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Combinatorial Patchworking, Real Tropical Curves and Hyperbolic Varieties

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Preface

The thesis is a collection of three papers in the field of real algebraic geometry, presented in chronological order of writing. The first paper is about hyperbolic varieties, the second one about hyperbolic curves near the tropical limit, and the third one about real algebraic curves near the tropical limit, so that the second paper connects the two others. The research is funded by the Trond Mohn Stiftelse (TMS) project "Algebraic and topological cycles in complex and tropical geometry".

Cédric Le Texier
Oslo, November 2021

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Oslo, November 2021

List of Papers

Paper I

M. Kummer, C. Le Texier and M. Manzaroli. 'Real fibered morphisms of del Pezzo surfaces and conic bundles'. *arXiv preprint arXiv:2101.08703* , submitted for publication.

Paper II

C. Le Texier 'Hyperbolic plane curves near the non-singular tropical limit'. *arXiv preprint arXiv:2109.14961*

Paper III

C. Le Texier 'Topology of real algebraic curves near the non-singular tropical limit'. *arXiv preprint arXiv:2111.08607*

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Chapter 1

Introduction

In this thesis, we study hyperbolic varieties and the topology of real algebraic varieties, which are two important subjects in real algebraic geometry, and the link between those subjects and tropical geometry.

1.1 Real algebraic geometry

Real algebraic geometry is the study of geometric objects which are the set of solutions of a system of polynomial equations with real coefficients. This subject has connections to many areas in mathematics such as analytic geometry, algebraic topology and analysis, as well as many applications in interdisciplinary fields such as computer-aided design, optimisation, computer vision and robotics. The three papers presented in this thesis have real algebraic geometry as their common theme.

The ground field will be a *real closed field* $\mathfrak{R}$, which is a field such that its extension by the square root of -1 is algebraically closed [BCR13, Theorem 1.2.2]. By Tarski-Seindenberg's principle [BCR13, Theorem 1.4.2], a statement of the first order of logic is true over every real closed field if and only if it is true over the field of real numbers $\mathbb{R}$. Therefore, a lot of questions over any real closed field can be treated as questions over $\mathbb{R}$, and vice versa.

An *affine real algebraic variety* over a real closed field $\mathfrak{R}$ is a topological space X isomorphic to a real algebraic set $V(f_1, \dots, f_m) \subset \mathfrak{R}^n$ equipped with both Zariski and Euclidean topology, where

$$V(f_1, \dots, f_m) := \{x \in \mathfrak{R}^n \mid f_1(x) = 0, \dots, f_m(x) = 0\},$$

for $f_1, \dots, f_m$ polynomials in $\mathfrak{R}[x_1, \dots, x_n]$. For instance, by [BCR13, Proposition 3.2.10], the Zariski-open subset $U := V \cap (\mathfrak{R}^\times)^n$ of a real algebraic set $V \subset \mathfrak{R}^n$ is an affine real algebraic variety. A *real algebraic variety* over $\mathfrak{R}$ is a topological space X such that there exists a finite Zariski-open cover $(U_i)_{i \in I}$ of X with each U_i being an affine real algebraic variety. For instance, for X a real algebraic variety in the real algebraic torus $(\mathfrak{R}^\times)^n$, the compactification $\overline{X}$ of X inside a projective space $\mathbb{P}^n$ is a real algebraic variety. More generally, the compactification $\overline{X}$ of X inside a projective toric variety $\mathbb{P}_\Sigma$ defined by a fan Σ of dimension n is a real algebraic variety.

In this thesis, we will consider real algebraic varieties over $\mathfrak{R} = \mathbb{R}$ and over $\mathfrak{R} = \mathbb{K}_\mathbb{R}$, for $\mathbb{K}_\mathbb{R}$ the real closed field of *locally convergent generalised Puiseux series with real coefficients* [IMS09]. An element of $\mathbb{K}_\mathbb{R}$ is of the form

$$\alpha(t) := \sum_{r \in R} \alpha_r t^r,$$

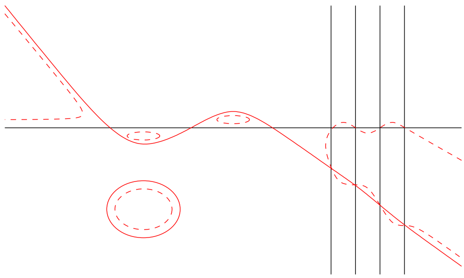


Figure 1.1: Harnack's construction of a maximal quartic curve.

with $R \subset \mathbb{R}$ a well-ordered set, all coefficients α_r belong to the real numbers $\mathbb{R}$ and the series is convergent for $t \in \mathbb{R}_{>0}$ small enough. A real algebraic variety X over $\mathbb{K}_{\mathbb{R}}$ can be seen as a family of real algebraic varieties $(X_t)_t$ over $\mathbb{R}$ parametrised by t in the interval $]0, \varepsilon[\subset \mathbb{R}_{>0}$, with ε chosen so that all the coefficients $\alpha(t) \in \mathbb{K}_{\mathbb{R}}$ of the polynomials defining X are convergent on $]0, \varepsilon[$. Similarly, let $\mathbb{K}$ be the algebraic closure of $\mathbb{K}_{\mathbb{R}}$, corresponding to the *locally convergent generalised Puiseux series with complex coefficients* [IMS09]. Then an algebraic variety X over $\mathbb{K}$ can be seen as a family of algebraic varieties $(X_t)_t$ over $\mathbb{C}$. This time, the parameter t belongs to the punctured open disc $D(\varepsilon) - \{0\} \subset \mathbb{C}^{\times}$ of radius $\varepsilon \in \mathbb{R}_{>0}$, with ε chosen so that all the coefficients $\alpha(t) \in \mathbb{K}$ of the polynomials defining X are convergent on $D(\varepsilon) - \{0\}$.

Any real closed field $\mathfrak{R}$ possesses *orderings* [BCR13], so that we can consider polynomial inequalities in addition to polynomial equalities. A *basic open semi-algebraic set* is a subset $W(f_1, \dots, f_m) \subset \mathfrak{R}^n$ defined as

$$W(f_1, \dots, f_m) := \{\underline{x} \in \mathfrak{R}^n \mid f_1(\underline{x}) > 0, \dots, f_m(\underline{x}) > 0\},$$

for polynomials $f_1, \dots, f_m \in \mathfrak{R}[x_1, \dots, x_n]$ in the ordered real closed field $(\mathfrak{R}, \leq)$. A general semi-algebraic set is a finite sequence of unions, intersections and complements of basic open semi-algebraic sets.

For X a real algebraic variety over $\mathfrak{R}$ a real closed field, we denote by $X(\mathfrak{R})$ the *set of real points* of X and by $X(\overline{\mathfrak{R}})$ the *set of complex points* of X, for $\overline{\mathfrak{R}}$ the algebraic closure of $\mathfrak{R}$. We call an algebraic variety defined by a single polynomial a *hypersurface*. If a hypersurface has defining polynomial of degree 1, we call it a *hyperplane*.

1.2 Topology of real algebraic varieties

The study of the Euclidean topology of real algebraic varieties dates back to the 19th century, with *Harnack-Klein's inequality* ([Har76], [Kle73]), which states that for $\mathcal{C}$ a non-singular real algebraic curve of genus g, the number of connected components of $\mathcal{C}(\mathbb{R})$ is less than or equal to $g+1$. Moreover, Harnack gave a construction of *maximal* non-singular real algebraic curves in $\mathbb{P}^2$ for any degree d [Har76] (see for instance Figure 1.1 for Harnack's construction of a maximal

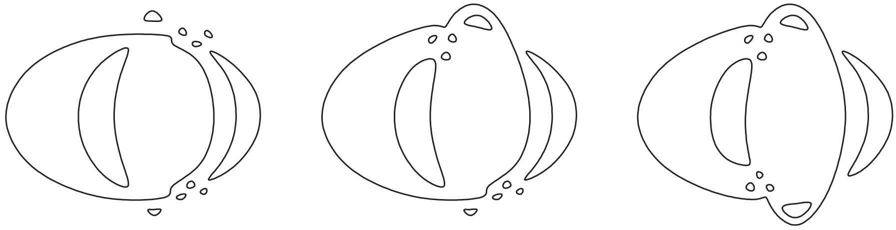


Figure 1.2: The Harnack, Gudkov and Hilbert sextics

quartic curve from a given maximal cubic curve). That is he constructed for each degree d a non-singular real algebraic curve of degree d with $g+1$ connected components in its real part. The bound has been generalised to any dimension as *Smith-Thom inequality*, stating that for X a non-singular real algebraic variety, the sum of Betti numbers of the real part $X(\mathbb{R})$ is bounded by the sum of Betti numbers of the complex part $X(\mathbb{C})$. However, the existence of a non-singular real algebraic variety with a maximal prescribed Betti number is an open question in most cases, see for instance [Bih99] in the case of degree 5 surfaces in $\mathbb{P}^3$, and [Bih01], [Ren15] for the case of degree 6 surfaces in $\mathbb{P}^3$.

Inspired by Harnack-Klein inequality, Hilbert asked in his 16th problem ([Hil91], [Hil00]) for a classification of the possible isotopy types for the real part of a degree 6 non-singular real algebraic curve in $\mathbb{P}^2$. In the same problem, Hilbert asked for a classification of the possible isotopy types for the real part of a degree 4 non-singular real algebraic surface in $\mathbb{P}^3$. A generalisation of this question would be to classify the possible topological pairs $(\mathbb{P}^n(\mathbb{R}), X(\mathbb{R}))$, for X a non-singular real hypersurface of fixed degree d in $\mathbb{P}^n_{\mathbb{R}}$. In the case of non-singular real algebraic curves, the real part is homeomorphic to a disjoint union of circles S^1 [BR91]. If, in addition, such a curve is of degree d in $\mathbb{P}^2_{\mathbb{R}}$, the real part consists only of *ovals*, which are connected components disconnecting $\mathbb{P}^2(\mathbb{R})$), and a *pseudo-line* if d is odd, which is a connected component that does not disconnect $\mathbb{P}^2(\mathbb{R})$ [BCR13, Proposition 11.6.1].

The initial question for degree 6 curves was solved 70 years later by Gudkov [Gud69], by constructing the last missing isotopy types (see Figure 1.2 for the missing maximal isotopy type), and the analogue in degree 7 was completed by Viro [Vir80] using *patchworking*. The idea of Viro's patchworking in [Vir80] was to construct a real algebraic curve by gluing several real algebraic curves with some specified topological types and singularities, and classify the topological types of the possible smoothings of the constructed curve.

Nowadays, we still do not know the complete classification of isotopy types in degree 8, but a relaxation of this problem to pseudo-holomorphic maximal curves instead of maximal algebraic curves has been solved by Orevkov [Ore02]. Similarly, the initial question for degree 4 surfaces was solved by Kharlamov [Kha76]. As we saw, we do not have a complete classification of topological types starting from degree 5 surfaces, hence we cannot have a complete classification of isotopy types in those cases.

In relation to the 16th Hilbert problem, one can study the semi-algebraic sets defined by a unique polynomial. Indeed, if f is a homogeneous real polynomial defining a non-singular hypersurface $X \subset \mathbb{R}^n$, then the basic open semi-algebraic set $W(f)$ is the subset of $\mathbb{R}^n \backslash X(\mathbb{R})$ where the polynomial f is positive. If f is of even degree, we say that the homogenisation $\tilde{f}$ of f is positive at a point $p \in \mathbb{P}^n(\mathbb{R})$ if p belongs to the projective compactification $W(\tilde{f}) \subset \mathbb{P}^n(\mathbb{R}) \backslash \tilde{X}(\mathbb{R})$ of the semi-algebraic set $W(f)$, for $\tilde{X}$ the projective compactification of X in $\mathbb{P}^n_{\mathbb{R}}$. Then obtaining topological information on $W(\tilde{f})$ allows us to obtain topological information on the pair $(\mathbb{P}^n(\mathbb{R}), \tilde{X}(\mathbb{R}))$.

In the case of non-singular real algebraic curves of even degree in $\mathbb{P}^2_{\mathbb{R}}$, the question of characterising a semi-algebraic set of the form $W(\tilde{f})$ can be reformulated into studying even and odd ovals. An oval is said to be *even* if it lies in the interior of an even number of ovals, and is said to be *odd* otherwise. Let $\mathcal{C}$ be a smooth real algebraic curve in $\mathbb{P}^2$ of degree $2k$ defined by a homogeneous polynomial $\tilde{f}$, such that $\tilde{f}$ is negative on the connected component of $\mathbb{P}^2(\mathbb{R}) \backslash \mathcal{C}(\mathbb{R})$ outside every oval of $\mathcal{C}(\mathbb{R})$. Then the number p of even ovals of the real part $\mathcal{C}(\mathbb{R})$ is the number of connected components of $W(\tilde{f})$, and the number n of odd ovals of $\mathcal{C}(\mathbb{R})$ is one less than the number of connected component of $\mathbb{P}^2(\mathbb{R}) \backslash W(\tilde{f})$.

For $\mathcal{C}$ a non-singular real algebraic curve of degree $2k$ in $\mathbb{P}^2_{\mathbb{R}}$, with p and n its number of even and odd ovals, Ragsdale [Rag06] made the conjecture that

$$p \leq \frac{3k^2 - 3k}{2} + 1, \quad n \leq \frac{3k^2 - 3k}{2}.$$

Independently, Petrowsky [Pet38] proved that

$$|p - n| \leq \frac{3k^2 - 3k}{2} + 1$$

and that there exists some curves reaching this bound for any degree $2k$. He then conjectured, in a similar fashion as Ragsdale, that

$$p \leq \frac{3k^2 - 3k}{2} + 1, \quad n \leq \frac{3k^2 - 3k}{2} + 1.$$

Since we know the complete isotopy classifications up to degree 7, we can see that both Ragsdale's and Petrowsky's conjectures are true in degree $2k$ for $k \leq 3$. Viro constructed (again by patchworking) some non-singular real algebraic curves of degree 8 in $\mathbb{P}^2$ satisfying $n = \frac{3k^2-3k}{2}+1$ [Vir80], therefore contradicting Ragsdale's conjecture but still satisfying Petrowsky's conjecture. Using a combinatorial version of Viro's patchworking, which consists in constructing a piecewise-linear set isotopic to the real part of a real algebraic curve, Itenberg [Ite93] found examples of curves of degree $2k, k \geq 5$ satisfying

$$p = \frac{3k^2 - 3k}{2} + 1 + \left\lfloor \frac{(k-3)^2 + 4}{8} \right\rfloor, \text{ or } n = \frac{3k^2 - 3k}{2} + \left\lfloor \frac{(k-3)^2 + 4}{8} \right\rfloor.$$

Note that Viro's combinatorial patchworking is one of the first motivations for the development of tropical geometry, as we will see later. On Figure 1.3, we

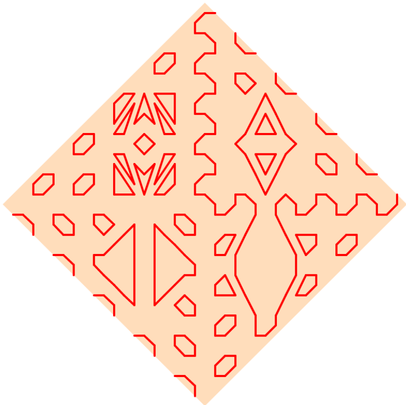


Figure 1.3: Itenberg's counter-example for even ovals in degree 10

see Itenberg's counter-example of Ragsdale conjecture for even ovals in degree 10, constructed thanks to the Combinatorial Patchworking Tool of El-Hilany, Rau and Renaudineau (this could also be constructed with the Sage package Viro.sage from De Wolff, O'Neill and Owusu Kwaakwah). In particular, both Ragsdale's and Petrowsky's conjectures are false. Haas and Itenberg gave other counter-examples afterwards with more even and odd ovals in [Haa95], [Ite01].

However, several questions are still open around Ragsdale and Petrowsky conjectures. For instance, given a non-singular real algebraic curve $\mathcal{C}$ of degree $2k$ in $\mathbb{P}^2$, with p even ovals and n odd ovals, the combination of the Harnack-Klein inequality with Petrowsky's inequality gives the bounds

$$p \leq \frac{7k^2 - 9k + 6}{4}, \quad n \leq \frac{7k^2 - 9k + 4}{4}.$$

One open question is to know whether the Harnack-Klein-Petrowsky bound is sharp. The best result in this direction has been obtained by Brugallé [Bru+06], who showed that the bound is asymptotically sharp.

We also do not know if there exists a maximal curve not satisfying Petrowsky's conjecture. Note that the examples of Viro [Vir80] were maximal curves, justifying why we consider only Petrowsky's conjecture and not Ragsdale's. This second question motivated Haas in his thesis [Haa97] to study the properties of combinatorial patchworking. As a concluding result, Haas proved that a maximal curve $\mathcal{C}$ of degree $2k$ obtained via combinatorial patchworking satisfies

$$p \leq R(k) + 1, \text{ and } n \leq R(k) + 4,$$

for p and n the number of even and odd ovals of $\mathcal{C}(\mathbb{R})$ and for $R(k) = \frac{3k^2-3k}{2}$ [Haa97, Theorem 12.4.0.12]. Moreover, there exists for each $k \geq 1$ a maximal curve of degree $2k$ obtained via combinatorial patchworking satisfying $p = R(k) + 1$, and those curves have the isotopy type of the curves constructed by Harnack [Har76] (see Figure 1.1 for the degree $2k = 4$). However, to the

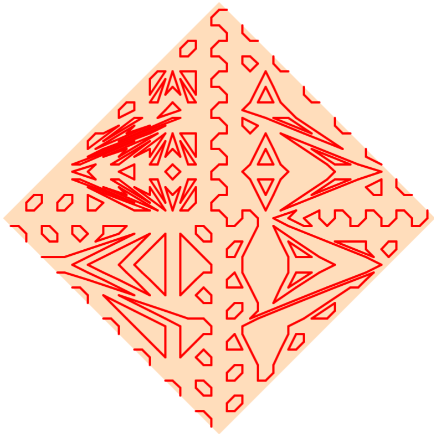


Figure 1.4: Counter-example from 1.2.1 in degree 14

knowledge of the author, no example with $n \geq R(k)+1$ has been found via combinatorial patchworking.

In Paper III, we construct new counter-examples to Ragsdale conjecture in a similar fashion as Itenberg. Those counter-examples satisfy the following result.

Theorem 1.2.1 (Theorem III.1.7)**.** *There exist non-singular dividing* $(M - 2\left\lfloor \frac{k-3}{2} \right\rfloor)$ *real algebraic curves of degree* $2k$ *in* $\mathbb{P}^2$*, with* $k \geq 5$*, such that their real parts have* $R(k) + 1 + \frac{k^2-5k+s(k)}{6}$ *even ovals, with* $0 \leq s(k) \leq 10$ *determined by the value of* k *modulo 6. The number of even ovals depending on* k *modulo 6 are listed in Table III.3.*

For instance, Figure 1.4 represents the isotopy type of the counter-example of Theorem 1.2.1 in degree 14, with $R(7) + 5 = 68$ even ovals, constructed again via the Combinatorial Patchworking Tool.

1.3 Hyperbolic varieties

Among real algebraic varieties, a particularly interesting class is given by hyperbolic varieties. A real algebraic variety $X \subset \mathbb{P}^n_{\mathfrak{R}}$ is *hyperbolic with respect to* a real linear subspace $E \subset \mathbb{P}^n_{\mathfrak{R}}$ of dimension $n - \dim(X) - 1$ if $X \cap E = \emptyset$ and for every real linear subspace $H \subset \mathbb{P}^n_{\mathfrak{R}}$ of dimension $n - \dim(X)$ containing E, the intersection $X(\overline{\mathfrak{R}}) \cap H(\overline{\mathfrak{R}})$ is contained in the set of real points $X(\mathfrak{R})$ (extension of the definition from [SV18] to any real closed field). We say that $X \subset \mathbb{P}^n_{\mathfrak{R}}$ is *hyperbolic* if there exists a real linear subspace $E \subset \mathbb{P}^n_{\mathfrak{R}}$ such that X is hyperbolic with respect to E. The set

$$\mathcal{H}_X := \{E \subset \mathbb{P}^n_{\mathfrak{R}} \mid X \text{ is hyperbolic with respect to } E\}$$

is called the *hyperbolicity locus* of X. We say that a real algebraic hypersurface $X \subset \mathbb{P}^n_{\mathfrak{R}}$ is *stable* [GW96], or more generally a real algebraic variety

$X \subset \mathbb{P}^n_{\mathfrak{R}}$ is *positively hyperbolic* [RVY21, Proposition 2.11], if the hyperbolicity locus of X contains the whole set parametrised by the positive Grassmannian $\mathbb{G}_+(\mathrm{codim}(\mathbb{P}^n, X), n+1)$. Hyperbolic varieties are a higher dimensional analogue of hyperbolic polynomials in $\mathfrak{R}[x]$, which are real polynomials with set of roots contained in $\mathfrak{R}$ [Går51]. In particular, we can obtain a (homogenisation of a) hyperbolic polynomial in $\mathfrak{R}[x_0, x_1]$ from defining polynomials of a hyperbolic complete intersection and defining polynomials of a real linear space intersecting that complete intersection in only real points.

Hyperbolic varieties are interesting for many areas of mathematics and interdisciplinary fields such as partial differential equations [Går51], topology of real algebraic varieties [Rok78] and convex optimisation [Gül97], [Ren04]. Positively hyperbolic varieties have a strong connection with the theory of matroids [Cho+04],[Brä07], valuated matroids [Brä10] and positroids [RVY21]. As we will see later, Speyer used the *tropicalisation* of stable curves in $\mathbb{P}^2$ in order to solve Horn's problem [Spe05]. Similarly, Brändén used tropicalisation to show the link between stable hypersurfaces and valuated matroids [Brä10]. Finally, Rincón, Vinzant and Yu used again tropicalisation to relate positively hyperbolic varieties and positroids [RVY21].

Recall that the real part of a non-singular algebraic curve in $\mathbb{P}^2_{\mathbb{R}}$ is a disjoint union of ovals with possibly a pseudo-line. For X a non-singular hypersurface in $\mathbb{P}^{n+1}_{\mathbb{R}}$, with $n \geq 2$, we say that a connected component X_0 of $X(\mathbb{R})$ is an *ovaloid* if X_0 is homeomorphic to the n-sphere S^n (and so disconnects $\mathbb{P}^n(\mathbb{R})$ since $n \geq 2$), and we say that X_0 is a *pseudo-hyperplane* if X_0 is homeomorphic to the topological real projective space $\mathbb{RP}^n$. We extend these definitions to $n = 1$ by saying that an oval is an ovaloid and a pseudo-line is a pseudo-hyperplane. Note that for $n \geq 2$, ovaloids and pseudo-hyperplanes are not the only possible homeomorphism types for a connected component X_0 of $X(\mathbb{R})$. Helton and Vinnikov [HV07, Theorem 5.2] showed that a non-singular hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{R}}$ of degree d is hyperbolic if and only if the set of real points $X(\mathbb{R})$ consists of $\lfloor \frac{d}{2} \rfloor$ *nested* ovaloids, plus a pseudo-hyperplane if d is odd. More precisely, we have a chain of inclusions of the $(n+1)$-dimensional open discs bounded by each ovaloid of $X(\mathbb{R})$. If $d \geq 2$, the hyperbolicity locus of X is the $(n+1)$-dimensional open disc bounded by an ovaloid of $X(\mathbb{R})$ which is included in all other discs bounded by ovaloids of $X(\mathbb{R})$. Note that if $d = 1$ (and $n > 0$), the hyperbolicity locus of X is the entire set $\mathbb{P}^{n+1}(\mathbb{R}) \backslash X(\mathbb{R})$. In particular, the hyperbolicity locus of a non-singular hyperbolic hypersurface in $\mathbb{P}^{n+1}_{\mathbb{R}}$ is connected. Moreover, Helton and Vinnikov proved that in the hypersurface case, the hyperbolicity locus is convex [HV07, Property 5.3(3)]. In higher codimension, the hyperbolicity locus is not connected in general, see [KS20b, §5].

A *determinantal representation* of a homogeneous polynomial $P \in \overline{\mathfrak{R}}[x_0, \dots x_n]$ of degree d is an expression

$$M := M(\underline{x}) := M_0 x_0 + \ldots + M_n x_n,$$

where the M_i's are $(d \times d)$-matrices with coefficients in $\overline{\mathfrak{R}}$, such that $\det(M) = cF$ for some $c \in \overline{\mathfrak{R}}^{\times}$. The representation is *Hermitian* if the matrices M_i are

Hermitian. If P has coefficients over a real closed field $\mathfrak{R}$, we say that the representation M is *definite* if the matrix $M(E)$ is positive definite for some point $E \in \mathbb{P}^n_{\mathfrak{R}}$. If a real algebraic hypersurface $X \subset \mathbb{P}^n_{\mathfrak{R}}$ admits a definite Hermitian representation M, then X is hyperbolic with respect to every real point $E \subset \mathbb{P}^n_{\mathfrak{R}}$ such that $M(E)$ is positive definite. The Lax conjecture asks conversely if every hyperbolic curve in $\mathbb{P}^2_{\mathfrak{R}}$ admits a real symmetric definite determinantal representation. The conjecture was proved by Lewis, Parrilo and Ramana [LPR05]. However, the generalised Lax conjecture for any hyperbolic hypersurface in $\mathbb{P}^n_{\mathfrak{R}}$ does not hold [Brä11].

In a more general setting, if a real algebraic variety $X \subset \mathbb{P}^n_{\mathfrak{R}}$ admits a definite Hermitian representation M satisfying some additional conditions [SV18, Section 2], then X is hyperbolic with respect to every real linear space $E \subset \mathbb{P}^n_{\mathfrak{R}}$ such that the $M(E)$ is positive definite. The converse statement is true for hyperbolic curves [SV18, Theorem 7.2]

An alternate way to define hyperbolic varieties is through real-fibered morphisms. A morphism $f : X \to Y$ between two real algebraic varieties X and Y is said to be *real-fibered* if $f^{-1}(Y(\mathfrak{R})) = X(\mathfrak{R}) \neq \emptyset$. Then a real algebraic variety $X \subset \mathbb{P}^N_{\mathfrak{R}}$ of dimension n is hyperbolic with respect to a real linear space $E \subset \mathbb{P}^N_{\mathfrak{R}}$ if and only if there exists a real-fibered morphism $f : X \to \mathbb{P}^n_{\mathfrak{R}}$ given as a composition $\pi_E \circ i$, with i an embedding of X in $\mathbb{P}^N_{\mathfrak{R}}$ for some $N > n$ and $\pi_E : \mathbb{P}^N_{\mathfrak{R}} \to \mathbb{P}^n_{\mathfrak{R}}$ the projection with center E [KS20a]. In that case, we call $f = \pi_E \circ i$ a *hyperbolic morphism*. From results of Ahlfors [Ahl50, Theorem 10], Gabard [Gab06, Theorem 7.1], Kummer and Shamovich [KS20a, Theorem 2.8], we obtain that a non-singular real algebraic curve C admits a hyperbolic morphism $f : C \to \mathbb{P}^1_{\mathbb{R}}$ if and only if C is *dividing*, meaning that the set $C(\mathbb{C})\backslash C(\mathbb{R})$ is disconnected. Using this, Kummer and Shaw [KS20b, Examples 5.1 and 5.2] found examples of hyperbolic curves with disconnected hyperbolicity locus.

For X a non-singular real algebraic variety of dimension $n \geq 2$, the existence of a real-fibered morphism $f : X \to \mathbb{P}^n_{\mathbb{R}}$ constrains the topology of the real part $X(\mathbb{R})$, as f must restrict to a covering map $\tilde{f} : X(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$ [KS20a, Theorem 2.19]. In particular, Kummer and Shamovich [KS20a, Corollary 2.20] proved that the real part $X(\mathbb{R})$ consists of s ovaloids and r pseudo-hyperplanes, for $r + 2s = \deg(f)$.

In Paper I, we characterise the real-fibered and hyperbolic morphisms to $\mathbb{P}^2_{\mathbb{R}}$ of real *del Pezzo surfaces*, which are real surfaces with ample anti-canonical class, and minimal conic bundles over $\mathbb{P}^1_{\mathbb{R}}$, which are surfaces given as union of plane conic fibres coming from $\mathbb{P}^1_{\mathbb{R}}$ and with Picard rank 2. This work is motivated by the fact that the real Picard group of those surfaces are completely classified [Com14], [Rus02], hence we can do a systematic study of real-fibredness and hyperbolicity of the morphisms. The classification of divisors on del Pezzo surfaces inducing a real-fibered morphism, possibly hyperbolic, is given in Table I.1. We obtain this classification thanks to one of the main results of Paper I, allowing to check the hyperbolicity of a real algebraic variety of dimension greater than or equal to 2 in terms of the hyperbolicity of a hyperplane section.

Theorem 1.3.1 (Theorem I.1.5)**.** *Let $X \subset \mathbb{P}^n_{\mathbb{R}}$ be a smooth variety of dimension*

$k \geq 2$. Let $H \subset \mathbb{P}^n_{\mathbb{R}}$ be a hyperplane such that $C = X \cap H$ is a smooth $(k-1)$-variety. Assume that each connected component of $X(\mathbb{R})$ contains exactly one connected component of $C(\mathbb{R})$. Moreover, let $E \subset H$ be a linear space of dimension $n-k-1$ with $X \cap E = \emptyset$. Then the following are equivalent:

1. *X is hyperbolic with respect to E.*

2. *X satisfies $X(\mathbb{R}) \simeq sS^k \sqcup r\mathbb{RP}^k$ such that $\deg(X) = 2s + r$. The class of each connected component that is homeomorphic to a real projective space is non-trivial in $H_k(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}_2)$ and $C \subset H = \mathbb{P}^{n-1}_{\mathbb{R}}$ is hyperbolic with respect to E.*

We extract from Table I.1 the following other main results of Paper I, first concerning the finite real-fibered morphisms.

Theorem 1.3.2 (Theorem I.1.2). *Let X be a del Pezzo surface such that each connected component of $X(\mathbb{R})$ is homeomorphic to either the sphere or the real projective plane. There is a finite real-fibered morphism $X \to \mathbb{P}^2$ if and only if we have one of the following:*

1. *X has real Picard rank 1;*

2. *X is a conic bundle of real Picard rank 2;*

3. *X is the blow-up of one of the above surfaces at one or two real points.*

Only one of the del Pezzo surfaces in Table I.1 admits a real-fibered morphism which is not hyperbolic. We obtain in this way an analogue of Theorem 1.3.2 for hyperbolic morphisms.

Theorem 1.3.3 (Theorem I.1.6). *Let X be a del Pezzo surface such that each connected component of $X(\mathbb{R})$ is homeomorphic to either the sphere or the real projective plane. There is an embedding $X \hookrightarrow \mathbb{P}^n$ such that the image is a hyperbolic variety if and only if we have one of the following:*

1. *X has real Picard rank 1;*

2. *X is a conic bundle of real Picard rank 2;*

3. *X is the blow-up of one of the above surfaces at one real point.*

Motivated by the conic bundle case of 1.3.3, we go further by studying hyperbolic minimal conic bundles, and obtain a construction of many possible topological types for hyperbolic surfaces.

Proposition 1.3.4 (Proposition I.7.12). *Let $s \geq 3$ and $r \geq 0$. There exists a smooth irreducible hyperbolic surface $X \subset \mathbb{P}^n_{\mathbb{R}}$ such that $X(\mathbb{R})$ is homeomorphic to the disjoint union of s spheres and r real projective planes.*

1.4 Tropical geometry

Tropical geometry is a way to study algebraic varieties over a field equipped with a non-archimedean valuation, such as the field of locally convergent generalised Puiseux series $\mathbb{K}$, by looking at properties of the polyhedral complexes obtained as image via the valuation (or *tropicalisation*) of those algebraic varieties. Tropical geometry has applications in many areas such as topology of real algebraic varieties [Vir01], matroid theory [Stu02], convex optimisation [DS03], enumerative geometry [Mik05] and Hodge theory [Ite+19].

We call *tropical numbers* the semi-field with ground set $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$, *tropical addition* $\oplus :=$ max and *tropical multiplication* $\otimes := +$. From now on, we will drop the $\otimes$ notation whenever the context is clear. A *tropical polynomial* $P \in \mathbb{T}[x_1, \dots, x_n]$ of the form

$$P(x_1, \dots, x_n) = \bigoplus_{a \in A} c_a x_1^{a_1} \dots x_n^{a_n}.$$

for A a finite subset of $(\mathbb{Z}_{\geq 0})^n$ and all coefficients $c_a \in \mathbb{T}$.

From the notions of tropical polynomials, we obtain an analogue of algebraic varieties, defined over the semi-field $\mathbb{T}$ instead of a field k. For $P \in \mathbb{T}[x_1, \dots x_n]$ a tropical polynomial, the polyhedral complex V_P is defined as

$$V_P := \{\underline{x} \in (\mathbb{T}^\times)^n \;:\; \exists a \neq b \in A, P(\underline{x}) = c_a \underline{x}^a = c_b \underline{x}^b\}.$$

A *tropical hypersurface* X in $(\mathbb{T}^\times)^n \simeq \mathbb{R}^n$ is a polyhedral complex of the form V_P, equipped with the weight function on facets

$$\begin{aligned} w : \mathrm{Facets}(V_P) &\to \mathbb{N}_{>0} \\ \sigma &\mapsto \max_{i \in \{1, \dots, n\}} (\gcd(|a_i - b_i|)), \end{aligned}$$

for $a \neq b \in A$ so that $P(\underline{x}) = c_a \underline{x}^a = c_b \underline{x}^b$ for all points $\underline{x}$ in the face σ. We can extend the definition of tropical hypersurface in other ambient spaces, such as the affine tropical space $\mathbb{T}^n$ or a projective tropical toric variety $\mathbb{TP}_\Sigma$ [Pay09], [MR09, §3.2] of dimension n, by taking the closure of V_P inside those spaces. More generally, a *tropical variety* in $\mathbb{T}^n$ is a rational weighted polyhedral complex in $\mathbb{T}^n$ satisfying a balancing condition around all codimension 1 faces.

From an algebraic variety $\mathcal{X}$ in $\mathbb{K}^n$, we can associate a tropical variety X via *tropicalisation.* Let val be a non-archimedean valuation on $\mathbb{K}$ defined as

$$\begin{aligned} \mathrm{val} : \mathbb{K}^\times &\to \mathbb{R} \\ \alpha = \sum_{r \in R} \alpha_r t^r &\mapsto \min_R \{r \;:\; \alpha_r \neq 0\} \end{aligned}$$

for $R \subset \mathbb{R}$ a well-ordered set and $\mathrm{val}(0) = +\infty$. Then the *tropicalisation* $\mathrm{Trop}(V)$ of a subset $V \subset \mathbb{K}^n$ is the image of V via the map

$$\begin{aligned} \mathrm{Trop}_0 : \mathbb{K}^n &\to \mathbb{T}^n \\ \underline{\beta} &\mapsto (-\mathrm{val}(\beta_1), \dots, -\mathrm{val}(\beta_n)), \end{aligned}$$

equipped with the weight function w on facets if the image is a tropical variety.

The following theorem from Kapranov is sometimes considered as the *fundamental theorem of tropical geometry*.

Theorem 1.4.1 ([Kap00]). *Let $\mathcal{X}$ be an algebraic hypersurface with defining polynomial $\mathcal{P} = \sum \alpha_a \underline{x}^a \in \mathbb{K}[x_1, \ldots, x_n]$. Let X be the tropical hypersurface defined by the tropical polynomial $P = \bigoplus(-\operatorname{val}(\alpha_a))\underline{x}^a \in \mathbb{T}[x_1, \ldots, x_n]$. Then*

$$\operatorname{Trop}(\mathcal{X}) = X.$$

Theorem 1.4.1 can be extended to complete intersections $\mathcal{X}_1 \cap \ldots \cap \mathcal{X}_n$, with each $\mathcal{X}_i$ an algebraic hypersurface, if and only if the intersection $X_1 \cap \ldots \cap X_n$ of the tropicalisations $X_i := \operatorname{Trop}(\mathcal{X}_i)$ is *transverse*. In the non-transverse case, we obtain

$$\operatorname{Trop}(\mathcal{X}_1 \cap \ldots \cap \mathcal{X}_n) \subsetneq X_1 \cap \ldots \cap X_n.$$

This fact justifies the development of *stable intersection*, or *tropical intersection*, of tropical varieties, see for instance [RST05],[BD12]. In Paper II, we study the intersections of the form $\operatorname{Trop}(\mathcal{X}_1 \cap \mathcal{X}_2)$, for $\mathcal{X}_1$ and $\mathcal{X}_2$ non-singular real algebraic curves in $(\mathbb{K}_{\mathbb{R}}^{\times})^2$, in terms of the tropicalisations X_i and some *real structure* on X_i induced by $\mathcal{X}_i$.

1.5 Real tropical geometry

In order to obtain information about the real part of real algebraic varieties over $\mathbb{K}_{\mathbb{R}}$ from their tropicalisations, we need to add some additional structure either on the tropical semi-field or directly on the tropical varieties. Several strategies are used in the literature. For instance, Viro introduced the *tropical real hyperfield* $\mathcal{T}\mathbb{R}$ [Vir10], where the ground set is $\mathbb{R}$, the multiplication is the usual multiplication over $\mathbb{R}$ and the addition is a multi-valued operation, equal to the maximum of the summands if their absolute value are distinct. This hyperfield has been used in particular in [JSY18] to define the *real tropicalisation* of semi-algebraic sets. A similar strategy consists in looking at the *signed tropical numbers* $\mathbb{T}_{\pm}$, see for instance [AGS20], given by taking two symmetric copies of $\mathbb{T}$ glued along their copy of the element $-\infty$. The addition and multiplication on $\mathbb{T}_{\pm}$ are defined only on elements of the same copy. In order to obtain operations defined on the whole set, either we can extend $\mathbb{T}_{\pm}$ to the *symmetrised semi-ring* introduced in [Aki+90] and used in [LV19] to define *signed tropical convexity*, or we can extend $\mathbb{T}_{\pm}$ to a hyperfield, that we can obtain from $\mathcal{T}\mathbb{R}$ via a logarithmic map, see [Vir10] for this strategy in the complex case.

In this thesis, we will work directly with tropical varieties and recover their *realisations*, that is the real algebraic varieties over $\mathbb{K}_{\mathbb{R}}$ that tropicalise to the considered tropical varieties, hence we will choose to add a real structure directly on the tropical varieties instead of using a real analogue of the tropical numbers.

1.5.1 Real phase structure on a non-singular tropical variety

We restrict to the case of a *non-singular* tropical hypersurface X, meaning that X is locally around each vertex the image by a translation and a map in $\mathbf{GL}(n,\mathbb{Z})$ of a tropical hyperplane. The following is defined in [Ren17], [RS21].

For $\mathrm{Aff}_n(\mathbb{Z}_2^{n+1})$ the set of n-dimensional $\mathbb{Z}_2$-affine subspaces of $\mathbb{Z}_2^{n+1}$, a *real phase structure* $\mathcal{E}$ on a non-singular tropical hypersurface $X \subset \mathbb{R}^{n+1}$ is a map

$$\mathcal{E} : \mathrm{Facets}(X) \to \mathrm{Aff}_n(\mathbb{Z}_2^{n+1})$$

such that $\mathcal{E}$ satisfies the following properties. For each facet σ of X, the $\mathbb{Z}_2$-affine space $\mathcal{E}(\sigma)$ is parallel to the $\mathbb{Z}_2$-vector space generated by the primitive integer directions modulo 2 in σ. Moreover, for each codimension 1 face τ of X, with adjacent facets $\sigma_1, \dots, \sigma_k$, an element $\varepsilon \in \mathbb{Z}_2^{n+1}$ appearing in a set $\mathcal{E}(\sigma_i)$ must appear in a single distinct set $\mathcal{E}(\sigma_j)$.

The *real part* $\mathbb{R}X_{\mathcal{E}}$ of a non-singular tropical hypersurface $X \subset \mathbb{R}^{n+1}$ equipped with the real phase structure $\mathcal{E}$ is given as

$$\mathbb{R}X_{\mathcal{E}} := \overline{\bigcup_{\sigma \in \mathrm{Facets}(X)} \left(\bigcup_{\varepsilon \in \mathcal{E}(\sigma)} \sigma_\varepsilon \right)} \subset \bigcup_{\varepsilon \in \mathbb{Z}_2^{n+1}} \mathbb{R}_\varepsilon^{n+1},$$

for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n+1}) \in \mathbb{Z}_2^{n+1}$ and σ_ε the symmetric copy of σ lying in the symmetric copy $\mathbb{R}_\varepsilon^{n+1}$ of $\mathbb{R}^{n+1}$.

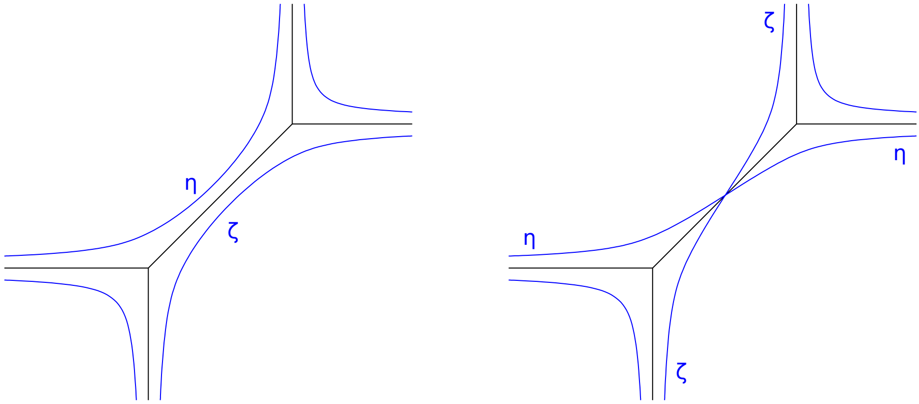
If X is a non-singular tropical hypersurface in a non-singular projective tropical toric variety $\mathbb{TP}_\Sigma$, where the fan Σ is dual to the Newton polygon Δ of X, the definition of real phase structure $\mathcal{E}$ on X extends naturally, and the real part $\mathbb{R}X_{\mathcal{E}}$ lies in a topological space homeomorphic to $\mathbb{RP}_\Sigma$, for $\mathbb{RP}_\Sigma$ the real part of the toric variety $\mathbb{P}_\Sigma$ constructed in [GKZ14, Theorem 11.5.4]. The couple $(X, \mathcal{E})$ will be called a *non-singular real tropical hypersurface*. We obtain the following reformulation of Viro's combinatorial patchworking theorem.

Theorem 1.5.1 ([Vir01])**.** *Let $(X, \mathcal{E})$ be a non-singular real tropical hypersurface in $\mathbb{TP}_\Sigma$. There exists a non-singular real algebraic hypersurface $\mathcal{X} := (\mathcal{X}_t)_t \subset \mathbb{P}_\Sigma^{\mathbb{K}_\mathbb{R}}$ with* $\mathrm{Trop}(\mathcal{X}) = X$ *such that we have a homeomorphism of pairs*

$$(\mathbb{P}_\Sigma(\mathbb{R}), \mathcal{X}_t(\mathbb{R})) \simeq (\mathbb{RP}_\Sigma, \mathbb{R}X_{\mathcal{E}})$$

for $t > 0$ small enough.

Moreover, the real phase structure $\mathcal{E}$ determines the distribution of signs of the defining polynomial $\mathcal{P}$ of $\mathcal{X}$, up to multiplying $\mathcal{P}$ by -1, see [RS21, Remark 3.8]. In particular, we can easily construct an example of non-singular real algebraic hypersurface $\mathcal{X}$ satisfying Theorem 1.5.1 for a non-singular real tropical hypersurface $(X, \mathcal{E})$, by letting $\mathcal{X}$ be defined by a *Viro polynomial* [Vir01, §4.1], that is a polynomial in $\mathbb{K}_\mathbb{R}[x_1, \dots, x_n]$ with coefficients of the form $\pm t^{-r}$. Sturmfels [Stu94] generalised Theorem 1.5.1 to the case when $(X, \mathcal{E})$ is a non-singular real tropical transverse complete intersection, and we can determine in the same manner as above the distributions of signs of the defining polynomials of the hypersurfaces involved in the intersection.



(a) Amoeba around a non-twisted edge. (b) Amoeba around a twisted edge.

Figure 1.5: Local views of real amoeba near a bounded edge.

1.5.2 Topology of the real part of a tropical curve

In Paper II and Paper III, we investigate the case of non-singular real tropical curves $(C, \mathcal{E})$ inside a non-singular projective tropical toric surface $\mathbb{TP}_\Sigma$. In this case, by Theorem 1.5.1, the real part $\mathbb{R}C_\mathcal{E}$ must be homeomorphic to a non-empty disjoint union of circles S^1. By [RS21, Theorem 7.2], the real phase structure $\mathcal{E}$ on C contains all the information necessary to compute the number of connected components of $\mathbb{R}C_\mathcal{E}$. More precisely, the real phase structure $\mathcal{E}$ determines a set of *twisted edges* on the tropical curve C, which we will use to give criterions to obtain a prescribed number of connected components, as follows. Let e be a bounded edge of C with vertices v_1 and v_2, let e_i be the edge adjacent to v_i on one side of the affine line containing e, and let e'_i be the edge adjacent to v_i on the other side of the affine line containing e, so that e_1 and e_2 are on the same side (and similarly for e'_1 and e'_2). A bounded edge e of C is said to be *twisted* if the intersections $\mathcal{E}(e) \cap \mathcal{E}(e_i) \cap \mathcal{E}(e'_j)$ are non-empty, for $i \neq j \in \{1, 2\}$.

The notion of twisted edges relates real phase structures with *real amoebas* [Bru+15, Section 3.2], which are the sets $\mathcal{A}_t := \mathrm{Log}_t(\mathcal{C}_t(\mathbb{R}))$ for $t \in \mathbb{R} > 0$ small enough when the family of non-singular real algebraic curves $\mathcal{C} := (\mathcal{C}_t)_t$ satisfies Theorem 1.5.1 for the non-singular real tropical curve $(C, \mathcal{E})$. Indeed, locally around each twisted edge e of C, the sets $\mathcal{A}_t$ (for $t > 0$ small enough) consist of two branches intersecting in one point (Figure 1.5b). In particular, the set of twisted edges T on C determines the real part $\mathbb{R}C_\mathcal{E}$ up to symmetry with respect to a 1-dimensional stratum of the real part $\mathbb{RP}_\Sigma$ of $\mathbb{TP}_\Sigma$ [Bru+15, §3.2].

In his thesis, Haas gave a necessary and sufficient condition for non-singular real algebraic curves obtained via Viro's combinatorial patchworking to be maximal [Haa97, Theorem 7.3.0.10]. In the following theorems, a *primitive cycle* of C is a cycle of C (seen as a graph) bounding a connected component of the complement $\mathbb{TP}_\Sigma \backslash C$, and an *exposed edge* of C is an edge belonging to the boundary of a connected component D of $\mathbb{TP}_\Sigma \backslash C$ meeting a 1-dimensional stratum of $\mathbb{TP}_\Sigma$. Haas' result can be reformulated in terms of twisted edges on a non-singular real tropical curve as follows.

Theorem 1.5.2 ([Haa97]). *Let $(C, \mathcal{E})$ be a non-singular real tropical curve in a non-singular projective tropical toric surface $\mathbb{TP}_\Sigma$, and let T be the set of twisted edges on C. Let g be the number of primitive cycles on C. The real part $\mathbb{R}C_\mathcal{E}$ has exactly $g+1$ connected components if and only if*

- *every cycle in C has an even number of edges belonging to T;*
- *every edge of T is exposed.*

In Paper III, we obtain two new criteria for constructing a real part $\mathbb{R}C_\mathcal{E}$ with prescribed number of connected components, in terms of twisted edges on the non-singular tropical curve C.

Theorem 1.5.3 (Theorem III.4.16). *Let $(C, \mathcal{E})$ be a non-singular real tropical curve in a non-singular projective tropical toric surface $\mathbb{TP}_\Sigma$, and let T be the set of twisted edges on C induced by $\mathcal{E}$. Let g be the number of primitive cycles on C. The real part $\mathbb{R}C_\mathcal{E}$ has exactly g connected components if and only if the graph dual to T contains a complete subgraph K_n on $1 \leq n \leq 4$ vertices such that:*

1. *the non-exposed edges in T are those dual to the edges of K_n;*
2. *every primitive cycle γ in C dual to a vertex of K_n has an odd number of edges in T;*
3. *every primitive cycle γ in C not dual to a vertex of K_n has an even number of edges in T.*

In Corollary III.4.21, we use Theorem 1.5.3 in order to give a sufficient condition to construct a real part $\mathbb{R}C_\mathcal{E}$ with $g-r$ connected components, for any $r \leq g$. Moreover, the curves $\mathcal{C}_t$ are non-dividing for all $t \in \mathbb{R}_{>0}$ small enough, with $\mathcal{C} := (\mathcal{C}_t)_t$ a non-singular algebraic curve over $\mathbb{K}_\mathbb{R}$ satisfying Theorem 1.5.1 for the non-singular real tropical curve $(C, \mathcal{E})$. Indeed, Haas showed (in terms of combinatorial patchworking) that a non-singular real tropical curve $(C, \mathcal{E})$ is dividing, that is the curves $\mathcal{C}_t$ from Theorem 1.5.1 are dividing for all $t \in \mathbb{R}_{>0}$ small enough, if and only if each cycle on C (seen as a graph) has an even number of twisted edges [Haa97, Section 5.4]. This motivates the following new criterion for dividing $(M-2)$-curves.

Theorem 1.5.4 (Theorem III.4.22). *Let $(C, \mathcal{E})$ be a non-singular dividing real tropical curve in a non-singular projective tropical toric surface $\mathbb{TP}_\Sigma$, and let T be the set of twisted edges on C. Let g be the number of primitive cycles on C. The real part $\mathbb{R}C_\mathcal{E}$ has exactly $g-1$ connected components if and only if for $T' \subset T$ the subset of non-exposed twisted edges, the graph dual to T' is either a complete bipartite planar graph or a complete tripartite planar graph.*

In Corollary III.4.25, we use Theorem 1.5.4 in order to give a sufficient condition to construct a real part $\mathbb{R}C_{\mathcal{E}}$ with $g-2s$ connected components such that $(C,\mathcal{E})$ is dividing, for any $s \leq \lfloor \frac{g}{2} \rfloor$.

1.5.3 Relative topology of the real part of a tropical curve

In Section 1.5.2, we provided information on the intrinsic topology of the real part of a non-singular real tropical curve in terms of twisted edges. The next natural problem is to determine how the data of twisted edges gives information on the relative topology of the real part of a non-singular real tropical curve.

Let $(C,\mathcal{E})$ be a non-singular real tropical curve in a non-singular projective tropical toric surface $\mathbb{TP}_\Sigma$, with set of twisted edges T. We extend naturally the definitions of ovals, even ovals and odd ovals of a non-singular real algebraic curve $\mathcal{C}$ in a non-singular projective toric surface $\mathbb{P}_\Sigma$ to the real part $\mathbb{R}C_{\mathcal{E}} \subset \mathbb{RP}_\Sigma$ of the non-singular real tropical curve $(C,\mathcal{E})$, see Definition III.1.5, Section III.5.2 and Section III.6. In Proposition III.5.10, we show that if the Newton polygon of C has all edges of even lattice length and if all the edges in T are non-exposed, then the real part $\mathbb{R}C_{\mathcal{E}}$ consists only of ovals, no matter which non-singular projective tropical toric surface $\mathbb{TP}_\Sigma$ we started with.

In the case of a non-singular real tropical curve $(C,\mathcal{E})$ of even degree in $\mathbb{TP}^2$, with set of twisted edges T satisfying Theorem 1.5.2, Haas gave a count of the number of even and odd ovals of $\mathbb{R}C_{\mathcal{E}}$ in terms of T and the parity of the integer points inside the Newton polygon of C [Haa97, Corollary 11.4.2.2]. We show in Theorem III.1.6 a similar result for $(C,\mathcal{E})$ a non-singular dividing real tropical curve in a non-singular projective tropical toric surface $\mathbb{TP}_\Sigma$, so that the associated set of twisted edges T is dual to a disjoint union of complete bipartite graphs of the form $K_{2,2l}$. In particular, we can count the number of even and odd ovals of the counter-examples to Ragsdale's conjecture constructed by Itenberg [Ite93] and the new counter-examples we construct in Theorem 1.2.1 using Theorem III.1.6. Moreover, Theorem III.1.6 provides a similar framework as Haas used to prove his upper bound on the number of even and odd ovals of a maximal tropical curve in $\mathbb{TP}^2$ [Haa97, Theorem 12.4.0.12]. As a future direction of research, it would then be interesting to prove upper bounds on the number of even and odd ovals of a dividing tropical curve satisfying the assumptions of Theorem III.1.6, or more generally the assumptions of Proposition III.6.11, depending on the ambient non-singular projective tropical toric surface $\mathbb{TP}_\Sigma$.

1.5.4 Intersection of real tropical curves

In Paper II, we study non-singular hyperbolic curves in $\mathbb{P}^2_{\mathbb{K}_\mathbb{R}}$ in terms of their tropicalisations and their associated real phase structures. We can use the criteria from Section 1.5.2 and Section 1.5.3 in order to obtain a characterisation of the real phase structure induced by a non-singular hyperbolic curve in $\mathbb{P}^2_\mathbb{R}$, based on Orevkov's characterisation [Ore02, Proposition 1.1], as we prove in Proposition II.1.2. However, we do not obtain any information on the hyperbolicity locus this way. Since hyperbolic curves are defined in terms of their

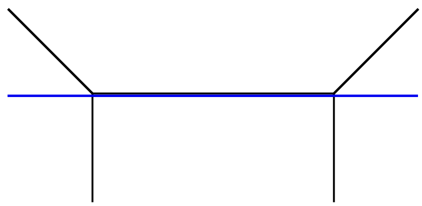


Figure 1.6: Non-transverse intersection in an edge.

intersection with members of a pencil of real lines, we then want to study the intersections of non-singular real tropical curves, and what are the possible realisations in $(\mathbb{K}^\times)^2$ of those real tropical intersections. A *realisation* of a non-singular real tropical curve $(C, \mathcal{E})$ in $\mathbb{R}^2$ is a non-singular real algebraic curve in $(\mathbb{K}_\mathbb{R}^\times)^2$ satisfying Theorem 1.5.1 for $(C, \mathcal{E})$. Then a *realisation* of an intersection component E of two non-singular real tropical curves $(C, \mathcal{E})$ and $(C', \mathcal{E}')$ is a set $\{p_1, \ldots, p_m\} \subset (\mathbb{K}^\times)^2$ of intersection points of realisations $\mathcal{C}, \mathcal{C}'$ of $(C, \mathcal{E})$ and $(C', \mathcal{E}')$ that tropicalise in E. For our purpose, we only need to consider the intersections types arising when intersecting a non-singular real tropical curve with a pencil of real tropical lines. If we choose generically the base point of the pencil, this reduces to three possible intersection types.

Let $(C, \mathcal{E})$ and $(C', \mathcal{E}')$ be two non-singular real tropical curves in $\mathbb{R}^2$. The first intersection type we consider is the case of *transverse intersection points*, which are isolated intersection points of C and C' in the interior of both an edge of C and an edge of C' (considering the edges as closed in Euclidean topology). Depending on the direction modulo 2 of those edges and the real phase structures $\mathcal{E}$ and $\mathcal{E}'$, we can say that $(C, \mathcal{E})$ and $(C, \mathcal{E}')$ intersect in either 0, 1 or 2 *real tropical points* of the form (p, ε), for $\varepsilon \in \mathbb{Z}_2^2$. We show in Proposition II.4.6 that for any realisation of $(C, \mathcal{E})$ and $(C', \mathcal{E}')$, each of these real tropical intersection points is realised by a single real intersection point in $\mathcal{C} \cap \mathcal{C}'$ of intersection multiplicity 1, and the number of pairs of complex conjugated points in $\mathcal{C} \cap \mathcal{C}'$ with tropicalisation p is determined by the direction of the edges.

The second intersection type correspond to the case when an edge of C' contains a bounded edge of C in its interior, see Figure 1.6. In that case, we obtain the following result.

Theorem 1.5.5 (Theorem II.1.3)**.** *Let $(C, \mathcal{E})$ and $(C', \mathcal{E}')$ be two non-singular real tropical curves in $\mathbb{R}^2$ such that there exists a closed bounded edge e of C contained in the interior of a closed edge e' of C' (see Figure 1.6).*

1. *If $\mathcal{E}(e) \neq \mathcal{E}'(e')$, or is $\mathcal{E}(e) = \mathcal{E}'(e')$ and e is twisted, then all realisations of $(C, \mathcal{E})$ and $(C', \mathcal{E}')$ intersect in two distinct real intersection points of multiplicity 1 with tropicalisation in e.*

2. *If $\mathcal{E}(e) = \mathcal{E}'(e')$ and e is non-twisted, then a realisation of $(C, \mathcal{E})$ can intersect a realisation of $(C', \mathcal{E}')$ in either two distinct real points of multiplicity 1 with tropicalisation in e, two distinct complex conjugated points*

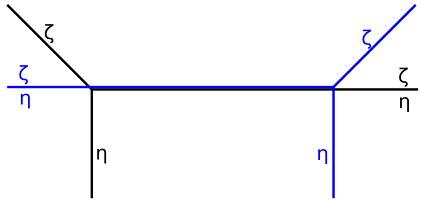


Figure 1.7: Relatively non-twisted non-transverse intersection component.

of multiplicity 1 with tropicalisation in e, or a multiplicity 2 real point with tropicalisation in e. Moreover, there exist infinitely many realisations satisfying the first two cases, and there exist exactly two pairs of realisations satisfying the third case.

The last type of intersection considered corresponds to the case when an edge e of C and an edge e' of C' intersect in a segment E strictly contained in both edges, see Figure 1.7. In order to treat this case, we introduce *relatively twisted* intersection components. Let E be a segment as above, with vertices v_1 and v_2. Let e_i be the edge of $C \cup C'$ adjacent to v_i on one side of the affine line containing E, and let e_i' be the edge of $C \cup C'$ adjacent to v_i on the other side of the affine line containing E, so that e_1 and e_2 are on the same side (and similarly for e_1' and e_2'). Up to renumbering the vertices, the edges e_1 and e_1' belong to C, and the edges e_2 and e_2' belong to C'. Let $\mathcal{E}$ and $\mathcal{E}'$ be real phase structures on C and C' so that $\mathcal{E}(e) = \mathcal{E}'(e')$. The segment E is said to be relatively twisted with respect to $(C, \mathcal{E})$ and $(C', \mathcal{E}')$ if the intersections $\mathcal{E}(e) \cap \mathcal{E}(e_1) \cap \mathcal{E}'(e_2')$ and $\mathcal{E}(e) \cap \mathcal{E}(e_1') \cap \mathcal{E}'(e_2)$ are non-empty. Otherwise, the segment E is said to be relatively non-twisted, and the real phase structures satisfy the condition pictured in Figure 1.7. We obtain the following result, where the roles of relatively twisted and non-twisted are switched compared to Theorem 1.5.5.

Theorem 1.5.6 (Theorem II.1.4)**.** *Let $(C, \mathcal{E})$ and $(C', \mathcal{E}')$ be two non-singular real tropical curves in $\mathbb{R}^2$ such that there exists a non-transverse connected component $E \subset C \cap C'$ which is a segment, non-reduced to a point, strictly contained in both a closed edge e of C and a closed edge e' of C' (see Figure 1.7).*

1. *If $\mathcal{E}(e) \neq \mathcal{E}'(e')$, or if $\mathcal{E}(e) = \mathcal{E}'(e')$ and E is relatively non-twisted, then all realisations of $(C, \mathcal{E})$ and $(C', \mathcal{E}')$ intersect in two distinct real points of multiplicity 1 with tropicalisation in E.*

2. *If $\mathcal{E}(e) = \mathcal{E}'(e')$ and E is relatively twisted, then a realisation $(C, \mathcal{E})$ can intersect a realisation of $(C', \mathcal{E}')$ in either two distinct real points of multiplicity 1 with tropicalisation in E, two distinct complex conjugated points of multiplicity 1 with tropicalisation in E, or a multiplicity 2 real point with tropicalisation in E. Moreover, there exist infinitely many realisations satisfying the first two cases, and there exist exactly two pairs of realisations satisfying the third case.*

1.5.5 Hyperbolic tropical curves

Speyer characterised the tropicalisation of stable curves in $\mathbb{P}^2_{\mathbb{K}}$ in order to solve Horn's problem [Spe05]. His result can be reformulated as follows. A non-singular real algebraic curve $\mathcal{C} \subset \mathbb{P}^2_{\mathbb{K}}$ is stable if and only if the tropicalisation $C := \mathrm{Trop}(\mathcal{C})$ is a honeycomb, that is every edge of C has direction $(1,0),(0,1)$ or $(1,1)$, and the vector $(0,0) \in \mathbb{Z}_2^2$ does not appear in the real phase structure $\mathcal{E}$ on C such that $\mathcal{C}$ is a realisation of $(C,\mathcal{E})$.

In Paper II, we generalise Speyer's result to the case of non-singular hyperbolic curves in $\mathbb{P}^2_{\mathbb{K}}$, first using a topological criterion and then an intersection-theoretic criterion. We say that a non-singular real tropical curve $(C,\mathcal{E})$ of degree d in $\mathbb{TP}^2$ is *hyperbolic* if the pair $(\mathbb{RP}^2, \mathbb{R}C_{\mathcal{E}})$ is homeomorphic (via Theorem 1.5.1) to a pair $(\mathbb{P}^2(\mathbb{R}), \mathcal{C}_t(\mathbb{R}))$, for $\mathcal{C} := (\mathcal{C}_t)_t \subset \mathbb{P}^2_{\mathbb{K}_R}$ a non-singular curve of degree d such that $\mathcal{C}_t$ is hyperbolic for $t > 0$ small enough. Thanks to a result from Orevkov [Ore07, Proposition 1.1] and Haas criterion for dividing real tropical curves, we obtain the following proposition.

Proposition 1.5.7 (Proposition II.1.2)**.** *Let $(C,\mathcal{E})$ be a non-singular real tropical curve of degree d in $\mathbb{TP}^2$, with set of twisted edges T. Then $(C,\mathcal{E})$ is hyperbolic if and only if every (graph-theoretic) cycle on C has an even number of edges in T and the real part $\mathbb{R}C_{\mathcal{E}}$ has $\lceil \frac{d}{2} \rceil$ connected components.*

In particular, we can use Theorem 1.5.4 and Corollary III.4.25 to construct many examples of non-singular hyperbolic tropical curves. However, we do not recover information on the hyperbolicity locus via these results.

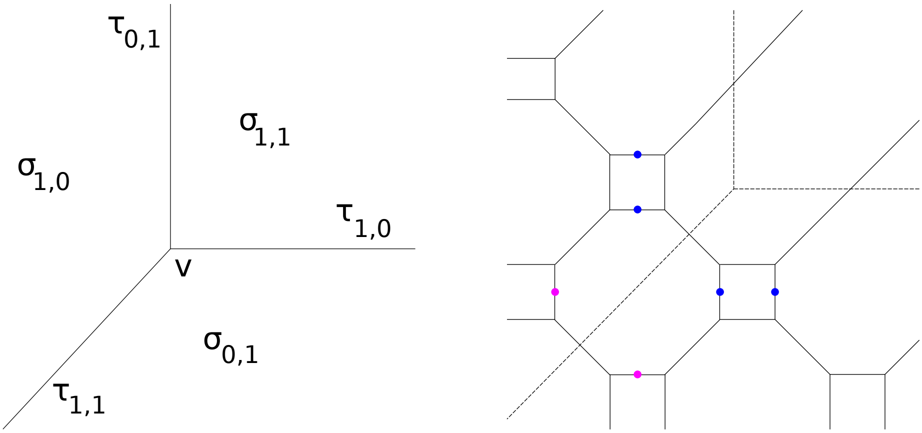
Thanks to the results of Section 1.5.4, we can characterise intersection-theoretically the hyperbolicity of non-singular real tropical curves in $\mathbb{TP}^2$. We say that a non-singular real tropical curve $(C,\mathcal{E})$ of degree d in $\mathbb{TP}^2$ is *hyperbolic with respect to* a real tropical point (p,ε) if $p \notin C$ and for every realisation $\mathcal{C}$ of $(C,\mathcal{E})$ in $\mathbb{P}^2_{\mathbb{K}_\mathbb{R}}$, every real line $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{K}_\mathbb{R}}$ going through a realisation of (p,ε) intersect $\mathcal{C}$ in d distinct points in $\mathbb{P}^2(\mathbb{K}_\mathbb{R})$.

We can parametrise a pencil of tropical lines through a point $v \in \mathbb{TP}^2$ via Figure 1.8a, so that each point lying on a 1-dimensional face τ_η is the vertex of a tropical line through v. Then the hyperbolicity of a non-singular real tropical curve with respect to a real tropical point (v,ε) is expressed in terms of this parametrisation as follows.

Theorem 1.5.8 (Theorem II.1.5)**.** *Let $(C,\mathcal{E})$ be a non-singular real tropical curve of degree d in $\mathbb{TP}^2$, let v' be a point of $\mathbb{TP}^2 \backslash C$ and let $\varepsilon \in \mathcal{R}^2$. Then $(C,\mathcal{E})$ is hyperbolic with respect to the real tropical point (v',ε) if and only if for Σ_v the subdivision of $\mathbb{TP}^2$ with respect to a point v generic with respect to C in the same connected component of $\mathbb{TP}^2 \backslash C$ as v', we have:*

1. *Every vertex of C lying in the interior of a face σ_η of Σ_v is incident to an edge of primitive integer direction η.*

2. *Every edge e of C intersecting a face τ_ζ of Σ_v and such that $|\det(\overrightarrow{e}|\zeta)| = 2$, for $\overrightarrow{e}$ the primitive integer direction of e, satisfies $\varepsilon \in \mathcal{E}_e$.*



(a) Polyhedral subdivision Σ_v of $\mathbb{TP}^2$.

(b) Non-singular real tropical curve hyperbolic with respect to a real tropical point.

Figure 1.8

3. *For every bounded edge e of C of primitive integer direction η intersecting the face σ_η of Σ_v, the edge e is twisted if $e \subset \sigma_\eta$, and otherwise the segment $e \cap \sigma_\eta$ is relatively non-twisted with respect to the unique real tropical line $(L, \mathcal{E}')$ through (v, ε) with $(e \cap \sigma_\eta) \subset e' \subset L$ and $\mathcal{E}_e = \mathcal{E}'_{e'}$.*

If C is a honeycomb in $\mathbb{TP}^2$, then Item 1 and Item 2 of Theorem 1.5.8 are automatically satisfied. Thus, we can go further in the honeycomb case, see Corollary II.1.6 and Figure 1.9. The idea is that given an integer point p (such as the purple point in Figure 1.9) in the Newton polygon of a non-singular real honeycomb $(C, \mathcal{E})$, the dual connected component $p^\vee \subset \mathbb{TP}^2 \setminus C$ has a symmetric copy contained in the hyperbolicity locus of $\mathbb{R}C_\mathcal{E}$ if and only if the edges on the left, below and diagonally above p are dual to twisted edges of C (see the blue edges in Figure 1.9).

1.6 Summary of Papers

Paper I gives a classification of real-fibered and hyperbolic morphisms from smooth real del Pezzo surfaces to the projective plane $\mathbb{P}^2_\mathbb{R}$, using the hyperplane section criterion for hyperbolicity given in Theorem 1.3.1. Some common properties of the del Pezzo surfaces admitting such a real-fibered morphism are listed in Theorem 1.3.2. Similarly, some common properties of the del Pezzo surfaces admitting a hyperbolic morphism are listed in Theorem 1.3.3. For instance, the real del Pezzo surfaces that can be defined as minimal conic bundles admit a hyperbolic morphism. This motivates the last section, which begins to characterise hyperbolic morphisms of

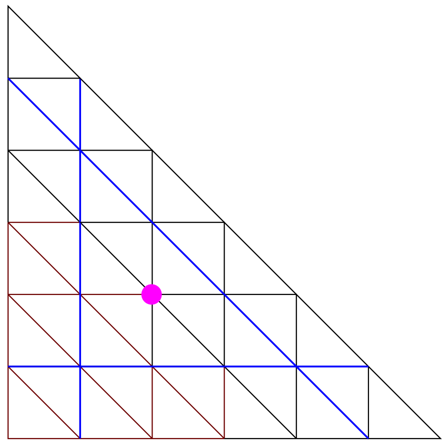


Figure 1.9: Dual Subdivision of a hyperbolic honeycomb.

minimal conic bundles (Corollary I.7.3, Proposition I.7.6). Using those minimal conic bundles, we construct almost all possible homeomorphism types of smooth hyperbolic surfaces (Proposition 1.3.4).

Paper II is a study of hyperbolic curves near the non-singular tropical limit. Non-singular hyperbolic real tropical curves in $\mathbb{TP}^2$ are characterised via a combinatorial analogue of a topological criterion for non-singular hyperbolic curves in $\mathbb{P}^2$ (Proposition II.1.2). The tropical and real tropical analogues of the hyperbolicity locus of a non-singular hyperbolic curve in $\mathbb{P}^2$ are introduced, given as subsets of $\mathbb{TP}^2$ and $\mathbb{RP}^2$ respectively (Definition II.5.10), and we relate these to the notions of signed tropical convexity (Remark II.5.13) and tropical spectrahedra (Remark II.5.17). Using those analogues to make the correspondence between the topological and the intersection-theoretic characterisation of hyperbolicity in the tropical setting, we show the necessary and sufficient conditions for a non-singular real tropical curve of degree d in $\mathbb{TP}^2$ to be hyperbolic with respect to a fixed real tropical point (Theorem II.1.5). We go further in the case of honeycombs, using $\mathbb{Z}_2$-vector spaces and $\mathbb{Z}_2$-affine spaces structures on subsets of *configurations of twists* (Theorem II.5.29).

Paper III is a continuation of Itenberg's work [Ite95] and Haas' thesis [Haa97] on the understanding of combinatorial patchworking of real algebraic curves, using the recent developments from [RS21] on the topology of non-singular real tropical hypersurfaces. We introduce the notion of *twisted cycle*, which allows to describe the connected components of the real part of a non-singular tropical curve directly from the real phase structure. We give a characterisation of the set of twisted edges on a non-singular $(M-1)$ real tropical curve (Theorem 1.5.3) and on a non-singular dividing $(M-2)$ real tropical curve (Theorem 1.5.4), extending Haas' result [Haa97, Theorem

7.3.0.10] for maximal curves. We then obtain a sufficient condition on a non-singular real tropical curve in an arbitrary non-singular projective tropical toric surface so that its real part consists only of ovals (Proposition III.5.10). From the real tropical curves satisfying the latter condition, we count the number of even and odd ovals in their real part in terms of the dual subdivision (Theorem III.1.6), in a similar manner as the count for maximal tropical curves in $\mathbb{TP}^2$ given by Itenberg [Ite95] and Haas [Haa97]. Using Theorem III.1.6, we give a construction of new counter-examples to Ragsdale's conjecture (Theorem 1.2.1) in a similar fashion as Itenberg's construction [Ite01].

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Appendices

Appendix A

Real structure above trivalent graphs

A.1 Definitions

Definition A.1.1. A planar graph C is said to be *trivalent* if every vertex v of C is either 3-valent or 1-valent. An edge e of C is said to be *bounded* if its two incident vertices v, v' are 3-valent. The subset of bounded edges of C is denoted $\mathrm{Edge}^0(C)$.

Remark A.1.2. We can put a $\mathbb{Z}_2$-vector space structure on $\mathrm{Edge}^0(C)$ by identifying it with $\mathbb{Z}_2^{|\mathrm{Edge}^0(C)|}$.

Remark A.1.3. A non-singular tropical curve C in a non-singular projective tropical toric surface $\mathbb{TP}_\Sigma$ is a trivalent graph.

Given a trivalent graph C, we construct a topological surface S as follows:

1. To each vertex $v \in \mathrm{Vertex}(C)$, we associate a topological surface S_v, which is an oriented pair of pants if v is 3-valent and an oriented closed disk if v is 1-valent. We choose a one-to-one correspondence between the boundary components of S_v and edges e of C such that v is incident to e.

2. To each edge $e \in \mathrm{Edge}(C)$, we associate an oriented cylinder S_e. We choose a one-to-one correspondence between the boundary components of S_e and the vertices v of C incident to e.

3. To each pair $(v, e) \in \mathrm{Vertex}(C) \times \mathrm{Edge}(C)$ such that v is incident to e, we associate an orientation reversing homeomorphism between the boundary component of S_v associated to e and the boundary component of S_e associated to v.

The surface S is then obtained by gluing all the surfaces S_e and S_v according to the homeomorphisms associated to the pairs (v, e).

For C a trivalent graph and S a surface constructed from C, we get that if $\dim H_1(C; \mathbb{Z}_2) = g$, then $\dim H_1(S; \mathbb{Z}_2) = 2g$. Moreover, the homology group $H_1(S; \mathbb{Z}_2)$ is given as the direct sum

$$H_1(S; \mathbb{Z}_2) = H_{1,0}(S; \mathbb{Z}_2) \oplus H_{0,1}(S; \mathbb{Z}_2)$$

with $H_{1,0}(S; \mathbb{Z}_2)$ generated by the non-contractible loops $\gamma_e \subset S_e$, and $H_{0,1}(S; \mathbb{Z}_2)$ is generated by the loops γ_α lifted from the cycles α on C. Both the groups $H_{1,0}(S; \mathbb{Z}_2)$ and $H_{0,1}(S; \mathbb{Z}_2)$ have then dimension g. Furthermore, by [RBB17, Lemma 2.3], the intersection form on $H_1(S; \mathbb{Z}_2)$ vanishes on $H_{1,0}(S; \mathbb{Z}_2)$.

Definition A.1.4 ([RBB17])**.** A *real trivalent graph* is a pair (C, τ) where C is a trivalent graph and τ is a continuous involution (called *real structure*), such that the restriction of τ on any (open) edge of C is either the identity or has no fixed points.

Remark A.1.5. Any trivalent graph C has a canonical real structure given by $\tau = \mathrm{id}$. In the following, whenever we do not precise which real structure we put on a trivalent graph C, we will mean the identity real structure.

Definition A.1.6. Let S be a surface constructed from a real trivalent graph C. A *real structure above* C is a continuous orientation-reversing involution $\tau_C : S \to S$ such that for each edge e of C, the restriction $(\tau_C)|_{S_e}$ is (up to isotopy and composition with a power of the Dehn twist DT) either the complex conjugation conj or the "half" Dehn twist $\mathrm{DT} \circ \mathrm{conj}$.

Definition A.1.7. Let C be a real trivalent graph, and let $T \subset \mathrm{Edge}^0(C)$ be a subset of bounded edges of C. The real structure τ_C above C *induces* T if, for any bounded edge $e \in \mathrm{Edge}^0(C)$, the restriction of τ_C to S_e satisfies:

$$(\tau_C)|_{S_e} \simeq \mathrm{DT} \circ \mathrm{conj} \Leftrightarrow e \in T.$$

The subset T will then be called a *set of twisted edges* on C.

Remark A.1.8. With the definition above, a set of twisted edges $T \in \mathrm{Edge}^0(C)$ does not need to satisfy some admissibility condition. If we assume that C is piecewise-linear and $T \in \mathrm{Adm}(C)$ is twist-admissible, then we recover the notion of set of twisted edges induced from a real phase structure on a tropical curve.

A.2 Computation of number of connected components

We want to generalise the computation of number of real components to real structures above trivalent graphs.

Theorem A.2.1. *Let C be a real trivalent graph, let S be a surface constructed from C and let $\tau_C : S \to S$ be a real structure above C. Let $T \in \mathrm{Edge}^0(C)$ be the set of twisted edges induced by τ_C. We have a homomorphism*

$$\partial_T : H_{0,1}(S; \mathbb{Z}_2) \to H_{1,0}(S, \mathbb{Z}_2)$$

sending each cycle $\gamma_\alpha \in H_{0,1}(S_C; \mathbb{Z}_2)$ (lifted from the cycle $\alpha \in H_1(C; \mathbb{Z}_2)$) to the cycle

$$\sum_{e \in \alpha \cap T} \gamma_e.$$

In particular, the number of connected components of $S^{\tau_C} := \mathrm{Fix}(\tau_C)$ is equal to $1 + \dim \ker \partial_T$.

Proof. The connected components of S^{τ_C} are homeomorphic to S^1, therefore the number of connected components of S^{τ_C} is given by $\dim H_1(S^{\tau_C}; \mathbb{Z}_2)$. By [BR91, Lemma C.3 , Proposition C.4], for $\sigma_{\#} := (\tau_{C,\#} + \mathrm{id}) : C_{\bullet}(S;\mathbb{Z}_2) \to C_{\bullet}(S;\mathbb{Z}_2)$ we have an exact sequence of chain complexes

$$0 \to C_{\bullet}(S^{\tau_C};\mathbb{Z}_2) \oplus \sigma_{\#}C_{\bullet}(S;\mathbb{Z}_2) \to C_{\bullet}(S;\mathbb{Z}_2) \to \sigma_{\#}C_{\bullet}(S;\mathbb{Z}_2) \to 0;$$

which induces the long exact sequence in homology

$$\begin{aligned} \cdots \to H_2(S/\tau_C, S^{\tau_C};\mathbb{Z}_2) &\overset{\partial_2}{\to} H_1(S^{\tau_C};\mathbb{Z}_2) \oplus H_1(S/\tau_C, S^{\tau_C};\mathbb{Z}_2) \\ \overset{(inc_*, inc'_*)}{\to} H_1(S;\mathbb{Z}_2) &\xrightarrow{\sigma_*} H_1(S/\tau_C, S^{\tau_C};\mathbb{Z}_2) \to \cdots . \end{aligned}$$

Therefore, we get by exactness that

$$\dim H_1(S^{\tau_C};\mathbb{Z}_2) \leq \dim \operatorname{im} \partial_2 + \dim \ker \sigma_* - \dim \operatorname{im} inc'_*. \tag{A.1}$$

The connecting homomorphism ∂_2 sends the fundamental class

$$[S/\tau_C] \in H_2(S/\tau_C, S^{\tau_C};\mathbb{Z}_2)$$

to the fundamental class of its boundary

$$[S^{\tau_C}] \in H_1(S^{\tau_C};\mathbb{Z}_2).$$

Since $H_2(S/\tau_C, S^{\tau_C};\mathbb{Z}_2)$ is of dimension 1, we get that $\operatorname{im} \partial_2 \subset H_1(S^{\tau_C};\mathbb{Z}_2)$, so we actually have equality in Equation (A.1), and $\dim \operatorname{im} \partial_2 = 1$. By the decomposition $H_1(S;\mathbb{Z}_2) = H_{1,0}(S;\mathbb{Z}_2) \oplus H_{0,1}(S;\mathbb{Z}_2)$, we get that

$$\ker(\sigma_*) = \ker\left((\sigma_*)|_{H_{1,0}(S;\mathbb{Z}_2)}\right) \oplus \ker\left((\sigma_*)|_{H_{0,1}(S;\mathbb{Z}_2)}\right).$$

Let us first consider $\ker\left((\sigma_*)|_{H_{0,1}(S;\mathbb{Z}_2)}\right)$. The real structure τ'_C above C induced by the empty set of twists on C satisfy $\tau'_{C,*} = \mathrm{id}$ by [RBB17, Corollary 3.4]. For $\mathfrak{E}(\tau_C - \tau'_C)$ the set of edges e of C such that

$$(\tau_C)|_{S_e} \not\simeq (\tau'_C)|_{S_e},$$

we get that for any cycle $\gamma_\alpha \in H_{0,1}(S;\mathbb{Z}_2)$,

$$\begin{aligned} \sigma_*(\gamma_\alpha) &= (\tau_{C,*} + \mathrm{id})(\gamma_\alpha) \\ &= \sum_{e \in \mathfrak{E}(\tau_C - \tau'_C) \cap \alpha} \gamma_e \\ &= \sum_{e \in \alpha \cap T} \gamma_e \in H_{1,0}(S;\mathbb{Z}_2). \end{aligned}$$

We then denote $\partial_T := (\sigma_*)|_{H_{0,1}(S;\mathbb{Z}_2)}$, which is the required homomorphism. Therefore, we can decompose $\ker(\sigma_*)$ into

$$\ker(\sigma_*) = \ker\left((\sigma_*)|_{H_{1,0}(S;\mathbb{Z}_2)}\right) \oplus \ker \partial_T.$$

Now, as said in the proof of [RBB17, Theorem 3.2], any two involutions $\tau_{C,1*}, \tau_{C,2*}$ have the same restriction on $H_{1,0}(S;\mathbb{Z}_2)$. In particular, we get that

$$\ker\left((\sigma_*)|_{H_{1,0}(S;\mathbb{Z}_2)}\right) = \ker\left((\tau_{C,*}+\mathrm{id})|_{H_{1,0}(S;\mathbb{Z}_2)}\right) = H_{1,0}(S;\mathbb{Z}_2);$$

thus we get the decomposition

$$\ker(\sigma_*) = H_{1,0}(S;\mathbb{Z}_2) \oplus \ker \partial_T.$$

The homomorphism inc'_* induced by inclusion of chain complexes sends each cycle $[\gamma_i] \in H_1(S/\tau_C, S^{\tau_C};\mathbb{Z}_2)$ to the corresponding cycle $[\gamma_i] \in H_1(S;\mathbb{Z}_2)$. Since $H_1(S/\tau_C, S^{\tau_C};\mathbb{Z}_2)$ has dimension g, then $\mathrm{im}\, inc'_*$ has dimension g. Therefore, we get that

$$\begin{aligned}\dim H_1(S^{\tau_C};\mathbb{Z}_2) &= \dim \mathrm{im}\,\partial_2 - \dim \mathrm{im}\, inc'_* + \dim \ker(\sigma_*)\\ &= 1 - g + \dim H_{1,0}(S;\mathbb{Z}_2) + \dim \ker \partial_T\\ &= 1 - g + g + \dim \ker \partial_T\\ &= 1 + \dim \ker \partial_T.\end{aligned}$$

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The result above allows us to relate the count of connected components of $\mathbb{R}C_{\mathcal{E}}$, for $(C,\mathcal{E})$ a non-singular real tropical curve, to the count for real structures above trivalent graphs, in the following way:

Theorem A.2.2. *Let $(C,\mathcal{E})$ be a non-singular real tropical curve with admissible set of twisted edges $T \in \mathrm{Adm}(C)$, let S be a surface constructed from C and let τ_C be a real structure above C inducing T. Then the number of connected components of the real part $\mathbb{R}C_{\mathcal{E}}$ is equal to the number of connected components of S^{τ_C}.*

Proof. Let $\{\gamma_1, \ldots, \gamma_g\}$ be a basis of $H_{0,1}(S;\mathbb{Z}_2)$ induced by a basis $\{\alpha_1, \ldots, \alpha_g\}$ of $H_1(C;\mathbb{Z}_2) = H_1(C;\mathcal{F}_0)$. The intersection form $\langle\ ,\ \rangle$ on $H_1(S;\mathbb{Z}_2)$ is non-degenerate, therefore we can compute the dimension of $\ker \partial_T$ (in the sense of Theorem A.2.1) by computing the dimension of the kernel of the matrix

$$A := \left(\langle \partial_T(\gamma_i), \gamma_j\rangle\right)_{i,j=1,\ldots,g}.$$

Now, we have a non-degenerate pairing

$$\langle\ ,\ \rangle_{\mathrm{trop}} : H_0(C;\mathcal{F}_1) \times H_1(C;\mathcal{F}_0) \to \mathbb{Z}_2$$

induced from the pairing on integral homology groups for non-singular tropical curves in [Sha11] (a similar non-degenerate pairing defined between tropical homology and cohomology groups is also defined in [Bru+15] (Section 7.8) and [MZ14] (Section 3.2)). This allows us to compute the dimension of $\ker \partial_T$ (in the sense of Theorem III.4.5) by computing the dimension of the kernel of the matrix

$$B := (\langle \partial_T(\alpha_i), \alpha_j \rangle_{\text{trop}})_{i,j=1,\ldots,g}\,.$$

Now, for all $i = 1, \ldots, g$, we have

$$\begin{aligned} \langle \partial_T(\gamma_i), \gamma_i \rangle &= |\alpha_i \cap T| \mod 2 \\ &= \langle \partial_T(\alpha_i), \alpha_i \rangle_{\text{trop}}; \end{aligned}$$

and for all $i \neq j \in \{1, \ldots, g\}$, we have

$$\begin{aligned} \langle \partial_T(\gamma_i), \gamma_j \rangle &= |\alpha_i \cap \alpha_j \cap T| \mod 2 \\ &= \langle \partial_T(\alpha_i), \alpha_j \rangle_{\text{trop}}. \end{aligned}$$

Therefore the matrices A and B are equal, and the result follows. ∎