H \cong \prod_{t=0}^k \mathrm{GL}(n_t, \mathbb{C}) = G^c_{\delta'}$ representations. ■

To complete the proof of Theorem II.1.1 we will need the following result.

Lemma II.7.7. *[Kro89, Corollary 3.2] Let Q be an extended Dynkin quiver with minimal imaginary root δ. Let $\Gamma \subset \mathrm{SU}(2)$ be the finite subgroup associated with the underlying unoriented graph of Q under the MacKay correspondence. Then there is a homeomorphism*

$$\mathcal{M}_0(Q,\delta) \cong \mathbb{C}^2/\Gamma$$

that restricts to an isometry away from the singular point. In particular, $\mathcal{M}_0^{\mathrm{reg}}(Q,\delta) = \mathcal{M}_0(Q,\delta) - \{0\}$.

Proof of Theorem II.1.1. Let Q be an extended Dynkin quiver with vertex set $\{0,1,\cdots,n\}$ and minimal imaginary root $\delta = (1,d) \in \mathbb{Z}^{n+1}$, where d is the maximal positive root in the associated root system $\Phi \subset \mathbb{Z}^n$ of type ADE. Let $\lambda \in \mathbb{C}^{n+1}$ be a parameter satisfying $\lambda\cdot\delta = 0$ and write $\lambda = (\lambda_1,\tau) \in \mathbb{C}\oplus\mathbb{C}^n$. Then by Theorem II.6.3 there is a bijection between $\mathcal{M}^{\mathrm{sing}}_{(0,\lambda)}(Q,\delta)$ and the components in the root space decomposition

$$\Phi \cap \tau^{\perp} = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_q.$$

Write $\mathcal{M}^{\mathrm{sing}}_{(0,\lambda)}(Q,\delta) = \{x_1,\cdots,x_q\}$, where x_i corresponds to Φ_i for $1 \le i \le q$. For each $1 \le i \le q$, let $Q^{(i)}$ denote the extended Dynkin quiver associated with the root system Φ_i and let $\delta^{(i)}$ be the associated minimal positive imaginary root. Then, according to Proposition II.7.6 and Lemma II.7.2, there is for each $1 \le i \le q$ an open neighborhood U_i of $x_i \in \mathcal{M}_{(0,\lambda)}(Q,\delta)$, an open neighborhood V_i of $0 \in \mathcal{M}_0(Q^{(i)},\delta^{(i)})$ and a biholomorphism $\rho_i\colon U_i \to V_i$. Importantly, since the category of complex manifolds is a full subcategory of the category of complex analytic spaces, this biholomorphism restricts to a biholomorphism $\rho_i\colon U_i^{\mathrm{reg}} \cong V_i^{\mathrm{reg}}$ of complex manifolds.

Let $\Gamma_i \subset \mathrm{SU}(2)$ be the finite subgroup associated with $Q^{(i)}$ under the McKay correspondence. By the above lemma there is for each i, $1 \le i \le q$, a homeomorphism $\mathcal{M}_0(Q^{(i)},\delta^{(i)}) \cong \mathbb{C}^2/\Gamma_i$ that restricts to an isometry away from the singular point. This map restricts to a homeomorphism $\kappa_i\colon V_i \cong W_i \subset \mathbb{C}^2/\Gamma_i$ for some open neighborhood W_i around 0. By shrinking the U_i and V_i if necessary, we may assume that $W_i = B_r(0)/\Gamma_i$ for some $r > 0$ for each $1 \le i \le q$. The compositions $\phi_i \coloneqq \kappa_i \circ \rho_i\colon U_i \to B_r(0)/\Gamma_i$ are then the required homeomorphisms. Indeed, for each i both ρ_i and κ_i restrict to diffeomorphisms away from the singular point, so we deduce that the restriction

$$\phi_i = \kappa_i \circ \phi_i\colon \mathcal{M}^{\mathrm{reg}}_{(0,\lambda)}(Q,\delta) \cap U_i = U_i - \{x_i\} \cong (B_r(0) - \{0\})/\Gamma_i$$

is a diffeomorphism. This completes the proof. ■

II.8 Configurations of Singularities and the Proof of Theorem II.1.2

Let Q be an extended Dynkin quiver with vertex set $I = \{0,1,\cdots,n\}$ and minimal imaginary root $\delta \in \mathbb{Z}^{n+1}$. In this section we take up the question of what kind of configurations of singularities that can occur in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ by varying the parameter λ. Assume that $\lambda \cdot \delta = 0$ and write $\lambda = (\lambda_1, \tau) \in \mathbb{C} \oplus \mathbb{C}^n$. Then according to Theorem II.6.3 and the local structure result in the previous section, the configuration of singularities is uniquely determined by the root space decomposition

$$\Phi \cap \tau^{\perp} = \Phi_1 \cup \cdots \cup \Phi_r,$$

where $\Phi \subset \mathbb{Z}^n$ is the root system of type ADE associated with Q. The problem therefore reduces to determining the number and types of root systems that can occur in the above root space decomposition.

Give $\mathbb{C}$ the total ordering determined by $z \le w$ if and only if either $\mathrm{Re}(z) \le \mathrm{Re}(w)$ or $\mathrm{Re}\, z = \mathrm{Re}\, w$ and $\mathrm{Im}\, z \le \mathrm{Im}\, w$. Note that this ordering is additive, that is, $z \le w \implies z + c \le w + c$ for each $c \in \mathbb{C}$. We say that an element $\tau \in \mathbb{C}^n$ is dominant if $\tau_i \ge 0$ for each i. The value of this notion comes from the simple observation that if $\tau \in \mathbb{C}^n$ is dominant and $\theta \in \mathbb{Z}^n$ then $\tau \cdot \theta = 0$ if and only if $\mathrm{supp}(\theta) \cap \mathrm{supp}(\tau) = \emptyset$.

Lemma II.8.1. *Let K denote the Dynkin diagram associated with the root system $\Phi \subset \mathbb{Z}^n$. Suppose $\tau \in \mathbb{C}^n$ is dominant and let J be the complement of $\mathrm{supp}(\tau)$ in $\{1, 2, \cdots, n\}$. Let $K_J \subset K$ be the full subgraph of K with vertex set $J \subset \{1, 2, \cdots, n\}$. Let*

$$K_J = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_r$$

be the decomposition of K_J into connected components. Then

$$\Phi \cap \tau^{\perp} = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_r,$$

where Φ_i is the ADE root system associated with K_i for each $1 \le i \le r$.

Proof. Note first that every connected subgraph of a Dynkin graph of type ADE is again a Dynkin graph of type ADE. Let J_i be the set of vertices for K_i in the decomposition in the statement and put $S_i = \{\epsilon_j : j \in J_i\}$. We claim that $S = \cup_i S_i$ is a base for $\Phi \cap \tau^{\perp}$. Indeed, S clearly consists of linearly independent elements and every element $\alpha \in \Phi^+ \cap \tau^{\perp}$ satisfies $\mathrm{supp}(\alpha) \subset J$ so it can be written as a positive linear integral combination of the elements of S. Then, as in the proof of Lemma II.5.2, the root space decomposition

$$\Phi \cap \tau^{\perp} = \Phi_1 \cup \cdots \Phi_r$$

is obtained by decomposing S into minimal pairwise orthogonal sets $S = \cup_i S_i$ and letting Φ_i be the subsystem generated by S_i. Importantly, this decomposition $S = \cup_i S_i$ is precisely the decomposition introduced in the beginning. We conclude that Φ_i is the root system associated with the Dynkin graph K_i for each $1 \le i \le r$. ■

For completeness we also show that the decomposition for an arbitrary parameter τ can in fact be put in the above standard form. Recall that the Weyl group associated with Φ is the finite group $W \subset \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^n)$ generated by the simple reflections $s_i \colon \mathbb{Z}^n \to \mathbb{Z}^n$ in the coordinate vectors ϵ_i for $1 \le i \le n$. There is a unique action of W on $\mathbb{C}^n$ such that $(w\alpha)\cdot\tau = \alpha\cdot(w^{-1}\tau)$ for all $\alpha\in\mathbb{Z}^n$ and $\tau\in\mathbb{C}^n$. This is the complexification of the dual action, where we identify $(\mathbb{R}^n)^*\cong\mathbb{R}^n$ using the standard scalar product.

The following lemma follows essentially from the proof in [Hum78, p. 51], see also [CH98, Lemma 7.2].

Lemma II.8.2. *For every $\tau\in\mathbb{C}^n$ there exists $w\in W$ such that $w\tau$ is dominant.*

Proof. Write $\Phi = \Phi^+\cup\Phi^-$ and define $\gamma = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. By [Hum78, p. 50] one has $s_i(\gamma) = \gamma-\epsilon_i$ for each $1\le i\le n$. Choose $w\in W$ such that $\gamma\cdot w\tau\ge\gamma\cdot w'\tau$ for every $w'\in W$ with respect to the total ordering on $\mathbb{C}$. We claim that $\tau' \coloneqq w\cdot\tau$ is dominant. Indeed, for each $1\le i\le n$ it holds true that

$$\gamma\cdot\tau'\ge\gamma\cdot s_i\tau' = s_i\gamma\cdot\tau' = \gamma\cdot\tau'-\epsilon_i\cdot\tau'$$

or equivalently $\tau'_i = \epsilon_i\cdot\tau'\ge 0$. This shows that $w\tau=\tau'$ is dominant. ∎

Proposition II.8.3. *Let K denote the Dynkin diagram associated with the root system $\Phi\subset\mathbb{Z}^n$. Given $\tau\in\mathbb{C}^n$ let*

$$\Phi\cap\tau^\perp = \Phi_1\cup\cdots\cup\Phi_r$$

be the corresponding decomposition into ADE root systems. Then there exists a full subgraph $K'\subset K$ and a decomposition $K' = K_1\sqcup\cdots\sqcup K_r$ into connected components such that Φ_i is isomorphic to the root system associated with K_i for each i.

Proof. By the previous lemma there exists a Weyl transformation $w\in W$ such that $w\tau\in\mathbb{C}^n$ is dominant. From the relation $\tau\cdot\alpha = w\tau\cdot w\alpha$ we deduce that the isomorphism $w\colon\Phi\to\Phi$ restricts to an isomorphism $\tau^\perp\cap\Phi\to(w\tau)^\perp\cap\Phi$. As this is an isomorphism of root systems, it preserves the decomposition into irreducible components. The result therefore follows from Lemma II.8.1 as $w\tau$ is dominant. ∎

The final ingredient needed to complete the proof of Theorem II.1.2 is contained in the following proposition. We use the notation $B_r(x)\subset\mathbb{C}^2$ and $\overline{B}_r(x)\subset\mathbb{C}^2$ for the open and closed ball, respectively, with center $x\in\mathbb{C}^2$ and radius r.

Proposition II.8.4. *Let Q be an extended Dynkin quiver with minimal imaginary root δ. Let $\Gamma\subset\mathrm{SU}(2)$ be the finite subgroup associated with the underlying extended Dynkin graph under the McKay correspondence. Let $\lambda\in\mathbb{C}^{n+1}$ be a parameter with $\lambda\cdot\delta = 0$. Then there is an open subset $U\subset\mathcal{M}^{\mathrm{reg}}_{(0,\lambda)}(Q,\delta)$ with compact complement in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ and a diffeomorphism $\phi\colon U\to(\mathbb{C}^2-\overline{B}_R(0))/\Gamma$. Moreover, $\phi^{-1}((\mathbb{C}^2-B_{R'}(0))/\Gamma)$ is closed in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ for each $R'>R$.*

Remark II.8.5. The final assertion is included to explicitly state that there are no limit points in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ as $x\in(\mathbb{C}^2-\overline{B}_R(0))/\Gamma$ tends to ∞.

Proof. Choose a parameter $\zeta \in \mathbb{R}^{n+1}$ satisfying $\zeta \cdot \delta = 0$ and $\zeta \cdot \theta \neq 0$ for each $\theta \in R_+(\delta)$ (defined in Theorem II.3.6) and put $\xi = (0, \lambda)$ and $\widetilde{\xi} = (\zeta, \lambda)$. To simplify the notation write

$$\widetilde{X} = \mathcal{M}_{\widetilde{\xi}}(Q, \delta) \quad \text{and} \quad X = \mathcal{M}_{\xi}(Q, \delta).$$

Then according to Theorem II.3.10 there is a holomorphic map $\pi\colon \widetilde{X} \to X$ which is a resolution of singularities. Furthermore, by Kronheimer's result mentioned in the introduction [Kro89, Corollary 3.12], the smooth 4-dimensional hyper-Kähler manifold $\widetilde{X}$ is diffeomorphic to the minimal resolution of the quotient singularity $\mathbb{C}^2/\Gamma$. We may therefore assume that there is a continuous proper map $\hat{\pi}\colon \widetilde{X} \to \mathbb{C}^2/\Gamma$ that restricts to a diffeomorphism $\hat{\pi}^{-1}((\mathbb{C}^2 - \{0\})/\Gamma) \cong (\mathbb{C}^2 - \{0\})/\Gamma$. The situation is summarized in the following diagram

$$X \xleftarrow{\ \pi\ } \widetilde{X} \xrightarrow{\ \hat{\pi}\ } \mathbb{C}^2/\Gamma \ .$$

Since the open sets $\hat{\pi}^{-1}(B_R(0)/\Gamma)$ for $1 < R < \infty$ cover $\widetilde{X}$ and $\pi^{-1}(X^{\mathrm{sing}})$ is compact, there exists an R such that $\pi^{-1}(X^{sing}) \subset \hat{\pi}^{-1}(B_R(0)/\Gamma)$. Hence,

$$V \coloneqq \hat{\pi}^{-1}((\mathbb{C}^2 - \overline{B}_R(0)/\Gamma) \subset \pi^{-1}(X^{\mathrm{reg}}),$$

and as $\hat{\pi}$ is proper $X - V = \hat{\pi}^{-1}(\overline{B}_R(0)/\Gamma)$ is compact. The biholomorphism $\pi\colon \pi^{-1}(X^{\mathrm{reg}}) \cong X^{\mathrm{reg}}$ therefore maps V onto an open subset $U \subset X^{\mathrm{reg}}$. The composition of the restrictions $\pi^{-1}\colon U \to V$ and $\hat{\pi}\colon V \to (\mathbb{C}^2 - \overline{B}_R(0))/\Gamma$ gives the required diffeomorphism $\phi\colon U \cong (\mathbb{C}^2 - \overline{B}_R(0))/\Gamma$. Finally,

$$\phi^{-1}(\mathbb{C}^2 - B_{R'}(0)))/\Gamma = \pi(\hat{\pi}^{-1}(\mathbb{C}^2 - B_{R'}(0))/\Gamma)$$

is closed in X for each $R' > R$ because $\hat{\pi}$ is continuous and π is a closed map (as it is proper and X is locally compact Hausdorff). ■

Proof of Theorem II.1.2. Let $\Gamma_0, \Gamma_1, \cdots, \Gamma_q \subset \mathrm{SU}(2)$ be finite subgroups and let K_i denote the Dynkin diagram associated with K_i for each $0 \leq i \leq q$. Assume that $K' \coloneqq K_1 \sqcup K_2 \sqcup \cdots \sqcup K_q$ can be realized as a full subgraph of K_0. Identify the vertex set of K_0 with $\{1, 2, \cdots, n\}$ for some $n \in \mathbb{N}$ and let $J \subset \{1, \cdots, n\}$ be the vertices of the subgraph K'. Let $\Phi \subset \mathbb{Z}^n$ be the root system associated with K and specify $\tau \in \mathbb{C}^n$ by $\tau_j = 1$ if $j \notin J$ and $\tau_j = 0$ otherwise. Then τ is dominant and $\operatorname{supp} \tau$ is complementary to J. By Lemma II.8.1 we have a root space decomposition

$$\Phi \cap \tau^{\perp} = \Phi_1 \cup \cdots \cup \Phi_q, \tag{II.8}$$

where Φ_i is the ADE root system associated with the Dynkin graph K_i for each $1 \leq i \leq q$.

Let Q be an extended Dynkin quiver with underlying extended Dynkin graph corresponding to Γ_0 under the McKay correspondence (i.e., $\widetilde{K_0}$). We identify the set of vertices with $\{0, 1, \cdots, n\}$ such that the minimal imaginary root is given by $(1, d) \in \mathbb{Z}^{n+1}$ where $d \in \Phi \subset \mathbb{Z}^n$ is the maximal positive root. Then

$\lambda \coloneqq (-d \cdot \tau, \tau) \in \mathbb{C}^{n+1}$ satisfies $\lambda \cdot \delta = 0$. Set $X \coloneqq \mathcal{M}_{(0,\lambda)}(Q, \delta)$. Then, according to Theorem II.1.1, we may write $X^{\text{sing}} = \{x_1, x_2, \cdots, x_q\}$ and for each $1 \leq i \leq q$ there is an open neighborhood $x_i \subset V_i \subset X$ and a homeomorphism $\phi_i \colon V_i \to B_r(0)/\Gamma_i$, for some fixed r independent of i. Furthermore, each ϕ_i restricts to a diffeomorphism away from the singular point. Next, by Proposition II.8.4 there is an open subset $U' \subset X^{\text{reg}}$ with $X - U'$ compact and a diffeomorphism $\phi_0 \colon U' \cong (\mathbb{C}^2 - \overline{B}_{R'}(0))/\Gamma_0$ for some $R' > 0$. In addition, $\phi_0^{-1}((\mathbb{C}^2 - B_R(0))/\Gamma)$ is closed in X for each $R > R'$.

For part (i) we already know that X^{reg} is a smooth hyper-Kähler 4-manifold. The space X is connected by Lemma II.6.2 and, in view of the above local models around the singularities, it is clear that $X^{\text{reg}} = X - \{x_1, \cdots, x_q\}$ is connected as well.

For part (ii) and (iii) fix $R > R'$ and let $C \subset X$ be the closed subset $\phi^{-1}((\mathbb{C}^2 - B_R(0))/\Gamma)$. Since $C \subset X^{\text{reg}}$ and X is Hausdorff, we may assume after possibly shrinking the V_i (and hence $r > 0$) that the open sets $V_1, V_2, \cdots, V_q$ are pairwise disjoint and that $V_i \cap C = \emptyset$ for each i. Put

$$U_0 \coloneqq \phi^{-1}((\mathbb{C}^2 - \overline{B}_R(0))/\Gamma) \subset X^{\text{reg}} \text{ and } U_i \coloneqq V_i - \{x_i\} \subset X^{\text{reg}},\ 1 \leq i \leq q.$$

Then the open subset $U_0, U_1, U_2, \cdots, U_q$ are pairwise disjoint, the complement of their union is compact in X^{reg}, and we have diffeomorphisms $\phi_0 \colon U_0 \cong (\mathbb{C}^2 - \overline{B}_R(0))/\Gamma$ and $\phi_i \colon U_i \cong (B_r(0) - \{0\})/\Gamma$ for $1 \leq i \leq q$. We now decrease r and increase R slightly to ensure that each ϕ_i extends over a slightly bigger open set for each $0 \leq i \leq q$. The proof of part (ii) is completed by composing ϕ_0 with the evident diffeomorphism $(\mathbb{C}^2 - \overline{B}_R(0))/\Gamma \cong (R, \infty) \times S^3/\Gamma \cong (0, \infty) \times S^3/\Gamma$ and by composing ϕ_i with the diffeomorphism

$$(B_r(0) - \{0\})/\Gamma_i \cong (0, r) \times S^3/\Gamma_i \cong (0, \infty) \times S^3/\Gamma_i,$$

where the final diffeomorphism includes a time reversal, for each $1 \leq i \leq q$. Finally, $Y = X^{\text{reg}} - \cup_{i=0}^q U_i$ is compact a manifold with boundary components S^3/Γ_i, $0 \leq i \leq q$, because we arranged that ϕ_i actually extends to a diffeomorphism $\phi_i' \colon U_i' \cong (-t_0, \infty) \times S^3/\Gamma_i$ for some $t_0 > 0$ for each $0 \leq i \leq q$. This completes the verification of part (iii) and hence the proof. ■

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