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Symmetries of the Klein Quartic Curve

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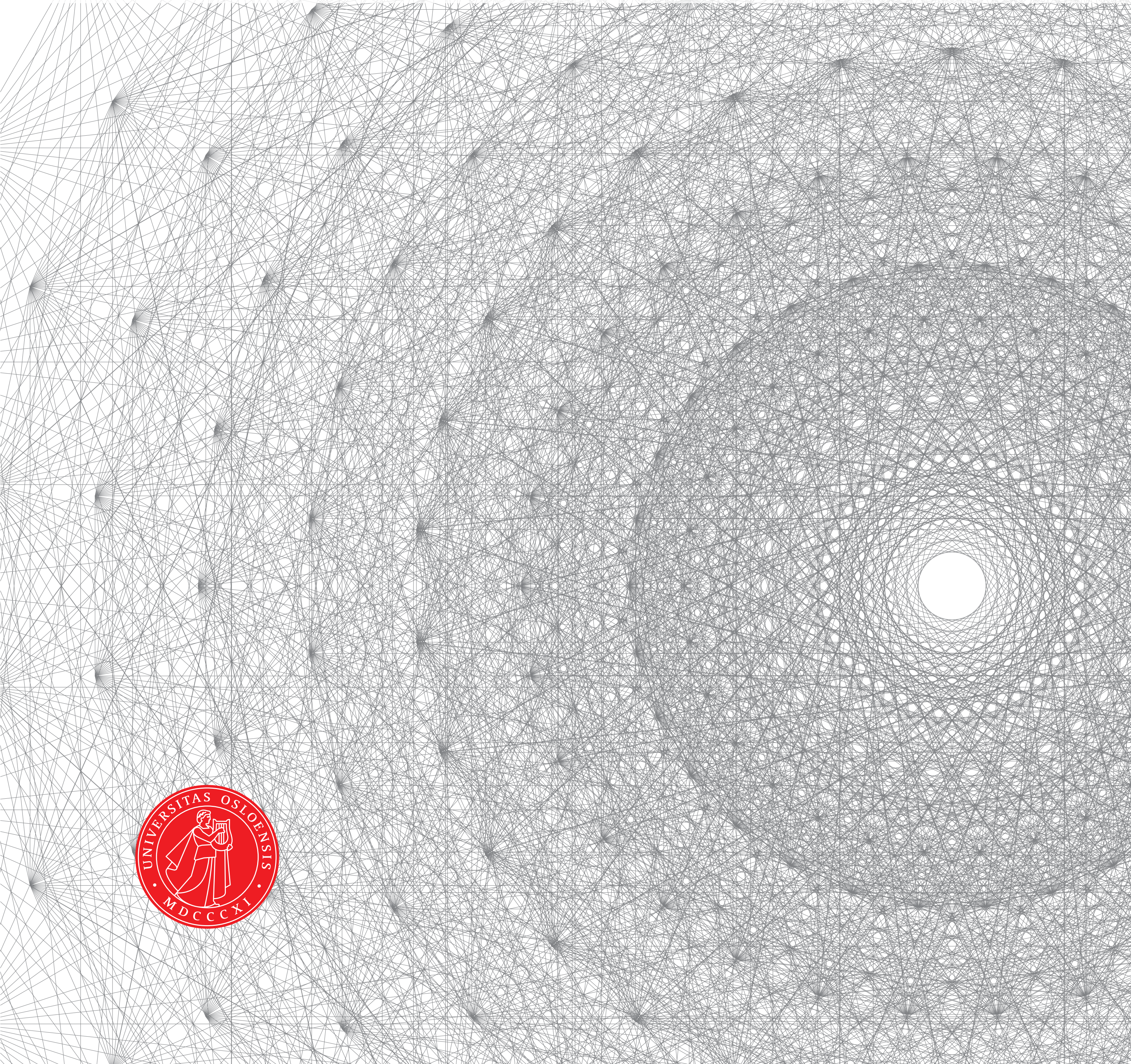

This master's thesis is submitted under the master's programme *Mathematics*, with programme option *Mathematics*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 60 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

We examine some of the symmetries of the Klein quartic curve by describing the fixed points of the subgroups of its automorphism group, and some orbits of fixed points on the quartic curve and on the curves of the covariants.

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CHAPTER 1

Introduction

The aim of this thesis is to examine some of the symmetries of the Klein quartic curve by describing the fixed points of the subgroups of its automorphism group, and some orbits of fixed points on the quartic curve and on the curves of the covariants.
In chapter 2, we define the Klein quartic invariant and its covariants.
In chapter 3, we describe generators and cyclic subgroups of the automorphism group of the Klein quartic curve, specifically the isomorphic groups the projective special linear group PSL(2,7) and the general linear group GL(3,2).
Next, in chapter 4, we examine the representation of the automorphism group in GL(3,$\mathbb{C}$) and the fixed points of its subgroups.
Finally, in chapter 5, by way of examination of some fixed points on the curve of an invariant of degree 21, we show that specific products of the fixed lines of groups of order 2 return an integral factoring of the degree-21 invariant.

CHAPTER 2

The Klein Quartic Curve

With homogeneous coordinates $[x : y : z]$ on $\mathbb{P}^2(\mathbb{C})$, we define the Klein quartic invariant:

Definition 2.0.1. The Klein quartic invariant: $K4 := x^3y + y^3z + z^3x$.

The zero set of this invariant is the Klein quartic curve. It was first described by Klein in [Kle99], and is a compact Riemann surface of genus 3 with an automorphism group of size 168.

Klein also described three covariants to the quartic: a sextic invariant, a degree-14 invariant and a degree-21 invariant.

Definition 2.0.2. The sextic invariant: $K6 := 5x^2y^2z^2 - xy^5 - yz^5 - zx^5$

Definition 2.0.3. The degree-14 invariant:

$$K14 := \frac{1}{9}\begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial g}{\partial x} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial g}{\partial y} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} & \frac{\partial g}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} & 0 \end{vmatrix}$$

$$= x^{14} + y^{14} + z^{14} - 34(x^{11}y^2z + x^2yz^{11} + xy^{11}z^2) - 250(x^9yz^4 + xy^4z^9 + x^4y^9z)$$

$$+ 375(x^8y^4z^2 + x^4y^2z^8 + x^2y^8z^4) + 18(x^7y^7 + x^7z^7 + y^7z^7)$$

$$+ 126(x^6y^3z^5 + x^3y^5z^6 + x^5y^6z^3)$$

Note: The term $\frac{1}{9}$ is not unique, but yields an integral polynomial.

Definition 2.0.4. The degree-21 invariant:

$$K21 := \frac{1}{14}\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} \\ \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} & \frac{\partial h}{\partial z} \end{vmatrix}$$

$$= x^{21} + y^{21} + z^{21} - 7(x^{18}y^2z + x^2yz^{18} + xy^{18}z^2)$$

$$+ 217(x^{16}yz^4 + xy^4z^{16} + x^4y^{16}z) - 308(x^{15}y^4z^2 + x^4y^2z^{15} + x^2y^{15}z^4)$$

$$- 57(x^{14}y^7 + x^7z^{14} + y^{14}z^7) - 289(x^{14}z^7 + y^7z^{14} + x^7y^{14})$$

$$+ 4018(x^{13}y^3z^5 + x^3y^5z^{13} + x^5y^{13}z^3) + 637(x^{12}y^6z^3 + x^6y^3z^{12} + x^3y^{12}z^6)$$

$$+ 1638(x^{11}y^9z + x^9yz^{11} + xy^{11}z^9) - 6279(x^{11}y^2z^8 + x^2y^8z^{11} + x^8y^{11}z^2)$$

$$+ 7007(x^{10}y^5z^6 + x^5y^6z^{10} + x^6y^{10}z^5) - 10010(x^9y^8z^4 + x^8y^4z^9 + x^4y^9z^8)$$

$$+ 10296x^7y^7z^7$$

Note: The term $\frac{1}{14}$ is not unique, but yields an integral polynomial.

CHAPTER 3

The automorphism group

In this chapter we study some aspects of two representations of the automorphism group of the Klein quartic curve; PSL(2,7) and GL(3,2).

3.1 Generators of PSL(2,7)

The projective special linear group PSL(2,7) consists of the quotient group of all 2×2 matrices with unit determinant over the finite field of 7 elements, identifying the identity matrix I and -I. It is well known that PSL(2,7) has 168 elements. This group is generated by the matrices S', T' and R':

$$S' := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$T' := \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

$$R' := \begin{pmatrix} 0 & 6 \\ 1 & 0 \end{pmatrix}$$

By calculation we establish the following proposition:

Proposition 3.1.1. $S'^7 = T'^3 = R'^2 = I$.

3.2 Cyclic subgroups of PSL(2,7)

The cyclic subgroups of PSL(2,7) are as follows: 28 cyclic subgroups of order 3, 21 cyclic subgroups of order 4, each with a further cyclic subgroup of order 2, and 8 cyclic subgroups of order 7. These cyclic subgroups with a generator are listed in the tables below.

Table 3.1: Cyclic subgroups of order 7 in PSL(2,7) with a generator

Group	Generator
A'_1	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
A'_2	$\begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$
A'_3	$\begin{pmatrix} 0 & 1 \\ 6 & 2 \end{pmatrix}$
A'_4	$\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$
A'_5	$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$
A'_6	$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$
A'_7	$\begin{pmatrix} 2 & 1 \\ 6 & 0 \end{pmatrix}$
A'_8	$\begin{pmatrix} 3 & 4 \\ 6 & 6 \end{pmatrix}$

Table 3.2: Cyclic subgroups of order 3 in PSL(2,7) with a generator

Group	Generator	Group	Generator	Group	Generator
B'_1	$\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$	B'_2	$\begin{pmatrix} 3 & 5 \\ 0 & 5 \end{pmatrix}$	B'_3	$\begin{pmatrix} 3 & 3 \\ 0 & 5 \end{pmatrix}$
B'_4	$\begin{pmatrix} 3 & 1 \\ 0 & 5 \end{pmatrix}$	B'_5	$\begin{pmatrix} 3 & 6 \\ 0 & 5 \end{pmatrix}$	B'_6	$\begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix}$
B'_7	$\begin{pmatrix} 3 & 2 \\ 0 & 5 \end{pmatrix}$	B'_8	$\begin{pmatrix} 0 & 1 \\ 6 & 6 \end{pmatrix}$	B'_9	$\begin{pmatrix} 1 & 6 \\ 1 & 0 \end{pmatrix}$
B'_{10}	$\begin{pmatrix} 1 & 3 \\ 6 & 5 \end{pmatrix}$	B'_{11}	$\begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix}$	B'_{12}	$\begin{pmatrix} 1 & 4 \\ 5 & 0 \end{pmatrix}$
B'_{13}	$\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$	B'_{14}	$\begin{pmatrix} 1 & 5 \\ 5 & 5 \end{pmatrix}$	B'_{15}	$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$
B'_{16}	$\begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}$	B'_{17}	$\begin{pmatrix} 1 & 6 \\ 3 & 5 \end{pmatrix}$	B'_{18}	$\begin{pmatrix} 1 & 5 \\ 4 & 0 \end{pmatrix}$
B'_{19}	$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$	B'_{20}	$\begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$	B'_{21}	$\begin{pmatrix} 3 & 0 \\ 3 & 5 \end{pmatrix}$
B'_{22}	$\begin{pmatrix} 3 & 0 \\ 1 & 5 \end{pmatrix}$	B'_{23}	$\begin{pmatrix} 3 & 0 \\ 6 & 5 \end{pmatrix}$	B'_{24}	$\begin{pmatrix} 3 & 0 \\ 5 & 5 \end{pmatrix}$
B'_{25}	$\begin{pmatrix} 3 & 0 \\ 2 & 5 \end{pmatrix}$	B'_{26}	$\begin{pmatrix} 3 & 5 \\ 3 & 3 \end{pmatrix}$	B'_{27}	$\begin{pmatrix} 3 & 3 \\ 5 & 3 \end{pmatrix}$
B'_{28}	$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$				

Table 3.3: Cyclic subgroups of order 4 in PSL(2,7) with a generator

Group	Generator	Name	Generator	Name	Generator
C'_1	$\begin{pmatrix}0 & 2\\3 & 3\end{pmatrix}$	C'_2	$\begin{pmatrix}3 & 4\\5 & 0\end{pmatrix}$	C'_3	$\begin{pmatrix}3 & 3\\2 & 0\end{pmatrix}$
C'_4	$\begin{pmatrix}3 & 2\\3 & 0\end{pmatrix}$	C'_5	$\begin{pmatrix}2 & 5\\2 & 2\end{pmatrix}$	C'_6	$\begin{pmatrix}2 & 6\\4 & 2\end{pmatrix}$
C'_7	$\begin{pmatrix}0 & 1\\6 & 4\end{pmatrix}$	C'_8	$\begin{pmatrix}0 & 1\\6 & 3\end{pmatrix}$	C'_9	$\begin{pmatrix}1 & 5\\6 & 3\end{pmatrix}$
C'_{10}	$\begin{pmatrix}1 & 6\\6 & 2\end{pmatrix}$	C'_{11}	$\begin{pmatrix}2 & 3\\1 & 2\end{pmatrix}$	C'_{12}	$\begin{pmatrix}1 & 3\\5 & 2\end{pmatrix}$
C'_{13}	$\begin{pmatrix}1 & 6\\5 & 3\end{pmatrix}$	C'_{14}	$\begin{pmatrix}1 & 2\\4 & 2\end{pmatrix}$	C'_{15}	$\begin{pmatrix}1 & 4\\4 & 3\end{pmatrix}$
C'_{16}	$\begin{pmatrix}1 & 3\\3 & 3\end{pmatrix}$	C'_{17}	$\begin{pmatrix}1 & 5\\3 & 2\end{pmatrix}$	C'_{18}	$\begin{pmatrix}1 & 1\\2 & 3\end{pmatrix}$
C'_{19}	$\begin{pmatrix}1 & 4\\2 & 2\end{pmatrix}$	C'_{20}	$\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$	C'_{21}	$\begin{pmatrix}1 & 2\\1 & 3\end{pmatrix}$

The cyclic subgroups of order 2 are generated by the generators of order 4 squared. We notice that the square of the generator of C'_5 equals R'.

3.3 Generators of GL(3,2)

The general linear group GL(3,2) is the set of all invertible 3x3 matrices under multiplication over the finite field of 2 elements. GL(3,2) is isomorphic to PSL(2,7), and hence has the same structure when it comes to cyclic subgroups. The corresponding generators are S”, T” and R”:

$$S'' := \begin{pmatrix}0 & 0 & 1\\1 & 0 & 1\\0 & 1 & 0\end{pmatrix}$$

$$T'' := \begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 1\end{pmatrix}$$

$$R'' := \begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix}$$

3.4 Cyclic subgroups of GL(3,2)

The cyclic subgroups with a generator are listed in the tables below.

Table 3.4: Cyclic subgroups of order 7 in PSL(3,2) with a generator

Group	Generator
A_1''	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
A_2''	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$
A_3''	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
A_4''	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
A_5''	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$
A_6''	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
A_7''	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$
A_8''	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Table 3.5: Cyclic subgroups of order 3 in PSL(3,2) with a generator

Group	Generator	Group	Generator	Group	Generator
B''_1	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&1\end{pmatrix}$	B''_2	$\begin{pmatrix}0&1&1\\1&1&1\\0&0&1\end{pmatrix}$	B''_3	$\begin{pmatrix}0&0&1\\0&1&0\\1&1&1\end{pmatrix}$
B''_4	$\begin{pmatrix}1&1&1\\1&1&0\\0&1&0\end{pmatrix}$	B''_5	$\begin{pmatrix}0&1&0\\1&0&1\\1&1&0\end{pmatrix}$	B''_6	$\begin{pmatrix}1&1&0\\1&0&0\\1&0&1\end{pmatrix}$
B''_7	$\begin{pmatrix}1&0&1\\0&1&1\\1&0&0\end{pmatrix}$	B''_8	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}$	B''_9	$\begin{pmatrix}0&1&0\\1&1&0\\0&0&1\end{pmatrix}$
B''_{10}	$\begin{pmatrix}1&1&1\\0&1&1\\0&1&0\end{pmatrix}$	B''_{11}	$\begin{pmatrix}1&0&0\\0&1&1\\1&1&0\end{pmatrix}$	B''_{12}	$\begin{pmatrix}1&1&1\\1&0&1\\0&0&1\end{pmatrix}$
B''_{13}	$\begin{pmatrix}0&0&1\\1&1&0\\1&0&1\end{pmatrix}$	B''_{14}	$\begin{pmatrix}1&1&0\\0&1&1\\0&1&0\end{pmatrix}$	B''_{15}	$\begin{pmatrix}1&0&0\\1&1&1\\0&1&0\end{pmatrix}$
B''_{16}	$\begin{pmatrix}1&0&0\\1&1&1\\1&1&0\end{pmatrix}$	B''_{17}	$\begin{pmatrix}1&0&1\\0&1&1\\0&1&0\end{pmatrix}$	B''_{18}	$\begin{pmatrix}0&1&1\\0&1&0\\1&0&1\end{pmatrix}$
B''_{19}	$\begin{pmatrix}1&1&0\\1&0&0\\1&1&1\end{pmatrix}$	B''_{20}	$\begin{pmatrix}0&0&1\\0&1&1\\1&0&1\end{pmatrix}$	B''_{21}	$\begin{pmatrix}1&1&1\\1&0&0\\0&0&1\end{pmatrix}$
B''_{22}	$\begin{pmatrix}0&1&1\\1&0&1\\0&1&0\end{pmatrix}$	B''_{23}	$\begin{pmatrix}1&1&0\\1&1&1\\1&0&0\end{pmatrix}$	B''_{24}	$\begin{pmatrix}1&0&1\\0&1&0\\1&1&0\end{pmatrix}$
B''_{25}	$\begin{pmatrix}0&1&0\\1&1&0\\1&1&1\end{pmatrix}$	B''_{26}	$\begin{pmatrix}0&1&0\\0&1&1\\1&1&1\end{pmatrix}$	B''_{27}	$\begin{pmatrix}0&1&1\\1&0&1\\1&0&0\end{pmatrix}$
B''_{28}	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$				

Table 3.6: Cyclic subgroups of order 4 in PSL(3,2) with a generator

Group	Generator	Group	Generator	Group	Generator
C''_1	$\begin{pmatrix}0&0&1\\1&1&1\\0&1&0\end{pmatrix}$	C''_2	$\begin{pmatrix}0&1&0\\0&0&1\\1&1&1\end{pmatrix}$	C''_3	$\begin{pmatrix}1&0&1\\1&0&0\\1&1&0\end{pmatrix}$
C''_4	$\begin{pmatrix}0&0&1\\1&0&1\\0&1&1\end{pmatrix}$	C''_5	$\begin{pmatrix}1&1&1\\1&1&0\\1&0&1\end{pmatrix}$	C''_6	$\begin{pmatrix}1&1&0\\0&1&0\\1&0&1\end{pmatrix}$
C''_7	$\begin{pmatrix}1&0&1\\0&1&1\\1&1&1\end{pmatrix}$	C''_8	$\begin{pmatrix}0&1&1\\1&1&1\\1&1&0\end{pmatrix}$	C''_9	$\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}$
C''_{10}	$\begin{pmatrix}0&0&1\\1&0&0\\1&1&1\end{pmatrix}$	C''_{11}	$\begin{pmatrix}1&0&1\\1&1&0\\0&0&1\end{pmatrix}$	C''_{12}	$\begin{pmatrix}0&1&1\\0&1&0\\1&0&0\end{pmatrix}$
C''_{13}	$\begin{pmatrix}1&0&0\\1&1&1\\1&0&1\end{pmatrix}$	C''_{14}	$\begin{pmatrix}0&0&1\\1&1&0\\1&0&0\end{pmatrix}$	C''_{15}	$\begin{pmatrix}1&0&1\\0&0&1\\0&1&0\end{pmatrix}$
C''_{16}	$\begin{pmatrix}1&0&0\\0&0&1\\1&1&0\end{pmatrix}$	C''_{17}	$\begin{pmatrix}0&1&0\\1&0&1\\0&0&1\end{pmatrix}$	C''_{18}	$\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$
C''_{19}	$\begin{pmatrix}0&1&0\\1&0&0\\0&1&1\end{pmatrix}$	C''_{20}	$\begin{pmatrix}0&1&1\\0&0&1\\1&0&1\end{pmatrix}$	C''_{21}	$\begin{pmatrix}1&1&1\\0&1&0\\0&1&1\end{pmatrix}$

Again, a generator from the table above squared yields a generator of a cyclic subgroup of order 2.

CHAPTER 4

Representation in GL(3,$\mathbb{C}$)

4.1 Generators and subgroups

Following [Elk99, p. 54], PSL(2,7) has a faithful 3-dimensional representation in GL(3,$\mathbb{C}$), the set of all invertible 3x3 matrices over the field of complex numbers, generated by the three matrices S, T and R:

$$S := \begin{pmatrix} \zeta^4 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta \end{pmatrix}$$

$$T := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$R := \alpha \begin{pmatrix} \zeta - \zeta^6 & \zeta^2 - \zeta^5 & \zeta^4 - \zeta^3 \\ \zeta^2 - \zeta^5 & \zeta^4 - \zeta^3 & \zeta - \zeta^6 \\ \zeta^4 - \zeta^3 & \zeta - \zeta^6 & \zeta^2 - \zeta^5 \end{pmatrix}$$

$$\zeta := e^{2\pi i/7}$$

$$\alpha := -\frac{1}{\sqrt{-7}}$$

In this representation, the generating matrices S, T and R correspond to the generating matrices S', T' and R' of PSL(2,7) respectively. We name the subgroup in GL(3,$\mathbb{C}$) generated by S,T,R G. It is isomorphic to PSL(2,7) and GL(3,2).

We want to examine the nontrivial proper subgroups of G in order to establish if they have fixed points in common, whether the fixed points are on any of our curves, and whether the fixed points constitute orbits. Following [Kle99], the nontrivial proper subgroups of G are:

a) 8 conjugate elementary abelian groups of order 7

b) 28 conjugate cyclic groups of order 3

c) 21 conjugate cyclic groups of order 4

d) 21 conjugate cyclic groups of order 2

e) two classes of 7 conjugate dihedral abelian Klein 4-groups of order 4

f) 28 dihedral nonabelian groups of order 6

g) 21 dihedral nonabelian groups of order 8

h) 8 nonabelian groups of order 21

i) two classes of 7 nonabelian conjugates of
the symmetric group of degree 4

j) two classes of 7 nonabelian conjugates of
the alternating group of degree 4

When seeing points in $\mathbb{CP}^2$ as one-dimensional subspaces of a three-dimensional space, a non-zero eigenvector represents a fixed point. [MAT21] has been a useful aid in finding the fixed points.

4.2 Conjugate groups of order 7

The subgroups of order 7 of G, with generators and fixed points, are as follows:

Table 4.1: Conjugate subgroups of order 7 of G with a generator and fixed points

Name	Generator	Fixed point 1	Fixed point 2	Fixed point 3
A_1	S	$v_{A_{1,1}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$v_{A_{1,2}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$v_{A_{1,3}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
A_2	S^6RS^6	$v_{A_{2,1}} = \alpha \begin{bmatrix} \zeta^6-\zeta \\ \zeta^5-\zeta^2 \\ \zeta^3-\zeta^4 \end{bmatrix}$	$v_{A_{2,2}} = \alpha \begin{bmatrix} \zeta^3-\zeta^4 \\ \zeta^6-\zeta \\ \zeta^5-\zeta^2 \end{bmatrix}$	$v_{A_{2,3}} = \alpha \begin{bmatrix} \zeta^5-\zeta^2 \\ \zeta^3-\zeta^4 \\ \zeta^6-\zeta \end{bmatrix}$
A_3	RS^5	$v_{A_{3,1}} = \alpha \begin{bmatrix} \zeta-\zeta^3 \\ \zeta^5-\zeta^2 \\ \zeta^2-\zeta^3 \end{bmatrix}$	$v_{A_{3,2}} = \alpha \begin{bmatrix} \zeta^6-1 \\ 1-\zeta^2 \\ \zeta^5-\zeta^2 \end{bmatrix}$	$v_{A_{3,3}} = \alpha \begin{bmatrix} \zeta^5-\zeta^2 \\ \zeta-\zeta^2 \\ \zeta^3-\zeta^5 \end{bmatrix}$
A_4	SRS^4	$v_{A_{4,1}} = \alpha \begin{bmatrix} \zeta^3-\zeta^5 \\ \zeta^5-\zeta^2 \\ \zeta-\zeta^2 \end{bmatrix}$	$v_{A_{4,2}} = \alpha \begin{bmatrix} \zeta^2-\zeta^3 \\ \zeta-\zeta^3 \\ \zeta^5-\zeta^2 \end{bmatrix}$	$v_{A_{4,3}} = \alpha \begin{bmatrix} \zeta^5-\zeta^2 \\ \zeta^6-1 \\ 1-\zeta^2 \end{bmatrix}$
A_5	S^3RS^2	$v_{A_{5,1}} = \alpha \begin{bmatrix} 1-\zeta^2 \\ \zeta^5-\zeta^2 \\ \zeta^6-1 \end{bmatrix}$	$v_{A_{5,2}} = \alpha \begin{bmatrix} \zeta-\zeta^2 \\ \zeta^3-\zeta^5 \\ \zeta^5-\zeta^2 \end{bmatrix}$	$v_{A_{5,3}} = \alpha \begin{bmatrix} \zeta^5-\zeta^2 \\ \zeta^2-\zeta^3 \\ \zeta-\zeta^3 \end{bmatrix}$
A_6	S^2RS^3	$v_{A_{6,1}} = \alpha \begin{bmatrix} \zeta^5-1 \\ \zeta^5-\zeta^2 \\ 1-\zeta \end{bmatrix}$	$v_{A_{6,2}} = \alpha \begin{bmatrix} \zeta^5-\zeta^6 \\ \zeta^2-\zeta^4 \\ \zeta^5-\zeta^2 \end{bmatrix}$	$v_{A_{6,3}} = \alpha \begin{bmatrix} \zeta^5-\zeta^2 \\ \zeta^4-\zeta^5 \\ \zeta^4-\zeta^6 \end{bmatrix}$
A_7	S^5R	$v_{A_{7,1}} = \alpha \begin{bmatrix} \zeta^4-\zeta^6 \\ \zeta^5-\zeta^2 \\ \zeta^4-\zeta^5 \end{bmatrix}$	$v_{A_{7,2}} = \alpha \begin{bmatrix} 1-\zeta \\ \zeta^5-1 \\ \zeta^5-\zeta^2 \end{bmatrix}$	$v_{A_{7,3}} = \alpha \begin{bmatrix} \zeta^5-\zeta^2 \\ \zeta^5-\zeta^6 \\ \zeta^2-\zeta^4 \end{bmatrix}$
A_8	S^4RS	$v_{A_{8,1}} = \alpha \begin{bmatrix} \zeta^2-\zeta^4 \\ \zeta^5-\zeta^2 \\ \zeta^5-\zeta^6 \end{bmatrix}$	$v_{A_{8,2}} = \alpha \begin{bmatrix} \zeta^4-\zeta^5 \\ \zeta^4-\zeta^6 \\ \zeta^5-\zeta^2 \end{bmatrix}$	$v_{A_{8,3}} = \alpha \begin{bmatrix} \zeta^5-\zeta^2 \\ 1-\zeta \\ \zeta^5-1 \end{bmatrix}$

To show that $A1-A8$ are conjugate, consider the automorphism $i_g : G \to G$ where $i_g(x) = gxg^{-1}$ for all $x \epsilon G$. Let $x = S$, a generator of $A1$. Let $g = RS^6RS$, with inverse $g^{-1} = S^6RSR$.
Then $i_g(S) = RS^6RSSS^6RSR = RS^6RSRSR = RS^5SRSRSR = RS^5(SR)^3$.

Now, (SR) generates a (cyclic) group of order 3, the group $B9$ in table 4.3 below. Hence $(SR)^3 = 1$ and $i_g(S) = RS^5$. RS^5 is an element of order

7 and generates $A7$, so $i_g[A1] = A7$. Furthermore, by conjugating RS^5 by $h = S$, with inverse $h^{-1} = S^6$, we get SRS^4, which generates $A4$. Repeatedly conjugating by $h = S$ we find that all the elements of order 7 are covered, until conjugation of S^6RS^6 gets us back to RS^5. This shows that all the groups of order 7 are conjugate, hence they are all isomorphic to each other.

We examine whether the fixed points in table 4.1 satisfy the Klein quartic equation, and consider $v_{A_{2,1}}$ as an example.
Let $x = \alpha(\zeta^6 - \zeta)$, $y = \alpha(\zeta^5 - \zeta^2)$ and $z = \alpha(\zeta^3 - \zeta^4)$.
We substitute these values into f, and calculate, recalling that $\zeta^7 = (e^{2\pi i/7})^7 = 1$.

$$f = x^3y + y^3z + z^3x$$

$$= \alpha^4((\zeta^6 - \zeta)^3(\zeta^5 - \zeta^2) + (\zeta^5 - \zeta^2)^3(\zeta^3 - \zeta^4) + (\zeta^3 - \zeta^4)^3(\zeta^6 - \zeta))$$

$$= \alpha^4((2\zeta^6 + \zeta^5 - 3\zeta^4 - 3\zeta^3 + \zeta^2 + 2\zeta)$$
$$+ (-3\zeta^6 + 2\zeta^5 + \zeta^4 + \zeta^3 + 2\zeta^2 - 3\zeta) + (\zeta^6 - 3\zeta^5 + 2\zeta^4 + 2\zeta^3 - 3\zeta^2 + \zeta))$$

$$= 0$$

Proceeding in the same manner for all the fixed points, we get a general result.

Given $v_{A_{i,j}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we find that:

Proposition 4.2.1. *All the fixed points of the groups of order 7 are on the Klein quartic curve* $K4 = x^3y + y^3z + z^3x = 0$.

We find the following relations between these fixed points and the group generators:

Table 4.2: Relations between the fixed points of the subgroups of order 7 of G and the group generators

$v_{A_{i,j}}$	$Rv_{A_{i,j}}$	$Sv_{A_{i,j}}$	$Tv_{A_{i,j}}$
$v_{A_{1,1}}$	$v_{A_{2,1}}$	$v_{A_{1,1}}$	$v_{A_{1,2}}$
$v_{A_{1,2}}$	$v_{A_{2,3}}$	$v_{A_{1,2}}$	$v_{A_{1,3}}$
$v_{A_{1,3}}$	$v_{A_{2,2}}$	$v_{A_{1,3}}$	$v_{A_{1,1}}$
$v_{A_{2,1}}$	$v_{A_{1,1}}$	$v_{A_{3,1}}$	$v_{A_{2,2}}$
$v_{A_{2,2}}$	$v_{A_{1,3}}$	$v_{A_{3,2}}$	$v_{A_{2,3}}$
$v_{A_{2,3}}$	$v_{A_{1,2}}$	$v_{A_{3,3}}$	$v_{A_{2,1}}$
$v_{A_{3,1}}$	$v_{A_{7,1}}$	$v_{A_{4,1}}$	$v_{A_{4,2}}$
$v_{A_{3,2}}$	$v_{A_{7,2}}$	$v_{A_{4,2}}$	$v_{A_{4,3}}$
$v_{A_{3,3}}$	$v_{A_{7,3}}$	$v_{A_{4,3}}$	$v_{A_{4,1}}$
$v_{A_{4,1}}$	$v_{A_{6,2}}$	$v_{A_{6,1}}$	$v_{A_{5,2}}$
$v_{A_{4,2}}$	$v_{A_{6,3}}$	$v_{A_{6,2}}$	$v_{A_{5,3}}$
$v_{A_{4,3}}$	$v_{A_{6,1}}$	$v_{A_{6,3}}$	$v_{A_{5,1}}$
$v_{A_{5,1}}$	$v_{A_{8,3}}$	$v_{A_{8,1}}$	$v_{A_{3,2}}$
$v_{A_{5,2}}$	$v_{A_{8,1}}$	$v_{A_{8,2}}$	$v_{A_{3,3}}$
$v_{A_{5,3}}$	$v_{A_{8,2}}$	$v_{A_{8,3}}$	$v_{A_{3,1}}$
$v_{A_{6,1}}$	$v_{A_{4,3}}$	$v_{A_{5,1}}$	$v_{A_{7,2}}$
$v_{A_{6,2}}$	$v_{A_{4,1}}$	$v_{A_{5,2}}$	$v_{A_{7,3}}$
$v_{A_{6,3}}$	$v_{A_{4,2}}$	$v_{A_{5,3}}$	$v_{A_{7,1}}$
$v_{A_{7,1}}$	$v_{A_{3,1}}$	$v_{A_{2,1}}$	$v_{A_{8,2}}$
$v_{A_{7,2}}$	$v_{A_{3,2}}$	$v_{A_{2,2}}$	$v_{A_{8,3}}$
$v_{A_{7,3}}$	$v_{A_{3,3}}$	$v_{A_{2,3}}$	$v_{A_{8,1}}$
$v_{A_{8,1}}$	$v_{A_{5,2}}$	$v_{A_{7,1}}$	$v_{A_{6,2}}$
$v_{A_{8,2}}$	$v_{A_{5,3}}$	$v_{A_{7,2}}$	$v_{A_{6,3}}$
$v_{A_{8,3}}$	$v_{A_{5,1}}$	$v_{A_{7,3}}$	$v_{A_{6,1}}$

Since the group generators R, S, T can map all the fixed points to every other fixed point in the same set, and only to these, we conclude:

Proposition 4.2.2. *The 24 fixed points of the groups of order 7 form an orbit.*

Klein shows in [Kle99] that these fixed points are found where the zero set of the sextic invariant $K6 = xy^5 + yz^5 + zx^5 - 5x^2y^2z^2$ intersects the Klein quartic curve. By calculation we conclude that none of the points are in the zero sets of $K14$ or $K21$.

Next we examine whether a tangent to the Klein quartic curve through a fixed point of a group of order 7 intersects the quartic curve somewhere else. In general we know that such a tangent must satisfy

$$\frac{\partial K4}{\partial x}(a,b,c)(x-a) + \frac{\partial K4}{\partial y}(a,b,c)(y-b) + \frac{\partial K4}{\partial z}(a,b,c)(z-c)$$

$$= (3a^2b + c^3)(x - a) + (3b^2c + a^3)(y - b) + (3c^2a + b^3)(z - c) = 0,$$

where (a,b,c) is the fixed point.

We examine $v_{A_{1,1}}$ and set $(a, b, c) = (1, 0, 0)$.
The equation above yields $z = 0$, and inserting this into $K4$, we get $x^3y = 0$.
$y = 0$ gives us the point we started with, while $x = 0$ gives us $v_{A_{1,3}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Here $v_{A_{1,3}}$ is a zero of multiplicity 3, which implies that is a simple inflection point. Similarly, all the tangents to $K4$ through a point of the orbit of the fixed points of the groups of order 7 passes through another point of the orbit, which is a simple inflection point. According to [Mir95, p. 241], a smooth algebraic curve of degree d has exactly $3d(d - 2)$ inflection points (assuming they are all simple). In our case, $K4$ has 24 inflection points.

Proposition 4.2.3. *For every fixed point of a group of order 7, there is a tangent to the Klein quartic curve through that point which intersects the curve in another point, and only there. That point is also a fixed point of a group of order 7. The fixed points of the groups of order 7 are all the inflection points of the Klein quartic curve.*

Proceeding in the same manner for the sextic curve $K6$, we get

$$\tfrac{\partial K6}{\partial x}(a, b, c)(x - a) + \tfrac{\partial K6}{\partial y}(a, b, c)(y - b) + \tfrac{\partial K6}{\partial z}(a, b, c)(z - c)$$

$$= (10ab^2c^2 - b^5)(x - a) + (10a^2bc^2 - c^5)(y - b) + (10a^2b^2c - a^5)(z - c) = 0.$$

Again we insert $(a, b, c) = (1, 0, 0)$, which yields $z = 0$. Applying this to $K6$, we get $xy^5 = 0$.
We see that the tangent in question intersects the sextic curve in $v_{A_{1,2}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.
$v_{A_{1,2}}$ is a zero of multiplicity 5. Again by [Mir95, p. 241], $K6$ has $3d(d-2) = 72$ inflection points, where an inflection point where the tangent meets the curve at the point with multiplicity v is counted $v - 2$ times. Counting the 24 points of the orbit of the fixed points of the groups of order 7 three times, we see that these are all the inflection points of $K6$.
By the properties of symmetry we establish a propostion similar to the previous one.

Proposition 4.2.4. *For every fixed point of a group of order 7, there is a tangent to the sextic curve through that point which intersects the curve in another point, and only there. That point is also a fixed point of a group of order 7. The fixed points of the groups of order 7 are all the inflection points of the sextic curve K6.*

4.3 Conjugate groups of order 3

The subgroups of order 3, with generators, are as follows:

Table 4.3: Conjugate subgroups of order 3 of G with a generator

Group	Generator	Group	Generator
B1	T	B15	RS^5RS^2
B2	ST	B16	RS^3RS
B3	S^2T	B17	RS^4RS^6
B4	S^3T	B18	RS^3RS^5
B5	S^4T	B19	RS^4RS^2
B6	S^5T	B20	$RSRT$
B7	S^6T	B21	RS^6RT
B8	RS	B22	RS^2RT
B9	SR	B23	RS^5RT
B10	$RSRS^3$	B24	RS^3RT
B11	RS^6RS^4	B25	RS^4RT
B12	RS^2RS^4	B26	$SRTS$
B13	RS^5RS^3	B27	$S^2RT^2S^2$
B14	RS^2RS^5	B28	S^3RS^3

Conjugation by S yields four sets of subgroups with seven subgroups each, such that $B_j = S^n[B_i]S^{-n}$ for B_i, B_j subgroups in the same set and $n\epsilon\mathbb{Z}$. These sets are $\{B1 - B7\}$, $\{B8 - B11, B22, B23, B28\}$, $\{B12 - B15, B24, B25, B27\}$ and $\{B16 - B21, B26\}$.
Similarly, conjugation by T yields one set $\{B1\}$ (since $TTT^{-1} = T$) and the nine sets $\{B2, B3, B5\}$, $\{B4, B6, B7\}$, $\{B8, B12, B19\}$, $\{B9, B13, B18\}$, $\{B10, B14, B17\}$, $\{B11, B15, B16\}$, $\{B20, B22, B25\}$, $\{B21, B23, B24\}$ and $\{B26, B27, B28\}$.
Finally, conjugation by R gives us four sets with one subgroup each: $\{B1\}$, $\{B14\}$, $\{B15\}$ and $\{B28\}$, and twelve sets of two subgroups each: $\{B2, B24\}$, $\{B3, B21\}$, $\{B4, B22\}$, $\{B5, B23\}$, $\{B6, B20\}$, $\{B7, B25\}$, $\{B8, B9\}$, $\{B10, B17\}$, $\{B11, B16\}$, $\{B12, B18\}$, $\{B13, B19\}$ and $\{B26, B27\}$.
This means that all the 28 subgroups are conjugate, i.e. there is some $g\epsilon G$ such that $B_j = gB_ig^{-1}$ for any pair B_i, B_j of subgroups. For example $(RTS)[B1](RTS)^{-1} = B20$.

The groups all have the eigenvalues $e^{2\pi i/3}$, $e^{4\pi i/3}$ and 1. We designate the fixed points on the Klein quartic curve as $v_{B_{i,j}}$, i=1,2,...,28, j=1,2. We have used [MAT21] to find the fixed points, so the coordinates of the points are approximations. For instance we get

$$v_{B_{2,1}} = \begin{bmatrix} 0.2974 + 1.3025i \\ 0.6410 + 0.3086i \\ 1 \end{bmatrix}.$$

These values yield $f = -1.2 \cdot 10^{-5} - 2.1 \cdot 10^{-4}i$. We claim that the point is on the Klein quartic curve ($f = 0$) under the assumption that a more powerful program would give us that result, and based on what is stated by Klein and others about the fixed points. Under this assumption, we proceed. The fixed points with the first two eigenvalues are on the Klein quartic curve $K4$, while the fixed points with eigenvalue 1 are not. We calculate the effect of the group generators on these fixed points, and get the following results:

Table 4.4: Relations between the fixed points on the Klein quartic curve of the subgroups of order 3 of G and the group generators

$v_{B_{i,j}}$	$Rv_{B_{i,j}}$	$Sv_{B_{i,j}}$	$Tv_{B_{i,j}}$
$v_{B_{1,1}}$	$v_{B_{1,2}}$	$v_{B_{7,1}}$	$v_{B_{1,1}}$
$v_{B_{1,2}}$	$v_{B_{1,1}}$	$v_{B_{7,2}}$	$v_{B_{1,2}}$
$v_{B_{2,1}}$	$v_{B_{24,2}}$	$v_{B_{1,1}}$	$v_{B_{3,1}}$
$v_{B_{2,2}}$	$v_{B_{24,1}}$	$v_{B_{1,2}}$	$v_{B_{3,2}}$
$v_{B_{3,1}}$	$v_{B_{21,2}}$	$v_{B_{2,1}}$	$v_{B_{5,1}}$
$v_{B_{3,2}}$	$v_{B_{21,1}}$	$v_{B_{2,2}}$	$v_{B_{5,2}}$
$v_{B_{4,1}}$	$v_{B_{22,2}}$	$v_{B_{3,1}}$	$v_{B_{7,1}}$
$v_{B_{4,2}}$	$v_{B_{22,1}}$	$v_{B_{3,2}}$	$v_{B_{7,2}}$
$v_{B_{5,1}}$	$v_{B_{23,2}}$	$v_{B_{4,1}}$	$v_{B_{2,1}}$
$v_{B_{5,2}}$	$v_{B_{23,1}}$	$v_{B_{4,2}}$	$v_{B_{2,2}}$
$v_{B_{6,1}}$	$v_{B_{20,2}}$	$v_{B_{5,1}}$	$v_{B_{4,1}}$
$v_{B_{6,2}}$	$v_{B_{20,1}}$	$v_{B_{5,2}}$	$v_{B_{4,2}}$
$v_{B_{7,1}}$	$v_{B_{25,2}}$	$v_{B_{6,1}}$	$v_{B_{6,1}}$
$v_{B_{7,2}}$	$v_{B_{25,1}}$	$v_{B_{6,2}}$	$v_{B_{6,2}}$
$v_{B_{8,1}}$	$v_{B_{9,1}}$	$v_{B_{9,1}}$	$v_{B_{19,2}}$
$v_{B_{8,2}}$	$v_{B_{9,2}}$	$v_{B_{9,2}}$	$v_{B_{19,1}}$
$v_{B_{9,1}}$	$v_{B_{8,1}}$	$v_{B_{11,2}}$	$v_{B_{18,1}}$
$v_{B_{9,2}}$	$v_{B_{8,2}}$	$v_{B_{11,1}}$	$v_{B_{18,2}}$
$v_{B_{10,1}}$	$v_{B_{17,2}}$	$v_{B_{8,1}}$	$v_{B_{17,1}}$
$v_{B_{10,2}}$	$v_{B_{17,1}}$	$v_{B_{8,2}}$	$v_{B_{17,2}}$
$v_{B_{11,1}}$	$v_{B_{16,2}}$	$v_{B_{22,2}}$	$v_{B_{16,1}}$
$v_{B_{11,2}}$	$v_{B_{16,1}}$	$v_{B_{22,1}}$	$v_{B_{16,2}}$
$v_{B_{12,1}}$	$v_{B_{18,2}}$	$v_{B_{15,1}}$	$v_{B_{8,2}}$
$v_{B_{12,2}}$	$v_{B_{18,1}}$	$v_{B_{15,2}}$	$v_{B_{8,1}}$
$v_{B_{13,1}}$	$v_{B_{19,2}}$	$v_{B_{25,1}}$	$v_{B_{9,1}}$
$v_{B_{13,2}}$	$v_{B_{19,1}}$	$v_{B_{25,2}}$	$v_{B_{9,2}}$
$v_{B_{14,1}}$	$v_{B_{14,2}}$	$v_{B_{13,1}}$	$v_{B_{10,1}}$
$v_{B_{14,2}}$	$v_{B_{14,1}}$	$v_{B_{13,2}}$	$v_{B_{10,2}}$

$v_{B_{i,j}}$	$Rv_{B_{i,j}}$	$Sv_{B_{i,j}}$	$Tv_{B_{i,j}}$
$v_{B_{15,1}}$	$v_{B_{15,2}}$	$v_{B_{27,2}}$	$v_{B_{11,1}}$
$v_{B_{15,2}}$	$v_{B_{15,1}}$	$v_{B_{27,1}}$	$v_{B_{11,2}}$
$v_{B_{16,1}}$	$v_{B_{11,2}}$	$v_{B_{17,2}}$	$v_{B_{15,1}}$
$v_{B_{16,2}}$	$v_{B_{11,1}}$	$v_{B_{17,1}}$	$v_{B_{15,2}}$
$v_{B_{17,1}}$	$v_{B_{10,2}}$	$v_{B_{20,1}}$	$v_{B_{14,1}}$
$v_{B_{17,2}}$	$v_{B_{10,1}}$	$v_{B_{20,2}}$	$v_{B_{14,2}}$
$v_{B_{18,1}}$	$v_{B_{12,2}}$	$v_{B_{21,2}}$	$v_{B_{13,1}}$
$v_{B_{18,2}}$	$v_{B_{12,1}}$	$v_{B_{21,1}}$	$v_{B_{13,2}}$
$v_{B_{19,1}}$	$v_{B_{13,2}}$	$v_{B_{26,2}}$	$v_{B_{12,1}}$
$v_{B_{19,2}}$	$v_{B_{13,1}}$	$v_{B_{26,1}}$	$v_{B_{12,2}}$
$v_{B_{20,1}}$	$v_{B_{6,2}}$	$v_{B_{19,2}}$	$v_{B_{25,1}}$
$v_{B_{20,2}}$	$v_{B_{6,1}}$	$v_{B_{19,1}}$	$v_{B_{25,2}}$
$v_{B_{21,1}}$	$v_{B_{3,2}}$	$v_{B_{16,1}}$	$v_{B_{24,1}}$
$v_{B_{21,2}}$	$v_{B_{3,1}}$	$v_{B_{16,2}}$	$v_{B_{24,2}}$
$v_{B_{22,1}}$	$v_{B_{4,2}}$	$v_{B_{28,2}}$	$v_{B_{20,1}}$
$v_{B_{22,2}}$	$v_{B_{4,1}}$	$v_{B_{28,1}}$	$v_{B_{20,2}}$
$v_{B_{23,1}}$	$v_{B_{5,2}}$	$v_{B_{10,2}}$	$v_{B_{21,1}}$
$v_{B_{23,2}}$	$v_{B_{5,1}}$	$v_{B_{10,1}}$	$v_{B_{21,2}}$
$v_{B_{24,1}}$	$v_{B_{2,2}}$	$v_{B_{12,1}}$	$v_{B_{23,1}}$
$v_{B_{24,2}}$	$v_{B_{2,1}}$	$v_{B_{12,2}}$	$v_{B_{23,2}}$
$v_{B_{25,1}}$	$v_{B_{7,2}}$	$v_{B_{24,2}}$	$v_{B_{22,1}}$
$v_{B_{25,2}}$	$v_{B_{7,1}}$	$v_{B_{24,1}}$	$v_{B_{22,2}}$
$v_{B_{26,1}}$	$v_{B_{27,2}}$	$v_{B_{18,1}}$	$v_{B_{27,1}}$
$v_{B_{26,2}}$	$v_{B_{27,1}}$	$v_{B_{18,2}}$	$v_{B_{27,2}}$
$v_{B_{27,1}}$	$v_{B_{26,2}}$	$v_{B_{14,1}}$	$v_{B_{28,2}}$
$v_{B_{27,2}}$	$v_{B_{26,1}}$	$v_{B_{14,2}}$	$v_{B_{28,1}}$
$v_{B_{28,1}}$	$v_{B_{28,2}}$	$v_{B_{23,1}}$	$v_{B_{26,2}}$
$v_{B_{28,2}}$	$v_{B_{28,1}}$	$v_{B_{23,2}}$	$v_{B_{26,1}}$

By the same reasoning as in the previous proposition, we conclude:

Proposition 4.3.1. *The groups of order 3 each have 3 fixed points, of which 2 are on the Klein quartic curve. The 56 fixed points on the curve form an orbit.*

The fixed points of this orbit are found where the degree-14 invariant $K14$ intersects the Klein quartic curve. They do not satisfy $K6 = 0$ or $K21 = 0$. However, the fixed points of the groups of order 3 that are not on the Klein quartic curve, satisfy $K21 = 0$, but not $K6 = 0$ or $K14 = 0$.

We examine the lines between the two fixed points of the groups of order 3 that are on $K4$.
For the 3-group $B1$ these are the fixed points

$$v_{B_{1,1}} = \begin{bmatrix} 1 \\ e^{2\pi i/3} \\ e^{4\pi i/3} \end{bmatrix} \text{ and } v_{B_{1,2}} = \begin{bmatrix} 1 \\ e^{4\pi i/3} \\ e^{2\pi i/3} \end{bmatrix}.$$

Parametrizing the line through them, we get
$l_{B_1} := v_{B_{1,1}} + k(v_{B_{1,2}} - v_{B_{1,1}})$

$$= \begin{bmatrix} 1 \\ e^{2\pi i/3} + k(e^{4\pi i/3} - e^{2\pi i/3}) \\ e^{4\pi i/3} + k(e^{2\pi i/3} - e^{4\pi i/3}) \end{bmatrix}.$$

We define:
$x_{l_{B_1}} := 1$,
$y_{l_{B_1}} := e^{2\pi i/3} + k(e^{4\pi i/3} - e^{2\pi i/3})$
and $z_{l_{B_1}} := e^{4\pi i/3} + k(e^{2\pi i/3} - e^{4\pi i/3})$.

The following holds for the line l_{B_1}:
$x_{l_{B_1}} + y_{l_{B_1}} + z_{l_{B_1}}$
$= 1 + e^{2\pi i/3} + k(e^{4\pi i/3} - e^{2\pi i/3}) + e^{4\pi i/3} + k(e^{2\pi i/3} - e^{4\pi i/3})$
$= 1 + e^{2\pi i/3} + e^{4\pi i/3} = 0$.

The relation $x + y + z = 0$ defines l_{B_1}. Substituting $z = -(x + y)$ into $K4 = x^3y + y^3z + z^3x$, we find that l_{B_1} and $K4$ have common points where $l_{B_1}* := -(x^2 + xy + y^2)^2 = 0$. This yields the result that the fixed points $v_{B_{1,1}}$ and $v_{B_{1,2}}$ are the only common points of l_{B_1} and $K4$. Both points are solutions of $l_{B_1}* = 0$ with multiplicity 2, implying that l_{B_1} is tangent to $K4$ at the two points. By the properties of symmetry this must hold for all the similar lines through fixed points of groups of order 3 that are on the Klein quartic curve.

We cite Bezout's Theorem in order to introduce a further proposition.

Theorem 4.3.2. *(Bezout's Theorem) Let C and C' be two curves in $\mathbb{P}^2$ without common components, of degree d and d' respectively. Then the number of points of $C \cap C'$, counting intersection multiplicity, equals dd'.*

By Bezout's theorem, there are (at most) 56 intersection points of K4 and K14. These are the 56 points of our orbit, yielding the 28 bitangents. Furthermore, it is known from the theory of algebraic plane curves that a general quartic plane curve has 28 bitangents, so these are all the bitangents of the Klein quartic curve.

Proposition 4.3.3. *The lines through the two fixed points of a group of order 3 that intersect the Klein quartic curve are bitangents of the curve, and these are all the bitangents of the Klein quartic curve. No other points of the lines are on the curve.*

4.4 Conjugate cyclic groups of order 4

The cyclic groups of order 4, with generators, are as follows:

Table 4.5: Conjugate cyclic subgroups of order 4 of G with a generator

Name	Generator	Name	Generator	Name	Generator
C1	RTS	C8	RS^4	C15	RS^3RS^4
C2	TSR	C9	$RSRS^5$	C16	RS^4RS^3
C3	TS^6R	C10	$RSRS^6$	C17	RS^4RS^5
C4	SRT	C11	S^2RS^2	C18	RS^5RS
C5	$STRS$	C12	RS^2RS^3	C19	RS^5RS^4
C6	$S^4T^2RS^4$	C13	RS^2RS^6	C20	RS^6RS
C7	RS^3	C14	RS^3RS^2	C21	RS^6RS^2

To verify that the subgroups are conjugate, we can use the same procedure as in section 4.3. We just leave as an example conjugation of $C1$ by the generators of G. Repeated conjugation by R yields the subset $\{C1, C2\}$. When it comes to T we get the subset $\{C1, C3, C8\}$. Lastly, repeated conjugation by S produces the subset $\{C1, C4, C6, C14, C15, C16, C17\}$.

The groups all have the eigenvalues i, $-i$ and 1. None of the fixed points satisfies $K4 = 0$, and they are thus not on the Klein quartic curve. We find that the fixed points with eigenvalues i and $-i$ are on the sextic curve.

Let $v_{C_{1,1}} = \begin{bmatrix} -0.1274 + 0.5583i \\ -0.3444 + 0.1659i \\ 0.7252 \end{bmatrix}$ be the fixed point of $C1$ with eigenvalue i. Then $K6 = 5x^2y^2z^2 - xy^5 - yz^5 - zx^5$
$= 5(-0.1274 + 0.5583i)^2(-0.3444 + 0.1659i)^2(0.7252)^2$
$- (-0.1274 + 0.5583i)(-0.3444 + 0.1659i)^5$
$- (-0.3444 + 0.1659i)(0.7252)^5$
$- (0.7252)(-0.1274 + 0.5583i)^5$
$= 0.$

None of the fixed points are on the degree-14 curve, but they are all on the degree-21 curve.

In the following table we see the products of each generator with each fixed point, where $v_{C_{1,1}}$ is the fixed point with eigenvalue i for C1, $v_{C_{1,2}}$ the fixed point with eigenvalue $-i$ for C1, and similarly for the other subgroups.

Table 4.6: Relations between the fixed points on the sextic curve of the cyclic subgroups of order 4 of G and the group generators

$v_{C_{i,j}}$	$Rv_{C_{i,j}}$	$Sv_{C_{i,j}}$	$Tv_{C_{i,j}}$
$v_{C_{1,1}}$	$v_{C_{2,1}}$	$v_{C_{4,1}}$	$v_{C_{3,2}}$
$v_{C_{1,2}}$	$v_{C_{2,2}}$	$v_{C_{4,2}}$	$v_{C_{3,1}}$
$v_{C_{2,1}}$	$v_{C_{1,1}}$	$v_{C_{12,1}}$	$v_{C_{7,2}}$
$v_{C_{2,2}}$	$v_{C_{1,2}}$	$v_{C_{12,2}}$	$v_{C_{7,1}}$
$v_{C_{3,1}}$	$v_{C_{4,2}}$	$v_{C_{5,2}}$	$v_{C_{8,2}}$
$v_{C_{3,2}}$	$v_{C_{4,1}}$	$v_{C_{5,1}}$	$v_{C_{8,1}}$
$v_{C_{4,1}}$	$v_{C_{3,2}}$	$v_{C_{15,1}}$	$v_{C_{2,1}}$
$v_{C_{4,2}}$	$v_{C_{3,1}}$	$v_{C_{15,2}}$	$v_{C_{2,2}}$
$v_{C_{5,1}}$	$v_{C_{5,1}}$	$v_{C_{2,1}}$	$v_{C_{11,1}}$
$v_{C_{5,2}}$	$v_{C_{5,2}}$	$v_{C_{2,2}}$	$v_{C_{11,2}}$
$v_{C_{6,1}}$	$v_{C_{11,1}}$	$v_{C_{17,1}}$	$v_{C_{5,1}}$
$v_{C_{6,2}}$	$v_{C_{11,2}}$	$v_{C_{17,2}}$	$v_{C_{5,2}}$
$v_{C_{7,1}}$	$v_{C_{8,2}}$	$v_{C_{20,1}}$	$v_{C_{4,2}}$
$v_{C_{7,2}}$	$v_{C_{8,1}}$	$v_{C_{20,2}}$	$v_{C_{4,1}}$
$v_{C_{8,1}}$	$v_{C_{7,2}}$	$v_{C_{21,1}}$	$v_{C_{1,1}}$
$v_{C_{8,2}}$	$v_{C_{7,1}}$	$v_{C_{21,2}}$	$v_{C_{1,2}}$
$v_{C_{9,1}}$	$v_{C_{13,2}}$	$v_{C_{7,1}}$	$v_{C_{16,1}}$
$v_{C_{9,2}}$	$v_{C_{13,1}}$	$v_{C_{7,2}}$	$v_{C_{16,2}}$
$v_{C_{10,1}}$	$v_{C_{10,2}}$	$v_{C_{8,1}}$	$v_{C_{17,1}}$
$v_{C_{10,2}}$	$v_{C_{10,1}}$	$v_{C_{8,2}}$	$v_{C_{17,2}}$
$v_{C_{11,1}}$	$v_{C_{6,1}}$	$v_{C_{9,2}}$	$v_{C_{6,1}}$
$v_{C_{11,2}}$	$v_{C_{6,2}}$	$v_{C_{9,1}}$	$v_{C_{6,2}}$
$v_{C_{12,1}}$	$v_{C_{17,2}}$	$v_{C_{18,1}}$	$v_{C_{10,1}}$
$v_{C_{12,2}}$	$v_{C_{17,1}}$	$v_{C_{18,2}}$	$v_{C_{10,2}}$
$v_{C_{13,1}}$	$v_{C_{9,2}}$	$v_{C_{19,1}}$	$v_{C_{9,1}}$
$v_{C_{13,2}}$	$v_{C_{9,1}}$	$v_{C_{19,2}}$	$v_{C_{9,2}}$
$v_{C_{14,1}}$	$v_{C_{19,2}}$	$v_{C_{6,2}}$	$v_{C_{19,1}}$
$v_{C_{14,2}}$	$v_{C_{19,1}}$	$v_{C_{6,1}}$	$v_{C_{19,2}}$
$v_{C_{15,1}}$	$v_{C_{15,2}}$	$v_{C_{14,2}}$	$v_{C_{18,1}}$
$v_{C_{15,2}}$	$v_{C_{15,1}}$	$v_{C_{14,1}}$	$v_{C_{18,2}}$
$v_{C_{16,1}}$	$v_{C_{16,2}}$	$v_{C_{1,2}}$	$v_{C_{13,1}}$
$v_{C_{16,2}}$	$v_{C_{16,1}}$	$v_{C_{1,1}}$	$v_{C_{13,2}}$
$v_{C_{17,1}}$	$v_{C_{12,2}}$	$v_{C_{16,2}}$	$v_{C_{12,1}}$
$v_{C_{17,2}}$	$v_{C_{12,1}}$	$v_{C_{16,1}}$	$v_{C_{12,2}}$
$v_{C_{18,1}}$	$v_{C_{21,2}}$	$v_{C_{13,2}}$	$v_{C_{21,1}}$
$v_{C_{18,2}}$	$v_{C_{21,1}}$	$v_{C_{13,1}}$	$v_{C_{21,2}}$
$v_{C_{19,1}}$	$v_{C_{14,2}}$	$v_{C_{3,1}}$	$v_{C_{20,1}}$
$v_{C_{19,2}}$	$v_{C_{14,1}}$	$v_{C_{3,2}}$	$v_{C_{20,2}}$
$v_{C_{20,1}}$	$v_{C_{20,2}}$	$v_{C_{10,2}}$	$v_{C_{14,1}}$
$v_{C_{20,2}}$	$v_{C_{20,1}}$	$v_{C_{10,1}}$	$v_{C_{14,2}}$
$v_{C_{21,1}}$	$v_{C_{18,2}}$	$v_{C_{11,1}}$	$v_{C_{15,1}}$
$v_{C_{21,2}}$	$v_{C_{18,1}}$	$v_{C_{11,2}}$	$v_{C_{15,2}}$

Again we see that the generators of G maps the fixed points to every other fixed point in the set, and nowhere else. We conclude that the 42 fixed points on the sextic curve form an orbit.

Proposition 4.4.1. *The cyclic groups of order 4 each have 3 fixed points, of which 2 are on the sextic curve. The 42 fixed points on the curve form an orbit.*

4.5 Conjugate groups of order 2

The generators of the order 2 groups are the squares of the generators in the table above, but for completeness and since we have found some simplifactions, we leave a table of generators:

Table 4.7: Conjugate subgroups of order 2 of G with a generator

Name	Generator	Name	Generator	Name	Generator
D1	RS^2RS	D8	$(RS^4)^2$	D15	$(RS^3RS^4)^2$
D2	SRS^6	D9	S^3RS^4RS	D16	$(RS^4RS^3)^2$
D3	S^6RS	D10	S^2RS^3RS	D17	$(RS^4RS^5)^2$
D4	RS^5RS^6	D11	RT	D18	S^3RS^4
D5	R	D12	S^2RS^5	D19	S^5RS^2
D6	TR	D13	S^4RS^3	D20	SRS^3RS^2
D7	$(RS^3)^2$	D14	$(RS^3RS^2)^2$	D21	SRS^4RS^4

Conjugation follows the pattern of the groups of order 4 in section 4.4. To show this, let M be an element of G that generates a cyclic 4-group, and conjugate M by an element g such that $gMg^{-1} = N$. Then $N = g^{-1}Mg$ and $N^2 = g^{-1}M^2g$. It follows that M^2 is an element that generates a 2-group and that $gM^2g^{-1} = N^2$.

The groups have the eigenvalue 1 with multiplicity 1 and the eigenvalue -1 with multiplicity 2. This means that any point on the line through the fixed points with eigenvalue -1 is fixed; we have a fixed line. To show this, we let $v_{D_{2,1}}$ and $v_{D_{2,2}}$ be the two fixed points of $D2$ with eigenvalue -1. Any point on the line through them can be expressed as $v_{D_{2,1}} + k(v_{D_{2,2}} - v_{D_{2,1}})$, where $k \in \mathbb{C}$ is a constant. In this case we have:

$$v_{D_{2,1}} = \begin{bmatrix} 0.8919 \\ -0.0919 - 0.4028i \\ 0.1657 + 0.0798i \end{bmatrix}$$

$$v_{D_{2,2}} = \begin{bmatrix} -0.0475 - 0.2199i \\ 0.1349 + 0.3638i \\ 0.8938 \end{bmatrix}$$

The line:
$v_{D_2} := v_{D_{2,1}} + k(v_{D_{2,2}} - v_{D_{2,1}})$

$$= \begin{bmatrix} 0.8919 \\ -0.0919 - 0.4028i \\ 0.1657 + 0.0798i \end{bmatrix} + k \begin{bmatrix} -0.9394 - 0.2199i \\ 0.2268 + 0.7666i \\ 0.7281 - 0.0798i \end{bmatrix}$$

$$= \begin{bmatrix} 0.8919 - k(0.9394 + 0.2199i) \\ -0.0919 - 0.4028i + k(0.2268 + 0.7666i) \\ 0.1657 + 0.0798i + k(0.7281 - 0.0798i) \end{bmatrix}$$

We multiply v_{D_2} with D_2 from the left:

$D_2 v_{D_2} =$

$$\begin{pmatrix} -0.5910 & 0.1640 - 0.7185i & -0.2955 + 0.1423i \\ 0.1640 + 0.7185i & 0.3280 & -0.3685 - 0.4621i \\ -0.2955 - 0.1423i & -0.3685 + 0.4621i & -0.7370 \end{pmatrix} \cdot \begin{bmatrix} 0.8919 - k(0.9394 + 0.2199i) \\ -0.0919 - 0.4028i + k(0.2268 + 0.7666i) \\ 0.1657 + 0.0798i + k(0.7281 - 0.0798i) \end{bmatrix}$$
$$= \begin{bmatrix} -0.8919 + k(0.9394 + 0.2199i) \\ 0.0919 + 0.4027i - k(0.2268 + 0.7666i) \\ -0.1657 - 0.0798i - k(0.7281 - 0.0798i) \end{bmatrix}$$
$$= (-1) \begin{bmatrix} 0.8919 - k(0.9394 + 0.2199i) \\ -0.0919 - 0.4027i + k(0.2268 + 0.7666i) \\ 0.1657 + 0.0798i + k(0.7281 - 0.0798i) \end{bmatrix}$$

We get the expected eigenvalue of -1, and conclude that any point on the line is a fixed point. Again we have assumed that the lack of accuracy is due to the insufficient power of [MAT21].

None of the fixed points with eigenvalue 1 of the groups of order 2 are on the quartic, sextic or degree-14 curves. However, they are all on the degree-21 curve.

Solving the Klein quartic equation with the parametrization of v_{D_2} above, i.e. $x_{v_{D_2}} = 0.8919 - k(0.9394 + 0.2199i)$, $y_{v_{D_2}} = -0.0919 - 0.4028i + k(0.2268 + 0.7666i)$ and $z_{v_{D_2}} = 0.1657 + 0.0798i + k(0.7281 - 0.0798i)$ in $x^3_{v_{D_2}} y_{v_{D_2}} + y^3_{v_{D_2}} z_{v_{D_2}} + z^3_{v_{D_2}} x_{v_{D_2}} = 0$, [MAT21] gives us four distinct solution.

Proposition 4.5.1. *The fixed lines of the groups of order 2 intersect the Klein quartic curve in four points.*

[Elk99] shows that these 84 points is an orbit, and the intersection of the zero sets of K4 and K21.

4.6 Two classes of 7 Klein 4-groups

These groups consist of 3 matrices from 3 different groups of order 2, other then the identity matrix. Following Klein's exposition for the case of PSL(2,7), the corresponding subgroups of G are:
$E1 := D1, D2, D10$
$E2 := D1, D7, D19$
$E3 := D2, D8, D14$
$E4 := D3, D4, D20$
$E5 := D3, D7, D17$
$E6 := D4, D8, D12$
$E7 := D5, D10, D15$
$E8 := D5, D16, D20$
$E9 := D6, D9, D19$
$E10 := D6, D12, D21$
$E11 := D9, D13, D16$
$E12 := D11, D13, D14$
$E13 := D11, D17, D18$
$E14 := D15, D18, D21$

Each 2-group is in two of the above 4-groups, one from each class. For instance $E7$ and $E8$, both containing $D5$, must be from different classes.
Sorting the Klein 4-groups into the two classes, one class consists of $E1$, $E5$, $E6$, $E8$, $E9$, $E12$ and $E14$, and the other class consists of the rest of the Klein 4-groups.
Conjugation of $E1$ by R gives us $RD1R^{-1} = D2$, $RD2R^{-1} = D1$ and $RD10R^{-1} = D10$, so $RE1R^{-1} = E1$. Conjugation by T yields $TD1T^{-1} = D3$, $TD2T^{-1} = D7$ and $TD10T^{-1} = D17$, which means that $TE1T^{-1} = E5$. Finally, conjugation by S results in $SD1S^{-1} = D4$, $SD2S^{-1} = D12$ and $SD10S^{-1} = D8$, yielding $SE1S^{-1} = E6$. As we can see, $E1$, $E5$ and $E6$ belong to the same class.

Associated to each 2-group there is a fixed point and a fixed line. In the following, let $P5$ be the fixed point of $D5$ and $L5$ be the fixed line of $D5$, and let the fixed points and lines of the other 2-groups be assigned in the same way. The eigenvectors defining the fixed lines are those whose corresponding eigenvalues equal -1. Now $D10P5 = -P5$ and $D15P5 = -P5$, implying that $P5$ is a point both on $L10$ and $L15$. We check this by parametrizing the lines.

The relevant eigenvectors of $D10$ are

$$v_{D_{10,1}} = \begin{bmatrix} 0.9319 \\ -0.1097 + 0.1376i \\ -0.0706 + 0.3091i \end{bmatrix} \text{ and } v_{D_{10,2}} = \begin{bmatrix} 0.1617 + 0.1042i \\ 0.8211 \\ -0.4717 + 0.2575i \end{bmatrix},$$

and for $D15$ they are

$$v_{D_{15,1}} = \begin{bmatrix} 0.3614 + 0.4532i \\ -0.1134 - 0.4970i \\ 0.6357 \end{bmatrix} \text{ and } v_{D_{15,2}} = \begin{bmatrix} -0.2329 + 0.0040i \\ 0.8953 \\ -0.0963 - 0.3673i \end{bmatrix}.$$

For convenience of calculation we divide the coordinates of each vector by the third coordinate, yielding

$$v_{D_{10,1}} = \begin{bmatrix} 0.9319 \\ -0.1097 + 0.1376i \\ -0.0706 + 0.3091i \end{bmatrix} = \begin{bmatrix} -0.6545 - 2.8654i \\ 0.5001 + 0.2407i \\ 1 \end{bmatrix},$$

$$v_{D_{10,2}} = \begin{bmatrix} 0.1617 + 0.1042i \\ 0.8211 \\ -0.4717 + 0.2575i \end{bmatrix} = \begin{bmatrix} -0.1712 - 0.3144i \\ -1.3411 - 0.7321i \\ 1 \end{bmatrix},$$

$$v_{D_{15,1}} = \begin{bmatrix} 0.3614 + 0.4532i \\ -0.1134 - 0.4970i \\ 0.6357 \end{bmatrix} = \begin{bmatrix} 0.5685 + 0.7129i \\ -0.1784 - 0.7818i \\ 1 \end{bmatrix},$$

and

$$v_{D_{15,2}} = \begin{bmatrix} -0.2329 + 0.0040i \\ 0.8953 \\ -0.0963 - 0.3673i \end{bmatrix} = \begin{bmatrix} 0.1454 - 0.5960i \\ -0.5980 + 2.2807i \\ 1 \end{bmatrix}.$$

Parametrizing the fixed lines of $D10$ and $D15$, we get:
$L10 = v_{D_{10,1}} + s(v_{D_{10,2}} - v_{D_{10,1}})$

$$= \begin{bmatrix} -0.6545 - 2.8654i \\ 0.5001 + 0.2407i \\ 1 \end{bmatrix}$$

$$+ s(\begin{bmatrix} -0.1712 - 0.3144i \\ -1.3411 - 0.7321i \\ 1 \end{bmatrix} - \begin{bmatrix} -0.6545 - 2.8654i \\ 0.5001 + 0.2407i \\ 1 \end{bmatrix})$$

and

$L15 = v_{D_{15,1}} + t(v_{D_{15,2}} - v_{D_{15,1}})$

$$= \begin{bmatrix} 0.5685 + 0.7129i \\ -0.1784 - 0.7818i \\ 1 \end{bmatrix}$$
$$+ t(\begin{bmatrix} 0.1454 - 0.5960i \\ -0.5980 + 2.2807i \\ 1 \end{bmatrix} - \begin{bmatrix} 0.5685 + 0.7129i \\ -0.1784 - 0.7818i \\ 1 \end{bmatrix}).$$

Let $p = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} = L10 \cap L15.$

Then $x_p = -0.6545 - 2.8654i + s(-0.1712 - 0.3144i - (-0.6545 - 2.8654i))$
$= 0.5685 + 0.7129i + t(0.1454 - 0.5960i - (0.5685 + 0.7129i)),$

$y_p = 0.5001 + 0.2407i + s(-1.3411 - 0.7321i - (0.5001 + 0.2407i))$
$= -0.1784 - 0.7818i + t(-0.5980 + 2.2807i - (-0.1784 - 0.7818i))$,
and finally $z_p = 1$.

Cleaning up the expressions we have that
$x_p = -0.6545 - 2.8654i + s(0.4833 + 2.5510i)$
$= 0.5685 + 0.7129i + t(-0.4231 - 1.3089i)$
and $y_p = 0.5001 + 0.2407i + s(-1.8412 - 0.9728i)$
$= -0.1784 - 0.7818i + t(-0.4196 + 3.0625i)$.

Solving the two equations x_p and y_p for t, we get that

$$t = \frac{(-0.6545-2.8654i)-(0.5685+0.7129i)+s(0.4833+2.5510i)}{-0.4231-1.3089i}$$

$$= \frac{(0.5001+0.2407i)-(-0.1784-0.7818i)+s(-1.8412-0.9728i)}{-0.4196+3.0625i}.$$

Solving this for s, yields

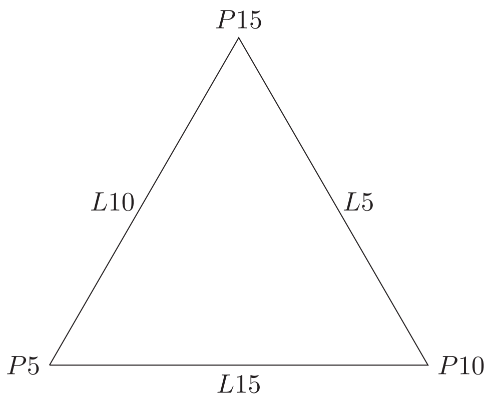
$$s = \frac{(-0.4231-1.3089i)((0.5001+0.2407i)-(-0.1784-0.7818i))-(-0.4196+3.0625i)((-0.6545-2.8654i)-(0.5685+0.7129i))}{(0.4833+2.5510i)(-0.4196+3.0625i)-(-1.8412-0.9728i)(-0.4231-1.3089i)}$$

$= 1.2206 - 0.5142i$.

It follows that $x_p = 1.2471 - 0.001i$, $y_p = -2.2475$ and $p = \begin{bmatrix} 1.2471 - 0.001i \\ -2.2475 \\ 1 \end{bmatrix}$.
Comparing this to
$P5 = \begin{bmatrix} -0.4522 \\ 0.8149 \\ -0.3626 \end{bmatrix} = \begin{bmatrix} 1.2471 \\ -2.2474 \\ 1 \end{bmatrix}$
and taking into account the inaccuracies introduced by the limitations of our software, this seems to confirm that $L10 \cap L15 = P5$. Proceeding in the same way for the other points and lines of $E7$, we find that $L5 \cap L15 = P10$ and $L5 \cap L10 = P15$.

Similarly for $E8$, $L5 \cap L16 = P20$, $L5 \cap L20 = P16$, and $L16 \cap L20 = P5$. The results hold for both classes. We conclude that the fixed points of the 2-groups are also fixed points of the dihedral abelian 4-groups they are part of. The fixed points are the vertices of triangles where the sides are segments of the fixed lines of the 2-groups, as illustrated in figure 4.1. We have seen before that these fixed points are on the degree-21 curve.

Figure 4.1: The fixed points of E7



Proposition 4.6.1. *The fixed points of the dihedral abelian 4-groups are the fixed points of the 2-groups they consist of. There are three fixed points for each dihedral abelian 4-group. The fixed points are not on the Klein quartic curve, the sextic curve or the degree-14 curve. They are on the degree-21 curve.*

Since every 2-group is part of two Klein 4-groups, we know that every line is part of two triangles like the one in figure 4.1. This means that every one of the fixed lines of the 2-groups intersects four different fixed points on the degree-21 curve. For instance, since $D1$ is part of $E1$ and $E2$, the fixed points of the other 2-groups in those Klein 4-groups must be on the fixed line $L1$ of $D1$. We list which points are on which lines:
L1: P2, P7, P10, P19
L2: P1, P8, P10, P14
L3: P4, P7, P17, P20
L4: P3, P8, P12, P20
L5: P10, P15, P16, P20
L6: P9, P12, P19, P21
L7: P1, P3, P17, P19
L8: P2, P4, P12, P14
L9: P6, P13, P16, P19
L10: P1, P2, P5, P15
L11: P13, P14, P17, P18
L12: P4, P6, P8, P21
L13: P9, P11, P14, P16
L14: P2, P8, P11, P13
L15: P5, P10, P18, P21
L16: P5, P9, P13, P20
L17: P3, P7, P11, P18
L18: P11, P15, P17, P21
L19: P1, P6, P7, P9
L20: P3, P4, P5, P16
L21: P6, P12, P15, P18

The other way around, every fixed point is on four different lines, for instance the point $P1$ is on $L2$, $L7$, $L10$ and $L19$.

4.7 Dihedral groups of order 6

These groups consist of the matrices of a group of order 3 together with three matrices from three different groups of order 2:
$F1 := B1, D5, D6, D11$
$F2 := B2, D3, D14, D21$
$F3 := B3, D8, D15, D19$
$F4 := B4, D4, D10, D13$
$F5 := B5, D1, D18, D20$
$F6 := B6, D7, D12, D16$
$F7 := B7, D2, D9, D17$
$F8 := B8, D8, D9, D17$
$F9 := B9, D7, D13, D21$
$F10 := B10, D10, D11, D12$
$F11 := B11, D11, D19, D20$
$F12 := B12, D3, D13, D15$
$F13 := B13, D2, D16, D18$
$F14 := B14, D5, D12, D17$
$F15 := B15, D5, D14, D19$
$F16 := B16, D6, D14, D20$
$F17 := B17, D6, D10, D17$
$F18 := B18, D4, D9, D15$
$F19 := B19, D1, D16, D21$
$F20 := B20, D8, D16, D17$
$F21 := B21, D7, D14, D15$
$F22 := B22, D3, D9, D10$
$F23 := B23, D2, D20, D21$
$F24 := B24, D4, D18, D19$
$F25 := B25, D1, D12, D13$
$F26 := B26, D1, D4, D11$
$F27 := B27, D2, D3, D6$
$F28 := B28, D5, D7, D8$

All these subgroups are conjugate. Conjugation follows the same pattern as the one described in section 4.3 for the groups of order 3. By this we mean that conjugation of one of these dihedral groups by one of the generators of G yields the dihedral group which contains the 3-group which is conjugate to the 3-group in the first dihedral group by the same generator. In particular, to follow the example in 4.3, $(RTS)[F1](RTS)^{-1} = F20$.

We examine $F1$, consisting of $B1$, $D5$, $D6$ and $D11$. $B1$ has three fixed points; $v_{B_{1,1}} = \begin{bmatrix} e^{4\pi i/3} \\ e^{2\pi i/3} \\ 1 \end{bmatrix}$ and $v_{B_{1,2}} = \begin{bmatrix} e^{2\pi i/3} \\ e^{4\pi i/3} \\ 1 \end{bmatrix}$, both on the Klein quartic curve, and $v_{B_{1,3}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ not on the curve.

The fixed points of $D5$, $D6$ and $D11$ respectively are $P5 = \begin{bmatrix} 1.2471 \\ -2.2474 \\ 1 \end{bmatrix}$,
$P6 = \begin{bmatrix} -1.8021 \\ 0.8019 \\ 1 \end{bmatrix}$ and $P11 = \begin{bmatrix} -0.4450 \\ -0.5549 \\ 1 \end{bmatrix}$.

We see that the fixed points of the 2-groups are all distinct from each other and from the fixed points of the 3-group. The only possibility for any fixed points of $F1$ is if any of the fixed points of $B1$ is on all the fixed lines of the 2-groups. We know that this means that multiplying a fixed point from the left with each of the generating matrices of the 2-groups yields the negative of the fixed point. Doing this, we get $D5v_{B_{1,1}} = D6v_{B_{1,1}} = D11v_{B_{1,1}} = v_{B_{1,2}} \neq -v_{B_{1,1}}$,
$D5v_{B_{1,2}} = D6v_{B_{1,2}} = D11v_{B_{1,2}} = v_{B_{1,1}} \neq -v_{B_{1,2}}$,
and finally $D5v_{B_{1,3}} = D6v_{B_{1,3}} = D11v_{B_{1,3}} = -v_{B_{1,3}}$.
We see that $v_{B_{1,3}}$ is on all three lines. However it is not on the Klein quartic curve.

Let us check in another way if $v_{B_{1,3}}$ is on $L5$:

$$L5 = v_{D_{5,1}} + k(v_{D_{5,2}} - v_{D_{5,1}})$$

$$= \begin{bmatrix} -4.8499 \\ -2.2463 \\ 1 \end{bmatrix} + k(\begin{bmatrix} -0.0686 - 0.2127i \\ 0.4069 - 0.1181i \\ 1 \end{bmatrix} - \begin{bmatrix} -4.8499 \\ -2.2463 \\ 1 \end{bmatrix})$$

$$= \begin{bmatrix} -4.8499 + k(4.7813 - 0.2127i) \\ -2.2463 + k(2.6532 - 0.1181i) \\ 1 \end{bmatrix}.$$

Solving for the first coordinate:

$$-4.8499 + k(4.7813 - 0.2127i) = 1$$

$$k = \frac{1+4.8499}{4.7813-0.2127i} = 1.2211 + 0.0543i.$$

Applying the result to the second coordinate:

$$-2.2463 + k(2.6532 - 0.1181i)$$

$= -2.2463 + (1.2211 + 0.0543i)(2.6532 - 0.1181i)$
$= 0.9999 - 0.0001i.$

Taking into account the inaccuracies, this also indicates that $v_{B_{1,3}}$ is on the line.

In the same way, $v_{B_{2,3}}$ is a fixed point of $F2$ since $D3v_{B_{2,3}} = D14v_{B_{2,3}} = D21v_{B_{2,3}} = -v_{B_{2,3}}$. $v_{B_{1,3}}$ and $v_{B_{2,3}}$ are not on either of the quartic, sextic or degree-14 curves, but they are both on the degree-21 curve. All the groups in this category are in the same class, so we draw a general conclusion.

Proposition 4.7.1. *The 28 dihedral nonabelian groups of order 6 each has one fixed point. The fixed points are not on the Klein quartic curve, the sextic curve or the degree-14 curve. They are on the degree-21 curve.*

In table 4.8 we see what happens when we multiply the generator of a subgroup of $F1$ with the fixed points of the subgroups $B1$, $D1$, $D6$ and $D11$. In the table v_{D_5}, v_{D_6} or $v_{D_{11}}$ is any fixed point of $D5$, $D6$ or $D11$ respectively, whether it is an isolated fixed point or a point on a fixed line. The relation is the same. The columns under each fixed point consists of the points the fixed points are sent to by the action of the different subgroups of $F1$, thus giving us the orbits of $F1$ in the complex projective plane.

Table 4.8: Orbits of the fixed points of the subgroups of $F1$

	$v_{B_{1,1}}$	$v_{B_{1,2}}$	$v_{B_{1,3}}$	v_{D_5}	v_{D_6}	$v_{D_{11}}$
$B1$	$v_{B_{1,1}}$	$v_{B_{1,2}}$	$v_{B_{1,3}}$	$v_{D_{11}}$	v_{D_5}	v_{D_6}
$D5$	$v_{B_{1,2}}$	$v_{B_{1,1}}$	$v_{B_{1,3}}$	v_{D_5}	$v_{D_{11}}$	v_{D_6}
$D6$	$v_{B_{1,2}}$	$v_{B_{1,1}}$	$v_{B_{1,3}}$	$v_{D_{11}}$	v_{D_6}	v_{D_5}
$D11$	$v_{B_{1,2}}$	$v_{B_{1,1}}$	$v_{B_{1,3}}$	v_{D_6}	v_{D_5}	$v_{D_{11}}$

We see that $F1$ has orbits of order 1, 2 and 3.

4.8 Dihedral groups of order 8

To construct these, take a cyclic group of order 4 and add to it all the matrices that appear together with its element of order 2 in the Klein 4-groups:
$H1 := C1, D2, D7, D10, D19$
$H2 := C2, D1, D8, D10, D14$
$H3 := C3, D4, D7, D17, D20$
$H4 := C4, D3, D8, D12, D20$
$H5 := C5, D10, D15, D16, D20$
$H6 := C6, D9, D12, D19, D21$
$H7 := C7, D1, D3, D17, D19$
$H8 := C8, D2, D4, D12, D14$
$H9 := C9, D6, D13, D16, D19$
$H10 := C10, D1, D2, D5, D15$
$H11 := C11, D13, D14, D17, D18$
$H12 := C12, D4, D6, D8, D21$
$H13 := C13, D9, D11, D14, D16$
$H14 := C14, D2, D8, D11, D13$
$H15 := C15, D5, D10, D18, D21$
$H16 := C16, D5, D9, D13, D20$
$H17 := C17, D3, D7, D11, D18$
$H18 := C18, D11, D15, D17, D21$
$H19 := C19, D1, D6, D7, D9$
$H20 := C20, D3, D4, D5, D16$
$H21 := C21, D6, D12, D15, D18$

These subgroups are also all conjugate, and conjugation follows the pattern of the cyclic 4-groups of section 4.4. For example, conjugation of $C1$, $D2$, $D7$, $D10$ and $D19$ respectively by R, we get $C2$, $D1$, $D8$, $D10$ and $D19$. This means that conjugation of $H1$ by R yields $H2$, just as conjugation of $C1$ by R yields $C2$. This pattern holds throughout.

We produce a table showing the orbits of the fixed points of the subgroups of $H1$.

Table 4.9: Orbits of the fixed points of the subgroups of $H1$

	$v_{C_{1,1}}$	$v_{C_{1,2}}$	$v_{C_{1,3}}$	v_{D_2}	v_{D_7}	$v_{D_{10}}$	$v_{D_{19}}$
$C1$	$v_{C_{1,1}}$	$v_{C_{1,2}}$	$v_{C_{1,3}}$	$v_{D_{10}}$	$v_{D_{19}}$	v_{D_2}	v_{D_7}
$D2$	$v_{C_{1,2}}$	$v_{C_{1,1}}$	$v_{C_{1,3}}$	v_{D_2}	$v_{D_{19}}$	$v_{D_{10}}$	v_{D_7}
$D7$	$v_{C_{1,2}}$	$v_{C_{1,1}}$	$v_{C_{1,3}}$	$v_{D_{10}}$	v_{D_7}	v_{D_2}	$v_{D_{19}}$
$D10$	$v_{C_{1,2}}$	$v_{C_{1,1}}$	$v_{C_{1,3}}$	v_{D_2}	$v_{D_{19}}$	$v_{D_{10}}$	v_{D_7}
$D19$	$v_{C_{1,2}}$	$v_{C_{1,1}}$	$v_{C_{1,3}}$	$v_{D_{10}}$	v_{D_7}	v_{D_2}	$v_{D_{19}}$

We see that $H1$ has orbits of order 1 and 2, and that $v_{C_{1,3}}$ is a fixed point for $H1$. This is the fixed point of $C1$ that is not on the Klein quartic curve. It is on the degree-21 curve however. Since all the groups in this category are of the same class, we can draw a conclusion.

Proposition 4.8.1. *The 21 dihedral nonabelian groups of order 8 each has one fixed point. The fixed points are not on the Klein quartic curve, the sextic curve or the degree-14 curve. They are on the degree-21 curve.*

4.9 Nonabelian groups of order 21

These subgroups consist of the matrices of a group of order 7 together with 14 matrices from the groups of order 3. Klein describes one such subgroup in PSL(2,7) as consisting of the matrices $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & k \\ 0 & 4 \end{pmatrix}$ and $\begin{pmatrix} 3 & k \\ 0 & 5 \end{pmatrix}$ for $k = 0, 1, ..., 6$.

In PSL(2,7) these are the matrices of the subgroups $A1'$, $B1'$, $B2'$, $B3'$, $B4'$, $B5'$, $B6'$ and $B7'$, so for G we can define $J1 := A1, B1, B2, B3, B4, B5, B6, B7$ as one of these nonabelian groups.

As for conjugation, we have seen in section 4.2 that the 7-groups are conjugate. Conjugation of $A1$ by RS^6RS yields $A8$, and conjugation of the remaining subgroups $B1 - B7$ of $J1$ by the same element, gives us $B19$, $B24$, $B7$, $B12$, $B16$, $B28$ and $B10$ respectively. This is another nonabelian group of order 21, $J8$. That the same group properties hold for $J8$ can be seen considering a relation $AB = C$ for A,B,$C \epsilon J1$. Conjugation of A, B and C by $g \epsilon G$ leads to $gAg^{-1}gBg{-}1 = gABg{-}1 = gCg{-}1$. Starting from $J8$ and conjugating repeatedly by S returns the remaining nonabelian groups of order 21, so all these subgroups are conjugate.

We have seen in 4.2 that the fixed points of the subgroups of order 7 are on the intersection of the quartic and sextic curves, inflection points on both curves. In 4.3 we saw that two of the fixed points of the 3-groups are on the quartic curve, but not on the sextic curve. The other fixed points are on the degree-21 curve, but not on any of the other curves we study. Since all of the fixed points of the 7-groups are on the sextic curve, and none of the fixed points of the 3-groups are, we conclude:

Proposition 4.9.1. *None of the 8 nonabelian groups of order 21 have any fixed points in common.*

4.10 Two classes of 7 nonabelian conjugates of the symmetric group of degree 4

We follow Kleins description of these subgroups in PSL(2,7). We start with a Klein 4-group and add to it six matrices from three cyclic 4-groups whose second iterates belong to the Klein 4-group. Then add three pairs of 2-groups that commute with some 2-group in the Klein 4-group. Finally, add the compositions of the matrices of the three pairs just mentioned, which yield four new pairs of matrices from 3-groups. We look more closely at these compositions as described by Klein. He starts with the Klein 4-group consisting of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 3 & -2 \\ -2 & -3 \end{pmatrix}$, and $\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}$. The six 2-groups that commute with some 2-group in the Klein 4-group are $\begin{pmatrix} 2 & -3 \\ -3 & -2 \end{pmatrix}$, $\begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$,

$\begin{pmatrix} 3 & -1 \\ 3 & -3 \end{pmatrix}$, and $\begin{pmatrix} -3 & -3 \\ 1 & 3 \end{pmatrix}$. According to Klein, compositions of these matrices,

other then compositions of matrices belonging to the same pair, yield eight matrices from four different 3-groups. Compositions of matrices from the same pair already belong to the Klein 4-group we started with. The other eight matrices in Klein's example are given as $\begin{pmatrix} -3 & -1 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} -2 & -1 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}$,

$\begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & -3 \\ 2 & 1 \end{pmatrix}$.

There are in total 24 compositions to consider, and by performing them we get every one of the eight matrices expected three times, except the last one.
In stead, we have that

$$\begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -3 & -3 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -3 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}.$$

There seems to be an error or a misprint in Klein's paper.
Correcting that and considering the corresponding subgroups of G rather than of PSL(2,7), we can define one of the subgroups in this section as $M8 := E8, C5, C16, C20, D3, D4, D9, D10, D13, D15, B4, B12, B18, B22$.
From 4.6 we know that $E8 := D5, D16, D20$. Since every matrix of $D5$ is in $C5$, every matrix of $D16$ is in $C16$ and every matrix of $D20$ is in $C20$, we can simplify and define
$M8 := C5, C16, C20, D3, D4, D9, D10, D13, D15, B4, B12, B18, B22$.

Conjugation of each of the subgroups making up $M8$ by S in the same order as above, returns $C2$, $C1$, $C10$, $D5$, $D15$, $D7$, $D8$, $D19$, $D14$, $B2$, $B15$, $B21$ and $B28$ respectively. The group containing these subgroups also contains $E1 = D1, D2, D10$, so it is natural to designate it as $M1$. Repetitive conjugation by S yields all the groups of the class, while the same procedure applied to a group in the other class, for example $M7$ containing $E7$, gives us the groups of the other class.

$E8$ has three fixed points, the ones with eigenvalue 1 for each of the 2-groups $D5$, $D16$ and $D20$. These are

$P5 = \begin{bmatrix} 1.2471 \\ -2.2474 \\ 1 \end{bmatrix}$, $P16 = \begin{bmatrix} -1.1233 + 1.4090i \\ -0.1784 + 0.7817i \\ 1 \end{bmatrix}$

and $P20 = \begin{bmatrix} 0.0990 - 0.4339i \\ 0.4999 - 0.2408i \\ 1 \end{bmatrix}$.

These points are fixed points for $C5$, $C16$ and $C20$ by the above.
Multiplication of the generating matrices of the other 2-groups and the 3-groups in $M8$ respectively with the three fixed points, yields the following results:

Table 4.10: Orbits of the fixed points of $E8$

	$P5$	$P16$	$P20$
$D3$	$P16$	$P5$	$P20$
$D4$	$P16$	$P5$	$P20$
$D9$	$P20$	$P16$	$P5$
$D10$	$P5$	$P20$	$P16$
$D13$	$P20$	$P16$	$P5$
$D15$	$P5$	$P20$	$P16$
$B4$	$P20$	$P5$	$P16$
$B12$	$P20$	$P5$	$P16$
$B18$	$P16$	$P20$	$P5$
$B22$	$P16$	$P20$	$P5$

We see that the three fixed points of $E8$ is an orbit.

We construct the group $M7$ from the other class. It consists of $E7 = D5, D10, D15$ and then also of the cyclic 4-groups $C5$, $C10$ and $C15$. $D5$ commutes with $D16$ and $D20$, $D10$ commutes with $D1$ and $D2$, and $D15$ commutes with $D18$ and $D21$. These are also part of $M7$. Finally, compositions of the last six 2-groups show that $B5$, $B13$, $B19$ and $B23$ belong to $M7$. By checking the products of the subgroups contained in $M7$ with the three fixed points of $E7$, we find the same type of orbits as in the former case.

Proposition 4.10.1. *None of the nonabelian conjugates of the symmetric group of degree 4 have any fixed points in common.*

4.11 Two classes of 7 nonabelian conjugates of the alternating group of degree 4

The alternating group of degree 4 consists of the even permutations of the symmetric group of degree 4. The even permutations are the identity, the 3-cycles and the double-transpositions. In our cases, this means that the alternating group $N8$ of $M8$ consists of the elements of $E8$ (i.e. $D5$, $D16$ and $D20$), $B4$, $B12$, $B18$ and $B22$. Similarly, $N7 := E7(D5, D10, D15), B5, B13, B19, B23$. Conjugation follows the pattern of the symmetric groups, in that conjugation by S yields all the alternating groups of both classes. Checking back with the four bottom rows of table 4.10 we establish that the fixed points of the Klein 4-groups form three-point orbits under the alternating groups, and that the alternating groups do not have any fixed points.

Proposition 4.11.1. *None of the nonabelian conjugates of the alternating group of degree 4 have any fixed points in common.*

4.12 Summary

Summing up what we know about the fixed points and orbits on the Klein quartic curve:

Proposition 4.12.1. *The 24 fixed points of the groups of order 7 are on the curve. They constitute an orbit and is the intersection of the quartic and the sextic curve.*
2 fixed points from each of the 28 groups of order 3 are on the curve. These 56 fixed points constitute an orbit and is the intersection of the quartic and the degree-14 curve.
The fixed lines of the groups of order 2 intersect the quartic curve in four points each. These 84 points form an orbit and is the intersection of the quartic and degree-21 curve.

CHAPTER 5

Factoring K21

5.1 Examination of some fixed points on K21

As we have seen, every 3-group has three fixed points, with one of these satisfying K21=0, in total 28 points. Factoring with [MAT21] reveals that K21 can be factored into four factors over $\mathbb{Z}$; one polynomial of degree 3, and three polynomials of degree 6, which we define as

Q1:=$x^3 - x^2y - 2x^2z - 2xy^2 + 6xyz - xz^2 + y^3 - y^2z - 2yz^2 + z^3$

Q2:=$x^6 + 5x^5y + 3x^5z + 11x^4y^2 + 16x^4yz + 9x^4z^2 + 13x^3y^3 + 36x^3y^2z + 37x^3yz^2 + 13x^3z^3 + 9x^2y^4 + 37x^2y^3z + 55x^2y^2z^2 + 36x^2yz^3 + 11x^2z^4 + 3xy^5 + 16xy^4z + 36xy^3z^2 + 37xy^2z^3 + 16xyz^4 + 5xz^5 + y^6 + 5y^5z + 11y^4z^2 + 13y^3z^3 + 9y^2z^4 + 3yz^5 + z^6$

Q3:=$x^6 - 2x^5y + 3x^5z + 4x^4y^2 - 5x^4yz + 2x^4z^2 - x^3y^3 + 8x^3y^2z + 9x^3yz^2 - x^3z^3 + 2x^2y^4 + 9x^2y^3z - x^2y^2z^2 + 8x^2yz^3 + 4x^2z^4 + 3xy^5 - 5xy^4z + 8xy^3z^2 + 9xy^2z^3 - 5xyz^4 - 2xz^5 + y^6 - 2y^5z + 4y^4z^2 - y^3z^3 + 2y^2z^4 + 3yz^5 + z^6$

Q4:=$x^6 - 2x^5y - 4x^5z + 4x^4y^2 + 2x^4yz + 9x^4z^2 - 8x^3y^3 - 6x^3y^2z + 2x^3yz^2 - 8x^3z^3 + 9x^2y^4 + 2x^2y^3z + 13x^2y^2z^2 - 6x^2yz^3 + 4x^2z^4 - 4xy^5 + 2xy^4z - 6xy^3z^2 + 2xy^2z^3 + 2xyz^4 - 2xz^5 + y^6 - 2y^5z + 4y^4z^2 - 8y^3z^3 + 9y^2z^4 - 4yz^5 + z^6$

In order to find which factor(s) are zero for each point, we calculate every factor with [MAT21] for every fixed point. In each case we get a complex number. Determining its distance from 0, we get the results in the following table. The numbers in the table are the exponent of the result in standard form notation, so for instance "-16" in the table means that the result is larger than or equal to 10^{-16} but smaller than 10^{-15}. b_1 designates the fixed point (on K21) of the 3-group $B1$, and so on.

Table 5.1: Distance from 0 as calculated by MATLAB

point	Q1	Q2	Q3	Q4
b_1	-16	1	0	-1
b_2	0	-5	-5	-5
b_3	0	-5	-5	-4
b_4	0	-5	-5	-4
b_5	0	-5	-5	-4
b_6	0	-5	-5	-4
b_7	0	-5	-5	-5
b_8	0	0	-8	-4
b_9	0	0	-8	-4
b_{10}	-4	-8	0	0
b_{11}	-4	-8	0	0
b_{12}	0	0	-8	-4
b_{13}	0	0	-8	-4
b_{14}	-4	-8	0	0
b_{15}	-4	-8	0	0
b_{16}	-4	-8	0	0
b_{17}	-4	-8	0	0
b_{18}	0	0	-8	-4
b_{19}	0	0	-8	-4
b_{20}	0	-4	-4	-5
b_{21}	0	-4	-4	-5
b_{22}	0	-4	-4	-5
b_{23}	0	-4	-4	-5
b_{24}	0	-4	-4	-5
b_{25}	0	-4	-4	-5
b_{26}	-5	0	0	-9
b_{27}	-5	0	0	-9
b_{28}	-5	0	0	-9

It is hard to tell which points are exactly zero for each factor. Assuming that -4 or lower indicates a true zero, preliminarily we seem to have one point (b_1) on Q1, twelve points simultaneously on Q2, Q3 and Q4, six points on Q1 and Q2, six points on Q3 and Q4, and lastly three points on both Q1 and Q4.

5.2 Linear factoring

To find the real symmetries we seem to need a complete factoring of K21 into linear factors. From [Kle99, p. 304] and [Adl99, p. 265] we learn that K21 is a product of the fixed lines of the 2-groups. The fixed lines have the eigenvalue -1, so by multiplying each involution by the fixed points, we can tell that the fixed point is on the line if the result of the multiplication is the negative of the fixed point. We get the following allocation:

L1: b_5, b_{19}, b_{25}, b_{26}

L2: b_7, b_{13}, b_{23}, b_{27}
L3: b_2, b_{12}, b_{22}, b_{27}
L4: b_4, b_{18}, b_{24}, b_{26}
L5: b_1, b_{14}, b_{15}, b_{28}
L6: b_1, b_{16}, b_{17}, b_{27}
L7: b_6, b_9, b_{21}, b_{28}
L8: b_3, b_8, b_{20}, b_{28}
L9: b_7, b_8, b_{18}, b_{22}
L10: b_4, b_{10}, b_{17}, b_{22}
L11: b_1, b_{10}, b_{11}, b_{26}
L12: b_6, b_{10}, b_{14}, b_{25}
L13: b_4, b_9, b_{12}, b_{25}
L14: b_2, b_{15}, b_{16}, b_{21}
L15: b_3, b_{12}, b_{18}, b_{21}
L16: b_6, b_{13}, b_{19}, b_{20}
L17: b_7, b_{14}, b_{17}, b_{20}
L18: b_5, b_8, b_{13}, b_{24}
L19: b_3, b_{11}, b_{15}, b_{24}
L20: b_5, b_{11}, b_{16}, b_{23}
L21: b_2, b_9, b_{19}, b_{23}

There are four points on every line, and three lines through every point, as follows:

b_1: L5, L6, L11
b_2: L3, L14, L21
b_3: L8, L15, L19
b_4: L4, L10, L13
b_5: L1, L18, L20
b_6: L7, L12, L16
b_7: L2, L9, L17
b_8: L8, L9, L18
b_9: L7, L13, L21
b_{10}: L10, L11, L12
b_{11}: L11, L19, L20
b_{12}: L3, L13, L15
b_{13}: L2, L16, L18
b_{14}: L5, L12, L17
b_{15}: L5, L14, L19
b_{16}: L6, L14, L20
b_{17}: L6, L10, L17
b_{18}: L4, L9, L15
b_{19}: L1, L16, L21
b_{20}: L8, L16, L17
b_{21}: L7, L14, L15
b_{22}: L3, L9, L10
b_{23}: L2, L20, L21

b_{24}: L4, L18, L19
b_{25}: L1, L12, L13
b_{26}: L1, L4, L11
b_{27}: L2, L3, L6
b_{28}: L5, L7, L8

5.3 Factoring the degree-3 factor

Looking back at table 5.1, we see that b_1 is by far the point that comes closest to zero for a specific factor, namely Q1. We hypothesize that this may be because b_1 is a zero for all the three fixed lines of L5, L6 and L11 simultaneously, i.e that Q1 is a product of the three lines. To check this, we need to find the equations of the lines. For any 2-group, let $v_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be the fixed points with eigenvalue -1. Then $v := \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 + k(x_2 - x_1) \\ y_1 + k(y_2 - y_1) \\ z_1 + k(z_2 - z_1) \end{bmatrix}$, with $k \in \mathbb{C}$ a constant, is any point on the line.
Let M be the matrix with v_1, v_2, v respectively as column vectors. Then its determinant $detM = \begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \\ z_1 & z_2 & z \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_1 + k(x_2 - x_1) \\ y_1 & y_2 & y_1 + k(y_2 - y_1) \\ z_1 & z_2 & z_1 + k(z_2 - z_1) \end{vmatrix} = 0$, which means that $(y_1z_2 - y_2z_1)x + (x_2z_1 - x_1z_2)y + (x_1y_2 - x_2y_1)z = 0$ is an expression for the fixed line.
We utilize this to find the fixed line $L5$ of $D5$. The fixed points of $D5$ with eigenvalue -1 are
$v_{D_{5,1}} = \begin{bmatrix} 0.8919 \\ 0.4131 \\ -0.1839 \end{bmatrix}$ and $v_{D_{5,2}} = \begin{bmatrix} -0.0619 - 0.1918i \\ 0.3670 - 0.1065i \\ 0.9019 \end{bmatrix}$,
so $L5 = (0.4131 \cdot 0.9019 - (0.3670 - 0.1065i)(-0.1839))x$
$+ ((-0.0619 - 0.1918i)(-0.1839) - 0.8919 \cdot 0.9019)y$
$+ (0.8919(0.3670 - 0.1065i) - (-0.0619 - 0.1918i)0.4131)z$
$= (0.4401 - 0.0196i)x + (-0.7930 + 0.0353i)y + (0.3529 - 0.0158i)z$
$= 0.$

In the same way we find that
$L6 = (-0.8063 + 0.0533i)x + (0.3589 - 0.0237i)y + (0.4475 - 0.0296i)z = 0$ and
$L11 = (-0.3622 + 0.0065i)x + (-0.4517 + 0.0081i)y + (0.8139 - 0.0147i)z = 0.$

We calculate the product of the fixed lines of our hypothesis:
$L5 \cdot L6 \cdot L11$
$= (0.4401 - 0.0196i)x + (-0.7930 + 0.0353i)y + (0.3529 - 0.0158i)z$
$\cdot (-0.8063 + 0.0533i)x + (0.3589 - 0.0237i)y + (0.4475 - 0.0296i)z$
$\cdot (-0.3622 + 0.0065i)x + (-0.4517 + 0.0081i)y + (0.8139 - 0.0147i)z$

$= (0.1279 - 0.0165i)x^3 + (-0.1279 + 0.0165i)x^2y + (-0.2558 + 0.0331i)x^2z$
$+ (-0.2558 + 0.0331i)xy^2 + (0.7674 - 0.0992i)xyz + (-0.1279 + 0.0166i)xz^2$
$+ (0.1279 - 0.0165i)y^3 + (-0.1279 + 0.0165i)y^2z + (-0.2558 + 0.0331i)yz^2$
$+ (0.1279 - 0.0166i)z^3$

Multiplying our expression for $Q1$ by a factor of $(0.1279 - 0.0165i)$, the coefficient of x^3 in the product above, we get:
$= (0.1279 - 0.0165i)x^3 + (-0.1279 + 0.0165i)x^2y + (-0.2558 + 0.0330i)x^2z$
$+ (-0.2558 + 0.0330i)xy^2 + (0.7674 - 0.0990i)xyz + (-0.1279 + 0.0165i)xz^2$
$+ (0.1279 - 0.0165i)y^3 + (-0.1279 + 0.0165i)y^2z + (-0.2558 + 0.0330i)yz^2$
$+ (0.1279 - 0.0165i)z^3$

Taking into account the compounded inaccuracies inherent in many consecutive MATLAB calculations limited to four decimals, we conclude that $Q1$ is a product of the three lines $L5$, $L6$ and $L11$.

5.4 Equations for the fixed lines

Following the same method, we derive all the 21 fixed lines.
They are:

L1: (-0.5481 + 0.2041i)x+(0.0811 - 0.2474i)y+(0.0428 + 0.3217i)z=0
L2: (-0.0755 - 0.4311i)x+(-0.7875 - 0.0402i)y+(0.2045 + 0.2851i)z=0
L3: (-0.1218 + 0.2800i)x+(-0.5407 - 0.1017i)y+(0.1853 - 0.1600i)z=0
L4: (-0.7210 + 0.0393i)x+(0.2137 + 0.2399i)y+(-0.1104 - 0.3852i)z=0
L5: (0.4401 - 0.0196i)x+(-0.7930 + 0.0353i)y+(0.3529 - 0.0158i)z=0
L6: (-0.8063 + 0.0533i)x+(0.3589 - 0.0237i)y+(0.4475 - 0.0296i)z=0
L7: (-0.2110 - 0.2632i)x+(0.0946 + 0.4099i)y+(0.7579 - 0.0018i)z=0
L8: (-0.2117 + 0.2915i)x+(0.0801 - 0.4420i)y+(0.8087 - 0.0366i)z=0
L9: (0.2977 + 0.1329i)x+(-0.2441 + 0.3249i)y+(0.7321 - 0.0211i)z=0
L10: (0.0743 - 0.3470i)x+(0.3960 - 0.1973i)y+(0.7973 - 0.0108i)z=0
L11: (-0.3622 + 0.0065i)x+(-0.4517 + 0.0081i)y+(0.8139 - 0.0147i)z=0
L12: (-0.0879 + 0.4174i)x+(-0.4691 + 0.6090i)y+(0.2178 + 0.2638i)z=0
L13: (0.2624 - 0.3661i)x+(-0.8105 + 0.0417i)y+(-0.3331 - 0.1397i)z=0
L14: (-0.8100 - 0.0219i)x+(-0.0707 - 0.3537i)y+(-0.3997 - 0.2060i)z=0
L15: (-0.7408 + 0.0895i)x+(-0.2797 + 0.1789i)y+(0.2952 + 0.2905i)z=0
L16: (-0.7922 + 0.0227i)x+(-0.3220 - 0.1438i)y+(0.2643 - 0.3516i)z=0
L17: (-0.7967 + 0.0868i)x+(-0.0412 + 0.3543i)y+(-0.3775 + 0.2352i)z=0
L18: (0.3186 + 0.3035i)x+(-0.7855 + 0.1080i)y+(-0.2941 + 0.1949i)z=0
L19: (-0.4050 - 0.1983i)x+(-0.8125 - 0.0052i)y+(-0.0782 - 0.3531i)z=0
L20: (0.1089 + 0.3265i)x+(0.4030 + 0.1477i)y+(0.7698 - 0.0755i)z=0
L21: (0.3281 - 0.1531i)x+(-0.2771 - 0.3564i)y+(0.8134 + 0.0100i)z=0

5.5 Factoring the degree-6 factors

We have seen that the product of the fixed lines of D5, D6 and D11 is the degree-3 factor Q1. We expect the remaining 18 fixed lines to consist of three sets of six lines each, where the product of lines in each set is a degree-6 factor. Looking back at table 5.1, we seem to have clear and similar zeros for Q3 for the points b_8, b_9, b_{12}, b_{13}, b_{18} and b_{19}. We examine the lines going through these points:

b_8: L8, L9, L18
b_9: L7, L13, L21
b_{12}: L3, L13, L15
b_{13}: L2, L16, L18
b_{18}: L4, L9, L15
b_{19}: L1, L16, L21

We notice that six lines occur twice in the list above; L9, L13, L15, L16, L18 and L21. We hypothesize that the product of these lines is the degree-6 factor Q3. Initially we find that the product of the x-coefficients of the six lines is equal to the product of the y-coefficients and equal to the product of the z-coefficients. This means that the coefficients of x^6, y^6 and z^6 in the product are equal, as they must be in Q3. Performing the multiplication of the six lines, we get:

$L9 \cdot L13 \cdot L15 \cdot L16 \cdot L18 \cdot L21$

$$
\begin{aligned}
&= ((0.2977 + 0.1329i)x + (-0.2441 + 0.3249i)y + (0.7321 - 0.0211i)z)\\
&\cdot ((0.2624 - 0.3661i)x + (-0.8105 + 0.0417i)y + (-0.3331 - 0.1397i)z)\\
&\cdot ((-0.7408 + 0.0895i)x + (-0.2797 + 0.1789i)y + (0.2952 + 0.2905i)z)\\
&\cdot ((-0.7922 + 0.0227i)x + (-0.3220 - 0.1438i)y + (0.2643 - 0.3516i)z)\\
&\cdot ((0.3186 + 0.3035i)x + (-0.7855 + 0.1080i)y + (-0.2941 + 0.1949i)z)\\
&\cdot ((0.3281 - 0.1531i)x + (-0.2771 - 0.3564i)y + (0.8134 + 0.0100i)z)
\end{aligned}
$$

$$
\begin{aligned}
&= (0.0130 - 0.0048i)x^6 + (-0.0260 + 0.0096i)x^5y + (0.0389 - 0.0144i)x^5z\\
&+ (0.0519 - 0.0191i)x^4y^2 + (-0.0649 + 0.0239i)x^4yz + (0.0259 - 0.0096i)x^4z^2\\
&+ (-0.0130 + 0.0048i)x^3y^3 + (0.1038 - 0.0383i)x^3y^2z + (0.1168 - 0.0431i)x^3yz^2\\
&+ (-0.0130 + 0.0048i)x^3z^3 + (0.0259 - 0.0096i)x^2y^4 + (0.1168 - 0.0431i)x^2y^3z\\
&+ (-0.0130 + 0.0048i)x^2y^2z^2 + (0.1038 - 0.0383i)x^2yz^3 + (0.0519 - 0.0191i)x^2z^4\\
&+ (0.0389 - 0.0144i)xy^5 + (-0.0649 + 0.0239i)xy^4z + (0.1038 - 0.0383i)xy^3z^2\\
&+ (0.1168 - 0.0431i)xy^2z^3 + (-0.0649 + 0.0239i)xyz^4 + (-0.0260 + 0.0096i)xz^5\\
&+ (0.0130 - 0.0048i)y^6 + (-0.0259 + 0.0096i)y^5z + (0.0519 - 0.0191i)y^4z^2\\
&+ (-0.0130 + 0.0048i)y^3z^3 + (0.0259 - 0.0096i)y^2z^4 + (0.0389 - 0.0144i)yz^5\\
&+ (0.0130 - 0.0048i)z^6
\end{aligned}
$$

Multiplying our expression for $Q3$ by a factor of $(0.0130 - 0.0048i)$, the coefficient of x^6 in the product above, we get:

$$(0.0130 - 0.0048i)x^6 + (-0.0260 + 0.0096i)x^5y + (0.0390 - 0.0144i)x^5z$$
$$+ (0.0520 - 0.0192i)x^4y^2 + (-0.0650 + 0.0240i)x^4yz + (0.0260 - 0.0096i)x^4z^2$$
$$+ (-0.0130 + 0.0048i)x^3y^3 + (0.1040 - 0.0384i)x^3y^2z + (0.1170 - 0.0432i)x^3yz^2$$
$$+ (-0.0130 + 0.0048i)x^3z^3 + (0.0260 - 0.0096i)x^2y^4 + (0.1170 - 0.0432i)x^2y^3z$$
$$+ (-0.0130 + 0.0048i)x^2y^2z^2 + (0.1040 - 0.0384i)x^2yz^3 + (0.0520 - 0.0192i)x^2z^4$$
$$+ (0.0390 - 0.0144i)xy^5 + (-0.0650 + 0.0240i)xy^4z + (0.1040 - 0.0384i)xy^3z^2$$
$$+ (0.1170 - 0.0432i)xy^2z^3 + (-0.0650 + 0.0240i)xyz^4 + (-0.0260 + 0.0096i)xz^5$$
$$+ (0.0130 - 0.0048i)y^6 + (-0.0260 + 0.0096i)y^5z + (0.0520 - 0.0192i)y^4z^2$$
$$+ (-0.0130 + 0.0048i)y^3z^3 + (0.0260 - 0.0096i)y^2z^4 + (0.0390 - 0.0144i)yz^5$$
$$+ (0.0130 - 0.0048i)z^6$$

By the same reasoning as for $Q1$, we claim that $Q3$ is the product of the six lines L9, L13, L15, L16, L18 and L21.

Looking at table 5.1 again, we have clear and similar zeros for Q2 for the points b_{10}, b_{11}, b_{14}, b_{15}, b_{16} and b_{17}. We examine the lines going through these points:

b_{10}: L10, L11, L12
b_{11}: L11, L19, L20
b_{14}: L5, L12, L17
b_{15}: L5, L14, L19
b_{16}: L6, L14, L20
b_{17}: L6, L10, L17

All the lines occur twice in this list, but we know that L5, L6 and L11 make up Q1. Checking the product of the remaining six lines L10, L12, L14, L17, L19 and L20, we find that the coefficients of x^6, y^6 and z^6 in the product are equal by the method we used before, and perform the full multiplication. We get:

$$L10 \cdot L12 \cdot L14 \cdot L17 \cdot L19 \cdot L20$$

$$= ((0.0743 - 0.3470i)x + (0.3960 - 0.1973i)y + (0.7973 - 0.0108i)z)$$
$$\cdot ((-0.0879 + 0.4174i)x + (-0.4691 + 0.6090i)y + (0.2178 + 0.2638i)z)$$
$$\cdot ((-0.8100 - 0.0219i)x + (-0.0707 - 0.3537i)y + (-0.3997 - 0.2060i)z)$$
$$\cdot ((-0.7967 + 0.0868i)x + (-0.0412 + 0.3543i)y + (-0.3775 + 0.2352i)z)$$
$$\cdot ((-0.4050 - 0.1983i)x + (-0.8125 - 0.0052i)y + (-0.0782 - 0.3531i)z)$$
$$\cdot ((0.1089 + 0.3265i)x + (0.4030 + 0.1477i)y + (0.7698 - 0.0755i)z)$$

$$= (0.0069 - 0.0136i)x^6 + (0.0346 - 0.0680i)x^5y + (0.0207 - 0.0408i)x^5z$$
$$+ (0.0760 - 0.1496i)x^4y^2 + (0.1106 - 0.2176i)x^4yz + (0.0622 - 0.1224i)x^4z^2$$
$$+ (0.0899 - 0.1768i)x^3y^3 + (0.2489 - 0.4896i)x^3y^2z + (0.2557 - 0.5032i)x^3yz^2$$

$+ (0.0898 - 0.1768i)x^3z^3 + (0.0622 - 0.1224i)x^2y^4 + (0.2558 - 0.5032i)x^2y^3z$
$+ (0.3802 - 0.7480i)x^2y^2z^2 + (0.2489 - 0.4897i)x^2yz^3 + (0.0760 - 0.1497i)x^2z^4$
$+ (0.0207 - 0.0408i)xy^5 + (0.1106 - 0.2176i)xy^4z + (0.2489 - 0.4896i)xy^3z^2$
$+ (0.2558 - 0.5033i)xy^2z^3 + (0.1106 - 0.2176i)xyz^4 + (0.0346 - 0.0680i)xz^5$
$+ (0.0069 - 0.0136i)y^6 + (0.0346 - 0.0680i)y^5z + (0.0761 - 0.1496i)y^4z^2$
$+ (0.0899 - 0.1768i)y^3z^3 + (0.0622 - 0.1224i)y^2z^4 + (0.0207 - 0.0408i)yz^5$
$+ (0.0069 - 0.0136i)z^6$

Multiplying our expression for $Q2$ by a factor of $(0.0069 - 0.0136i)$, the coefficient of x^6 in the product above, we get:

$(0.0069 - 0.0136i)x^6 + (0.0345 - 0.0680i)x^5y + (0.0207 - 0.0408i)x^5z$
$+ (0.0759 - 0.1496i)x^4y^2 + (0.1104 - 0.2176i)x^4yz + (0.0621 - 0.1224i)x^4z^2$
$+ (0.0897 - 0.1768i)x^3y^3 + (0.2484 - 0.4896i)x^3y^2z + (0.2553 - 0.5032i)x^3yz^2$
$+ (0.0897 - 0.1768i)x^3z^3 + (0.0621 - 0.1224i)x^2y^4 + (0.2553 - 0.5032i)x^2y^3z$
$+ (0.3795 - 0.7480i)x^2y^2z^2 + (0.2484 - 0.4896i)x^2yz^3 + (0.0759 - 0.1496i)x^2z^4$
$+ (0.0207 - 0.0408i)xy^5 + (0.1104 - 0.2176i)xy^4z + (0.2484 - 0.4896i)xy^3z^2$
$+ (0.2553 - 0.5032i)xy^2z^3 + (0.1104 - 0.2176i)xyz^4 + (0.0345 - 0.0680i)xz^5$
$+ (0.0069 - 0.0136i)y^6 + (0.0345 - 0.0680i)y^5z + (0.0759 - 0.1496i)y^4z^2$
$+ (0.0897 - 0.1768i)y^3z^3 + (0.0621 - 0.1224i)y^2z^4 + (0.0207 - 0.0408i)yz^5$
$+ (0.0069 - 0.0136i)z^6$

Again, we hold that even though the product deviates slightly from a perfect match with the modified expression for $Q2$, the conclusion that $Q2$ is a product of the six lines L10, L12, L14, L17, L19 and L20 is justified.

Left over now are the fixed lines of D1, D2, D3, D4, D7 and D8. Again we confirm that the coefficients of x^6, y^6 and z^6 in the product are equal, and proceed to perform the full multiplication. We get:

$L1 \cdot L2 \cdot L3 \cdot L4 \cdot L7 \cdot L8$

$= ((-0.5481 + 0.2041i)x + (0.0811 - 0.2474i)y + (0.0428 + 0.3217i)z)$
$\cdot ((-0.0755 - 0.4311i)x + (-0.7875 - 0.0402i)y + (0.2045 + 0.2851i)z)$
$\cdot ((-0.1218 + 0.2800i)x + (-0.5407 - 0.1017i)y + (0.1853 - 0.1600i)z)$
$\cdot ((-0.7210 + 0.0393i)x + (0.2137 + 0.2399i)y + (-0.1104 - 0.3852i)z)$
$\cdot ((-0.2110 - 0.2632i)x + (0.0946 + 0.4099i)y + (0.7579 - 0.0018i)z)$
$\cdot ((-0.2117 + 0.2915i)x + (0.0801 - 0.4420i)y + (0.8087 - 0.0366i)z)$

$= (0.0067 - 0.0015i)x^6 + (-0.0134 + 0.0030i)x^5y + (-0.0268 + 0.0060i)x^5z$
$+ (0.0268 - 0.0060i)x^4y^2 + (0.0134 - 0.0030i)x^4yz + (0.0602 - 0.0136i)x^4z^2$
$+ (-0.0535 + 0.0121i)x^3y^3 + (-0.0402 + 0.0090i)x^3y^2z + (0.0134 - 0.0030i)x^3yz^2$
$+ (-0.0535 + 0.0121i)x^3z^3 + (0.0602 - 0.0136i)x^2y^4 + (0.0134 - 0.0030i)x^2y^3z$
$+ (0.0870 - 0.0196i)x^2y^2z^2 + (-0.0401 + 0.0090i)x^2yz^3 + (0.0267 - 0.0060i)x^2z^4$
$+ (-0.0268 + 0.0060i)xy^5 + (0.0134 - 0.0030i)xy^4z + (-0.0401 + 0.0091i)xy^3z^2$

$+ (0.0134 - 0.0030i)xy^2z^3 + (0.0134 - 0.0030i)xyz^4 + (-0.0134 + 0.0030i)xz^5$
$+ (0.0067 - 0.0015i)y^6 + (-0.0134 + 0.0030i)y^5z + (0.0267 - 0.0060i)y^4z^2$
$+ (-0.0535 + 0.0121i)y^3z^3 + (0.0602 - 0.0136i)y^2z^4 + (-0.0267 + 0.0060i)yz^5$
$+ (0.0067 - 0.0015i)z^6$

Multiplying our expression for $Q4$ by a factor of $(0.0067 - 0.0015i)$, the coefficient of x^6 in the product above, we get:

$(0.0067 - 0.0015i)x^6 + (-0.0134 + 0.0030i)x^5y + (-0.0268 + 0.0060i)x^5z$
$+ (0.0268 - 0.0060i)x^4y^2 + (0.0134 - 0.0030i)x^4yz + (0.0603 - 0.0135i)x^4z^2$
$+ (-0.0536 + 0.0120i)x^3y^3 + (-0.0402 + 0.0090i)x^3y^2z + (0.0134 - 0.0030i)x^3yz^2$
$+ (-0.0536 + 0.0120i)x^3z^3 + (0.0603 - 0.0135i)x^2y^4 + (0.0134 - 0.0030i)x^2y^3z$
$+ (0.0871 - 0.0195i)x^2y^2z^2 + (-0.0402 + 0.0090i)x^2yz^3 + (0.0268 - 0.0060i)x^2z^4$
$+ (-0.0268 + 0.0060i)xy^5 + (0.0134 - 0.0030i)xy^4z + (-0.0402 + 0.0090i)xy^3z^2$
$+ (0.0134 - 0.0030i)xy^2z^3 + (0.0134 - 0.0030i)xyz^4 + (-0.0134 + 0.0030i)xz^5$
$+ (0.0067 - 0.0015i)y^6 + (-0.0134 + 0.0030i)y^5z + (0.0268 - 0.0060i)y^4z^2$
$+ (-0.0536 + 0.0120i)y^3z^3 + (0.0603 - 0.0135i)y^2z^4 + (-0.0268 + 0.0060i)yz^5$
$+ (0.0067 - 0.0015i)z^6$

By the same reasoning as before, we conclude that $Q4$ is a product of the six lines L1, L2, L3, L4, L7 and L8.

5.6 Summary

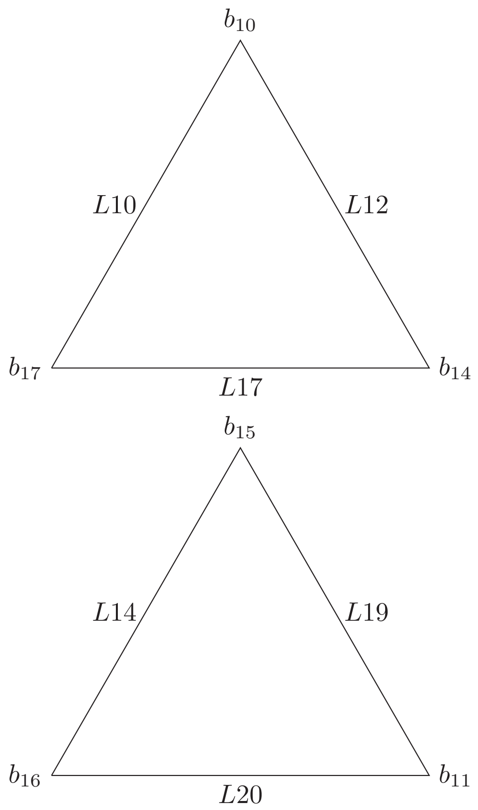
We started out this chapter with examining the fixed points of the 3-groups, those on the degree-21 curve. This led us to consider the factoring of K21. By examination, hypothesis and calculation, we achieved a factoring of the four factors of K21 over $\mathbb{Z}$.

$Q1 = L5 \cdot L6 \cdot L11$
$Q2 = L9 \cdot L13 \cdot L15 \cdot L16 \cdot L18 \cdot L21$
$Q3 = L10 \cdot L12 \cdot L14 \cdot L17 \cdot L19 \cdot L20$
$Q4 = L1 \cdot L2 \cdot L3 \cdot L4 \cdot L7 \cdot L8$

Out of the many intersections of the fixed lines of the 2-groups, we present a few related to our factoring.

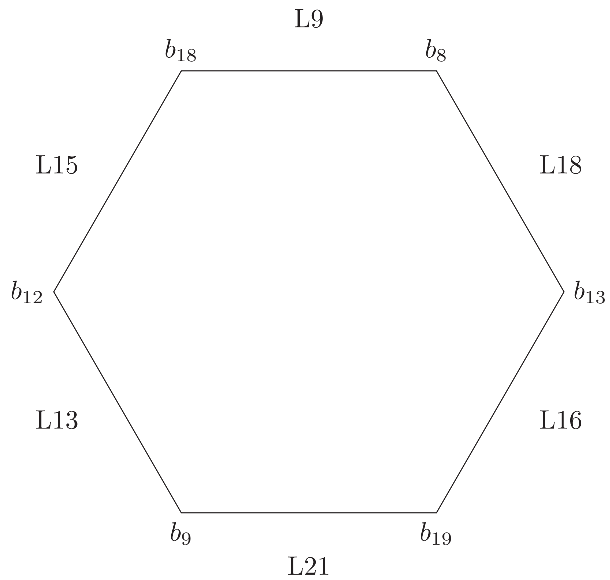
The lines in $Q2$ meet pairwise in fixed points of 3-groups, with which we started, making up two triangles.

Figure 5.1: The triangles of Q2



The lines in $Q3$ make up a hexagon with vertices in fixed points of 3-groups.

Figure 5.2: The hexagon of Q3



The lines in $Q1$ and $Q4$ do not make up similar polygons.

5.7 Another factoring of K21

There are many ways to factor our degree-21 invariant. Referring back to section 4.6, we know that the fixed lines of the 2-groups intersect each other in fixed points of other 2-groups, making up triangles. For instance, we have the fixed points of $E7$ in figure 4.1. Conjugation of $E7$ by S returns $E3$, and then $E10$, $E13$, $E11$, $E2$, $E4$ and back to $E7$ by consecutive conjugations. Starting with $E8$ in the other class of Klein 4-groups, consecutive conjugations by S gives us $E1$, $E6$, $E14$, $E12$, $E9$, $E5$ and back to $E8$, in that order. For every class we can draw seven triangles and every fixed line occurs only once in a triangle for each class. The product of the three fixed lines in a triangle is a factor of degree 3, and in the same way for the other triangles. The product of the seven such degree-3 factors for each of the two classes is K21. We may say that the triangles or lines are related by conjugation.

Bibliography

[Adl99] Adler, A. *Hirzebruch's Curves F1, F2, F4, F14, F28 for* $\mathbb{Q}(\sqrt{7})$*, The Eightfold Way: The Beauty of Klein's Quartic Curve, ed. Sylvio Levi, 221-285. Mathematical Sciences Research Institute publications, 35*. Cambridge: Cambridge University Press, 1999.

[Elk99] Elkies, N. D. *The Klein quartic in number theory, The Eightfold Way: The Beauty of Klein's Quartic Curve, ed. Sylvio Levi, 51-102. Mathematical Sciences Research Institute publications, 35*. Cambridge: Cambridge University Press, 1999.

[Kle99] Klein, F. *On the Order-Seven Transformation of Elliptic Functions, The Eightfold Way: The Beauty of Klein's Quartic Curve, ed. Sylvio Levi, 287-328. Mathematical Sciences Research Institute publications, 35*. Cambridge: Cambridge University Press, 1999.

[MAT21] MATLAB. *v9.10.0.1739362 (x64)*. Natick, Massachusetts: The MathWorks Inc., 2021.

[Mir95] Miranda, R. *Algebraic curves and Riemann surfaces. Vol. 5. Graduate Studies in Mathematics*. Providence, RI: American Mathematical Society, 1995.