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LEFSCHETZ SECTION THEOREMS FOR TROPICAL HYPERSURFACES

THÉORÈME DE LA SECTION HYPERPLANE DE LEFSCHETZ POUR LES HYPERSURFACES TROPICALES

Abstract. — We establish variants of the Lefschetz section theorem for the integral tropical homology groups of tropical hypersurfaces of tropical toric varieties. It follows from these theorems that the integral tropical homology groups of non-singular tropical hypersurfaces which are compact or contained in $\mathbb{R}^n$ are torsion free. We prove a relationship between the coefficients of the χ_y genera of complex hypersurfaces in toric varieties and Euler characteristics of the integral tropical cellular chain complexes of their tropical counterparts. It follows that the integral tropical homology groups give the Hodge numbers of compact non-singular hypersurfaces of complex toric varieties. Finally for tropical hypersurfaces in certain affine toric varieties, we relate the ranks of their tropical homology groups to the Hodge–Deligne numbers of their complex counterparts.

Keywords: tropical geometry, tropical homology, Lefschetz section theorems, Hodge theory.
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Résumé. — Nous démontrons des analogues au théorème de la section hyperplane de Lefschetz pour l'homologie tropicale entière d'hypersurfaces tropicales dans des variétés toriques tropicales. Nous en déduisons que les groupes d'homologie des hypersurfaces tropicales non-singulières compactes (ou contenues dans $\mathbb{R}^n$) sont sans torsion. Nous en déduisons également une relation entre les coefficients du genre χ_y des hypersurfaces complexes dans les variétés toriques et les caractéristiques d'Euler des complexes de chaînes cellulaires tropicales des hypersurfaces tropicales. Il s'ensuit que les groupes d'homologies tropicales à coefficient entier ont pour rang les nombres de Hodges d'hypersurfaces compactes non-singulières dans des variétés toriques complexes. Finalement pour les hypersurfaces tropicales dans certaines variétés toriques affines, nous relions les rangs de leurs groupes d'homologie tropicale aux nombres de Hodge–Deligne des hypersurfaces complexes correspondantes.

1. Introduction

Tropical homology is a homology theory with non-constant coefficients for polyhedral spaces. Itenberg, Katzarkov, Mikhalkin, and Zharkov, show that under suitable conditions, the $\mathbb{Q}$-tropical Betti numbers of the tropical limit of a family of complex projective varieties are equal to the corresponding Hodge numbers of a generic member of the family [IKMZ19]. This explains the particular interest of these homology groups in tropical and complex algebraic geometry.

In this paper we consider the integral versions of tropical homology groups for hypersurfaces in tropical toric varieties. The $(p,q)^{\text{th}}$ tropical homology group of a rational polyhedral complex Z is denoted $H_q(Z;\mathcal{F}_p^Z)$ and the Borel–Moore homology group is denoted $H_q^{BM}(Z;\mathcal{F}_p^Z)$. To avoid ambiguity we will often also refer to $H_q(Z;\mathcal{F}_p^Z)$ as a standard tropical homology group. When a rational polyhedral complex Z is compact then $H_q(Z;\mathcal{F}_p^Z) = H_q^{BM}(Z;\mathcal{F}_p^Z)$. These homology groups are defined in Section 2 as the cellular tropical homology groups [KS17, MZ14]. For a comparison between cellular homology and singular homology, see Remark 2.20.

Our main goal is to prove that these homology groups are torsion free for a compact non-singular tropical hypersurface in a compact non-singular tropical toric variety. The road to the proof of this statement is quite similar to the one followed to prove that the integral homology of a complex projective hypersurface is torsion free. Namely, in order to prove that the integral tropical homology groups are without torsion, we first establish a tropical variant of the Lefschetz section theorem. Ultimately however, the techniques used in the proofs are quite different from the complex setting, since we are working with polyhedral spaces instead of algebraic varieties. Also notice that in the tropical version of the Lefschetz section theorem stated below the tropical hypersurface is not required to be compact. However, the tropical hypersurface is required to be combinatorially ample in the tropical toric variety, see Definition 2.5. For the notion of cellular pair see Definition 2.7 and for the notion of parent face see Definition 2.9.

Theorem 1.1. — *Let X be a non-singular and combinatorially ample tropical hypersurface of an $n+1$ dimensional non-singular tropical toric variety Y. Then the map induced by inclusion*

$$i_*\colon H_q^{BM}\left(X;\mathcal{F}_p^X\right) \to H_q^{BM}\left(Y;\mathcal{F}_p^Y\right)$$

is an isomorphism when $p+q<n$ and a surjection when $p+q=n$.

If additionally, the pair (Y,X) is a cellular pair and every parent face of a compact face of Y is compact, then the map induced by inclusion

$$i_*\colon H_q\left(X;\mathcal{F}_p^X\right)\to H_q\left(Y;\mathcal{F}_p^Y\right)$$

is an isomorphism when $p+q<n$ and a surjection when $p+q=n$.

Tropical homology with real or rational coefficients is the homology of the cosheaf of real vector spaces $\mathcal{F}_p\otimes\mathbb{R}$ or $\mathcal{F}_p\otimes\mathbb{Q}$, respectively. Theorem 1.1 holds in the case of tropical homology with real coefficients when we remove the assumption that the tropical hypersurface X is non-singular, provided that the tropical toric variety Y is non-singular, combinatorially ample and also that X is proper in Y, see Definition 2.3. Below we state the theorems in the case of real coefficients.

Theorem 1.2. — *Let X be a combinatorially ample tropical hypersurface of an $n+1$ dimensional non-singular tropical toric variety Y such that X is proper in Y. Then the maps induced by inclusion*

$$i_*\colon H_q^{BM}\left(X;\mathcal{F}_p^X\otimes\mathbb{R}\right)\to H_q^{BM}\left(Y;\mathcal{F}_p^Y\otimes\mathbb{R}\right)$$

are isomorphisms when $p+q<n$ and surjections when $p+q=n$. If additionally, the pair (Y,X) is a cellular pair and every parent face of a compact face of Y is compact, then the maps induced by inclusion

$$i_*\colon H_q\left(X;\mathcal{F}_p^X\otimes\mathbb{R}\right)\to H_q\left(Y;\mathcal{F}_p^Y\otimes\mathbb{R}\right)$$

are isomorphisms when $p+q<n$ and surjections when $p+q=n$.

The above theorem in tropical geometry is inspired by the Lefschetz hyperplane section theorem in complex algebraic geometry. When a toric variety is embedded by way of a very ample line bundle, a hyperplane section of this embedding corresponds to a hypersurface in the toric variety. Applying the classical Lefschetz hyperplane section theorem in this situation relates the homology of the hypersurface to that of the toric variety in a way analogous to the tropical theorem stated above.

Adiprasito and Björner present tropical variants of the Lefschetz hyperplane section theorem in [AB14]. Their theorems relate the tropical homology with real coefficients of a non-singular tropical variety X contained in a tropical toric variety to the tropical homology groups of the intersection of X with a so-called "chamber complex". A chamber complex is a codimension one polyhedral complex in a tropical toric variety whose complement consists of pointed polyhedra, in particular it need not to be balanced. Adiprasito and Björner first establish some topological properties of filtered geometric lattices and then use Morse theory to prove their tropical versions of the Lefschetz theorem. The proof we present here does not utilise Morse theory but instead proves vanishing theorems for the homology of cosheaves that arise in short exact sequences relating the cosheaves for the tropical homology of X and the ambient space. Furthermore, we relate the integral tropical homology groups of a non-singular tropical hypersurface with the integral tropical homology groups of the ambient tropical toric variety. Another result which follows from the Lefschetz section theorem for the integral tropical homology groups of hypersurfaces is that under the correct hypotheses on the ambient space these homology groups are torsion free. At the end of the introduction we discuss the implications of torsion freeness to recent results on the Betti numbers of real algebraic hypersurfaces arising from patchworking.

The tropical (co)homology groups with integral coefficients of a non-singular tropical hypersurface satisfy a variant of Poincaré duality [JRS18]. Using this we deduce in Section 4 that the tropical homology groups of a non-singular tropical hypersurface in a non-singular tropical toric variety which satisfy the assumptions below are torsion free, as long as the homology of the tropical toric variety is also torsion free.

Theorem 1.3. — *Let X be a non-singular and combinatorially ample tropical hypersurface in a non-singular tropical toric variety Y such that (Y, X) is a cellular pair and every parent face of a compact face of Y is compact. If the tropical homology groups $H_q(Y; \mathcal{F}_p)$ are torsion free for all p and q, then both the Borel–Moore and standard tropical homology groups of X are also torsion free.*

Proposition 4.1 proves that the tropical homology groups $H_q(Y; \mathcal{F}_p)$ of a compact non-singular tropical toric variety are all torsion free. If Y is a non-singular tropical toric variety which is not necessarily compact, and for all p and q the groups $H_q(Y; \mathcal{F}_p)$ are torsion free, then so are the Borel–Moore groups $H_q^{BM}(Y; \mathcal{F}_p)$. This is proved in the proof of Theorem 1.3.

Corollary 1.4. — *If Y is a compact non-singular tropical toric variety and X is a combinatorially ample non-singular tropical hypersurface in Y, then all integral tropical homology groups of X are torsion free.*

Corollary 1.5. — *Let Y be a non-singular tropical toric variety associated to a fan whose support is a convex cone and such that the complex toric variety $Y_{\mathbb{C}}$ is quasi-projective. Let X be a combinatorially ample non-singular tropical hypersurface in Y such that (Y, X) is a cellular pair and every parent face of a compact face of Y is compact. Then both the standard and Borel–Moore integral tropical homology groups of X are torsion free.*

Corollary 1.6. — *The tropical homology groups of a non-singular tropical hypersurface in $\mathbb{R}^{n+1}$ are torsion free.*

The above theorem and corollaries follow from the tropical Lefschetz section theorems established here for hypersurfaces. That is why we require in Theorem 1.3 that (Y, X) be a cellular pair, the assumptions that the hypersurface X is combinatorially ample in Y, and that every parent face of a compact face of Y is compact. We do not know if these assumptions are necessary, or if an alternate more direct proof of torsion freeness exists. In Corollary 1.6, there is no assumption that $(\mathbb{R}^{n+1}, X)$ is a cellular pair. This is because if the Newton polytope of X is full dimensional then $(\mathbb{R}^{n+1}, X)$ is a cellular pair. If not, then X is a product $\mathbb{R}^k \times X'$ for a tropical hypersurface X' in $\mathbb{R}^{n+1-k}$ which has full dimensional Newton polytope. Then we can apply the Künneth formula for tropical homology [GS19, Theorem B] to the product and obtain that the tropical homology groups of X are torsion free.

Question 1.7. — Are the integral tropical homology groups of any non-singular tropical hypersurface of a tropical toric variety torsion free?

In Section 5, we first find that the Euler characteristics of the cellular chain complexes for Borel–Moore tropical homology of a non-singular tropical hypersurface give the coefficients of the χ_y genus of a torically non-degenerate complex hypersurface with the same Newton polytope (see Definition 5.1 for the definition of torically non-degenerate). The Hodge–Deligne numbers of a complex variety $X_{\mathbb{C}}$ are denoted by $h^{p,q}(H^k_c(X_{\mathbb{C}}))$ (see for example [DK86]).The coefficients of the χ_y genus of $X_{\mathbb{C}}$ are given in terms of the numbers

$$e^{p,q}_c(X_{\mathbb{C}}) := \sum_k (-1)^k h^{p,q}\left(H^k_c(X_{\mathbb{C}})\right),$$

and the χ_y genus is then defined as

$$\chi_y(X_{\mathbb{C}}) := \sum_{p,q} e^{p,q}_c(X_{\mathbb{C}}) y^p.$$

Theorem 1.8. — *Let X be an n-dimensional non-singular tropical hypersurface in a non-singular tropical toric variety Y. Let $X_{\mathbb{C}}$ be a complex hypersurface torically non-degenerate in the complex toric variety $Y_{\mathbb{C}}$ such that X and $X_{\mathbb{C}}$ have the same Newton polytope. Then*

$$(-1)^p \chi\left(C^{BM}_\bullet\left(X; \mathcal{F}^X_p\right)\right) = \sum_{q=0}^{n} e^{p,q}_c(X_{\mathbb{C}}),$$

and thus

$$\chi_y(X_{\mathbb{C}}) = \sum_{p=0}^{n} (-1)^p \chi\left(C^{BM}_\bullet\left(X; \mathcal{F}^X_p\right)\right) y^p.$$

From the above theorem we obtain an immediate relation between the dimensions of the $\mathbb{R}$-tropical homology groups of a tropical hypersurface and the χ_y genus of corresponding complex hypersurface. Namely, in the situation of the above theorem we have

$$(1.1) \qquad (-1)^p \sum_{q=0}^{n} \operatorname{rank} H^{BM}_q\left(X; \mathcal{F}^X_p\right) = \sum_{q=0}^{n} e^{p,q}_c(X_{\mathbb{C}}).$$

We combine the results from Section 4 and Equation (1.1) to calculate the ranks of the tropical homology groups of non-singular tropical hypersurfaces in compact tropical toric varieties in Corollary 1.9.

Corollary 1.9. — *Let X be a non-singular and combinatorially ample compact tropical hypersurface in a non-singular compact tropical toric variety Y and assume that X has Newton polytope Δ. Let $X_{\mathbb{C}}$ be a torically non-degenerate complex hypersurface in the compact complex toric variety $Y_{\mathbb{C}}$ also with Newton polytope Δ. Then for all p and q we have*

$$\dim H^{p,q}(X_{\mathbb{C}}) = \operatorname{rank} H_q\left(X; \mathcal{F}^X_p\right).$$

In the situation of Corollary 1.5, if the toric variety is affine and constructed from a fan whose support is a convex cone of maximal dimension, we also determine the ranks of the Borel–Moore tropical homology groups of tropical hypersurfaces in terms of the Hodge–Deligne numbers with compact support of complex hypersurfaces in

Corollary 5.2. In Corollary 5.3, we again apply Theorem 1.8 to compute the ranks of the Borel–Moore tropical homology groups of tropical hypersurfaces in the torus.

Our main motivation establishing torsion freeness of the tropical homology groups of tropical hypersurfaces and a relation between their ranks and the Hodge–Deligne numbers of complex hypersurfaces comes from a recently established relation between the $\mathbb{Z}_2$-tropical homology groups of tropical hypersurfaces and Betti numbers of patchworked real algebraic hypersurfaces. In the theorem below $H_q(X;\mathcal{F}_p^{X,\mathbb{Z}_2})$ denotes the tropical homology groups considered with coefficients in $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

Theorem 1.10. — [RS18, Theorem 1.5] *If V is a non-singular real algebraic hypersurface in a toric variety obtained from a primitive patchworking of the tropical hypersurface X equipped with a real structure then for all q we have,*

$$b_q(\mathbb{R}V) \leqslant \sum_{p=1}^{\dim X} \dim H_q\left(X;\mathcal{F}_p^{X,\mathbb{Z}_2}\right).$$

When the integral tropical homology groups are torsion free then we have

$$\operatorname{rank} H_q\left(X;\mathcal{F}_p^{X}\right) = \dim H_q\left(X;\mathcal{F}_p^{X,\mathbb{Z}_2}\right)$$

for all p and q. This together with Corollaries 1.9 and 5.3 allow the bounds in Theorem 1.10 on the Betti numbers of the real points of a patchworked algebraic variety to be written in terms of Hodge–Deligne numbers of the complexification. For instance, one obtains the following result, conjectured by Itenberg around 2005, and later appeared in [Ite17].

Theorem 1.11 ([RS18, Theorem 1.4]). — *Let V be a real hypersurface in $\mathbb{P}^{n+1}$ arising from a primitive patchworking. Then for any integer $q = 0, \dots, n$ we have*

$$b_q(\mathbb{R}V) \leqslant \begin{cases} h^{q,q}(\mathbb{C}V) & \text{for } q = n/2, \\ h^{q,n-q}(\mathbb{C}V) + 1 & \text{otherwise.} \end{cases}$$

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2. Preliminaries

2.1. Tropical toric varieties

In this text we will always use the standard lattice $\mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}$. The tropical numbers are $\mathbb{T} = [-\infty, +\infty)$. We equip the set $\mathbb{T}$ with a topology so that it is isomorphic to a half open interval. Tropical affine space of dimension n is $\mathbb{T}^n$ and is equipped with the product topology.

In algebraic geometry over a field a rational polyhedral fan in $\mathbb{R}^{n+1}$ produces an $n+1$ dimensional toric variety, see for example [Ful93]. The same fact is true in tropical geometry. Given a rational polyhedral fan in $\mathbb{R}^{n+1}$ we can construct a tropical toric variety, see [MS15, Section 6.2], [MR18, Section 3.2]. We will briefly recall this construction. Let Σ be a rational polyhedral pointed fan. For any cone ρ of Σ, denote by $\mathbb{L}(\rho)$ the subspace of $\mathbb{R}^{n+1}$ spanned by ρ, and set $\mathbb{L}_{\mathbb{Z}}(\rho) := \mathbb{L}(\rho) \cap \mathbb{Z}^{n+1}$.

Definition 2.1. — *The tropical toric variety associated to the fan Σ is the set*

$$Y_\Sigma := \bigsqcup_{\rho \in \Sigma} \mathbb{R}^{n+1}/\mathbb{L}(\rho),$$

equipped with the topology from [MR18, Remark 3.2.5] *and* [Pay09, Section 3]*, which is the unique topology satisfying*

- *The inclusions $\mathbb{R}^{n+1}/\mathbb{L}(\rho) \hookrightarrow Y_\Sigma$ are continuous for any cone $\rho \in \Sigma$.*
- *For any $x \in \mathbb{R}^{n+1}$ and any $v \in \mathbb{R}^{n+1}$, the sequence $(x+nv)_{n\in\mathbb{N}} \in \mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}/\mathbb{L}(0)$ converges in Y_Σ if and only if v is contained in the support of the fan Σ.*

Given a toric variety Y and a cone ρ of the associated fan, we denote by Y_ρ the stratum $\mathbb{R}^{n+1}/\mathbb{L}(\rho) \subset Y$ (using the same notations as above). A rational polyhedral fan Σ is *simplicial* if each of its cones is the cone over a simplex. A simplicial rational polyhedral fan is unimodular if the primitive integer directions of the rays of each cone can be completed to a basis of $\mathbb{Z}^{n+1}$. Just as in the case over a field, a tropical toric variety is *non-singular* if it is built from a simplicial unimodular rational polyhedral fan, The tropical toric varieties considered in this text are always non-singular. A tropical toric variety is compact if and only if the corresponding fan is complete.

A tropical toric variety Y has a stratification and the combinatorics of the stratification is governed by its fan Σ. A stratum of dimension k of Y corresponds to a cone ρ of dimension $n+1-k$ of Σ. We denote the vertex of the fan by ρ_0 and the corresponding open stratum of Y by simply Y_0. For any point $y \in Y$, the order of sedentarity of y, denoted $\mathrm{sed}(y)$, is defined to be the codimension in Y of the stratum containing y.

A tropical toric variety is naturally equipped with a lattice on each stratum. More precisely, the stratum Y_ρ is equipped with the lattice $\mathbb{Z}^{n+1}/\mathbb{L}_{\mathbb{Z}}(\rho)$. When ρ is of dimension k, there is a lattice preserving isomorphism of vector spaces $Y_\rho \cong \mathbb{R}^{n+1-k}$. For two cones ρ and ρ' of Σ we have $Y_{\rho'} \subset \overline{Y_\rho}$ if and only if ρ is a face of ρ' in Σ. If ρ is a cone of Σ, then there is a projection map $\pi_\rho \colon \mathbb{R}^{n+1} \to Y_\rho$. If ρ' is a face of ρ in the fan Σ, then $\mathbb{L}(\rho') \subseteq \mathbb{L}(\rho)$ and there is a projection map $\pi_{\rho',\rho} \colon Y_{\rho'} \to Y_\rho$.

Example 2.2. — The tropical projective space $\mathbb{TP}^n$ is the tropical toric variety constructed from the fan consisting of cones

$$\mathbb{R}_{\geqslant 0}\, e_{i_1} + \cdots + \mathbb{R}_{\geqslant 0}\, e_{i_k},$$

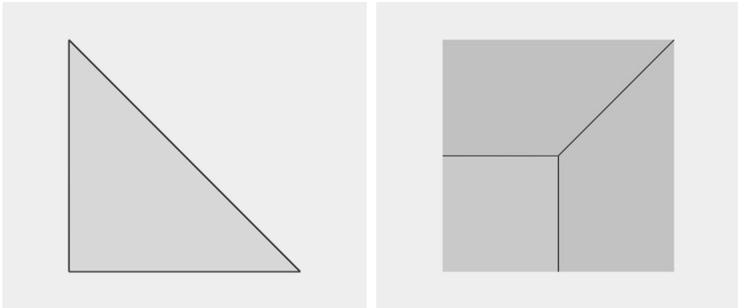

Figure 2.1. The tropical projective plane $\mathbb{TP}^2$ on the left and its normal fan on the right.

for all $\{i_1 \cdots i_k\} \subsetneq \{0, \cdots, n\}$, where $e_1, \cdots, e_n$ is the standard basis of $\mathbb{R}^n$ and $e_0 = -\sum_{k=1}^n e_k$. It can also be described as the quotient

$$\frac{\mathbb{T}^{n+1} \backslash (-\infty, \ldots, -\infty)}{[x_0 : \cdots : x_n] \sim [a + x_0 : \cdots : a + x_n]},$$

where $a \in \mathbb{T} \backslash -\infty$. The stratification of $\mathbb{TP}^n$ can be described using homogeneous coordinates. For a subset $I \subset \{0, \ldots, n\}$ define

$$\mathbb{TP}^n_I = \{x \in \mathbb{TP}^n \mid x_i = -\infty \text{ if and only if } i \in I\}.$$

The set $\mathbb{TP}^n_I$ corresponds to the cone

$$\sum_{i \in I} \mathbb{R}_{\geqslant 0}\, e_i.$$

The order of sedentarity of a point $x = [x_0 : \cdots : x_n] \in \mathbb{TP}^n$ is

$$\operatorname{sed}(x) = \#\{i \mid x_i = -\infty\}.$$

A rational polyhedron in a tropical toric variety Y is the closure in Y of a rational polyhedron in some stratum Y_ρ. A *rational polyhedral complex* Z in Y is a collection of rational polyhedra in Y such that $Z \cap Y_\rho$ is a rational polyhedral complex in $Y_\rho \cong \mathbb{R}^{\operatorname{codim} \rho}$ for every cone ρ of Σ and satisfying:

(1) for a polyhedron $\sigma \in Z$, if τ is a face of σ, which is denoted $\tau \subset \sigma$, we have $\tau \in Z$;
(2) for $\sigma, \sigma' \in Z$, if $\tau = \sigma \cap \sigma'$ is non-empty then τ is a face of both σ and σ'.

A rational polyhedral complex Z' is a subpolyhedral complex of Z if any polyhedron of Z' is a polyhedron of Z. For a polyhedron σ in Y we define $\operatorname{sed}(\sigma)$ to be $\operatorname{sed}(y)$ for any y in the relative interior of σ. This is a generalization of the notion of sedentarity from [BIMS15, Section 5.5] to tropical toric varieties beyond $\mathbb{TP}^{n+1}$. Two rational polyhedral complexes are *combinatorially isomorphic* if they are isomorphic as posets under inclusion.

Definition 2.3. — *A rational polyhedral complex Z is* proper *in Y if for each cell σ and each cone ρ such that $\sigma \cap Y_\rho \neq \varnothing$, one has* $\dim(\sigma \cap Y_\rho) = \dim(\sigma) - \dim(\rho)$.

If σ is a polyhedron in Y which is the closure of a polyhedron in Y_0 then $\sigma \cap Y_\rho \neq \varnothing$ if and only if the recession cone of σ intersects $\operatorname{relint}(\rho)$ [OR13, Lemma 3.9]. The same lemma also shows that if $\sigma \cap Y_\rho \neq \varnothing$, then $\sigma \cap Y_\rho = \pi_\rho(\sigma \cap Y_0)$. In particular, this implies that all faces in the closure in Y of a rational polyhedral complex in Y_0 are again rational.

Therefore, if a rational polyhedral complex Z is proper in Y and σ is a face of Z such that $\operatorname{relint} \sigma \subset Y_\rho$ where $\rho \neq 0$, there exists at most one face of sedentarity 0 of Z of dimension $\dim \sigma + \dim \rho$ containing σ as a face. This is because if Z is proper in Y any face of sedentarity 0 intersecting Y_ρ in σ must be contained in a unique affine subspace of Y_0 of dimension $\dim \sigma + \dim \rho$. If there are two distinct faces of Z with sedentarity 0 intersecting Y_ρ in σ their intersection is also of dimension $\dim \sigma + \dim \rho$. Since Z is a rational polyhedral complex the intersection of these two faces must then be a face of Z so it is one, and hence both, of the faces.

2.2. Tropical hypersurfaces

A tropical hypersurface X in $\mathbb{R}^{n+1}$ is a weighted rational polyhedral complex of codimension one which satisfies the balancing condition well-known in tropical geometry. Any tropical hypersurface in $\mathbb{R}^{n+1}$ is defined by a tropical polynomial [MR18, Theorem 2.4.10]. As a rational polyhedral complex, a tropical hypersurface X is dual to a regular subdivision of the Newton polytope of its defining polynomial, and this subdivision is also induced by the polynomial. A tropical hypersurface X in $\mathbb{R}^{n+1}$ is *non-singular* if it is dual to a *primitive* regular triangulation of its Newton polytope. In this case, the weights of all top dimensional faces of X are equal to one. For the definitions and properties of tropical hypersurfaces in $\mathbb{R}^{n+1}$ and the dual subdivisions of their Newton polytopes we refer the reader to [MS15, Chapter 3] and [BIMS15, Section 5.1].

Definition 2.4. — *A tropical hypersurface X in a tropical toric variety Y of dimension $n+1$ is a weighted rational polyhedral complex in Y which is the closure of a tropical hypersurface $X_0 \subset \mathbb{R}^{n+1}$. The weights on top dimensional faces of X are inherited directly from X_0. A tropical hypersurface X is* non-singular *in Y if for every open toric stratum Y_ρ the intersection $X_\rho := X \cap Y_\rho$ is a non-singular tropical hypersurface in $Y_\rho \cong \mathbb{R}^{n+1-\dim \rho}$.*

In particular, if X is non-singular in Y, then $X_\rho := X \cap Y_\rho$ is of dimension $n - \dim \rho$ and X is proper in Y. Moreover, there is a tropical polynomial f_ρ defining X_ρ and X_ρ is dual to a *primitive* regular triangulation of the Newton polytope of f_ρ. We always consider the polyhedral structure on $X \cap \mathbb{R}^{n+1}$ which is dual to the regular subdivision of its Newton polytope.

When considering a tropical hypersurface X contained in a tropical toric variety Y, we always use the polyhedral structure on Y obtained from refining by X.

Let γ be a polyhedron of dimension s and $\operatorname{sed}(\gamma) = 0$ in a tropical toric variety Y. For each cone ρ in the fan Σ defining Y, set $\gamma_\rho := \gamma \cap Y_\rho$ and define

$$\gamma^\circ := \bigsqcup_\rho \operatorname{relint} \gamma_\rho.$$

If we assume that γ intersects the boundary of Y properly, a face σ of γ^o of dimension q is necessarily of sedentarity order $\operatorname{sed}(\sigma) = \dim \gamma - q$.

To prove the tropical version of the Lefschetz hyperplane section theorem we require the following additional assumption on X. With the exception of Theorem 1.8, we will always require that X is combinatorially ample in Y.

Definition 2.5. — *A tropical hypersurface X in an $n+1$ dimensional tropical toric variety Y is* combinatorially ample *if for every face γ of dimension $n+1$ of Y, considered with the refinement given by X, the rational polyhedral complex γ^o is combinatorially isomorphic to a product of copies of $\mathbb{T}$ and $\mathbb{R}$.*

Suppose that a tropical polynomial f defines a non-singular tropical hypersurface X_0 in $\mathbb{R}^{n+1}$. If the Newton polytope of f is full dimensional and the dual fan of the polytope defines a non-singular tropical toric variety Y, then the compactification of X_0 in Y is non-singular and combinatorially ample.

Remark 2.6. — Here we explain why, if Y is a compact tropical toric variety and X a combinatorially ample non-singular tropical hypersurface in Y, then the normal fan of the Newton polytope of X is the same as the fan defining the tropical toric variety Y.

First note that if X is combinatorial ample in Y then, for every 1-dimensional stratum Y_ρ of Y we have $X \cap Y_\rho \neq \varnothing$. Let $Y_{\mathbb{C}}$ denote the compact complex toric variety obtained from the same fan as Y. If $X_{\mathbb{C}}$ is any non-singular complex hypersurface in $Y_{\mathbb{C}}$ with the same Newton polytope as the tropical hypersurface X, then $X_{\mathbb{C}}$ also has non-empty intersection with every 1-dimensional stratum $Y_{\mathbb{C},\rho}$ of $Y_{\mathbb{C}}$. By the Kleinman condition and the Toric Cone Theorem [Wis02, p. 254], the hypersurface $X_{\mathbb{C}}$ is an ample Cartier divisor. If in addition, the tropical hypersurface is proper in Y, then we may choose $X_{\mathbb{C}}$ to be a T-Cartier divisor. By [Ful93, Exercise 2, p. 72], the normal fan of the Newton polytope of the defining equation for $X_{\mathbb{C}}$ must be the same as the fan defining the toric variety $Y_{\mathbb{C}}$. Therefore, we can conclude that the normal fan of the Newton polytope of X is the same as the fan defining the tropical toric variety Y.

To prove Lefschetz hyperplane section theorem in the case of standard tropical homology, we also need the following assumption on the topological pair (Y, X).

Definition 2.7. — *Let Y be a tropical toric variety and let $X \subset Y$ be a tropical hypersurface. We say that the pair (Y, X) is a cellular pair if the cellular structure induced by X on the one-point compactification $\widehat{Y}$ of Y is a regular CW-complex. More precisely, for any cell σ of $\widehat{Y}$ of dimension k, the pair $(\sigma, \operatorname{int}(\sigma))$ is homeomorphic to the pair $(B^k, \operatorname{int}(B^k))$, where B^k is the closed Euclidean ball of dimension k.*

Requiring (Y, X) to be a cellular pair implies that X and Y equipped with the polyhedral structure induced by X are both cellular complexes in the sense of [She85] and [Cur14, Chapter 4]. This topological condition is required to use the cellular description of cosheaf homology groups from [Cur14].

Example 2.8. — There are examples of tropical hypersurfaces in tropical toric varieties which are not cellular pairs. For example, consider X to be supported on the line $x = 0$ in $\mathbb{R}^2$. Then the one point compactification of $(\mathbb{R}^2, X)$ is not a regular CW-complex. In fact, if X is a tropical hypersurface in $\mathbb{R}^{n+1}$, then $(\mathbb{R}^{n+1}, X)$ is a cellular pair if and only if the Newton polytope of X is full dimensional.

There exist tropical hypersurfaces in $\mathbb{T}^{n+1}$ which do not intersect the boundary of $\mathbb{T}^{n+1}$. For example, let $X \subset \mathbb{T}^2$ be the tropical curve with three rays in directions $(-2,1)$, $(1,-2)$ and $(1,1)$. In this case, the pair $(\mathbb{T}^{n+1}, X)$ is not a cellular pair, though X may be combinatorially ample in $\mathbb{T}^{n+1}$. However, if Y is a compact tropical toric variety and X is a hypersurface which intersects the boundary of Y transversally then (Y, X) is a cellular pair.

Consider in $\mathbb{T}^3$ the two dimensional tropical hypersurface X whose support is a fan with rays $(-1,0,1)$, $(0,-1,1)$, $(1,1,-3)$, and $(0,0,1)$ and a two dimensional face generated by each pair of rays. The hypersurface X does not intersect the boundary of $\mathbb{T}^3$ since all the faces of X have an empty intersection with the cones defining the fan dual to $\mathbb{T}^3$ [OR13, Lemma 3.9]. The hypersurface X is non-singular since its Newton polytope is a tetrahedron of normalized volume equal to one. The pair $(\mathbb{T}^3, X)$ is not a cellular pair, and Theorem 1.2 does not hold in this case since for example $H_0(X; \mathcal{F}_1) = \mathbb{R}^3$ but $H_0(\mathbb{T}^3; \mathcal{F}_1) = 0$

Definition 2.9. — *Let Y be a tropical toric variety and X be a tropical hypersurface that is proper in Y and σ a cell of the subdivision of Y induced by X of dimension q. The parent face of σ is the unique face of Y of sedentarity order 0 and dimension $q + \operatorname{sed}(\sigma)$ which contains σ.*

2.3. Tropical homology

A rational polyhedral complex Z has the structure of a category. The objects of this category are the cells of Z and there is a morphism $\tau \to \sigma$ if the cell τ is included in σ. We use the notation Z^{op} to denote the category that has the same objects as Z, and with morphisms corresponding to the morphisms of Z but with their directions reversed. Let $\mathrm{Mod}_{\mathbb{Z}}$ denote the category of modules over $\mathbb{Z}$. We now define cellular sheaves and cosheaves of $\mathbb{Z}$-modules on Z.

Definition 2.10. — *Given a rational polyhedral complex Z, a* cellular cosheaf *$\mathcal{G}$ is a functor*

$$\mathcal{G}\colon Z^{\mathrm{op}} \to \mathrm{Mod}_{\mathbb{Z}}\,.$$

More explicitly, a cellular cosheaf consists of a $\mathbb{Z}$-module $\mathcal{G}(\sigma)$ for each cell σ in Z together with a morphism $\iota_{\sigma\tau}\colon \mathcal{G}(\sigma) \to \mathcal{G}(\tau)$ for each pair τ, σ when τ is a face of σ. Since $\mathcal{G}$ is a functor, for any triple of cells $\gamma \subset \tau \subset \sigma$ the morphisms ι commute in the sense that

$$\iota_{\sigma\gamma} \;=\; \iota_{\tau\gamma} \circ \iota_{\sigma\tau}$$

Dually, a cellular sheaf $\mathcal{H}$ is a morphism $\mathcal{H}\colon Z \to \mathrm{Mod}_{\mathbb{Z}}$. Therefore, for each σ there is a $\mathbb{Z}$-module $\mathcal{H}(\sigma)$ and there are morphisms $\rho_{\tau\sigma}\colon \mathcal{H}(\tau) \to \mathcal{H}(\sigma)$ when τ is a face of σ.

The cosheaves that we use throughout the text will always be free $\mathbb{Z}$-modules unless it is otherwise stated. We will now define the integral multi-tangent modules. We refer the reader to [BIMS15, KS17, MZ14] for the definitions of the multi-tangent spaces with rational and real coefficients.

Let Y be the non-singular tropical toric variety corresponding to a fan Σ. Let ρ be a simplicial cone of Σ which has rays in primitive integer directions $r_1, \ldots, r_s$. Then we define

$$T(Y_\rho) := \frac{\mathbb{R}^{n+1}}{\langle r_1, \ldots, r_s \rangle} \qquad \text{and} \qquad T_{\mathbb{Z}}(Y_\rho) := \frac{\mathbb{Z}^{n+1}}{\langle r_1, \ldots, r_s \rangle}.$$

If Y_ρ and Y_η are a pair of strata such that $Y_\eta \subset \overline{Y}_\rho$ then the generators of the cone η contain the generators of the cone ρ and thus we get projection maps:

$$\pi_{\rho\eta} \colon T(Y_\rho) \to T(Y_\eta) \qquad \text{and} \qquad \pi_{\rho\eta} \colon T_{\mathbb{Z}}(Y_\rho) \to T_{\mathbb{Z}}(Y_\eta). \tag{2.1}$$

Recall that a polyhedron in Y is the closure in Y of a rational polyhedron in some stratum $Y_\rho \cong \mathbb{R}^k$. Therefore, if σ is a polyhedron in Y, then $\operatorname{relint} \sigma$ is contained in some stratum Y_ρ of Y. Let $T(\sigma)$ denote the tangent space to the relative interior of σ in $T(Y_\rho)$ when $\operatorname{relint} \sigma$ is contained in Y_ρ. Since σ is rational there is a full rank lattice $T_{\mathbb{Z}}(\sigma) \subset T(\sigma)$.

Definition 2.11. — *Let Z be a rational polyhedral complex in a tropical toric variety Y. The integral p-multi-tangent cosheaf of Z is a cellular cosheaf $\mathcal{F}_p$ of $\mathbb{Z}$-modules on Z. For a face τ of Z such that* $\operatorname{relint} \tau$ *is contained in the stratum Y_ρ we have*

$$\mathcal{F}_p^Z(\tau) = \sum_{\tau \subset \sigma \subset Z_\rho} \bigwedge^p T_{\mathbb{Z}}(\sigma). \tag{2.2}$$

For $\tau \subset \sigma$, the maps of the cellular cosheaf $i_{\sigma\tau} \colon \mathcal{F}_p^Z(\sigma) \to \mathcal{F}_p^Z(\tau)$ are induced by natural inclusions when $\operatorname{int}(\sigma)$ *and* $\operatorname{int}(\tau)$ *are in the same stratum of Y. Otherwise are induced by the quotients $\pi_{\rho\eta}$ composed with inclusions when* $\operatorname{int}(\sigma) \subset Y_\rho$ *and* $\operatorname{int}(\tau) \subset Y_\eta$.

Example 2.12. — Let Y be a tropical toric variety. Consider the polyhedral structure on Y given by $Y = \bigcup \overline{Y}_\rho$ induced by the toric stratification. One has

$$\mathcal{F}_p^Y(\overline{Y}_\rho) = \bigwedge^p T_{\mathbb{Z}}(Y_\rho) \cong \bigwedge^p \mathbb{Z}^{\operatorname{codim} \rho},$$

and the cosheaf maps are the maps induced by the projection maps $\pi_{\rho\eta}$ defined in (2.1).

Example 2.13. — Let $H_n \subset \mathbb{R}^{n+1}$ denote the standard tropical hyperplane in $\mathbb{R}^{n+1}$. Then H_n is the tropical variety defined by the tropical polynomial function

$$f(x_1, \ldots, x_{n+1}) = \max\{0, x_1, \ldots, x_{n+1}\}.$$

Its Newton polytope is the standard simplex in $\mathbb{R}^{n+1}$.

The tropical hypersurface H_n is a fan of dimension n, it has $n+2$ rays that are in the directions $-e_1, \ldots, -e_{n+1}$, and $e_1 + \ldots + e_{n+1}$. See the left hand side of Figure 2.3 for the standard hyperplane in $\mathbb{R}^3$. Every subset of the rays of size less than or equal to n spans a cone of H_n. If v is the vertex of H_n, then $\mathcal{F}_p^{H_n}(v) = \Lambda^p \mathbb{Z}^{n+1}$, for $0 \leqslant p \leqslant n$ and $\mathcal{F}_{n+1}^{H_n}(v) = 0$. Moreover, we have

$$\chi_v(\lambda) := \sum_{p=0}^{n} (-1)^p \operatorname{rank} \mathcal{F}_p^{H_n}(v) \lambda^p = (1-\lambda)^{n+1} - (-\lambda)^{n+1}.$$

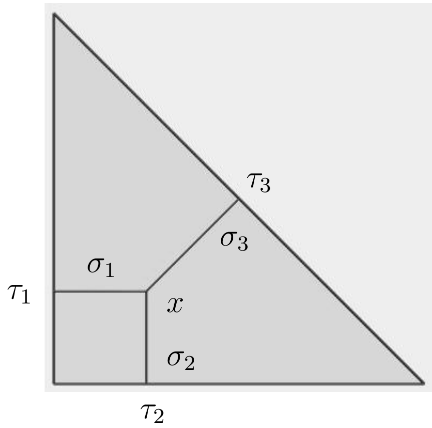

Figure 2.2. The tropical line X in $\mathbb{TP}^2$ from Example 2.14.

Example 2.14. — Figure 2.2 shows a tropical line X contained in the tropical projective plane $\mathbb{TP}^2$ from Example 2.2. The polyhedral structure on $\mathbb{TP}^2$ induced by X has 7 vertices, 9 edges, and 3 faces of dimension 2.

For any face σ of this polyhedral structure on $\mathbb{TP}^2$, the rank of $\mathcal{F}_p^{\mathbb{TP}^2}(\sigma)$ depends only on the dimension of the stratum of $\mathbb{TP}^2$ which contains $\operatorname{relint}(\sigma)$. If $\operatorname{relint}(\sigma)$ is contained in a stratum of $\mathbb{TP}^2$ of dimension k then $\mathcal{F}_p^{\mathbb{TP}^2}(\sigma) \cong \bigwedge^p \mathbb{Z}^k$.

The directions of the rays of the fan for $\mathbb{TP}^2$ are

$$v_1 = (-1, 0), \quad v_2 = (0, -1), \quad \text{and} \quad v_3 = (1, 1).$$

Referring to the labeling in Figure 2.2, we have

$$\mathcal{F}_1^X(x) = \langle v_1, v_2, v_3 \rangle \cong \mathbb{Z}^2, \quad \mathcal{F}_1^X(\sigma_i) = \langle v_i \rangle, \quad \text{and} \quad \mathcal{F}_1^X(\tau_i) = 0.$$

When $p = 0$, we have $\mathcal{F}_0^X(\gamma) = \mathbb{Z}$ for all γ in X and $\mathcal{F}_p^X(\gamma) = 0$ for all γ in X when $p \geqslant 2$.

The following lemma about the structure of the cosheaves in the case of a non-singular tropical hypersurface will be useful later on.

Lemma 2.15. — *Let X be a non-singular tropical hypersurface in a tropical toric variety Y. If τ is a face of X of dimension q whose relative interior is contained in a stratum Y_ρ of dimension m, then*

$$\mathcal{F}_p^X(\tau) \cong \bigoplus_{l=0}^{p} \mathcal{F}_{p-l}^{H_{m-q-1}}(v) \otimes \bigwedge^{l} T_{\mathbb{Z}}(\tau),$$

where H_{m-q-1} is the standard tropical hyperplane of dimension $m-q-1$ in $\mathbb{R}^{m-q}$ and v denotes its vertex.

If τ is a codimension one face of σ in X and $\operatorname{relint}(\tau)$ and $\operatorname{relint}(\sigma)$ are contained in the distinct strata Y_ρ and Y_η, respectively, then the cosheaf map $i_{\sigma\tau} \colon \mathcal{F}_p^X(\sigma) \to \mathcal{F}_p^X(\tau)$ together with the above isomorphisms commute with the map

$$\bigoplus_{l=0}^{p} \mathcal{F}_{p-l}^{H_{m-q-1}}(v) \otimes \bigwedge^{l} T_{\mathbb{Z}}(\sigma) \to \bigoplus_{l=0}^{p} \mathcal{F}_{p-l}^{H_{m-q-1}}(v) \otimes \bigwedge^{l} T_{\mathbb{Z}}(\tau), \tag{2.3}$$

which is induced by the map $\mathrm{id} \otimes \pi_{\eta\rho}$ *on each factor of the direct sum, where* $\pi_{\eta\rho} \colon \bigwedge^l T_{\mathbb{Z}}(\sigma) \to \bigwedge^l T_{\mathbb{Z}}(\tau)$ *is from Equation* (2.1).

Proof. — Recall that $T_{\mathbb{Z}}(\tau)$ denotes the integral points in the tangent space of the face τ. Now let L be a $m-q$ dimensional affine subspace of $\mathbb{R}^m \cong Y_\rho$ defined over $\mathbb{Z}$ such that L intersects all faces of X_ρ that contain $\mathrm{relint}(\tau)$ transversally and that together $T_{\mathbb{Z}}(L)$ and $T_{\mathbb{Z}}(\tau)$ generate the lattice $T_{\mathbb{Z}}(Y_\rho)$. By the above transversality assumption, the intersection $L' = L \cap X$ has a single vertex v' contained in τ.

For every l there is a map

$$i_l \colon \mathcal{F}^{L'}_{p-l}(v') \otimes \bigwedge^l T_{\mathbb{Z}}(\tau) \to \mathcal{F}^X_p(\tau),$$

given by taking the wedge product of the vectors in $\mathcal{F}^{L'}_{p-l}(v)$ and $\bigwedge^l T_{\mathbb{Z}}(\tau)$. Taking the direct sum of the maps i_l for all $0 \leqslant l \leqslant p$ gives a map

$$\bigoplus_{l=0}^{p} \mathcal{F}^{L'}_{p-l}(v') \otimes \bigwedge^l T_{\mathbb{Z}}(\tau) \to \mathcal{F}^X_p(\tau). \tag{2.4}$$

If σ is a facet of $X \cap Y_\rho$ containing the face τ, then by our assumptions on L', we have

$$T_{\mathbb{Z}}(\sigma) \cong T_{\mathbb{Z}}(\tau) \oplus T_{\mathbb{Z}}(L' \cap \sigma).$$

Therefore,

$$\mathcal{F}^X_p(\sigma) \cong \bigoplus_{l=0}^{p} \mathcal{F}^{L' \cap \sigma}_{p-l}(v') \otimes \bigwedge^l T_{\mathbb{Z}}(\tau).$$

Now since $\mathcal{F}^X_p(\tau)$ is generated by all $\mathcal{F}^X_p(\sigma)$ for σ a facet containing τ, the map in Equation (2.4) is an isomorphism.

By the assumption that X is non-singular in Y, every non-empty stratum $X_\rho = Y_\rho \cap X$ is a non-singular tropical hypersurface in $\mathbb{R}^m$, where $m = n+1-\dim \rho$. Therefore, the hypersurface X_ρ is defined by a tropical polynomial f_ρ and it is dual to a primitive regular subdivision of the Newton polytope of f_ρ which is induced by f_ρ. A face σ of X whose relative interior is contained in X_ρ is dual to a face of the dual subdivision of $\Delta(f_\rho)$, and since this dual subdivision is primitive, the face dual to σ is a simplex. Therefore, near the vertex v' the rational polyhedral complex L' is up to an integral affine transformation the same as a neighborhood of the vertex v of the tropical hyperplane H_{m-q-1} and we have $\mathcal{F}^{L'}_{p-l}(v') \cong \mathcal{F}^{H_{m-q-1}}_{p-l}(v)$. This proves the isomorphism stated in the Lemma 2.15.

If τ is a face of σ, and τ and σ are contained in Y_η and Y_ρ respectively, for $\eta \neq \rho$, then we can write $T_{\mathbb{Z}}(Y_\rho) = T_{\mathbb{Z}}(L_\sigma) \oplus T_{\mathbb{Z}}(\sigma)$ and $T_{\mathbb{Z}}(Y_\eta) = T_{\mathbb{Z}}(L_\tau) \oplus T_{\mathbb{Z}}(\tau)$, where L_σ and L_τ are the linear spaces chosen in the argument above to intersect σ and τ, respectively. Since the polyhedral structure on X is proper in Y, the map $\pi_{\rho\eta} \colon T_{\mathbb{Z}}(Y_\rho) \to T_{\mathbb{Z}}(Y_\eta)$ restricts to an isomorphism between $T_{\mathbb{Z}}(L_\sigma)$ and $T_{\mathbb{Z}}(L_\tau)$. Therefore, it also restricts to an isomorphism between $\mathcal{F}^{L_\sigma \cap X}_p(v_\sigma)$ and $\mathcal{F}^{L_\tau \cap X}_p(v_\tau)$ for all p. The claim about the commutativity of the above isomorphisms with the maps in Equation (2.3) and $i_{\sigma\tau} \colon \mathcal{F}^X_p(\sigma) \to \mathcal{F}^X_p(\tau)$ follows since $i_{\sigma\tau}$ is induced by projecting along a direction $\pi_{\rho\eta}$. □

Corollary 2.16. — *Let X be a non-singular tropical hypersurface of a tropical toric variety Y. Let σ be a face of X of dimension q whose relative interior is contained in stratum Y_ρ of dimension m. Then the polynomial defined by*

$$\chi_\sigma(\lambda) := \sum_{p=0}^{n} (-1)^p \operatorname{rank} \mathcal{F}_p^X(\sigma)\lambda^p.$$

is

$$\chi_\sigma(\lambda) = (1-\lambda)^m - (1-\lambda)^q(-\lambda)^{m-q}.$$

Proof. — Using the isomorphism in Lemma 2.15, together with the formula for the ranks of $\mathcal{F}_p^{H_{n-q}}(v)$ from Example 2.13, we obtain

$$\chi_\sigma(\lambda) = (1-\lambda)^q \left[(1-\lambda)^{m-q} - (-\lambda)^{m-q}\right].$$

The statement of the corollary follows upon simplification. ☐

In order to define the cellular tropical homology groups of a rational polyhedral complex Z we must first fix orientations of each of its cells. Let Z^q denote the cells of dimension q of Z. We define an *orientation map* on pairs of cells, $\mathcal{O}\colon Z^q \times Z^{q-1} \to \{0, 1, -1\}$ by:

(2.5)
$$\mathcal{O}(\sigma,\tau) := \begin{cases} 0 & \text{if } \tau \not\subset \sigma, \\ 1 & \text{if the orientation of } \tau \text{ coincides with its orientation in } \partial\sigma, \\ -1 & \text{if the orientation of } \tau \text{ differs from its orientation in } \partial\sigma. \end{cases}$$

Definition 2.17. — *Let Z be a rational polyhedral complex and $\mathcal{G}$ a cellular cosheaf on Z. The groups of* cellular q-chains in Z with coefficients in $\mathcal{G}$ *are*

$$C_q(Z;\mathcal{G}) = \bigoplus_{\substack{\dim \sigma = q \\ \sigma \text{ compact}}} \mathcal{G}(\sigma).$$

The boundary maps $\partial\colon C_q(Z;\mathcal{G}) \to C_{q-1}(Z;\mathcal{G})$ are given by the direct sums of the cosheaf maps $i_{\sigma\tau}$ for $\tau \subset \sigma$ composed with the orientation maps $\mathcal{O}_{\sigma\tau}$ for all τ and σ. The q^{th} homology group of $\mathcal{G}$ *is*

$$H_q(Z;\mathcal{G}) = H_q(C_\bullet(Z;\mathcal{G})).$$

Definition 2.18. — *Let Z be a rational polyhedral complex and $\mathcal{G}$ a cellular cosheaf on Z. The groups of* Borel–Moore cellular q-chains in Z with coefficients in $\mathcal{G}$ *are*

$$C_q^{BM}(Z;\mathcal{G}) = \bigoplus_{\dim \sigma = q} \mathcal{G}(\sigma).$$

The boundary maps $\partial\colon C_q^{BM}(Z;\mathcal{G}) \to C_{q-1}^{BM}(Z;\mathcal{G})$ are given by the direct sums of the cosheaf maps $i_{\sigma\tau}$ for $\tau \subset \sigma$ with the orientation maps $\mathcal{O}_{\sigma\tau}$ for all τ and σ. The q^{th} homology group of $\mathcal{G}$ *is*

$$H_q^{BM}(Z;\mathcal{G}) = H_q\left(C_\bullet^{BM}(Z;\mathcal{G})\right).$$

Definition 2.19. — *The* $(p,q)^{\text{th}}$ tropical homology group *is*

$$H_q\left(Z;\mathcal{F}_p^Z\right) = H_q\left(C_\bullet\left(Z;\mathcal{F}_p^Z\right)\right). \tag{2.6}$$

The $(p,q)^{\text{th}}$ Borel–Moore tropical homology group *is*

$$H_q^{BM}\left(Z;\mathcal{F}_p^Z\right) = H_q\left(C_\bullet^{BM}\left(Z;\mathcal{F}_p^Z\right)\right).$$

Remark 2.20. — Both the Borel–Moore and the standard tropical cellular homology groups of cosheaves are defined with respect to a fixed polyhedral structure. Let X be a tropical hypersurface in a tropical toric variety Y, and consider the polyhedral structure on X coming from the dual subdivision of its Newton polytope and the polyhedral structure on Y induced by X. When (Y,X) is a cellular pair in the sense of Definition 2.7 then the cellular homology groups from (2.6) of X or Y are isomorphic to singular tropical homology groups of X or Y, respectively [Cur14, Theorem 7.3.2].

On the other hand, even when (Y,X) is not a cellular pair, the Borel–Moore tropical cellular homology groups of X and Y are always isomorphic to the Borel–Moore singular homology groups of X and Y, respectively. In fact, one can always find a compactification of the pair (Y,X) such that $(\overline{X},\overline{X}\backslash X)$, $(\overline{Y},\overline{Y}\backslash Y)$ and $(\overline{Y},\overline{X})$ are cellular pairs. The Borel–Moore homology groups of X are isomorphic to the relative homology groups of the pair $(\overline{X},\overline{X}\backslash X)$, and similarly for Y and $(\overline{Y},\overline{Y}\backslash Y)$.

If $\mathcal{G}$ is a cellular sheaf on a rational polyhedral complex Z, then the group of q cochains and q cochains with compact support of $\mathcal{G}$ are respectively,

$$C^q(Z;\mathcal{G}) = \bigoplus_{\substack{\dim\sigma=q\\ \sigma\text{ compact}}} \mathcal{G}(\sigma) \quad\text{and}\quad C_c^q(Z;\mathcal{G}) = \bigoplus_{\dim\sigma=q} \mathcal{G}(\sigma).$$

The complex of cochains and cochains with compact support of $\mathcal{G}$ are formed from the cochain groups together with the restriction maps $r_{\tau\sigma}$ combined with the orientation map $\mathcal{O}$ as in the case for a cosheaf. The cohomology groups of $\mathcal{G}$ are defined as the cohomology of these complexes.

Definition 2.21. — *Let Z be a rational polyhedral complex and $\mathcal{G}$ a cellular sheaf on Z. The cohomology groups and cohomology groups with compact support of $\mathcal{G}$ are respectively,*

$$H^q(Z;\mathcal{G}) := H^q(C^\bullet(Z;\mathcal{G})) \quad\textit{and}\quad H_c^q(Z;\mathcal{G}) := H^q(C_c^\bullet(Z;\mathcal{G})).$$

Remark 2.22. — Given a rational polyhedral complex Z we can define a collection of cellular sheaves $\mathcal{F}_Z^p$ from the cosheaves $\mathcal{F}_p^Z$. For a face σ of Z set $\mathcal{F}_Z^p(\sigma) = \operatorname{Hom}(\mathcal{F}_p^Z(\sigma),\mathbb{Z})$. For τ a face of σ the map $\rho_{\tau\sigma}:\mathcal{F}_Z^p(\tau)\to\mathcal{F}_Z^p(\sigma)$ is given by dualizing the corresponding map from the cosheaf $\mathcal{F}_p^Z$. Then we have

$$C^q\left(Z;\mathcal{F}_Z^p\right) = \operatorname{Hom}\left(C_q\left(Z;\mathcal{F}_p^Z\right),\mathbb{Z}\right) \quad\text{and}\quad C_c^q\left(Z;\mathcal{F}_Z^p\right) = \operatorname{Hom}\left(C_q^{BM}\left(Z;\mathcal{F}_p^Z\right),\mathbb{Z}\right).$$

Therefore for Z a non-singular tropical toric variety or a non-singular tropical hypersurface in a tropical toric variety, the tropical cohomology groups and cohomology groups with compact support are respectively,

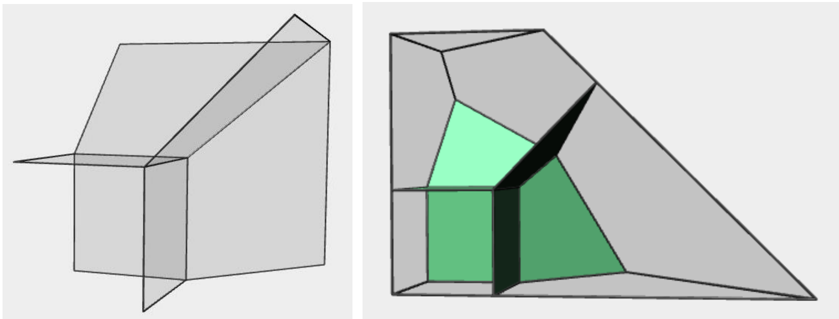

Figure 2.3. The standard tropical hyperplane in $\mathbb{R}^3$ on the left its closure in the tropical toric variety described in Example 2.23 on the right.

$$H^q(Z;\mathcal{F}^p) := H^q\left(\mathrm{Hom}\left(C_\bullet\left(Z;\mathcal{F}_p^Z\right),\mathbb{Z}\right)\right)$$

and

$$H_c^q\left(Z;\mathcal{F}^p\right) := H^q\left(\mathrm{Hom}\left(C_\bullet^{BM}\left(Z;\mathcal{F}_p^Z\right),\mathbb{Z}\right)\right).$$

Let us end the preliminaries with two examples that show the necessity of the assumptions in Theorem 1.1

Example 2.23. — Here is a counter example to Theorem 1.1 when we drop the condition of combinatorial ampleness from Definition 2.5. Consider the standard tropical hyperplane $X^o \subset \mathbb{R}^{n+1}$. The case when $n = 2$ is depicted in the left of Figure 2.3. Let Σ be the fan for $n+1$ dimensional projective space blown up in a toric fixed point, and let Y be the tropical toric variety defined by Σ. Let X denote the compactification of X^o in Y. Then it can be computed that rank $H_1(X;\mathcal{F}_1^X) = 1$ and rank $H_1(Y;\mathcal{F}_1^Y) = 2$, so the map $H_1(X;\mathcal{F}_1^X) \to H_1(Y;\mathcal{F}_1^Y)$ is not an isomorphism when $n > 2$. The connected component of $Y \backslash X$ containing the stratum of Y dual to the ray of Σ corresponding to the exceptional divisor of the blow up does not satisfy the condition to be combinatorially ample.

The complex geometric version of the same scenario also fails the Lefschetz hyperplane section theorem, since the hypersurface of the toric variety is not ample.

When (Y, X) is not a cellular pair, the standard cellular tropical homology groups are not isomorphic to the standard singular tropical homology groups from [IKMZ19]. Upon subdividing the spaces Y and X to form a cellular pair we can nevertheless use the cellular chain complexes to compute the tropical homology. However, the next example shows that the condition that (Y, X) equipped with their inherent subdivisions form a cellular pair is required for the Lefschetz theorem to hold for standard tropical homology, even when we compute the groups after refining the polyhedral structure or using singular tropical homology.

Example 2.24. — Consider the case when the Newton polytope of X is an interval of lattice length equal to 1. Then the tropical hypersurface X is a (classical) $\mathbb{Z}$-affine subspace of $Y = \mathbb{R}^{n+1}$ of dimension n. Upon subdividing X and Y so that they form a cellular pair, or using singular tropical homology, we can compute the standard tropical homology groups to be:

$$H_q\left(X;\mathcal{F}_p^X\right)=\begin{cases}\bigwedge^p\mathbb{Z}^n & \text{if } q=0,\\ 0 & \text{if } q\neq 0\end{cases}\quad\text{and}\quad H_q\left(Y;\mathcal{F}_p^Y\right)=\begin{cases}\bigwedge^p\mathbb{Z}^{n+1} & \text{if } q=0,\\ 0 & \text{if } q\neq 0.\end{cases}$$

Whereas, the Borel–Moore homology groups are

$$H_q^{BM}\left(X;\mathcal{F}_p^X\right)=\begin{cases}\bigwedge^p\mathbb{Z}^n & \text{if } q=n,\\ 0 & \text{if } q\neq n\end{cases}$$

and

$$H_q^{BM}\left(Y;\mathcal{F}_p^Y\right)=\begin{cases}\bigwedge^p\mathbb{Z}^{n+1} & \text{if } q=n+1,\\ 0 & \text{if } q\neq n+1.\end{cases}$$

We see that the conclusion of the Lefschetz section theorem as stated in Theorem 1.1 does not hold for the standard tropical homology groups, however there is no contradiction for the Borel–Moore homology groups.

3. Tropical Lefschetz section theorem

A tropical hypersurface X in a tropical toric variety Y induces a polyhedral structure on Y. Unless it is explicitly mentioned we will use this polyhedral structure on Y to compute its cellular tropical homology groups. Following Remark 2.20, we obtain the same homology groups using this polyhedral structure as if we chose the polyhedral structure from the stratification of Y dual to the polyhedral fan defining it, see Example 2.12.

If $Z'\subset Z$ is a rational subpolyhedral complex and $\mathcal{G}$ is a cosheaf on Z, then the restriction cosheaf $\mathcal{G}|_{Z'}$ is a cosheaf on Z' which assigns the $\mathbb{Z}$-module $\mathcal{G}(\sigma)$ for σ a face of Z'. The cosheaf $\mathcal{G}|_{Z'}$ can also be considered as a cosheaf on Z. In this case, it assigns $\mathcal{G}(\sigma)$ if σ is a face of Z' and 0 otherwise.

Since we consider the polyhedral structure on Y induced by X, the tropical hypersurface X is a rational subpolyhedral complex of Y and we have the cosheaves $\mathcal{F}_p^Y|_X$, which can be considered on X or Y as described above.

To prove Theorems 1.1 and 1.2, we consider two exact sequences of cosheaves. The first is the exact sequence of cosheaves on Y given by,

$$0\to\mathcal{F}_p^Y|_X\to\mathcal{F}_p^Y\to\mathcal{Q}_p\to 0. \tag{3.1}$$

The second one consists of cosheaves on X and is given by,

$$0\to\mathcal{F}_p^X\to\mathcal{F}_p^Y|_X\to\mathcal{N}_p\to 0. \tag{3.2}$$

The injective maps on the left hand side of both cosheaf sequences are both natural inclusions on the stalks over faces. The cosheaves $\mathcal{Q}_p$ and $\mathcal{N}_p$ are defined as the cokernel cosheaves in both short exact sequences. The cosheaves $\mathcal{F}_p^Y|_X$, $\mathcal{F}_p^Y$, and $\mathcal{F}_p^X$ are all free $\mathbb{Z}$-modules. Moreover, since X is a non-singular tropical hypersurface, the cosheaves $\mathcal{Q}_p$ and $\mathcal{N}_p$ are also cosheaves of free $\mathbb{Z}$-modules.

Proposition 3.1. — *If X is a non-singular tropical hypersurface in Y, the cosheaves $\mathcal{Q}_p$ and $\mathcal{N}_p$ from* (3.1) *and* (3.2) *are cosheaves of free $\mathbb{Z}$-modules.*

Proof. — The cosheaf $\mathcal{Q}_p$ satisfies $\mathcal{Q}_p(\sigma) = 0$ if σ is a face of X and $\mathcal{Q}_p(\sigma) = \mathcal{F}_p(\sigma)$ for σ a face of Y and not a face of X. Therefore, the cosheaf $\mathcal{Q}_p$ consists of torsion free $\mathbb{Z}$-modules.

Given a face σ of dimension q of X, the $\mathbb{Z}$-module $\mathcal{F}_p^X(\sigma)$ is a submodule of $\mathcal{F}_p^Y(\sigma)$, and the map $\mathcal{F}_p^X(\sigma) \to \mathcal{F}_p^Y(\sigma)$ is simply the inclusion map.

Let Y_ρ be the minimal stratum of Y such that σ is contained in Y_ρ. Let m be the dimension of Y_ρ (as X is non-singular, we have $m \geqslant 1$). By Lemma 2.15, one has

$$\mathcal{F}_p^X(\sigma) \cong \bigoplus_{l=0}^{p} \mathcal{F}_l^{H_{m-q-1}}(v) \otimes \bigwedge^{p-l} \mathbb{Z}^q \subseteq \bigoplus_{l=0}^{p} \bigwedge^{l} \mathbb{Z}^{m-q} \otimes \bigwedge^{p-l} \mathbb{Z}^q \cong \bigwedge^{p} \mathbb{Z}^m = \mathcal{F}_p^Y(\sigma).$$

Using the canonical base $\{e_1 \, \dots, \, e_{m-q}, \, e_{m-q+1}, \, \dots, \, e_m\}$ of $\mathbb{Z}^{m-q} \times \mathbb{Z}^q \cong \mathbb{Z}^m$ and the associated base $\{e_{i_1} \wedge \, \dots \, \wedge e_{i_p}\}_{0 \leqslant i_1 < \dots i_p \leqslant m}$ of $\bigwedge^p \mathbb{Z}^m$, we immediately see from that description and the definitions of the standard tropical hyperplane H_{m-q-1} (see Example 2.13) and the cosheaves $\mathcal{F}_l^{H_{m-q-1}}$ that $\mathcal{F}_p^X(\sigma)$ is the free sub-$\mathbb{Z}$-module of $\mathcal{F}_p^Y(\sigma) \cong \bigwedge^p \mathbb{Z}^m$ spanned by all the elements $e_{i_1} \wedge \, \dots \, \wedge e_{i_l} \wedge e_{i_{l+1}} \wedge \, \dots \, \wedge e_{i_p}$ such that $i_1 < \, \dots \, < i_p$, that $i_l \leqslant m - q$ and that $l \leqslant m - q - 1$ (for $l = 0, \dots, p$). Therefore, the quotient $\mathcal{F}_p^Y(\sigma)/\mathcal{F}_p^X(\sigma)$ is free. □

Example 3.2. — If we drop the non-singularity assumption, the cosheaves $\mathcal{N}_p$ may have torsion. Consider for example the tropical hypersurface in $\mathbb{R}^3$ dual to the simplex of volume 2 with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,2)$, and consider the face σ of X adjacent to the edges of X of direction $(-2,0,1)$ and $(0,-2,1)$. Then the class in $\mathcal{N}_1(\sigma)$ represented by the vector $(-1,-1,1)$ is a 2-torsion class, since $(-1,-1,1) \in \mathcal{F}_1^Y|_X(\sigma)$ but $(-1,-1,1) \notin \mathcal{F}_1^X(\sigma)$ and $(-2,-2,2) \in \mathcal{F}_1^X(\sigma)$.

Example 3.3. — Consider again the tropical line X in $\mathbb{TP}^2$ from Example 2.14 and Figure 2.2. Then the cosheaf $\mathcal{Q}_p$ on $\mathbb{TP}^2$ assigns the trivial $\mathbb{Z}$-module to any face of $\mathbb{TP}^2$ which is also a face of X. For σ a face of $\mathbb{TP}^2$ and not a face of X, then $\mathcal{Q}_p(\sigma) = \mathcal{F}_p^{\mathbb{TP}^2}(\sigma)$. The inclusion maps $\mathcal{Q}_p(\sigma) \to \mathcal{Q}_p(\tau)$ are either 0 or equal to $\iota_{\sigma\tau} \colon \mathcal{F}_p^{\mathbb{TP}^2}(\sigma) \to \mathcal{F}_p^{\mathbb{TP}^2}(\tau)$.

For x the unique vertex of sedentarity 0 of X, the cosheaf $\mathcal{N}_p$ assigns $\mathcal{N}_p(x) = 0$, for all $p < 2$. When $p = 2$, we have $\mathcal{N}_p(x) = \bigwedge^2 \mathbb{Z}^2$.

For an edge σ_i of X the $\mathbb{Z}$-module $\mathcal{N}_p(\sigma_i)$ is a free module of rank 1, similarly for the three other vertices τ_i of X that have non-zero sedentarity.

To prove the Lefschetz section theorem for hypersurfaces, we first prove some useful lemmas, then some statements about the vanishing of both the standard and Borel–Moore homology with coefficients in $\mathcal{Q}_p$ and with coefficients in $\mathcal{N}_p$.

We recall the definition of γ^o for a face γ of X of dimension s and $\mathrm{sed}(\gamma) = 0$. For each cone ρ in the fan Σ defining Y, set $\gamma_\rho := \gamma \cap Y_\rho$ and define

$$\gamma^\circ := \bigsqcup_\rho \operatorname{relint} \gamma_\rho.$$

The set γ° is not a rational polyhedral complex since the strata are not closed polyhedra, however $\overline{\gamma^o}$ is a rational subpolyhedral complex of Y. The set γ^o is a stratified subset of Y and it can be viewed as a poset with the order relations given by inclusions. By Lemma 3.5, the set γ^o has a unique minimal strata.

If γ is a face of a rational polyhedral complex Z and $\mathcal{G}$ is a cellular cosheaf on Z, we can consider the cosheaf $\mathcal{G}$ restricted to γ^o even though γ^o is not a rational polyhedral complex. Similarly to Definition 2.10, the restriction $\mathcal{G}|_{\gamma^o}$ is a functor from γ^o considered as a poset to the category of $\mathbb{Z}$-modules. The groups of Borel–Moore chains of $\mathcal{G}$ restricted to γ^o are

$$C_q^{BM}(\gamma^o;\mathcal{G}|\gamma^o) := \bigoplus_{\dim\rho=q} \mathcal{G}(\gamma_\rho). \tag{3.3}$$

The chain groups form a complex with the boundary map

$$\partial\colon C_q^{BM}(\gamma^o;\mathcal{G}|_{\gamma^o}) \to C_{q-1}^{BM}(\gamma^o;\mathcal{G}|_{\gamma^o}) \tag{3.4}$$

given by the cosheaf maps combined with the orientation map inherited from Z. The homology groups of this complex are denoted $H_q^{BM}(\gamma^o;\mathcal{G}|_{\gamma^o})$. For simplicity we denote the cosheaves $\mathcal{F}_p^{\overline{\gamma^o}}|_{\gamma^\circ}$ by simply $\mathcal{F}_p^{\gamma^o}$.

If X is a hypersurface in $\mathbb{R}^{n+1}$, then for every face γ of X the complex γ^o consists of a single open cell $\operatorname{relint}(\gamma)$. See Figure 3.1 and Example 3.4 for illustrations of γ^o.

Example 3.4. — Let X be a tropical hypersurface in a 3-dimensional tropical toric variety Y. We describe the sets γ^o for some faces γ of X. If γ is a face of X which does not intersect any of the strata Y_ρ for $\rho \neq 0$ then γ° consists of a single cell which is simply $\operatorname{relint}(\gamma)$. Therefore γ° is combinatorially isomorphic to $\mathbb{R}^q$ where q is the dimension of γ.

Suppose that γ is a 2-dimensional face of X and $\gamma \cap Y_\rho \neq \varnothing$ for a unique 1-dimensional stratum Y_ρ. There must be two 2-dimensional strata $Y_{\rho'}$ and $Y_{\rho''}$ of Y which contain Y_ρ, moreover γ has non-empty intersection with both $Y_{\rho'}$ and $Y_{\rho''}$. Therefore, γ^o consists of four open cells and is combinatorially isomorphic to $\mathbb{T}^2$, see the left hand side of Figure 3.1. If γ is 2-dimensional and intersects only a single 2-dimensional stratum Y_ρ, then γ^o consists of two open cells and is combinatorially isomorphic to $\mathbb{R} \times \mathbb{T}$.

Suppose γ is a 1-dimensional face of X of sedentarity 0 such that $\gamma \cap Y_\rho$ is non-empty for a unique stratum Y_ρ of codimension 1. Such a situation is depicted on the right hand side of Figure 3.1. Then γ^o consists of two open cells, the 1-dimensional cell $\gamma_0 = \gamma \cap \mathbb{R}^3$ and the point $\gamma_\rho := \gamma \cap Y_\rho$.

In the case of such a 1-dimensional face γ, we can describe the restriction cosheaves. Namely, we have $\mathcal{F}_p^Y|_{\gamma^o}(\gamma) \cong \bigwedge^p \mathbb{Z}^3$ and $\mathcal{F}_p^{\gamma^o}(\gamma^o) \cong \bigwedge^p \mathbb{Z}$ for all p. The restrictions of the cosheaves $\mathcal{F}_p^X$ to γ^o are

$$\mathcal{F}_0^X|_{\gamma^o}(\gamma) = \mathbb{Z} \quad \mathcal{F}_1^X|_{\gamma^o}(\gamma) = \mathbb{Z}^3 \quad \mathcal{F}_2^X|_{\gamma^o}(\gamma) \cong \mathbb{Z}^2$$

and $\mathcal{F}_p^X|_{\gamma^o}(\gamma) = 0$ otherwise.

Lemma 3.5. — *Let Y be a tropical toric variety whose defining fan is simplicial and X be a combinatorically ample tropical hypersurface that is proper in Y. Then*

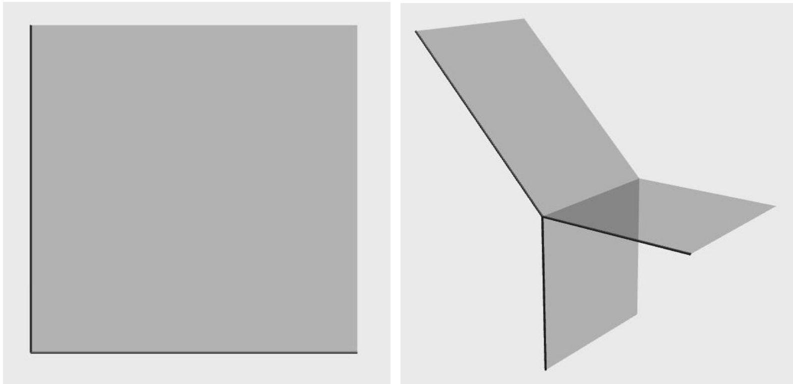

Figure 3.1. A depiction of the sets γ^o for two faces γ from Example 3.4

for every face γ of Y, considered with the polyhedral structure induced by X, the stratified set γ^o has a unique minimal face by inclusion.

Proof. — Let Σ be the simplicial fan defining Y. We show that for any face γ of Y, the collection S_γ of cones $\rho \in \Sigma$ such that $\gamma \cap Y_\rho \neq \varnothing$ forms a closed cone of Σ, which in particular implies the statement of the lemma, as the minimal face of γ^o will correspond to the intersection of γ with the stratum of Y of the maximal face of that closed cone.

Consider any top-dimensional cell $\widetilde{\gamma}$ of Y such that γ is a face of $\widetilde{\gamma}$. As we have assumed combinatorial ampleness, the stratified set $\widetilde{\gamma}^o$ is combinatorially isomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$ for some $0 \leqslant k \leqslant n+1$, where $n+1$ is the dimension of Y. Then $\widetilde{\gamma}^o$ has a minimal face which corresponds to the face $\{-\infty\}^k \times \mathbb{R}^{n+1-k} \subset \mathbb{T}^k \times \mathbb{R}^{n+1-k}$ under the isomorphism. The minimal face of $\widetilde{\gamma}^o$ corresponds to some simplicial cone $\widetilde{\rho}$ of Σ and every other face of $\widetilde{\gamma}^o$ corresponds to a face of $\widetilde{\rho}$. So the collection of cones in $S_{\widetilde{\gamma}}$ are the faces of $\widetilde{\rho}$.

By [OR13, Lemma 3.9], we have $\gamma \cap Y_\rho \neq \varnothing$ if and only if $C_\gamma \cap \operatorname{relint}(\rho) \neq \varnothing$, where C_γ is the recession cone of γ. Therefore, if a ray of Σ is in S_γ it is in $S_{\widetilde{\gamma}}$, and therefore a ray of $\widetilde{\rho}$. By convexity of the recession cone, if $C_\gamma \cap \operatorname{relint}(\rho_i) \neq \varnothing$ for a collection of rays $\rho_1, \ldots, \rho_k$, then $C_\gamma \cap \operatorname{relint}(\langle \rho_1, \ldots, \rho_k \rangle) \neq \varnothing$. Since $\widetilde{\rho}$ is a simplicial cone all faces are simplicial, and S_γ is a cone of $S_{\widetilde{\gamma}}$ which means it is a cone of Σ. Therefore it follows that γ^o has a unique minimal face. □

Lemma 3.6. — *Let X be a non-singular and* combinatorially ample *tropical hypersurface of an $n+1$ dimensional non-singular tropical toric variety Y. Consider the polyhedral structure on Y obtained by refinement by X. Let γ be a face of Y of sedentarity* 0*. Then for any p and all $q \neq \dim \gamma$*

$$H_q^{BM}\left(\gamma^o; \mathcal{F}_p^{\gamma^o}\right) = 0.$$

Proof. — Suppose γ is of dimension q and that the minimal face σ of γ^o is of dimension k. The Borel–Moore chain groups for γ^o are

$$C_q^{BM}\left(\gamma^o; \mathcal{F}_p^{\gamma^o}\right) = \bigoplus_{\substack{\gamma_\rho \cap \gamma^o \neq \varnothing \\ \dim \gamma_\rho = q}} \bigwedge^p T_{\mathbb{Z}}(\gamma_\rho).$$

The stratification on γ^o is isomorphic to the stratification of $\mathbb{R}^k \times \mathbb{T}^{q-k}$. Moreover, there are lattice isomorphisms between the integral tangent spaces $T_{\mathbb{Z}}(\gamma_\rho)$ and the integral tangent spaces of the corresponding strata. These isomorphisms are compatible with the projection maps between the strata and hence the boundary maps of the chain complexes. Therefore, the chain complex $C^{BM}_\bullet(\gamma^o; \mathcal{F}^{\gamma^o}_p)$ is isomorphic to

$$C^{BM}_\bullet\left(\mathbb{R}^k \times \mathbb{T}^{q-k}; \mathcal{F}_p^{\mathbb{R}^k\times\mathbb{T}^{q-k}}\right)$$

and there are isomorphisms of the corresponding homology groups.

By [JRS18], the space $\mathbb{R}^k \times \mathbb{T}^{q-k}$ satisfies Poincaré duality for tropical homology and the Borel–Moore tropical homology groups of $\mathbb{R}^{q-k} \times \mathbb{T}^k$ are zero except in degree $q = \dim \gamma$. The statement of the Lemma 3.6 follows. □

Lemma 3.7. — *Let X be a n-dimensional non-singular tropical hypersurface in a tropical toric variety Y. For σ a face of X of dimension q and sedentarity* $\operatorname{sed}(\sigma)$*, we have $\mathcal{N}_p(\sigma) = 0$ if $p \leqslant n - q - \operatorname{sed}(\sigma)$.*

Proof. — The $\mathbb{Z}$-modules $\mathcal{N}_p(\sigma)$, $\mathcal{F}^Y_p|_X(\sigma)$, and $\mathcal{F}^X_p$ are all free, so it suffices to show that the ranks of $\mathcal{F}^Y_p|_X(\sigma)$ and $\mathcal{F}^X_p$ are equal when $p \leqslant n - q - \operatorname{sed}(\sigma)$. By Example 2.14,

$$\operatorname{rank} \mathcal{F}^Y_p|_X(\sigma) = \binom{n+1-\operatorname{sed}(\sigma)}{p}.$$

By Corollary 2.16, the polynomial defined by

$$\chi_\sigma(\lambda) := \sum_{p=0}^{n} (-1)^p \operatorname{rank} \mathcal{F}^X_p(\sigma)\lambda^p.$$

is

$$\chi_\sigma(\lambda) = (1-\lambda)^{n+1-\operatorname{sed}(\sigma)} - (1-\lambda)^q(-\lambda)^{n-q+1-\operatorname{sed}(\sigma)}.$$

So that

$$\operatorname{rank} \mathcal{F}^X_p(\sigma) = \binom{n+1-\operatorname{sed}(\sigma)}{p} \text{ if } p \leqslant n-q-\operatorname{sed}(\sigma).$$

Therefore $\mathcal{N}_p(\sigma) = 0$ when $p \leqslant n - q - \operatorname{sed}(\sigma)$, and the proof is completed. □

Lemma 3.8. — *Let X be a combinatorially ample n-dimensional non-singular tropical hypersurface in a tropical toric variety Y. For a face γ of X of sedentarity* 0 *we have*

$$H^{BM}_q\left(\gamma^\circ; \mathcal{F}^X_p|_{\gamma^\circ}\right) = 0$$

for all $q \neq \dim\gamma$.

Proof. — Denote by γ_m the unique minimal face of γ° and suppose it is contained in the stratum Y_{ρ_m}.. Let Γ denote the star of γ_m in X_{ρ_m}, that is,

$$\Gamma = \operatorname{star}_{X_{\rho_m}}(\gamma_m) = \{\operatorname{relint}(\sigma) \mid \gamma_m \subset \sigma \subset X_{\rho_m}\} \subset \mathbb{R}^{n+1-\operatorname{sed}(\gamma_m)}.$$

Then as a rational polyhedral complex $\Gamma \subset \mathbb{R}^{n+1-\operatorname{sed}(\gamma_m)}$ is, up to $\mathrm{GL}_{n+1-\operatorname{sed}(\gamma_m)}(\mathbb{Z})$, equal to a basic open set of $\Gamma' \times \mathbb{R}^{\dim\gamma_m}$ where $\Gamma' = H_{n-\operatorname{sed}(\gamma_m)-\dim\gamma_m}$ is the standard tropical hyperplane in $\mathbb{R}^{n+1-\operatorname{sed}(\gamma_m)-\dim\gamma_m}$. For the notion of basic open set see [JSS19, Definition 3.7].

Moreover, the star of any other face γ_ρ in γ^o is, up to $\mathrm{GL}_{n+1}(\mathbb{Z})$, equal to a basic open set of $\Gamma' \times \mathbb{R}^{\dim \gamma_\rho}$. Let v be the vertex of Γ', then by Lemma 2.15, for any face γ_ρ in γ^o we have

$$\mathcal{F}_p^X(\gamma_\rho) \cong \bigoplus_{l=0}^{p} \mathcal{F}_{p-l}^{\Gamma'}(v) \otimes \bigwedge^{l} \mathbb{Z}^{\dim \gamma_\rho}.$$

This isomorphism follows from the tensor product formula for the $\mathbb{Z}$-module $\mathcal{F}_p^X(\gamma_\rho)$ in Lemma 2.15.

For each l from 0 to p, let $C_\bullet^{l,p}$ denote the chain complex whose terms are

$$C_q^{l,p} = \bigoplus_{\substack{\rho \,|\, \gamma_\rho \neq \varnothing \\ \mathrm{sed}(\gamma_\rho) = \dim\gamma - q}} \mathcal{F}_{p-l}^{\Gamma'}(v) \otimes \bigwedge^{l} \mathbb{Z}^{\dim \gamma_\rho}.$$

We define the boundary maps of the complex on the direct summands. If $\gamma_{\rho'}$ is a face of γ_ρ then the map on the direct summand is

$$\mathrm{id} \otimes \pi_{\rho\rho'} \colon \mathcal{F}_{p-l}^{\Gamma'}(v) \otimes \bigwedge^{l} \mathbb{Z}^{\dim \gamma_\rho} \to \mathcal{F}_{p-l}^{\Gamma'}(v) \otimes \bigwedge^{l} \mathbb{Z}^{\dim \gamma_{\rho'}},$$

where $\pi_{\rho\rho'} \colon \bigwedge^l \mathbb{Z}^{\dim\gamma_\rho} \to \bigwedge^l \mathbb{Z}^{\dim \gamma_{\rho'}}$ is induced by the projection map

$$\pi_{\rho\rho'} \colon T_{\mathbb{Z}}(Y_\rho) \to T_{\mathbb{Z}}(Y_{\rho'})$$

from (2.1). If $\gamma_{\rho'}$ is not a face of γ_ρ, then the map is 0.

Following the description of the cosheaf maps from Lemma 2.15, there are isomorphisms of chain complexes

$$C_\bullet^{BM}\left(\gamma^o; \mathcal{F}_p^X|_{\gamma^o}\right) \cong \bigoplus_{l=0}^{p} C_\bullet^{p,l}.$$

By distributivity of tensor products we also have the isomorphisms

$$C_\bullet^{p,l} \cong \mathcal{F}_{p-l}^{\Gamma'}(v) \otimes C_\bullet^{BM}\left(\gamma^o; \mathcal{F}_l^{\gamma^o}\right).$$

Moreover, the homology of the chain complex $C_\bullet^{BM}(\gamma^o; \mathcal{F}_l^{\gamma^o})$ vanishes except in degree $q = \dim\gamma$ by Lemma 3.6, so we also have $H_q^{BM}(\gamma^o, \mathcal{F}_l^{\gamma^o}) = 0$ for all $q \neq \dim\gamma$. Because the tensor product is right exact, we have $H_q(C_\bullet^{p,l}) = 0$ for $q \neq \dim\gamma$ and all l and p. It now follows that $H_q^{BM}(\gamma^\circ; \mathcal{F}_p^X|_{\gamma^\circ}) = 0$ for $q \neq \dim\gamma$. □

Lemma 3.9. — *Let X be a combinatorially ample n-dimensional non-singular tropical hypersurface in a tropical toric variety Y. For a face γ of X of sedentarity 0 we have*

$$H_q^{BM}(\gamma^\circ; \mathcal{N}_p|_{\gamma^\circ}) = 0$$

for all $q \neq \dim\gamma$.

Proof. — The chain groups $C_q^{BM}(\gamma^\circ; \mathcal{N}_p|_{\gamma^\circ})$ are all zero for $q > \dim\gamma$, therefore it suffices to prove the vanishing of the homology of the cosheaf $\mathcal{N}_p|_{\gamma^\circ}$ in degrees strictly less than $\dim\gamma$. To do this we return to the short exact sequence from (3.2) but restricted to γ^o, namely

$$0 \to \mathcal{F}_p^X|_{\gamma^\circ} \to \mathcal{F}_p^Y|_{\gamma^\circ} \to \mathcal{N}_p|_{\gamma^\circ} \to 0.$$

By proving the vanishing of the appropriate homology groups of the cosheaves $\mathcal{F}_p^X|_{\gamma^\circ}$ and $\mathcal{F}_p^Y|_{\gamma^\circ}$ in order to show the vanishing of the homology groups as stated in the theorem. By Lemma 3.8 we have $H_q^{BM}(\gamma^\circ; \mathcal{F}_p^X|_{\gamma^\circ}) = 0$ for $q < \dim \gamma$.

Next we will show that

$$H_q^{BM}\left(\gamma^\circ; \mathcal{F}_p^Y|_{\gamma^\circ}\right) = 0 \text{ for } q < \dim \gamma\,.$$

By Lemma 3.5, there is a unique maximal cone ρ_m of the fan of Y such that $Y_{\rho_m} \cap \gamma^o \neq \varnothing$. Let $\widetilde{Y}$ be the tropical toric variety of dimension $n+1$ defined by the fan consisting of the single cone ρ_m. There is a correspondence between the strata of γ^o and the strata of $\widetilde{Y}$, where a q-dimensional stratum σ of γ° corresponds to a $(n+1-\dim\gamma+q)$-dimensional stratum $\widetilde{\sigma}$ of $\widetilde{Y}$. Moreover, under this correspondence we have $\mathcal{F}_p^Y|_{\gamma^o}(\sigma) = \mathcal{F}_p^{\tilde{Y}}(\widetilde{\sigma})$. The cellular chain complex $C_\bullet^{BM}(\gamma^\circ; \mathcal{F}_p^Y|_{\gamma^\circ})$ is isomorphic to the chain complex

$$C^{BM}_{\bullet+n+1-\dim\gamma}\left(\widetilde{Y}; \mathcal{F}_p^{\tilde{Y}}\right).$$

By Lemma 3.6 it follows that $H_q^{BM}(\widetilde{Y}; \mathcal{F}_p^{\tilde{Y}}) = 0$ for $q < n+1$ and therefore, $H_q^{BM}(\gamma^\circ; \mathcal{F}_p^Y|_{\gamma^\circ}) = 0$ for $q < \dim\gamma$.

By considering the long exact sequence in homology from the sequence (3.2) restricted to γ^o proves that $H_q(\gamma^o; \mathcal{N}_p|_{\gamma^o}) = 0$ for all $q \neq \dim\gamma$. □

Proposition 3.10. — *Let X be a combinatorially ample non-singular tropical hypersurface of an $n+1$ dimensional non-singular tropical toric variety Y. Then $H_q^{BM}(Y; \mathcal{Q}_p) = 0$ for all $q < n+1$, and therefore the map*

$$H_q^{BM}\left(X; \mathcal{F}_p^Y|_X\right) \to H_q^{BM}\left(Y; \mathcal{F}_p^Y\right)$$

is an isomorphism when $q < n$ and a surjection when $q = n$.

If in addition (Y, X) is a cellular pair and every parent face of a compact face of $Y \backslash X$ is compact, then $H_q(Y; \mathcal{Q}_p) = 0$ for all $q < n+1$, and therefore the map

$$H_q\left(X; \mathcal{F}_p^Y|_X\right) \to H_q\left(Y; \mathcal{F}_p^Y\right)$$

is an isomorphism when $q < n$ and a surjection when $q = n$.

Proof. — We consider the polyhedral structure on Y given by refinement by X. For any face σ of Y which is also a face of X we have $\mathcal{Q}_p(\sigma) = 0$. Therefore we have the following isomorphism of cellular chain complexes,

$$C_\bullet^{BM}(Y; \mathcal{Q}_p) = \bigoplus_{\sigma \in Y\backslash X} \mathcal{F}_p^Y(\sigma). \tag{3.5}$$

When (Y, X) is a cellular pair, the cellular chain groups compute the standard homology by Remark 2.20 and we also have the isomorphism

$$C_\bullet(Y; \mathcal{Q}_p) = \bigoplus_{\substack{\sigma \in Y\backslash X \\ \sigma \text{ compact}}} \mathcal{F}_p^Y(\sigma). \tag{3.6}$$

The complement $Y \backslash X$ consists of connected components each of dimension $n+1$. Each such connected component is equal to γ^o where γ is a $n+1$ dimensional face of Y with polyhedral structure induced by X. For γ a face of Y of dimension $n+1$, there is the equality of cosheaves $\mathcal{F}_p^{\gamma^o} \cong \mathcal{F}_p^Y|_{\gamma^o}$. Each face σ in $Y \backslash X$ is contained in

γ^o for a unique $n+1$-dimensional face γ of Y. Moreover, the boundary of the face σ contained in γ^o is also contained in γ^o. Therefore, the cellular chain complexes for $\mathcal{Q}_p$ split and we have the following isomorphisms,

$$C^{BM}_\bullet(Y;\mathcal{Q}_p) = \bigoplus_{\dim\gamma=n+1} C^{BM}_\bullet\left(\gamma^o;\mathcal{F}_p^{\gamma^o}\right) \tag{3.7}$$

and

$$C_\bullet(Y;\mathcal{Q}_p) = \bigoplus_{\substack{\dim\gamma=n+1\\ \gamma \text{ compact}}} C^{BM}_\bullet\left(\gamma^o;\mathcal{F}_p^{\gamma^o}\right). \tag{3.8}$$

This produces the isomorphisms

$$H^{BM}_q(Y;\mathcal{Q}_p) = \bigoplus_{\dim\gamma=n+1} H^{BM}_q\left(\gamma^o;\mathcal{F}_p^{\gamma^o}\right)$$

and

$$H_q(Y;\mathcal{Q}_p) = \bigoplus_{\substack{\dim\gamma=n+1\\ \gamma \text{ compact}}} H^{BM}_q\left(\gamma^o;\mathcal{F}_p^{\gamma^o}\right).$$

It follows from Lemma 3.6 that $H^{BM}_q(\gamma^o;\mathcal{F}_p^{\gamma^o}) = 0$ if $q \neq n+1$, and we obtain that $H_q(Y;\mathcal{Q}_p) = H^{BM}_q(Y;\mathcal{Q}_p) = 0$ for all $q < n+1$.

A direct comparison of the respective chain complexes gives isomorphisms

$$H_q\left(Y;\mathcal{F}_p^Y|_X\right) \cong H_q\left(X;\mathcal{F}_p^Y|_X\right) \text{ and } H^{BM}_q\left(Y;\mathcal{F}_p^Y|_X\right) \cong H^{BM}_q\left(X;\mathcal{F}_p^Y|_X\right).$$

Lastly, combining this with the long exact sequence in homology associated to the short exact sequence (3.1) and the vanishing of $H^{BM}_q(Y;\mathcal{Q}_p)$ and $H_q(Y;\mathcal{Q}_p)$ for all $q < n+1$ proves the statement of the proposition. □

Proposition 3.11. — *Let X be a combinatorially ample non-singular n-dimensional tropical hypersurface in a $n+1$ dimensional non-singular tropical toric variety Y. Then*

$$H^{BM}_q(X;\mathcal{N}_p) = 0$$

for all $p+q \leqslant n$, and therefore the map

$$H^{BM}_q\left(X;\mathcal{F}_p^X\right) \to H^{BM}_q\left(X;\mathcal{F}_p^Y|_X\right)$$

is an isomorphism when $p+q<n$ and a surjection when $p+q=n$.

If in addition (Y,X) is a cellular pair and every parent face of a compact face of X is compact, then $H_q(X;\mathcal{N}_p) = 0$ for all $p+q \leqslant n$, and therefore the map

$$H_q\left(X;\mathcal{F}_p^X\right) \to H_q\left(X;\mathcal{F}_p^Y|_X\right)$$

is an isomorphism when $p+q<n$ and a surjection when $p+q=n$.

Proof. — By Lemma 3.7, for a face σ of dimension q and sedentarity k, we have $\mathcal{N}_p(\sigma) = 0$ if $k < n-q-p+1$. Moreover, by assumption X is proper in Y so there are no faces of X of dimension q and which have order of sedentarity strictly greater than $n-q$. Therefore, the Borel–Moore cellular chain groups with coefficients in $\mathcal{N}_p$ can be written as,

$$C_q^{BM}(X;\mathcal{N}_p) := \bigoplus_{k=\max\{0,n-p-q+1\}}^{n-q} \bigoplus_{\substack{\dim\sigma=q\\ \mathrm{sed}(\sigma)=k}} \mathcal{N}_p(\sigma). \tag{3.9}$$

Perform the change of variables $k+q=m$:

$$C_q^{BM}(X;\mathcal{N}_p) := \bigoplus_{m=\max\{q,n-p+1\}}^{n} \bigoplus_{\substack{\dim\sigma=q\\ \mathrm{sed}(\sigma)=m-q}} \mathcal{N}_p(\sigma). \tag{3.10}$$

If in addition (Y,X) is a cellular pair, by Remark 2.20 the cellular chain complexes compute the standard homology of X and we also have the isomorphism

$$C_q(X;\mathcal{N}_p) := \bigoplus_{m=\max\{q,n-p+1\}}^{n} \bigoplus_{\substack{\dim\sigma=q\\ \mathrm{sed}(\sigma)=m-q\\ \sigma\text{ compact}}} \mathcal{N}_p(\sigma). \tag{3.11}$$

We now filter the cellular chain complex for $\mathcal{N}_p$ using the order of sedentarity of faces. Set,

$$C_{q,m}^{BM}(X;\mathcal{N}_p) := \bigoplus_{\substack{\dim\sigma=q\\ \mathrm{sed}(\sigma)\leqslant m-q}} \mathcal{N}_p(\sigma) \quad\text{and}\quad C_{q,m}(X;\mathcal{N}_p) := \bigoplus_{\substack{\dim\sigma=q\\ \mathrm{sed}(\sigma)\leqslant m-q\\ \sigma\text{ compact}}} \mathcal{N}_p(\sigma)$$

Notice that $C_{q,m}^{\blacksquare}(X;\mathcal{N}_p) \subset C_{q,m+1}^{\blacksquare}(X;\mathcal{N}_p)$, where the $\blacksquare$ in the exponent denotes either Borel–Moore or standard homology.

Since X intersects the boundary of Y properly, the boundary operator can only increase the order of sedentarity by at most 1. Therefore,

$$\partial C_{q,m}^{\blacksquare}(X;\mathcal{N}_p) \subset C_{q-1,m}^{\blacksquare}(X;\mathcal{N}_p),$$

and there is a filtration of the chain complex $C_\bullet^{\blacksquare}(X;\mathcal{N}_p)$:

$$C_\bullet^{\blacksquare}(X;\mathcal{N}_p) = C_{\bullet,n}^{\blacksquare}(X;\mathcal{N}_p) \supset C_{\bullet,n-1}^{\blacksquare}(X;\mathcal{N}_p) \supset \cdots \supset C_{\bullet,\tilde m}^{\blacksquare}(X;\mathcal{N}_p) \supset 0,$$

where $\tilde m = \max\{q, n-p+1\}$. The first and last terms of the filtration come from the bounds on the direct sum in Equation (3.10).

The spectral sequence associated to this filtration for the Borel–Moore complex has 0^{th} page consisting of the terms

$$E_{q,m}^0 \cong \bigoplus_{\substack{\dim\sigma=q\\ \mathrm{sed}(\sigma)=m-q}} \mathcal{N}_p(\sigma). \tag{3.12}$$

The differentials $\partial_0\colon E_{q,m}^0 \to E_{q-1,m}^0$ are induced by the usual cellular differentials. The complex $E_{\bullet,m}^0$ is then

$$0 \to E_{m,m}^0 \to E_{m-1,m}^0 \to \ldots \to E_{1,m}^0 \to E_{0,m}^0 \to 0. \tag{3.13}$$

Notice that the differential ∂_0 decreases the dimensions of the cells by one and also increases the sedentarity of the cell by one. A q-dimensional face of sedentarity $m-q$ is in the boundary of a unique face γ of X of dimension m and sedentarity 0. Moreover, the differential ∂_0 is defined on the direct summands from (3.12) and it restricts to a non-zero map $\mathcal{N}_p(\sigma) \to \mathcal{N}_p(\sigma')$ if and only if σ and σ' are contained in the same m-dimensional sedentarity 0 face γ of X. In this case, the map $\mathcal{N}_p(\sigma) \to \mathcal{N}_p(\sigma')$ is the same as the one defined in (3.4) for the complex $C^{BM}_\bullet(\gamma^\circ; \mathcal{N}_p|_{\gamma^o})$. Therefore, we have an isomorphism of complexes for every m:

$$E^0_{\bullet,m} = \bigoplus_{\substack{\dim\gamma=m\\ \mathrm{sed}(\gamma)=0}} C^{BM}_\bullet(\gamma^\circ; \mathcal{N}_p|_{\gamma^o}).$$

By Lemma 3.9, for a face γ of dimension m and sedentarity 0, we have $H^{BM}_q(\gamma^\circ; \mathcal{N}_p|_{\gamma^\circ}) = 0$ for $q \neq m$, and the second page of the spectral sequence associated to the filtration under consideration satisfies $E^1_{q,m} = 0$ if $q \neq m$. Moreover, for $m \leqslant n-p$ the entire complex $E^0_{\bullet,m}$ is 0 by Lemma 3.7, so $E^1_{q,m} = 0$ for all q when $m \leqslant n-p$.

The differentials at the r^{th} page of the spectral sequence are given by $\partial_r \colon E^r_{q,m} \to E^r_{q-1,m+r}$. Therefore, the spectral sequence $E^\bullet_{\bullet,\bullet}$ satisfies $E^r_{q,m} = 0$ for any $r \geqslant 1$ and $q \leqslant n-p$. Since $E^\bullet_{\bullet,\bullet}$ converges, we conclude that $H^{BM}_q(X; \mathcal{N}_p) = 0$ for $p+q \leqslant n$.

To obtain the analogous statement for $H_q(X; \mathcal{N}_p)$, consider the spectral sequence associated to the filtration of the chain complex for the standard homology. The first page of this spectral sequence has terms like in Equation (3.12), except that the sum is taken over the faces σ which are compact. In order to proceed with a similar argument to that used for Borel–Moore homology, we require the assumption that if σ is compact then the unique face γ of X of sedentarity 0 which contains σ is also compact. Then the rest of the argument is the same as in the case of the Borel–Moore homology except we restrict to only compact faces of X.

To complete the proof of the proposition, consider the long exact sequence in homology associated to the short exact sequence in (3.2). Applying the vanishing statements for $H^{BM}_q(X; \mathcal{N}_p)$ gives the isomorphisms $H^{BM}_q(X; \mathcal{F}^X_p) \cong H^{BM}_q(Y; \mathcal{F}^Y_p)$ for all $p+q < n$. This completes the proof of Proposition 3.11. ☐

Theorem 1.1, which we state again, is now a trivial consequence of the two previous propositions.

Theorem (Theorem 1.1). — *Let X be a non-singular and combinatorially ample tropical hypersurface of an $n+1$ dimensional non-singular tropical toric variety Y. Then the map induced by inclusion*

$$i_*\colon H^{BM}_q\left(X; \mathcal{F}^X_p\right) \to H^{BM}_q\left(Y; \mathcal{F}^Y_p\right)$$

is an isomorphism when $p+q<n$ and a surjection when $p+q=n$.

If additionally, the pair (Y, X) is a cellular pair and every parent face of a compact face of X is compact, then the map induced by inclusion

$$i_*\colon H_q\left(X; \mathcal{F}^X_p\right) \to H_q\left(Y; \mathcal{F}^Y_p\right)$$

is an isomorphism when $p+q<n$ and a surjection when $p+q=n$.

Proof. — The proof of the theorem follows by combining the statements in Propositions 3.10 and 3.11. ☐

We now present the proof of the Lefschetz section theorem for the tropical homology groups with real coefficients of tropical hypersurfaces which are not necessarily non-singular, but are still proper in a non-singular tropical toric variety.

Theorem (Theorem 1.2). — *Let X be a combinatorially ample tropical hypersurface of an $n+1$ dimensional non-singular tropical toric variety Y such that X is proper in Y. Then the maps induced by inclusion*

$$i_*\colon H_q^{BM}\left(X;\mathcal{F}_p^X\otimes\mathbb{R}\right)\to H_q^{BM}\left(Y;\mathcal{F}_p^Y\otimes\mathbb{R}\right)$$

are isomorphisms when $p+q<n$ and surjections when $p+q=n$. If additionally, the pair (Y,X) is a cellular pair and every parent face of a compact face of Y is compact, then the maps induced by inclusion

$$i_*\colon H_q\left(X;\mathcal{F}_p^X\otimes\mathbb{R}\right)\to H_q\left(Y;\mathcal{F}_p^Y\otimes\mathbb{R}\right)$$

are isomorphisms when $p+q<n$ and surjections when $p+q=n$.

Proof. — The proof follows the same strategy as the proof of the Lefschetz theorems for the integral tropical homology groups. First we tensor the two exact sequences of $\mathbb{Z}$-module cosheaves from (3.1) and (3.2) with $\mathbb{R}$ to obtain two exact sequences of cosheaves of $\mathbb{R}$-vector spaces.

The vanishing of the homology groups $H_q(Y;\mathcal{Q}_p)$ from Proposition 3.10 still holds for $H_q(Y;\mathcal{Q}_p\otimes\mathbb{R})$. The chain groups still decompose as in the proof over $\mathbb{Z}$, namely

$$C_\bullet^{BM}\left(Y;\mathcal{Q}_p\otimes\mathbb{R}\right)=\bigoplus_{\dim\gamma=n+1}C_\bullet^{BM}\left(\gamma^o;\mathcal{F}_p^{\gamma^o}\otimes\mathbb{R}\right)\tag{3.14}$$

and

$$C_\bullet\left(Y;\mathcal{Q}_p\otimes\mathbb{R}\right)=\bigoplus_{\substack{\dim\gamma=n+1\\ \gamma\text{ compact}}}C_\bullet^{BM}\left(\gamma^o;\mathcal{F}_p^{\gamma^o}\otimes\mathbb{R}\right).\tag{3.15}$$

Since Y is non-singular and X is combinatorially ample in Y for each $n+1$ dimensional face γ of Y we also have isomorphisms of chain complexes

$$C_\bullet^{BM}\left(\gamma^o;\mathcal{F}_p^{\gamma^o}\otimes\mathbb{R}\right)\cong C_\bullet^{BM}\left(\mathbb{R}^m\times\mathbb{T}^{n+1-m};\mathcal{F}_p^{\mathbb{R}^m\times\mathbb{T}^{n+1-m}}\otimes\mathbb{R}\right)$$

as in the proof of Lemma 3.6. The homology groups of the complex on the right hand side of the above isomorphism vanish as in the integral case by [JSS19, Section 4].

We claim that a variant of Proposition 3.11 holds for the cosheaf $\mathcal{N}_p\otimes\mathbb{R}$. In order to prove this we describe the dimensions of the vector spaces $\mathcal{F}_p(\sigma)$ when X is a tropical hypersurface. Consider the polyhedral decomposition of Y induced by X, and let v be a vertex of X of sedentarity 0. Then v is contained in some $n+1$ dimensional face γ of this polyhedral decomposition of Y. For $p\leqslant n$ we have

$$\mathcal{F}_p(v)\otimes\mathbb{R}=\sum_{\substack{v\subset\sigma\subset\gamma\\ \dim\sigma=n}}\bigwedge^p T_Y(\sigma),$$

where in the sum the faces of σ are faces of X. The number of faces σ of dimension n is at least $n+1$. Up to a linear transformation, we can assume that the first $n+1$ hyperplanes are the standard hyperplanes $x_i=0$ in $\mathbb{R}^{n+1}$. And then for $p\leqslant n$ we have

$$\mathcal{F}_p(v)\otimes\mathbb{R}=\sum_{\substack{v\subset\sigma\subset\gamma\\ \dim\sigma=n}}\bigwedge^p T_Y(\sigma)=\bigwedge^p\mathbb{R}^{n+1}.$$

For any $p \leqslant n$ we have $\mathcal{F}_p^X(v) \otimes \mathbb{R} = \bigwedge^p \mathbb{R}^{n+1}$, and for $p > n$ we have $\mathcal{F}_p^X(v) \otimes \mathbb{R} = 0$. Therefore, we have

$$\chi_v^{\mathbb{R}}(\lambda) := \sum_{p=0}^{n} (-1)^p \dim \left(\mathcal{F}_p^X(v) \otimes \mathbb{R}\right) \lambda^p = (1-\lambda)^{n+1} - (-\lambda)^{n+1}.$$

We can repeat the above argument for vertices of X of non-zero sedentarity and also apply the same argument for the Künneth type formula from Lemma 2.15. Therefore, if τ is a face of X of dimension q whose relative interior is contained in a stratum Y_ρ of dimension m, we obtain

$$\mathcal{F}_p^X(\tau) \cong \bigoplus_{l=0}^{p} \mathcal{F}_{p-l}^{H_{m-q-1}}(v) \otimes \bigwedge^l T_{\mathbb{Z}}(\tau),$$

so that, as in Corollary 2.16, we have

$$\chi_\tau(\lambda) = (1-\lambda)^s - (1-\lambda)^q (-\lambda)^{m-q}.$$

This description enables us to conclude that Lemma 3.7 holds for $\mathcal{N}_p \otimes \mathbb{R}$. Similarly the proofs of Lemmas 3.8 and 3.9 as well as Proposition 3.11 hold for a arbitrary hypersurface when using $\mathbb{R}$ coefficients. Then the proof of the Theorem 1.2 is completed in the same way as the proof of Theorem 1.1. □

4. The tropical homology of hypersurfaces is torsion free

We start this section with the proof of Theorem 1.3, which uses the Lefschetz section theorem for the integral homology of a non-singular tropical hypersurface. This proposition establishes that the integral tropical homology groups of the hypersurface are also torsion free if the integral tropical homology groups of the tropical toric variety are as well.

Theorem (Theorem 1.3). — *Let X be a non-singular and combinatorially ample tropical hypersurface in a non-singular tropical toric variety Y such that (Y, X) is a cellular pair and every parent face of a compact face of Y is compact. If the tropical homology groups $H_q(Y; \mathcal{F}_p)$ are torsion free for all p and q, then both the Borel–Moore and standard tropical homology groups of X are also torsion free.*

Proof. — Let X be a non-singular tropical hypersurface of a tropical toric variety Y such that the standard tropical homology of Y is torsion free.

By the universal coefficient theorem for cohomology [Hat02, Theorem 3.2] for every p and q we have the following short exact sequence:

$$\begin{aligned} 0 \to \operatorname{Ext}\left(H_{n-q-1}\left(X; \mathcal{F}_{n-p}^X\right), \mathbb{Z}\right) &\to H^{n-q}\left(X; \mathcal{F}_X^{n-p}\right) \\ &\to \operatorname{Hom}\left(H_{n-q}\left(X; \mathcal{F}_{n-p}^X\right), \mathbb{Z}\right) \to 0. \end{aligned}$$

Notice that the cohomology of the sheaf $\mathcal{F}_X^{n-p}$ appears in the middle term because

$$C^q\left(X; \mathcal{F}_X^p\right) \cong \operatorname{Hom}\left(C_q\left(X; \mathcal{F}_q^X\right), \mathbb{Z}\right).$$

If $p+q \geqslant n$, then $2n-p-q-1 < n$, and it follows from Theorem 1.1 that

$$H_{n-q-1}\left(X;\mathcal{F}^X_{n-p}\right) \cong H_{n-q-1}\left(Y,\mathcal{F}^Y_{n-p}\right).$$

Since $H_{n-q-1}(Y;\mathcal{F}^Y_{n-p})$ is a free $\mathbb{Z}$-module by hypothesis, we conclude that

$$\operatorname{Ext}\left(H_{n-q-1}\left(X;\mathcal{F}^X_{n-p}\right),\mathbb{Z}\right) = 0.$$

Also the $\mathbb{Z}$-module $\operatorname{Hom}(H_{n-q}(X;\mathcal{F}^X_{n-p}),\mathbb{Z})$ is free since it consists of maps to a free module. Therefore, for all $p+q \geqslant n$ we have

$$H^{n-q}\left(X;\mathcal{F}_X^{n-p}\right) \cong \operatorname{Hom}\left(H_{n-q}\left(X;\mathcal{F}^X_{n-p}\right),\mathbb{Z}\right)$$

and $H^{n-q}(X;\mathcal{F}_X^{n-p})$ is torsion free. The tropical hypersurface X is a non-singular tropical manifold, so by Poincaré duality for tropical homology with integral coefficients from [JRS18] we have

$$H^{n-q}\left(X;\mathcal{F}_X^{n-p}\right) \cong H_q^{BM}\left(X;\mathcal{F}^X_p\right)$$

for all p, q. This, combined with the torsion freeness of $H^{n-q}(X;\mathcal{F}_X^{n-p})$ established above, proves that $H_q^{BM}(X;\mathcal{F}^X_p)$ is torsion free for all $p+q \geqslant n$.

Notice that applying the above argument to the tropical homology of Y shows that if the groups $H_q(Y,\mathcal{F}_p)$ are torsion free for all p and q, then $H_q^{BM}(Y,\mathcal{F}_p)$ are also torsion free for all p and q. It follows from this and Theorem 1.1, that $H_q^{BM}(X;\mathcal{F}^X_p)$ is torsion free for $p+q<n$, so the Borel–Moore tropical homology groups of X are all torsion free.

To prove that the standard tropical homology groups of X are torsion free, we use the universal coefficient theorem for cohomology with compact support. For every p and q we have the following short exact sequence

$$0 \to \operatorname{Ext}\left(H_{q-1}^{BM}\left(X;\mathcal{F}^X_p\right),\mathbb{Z}\right) \to H^q_c\left(X;\mathcal{F}_X^p\right) \\ \to \operatorname{Hom}\left(H_q^{BM}\left(X;\mathcal{F}^X_p\right),\mathbb{Z}\right) \to 0.$$

Since $H_q^{BM}(Y,\mathcal{F}_p)$ are torsion free for all p and q, it follows from Theorem 1.1 that $H_q^{BM}(X,\mathcal{F}_p)$ are torsion free for all $p+q<n$. Then the $\mathbb{Z}$-modules $H_q^{BM}(X;\mathcal{F}^X_p)$ are torsion free for all p and q, and the $\mathbb{Z}$-modules $H^q_c(X;\mathcal{F}^p_X)$ are also torsion free for all p and q. Applying again Poincaré duality, we have

$$H^q_c\left(X;\mathcal{F}^p_X\right) \cong H_{n-q}\left(X;\mathcal{F}^X_{n-p}\right),$$

and $H_q(X;\mathcal{F}^X_p)$ are also torsion free for all p and q. □

We now establish that the integral tropical homology groups of a compact tropical toric variety are torsion free. For a non-singular compact complex toric variety $Y_{\mathbb{C}}$, we let $h^{p,q}(Y_{\mathbb{C}})$ denote its $(p,q)^{\text{th}}$ Hodge number. Recall that $h^{p,q}(Y_{\mathbb{C}}) = 0$ if $p \neq q$ and the numbers $h^{p,p}(Y_{\mathbb{C}})$ form the toric h-vector of the simple polytope Δ whose normal fan is the fan defining $Y_{\mathbb{C}}$ [Ful93, Section 5.2].

Proposition 4.1. — *The integral tropical homology groups of a non-singular compact tropical toric variety Y are torsion free. Moreover, we have*

$$\operatorname{rank} H_q\left(Y;\mathcal{F}^Y_p\right) = h^{p,q}\left(Y_{\mathbb{C}}\right)$$

where $Y_{\mathbb{C}}$ is the corresponding non-singular compact complex toric variety. In particular, we have $H_q(Y;\mathcal{F}_p^Y)=0$ unless $p=q$.

Proof. — We now switch to computing the cellular homology groups of Y using the polyhedral structure on Y which is dual to the polyhedral structure on the defining fan Σ. Notice that every stratum $\overline{Y}_\sigma$ is compact. Let us first show that $H_q(Y;\mathcal{F}_p^Y)=0$ for all $p>q$. With this cellular structure on Y, a face $\overline{Y}_\sigma$ of dimension q has sedentarity order $n+1-q$ where $\dim Y=n+1$. By Definition 2.11, we have that $\mathcal{F}_p^Y(\overline{Y}_\sigma)=\bigwedge^p\mathcal{F}_1^Y(\overline{Y}_\sigma)$ where $\dim\mathcal{F}_1^Y(\overline{Y}_\sigma)=q$. Therefore, we have $\mathcal{F}_p^Y(\overline{Y}_\sigma)=0$ if $p>q$. Hence the chain groups $C_q(Y;\mathcal{F}_p^Y)$ are equal to zero for any $q<p$, which implies that $H_q(Y;\mathcal{F}_p^Y)=0$ for $q<p$.

Recall by Remark 2.22 that the tropical cohomology groups are the cohomology of the complex dual to the tropical cellular complexes. Therefore we can apply the universal coefficient theorem for cohomology [Hat02, Theorem 3.2] to get the exact sequence

$$\begin{aligned}(4.1)\qquad 0\to \operatorname{Ext}\left(H_q\left(Y;\mathcal{F}_p^Y\right),\mathbb{Z}\right)&\to H^{q+1}\left(Y;\mathcal{F}_Y^p\right)\\ &\to \operatorname{Hom}\left(H_{q+1}\left(Y;\mathcal{F}_p^Y\right),\mathbb{Z}\right)\to 0.\end{aligned}$$

When $q<p$ we have $H_q(Y;\mathcal{F}_p^Y)=0$, so there is the isomorphism

$$H^{q+1}\left(Y;\mathcal{F}_Y^p\right)\cong \operatorname{Hom}(H_{q+1}(Y;\mathcal{F}_p^Y),\mathbb{Z}).$$

The tropical toric variety Y is a tropical manifold, thus Poincaré duality for tropical homology with integral coefficients from [JRS18] states that

$$H^{q+1}\left(Y;\mathcal{F}_Y^p\right)\cong H_{n-q}\left(Y;\mathcal{F}_{n+1-p}^Y\right).$$

If $q\geqslant p$, then $n-q<n+1-p$ and applying the isomorphism above we obtain

$$H^{q+1}\left(Y;\mathcal{F}_Y^p\right)=H_{n-q}\left(Y;\mathcal{F}_{n+1-p}^Y\right)=0.$$

This means that

$$\operatorname{Tor}\left(H_q\left(Y;\mathcal{F}_p^Y\right)\right)=\operatorname{Ext}\left(H_q\left(Y;\mathcal{F}_p^Y\right),\mathbb{Z}\right)=0,$$

and so $H_q(Y;\mathcal{F}_p^Y)$ is torsion-free for all $q\geqslant p$ and thus for all p,q. We also see from the sequence in (4.1) that $H_q(Y;\mathcal{F}_p^Y)=0$ for all $q\neq p$.

All of the chain groups for the cellular tropical homology of Y are also free so we have

$$\chi\left(C_\bullet\left(Y;\mathcal{F}_p^Y\right)\right):=\sum_{q=0}^{n+1}(-1)^q\operatorname{rank}C_q\left(Y;\mathcal{F}_p^Y\right)=(-1)^p\operatorname{rank}H_p\left(Y;\mathcal{F}_p^Y\right).$$

Let f_q denote the number of strata of Y of dimension q. Then $(f_0,\ldots,f_{n+1})$ is the f-vector of a polytope P_Y whose normal fan is the fan defining Y. Then for every p and q we have $\operatorname{rank}C_q(Y;\mathcal{F}_p^Y)=\binom{q}{p}f_q$. Therefore,

$$\chi\left(C_\bullet\left(Y;\mathcal{F}_p^Y\right)\right):=\sum_{q=0}^{n+1}(-1)^q\binom{q}{p}f_q=(-1)^p h_p,$$

where $(h_0, \ldots, h_{n+1})$ is the h-vector of the simple polytope P_Y. By [Ful93, Section 5.2], we have $h_p = \dim H^{2p}(Y_\mathbb{C}) = h_{p,p}(Y_\mathbb{C})$ which completes the proof of Proposition 4.1. □

The following example shows that a tropical toric variety, which is not assumed to be non-singular, may have torsion in its tropical homology groups.

Example 4.2. — The fan producing the simplest classical singular toric variety, produces a tropical toric variety with torsion in its tropical homology groups. Let Y be defined by a single cone in $\mathbb{R}^2$ having rays $(-1,0)$ and $(-1,-2)$. Then the chain complex $0 \to C_2^{BM}(Y;\mathcal{F}_1^Y) \to C_1^{BM}(Y;\mathcal{F}_1^Y) \to 0$ consists of terms

$$C_2^{BM}\left(Y;\mathcal{F}_1^Y\right) = \langle e_1, e_2\rangle \quad \text{and} \quad C_1^{BM}\left(Y;\mathcal{F}_1^Y\right) \cong \frac{\mathbb{Z}^2}{< e_1 >} \oplus \frac{\mathbb{Z}^2}{< (-1,-2) >},$$

where the differentials are the direct sums of the projection maps. Then under the differential we have $e_1 \mapsto (0,2)$ and $e_2 \mapsto (1,-1)$ and the image of the differential is a proper sublattice of rank 2 of $C_1^{BM}(Y;\mathcal{F}_1^Y)$. In fact we have $H_1^{BM}(Y;\mathcal{F}_1) = \mathbb{Z}_2$.

Corollary (Corollary 1.4). — *If Y is a compact non-singular tropical toric variety and X is a combinatorially ample non-singular tropical hypersurface in Y, then all integral tropical homology groups of X are torsion free.*

Proof. — By Proposition 4.1, if Y is compact, all its integral tropical homology groups are torsion free. Then by Theorem 1.3, all the integral tropical homology groups of X are torsion free. □

Corollary (Corollary 1.5). — *Let Y be a non-singular tropical toric variety associated to a fan whose support is a convex cone and such that the complex toric variety $Y_\mathbb{C}$ is quasi-projective. Let X be a combinatorially ample non-singular tropical hypersurface in Y such that (Y,X) is a cellular pair and every parent face of a compact face of Y is compact. Then both the standard and Borel–Moore integral tropical homology groups of X are torsion free.*

Proof. — Assume that the convex cone supporting the fan of Y is full dimensional in $\mathbb{R}^{n+1}$. We will first show that the tropical toric variety Y equipped with the polyhedral structure dual to the polyhedral structure on its defining fan is a regular CW-complex. Thus the cellular tropical chain complexes can compute the standard and Borel–Moore homology groups of Y. To prove this claim, consider $Y_\mathbb{C}$, the quasi-projective toric variety associated to a fan Σ. Let D be any ample Cartier divisor on $Y_\mathbb{C}$ and consider the associated polyhedron P (see for example [Ful93, Chapter 3]). The hypothesis on the support of Σ implies that it is the normal fan of P ([Mus04, Chapter 6]). Therefore, the polyhedron P is combinatorially isomorphic to Y, the tropical toric variety associated to Σ. Since P is a polyhedron, it is a cell-complex in the sense of [Cur14, Chapter 4], and one can use the cellular description to compute the standard homology groups of Y.

As in the proof of Proposition 4.1, both standard and Borel–Moore tropical homology groups of Y vanish if $p > q$. It follows again from Poincaré duality and universal coefficients theorem that both standard and Borel–Moore tropical homology groups of Y are torsion free. The statement for X follows again from Theorem 1.3. Now suppose that the convex cone supporting the fan Σ is of codimension s in $\mathbb{R}^{n+1}$. Then the tropical toric variety Y is a product $\mathbb{R}^s \times Y'$ where Y' is a tropical toric variety of dimension $n+1-s$ satisfying the assumptions above. The tropical toric variety Y' is then combinatorially isomorphic to a polyhedron P'. By the Künneth formula for Borel–Moore tropical homology [GS19, Theorem B] we have

$$H_q^{BM}\left(Y;\mathcal{F}_p^Y\right) = \bigoplus_{\substack{i+j=p\\k+l=q}} H_k^{BM}\left(\mathbb{R}^s;\mathcal{F}_i^{\mathbb{R}^s}\right) \otimes H_l^{BM}\left(Y';\mathcal{F}_j^{Y'}\right).$$

Therefore, the Borel–Moore tropical homology groups of Y are all torsion free and thus so are the standard tropical homology groups. This completes the proof of Corollary 1.5. ☐

5. Betti numbers of tropical homology and Hodge numbers

The k-compactly supported cohomology group of a complex hypersurface $X_{\mathbb{C}} \subset (\mathbb{C}^*)^{n+1}$ carries a mixed Hodge structure, see [DK86]. The numbers $e_c^{p,q}(X_{\mathbb{C}})$ are defined to be

$$e_c^{p,q}(X_{\mathbb{C}}) := \sum_k (-1)^k h^{p,q}\left(H_c^k(X_{\mathbb{C}})\right),$$

where $h^{p,q}(H_c^k(X_{\mathbb{C}}))$ denote the Hodge–Deligne numbers of $X_{\mathbb{C}}$. The numbers $e_c^{p,q}$ $(X_{\mathbb{C}})$ are the coefficients of the E-polynomial of $X_{\mathbb{C}}$,

$$E\left(X_{\mathbb{C}};u,v\right) := \sum_{p,q} e_c^{p,q}(X_{\mathbb{C}})u^p v^q.$$

The χ_y genus of $X_{\mathbb{C}}$ is defined to be

$$\chi_y(X_{\mathbb{C}}) = E(X_{\mathbb{C}};y,1) := \sum_{p,q} e_c^{p,q}(X_{\mathbb{C}})y^p.$$

Theorem 1.8 relates the coefficients of the χ_y genus and the Euler characteristics of the chain complexes $C_\bullet^{BM}(X;\mathcal{F}_p)$. For the proof of the theorem we require the notion of torically non-degenerate complex hypersurfaces.

Definition 5.1. — *If $Y_{\mathbb{C}}$ is a complex toric variety, a hypersurface $X_{\mathbb{C}} \subset Y_{\mathbb{C}}$ is* torically non-degenerate *if the intersection of $X_{\mathbb{C}}$ with any torus orbit of $Y_{\mathbb{C}}$ is non-singular and $X_{\mathbb{C}}$ intersects each torus orbit of $Y_{\mathbb{C}}$ transversally. If $Y_{\mathbb{C}}$ is the complex toric variety associated to the Newton polytope of $X_{\mathbb{C}}$, then the second condition follows from the first one (see for example* [Kho77]*).*

Theorem (Theorem 1.8). — *Let X be an n-dimensional non-singular tropical hypersurface in a non-singular tropical toric variety Y. Let $X_{\mathbb{C}}$ be a complex hypersurface torically non-degenerate in the complex toric variety $Y_{\mathbb{C}}$ such that X and $X_{\mathbb{C}}$ have the same Newton polytope. Then*

$$(-1)^p \chi\left(C_\bullet^{BM}\left(X;\mathcal{F}_p^X\right)\right) = \sum_{q=0}^{n} e_c^{p,q}(X_{\mathbb{C}}),$$

and thus

$$\chi_y(X_{\mathbb{C}}) = \sum_{p=0}^{n}(-1)^p \chi\left(C_\bullet^{BM}\left(X; \mathcal{F}_p^X\right)\right) y^p.$$

Proof. — Firstly, the variety $X_{\mathbb{C}}$ is stratified by its intersection with the open torus orbits of $Y_{\mathbb{C}}$. Moreover, the numbers $e_c^{p,q}(X_{\mathbb{C}})$ are additive along strata by [DK86, Proposition 1.6]. So we have

$$\sum_{q=0}^{n} e_c^{p,q}(X_{\mathbb{C}}) = \sum_{\rho}\sum_{q=0}^{n} e_c^{p,q}(X_{\mathbb{C},\rho})$$

for $X_{\mathbb{C}} = \sqcup_\rho X_{\mathbb{C},\rho}$, where $X_{\mathbb{C},\rho} := X_{\mathbb{C}} \cap Y_{\mathbb{C},\rho}$ and $Y_{\mathbb{C},\rho}$ is the open torus orbit corresponding to the face ρ of the fan Σ defining Y and $Y_{\mathbb{C}}$.

The tropical hypersurface X admits a stratification analogous to $X_{\mathbb{C}}$. The Euler characteristics of the chain complexes for cellular tropical Borel–Moore homology of X satisfy the same additivity property. Namely,

$$\chi\left(C_\bullet^{BM}\left(X; \mathcal{F}_p^X\right)\right) = \sum_\rho \chi\left(C_\bullet^{BM}\left(X_\rho; \mathcal{F}_p^{X_\rho}\right)\right).$$

Moreover, for any face ρ of the fan Σ defining Y and $Y_{\mathbb{C}}$, the Newton polytope of $X_{\mathbb{C},\rho}$ is equal to the Newton polytope of X_ρ. In fact, since X is proper in Y and $X_{\mathbb{C}}$ intersects the boundary of $Y_{\mathbb{C}}$ properly, it is enough to prove it for ρ a ray of Σ and then proceed by recurrence. Up to a toric change of coordinates, one can assume that ρ is a ray in direction $e_1 = (1, 0, \cdots, 0)$. Then the hypersurface $X_{\mathbb{C},\rho}$ is given by the polynomial $f^{\mathbb{C}}(0, x_2, \cdots, x_{n+1})$, where $f^{\mathbb{C}}$ is the polynomial defining $X_{\mathbb{C}}$. Similarly the tropical polynomial of X_ρ is obtained from the tropical polynomial of X by removing all monomials containing x_1. So, the fact that X and $X_{\mathbb{C}}$ have the same Newton polytope implies that $X_{\mathbb{C},\rho}$ and X_ρ do as well. Therefore, it suffices to prove the statement for $X \subset \mathbb{R}^{n+1}$ and $X_{\mathbb{C}} \subset (\mathbb{C}^*)^{n+1}$.

We now assume that X is in $\mathbb{R}^{n+1}$ and $X_{\mathbb{C}}$ is in $(\mathbb{C}^*)^{n+1}$. In [KS16, Section 5.2], Katz and Stapeldon give a formula for the χ_y genus of a torically non-degenerate hypersurface in the torus. Their formula utilizes regular subdivisions of polytopes to refine the formula in terms of Newton polytopes of Danilov and Khovanskii [DK86]. Note that they use the term schön in exchange for torically non-degenerate. Let Δ be the Newton polytope for $X_{\mathbb{C}}$ and $\widetilde{\Delta}$ a regular subdivision of the lattice polytope Δ. Then the formula is

$$\chi_y(X_{\mathbb{C}}) = \sum_{\substack{F \subset \tilde{\Delta} \\ F \not\subset \partial\Delta}} (-1)^{n+1-\dim F} \chi_y(X_{\mathbb{C},F}), \tag{5.1}$$

where $X_{\mathbb{C},F}$ is the hypersurface in the torus $(\mathbb{C}^*)^{n+1}$ defined by the polynomial obtained by restricting the polynomial defining $X_{\mathbb{C}}$ to the monomials corresponding to the lattice points in the face F of $\widetilde{\Delta}$. Notice our description of $X_{\mathbb{C},F}$ differs from the one in [KS16] up to the direct product with a torus.

Suppose that $\widetilde{\Delta}$ is a primitive regular subdivision of Δ. Then for each face F of $\widetilde{\Delta}$ the variety $X_{\mathbb{C},F}$ is the complement of a hyperplane arrangement. By [Sha93] its mixed Hodge structure is pure and

$$\chi_y(X_{\mathbb{C},F}) = \sum_{p=0}^{n}(-1)^{n+p}\dim H_c^{n+p}(X_{\mathbb{C},F})y^p.$$

In fact, this hyperplane arrangement complement is $\mathcal{C}_{n-q}\times(\mathbb{C}^*)^q$, where $\dim F = n+1-q$ and $\mathcal{C}_{n-q}$ is the complement of $n+2-q$ generic hyperplanes in $\mathbb{C}P^{n-q}$. By [Zha13], we have $\dim H^p(X_{\mathbb{C},F}) = \operatorname{rank}\mathcal{F}_p(\sigma_F)$ where σ_F is the face of the tropical hypersurface X dual to F. By Poincaré duality for $X_{\mathbb{C},F}$ we obtain $\dim H_c^p(X_{\mathbb{C},F}) = \operatorname{rank}\mathcal{F}_{n-p}(\sigma_F)$. Therefore, we obtain the formula

$$\chi_y(X_{\mathbb{C},F}) = y^{-1}(y-1)^q\left[(y-1)^{n+1-q}-(-1)^{n+1-q}\right].$$

Therefore when the subdivision is primitive $\chi_y(X_{\mathbb{C},F})$ only depends on the dimension of F. Moreover, if $\widetilde{\Delta}$ is the subdivision dual to the tropical hypersurface X then formula in Equation (5.1) can be expressed in terms of the f-vector of bounded faces of X. Namely,

$$\chi_y(X_{\mathbb{C}}) = \sum_{q=0}^{n}(-1)^q y^{-1}(y-1)^q\left[(y-1)^{n+1-q}-(-1)^{n+1-q}\right]f_q^b, \tag{5.2}$$

where f_q^b denotes the number of bounded faces of X of dimension q.

On the other hand we can compute the Euler characteristics of the Borel–Moore chain complexes

$$\chi(C_\bullet^{BM}(X;\mathcal{F}_p)) = \sum_{\tau\in X}(-1)^{\dim\tau}\operatorname{rank}\mathcal{F}_p(\tau). \tag{5.3}$$

The star of a face τ of X is a basic open subset and satisfies Poincaré duality from [JRS18]. Therefore, we have

$$\begin{aligned}\operatorname{rank}\mathcal{F}_p(\tau) &= \operatorname{rank}H_0(\operatorname{star}(\tau);\mathcal{F}_p) = \operatorname{rank}H_c^n(\operatorname{star}(\tau);\mathcal{F}^{n-p})\\ &= \sum_{\sigma\supset s\tau\,\dim\sigma=q}(-1)^{n-q}\operatorname{rank}\mathcal{F}_{n-p}(\sigma).\end{aligned}$$

since $\operatorname{rank}\mathcal{F}^{n-p}(\tau) = \operatorname{rank}\mathcal{F}_{n-p}(\tau)$ and also $H_c^n(\operatorname{star}(\tau);\mathcal{F}^{n-p})$ is torsion free. Swapping the order of the sum we obtain

$$\chi\left(C_\bullet^{BM}(X;\mathcal{F}_p)\right) = \sum_{\sigma\in X}(-1)^{n-\dim\sigma}\operatorname{rank}\mathcal{F}_{n-p}(\sigma)\sum_{\tau\subset\sigma}(-1)^{\dim\tau}.$$

If σ is a bounded face of X, then $\sum_{\tau\subset\sigma}(-1)^{\dim\tau} = 1$. If σ is an unbounded face of X then $\sum_{\tau\subset\sigma}(-1)^{\dim\tau} = 0$, since the one point compactification of σ has Euler characteristic equal to 1. Therefore, the sum in Equation (5.3) becomes

$$\chi\left(C_\bullet^{BM}(X;\mathcal{F}_p)\right) = \sum_{\substack{\tau\in X\\ \tau\text{ bounded}}}(-1)^{n-\dim\tau}\operatorname{rank}\mathcal{F}_{n-p}(\tau).$$

For a face τ of dimension q we have

$$\sum_{p=0}^{n}(-1)^p \operatorname{rank} \mathcal{F}_{n-p}(\tau) y^p = (-1)^n y^n \chi_\tau(\frac{1}{y})$$
$$= (-1)^n y^{-1}(y-1)^q \left[(y-1)^{n+1-q} - (-1)^{n+1-q}\right],$$

where χ_τ is the polynomial from Corollary 2.16. By comparing this with Equation (5.2) we obtain

$$\chi_y(X_\mathbb{C}) = \sum_{p=0}^{n}(-1)^p \chi\left(C_\bullet^{BM}\left(X; \mathcal{F}_p^X\right)\right) y^p,$$

and the proof of the Theorem 1.8 is complete. □

Corollary (Corollary 1.9). — *Let X be a non-singular and combinatorially ample compact tropical hypersurface in a non-singular compact tropical toric variety Y and assume that X has Newton polytope Δ. Let $X_\mathbb{C}$ be a torically non-degenerate complex hypersurface in the compact complex toric variety $Y_\mathbb{C}$ also with Newton polytope Δ. Then for all p and q we have*

$$\dim H^{p,q}(X_\mathbb{C}) = \operatorname{rank} H_q\left(X; \mathcal{F}_p^X\right).$$

Proof. — By combining Proposition 4.1 with the Lefschetz hyperplane section theorems for tropical homology and the homology of complex hypersurfaces of toric varieties, for $p + q < n$, we have

$$\operatorname{rank} H_q(X; \mathcal{F}_p) = \operatorname{rank} H_q\left(Y; \mathcal{F}_p^Y\right) = h^{p,q}(Y_\mathbb{C}) = h^{p,q}(X_\mathbb{C}). \tag{5.4}$$

The above equations combined with the Poincaré duality statements for all of $X, Y, X_\mathbb{C}$ and $Y_\mathbb{C}$ establishes the same equalities when $p + q > n$.

Therefore, it only remains to prove the statement when $q = n - p$. It follows from the tropical and complex versions of Lefschetz theorems and from Proposition 4.1 that

$$\chi\left(C_\bullet^{BM}\left(X; \mathcal{F}_p^X\right)\right) = (-1)^p \operatorname{rank} H_p\left(Y; \mathcal{F}_p^Y\right) + (-1)^{n-p} \operatorname{rank} H_{n-p}\left(X; \mathcal{F}_p^X\right),$$

and

$$\sum_q e_c^{p,q}(X_\mathbb{C}) = \dim H^{p,p}(Y_\mathbb{C}) + (-1)^n \dim H^{p,n-p}(X_\mathbb{C})$$

for $p \neq \frac{n}{2}$.

For $p = \frac{n}{2}$, we get

$$\chi\left(C_\bullet^{BM}\left(X; \mathcal{F}_{\frac{n}{2}}^X\right)\right) = (-1)^{\frac{n}{2}} \operatorname{rank} H_{\frac{n}{2}}\left(X; \mathcal{F}_{\frac{n}{2}}^X\right),$$

and

$$\sum_q e_c^{\frac{n}{2},q}(X_\mathbb{C}) = \dim H^{\frac{n}{2},\frac{n}{2}}(X_\mathbb{C}).$$

Again by Proposition 4.1 for tropical toric varieties we have

$$\operatorname{rank} H_p\left(Y; \mathcal{F}_p^Y\right) = \dim H^{p,p}(Y_\mathbb{C}).$$

The statement of the corollary follows after applying Theorem 1.8. □

The next corollary also follows from Theorem 1.8.

Corollary 5.2. — *Let Y be a non-singular tropical toric variety associated to a fan whose support is a convex cone of maximal dimension in $\mathbb{R}^{n+1}$ and such that the complex toric variety $Y_{\mathbb{C}}$ is affine. Let X be a combinatorially ample non-singular tropical hypersurface in Y such that (Y, X) is a cellular pair and every parent face of a compact face of Y is compact. If $X_{\mathbb{C}}$ is a torically non-degenerate complex hypersurface in $Y_{\mathbb{C}}$ with the same Newton polytope as X, then*

$$\operatorname{rank} H_q^{BM}(X; \mathcal{F}_p) = \begin{cases} \sum_{l=0}^{q} h^{p,l}\left(H_c^n(X_{\mathbb{C}})\right) & \text{if } p+q=n \\ h^{p,p}\left(H^{2p}(X_{\mathbb{C}})\right) & \text{if } p=q>\frac{n}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Corollary 5.2. — It follows from [MM18, Theorem 3.6] that if $p \neq q$ or $k \neq 2p$, then $h^{p,q}(H_c^k(Y_{\mathbb{C}})) = 0$. Therefore, if $p \neq q$, then $e^{p,q}(Y_{\mathbb{C}}) = 0$ and when $p = q$ we have

$$e_c^{p,p}(Y_{\mathbb{C}}) = h^{p,p}\left(H_c^{2p}(Y_{\mathbb{C}})\right).$$

From the proof of Corollary 1.5, we also have $H_q^{BM}(Y; \mathcal{F}_p) = 0$ if $p \neq q$. The equality in Theorem 1.8 also holds if we replace X and $X_{\mathbb{C}}$ with non-singular toric varieties Y and $Y_{\mathbb{C}}$. This is because it holds for $(\mathbb{C}^*)^k$ and the Euler characteristic of the Borel–Moore complexes and the χ_y genus are both additive. Therefore, we obtain

$$\operatorname{rank} H_p^{BM}\left(Y; \mathcal{F}_p^Y\right) = h^{p,p}\left(H_c^{2p}(Y_{\mathbb{C}})\right).$$

Notice that since $Y_{\mathbb{C}}$ is affine, the Andreotti–Frankel theorem imply that $h^{p,p}(H_c^{2p}(Y_{\mathbb{C}})) = 0$ if $2p < n$, and thus $\operatorname{rank} H_p^{BM}(Y; \mathcal{F}_p^Y) = 0$ if $2p < n$. Combining the tropical Lefschetz theorem and Poincaré duality, we obtain that if $p + q \neq n$

$$\operatorname{rank} H_q^{BM}\left(X; \mathcal{F}_p^X\right) = \begin{cases} \operatorname{rank} H_{p+1}^{BM}\left(Y; \mathcal{F}_{p+1}^Y\right) & \text{if } p=q>\frac{n}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $X_{\mathbb{C}}$ is affine, one has again that $h^{p,q}(H_c^k(X_{\mathbb{C}})) = 0$ if $k < n$. By the Lefschetz-type theorems for the Hodge Deligne numbers on $H_c^n(X_{\mathbb{C}})$ [DK86, Section 3], we get $h^{p,q}(H_c^k(X_{\mathbb{C}})) = 0$ if $k > n$ and $p \neq q$ and that if $2p > n$

$$h^{p,p}\left(H_c^{2p}(X_{\mathbb{C}})\right) = h^{p+1,p+1}\left(H_c^{2p+2}(Y_{\mathbb{C}})\right).$$

Therefore,

$$e_c^{p,q}(X_{\mathbb{C}}) = \begin{cases} (-1)^n h^{p,q}\left(H_c^n(X_{\mathbb{C}})\right) \text{ if } p+q \leqslant n \\ h^{p+1,p+1}\left(H_c^{2p+2}(Y_{\mathbb{C}})\right) \text{ if } p=q>\frac{n}{2} \\ 0 \text{ otherwise.} \end{cases}$$

Then by applying Theorem 1.8 and using the fact that the Borel–Moore tropical homology groups of X are torsion free by Corollary 1.5, we obtain the statement of Corollary 5.2. □

Theorem 1.8 can be used to calculate the ranks of the tropical homology groups of tropical hypersurfaces in $\mathbb{R}^{n+1}$.

Corollary 5.3. — *Let X be a non-singular tropical hypersurface in $\mathbb{R}^{n+1}$ with full-dimensional Newton polytope. If $X_{\mathbb{C}}$ is a non-singular torically non-degenerate complex hypersurface in $(\mathbb{C}^*)^{n+1}$ with the same Newton polytope as X, then*

$$\operatorname{rank} H_q^{BM}(X;\mathcal{F}_p) = \begin{cases} \sum_{l=0}^{q} h^{p,l}\left(H_c^n(X_{\mathbb{C}})\right) & \text{if } p+q=n \\ h^{p,p}\left(H^{n+p}(X_{\mathbb{C}})\right) & \text{if } q=n \\ 0 & \text{otherwise.} \end{cases}$$

The Hodge–Deligne numbers appearing in the above corollary can be calculated using the algorithms in [DK86]. For example, when $X_{\mathbb{C}}$ a non-singular torically non-degenerate complex hypersurface in $(\mathbb{C}^*)^{n+1}$ we have $h^{p,p}(H^{n+p}(X_{\mathbb{C}})) = \binom{n+1}{p+1}$.

Proof of Corollary 5.3. — The proof follows exactly the same lines as the proof of Corollary 5.2. It follows from [DK86] that

$$h^{p,q}\left(H_c^k\left((\mathbb{C}^*)^{n+1}\right)\right) = \begin{cases} \binom{n+1}{p} \text{ if } p=q \text{ and } k=n+1+p \\ 0 \text{ otherwise .} \end{cases}$$

The Borel–Moore tropical homology groups satisfy $H_q^{BM}(\mathbb{R}^{n+1};\mathcal{F}_p)=0$ if $q\neq n+1$ and

$$\operatorname{rank} H_{n+1}^{BM}\left(\mathbb{R}^{n+1};\mathcal{F}_p\right) = \binom{n+1}{p}.$$

Combining Theorem 1.1 and Poincaré duality for the tropical homology of X, when $p+q\neq n$ we have

$$\operatorname{rank} H_q^{BM}\left(X;\mathcal{F}_p^X\right) = \begin{cases} \binom{n+1}{p+1} & \text{if } q=n, \\ 0 & \text{if } q\neq n. \end{cases}$$

The hypersurface $X_{\mathbb{C}}$ is a non-singular affine variety, so the Andreotti–Frankel theorem and Poincaré duality imply $H_c^k(X_{\mathbb{C}})=0$ if $k<n$. By the Lefschetz-type theorems for the Hodge–Deligne numbers on $H_c^n(X_{\mathbb{C}})$ [DK86, Section 3], if $k>n$ one has

$$h^{p,q}\left(H_c^k(X_{\mathbb{C}})\right) = \begin{cases} \binom{n+1}{p+1} & \text{if } p=q \text{ and } k=n+p \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$e_c^{p,q}(X_{\mathbb{C}}) = \begin{cases} (-1)^n h^{p,q}\left(H_c^n(X_{\mathbb{C}})\right) & \text{if } p+q\leqslant n \text{ and } p\neq q \\ (-1)^n h^{p,q}\left(H_c^n(X_{\mathbb{C}})\right) + (-1)^{n+p}\binom{n+1}{p+1} & \text{if } p+q\leqslant n \text{ and } p=q \\ (-1)^{n+p}\binom{n+1}{p+1} & \text{if } p+q>n \text{ and } p=q \\ 0 & \text{otherwise.} \end{cases}$$

Then by applying Theorem 1.8 and using the fact that the Borel–Moore tropical homology groups of X are torsion free by Corollary 1.6, we obtain the statement of Corollary 5.3. □

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