CONVEX CONES IN MAPPING SPACES BETWEEN MATRIX ALGEBRAS

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Abstract. We introduce the notion of *one-sided mapping cones* of positive linear maps between matrix algebras. These are convex cones of maps that are invariant under compositions by completely positive maps from either the left or right side. The duals of such convex cones can be characterized in terms of ampliation maps, which can also be used to characterize many notions from quantum information theory—such as separability, entanglement-breaking maps, Schmidt numbers, as well as decomposable maps and k-positive maps in functional analysis. In fact, such characterizations hold if and only if the involved cone is a one-sided mapping cone. Through this analysis, we obtain mapping properties for compositions of cones from which we also obtain several equivalent statements of the PPT (positive partial transpose) square conjecture.

1. Introduction

A *mapping cone* is a closed convex cone of positive linear maps that is closed under compositions by completely positive linear maps from both sides. The notion of mapping cones was introduced by the third author [36] in the 1980s to study extension problems of positive linear maps and has been studied in the context of quantum information theory [30, 31, 37, 38, 39, 42, 43]. Various notions from quantum information theory—such as separability, Schmidt numbers, positive partial transposes, and entanglement-breaking maps—can be explained in terms of mapping cones. The study of mapping cones is also closely related to that of operator systems [21].

In this paper, we consider closed convex cones of positive linear maps between matrix algebras which are closed under compositions by completely positive maps from only one side. That is, cones K that satisfy either $K \circ \mathbb{CP} \subset K$ or $\mathbb{CP} \circ K \subset K$, respectively, where $\mathbb{CP}$ denotes the convex cone of all completely positive maps and $K_1 \circ K_2$ is the set of all maps of the form $\phi_1 \circ \phi_2$ for maps $\phi_1 \in K_1$ and $\phi_2 \in K_2$. Mapping cones satisfy many nice properties regarding the compositions of maps [39, 30, 41, 43]. We show that these properties can also be used to characterize one-sided mapping cones.

The notion of ampliation (i.e., $1 \otimes \phi$ or $\phi \otimes 1$) of a linear map ϕ by the identity map on matrix algebras plays an important role in operator algebras, as evidenced by

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Stinespring's representation theorem [32] and a characterization of decomposability due to the third author [35]. It was also shown by the second author [12] that k-positivity of linear maps can be characterized by ampliation. Moreover, ampliation is also useful for characterizing certain kinds of linear maps in quantum information theory—such as entanglement-breaking maps [15, 29, 18]—and is crucial for distinguishing several kinds of positive (semi-definite) matrices in tensor product spaces. For example, some criteria for separability of quantum states can be presented in terms of ampliation of positive maps [17]. In this paper, we provide a single framework that allows us to recover all of the above-mentioned results. The main idea is that these kinds of characterizations hold only when the involved convex cones are one-sided mapping cones.

For this purpose, we first study the dual cones of closed convex cones of (not-necessarily positive) maps and their compositions $K_1 \circ K_2$. We obtain various relations among dual cones and compositions of convex cones, from which we also obtain several equivalent statements to the PPT-square conjecture.

Throughout this paper, we denote by M_A the matrix algebra acting on the finite-dimensional Hilbert space $\mathbb{C}^A$. We will work in the real vector spaces $H(M_A, M_B)$ consisting of all Hermitian-preserving linear maps from M_A into another matrix algebra M_B. Recall that a linear map $\phi : M_A \to M_B$ is called *Hermitian-preserving* if $a = a^*$ implies $\phi(a) = \phi(a)^*$—or equivalently if $\phi(a^*) = \phi(a)^*$—holds for every $a \in M_A$.

We define a bilinear pairing on the matrix algebra M_A as

$$\langle a, b \rangle_A = \mathrm{Tr}(a^{\mathrm{t}} b) = \sum_{i,j} a_{ij} b_{ij} \tag{1}$$

for every $a = [a_{ij}]$ and $b = [b_{ij}]$ in M_A, where a^{t} denotes the transpose of a. The analogous bilinear pairings on M_B and $M_A \otimes M_B$ will be denoted by $\langle\ ,\ \rangle_B$ and $\langle\ ,\ \rangle_{AB}$, respectively. Note that this is not the Hilbert–Schmidt inner product typically used in the literature. Using the pairing defined in (1) allows us to simplify many identities involving duality of cones considered in this paper. The main results of this paper are not affected by this choice of bilinear pairing, as most of the cones of matrices that we consider satisfy $K = K^{\mathrm{t}} = \overline{K}$ (e.g., the cone of positive matrices). Moreover, this pairing is an inner product when restricted to the real vector space of Hermitian matrices in M_A.

In the next section we present relationships between the dual cones and compositions $K_1 \circ K_2$ for subsets K_1 and K_2 of linear maps that characterize one-sided mapping cones. In Section 3 we revisit properties of mapping cones in relation to one-sided mapping cones, after which we characterize the duals of one-sided mapping cones in terms of ampliation maps in Section 4. This allows us to recover several well-known results in quantum information theory. Finally, in Section 5 we present several equivalent formulations of the PPT-square conjecture.

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2. One-sided mapping cones

For a given linear map $\phi : M_A \to M_B$, its Choi matrix $\mathrm{C}_\phi \in M_A \otimes M_B$ is defined as

$$\mathrm{C}_\phi = \sum_{ij} e_{ij} \otimes \phi(e_{ij}) \in M_A \otimes M_B,$$

where $\{e_{ij}\}$ denote the matrix units in M_A. The mapping defined by $\phi \mapsto \mathrm{C}_\phi$ is a linear isomorphism between the vector space $L(M_A, M_B)$ of all linear maps from M_A to M_B onto the tensor product $M_A \otimes M_B$ and is usually called the Jamiołkowski-Choi isomorphism [11, 19, 4]. It is easy to see that a map ϕ is Hermitian-preserving if and only if C_ϕ is a self-adjoint matrix. Recall that ϕ is completely positive if and only if C_ϕ is positive (semi-definite) [4]. Note that, for all $a \in M_A$ and $b \in M_B$, we have the identity

$$\langle a \otimes b, \mathrm{C}_\phi \rangle_{AB} = \sum_{i,j} \langle a_{ij} b, \phi(e_{ij}) \rangle_B = \langle b, \phi(a) \rangle_B. \tag{2}$$

We refer the reader to Chapter 4 of [42] for further properties of Choi matrices.

For a given map $\phi \in L(M_A, M_B)$, we define its *adjoint* map $\phi^* : M_B \to M_A$ by the condition

$$\langle \phi(a), b \rangle_B = \langle a, \phi^*(b) \rangle_A, \qquad \text{for all } a \in M_A, b \in M_B.$$

It is easy to see that $\phi \in H(M_A, M_B)$ if and only if $\phi^* \in H(M_B, M_A)$. For every $a \in M_A$ and $b \in M_B$, note that

$$\langle b \otimes a, \mathrm{C}_{\phi^*} \rangle_{BA} = \langle a, \phi^*(b) \rangle_A = \langle b, \phi(a) \rangle_B = \langle a \otimes b, \mathrm{C}_\phi \rangle_{AB}, \tag{3}$$

and thus the Choi matrix $\mathrm{C}_{\phi^*} \in M_B \otimes M_A$ of ϕ^* is the flip of C_ϕ. Before proceeding, we remark that the adjoint defined in this way is not the same as the adjoint with respect to the Hilbert–Schmidt inner product that is commonly used in the literature with the same notation. However, as mentioned previously, using the adjoint defined in this manner allows us to simplify many of the main results.

We define a bilinear pairing on $L(M_A, M_B)$ as

$$\langle \phi, \psi \rangle = \langle \mathrm{C}_\phi, \mathrm{C}_\psi \rangle_{AB} = \sum_{i,j} \langle \phi(e_{ij}), \psi(e_{ij}) \rangle_B, \tag{4}$$

for every $\phi, \psi \in L(M_A, M_B)$ (see, e.g., [30]). It is clear that we have the identity

$$\langle \phi, \psi \rangle = \langle \psi^*, \phi^* \rangle. \tag{5}$$

For maps $\phi \in L(M_A, M_B)$, $\psi \in L(M_B, M_C)$ and $\sigma \in L(M_A, M_C)$, we also have that

$$\langle \psi \circ \phi, \sigma \rangle = \sum_{i,j} \langle \psi(\phi(e_{ij})), \sigma(e_{ij}) \rangle_C = \sum_{i,j} \langle \phi(e_{ij}), \psi^*(\sigma(e_{ij})) \rangle_B = \langle \phi, \psi^* \circ \sigma \rangle,$$

which further implies that $\langle \phi, \psi^* \circ \sigma \rangle = \langle \sigma^* \circ \psi, \phi^* \rangle = \langle \psi, \sigma \circ \phi^* \rangle$ by (5) and thus

$$\langle \psi \circ \phi, \sigma \rangle = \langle \phi, \psi^* \circ \sigma \rangle = \langle \psi, \sigma \circ \phi^* \rangle. \tag{6}$$

For a subset K of $H(M_A, M_B)$ of Hermitian-preserving maps, the corresponding *dual cone* K° with respect to the bilinear pairing in (4) is the set defined as

$$K^\circ = \{\psi \in H(M_A, M_B) : \langle \phi, \psi \rangle \geqslant 0 \text{ for every } \phi \in K\}.$$

We note that $K^{\circ\circ}$ is the smallest closed convex cone in $H(M_A, M_B)$ containing K. In particular, we have $K^{\circ\circ} = K$ if and only if K is a closed convex cone. For closed convex cones K_1 and K_2, we denote by $K_1 \vee K_2$ and $K_1 \wedge K_2$ the convex hull and the intersection of K_1 and K_2, respectively. Then we have

$$(K_1 \vee K_2)^\circ = K_1^\circ \wedge K_2^\circ \qquad \text{and} \qquad (K_1 \wedge K_2)^\circ = K_1^\circ \vee K_2^\circ.$$

(See [12, 23, 13] for further properties of cones in a more general setting). For a subset K of $H(M_A, M_B)$, we also define the set K^* as

$$K^* = \{\phi^* \in H(M_B, M_A) : \phi \in K\}.$$

By the identity in (5), we have

$$K^{*\circ} = K^{\circ *}.$$

For subsets $K_0 \subset H(M_A, M_B)$, $K_1 \subset H(M_B, M_C)$ and $K_2 \subset H(M_A, M_C)$, the identities in (6) yield the following equivalences:

$$\begin{aligned} K_1 \circ K_0 \subset K_2^\circ \iff K_1^* \circ K_2 \subset K_0^\circ \iff K_2 \circ K_0^* \subset K_1^\circ \\ \iff K_1 \subset (K_2 \circ K_0^*)^\circ \iff K_0 \subset (K_1^* \circ K_2)^\circ \iff K_2 \subset (K_1 \circ K_0)^\circ, \end{aligned} \tag{7}$$

where we define

$$K_1 \circ K_0 = \{\phi_1 \circ \phi_0 \in H(M_A, M_C) : \phi_1 \in K_1, \phi_0 \in K_0\}.$$

We denote by $\mathbb{CP}_A$ the convex cone of all completely positive linear maps of M_A into itself and note that $\mathbb{CP}_A^\circ = \mathbb{CP}_A^* = \mathbb{CP}_A$. For a given closed convex cone K in $H(M_A, M_B)$, we are interested in conditions on K that are equivalent to the conditions $K \circ \mathbb{CP}_A \subset K$ and $\mathbb{CP}_B \circ K \subset K$, respectively. To do this, we may plug $K_0 = \mathbb{CP}_A$, $K_1 = K$ and $K_2 = K^\circ$ into (7) to obtain the following equivalences:

$$\begin{aligned} K \circ \mathbb{CP}_A \subset K \iff K^* \circ K^\circ \subset \mathbb{CP}_A \iff K^\circ \circ \mathbb{CP}_A \subset K^\circ \\ \iff K \subset (K^\circ \circ \mathbb{CP}_A)^\circ \iff \mathbb{CP}_A \subset (K^* \circ K^\circ)^\circ \iff K^\circ \subset (K \circ \mathbb{CP}_A)^\circ. \end{aligned} \tag{8}$$

For a subset K of $H(M_A, M_B)$, we define the sets $K^{\ogreaterthan}$ and $K^{\olessthan}$ as

$$K^{\ogreaterthan} = (K \circ \mathbb{CP}_A)^\circ \qquad \text{and} \qquad K^{\olessthan} = (\mathbb{CP}_B \circ K)^\circ.$$

For an arbitrary convex cone K of $H(M_A, M_B)$ we have $K \subset K \circ \mathbb{CP}_A$ and $K \subset \mathbb{CP}_B \circ K$, since the identity maps 1_A and 1_B are contained in $\mathbb{CP}_A$ and $\mathbb{CP}_B$, respectively. This implies the inclusions

$$K^{\ogreaterthan} \subset K^\circ \qquad \text{and} \qquad K^{\olessthan} \subset K^\circ. \tag{9}$$

We also have $K^{\circ\ogreaterthan} \subset K^{\circ\circ}$ and $K^{\circ\olessthan} \subset K^{\circ\circ}$ by applying (9) to K°. The above analysis is summarized in (10) of the following proposition. The equivalences in (11) can be shown analogously by choosing $K_0 = K$, $K_1 = \mathbb{CP}_B$ and $K_2 = K^\circ$ in (7).

Proposition 2.1. *For a closed convex cone K in $H(M_A, M_B)$, we have the following equivalent conditions:*

$$\begin{aligned} &K^\circ = K^{\ogreaterthan} \iff K^\circ \subset K^{\ogreaterthan} \iff K \circ \mathbb{CP}_A \subset K \iff K^\circ \circ \mathbb{CP}_A \subset K^\circ \\ &\iff K^* \circ K^\circ \subset \mathbb{CP}_A \iff K \subset K^{\circ\ogreaterthan} \iff K = K^{\circ\ogreaterthan}. \end{aligned} \tag{10}$$

We also have the following:

$$\begin{aligned} &K^\circ = K^{\olessthan} \iff K^\circ \subset K^{\olessthan} \iff \mathbb{CP}_B \circ K \subset K \iff \mathbb{CP}_B \circ K^\circ \subset K^\circ \\ &\iff K^\circ \circ K^* \subset \mathbb{CP}_B \iff K \subset K^{\circ\olessthan} \iff K = K^{\circ\olessthan}. \end{aligned} \tag{11}$$

We call a closed convex cone K of positive linear maps in $H(M_A, M_B)$ a *left-mapping cone* if $\mathbb{CP}_B \circ K \subset K$ holds and a *right-mapping cone* if $K \circ \mathbb{CP}_A \subset K$. For a closed convex cone K of positive maps in $H(M_A, M_B)$, we denote by $\mathcal{M}_K^{\mathrm{L}}$ (respectively $\mathcal{M}_K^{\mathrm{R}}$) the smallest left- (respectively right-) mapping cone that contains K.

Proposition 2.2. *For a closed convex cone K of positive maps, we have the following:*

(i) $\mathcal{M}_K^{\mathrm{R}} = K^{\ogreaterthan\circ}$ *and this closed convex cone is generated by $K \circ \mathbb{CP}_A$.*
(ii) $\mathcal{M}_K^{\mathrm{L}} = K^{\olessthan\circ}$ *and this closed convex cone is generated by $\mathbb{CP}_B \circ K$.*

Proof. To prove (i), we first note that the convex hull $(K \circ \mathbb{CP}_A)^{\circ\circ}$ of $K \circ \mathbb{CP}_A$ is a right-mapping cone. For any other right-mapping cone L satisfying $K \subset L$, we have $K \circ \mathbb{CP}_A \subset L \circ \mathbb{CP}_A \subset L$ and thus $(K \circ \mathbb{CP}_A)^{\circ\circ} \subset L$. Hence $\mathcal{M}_K^{\mathrm{R}} = (K \circ \mathbb{CP}_A)^{\circ\circ} = K^{\ogreaterthan\circ}$ and, moreover, the cone $K^{\ogreaterthan\circ}$ is generated by $K \circ \mathbb{CP}_A$. The proof of statement (ii) is analogous. □

The convex cones $K^{\ogreaterthan}$ and $K^{\olessthan}$ can be described in terms of compositions of maps, as is shown in the following proposition.

Proposition 2.3. *For a subset K of $H(M_A, M_B)$, we have the following identities:*

$$K^{\ogreaterthan} = \{\phi \in H(M_A, M_B) : \psi^* \circ \phi \in \mathbb{CP}_A \text{ for every } \psi \in K\}$$
$$K^{\olessthan} = \{\phi \in H(M_A, M_B) : \phi \circ \psi^* \in \mathbb{CP}_B \text{ for every } \psi \in K\}.$$

Proof. To prove the first identity, note for every map $\phi \in H(M_A, M_B)$ that

$$\begin{aligned} \phi \in K^{\ogreaterthan} &\iff \langle \phi, \psi \circ \sigma \rangle \geqslant 0 \text{ for every } \psi \in K,\ \sigma \in \mathbb{CP}_A \\ &\iff \langle \psi^* \circ \phi, \sigma \rangle \geqslant 0 \text{ for every } \psi \in K,\ \sigma \in \mathbb{CP}_A \\ &\iff \psi^* \circ \phi \in \mathbb{CP}_A \text{ for every } \psi \in K \end{aligned}$$

by the identities in (6). The proof of the second statement is analogous. □

Theorem 2.4. *For a closed convex cone K of positive maps in $H(M_A, M_B)$, the following are equivalent:*

(i) *K is a right-mapping cone.*

(ii) *K^* is a left-mapping cone.*

(iii) *For all $\phi \in L(M_A, M_B)$, $\phi \in K^\circ$ if and only if $\psi^* \circ \phi \in \mathbb{CP}_A$ for every $\psi \in K$.*

(iv) *For all $\phi \in L(M_A, M_B)$, $\phi \in K$ if and only if $\psi^* \circ \phi \in \mathbb{CP}_A$ for every $\psi \in K^\circ$.*

We also have the following equivalent conditions:

(v) *K is a left-mapping cone.*

(vi) *K^* is a right-mapping cone.*

(vii) *For all $\phi \in L(M_A, M_B)$, $\phi \in K^\circ$ if and only if $\phi \circ \psi^* \in \mathbb{CP}_B$ for every $\psi \in K$.*

(viii) *For all $\phi \in L(M_A, M_B)$, $\phi \in K$ if and only if $\phi \circ \psi^* \in \mathbb{CP}_B$ for every $\psi \in K^\circ$.*

Proof. By Proposition 2.1, we see that K is a right-mapping cone if and only if $K^\circ = K^{\circledgtr}$, which holds if and only if $K = K^{\circ\circledgtr}$. We note that $K^\circ = K^{\circledgtr}$ is equivalent to (iii) and $K = K^{\circ\circledgtr}$ is equivalent to (iv) by Proposition 2.3. To prove the equivalence (i)$\Longleftrightarrow$(ii), we first note that a map ϕ is positive if and only if ϕ^* is positive. We also note that $K^{\circledgtr *} = K^{*\circledless}$ for every closed convex cone K by Proposition 2.3. If K is a right-mapping cone then $K^{*\circ} = K^{\circ *} = K^{\circledgtr *} = K^{*\circledless}$, and so we see that K^* is a left-mapping cone by Proposition 2.1. Similarly, if K is a left-mapping cone then K^* is a right-mapping cone. The equivalences of (v), (vi), (vii) and (viii) are analogous. □

The properties listed in (iii), (iv), (vii) and (viii) of Theorem 2.4 have been considered in various contexts in the literature [39, 30, 41, 43]. We have shown here that such properties actually characterize one-sided mapping cones. (See also Corollary 3.2 in the next section.) It should be noted that the dual cone K° of a left/right-mapping cone K may not be a left/right-mapping cone, even though, by the equivalences in (8), it holds that $K \circ \mathbb{CP}_A \subset K$ if and only if $K^\circ \circ \mathbb{CP}_A \subset K^\circ$. The dual of a left/right-mapping cone may contain a non-positive map, as we will see in Example 3.5 of the next section.

We now investigate the Choi matrix $\mathrm{C}_{\psi\circ\phi}$ of compositions for maps $\phi \in L(M_A, M_B)$ and $\psi \in L(M_B, M_C)$ in terms of the Choi matrices C_ϕ and C_ψ. This will be useful for the examples considered later in the paper. For matrices $a \in M_A$ and $c \in M_C$, we use the identity in (2) to see that

$$\begin{aligned}
\langle a \otimes c, \mathrm{C}_{\psi\circ\phi} \rangle_{AC} &= \langle c, \psi(\phi(a)) \rangle_C \\
&= \langle \psi^*(c), \phi(a) \rangle_B \\
&= \sum_{k,\ell} \langle \psi^*(c), e_{k\ell} \rangle_B \langle e_{k\ell}, \phi(a) \rangle_B \\
&= \sum_{k,\ell} \langle c, \psi(e_{k\ell}) \rangle_C \langle \phi^*(e_{k\ell}), a \rangle_A \\
&= \sum_{k,\ell} \langle a \otimes c, \phi^*(e_{k\ell}) \otimes \psi(e_{k\ell}) \rangle_{AC},
\end{aligned}$$

where $\{e_{k\ell}\}$ denote the matrix units in M_B. We therefore have that

$$\mathrm{C}_{\psi\circ\phi} = \sum_{k\ell} \phi^*(e_{k\ell}) \otimes \psi(e_{k\ell}) \in M_A \otimes M_C. \tag{12}$$

This is the block-wise summation of the block Schur product [16, 8]

$$\mathrm{C}_{\phi^*} \,\square\, \mathrm{C}_\psi \in M_B \otimes (M_A \otimes M_C)$$

of $\mathrm{C}_{\phi^*} \in M_B \otimes M_A$ and $\mathrm{C}_\psi \in M_B \otimes M_C$. We may also note that the Choi matrix of the composition may be given by

$$\mathrm{C}_{\psi\circ\phi} = \sum_{k,\ell} e_{k\ell} \otimes \psi(\phi(e_{k\ell})) = (1_A \otimes \psi)(\mathrm{C}_\phi). \tag{13}$$

Example 2.5. Consider the map σ on the algebra of 2×2 matrices defined by

$$\sigma = 1_{M_2} + \mathrm{Ad}_{e_{21}},$$

where 1_{M_2} is the identity map on M_2 and, for a fixed matrix a, the map Ad_a is the map defined by $\mathrm{Ad}_a(x) = a^* x a$. The Choi matrices of σ and σ^* are given by

$$\mathrm{C}_\sigma = \begin{pmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix} \qquad \text{and} \qquad \mathrm{C}_{\sigma^*} = \begin{pmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix},$$

respectively, where $\cdot$ denotes a zero. We denote by $\mathbb{P}_1$ the convex cone consisting of all positive linear maps. We have the relations $\mathbb{P}_1 \circ \mathbb{CP} \subset \mathbb{P}_1$ and $\mathbb{CP} \circ \mathbb{P}_1 \subset \mathbb{P}_1$, and thus $\mathbb{P}_1$ satisfies all the conditions in Proposition 2.1. Define the cone K as

$$K = (\mathbb{P}_1^\circ \vee \{\sigma\})^{\circ\circ}, \tag{14}$$

which is the convex hull of $\mathbb{P}_1^\circ \vee \{\sigma\}$. Note that $K^\circ = \mathbb{P}_1 \wedge \{\sigma\}^\circ = \{\phi \in \mathbb{P}_1 : \langle \phi, \sigma \rangle \geqslant 0\}$ and that

$$\begin{aligned} K^{\ogreaterthan} &= \mathbb{P}_1 \wedge \{\sigma\}^{\ogreaterthan} = \{\phi \in \mathbb{P}_1 : \sigma^* \circ \phi \in \mathbb{CP}\} \\ K^{\olessthan} &= \mathbb{P}_1 \wedge \{\sigma\}^{\olessthan} = \{\phi \in \mathbb{P}_1 : \phi \circ \sigma^* \in \mathbb{CP}\} \end{aligned} \tag{15}$$

by Proposition 2.3. We will show that $K^{\olessthan} \nsubseteq K^{\ogreaterthan}$ and $K^{\ogreaterthan} \nsubseteq K^{\olessthan}$.

Toward this goal, consider maps $\phi_{[a,b,c,d]} : M_2 \to M_2$ with Choi matrix having the form

$$\mathrm{C}_{\phi_{[a,b,c,d]}} = \begin{pmatrix} a & \cdot & \cdot & -1 \\ \cdot & b & \cdot & \cdot \\ \cdot & \cdot & c & \cdot \\ -1 & \cdot & \cdot & d \end{pmatrix}$$

for fixed non-negative numbers a, b, c and d. (This is a (2×2)-variant of the generalized Choi map considered in [3].) For every rank-one projection $|\xi\rangle\langle\xi|$ having the form $|\xi\rangle = (x, y)^{\mathrm{t}}$ for fixed constants $x, y \in \mathbb{C}$, the map $\phi_{[a,b,c,d]}$ sends $|\xi\rangle\langle\xi|$ to

$$\phi_{[a,b,c,d]}(|\xi\rangle\langle\xi|) = \begin{pmatrix} a|x|^2 + c|y|^2 & -x\overline{y} \\ -\overline{x}y & b|x|^2 + d|y|^2 \end{pmatrix}.$$

Therefore $\phi_{[a,b,c,d]} \in \mathbb{P}_1$ if and only if the above matrix is positive for every $x, y \in \mathbb{C}$, which holds if and only if $\sqrt{ad} + \sqrt{bc} \geqslant 1$. We also have that $\langle \phi_{[a,b,c,d]}, \sigma \rangle = a + c + d - 2$ and that

$$\mathrm{C}_{\sigma^* \circ \phi} = \begin{pmatrix} a & \cdot & \cdot & -1 \\ \cdot & a+b & \cdot & \cdot \\ \cdot & \cdot & c & \cdot \\ -1 & \cdot & \cdot & c+d \end{pmatrix} \qquad \text{and} \qquad \mathrm{C}_{\phi \circ \sigma^*} = \begin{pmatrix} a+c & \cdot & \cdot & -1 \\ \cdot & b+d & \cdot & \cdot \\ \cdot & \cdot & c & \cdot \\ -1 & \cdot & \cdot & d \end{pmatrix}.$$

The identities in (15) therefore yield the following equivalences:

- $\phi_{[a,b,c,d]} \in K^\circ$ if and only if $\sqrt{ad} + \sqrt{bc} \geqslant 1$ and $a + c + d \geqslant 2$.
- $\phi_{[a,b,c,d]} \in K^{\ogreaterthan}$ if and only if $\sqrt{ad} + \sqrt{bc} \geqslant 1$ and $a(c + d) \geqslant 1$.
- $\phi_{[a,b,c,d]} \in K^{\olessthan}$ if and only if $\sqrt{ad} + \sqrt{bc} \geqslant 1$ and $(a + c)d \geqslant 1$.

Finally, we may take $[a, b, c, d] = [\frac{1}{3}, 1, 1, 1]$ to conclude that $K^{\olessthan} \not\subset K^{\ogreaterthan}$ and take $[a, b, c, d] = [1, 1, 1, \frac{1}{3}]$ to conclude that $K^{\ogreaterthan} \not\subset K^{\olessthan}$. □

3. Mapping cones revisited

Because the identity map is completely positive, we see that a closed convex cone K satisfies $\mathbb{CP}_B \circ K \circ \mathbb{CP}_A \subset K$ if and only if it satisfies both $K \circ \mathbb{CP}_A \subset K$ and $\mathbb{CP}_B \circ K \subset K$. Combining the equivalences in Proposition 2.1 yields the following theorem.

Theorem 3.1. *For a closed convex cone K in $H(M_A, M_B)$, the following are equivalent:*

(i) $K^\circ = K^{\ogreaterthan} = K^{\olessthan}$.
(ii) $K = K^{\circ\ogreaterthan} = K^{\circ\olessthan}$.
(iii) $\mathbb{CP}_B \circ K \circ \mathbb{CP}_A \subset K$.
(iv) $\mathbb{CP}_B \circ K^\circ \circ \mathbb{CP}_A \subset K^\circ$.
(v) $K^* \circ K^\circ \subset \mathbb{CP}_A$ *and* $K^\circ \circ K^* \subset \mathbb{CP}_B$.

Following [36], we say that a closed convex cone $K \subset \mathbb{P}_1$ is a *mapping cone* if $\phi_1 \circ \phi \circ \phi_2 \in K$ for all choices of maps $\phi \in K$, $\phi_1 \in \mathbb{CP}_A$ and $\phi_2 \in \mathbb{CP}_A$ (that is, if K satisfies condition (iii) of Theorem 3.1). The following corollary now follows directly from Theorem 2.4.

Corollary 3.2. *For a closed convex cone $K \subset \mathbb{P}_1$, the following are equivalent:*

(i) *K is a mapping cone.*
(ii) *K is both a left- and right-mapping cone.*
(iii) *For all maps $\phi \in L(M_A, M_B)$, $\phi \in K^\circ$ if and only if $\psi^* \circ \phi \in \mathbb{CP}_A$ for every $\psi \in K$ if and only if $\phi \circ \psi^* \in \mathbb{CP}_B$ for every $\psi \in K$.*
(iv) *For all maps $\phi \in L(M_A, M_B)$, $\phi \in K$ if and only if $\psi^* \circ \phi \in \mathbb{CP}_A$ for every $\psi \in K^\circ$ if and only if $\phi \circ \psi^* \in \mathbb{CP}_B$ for every $\psi \in K^\circ$.*

The implications (i)$\Longrightarrow$(iii) and (i)$\Longrightarrow$(iv) of the above corollary are well known in various contexts [39, 30, 41, 43]. Corollary 3.2 tells us that the converses of these implications are also true.

The convex cone $\mathbb{P}_k$ consisting of all *k-positive* linear maps is an example of a mapping cone. Recall that a linear map $\phi \in L(M_A, M_B)$ is called *k-positive* if its ampliation $1_{M_k} \otimes \phi$ with the $k \times k$ matrices is positive. The convex cone $\mathbb{CP}_{AB}$ of all completely positive maps in $L(M_A, M_B)$ coincides with $\mathbb{P}_{A\wedge B}$, where $A \wedge B$ denotes the minimum of the dimensions of $\mathbb{C}^A$ and $\mathbb{C}^B$ (see Corollary 4.1.9 in [42]). The range of $\mathbb{P}_k$ under the Jamiołkowski-Choi isomorphism is denoted by $\mathcal{BP}_k$. Matrices in $\mathcal{BP}_k$ are called *k-blockpositive*. It was shown in [12] that a map ϕ is k-positive if and only if

$$\langle \mathrm{C}_\phi, |\xi\rangle\langle\xi|\rangle_{AB} \geqslant 0 \quad \text{for every } |\xi\rangle \in \mathbb{C}^A \otimes \mathbb{C}^B \text{ with Schmidt rank } \leqslant k.$$

We denote by $\mathcal{S}_k$ the convex cone in $M_A \otimes M_B$ that is generated by all matrices of the form $|\xi\rangle\langle\xi|$ for vectors $|\xi\rangle \in \mathbb{C}^A \otimes \mathbb{C}^B$ whose Schmidt rank is less than or equal to k. For a matrix $X \in \mathcal{P}_{AB}$, the smallest k such that $X \in \mathcal{S}_k$ is called the *Schmidt number* of X, where $\mathcal{P}_{AB}$ (respectively $\mathcal{P}_A$ and $\mathcal{P}_B$) denotes the set of of all positive matrices in $M_A \otimes M_B$ (respectively M_A and M_B). The corresponding convex cone in $L(M_A, M_B)$ of all maps whose Choi matrices have Schmidt number at most k will be denoted by $\mathbb{SP}_k$. Elements of $\mathbb{SP}_k$ are said to be *k-superpositive* [1, 31]. Note that the convex cones $\mathbb{P}_k$ and $\mathbb{SP}_k$ are dual to each other and that $\mathcal{S}_{A\wedge B} = \mathcal{BP}_{A\wedge B} = \mathcal{P}_{AB}$. This duality is summarized by the following diagram, where JC denotes the Jamiołkowsi-Choi isomorphism:

$$\begin{array}{ccccccccccc} L(M_A, M_B): & \mathbb{SP}_1 & \subset & \mathbb{SP}_k & \subset & \mathbb{CP}_{AB} & \subset & \mathbb{P}_k & \subset & \mathbb{P}_1 \\ \downarrow \mathrm{JC} & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M_A \otimes M_B: & \mathcal{S}_1 & \subset & \mathcal{S}_k & \subset & \mathcal{P}_{AB} & \subset & \mathcal{BP}_k & \subset & \mathcal{BP}_1 \end{array} \tag{16}$$

The convex cone $\mathcal{S}_1$ in the tensor product $M_A \otimes M_B$ and its corresponding convex cone $\mathbb{SP}_1$ in $L(M_A, M_B)$ play crucial roles in quantum information theory. Recall that a *state* is a positive unital linear functional. Every state on the matrix algebra $M_A \otimes M_B$ corresponds to a density matrix $\varrho \in M_A \otimes M_B$ by the mapping $x \mapsto \langle x, \varrho\rangle_{AB}$. In this sense, we may identify a state with its corresponding density matrix. A density matrix in $M_A \otimes M_B$ is called *separable* if it belongs to $\mathcal{S}_1$ and *entangled* if it is not separable. By the duality between $\mathbb{P}_1$ and $\mathbb{SP}_1$, we see that a state ϱ is entangled if and only if there exists $\phi \in \mathbb{P}_1$ such that $\langle \mathrm{C}_\phi, \varrho\rangle < 0$ [17]. Maps in $\mathbb{SP}_1$ are also called *entanglement-breaking* [29, 18, 27, 15]. It is easy to see that the convex cone $\mathbb{SP}_k$ is generated by maps of the form Ad_V for matrices V with $\mathrm{rank}(V) \leqslant k$.

The following results show that the condition $K \subset \mathbb{P}_1$ in the definition of a mapping cone may be weakened significantly. Note that for any matrices $x \in M_A$ and $y \in M_B$, the mapping $\psi \in L(M_A, M_B)$ defined as $\psi(z) = \langle x, z\rangle_A\, y$ for every $z \in M_A$ has Choi matrix equal to $\mathrm{C}_\psi = x \otimes y$.

Lemma 3.3. *Let K be a nonzero mapping cone in $H(M_A, M_B)$ (i.e., a closed convex cone that satisfies $\mathbb{CP}_B \circ K \circ \mathbb{CP}_A \subset K$) and suppose there exists a map $\phi \in K$ such that $\mathrm{Tr}(\mathrm{C}_\phi) > 0$. Then we have $\mathbb{SP}_1 \subset K$.*

Proof. Let $x \in \mathcal{P}_A$ and $y \in \mathcal{P}_B$ and define maps $\psi \in L(M_A, M_A)$ and $\sigma \in L(M_B, M_B)$ whose Choi matrices are $\mathrm{C}_\psi = x \otimes I_A$ and $\mathrm{C}_\sigma = I_B \otimes y$ respectively. These maps are completely positive and thus $\sigma \circ \phi \circ \psi \in K$ by assumption. By (13), we have that

$$\mathrm{C}_{\sigma\circ\phi\circ\psi} = \big(1_A \otimes (\sigma \circ \phi)\big)(\mathrm{C}_\psi) = x \otimes \sigma(\phi(I_A)) = \mathrm{Tr}(\phi(I_A))\, x \otimes y = \mathrm{Tr}(\mathrm{C}_\phi)\, x \otimes y.$$

The desired result now follows from the fact that $\mathcal{P}_A \otimes \mathcal{P}_B$ generates the cone $\mathcal{S}_1$. ▫

Proposition 3.4. *Let K be a proper nonzero closed convex cone in $H(M_A, M_B)$ that satisfies the condition $\mathbb{CP}_B \circ K \circ \mathbb{CP}_A \subset K$. The following are equivalent:*

(i) $K \subset \mathbb{P}_1$.

(ii) *there exists* $\phi \in K$ *with* $\mathrm{Tr}(\mathrm{C}_\phi) > 0$.

(iii) $\mathbb{SP}_1 \subset K$.

Proof. It is clear that (i) implies (ii), as every nonzero map $\phi \in \mathbb{P}_1$ satisfies $\mathrm{Tr}(\mathrm{C}_\phi) > 0$. The implication (ii)$\Rightarrow$(iii) follows from Lemma 3.3. Suppose now that (iii) holds. Note that K° is nonzero by the assumption that K is proper and that (iii) is equivalent to $K^\circ \subset \mathbb{P}_1$, so there exists a map $\phi \in K^\circ$ satisfying $\mathrm{Tr}(\mathrm{C}_\phi) > 0$. By Theorem 3.1, we may apply Lemma 3.3 to the convex cone K° to see that $\mathbb{SP}_1 \subset K^\circ$, which implies that $K \subset \mathbb{P}_1$. ▫

It is known that every mapping cone K satisfies $\mathbb{SP}_1 \subset K \subset \mathbb{P}_1$ (see Lemma 5.1.5 in [42]). From the equivalence (i)$\Longleftrightarrow$(iii) of Proposition 3.4 together with (iii)$\Longleftrightarrow$(iv) of Theorem 3.1, we recover the well-known fact [40, 30] that K is a mapping cone if and only if K° is a mapping cone (see Theorem 6.1.3 in [42]). In particular, the convex cone $\mathbb{SP}_k$ is also a mapping cone. The condition $\mathbb{CP}_B \circ K \circ \mathbb{CP}_A \subset K$ in Proposition 3.4 cannot be replaced by the weaker condition $\mathbb{CP}_B \circ K \subset K$, as is shown in the following example.

Example 3.5. For any fixed positive map $\sigma : M_A \to M_B$, the set $K = \mathbb{CP}_B \circ \{\sigma\}$ is a convex cone. Moreover, it is clear that K is a left-mapping cone and thus $\mathcal{M}^{\mathrm{L}}_K = K$. Consider now the map σ whose Choi matrix is $e_{11} \otimes e_{11} \in M_A \otimes M_B$. For an arbitrary $\phi \in L(M_A, M_B)$, we have

$$\mathrm{C}_{\phi\circ\sigma} = e_{11} \otimes \phi(e_{11}) \in M_A \otimes M_B,$$

by (13). If the dimension of $\mathbb{C}^A$ is greater than 1, it is clear that $\mathbb{SP}_1 \not\subset K$ and so we have that $K^\circ \not\subset \mathbb{P}_1$ even though $K \subset \mathbb{P}_1$.

If K is a mapping cone in $L(M_A, M_B)$ then it is easy to see that $\{\phi \circ \mathrm{t} : \phi \in K\}$ is also a mapping cone, where t denotes the transpose map. We therefore obtain the following further examples of mapping cones defined by

$$\mathbb{P}^k := \{\phi \circ \mathrm{t} : \phi \in \mathbb{P}_k\} \qquad \text{and} \qquad \mathbb{SP}^k := \{\phi \circ \mathrm{t} : \phi \in \mathbb{SP}_k\}. \tag{17}$$

We have that $\mathbb{P}^{A \wedge B} = \mathbb{SP}^{A \wedge B}$ and this cone will be denoted by $\mathbb{CCP}$, whose elements are called *completely copositive maps*. It is also clear that the convex hull $K_1 \vee K_2$ and the intersection $K_1 \wedge K_2$ are mapping cones whenever K_1 and K_2 are mapping cones. In particular, the mapping cones

$$\mathbb{DEC} := \mathbb{CP} \vee \mathbb{CCP} \qquad \text{and} \qquad \mathbb{PPT} := \mathbb{CP} \wedge \mathbb{CCP}$$

play important roles in the theory of quantum information. Elements of $\mathbb{DEC}$ are called *decomposable* positive maps [33, 45, 34]. It is known that $\mathbb{P}_1 = \mathbb{DEC}$ if and only if $(\dim \mathbb{C}^A, \dim \mathbb{C}^B)$ is $(2, 2)$, $(2, 3)$ or $(3, 2)$ [33, 45, 5]. For a given $X \in M_A \otimes M_B$, the matrix $X^\Gamma = (\mathrm{t} \otimes 1_B)(X)$ is called the *partial transpose* of X. For a map $\phi \in L(M_A, M_B)$, we see that

$$\mathrm{C}_{\phi \circ \mathrm{t}} = \sum_{ij} e^A_{ij} \otimes \phi(e^A_{ji}) = (\mathrm{C}_\phi)^\Gamma.$$

We therefore have that $\phi \in \mathbb{PPT}$ if and only if both C_ϕ and C_ϕ^Γ are positive. Such matrices are called *positive partial transpose* (PPT). Because $\mathcal{S}_1^\Gamma = \mathcal{S}_1$, we have that ϱ is PPT for all separable states $\varrho \in \mathcal{S}_1$ [6, 25]. This is precisely the dual of the statement that $\mathbb{DEC} \subset \mathbb{P}_1$. A map $\phi \in L(M_A, M_B)$ is called a *PPT* map if $\phi \in \mathbb{PPT}$ (or equivalently C_ϕ is PPT). Therefore, a map is PPT if and only if it is both completely positive and completely copositive.

Consider now the lattice generated by the mapping cones listed in (16) and (17) with respect to the following two operations: the convex hull $K_1 \vee K_2$ and the intersection $K_1 \wedge K_2$ of closed convex cones K_1 and K_2. Mapping cones belonging to this lattice are said to be *typical* [30]. If M_A is the set of 2×2 matrices, this lattice of inclusions may be drawn as one of the following diagrams (depending on the dimension of M_B):

$$\begin{tikzcd}
& \mathbb{P}_1 & \\
\mathbb{CP} \arrow[ur] & & \mathbb{CCP} \arrow[ul] \\
& \mathbb{SP}_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{P}_1 & \\
& \mathbb{DEC} \arrow[u] & \\
\mathbb{CP} \arrow[ur] & & \mathbb{CCP} \arrow[ul] \\
& \mathbb{PPT} \arrow[ul] \arrow[ur] & \\
& \mathbb{SP}_1 \arrow[u] &
\end{tikzcd}$$

If the dimension of $\mathbb{C}^B$ is 2 or 3 then we have the diagram on the left. If $\dim(\mathbb{C}^B) \geqslant 4$ then we have the diagram on the right. It is known that there exist mapping cones which are not typical [20].

Example 3.6. Consider again convex cone $K = (\mathbb{P}_1^\circ \vee \{\sigma\})^{\circ\circ}$ from Example 2.5. We will show that $K \subsetneqq K^{\circleddash\circleddash}$. Toward this goal, consider the map $\tau : M_2 \to M_2$ whose Choi matrix is given by

$$\mathrm{C}_\tau = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}.$$

Note that we have $\tau^* = \tau$. For a map $\phi : M_2 \to M_2$ with Choi matrix given by $\mathrm{C}_\phi = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ for matrices $x, y, z, w \in M_2$, the Choi matrices of the compositions $\phi \circ \tau^*$ and $\phi \circ \sigma^*$ are $\mathrm{C}_{\phi\circ\tau^*} = \begin{pmatrix} w & z \\ y & x+w \end{pmatrix}$ and $\mathrm{C}_{\phi\circ\sigma^*} = \begin{pmatrix} x+w & y \\ z & w \end{pmatrix}$ respectively, where σ is the map from Example 2.5. If it holds that $\phi \in K^{\circleddash}$ then $\mathrm{C}_{\phi\circ\sigma^*}$ is positive and it is clear that $\mathrm{C}_{\phi\circ\tau^*}$ is also positive. It follows that that $\mathrm{C}_{\tau\circ\phi^*}$ is positive for every $\phi \in K^{\circleddash}$, from which we conclude that $\tau \in K^{\circleddash\circleddash}$. Now, toward a contradiction, suppose that $\tau \in K$. Since K is the convex cone generated by $\mathbb{SP}_1$ and σ, there must exist a map $\phi \in \mathbb{SP}_1$ and a number $\lambda \geqslant 0$ such that $\tau = \phi + \lambda\sigma$. Comparing the Choi matrices C_τ and C_σ, we see that we must have $\lambda = 0$ and thus $\tau \in \mathbb{SP}_1$. However, it is clear C_τ is entangled as it is not PPT, and thus $\tau \notin \mathbb{SP}_1$. This is in contradiction to the assumption that $\tau \in K$. We therefore conclude that $K \subsetneqq K^{\circleddash\circleddash}$. □

Let K be an arbitrary closed convex cone of positive maps. Recall that the smallest left- (respectively right-) mapping cone $\mathcal{M}_K^{\mathrm{L}}$ (respectively $\mathcal{M}_K^{\mathrm{R}}$) containing K is given by $(\mathbb{CP} \circ K)^{\circ\circ} = K^{\circleddash\circ}$ (respectively $(K \circ \mathbb{CP})^{\circ\circ} = K^{\circleddash\circ}$). By the same argument as in Proposition 2.2, we see that the smallest mapping cone $\mathcal{M}_K$ containing K is given by $\mathcal{M}_K = (\mathbb{CP} \circ K \circ \mathbb{CP})^{\circ\circ}$. Moreover, it is clear that $(\mathbb{CP} \circ K) \cup (K \circ \mathbb{CP}) \subset \mathbb{CP} \circ K \circ \mathbb{CP}$. We therefore obtain the following lattice of (not necessarily strict) inclusions:

$$\begin{tikzcd} & & \mathcal{M}_K^{\mathrm{L}} \arrow[dr] & & \\ K \arrow[r] & \mathcal{M}_K^{\mathrm{L}} \wedge \mathcal{M}_K^{\mathrm{R}} \arrow[ur] \arrow[dr] & & \mathcal{M}_K^{\mathrm{L}} \vee \mathcal{M}_K^{\mathrm{R}} \arrow[r] & \mathcal{M}_K \\ & & \mathcal{M}_K^{\mathrm{R}} \arrow[ur] & & \end{tikzcd} \tag{18}$$

Even though a closed convex cone K is a mapping cone if and only if it is both a left- and right-mapping cone, we shall see in the following example that $\mathcal{M}_K^{\mathrm{L}} \vee \mathcal{M}_K^{\mathrm{R}}$ need not coincide with $\mathcal{M}_K$. In particular, we will see that every inclusion in (18) is strict for the convex cone K from Example 2.5. For this cone, $\mathcal{M}_K^{\mathrm{R}}$ is a right-mapping cone which is not a left-mapping cone, as it holds that $\mathcal{M}^{\mathrm{L}}_{\mathcal{M}_K^{\mathrm{R}}} = \mathcal{M}_K$ in general.

Example 3.7. Consider again the closed convex cone $K = (\mathbb{SP}_1 \vee \{\sigma\})^{\circ\circ}$, where σ is the map defined in Example 2.5. Taking the duals of the convex cones in (18) yields the following chain of inclusions:

$$
\begin{tikzcd}
 & & K^{\olessthan} \arrow[dr] & & \\
(\mathbb{CP} \circ K \circ \mathbb{CP})^{\circ} \arrow[r] & K^{\olessthan} \wedge K^{\ogreaterthan} \arrow[ur] \arrow[dr] & & K^{\olessthan} \vee K^{\ogreaterthan} \arrow[r] & K^{\circ} \\
 & & K^{\ogreaterthan} \arrow[ur] & &
\end{tikzcd}
\tag{19}
$$

We will show that every inclusion in the above diagram is strict. It is clear that every inclusion in the diamond part of the lattice is strict, since $K^{\olessthan} \nsubseteq K^{\ogreaterthan}$ and $K^{\ogreaterthan} \nsubseteq K^{\olessthan}$ by Example 2.5. To see that the first inclusion is strict, consider the map ϕ_α defined for a fixed positive number $\alpha > 0$ by

$$\phi_\alpha\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \begin{pmatrix} \alpha x_{11} & x_{12} \\ x_{21} & x_{22}/\alpha \end{pmatrix}.$$

For $\alpha > 0$ and $\beta > 0$, we have that

$$\mathrm{C}_{\phi_\beta \circ \sigma \circ \phi_\alpha} = \begin{pmatrix} \alpha\beta & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta/\alpha & \cdot \\ 1 & \cdot & \cdot & 1/\alpha\beta \end{pmatrix} \in \mathrm{C}_{\mathbb{CP} \circ K \circ \mathbb{CP}}. \tag{20}$$

Consider now the map having the form $\psi = \phi_{[a,b,c,d]}$ as defined in Example 2.5, where we choose $a = b = c = d = \frac{1}{\sqrt{2}}$. It is clear that $\psi \in K^{\olessthan} \wedge K^{\ogreaterthan}$. Toward a contradiction, suppose that $\psi \in (\mathbb{CP} \circ K \circ \mathbb{CP})^{\circ}$. It must be the case that

$$\langle \psi, \phi_\beta \circ \sigma \circ \phi_\alpha \rangle = \frac{1}{\sqrt{2}}\left(\alpha\beta + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right) \geqslant 2$$

for every $\alpha, \beta > 0$ by (20). However, the above inequality fails to hold for $\alpha = 2$ and $\beta = \frac{1}{\sqrt{5}}$. It follows that $\psi \notin (\mathbb{CP} \circ K \circ \mathbb{CP})^{\circ}$ and thus $(\mathbb{CP} \circ K \circ \mathbb{CP})^{\circ} \subsetneqq K^{\olessthan} \wedge K^{\ogreaterthan}$.

Now consider maps of the form $\phi_{[a]} := \phi_{[a,a,a,a]}$. We see that $\phi_{[a]} \in \mathbb{P}_1$ if and only if $a \geqslant \frac{1}{2}$ and that $\phi_{[a]} \in K^{\circ}$ if and only if $a \geqslant \frac{2}{3}$. On the other hand, we have $\phi_{[a]} \in K^{\olessthan}$ if and only if $\phi_{[a]} \in K^{\ogreaterthan}$ if and only if $a \geqslant \frac{1}{\sqrt{2}}$. It follows that the last inclusion in (19) is also strict. Finally, we note that the mapping cone $\mathcal{M}_K$ is not typical (see Theorem 18 of [20]). □

4. Duality through ampliation

In this section, we discuss relationships between one-sided mapping cones and ampliation maps $1_A \otimes \phi$ and $\phi \otimes 1_B$. We will see that many dual objects—such as $K^{\olessthan}$ and $K^{\ogreaterthan}$—can be described in terms of ampliation. This allows us to recover many results in quantum information theory—such as separability criteria through ampliation of positive maps [17] as well as characterizations of entanglement-breaking maps [18] and Schmidt number [44]—in a single framework. We also recover some characterizations of decomposable maps due to the third author [35] and properties of k-positive maps due to the second author [12]. We stress that the above-mentioned characterizations hold if and only if the involved convex cones are one-sided mapping cones. For a convex cone K in $H(M_A \otimes M_B)$, we denote by C_K the convex cone in the tensor product $M_A \otimes M_B$ defined as $\mathrm{C}_K = \{\mathrm{C}_\phi : \phi \in K\}$.

Proposition 4.1. *For a closed convex cone K in $H(M_A, M_B)$ and a map $\phi \in H(M_A, M_B)$, the following are equivalent:*

(i) $\phi \in K^{\ominus}$.
(ii) $(1_A \otimes \phi)(\mathrm{C}_\psi) \in \mathrm{C}_{K^\circ}$ *for every* $\psi \in \mathbb{CP}_A$.
(iii) $(1_B \otimes \psi)(\mathrm{C}_{\phi^*}) \in \mathrm{C}_{K^{*\circ}}$ *for every* $\psi \in \mathbb{CP}_A$.
(iv) $(1_A \otimes \sigma^*)(\mathrm{C}_\phi) \in \mathcal{P}_{AA}$ *for every* $\sigma \in K$.
(v) $(1_A \otimes \phi^*)(\mathrm{C}_\sigma) \in \mathcal{P}_{AA}$ *for every* $\sigma \in K$.

It is useful to remember the domains and the ranges of the ampliation maps in Proposition 4.1. We have

$$\begin{aligned}
1_A \otimes \phi &: M_A \otimes M_A \to M_A \otimes M_B \\
1_B \otimes \psi &: M_B \otimes M_A \to M_B \otimes M_A \\
1_A \otimes \sigma^* &: M_A \otimes M_B \to M_A \otimes M_A \\
1_A \otimes \phi^* &: M_A \otimes M_B \to M_A \otimes M_A
\end{aligned}$$

for $\sigma \in K$ and $\psi \in \mathbb{CP}_A$.

Proof. Note that statements (i) and (ii) are equivalent to the following two statements, respectively:

(i′) $\langle \phi, \sigma \circ \psi^* \rangle \geqslant 0$ for every $\psi \in \mathbb{CP}_A$ and $\sigma \in K$.
(ii′) $\langle (1_A \otimes \phi)(\mathrm{C}_\psi), \mathrm{C}_\sigma \rangle_{AB} \geqslant 0$ for every $\psi \in \mathbb{CP}_A$ and $\sigma \in K$.

From (13), for every map $\psi : M_A \to M_A$ we have that

$$\langle \phi, \sigma \circ \psi^* \rangle = \langle \phi \circ \psi, \sigma \rangle = \langle \mathrm{C}_{\phi \circ \psi}, \mathrm{C}_\sigma \rangle_{AB} = \langle (1_A \otimes \phi)(\mathrm{C}_\psi), \mathrm{C}_\sigma \rangle_{AB},$$

which proves the equivalence (i)$\Longleftrightarrow$(ii). Taking the flip of the above identity yields

$$\langle \phi, \sigma \circ \psi^* \rangle = \langle \sigma^*, \psi^* \circ \phi^* \rangle = \langle \mathrm{C}_{\sigma^*}, (1_B \otimes \psi^*)(\mathrm{C}_{\phi^*}) \rangle_{BA},$$

which proves the equivalence (i)$\Longleftrightarrow$(iii) since $\mathbb{CP}_A^* = \mathbb{CP}_A$. On the other hand, we also have that

$$\begin{aligned}
\langle \phi, \sigma \circ \psi^* \rangle = \langle \sigma^* \circ \phi, \psi^* \rangle &= \langle (1_A \otimes \sigma^*)(\mathrm{C}_\phi), \mathrm{C}_{\psi^*} \rangle_{AA} \\
= \langle \psi, \phi^* \circ \sigma \rangle &= \langle \mathrm{C}_\psi, (1_A \otimes \phi^*)(\mathrm{C}_\sigma) \rangle_{AA}.
\end{aligned}$$

This completes the proof. □

Suppose that $M_A = M_B$. For a fixed convex cone K in $H(M_A, M_A)$, the third author [36] defined the set $P_K \subset M_A \otimes M_A$ as

$$P_K = \{\varrho \in M_A \otimes M_A : (1_A \otimes \sigma)(\varrho) \text{ is positive for every } \sigma \in K\}.$$

A map $\phi : M_A \to M_A$ is called *K-positive* [36] if $\langle \varrho, \phi \rangle \geqslant 0$ holds for every $\varrho \in K$. By the equivalence of statements (i) and (iv) of Proposition 4.1, we see that $P_K = \mathrm{C}_{K^{*\circledcirc}}$ and moreover that a map ϕ is K-positive if and only if $\phi \in K^{*\circledcirc\circ}$. If it is the case that $K = K^*$, we see that a map $\phi : M_A \to M_A$ is K-positive if and only if $\phi \in K^{\circledcirc\circ}$, which is equivalent to the condition that ϕ is the sum of maps of the form $\sigma \circ \psi$ for maps $\sigma \in K$ and $\psi \in \mathbb{CP}$ by statement (i) of Proposition 2.2. This recovers a result from [36], where it was also shown that a map ϕ is $\mathbb{CP}$-positive if and only if $\phi \in \mathbb{CP}$. This is a special case of the following characterization of right-mapping cones in terms of K-positivity, which follows trivially form Proposition 2.1.

Corollary 4.2. *Suppose that K is a closed convex cone in $H(M_A, M_A)$ such that $K^* = K$. The following are equivalent:*

(i) *For all maps $\phi : M_A \to M_A$, $\phi \in K$ if and only if ϕ is K-positive.*
(ii) $K \circ \mathbb{CP}_A \subset K$.

We also have the following characterization of $K^{\circledcirc}$ by ampliation on the right.

Proposition 4.3. *For a closed convex cone K in $H(M_A, M_B)$ and a map $\phi \in H(M_A, M_B)$, the following are equivalent:*

(i) $\phi \in K^{\circledcirc}$.
(ii) $(\phi^* \otimes 1_B)(\mathrm{C}_\psi) \in \mathrm{C}_{K^\circ}$ *for every* $\psi \in \mathbb{CP}_B$.
(iii) $(\psi \otimes 1_A)(\mathrm{C}_{\phi^*}) \in \mathrm{C}_{K^{*\circ}}$ *for every* $\psi \in \mathbb{CP}_B$.
(iv) $(\sigma \otimes 1_B)(\mathrm{C}_\phi) \in \mathcal{P}_{BB}$ *for every* $\sigma \in K$.
(v) $(\phi \otimes 1_B)(\mathrm{C}_\sigma) \in \mathcal{P}_{BB}$ *for every* $\sigma \in K$.

Proof. For all maps $\sigma \in K$ and $\phi \in \mathbb{CP}_B$, we have the following maps:

$$\begin{aligned} \phi^* \otimes 1_B &: M_B \otimes M_B \to M_A \otimes M_B \\ \psi \otimes 1_A &: M_B \otimes M_A \to M_B \otimes M_A \\ \sigma \otimes 1_B &: M_A \otimes M_B \to M_B \otimes M_B \\ \phi \otimes 1_B &: M_A \otimes M_B \to M_B \otimes M_B. \end{aligned}$$

First recall from (13) that

$$(1_B \otimes \phi)(\mathrm{C}_{\sigma^*}) = \mathrm{C}_{\phi\circ\sigma^*} \in M_B \otimes M_B. \tag{21}$$

Recalling that $\mathrm{C}_{\sigma\circ\phi^*}$ is the flip of $\mathrm{C}_{\phi\circ\sigma^*}$ from (3), taking the flip the identity in (21) yields

$$(\phi \otimes 1_B)(\mathrm{C}_\sigma) = \mathrm{C}_{\sigma\circ\phi^*}.$$

We therefore obtain the following identities for all maps $\sigma \in K$ and $\phi \in \mathbb{CP}_B$:

$$\begin{aligned} \langle \phi, \psi \circ \sigma \rangle = \langle \psi^* \circ \phi, \sigma \rangle &= \langle (\phi^* \otimes 1_B)(\mathrm{C}_{\psi^*}), \mathrm{C}_\sigma \rangle_{AB}, \\ = \langle \sigma^*, \phi^* \circ \psi \rangle &= \langle \mathrm{C}_{\sigma^*} (\psi^* \otimes 1_A)(\mathrm{C}_{\phi^*}) \rangle_{BA} \\ = \langle \phi \circ \sigma^*, \psi \rangle &= \langle (\sigma \otimes 1_B)(\mathrm{C}_\phi), \mathrm{C}_\psi \rangle_{BB} \end{aligned}$$

$$= \langle \psi^*, \sigma \circ \phi^* \rangle = \langle \mathrm{C}_{\psi^*}, (\phi \otimes 1_B)(\mathrm{C}_\sigma) \rangle_{BB}.$$

Using the fact that $\mathbb{CP}_B = \mathbb{CP}_B^*$, applying an argument similar to the one in the proof of Proposition 4.1 yields the desired conclusion. □

We now apply Proposition 2.1 to Proposition 4.1 and Proposition 4.3 to obtain the following characterizations of one-sided mapping cones in terms of ampliation maps.

Theorem 4.4. *For a closed convex cone K in $H(M_A, M_B)$, the following are equivalent:*

(i) $K \circ \mathbb{CP}_{AA} \subset K$.

(ii) *For all maps ϕ, $\phi \in K^\circ$ if and only if $(1_A \otimes \phi)(\mathrm{C}_\psi) \in \mathrm{C}_{K^\circ}$ for every $\psi \in \mathbb{CP}_A$.*

(iii) *For all maps ϕ, $\phi \in K^\circ$ if and only if $(1_B \otimes \psi)(\mathrm{C}_{\phi^*}) \in \mathrm{C}_{K^{*\circ}}$ for every $\psi \in \mathbb{CP}_A$.*

(iv) *For all maps ϕ, $\phi \in K^\circ$ if and only if $(1_A \otimes \sigma^*)(\mathrm{C}_\phi) \in \mathcal{P}_{AA}$ for every $\sigma \in K$.*

(v) *For all maps ϕ, $\phi \in K^\circ$ if and only if $(1_A \otimes \phi^*)(\mathrm{C}_\sigma) \in \mathcal{P}_{AA}$ for every $\sigma \in K$.*

We also have the following equivalence statements:

(vi) $\mathbb{CP}_{BB} \circ K \subset K$.

(vii) *For all maps ϕ, $\phi \in K^\circ$ if and only if $(\phi^* \otimes 1_B)(\mathrm{C}_\psi) \in \mathrm{C}_{K^\circ}$ for every $\psi \in \mathbb{CP}_B$.*

(viii) *For all maps ϕ, $\phi \in K^\circ$ if and only if $(\psi \otimes 1_A)(\mathrm{C}_{\phi^*}) \in \mathrm{C}_{K^{*\circ}}$ for every $\psi \in \mathbb{CP}_B$.*

(ix) *For all maps ϕ, $\phi \in K^\circ$ if and only if $(\sigma \otimes 1_B)(\mathrm{C}_\phi) \in \mathcal{P}_{BB}$ for every $\sigma \in K$.*

(x) *For all maps ϕ, $\phi \in K^\circ$ if and only if $(\phi \otimes 1_B)(\mathrm{C}_\sigma) \in \mathcal{P}_{BB}$ for every $\sigma \in K$.*

If K is a mapping cone then statements (i) and (vi) in Theorem 4.4 are trivially true. We therefore have the following characterization of mapping cones.

Corollary 4.5. *Let K be a mapping cone in $H(M_A, M_B)$. For a map $\phi \in L(M_A, M_B)$, the following statements are equivalent to the statement that $\phi \in K$:*

(i) $1_A \otimes \phi$ *sends* $\mathcal{P}_{AA}$ *into* C_K,

(ii) $1_B \otimes \psi$ *sends* C_{ϕ^*} *into* C_{K^*} *for every* $\psi \in \mathbb{CP}_A$,

(iii) $1_A \otimes \sigma^*$ *sends* C_ϕ *into* $\mathcal{P}_{AA}$ *for every* $\sigma \in K^\circ$,

(iv) $1_A \otimes \phi^*$ *sends* C_{K° *into* $\mathcal{P}_{AA}$,

(v) $\phi^* \otimes 1_B$ *sends* $\mathcal{P}_{BB}$ *into* C_K,

(vi) $\psi \otimes 1_A$ *sends* C_{ϕ^*} *into* C_{K^*} *for every* $\psi \in \mathbb{CP}_B$,

(vii) $\sigma \otimes 1_B$ *sends* C_ϕ *into* $\mathcal{P}_{BB}$ *for every* $\sigma \in K^\circ$,

(viii) $\phi \otimes 1_B$ *sends* C_{K° *into* $\mathcal{P}_{BB}$.

Applying statement (viii) to the cone $K = \mathbb{DEC}$ allows us to recover the result in [35], which states that a map $\phi \in H(M_A, M_B)$ is decomposable if and only if $\phi \otimes 1_B$ sends PPT matrices in $M_A \otimes M_B$ into the cone of positive matrices $\mathcal{P}_{BB}$. Applying statement (viii) to the cone $K = \mathbb{P}_k$ shows that a map ϕ is k-positive if and only if $\phi \otimes 1_B$ sends every matrix with Schmidt number at most k to a positive matrix, which recovers a result in [12, Theorem 3.3]. The notion of k-positivity can be also characterized in terms of the left-side ampliation as shown by the following corollary.

Corollary 4.6. *Let $\phi \in L(M_A, M_B)$ be a map. The condition that ϕ is k-positive is equivalent to each of the following statements:*

(i) $1_A \otimes \phi : M_A \otimes M_A \to M_A \otimes M_B$ *sends* $\mathcal{P}_{AA}$ *into* $\mathcal{BP}_k$,

(ii) $1_B \otimes \psi : M_B \otimes M_A \to M_B \otimes M_A$ *sends* C_{ϕ^*} *into* $\mathcal{BP}_k$ *for every* $\psi \in \mathbb{CP}_A$,

(iii) $1_A \otimes \sigma^* : M_A \otimes M_B \to M_A \otimes M_A$ *sends* C_ϕ *into* $\mathcal{P}_{AA}$ *for every* $\sigma \in \mathbb{SP}_k$,

(iv) $1_A \otimes \phi^* : M_A \otimes M_B \to M_A \otimes M_A$ *sends* $\mathcal{S}_k$ *into* $\mathcal{P}_{AA}$.

Taking the cone $K = \mathbb{SP}_1$ and applying Corollary 4.5, we see that a map ϕ is entanglement breaking (i.e., C_ϕ is separable) if and only if $1_A \otimes \phi$ sends every state to a separable state. This recovers a result from [18]. A similar result for k-superpositive maps, which can be found in [9], can be stated as follows. A map ϕ is k-superpositive if and only if $\phi \otimes 1_B$ sends every state to a state with Schmidt number $\leqslant k$.

We may interpret Corollary 4.5 in terms of Choi matrices C_ϕ instead of the map ϕ itself. For example, for the convex cone $K = \mathbb{SP}_1$, from statement (vii) we have that $\varrho \in \mathrm{C}_{\mathbb{SP}_1} = \mathcal{S}_1$ if and only if $\sigma \otimes 1_B$ sends ϱ to a positive matrix for every $\sigma \in \mathbb{P}_1 = \mathbb{SP}_1^\circ$. That is, we see that a state ϱ is separable if and only if $(\sigma \otimes 1_B)(\varrho)$ is positive for every positive map σ [17]. Similarly, we also have that ϱ has Schmidt number at most k if and only if $(\phi \otimes 1_B)(\varrho)$ is positive for every k-positive map σ [44]. We conclude this section by presenting the following further characterizations of separability that are found by applying statements (ii), (iii), (vi) and (vii) of Corollary 4.5. (See, e.g., [17].)

Corollary 4.7. *Let $\varrho \in M_A \otimes M_B$ be a state. The condition that ϱ is separable is equivalent to each of the following statements:*

(i) $\psi \otimes 1_B : M_A \otimes M_B \to M_A \otimes M_B$ *sends* ϱ *into* $\mathcal{S}_1$ *for every* $\psi \in \mathbb{CP}_A$.

(ii) $1_A \otimes \sigma^* : M_A \otimes M_B \to M_A \otimes M_A$ *sends* ϱ *into* $\mathcal{P}_{AA}$ *for every* $\sigma \in \mathbb{P}_1$.

(iii) $1_A \otimes \psi : M_A \otimes M_B \to M_A \otimes M_B$ *sends* ϱ *into* $\mathcal{S}_1$ *for every* $\psi \in \mathbb{CP}_B$.

(iv) $\sigma \otimes 1_B : M_A \otimes M_B \to M_B \otimes M_B$ *sends* ϱ *into* $\mathcal{P}_{BB}$ *for every* $\sigma \in \mathbb{P}_1$.

5. PPT-square conjecture

The notion of positive partial transpose plays an important role in quantum information theory, as evidenced by the PPT criterion for separability ($\mathcal{S}_1 \subset \mathcal{PPT}$). The following conjecture was recently proposed by Christandl in [28].

Conjecture 5.1. *If ϕ and ψ are PPT maps in M_A then $\psi \circ \phi$ is entanglement breaking.*

This conjecture is called the PPT-square conjecture. In our notation, the conjecture can be stated as the following inclusion:

$$\mathbb{PPT} \circ \mathbb{PPT} \subset \mathbb{SP}_1.$$

The conjecture has been supported by the following results. If ϕ is a unital or trace preserving PPT map then $\lim_{k\to\infty} d(\phi^k, \mathbb{SP}_1) \to 0$ [22]. If ϕ is a unital PPT map then there is a positive integer n such that $\phi^n \in \mathbb{SP}_1$ [26]. Moreover, the conjecture has been shown recently to be true in the case when M_A is the set of 3×3 matrices [2, 7]. See also [10, 14, 24] for related results.

Choosing the cones $K_0 = K_1 = \mathbb{PPT}$ and $K_2 = \mathbb{P}_1$, applying the equivalences in (7) yields the following equivalences:

$$\mathbb{PPT} \circ \mathbb{PPT} \subset \mathbb{SP}_1 \iff \mathbb{PPT} \circ \mathbb{P}_1 \subset \mathbb{DEC} \iff \mathbb{P}_1 \circ \mathbb{PPT} \subset \mathbb{DEC}.$$

The equivalence of the first and the third of the above statements was shown in [7]. From the identity $\mathbb{PPT} = \mathbb{PPT} \circ \mathbb{CP} = \mathbb{PPT} \circ \mathbb{DEC}$, we also have the following equivalences:

$$\mathbb{PPT} \circ \mathbb{PPT} \subset \mathbb{SP}_1 \iff \mathbb{PPT} \circ \mathbb{CP} \circ \mathbb{PPT} \subset \mathbb{SP}_1 \iff \mathbb{PPT} \circ \mathbb{DEC} \circ \mathbb{PPT} \subset \mathbb{SP}_1.$$

By the identity in (6), we have that

$$\langle \phi_0 \circ \phi_1 \circ \phi_2, \phi_3 \rangle = \langle \phi_1, \phi_0^* \circ \phi_3 \circ \phi_2^* \rangle,$$

for all maps $\phi_0 \in K_0$, $\phi_1 \in K_1$, $\phi_2 \in K_2$, and $\phi_3 \in K_3$, which yields the equivalence

$$K_0 \circ K_1 \circ K_2 \subset K_3^\circ \iff K_0^* \circ K_3 \circ K_2^* \subset K_1^\circ$$

for arbitrary closed convex cones K_0, K_1 K_2 and K_3. This observation yields the equivalences

$$\mathbb{PPT} \circ \mathbb{CP} \circ \mathbb{PPT} \subset \mathbb{SP}_1 \iff \mathbb{PPT} \circ \mathbb{P}_1 \circ \mathbb{PPT} \subset \mathbb{CP},$$
$$\mathbb{PPT} \circ \mathbb{DEC} \circ \mathbb{PPT} \subset \mathbb{SP}_1 \iff \mathbb{PPT} \circ \mathbb{P}_1 \circ \mathbb{PPT} \subset \mathbb{PPT}.$$

The above equivalences may be summarized by the following theorem.

Theorem 5.2. *The following statements are equivalent:*

(i) $\mathbb{PPT} \circ \mathbb{PPT} \subset \mathbb{SP}_1$.
(ii) $\mathbb{PPT} \circ \mathbb{P}_1 \subset \mathbb{DEC}$.
(iii) $\mathbb{P}_1 \circ \mathbb{PPT} \subset \mathbb{DEC}$.
(iv) $\mathbb{PPT} \circ \mathbb{CP} \circ \mathbb{PPT} \subset \mathbb{SP}_1$.
(v) $\mathbb{PPT} \circ \mathbb{DEC} \circ \mathbb{PPT} \subset \mathbb{SP}_1$.
(vi) $\mathbb{PPT} \circ \mathbb{P}_1 \circ \mathbb{PPT} \subset \mathbb{CP}$.
(vii) $\mathbb{PPT} \circ \mathbb{P}_1 \circ \mathbb{PPT} \subset \mathbb{PPT}$.

Using the identity in (12), we may also formulate the PPT-square conjecture in terms of block matrices as follows.

Conjecture 5.3. *For all states $\varrho_1, \varrho_2 \in M_A \otimes M_A$, if both ϱ_1 and ϱ_2 are PPT then the block-wise summation of the block Schur product $\varrho_1 \square \varrho_2 \in M_A \otimes (M_A \otimes M_A)$ is separable.*

References

[1] T. Ando, *Cones and norms in the tensor product of matrix spaces,* Linear Alg. Appl. **379** (2004), 3–41.
[2] L. Chen, Y. Yang and W.-S. Tang, *Positive-partial-transpose square conjecture for $n = 3$,* Phys. Rev. A **99** (2019), 012337.

[3] S.-J. Cho, S.-H. Kye and S. G. Lee, *Generalized Choi maps in 3-dimensional matrix algebras,* Linear Alg. Appl. **171** (1992), 213–224.
[4] M.-D. Choi, *Completely positive linear maps on complex matrices,* Linear Alg. Appl. **10** (1975), 285–290.
[5] M.-D. Choi, *Positive semidefinite biquadratic forms,* Linear Alg. Appl. **12** (1975), 95–100.
[6] M.-D. Choi, *Positive linear maps,* Operator Algebras and Applications (Kingston, 1980), pp. 583–590, Proc. Sympos. Pure Math. Vol 38. Part 2, Amer. Math. Soc., 1982.
[7] M. Christandl, A. Müller-Hermes and M. M. Wolf, *When Do Composed Maps Become Entanglement Breaking?* Ann. Henri Poincaré **20** (2019), 2295–2322.
[8] E. Christensen, *Decomposition of block Schur products,* preprint. arXiv: 1811.03668
[9] D. Chruściński and A. Kossakowski, *On Partially Entanglement Breaking Channels,* Open Sys. Information Dyn. **13** (2006), 17–26.
[10] B. Collins, Z. Yin and P. Zhong, *The PPT-square conjecture holds generically for some classes of independent states,* J. Phys. A: Math. Theor. **51** (2018), 425301.
[11] J. de Pillis, *Linear transformations which preserve Hermitian and positive semidefinite operators,* Pacific J. Math. **23** (1967), 129–137.
[12] M.-H. Eom and S.-H. Kye, *Duality for positive linear maps in matrix lgebras,* Math. Scand. **86** (2000), 130–142.
[13] K. H. Han and S.-H. Kye, On the convex cones arising from classifications of partial entanglement in the three qubit system, J. Phys. A: Math. Theor., **53** (2020), 015301.
[14] E. P. Hanson, C. Rouze and D. S. Franca, *On entanglement breaking times for quantum Markovian evolutions and the PPT^2 conjecture,* preprint. arXiv 1902.08173.
[15] A. S. Holevo, *Entanglement-breaking channels in infinite dimensions,* Probl. Inform. Transmission **44** (2008), 171–184.
[16] R. A. Horn and R. Mathias, *Block-matrix geberalizations of Schur's basic theorems on Hadamard products,* Linear Alg. Appl. **172** (1992), 337–346.
[17] M. Horodecki, P. Horodecki and R. Horodecki, *Separability of mixed states: necessary and sufficient conditions,* Phys. Lett. A **223** (1996), 1–8.
[18] M. Horodecki, P. W. Shor and M. B. Ruskai, *Entanglement braking channels,* Rev. Math. Phys. **15** (2003), 629–641.
[19] A. Jamiołkowski, *Linear transformations which preserve trace and positive semidefinite operators,* Rep. Math. Math. **3** (1972), 275–278.
[20] N. Johnston, Ł. Skowronek and E. Størmer, *Generation of mapping cones from small sets* Linear Algebra Appl. **438** (2013), 3062–3075.
[21] N. Johnston and E. Størmer, *Mapping cones are operator systems* Bull. London Math. Soc. **44** (2012), 738–748.
[22] M. Kennedy, N. Manor and V. I. Paulsen, *Composition of PPT Maps,* Quantum Inform. Comput. **18** (2018), 0472–0480.
[23] S.-H. Kye, *Facial structures for various notions of positivity and applications to the theory of entanglement,* Rev. Math. Phys. **25** (2013), 1330002.
[24] A. Müller-Hermes, *Decomposability of linear maps under tensor powers,* J. Math. Phys. **59** (2018), 102203.
[25] A. Peres, *Separability Criterion for Density Matrices,* Phys. Rev. Lett. **77** (1996), 1413–1415.
[26] M. Rahaman, S. Jaques and V. I. Paulsen, *Eventually entanglement breaking maps,* J. Math. Phys. **59** (2018), 062201.
[27] M. B. Ruskai, *Qubit entanglement breaking channels,* Rev. Math. Phys. **15** (2003), 643–662.
[28] M. B. Ruskai, M. Junge, D. Kribs, P. Hayden, A. Winter, *Operator structures in quantum information theory,* Final Report, Banff International Research Station (2012).
[29] P. W. Shor, *Additivity of the classical capacity of entanglement-breaking quantum channels,* J. Math. Phys. **43** (2002), 4334–4340.
[30] Ł. Skowronek, *Cones with a mapping cone symmetry in the finite-dimensional case,* Linear Algebra Appl. **435** (2011), 361–370.
[31] Ł. Skowronek, E. Størmer, and K. Zyczkowski, *Cones of positive maps and their duality relations,* J. Math. Phys. **50**, (2009), 062106.
[32] W. F. Stinespring, *Positive functions on C^*-algebras,* Proc. Amer. Math. Soc. **6,** (1955), 211–216.
[33] E. Størmer, *Positive linear maps of operator algebras,* Acta Math. **110** (1963), 233–278.

[34] E. Størmer, *Decomposition of positive projections on C^*-algebras,* Math. Ann. **247** (1980), 21–41.
[35] E. Størmer, *Decomposable positive maps on C^*-algebras,* Proc. Amer. Math. Soc. **86** (1982), 402–404.
[36] E. Størmer, *Extension of positive maps into $B(\mathcal{H})$,* J. Funct. Anal. **66** (1986), 235-254.
[37] E. Størmer, *Separable states and positive maps,* J. Funct. Anal. **254** (2008), 2303–2312.
[38] E. Størmer, *Separable states and positive maps II,* Math. Scand. **105** (2009), 188–198.
[39] E. Størmer, *Duality of cones of positive maps,* Münster J. Math. **2** (2009), 299–309.
[40] E. Størmer, *Mappings of positive maps,* Math. Scand. **108** (2011), 223–231.
[41] E. Størmer, *Tensor products of Positive maps of matrix algebras,* Math. Scand. **111** (2012), 5–11.
[42] E. Størmer, Positive Linear Maps of Operator Algebras, Springer-Verlag, 2013.
[43] E. Størmer, *Mapping cones and separable states,* Positivity 22 (2018), 493–499.
[44] B. M. Terhal and P. Horodecki, *A Schmidt number for density matrices* Phys. Rev. A **61** (2000), 040301.
[45] S. L. Woronowicz, *Positive maps of low dimensional matrix algebras,* Rep. Math. Phys. **10** (1976), 165–183.

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