Importance Measures in Repairable Multistate Systems With Aging

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Within the field of reliability multistate systems represent a natural extension of the classical binary approach. For an extensive introduction to this topic, see Natvig (2011b). Repairable multistate systems quickly become too complex for exact analytical calculations. Fortunately, however, such systems can be studied efficiently using discrete event simulations. See Huseby and Natvig (2012). In the binary case importance is usually measured using the approach by Birnbaum (1969). Several authors have extended the notion of importance measures to multi-state systems. See e.g., Zio et al. (2007) and Huseby et al. (2020). In the latter paper the component state processes were modelled as homogenous semi-Markov processes. Such processes typically reach stationary states very quickly. Thus, most properties of the system can be analysed using asymptotic distributions which typically are determined by mean waiting times and the transition matrix of the built-in Markov chain. In the present paper we follow the approach suggested by Huseby et al. (2020). Here, however, we focus on the non-homogenous case. This is relevant in systems subject to e.g., seasonal variations or aging. In order to model this we use an approach similar to Lindqvist et al. (2003). When the component processes are not homogenous, the analysis should cover the entire time frame, not just the asymptotic properties. This makes comparison of importance more complicated. Several numerical examples are included in order to illustrate the methodology.

Keywords: Multistate Systems, Importance Measures, Semi-Markov Processes, Trend-Renewal Processes.

1. Introduction

Basic reliability theory for multistate systems was established in the mid 1980s, and has been developed continuously since then. In particular, many different methods for analysing importance of components in multistate systems have been developed. There are two main reasons for calculating importance of components in a system. Firstly, it permits the analyst to determine which components merit the most additional research and development to improve overall system reliability at minimum cost or effort. Secondly, it may be used in diagnostics as a way of generating a list of components ordered with respect to how likely they are to have caused the system failure.

In the present paper we adopt the framework introduced in Huseby et al. (2020) where a repairable multistate system is described in a way emphasising the physical properties of the components and the systems. Within this framework Huseby et al. (2020) introduced four different generalisations of the classical Birnbaum measure. All these measures can be viewed as generalisations of the classical Birnbaum measure in the binary case. The measures are characterised along two axes: *forward-looking* versus *backward-looking*, and with respect to how criticality is measured. Forward-looking importance measures focus on the *next* component states. According to this approach the most important component is the one that has the highest probability of changing the system state. Backward-looking importance measures focus on the *previous* component states: According to this approach the most important component is the one that has the highest probability of having changed the system state. Furthermore, two approaches to measuring criticality are considered: probability of criticality versus impact of criticality.

Huseby et al. (2020) considered a case where the component state processes were modelled as homogenous semi-Markov processes. Such processes typically reach stationary states very quickly. Thus, most properties of the system can be analysed using asymptotic distributions which typically are determined by mean waiting times and the transition matrix of the built-in Markov chain. Here, however, we focus on the non-homogenous case. This is relevant in systems subject to e.g., seasonal variations or aging. In order to model this we use an approach similar to Lindqvist et al. (2003). When the component processes are not homogenous, the analysis should cover the entire time frame, not just the asymptotic properties. This makes comparison of importance more complicated. Several numerical examples

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are included in order to illustrate the methodology.

By using the proposed simulation techniques it is easy to analyse much more complex systems than those considered in the paper without running into time issues. Software for doing such simulations in realistic cases have been developed.

2. Multistate systems

For an extensive introduction to multistate systems we refer to Natvig (2011a). In the present paper we define a multistate system as an ordered pair (C, ϕ), where $C = \{1, \ldots, n\}$ is the component set, and ϕ is the *structure function* expressing the state of the system as a function of the component states. We let $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$ denote the *state vector* of the components where $X_i(t)$ is the state variable of Component i at time t. For $i \in C$, we let $S_i = \{0, 1, \ldots, r_i\}$ denote the set of states for Component i.

2.1. *The component state processes*

The component state processes $\{X_i(t)\}$, $i = 1, \ldots, n$ are modelled as *semi-Markov* processes. In particular, the state transition for Component $i \in C$ follows Markov chain, referred to as the *built-in Markov chain* for this particular component.

For each Component $i \in C$ we let $X_i^+(t)$ and $X_i^-(t)$ denote respectively the *next* and *previous* state of Component i at time $t > 0$. Each time Component i enters a state $u \in S_i$, it remains in this state for a random amount of time, and then makes a transition to some other state $v \in S_i$. For $u, v \in S_i$ we assume that:

$$P(X_i^+(t) = v | X_i(t) = u) = P_{uv}^{(i)}. \qquad (1)$$

The matrix of transition probabilities for the built-in Markov chain for Component i is denoted by $\boldsymbol{P}^{(i)}$, $i \in C$. Note that $\boldsymbol{P}^{(i)}$ is assumed to be independent of the time $t > 0$.

In order to find a similar expression for the conditional distribution of $X_i^-(t)$, we need the transition matrix for the backwards version of the built-in Markov chain, which we denote by $\boldsymbol{Q}^{(i)}$. It then follows that for all $u, v \in S_i$ we have:

$$P(X_i^-(t) = v | X_i(t) = u) = Q_{uv}^{(i)}. \qquad (2)$$

For Component $i \in C$ we denote the stationary probabilities of the built-in Markov chain by $\pi_u^{(i)}$, $u \in S_i$. We then have the following well-known relation between the transition matrices $\boldsymbol{P}^{(i)}$ and $\boldsymbol{Q}^{(i)}$ (see e.g., Ross (2014)):

$$Q_{uv}^{(i)} = \frac{\pi_v^{(i)}}{\pi_u^{(i)}} P_{vu}^{(i)}, \quad u, v \in S_i. \qquad (3)$$

Note that if the stationary distribution of the built-in Markov chain is *uniform*, i.e., if $\pi_u^{(i)} = 1/(r_i + 1)$, for all $u \in S_i$, we have:

$$\boldsymbol{Q}^{(i)} = (\boldsymbol{P}^{(i)})^T, \quad i \in C.$$

It is well-known that an irreducible aperiodic finite Markov chain has a uniform stationary distribution if and only if $\boldsymbol{P}^{(i)}$ is a *doubly stochastic matrix*, i.e., all row sums and column sums are equal to 1.

The Semi-Markov model specification is completed by introducing the times spent in each state between the transitions:

$$W_{ks}^{(i)} = \text{The } k\text{th waiting time in state } s \text{ for Component } i,\ s \in S_i,\ i \in C.$$

In the homogenous case we assume that all the waiting times are independent, and that for all states $s \in S_i$ and all components $i \in C$ the waiting times $W_{1s}^{(i)}, W_{2s}^{(i)}, \ldots$ are *identically distributed* with finite mean $\mu_s^{(i)}$. Then it follows from standard renewal theory (see e.g., Ross (2014)) that the stationary distribution of X_i is given by:

$$P[X_i = s] = \frac{\pi_s^{(i)} \mu_s^{(i)}}{\sum_{s' \in S_i} \pi_{s'}^{(i)} \mu_{s'}^{(i)}},\ s \in S_i,\ i \in C. \qquad (4)$$

2.2. *The structure function*

For each component $i \in C$ we introduce a function $f_i : S_i \to \mathbb{R}$ representing the *physical state* of the component as a function of the state. Thus, if $X_i(t) = x_i \in S_i$, then the physical state of Component i at time t is $f_i(X_i(t)) = f_i(x_i)$. If e.g., Component i is a pipeline, then the physical state of the component at a given point of time may be the capacity of the pipeline at this point of time. Being a physical property of the pipeline, this may be any non-negative number, and the function f_i provides a convenient way of encoding this directly into the model.

Note that the functions $f_1, \ldots, f_n$ do not necessary need to be nondecreasing. By skipping this restriction additional useful modelling flexibility is gained. This allows e.g., for the inclusion of burn-in phases, maintenance as well as minimal or partial repairs of a components as part of its life cycle before it reaches its failure state.

The structure function ϕ represents the state of the system expressed as a function of the states of the components. It is common in multistate reliability theory to assume that ϕ also assumes values in a set of non-negative integers. In this context, however, the structure function represents

the physical state of the system and is expressed as:

$$\phi(\boldsymbol{X}(t)) = \phi(f_1(X_1(t)), \ldots, f_n(X_n(t)))$$

Thus, the physical state of the system is a function of the physical states of the components. The advantage with this approach is that the system state is expressed in terms of physical quantities rather than being encoded more abstractly as non-negative integers.

2.3. *Criticality and importance in multistate systems*

According to standard conventions from reliability theory, the notation $\phi(x_i, \boldsymbol{X})$ is defined as:

$$\phi(x_i, \boldsymbol{X}) = \phi(X_1, \ldots, X_{i-1}, x_i, X_{i+1}, \ldots, X_n).$$

In cases where the index i is obvious from the context, we simply write $\phi(x, \boldsymbol{X})$. Using $X_i^+(t)$ and $X_i^-(t)$ Huseby et al. (2020) introduced two notions of criticality. Component i is *n-critical* at time t if:

$$\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t)). \quad (5)$$

Hence, Component i is n-critical at time t if changing the component to its *next* state would result in a system state change as well. Similarly, Component i is *p-critical* at time t if:

$$\phi(X_i^-(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t)). \quad (6)$$

Hence, Component i is p-critical at time t if changing the component to its *previous* state would result in a system state change as well.

Based on these criticality concepts Huseby et al. (2020) proceeded by introducing the following four different measures of importance:

$$I_{NB}^{(i)}(t) = P[\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t))]$$
$$I_{PB}^{(i)}(t) = P[\phi(X_i^-(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t))]$$
$$I_{NB}^{*(i)}(t) = E|\phi(X_i(t), \boldsymbol{X}(t)) - \phi(X_i^+(t), \boldsymbol{X}(t))|$$
$$I_{PB}^{*(i)}(t) = E|\phi(X_i^-(t), \boldsymbol{X}(t)) - \phi(X_i(t), \boldsymbol{X}(t))|$$

We observe that $I_{NB}^{(i)}(t)$ is the probability that Component i is n-critical, while $I_{PB}^{(i)}(t)$ is the probability that Component i is p-critical. Moreover, $I_{NB}^{*(i)}(t)$ is the expected impact on the structure function if Component i is changed to its next state, while $I_{PB}^{*(i)}(t)$ is the expected impact on the structure function if Component i is changed back to its previous state.

By conditioning on $X_i(t)$ and $X_i^+(t)$ we obtain the following expression for $I_{NB}^{(i)}(t)$:

$$I_{NB}^{(i)}(t) = \sum_{s,s' \in S_i} P[\phi(s, \boldsymbol{X}(t)) \neq \phi(s', \boldsymbol{X}(t))] \cdot P[X_i(t) = s] \cdot P_{ss'}^{(i)} \quad (7)$$

Similarly, by conditioning on $X_i^-(t)$ and $X_i(t)$ the following expression for $I_{PB}^{(i)}(t)$ is obtained:

$$I_{PB}^{(i)}(t) = \sum_{s,s' \in S_i} P[\phi(s, \boldsymbol{X}(t)) \neq \phi(s', \boldsymbol{X}(t))] \cdot P[X_i(t) = s] \cdot Q_{ss'}^{(i)} \quad (8)$$

The corresponding expressions for $I_{NB}^{*(i)}(t)$ and $I_{PB}^{*(i)}(t)$ are:

$$I_{NB}^{*(i)}(t) = \sum_{s,s' \in S_i} E|\phi(s, \boldsymbol{X}(t)) \neq \phi(s', \boldsymbol{X}(t))| \cdot P[X_i(t) = s] \cdot P_{ss'}^{(i)} \quad (9)$$

$$I_{PB}^{*(i)}(t) = \sum_{s,s' \in S_i} E|\phi(s, \boldsymbol{X}(t)) \neq \phi(s', \boldsymbol{X}(t))| \cdot P[X_i(t) = s] \cdot Q_{ss'}^{(i)} \quad (10)$$

For further result on $I_{NB}^{(i)}(t)$, $I_{PB}^{(i)}(t)$, $I_{NB}^{*(i)}(t)$ and $I_{PB}^{*(i)}(t)$ in the homogenous case we refer to Huseby et al. (2020).

3. The trend-renewal model

We start this section by considering a regular pure jump process with jumps at times $T_1 < T_2 < \cdots$. We also let $T_0 = 0$ and introduce the waiting times between the event times:

$$W_k = T_k - T_{k-1}, \quad k = 1, 2, \ldots.$$

In a homogenous semi-Markov process the waiting times are independent and identically distributed. If the events represent state changes of some component in a system, this may not be a realistic model. In order to include aging and similar effects, a *trend-renewal process* offers more flexibility. Following Lindqvist et al. (2003) we define a general *trend-renewal process* as follows:

Definition 3.1. Let $\lambda(t)$ be a nonnegative *intensity function* defined for all $t \geq 0$, satisfying $\Lambda(t) \equiv \int_0^t \lambda(u)du < \infty$ for each $t \geq 0$ and $\Lambda(\infty) = \infty$. Furthermore, let F be a cumulative distribution distribution function such that $F(0) = 0$. The pure jump process is a trend-renewal process with respect to F and λ, and written as

$TRP(F, \lambda)$, if the waiting times between the *time-transformed* event times $\Lambda(T_0), \Lambda(T_1), \ldots$ are independent and identically distributed with cumulative distribution function F.

Note that for a given process $TRP(F, \lambda)$ and constant $c > 0$, we may define alternative functions $\tilde{\lambda} = c\lambda$ and $\tilde{\Lambda}(t) = \int_0^t \tilde{\lambda}(u)du = c\int_0^t \lambda(u)du = c\Lambda(t)$, for all $t \geq 0$. If $V_k = \Lambda(T_k) - \Lambda(T_{k-1})$ is the waiting time between $\Lambda(T_{k-1})$ and $\Lambda(T_k)$, it follows that $P(V_k \leq v) = F(v)$, for $k = 1, 2, \ldots$.

We then let $\tilde{V}_k = \tilde{\Lambda}(T_k) - \tilde{\Lambda}(T_{k-1})$, $k = 1, 2, \ldots$ be the waiting times obtained using the alternative time-transform $\tilde{\Lambda}$. Then we obviously have that $\tilde{V}_k = cV_k$, $k = 1, 2, \ldots$. This implies that $\tilde{F}(v) = P(\tilde{V}_k < v) = F(v/c)$, $k = 1, 2, \ldots$. Hence, it follows that $TRP(F, \lambda)$ can alternatively be written as $TRP(\tilde{F}, \tilde{\lambda})$. In order to avoid this ambiguity, Lindqvist et al. (2003) only considered trend-renewal processes where the waiting times between the time-transformed event times had expectation 1. In the present paper, however, we focus on using trend-renewal processes as a modelling tool. From this perspective the two models, $TRP(F, \lambda)$ and $TRP(\tilde{F}, \tilde{\lambda})$ are equivalent, and we may choose the one that is most convenient in a given situation. Thus, in this context, the ambiguity is not causing problems, so we simply skip the restriction suggested by Lindqvist et al. (2003).

The family of trend-renewal processes is a very rich class of non-homogenous pure jump processes. At the same time such processes are very easy to simulate due to the property that the waiting times between the time-transformed event times, $V_1, V_2, \ldots$ are independent and identically distributed. More specifically, the event times $T_1, T_2, \ldots$ can be calculated using the following relation for $k = 1, 2, \ldots$:

$$T_k = \Lambda^{-1}[\Lambda(T_{k-1}) + V_k]. \tag{11}$$

In Lindqvist et al. (2003) a trend-renewal process was used to model the failure times of a certain component operating in an environment where repair times are negligible compared to times between failures. In the present paper, however, we consider multistate components. In simple cases we might be able to justify that waiting times in the failed state may be negligible. However, we still need to consider waiting times for more than one state. In order to handle this we need a more general type of trend-renewal model.

At this stage we simplify the notation slightly by considering a *single* component with states in the set $S = \{1, \ldots, r\}$. For $s \in S$ and $k = 1, 2, \ldots$ we introduce:

T'_{ks} = The kth time the component enters state s
T_{ks} = The kth time the component leaves state s

Note that we always have $T_{k-1,s} < T'_{ks} < T_{ks}$, for $k = 1, 2, \ldots$.

For $s \in S$ and $k = 1, 2, \ldots$ we also introduce the waiting times spent in each state between the transitions:

$$\begin{aligned} W_{ks} &= \text{The } k\text{th waiting time in state } s \\ &= T_{ks} - T'_{ks} \end{aligned}$$

Finally, we let $\Lambda_s(t) = \int_0^t \lambda_s(u)du$ denote the time transform applied to event times affecting the state $s \in S$.

We now consider two different models:

3.1. *The global time model*

In this model we assume that the waiting times for all the states are affected by the same clock. Thus, e.g., if a component ages by time, this aging occurs regardless of the state the component is in. In particular, the component ages also when it is under repair. A global time model is also appropriate for components which are subject to seasonal effects, since such effects typically affects the component regardless of state.

In this case we let the waiting times between the time-transformed event times be defined as $V_{ks} = \Lambda_s(T_{ks}) - \Lambda_s(T'_{ks})$, for $k = 1, 2, \ldots$. Assuming that $V_{1s}, V_{2s}, \ldots$ are independent and identically distributed with cumulative distribution function F_s, these quantities can easily be generated using Monte Carlo simulation, and the event times, $T_{1s}, T_{2s}, \ldots$ can be calculated for all $s \in S$ using the following relation for $k = 1, 2, \ldots$:

$$T_{ks} = \Lambda_s^{-1}[\Lambda_s(T'_{ks}) + V_{ks}]. \tag{12}$$

Note, however, that the event times $T'_{1s}, T'_{2s}, \ldots$ depends on models for the other states as well as the matrix of transition probabilities for the built-in Markov chain. Hence, in this case the sequences of waiting times for the different states are stochastically dependent. As a result the event times $T_{1s}, T_{2s}, \ldots$ do *not* follow an ordinary trend-renewal process as defined in Definition 3.1. Still, when the entire model is specified, it is well-defined and well-suited for Monte Carlo simulations.

3.2. *The local time model*

In this model we assume that the waiting times for the states are affected by individual clocks for each state. Thus, e.g., aging in one given state is independent of the time spent in the other states. In particular, aging in each of the functioning states

is independent of repair times. Such a model, however, is not likely to fit a case where the component is subject to global effects.

In order to model this we let $U_{0s} = 0$, and introduce the following quantities for $k = 1, 2, \ldots$ and $s \in S$:

$$U_{ks} = W_{1s} + \cdots + W_{ks} = U_{k-1,s} + W_{ks}.$$

Thus, $U_{1s}, U_{2s}, \ldots$ are the points of time when the component leaves state s, given that we use a clock which is stopped when the component is not in state s. We will refer to these points of time as the *net event times* for state s.

We then let the waiting times between the time-transformed net event times be defined as $V_{ks} = \Lambda_s(U_{ks}) - \Lambda_s(U_{k-1,s})$, for $k = 1, 2, \ldots$. Assuming that $V_{1s}, V_{2s}, \ldots$ are independent and identically distributed with cumulative distribution function F_s, these quantities can easily be generated using Monte Carlo simulation, and thus, the net event times for state s can be calculated for all $s \in S$ using the following relation for $k = 1, 2, \ldots$:

$$U_{ks} = \Lambda_s^{-1}[\Lambda_s(U_{k-1,s}) + V_{ks}]. \qquad (13)$$

By comparing Eq. (13) to Eq. (11) we observe that this implies the net event times $U_{1s}, U_{2s}, \ldots$ follow the trend-renewal process $TRP(F_s, \lambda_s)$. In fact, when using this model, the component process alternates randomly between s independent trend-renewal processes, one for each state of the component.

In order to find the event times $T_{1s}, T_{2s}, \ldots$, we recall that $T_{ks} = T'_{ks} + W_{ks}$, $k = 1, 2, \ldots$. Hence, we get for $k = 1, 2, \ldots$:

$$T_{ks} = T'_{ks} + U_{ks} - U_{k-1,s}. \qquad (14)$$

More generally, it is possible to combine the two time models by letting some of the states follow the global time model, and while other states follow the local time model. In the present paper, we only consider cases where the same time model is used for all states. A more general study of the two models will be covered in an upcoming paper.

3.3. *Additive intensity functions*

We close this section by considering more specific intensity functions. The *power law model* considered by Lakey and Rigdon (1992) is a well-known model with two parameters, $\alpha, \beta > 0$, where $\lambda(t) = \alpha\beta t^{\alpha-1}$, $t \geq 0$. The cumulative intensity function is given by:

$$\Lambda(t) = \int_0^t \lambda(u)du = \beta t^{\alpha}.$$

We observe that if $\alpha > 1$, the intensity is increasing, while if $\alpha < 1$, the intensity is decreasing. To run simulations we also need the inverse of the cumulative intensity function. For the power law model this is given by:

$$\Lambda^{-1}(u) = (u/\beta)^{1/\alpha}.$$

A more general model could allow non-monotonic intensity functions as well. This can be accomplished by using an additive model, i.e., a model of the form:

$$\lambda(t) = \sum_{j=1}^{m} \gamma_j \lambda_j(t),$$

where $\lambda_1, \ldots, \lambda_m$ are intensity functions, and where $\gamma_1, \ldots, \gamma_m$ are non-negative constants. By applying appropriate scaling, it is easy and convenient to choose these constants so that:

$$\sum_{j=1}^{m} \gamma_j = 1.$$

If this is done, λ becomes a *convex combination* of the intensity functions $\lambda_1, \ldots, \lambda_m$. By integrating the intensity function we find that the cumulative intensity function is:

$$\Lambda(t) = \sum_{j=1}^{m} \gamma_j \Lambda_j(t),$$

For additive intensity functions it may not be possible to find analytical expressions for the inverse of Λ. However, if we can find analytical expressions for $\Lambda_1^{-1}, \ldots, \Lambda_m^{-1}$, then Λ^{-1} can easily be determined numerically. Assume more specifically, that for a given u we are able to compute $t_j = \Lambda_j^{-1}(u)$, $j = 1, \ldots, m$, and that:

$$t_{\min} = \min_{1 \leq j \leq m} t_j, \text{ and } t_{\max} = \max_{1 \leq j \leq m} t_j.$$

Then it is easy to show that:

$$t_{\min} \leq \Lambda^{-1}(u) \leq t_{\max}.$$

By using these lower and upper bounds we can easily find $\Lambda^{-1}(u)$ numerically using e.g., the bisection method.

4. Importance measures in the non-homogenous case

In this section we return to the importance measures presented in Subsection 2.3, and illustrate the trend-renewal model introduced in Section 3 by considering a few specific examples. More specifically, we consider a multistate system (C, ϕ) where $C = \{1, 2\}$, and where both components have only three possible states, i.e., $S_1 = S_2 = \{0, 1, 2\}$. For simplicity we let $f_i(s) = s$, for all $s \in S_i$, $i = 1, 2$. It should be noted, though, that the suggested framework can easily

be applied to more complex situations with many component states and arbirary transitions without any modifications. The structure function is given by:

$$\phi(X_1(t), X_2(t)) = \min(f_1(X_1(t)), f_2(X_2(t))).$$

The transition matrices of the built-in Markov chains are:

$$\boldsymbol{P}^{(1)} = \boldsymbol{P}^{(2)} = \begin{bmatrix} 0.0, & 0.1, & 0.9 \\ 0.9, & 0.0, & 0.1 \\ 0.1, & 0.9, & 0.0 \end{bmatrix},$$

For $s \in S_i$, $i \in C$ and $k = 1, 2, \ldots$ we introduce:

$$T_{ks}^{(i)'} = \text{The } k\text{th time Component } i \text{ enters state } s$$
$$T_{ks}^{(i)} = \text{The } k\text{th time Component } i \text{ leaves state } s$$

Moreover, we introduce the waiting times spent in each state between the transitions:

$$W_{ks}^{(i)} = \text{Component } i\text{'s } k\text{th waiting time in state } s = T_{ks}^{(i)} - T_{ks}^{(i)'},$$

as well as the net event times:

$$U_{ks}^{(i)} = W_{1s}^{(i)} + \cdots + W_{ks}^{(i)} = U_{k-1,s}^{(i)} + W_{ks}^{(i)}.$$

The time transform applied to event times affecting the state $s \in S_i$, $i \in C$ is denoted by:

$$\Lambda_s^{(i)}(t) = \int_0^t \lambda_s^{(i)}(u)du.$$

Finally, we assume that $V_{1s}^{(i)}, V_{2s}^{(i)}, \ldots$ are independent and identically distributed with cumulative distribution function $F_s^{(i)}$.

For the global time model we then have:

$$T_{ks}^{(i)} = (\Lambda_s^{(i)})^{-1}[\Lambda_s(T_{ks}^{(i)'}) + V_{ks}^{(i)}],$$

while for the local time model we have:

$$U_{ks}^{(i)} = (\Lambda_s^{(i)})^{-1}[\Lambda_s(U_{ks}^{(i)}) + V_{ks}^{(i)}].$$

More specifically we assume that $V_{1s}^{(i)}, V_{2s}^{(i)}, \ldots$ are independent and exponentially distributed with expected value $\mu_s^{(i)}$, where:

$$\mu_0^{(1)} = \mu_0^{(2)} = 0.5,$$
$$\mu_1^{(1)} = \mu_1^{(2)} = 2.0,$$
$$\mu_2^{(1)} = \mu_2^{(2)} = 7.5.$$

In order to see the effect of the trend-renewal model more clearly, we use the following intensity functions:

$$\lambda_0^{(1)}(t) = \lambda_1^{(1)}(t) = 1,$$
$$\lambda_0^{(2)}(t) = \lambda_1^{(2)}(t) = \lambda_2^{(2)}(t) = 1.$$

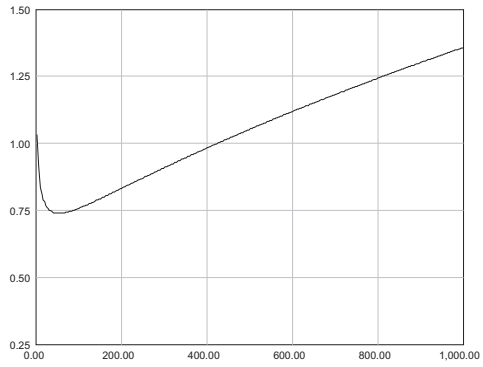

Fig. 1. The intensity function $\lambda_2^{(1)}(t)$

The corresponding cumulative intensity functions then become:

$$\Lambda_0^{(1)}(t) = \Lambda_1^{(1)}(t) = t,$$
$$\Lambda_0^{(2)}(t) = \Lambda_1^{(2)}(t) = \Lambda_2^{(2)}(t) = t.$$

For state 2 of Component 1, however, a more complicated intensity function is used:

$$\lambda_2^{(1)}(t) = \sum_{j=1}^{3} \gamma_j(\alpha_j \beta_j t^{\alpha_j - 1}),$$

where the parameters of $\lambda_2^{(1)}(t)$ are:

$$\alpha_1 = 0.75, \quad \alpha_2 = 1.0, \quad \alpha_3 = 1.6,$$
$$\beta_1 = 2.0, \quad \beta_2 = 1.0, \quad \beta_3 = 0.1,$$
$$\gamma_1 = 0.75, \quad \gamma_2 = 0.15, \quad \gamma_3 = 0.1.$$

The corresponding cumulative intensity function then becomes:

$$\Lambda_2^{(1)}(t) = \sum_{j=1}^{3} \gamma_j(\beta_j t^{\alpha_j}) = 0.75(2.0t^{0.75}) + 0.15t + 0.1(0.1t^{1.6}).$$

In Figure 1 we have plotted the intensity function $\lambda_2^{(1)}(t)$. We observe that $\lambda_2^{(1)}(t)$ is decreasing in the beginning, and then increasing later. This shape is often referred to as a *bath tub* shape.

The system is simulated using both the global and local time models over a time horizon of 1000 units. Figure 2 shows how the expected value of the component processes and system process develops as functions of time when the global time model is used, while Figure 3 shows the corresponding graph when the local time model is used. In both figures the red curve represents the expected state of Component 1. Due to the bath tub shape of the time transform, this curve is

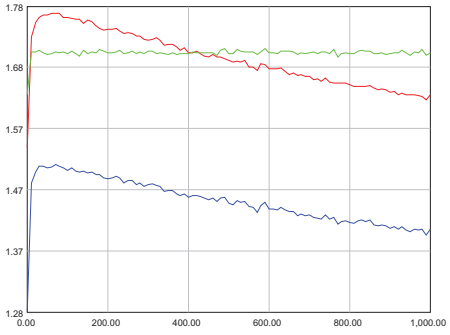

Fig. 2. E$[X_1(t)]$ (red), E$[X_2(t)]$ (green) and E$[\phi(t)]$ (blue) - Global time model

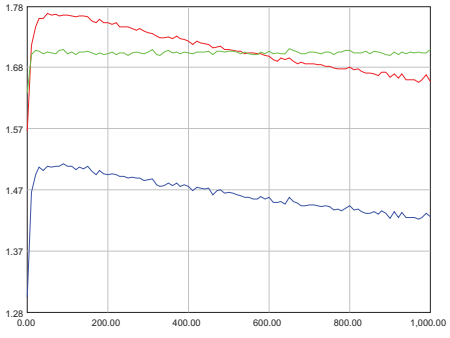

Fig. 3. E$[X_1(t)]$ (red), E$[X_2(t)]$ (green) and E$[\phi(t)]$ (blue) - Local time model

increasing in the beginning and decreasing when the aging effect starts to dominate. The green curve represents the expected state of Component 2. Since this component is not subject to any trend renewal effects, this curve stabilizes very fast. Finally, the blue curve represents the expected state of the system. This curve has more or less the same shape as the curve for the expected state of Component 1. However, due to influence from Component 2, the system curve lies below the curve for Component 1.

We observe that the two figures are very similar. The only difference is that the curves for Component 1 and the system decreases somewhat faster when the global time model is used, compared to the local time model. The reason for this is that the aging develops faster when a global time model is used since the clock runs uninterrupted.

We have also estimated the importance measures $I_{NB}^{(i)}(t)$ and $I_{NB}^{*(i)}(t)$, $i = 1, 2$ both for

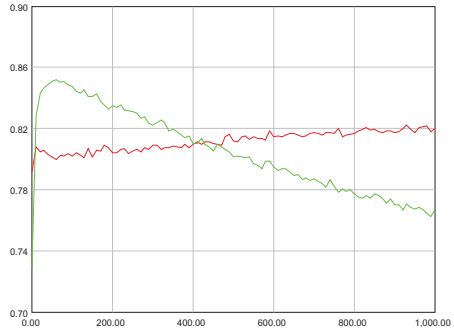

Fig. 4. $I_{NB}^{(1)}(t)$ (red) and $I_{NB}^{(2)}(t)$ (green) - Global time model

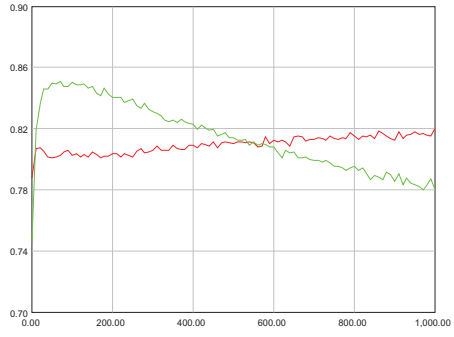

Fig. 5. $I_{NB}^{(1)}(t)$ (red) and $I_{NB}^{(2)}(t)$ (green) - Local time model

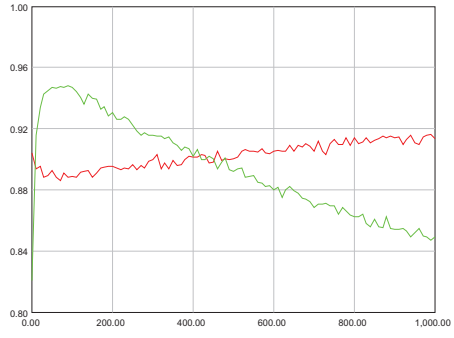

Fig. 6. $I_{NB}^{*(1)}(t)$ (red) and $I_{NB}^{*(2)}(t)$ (green) - Global time model

the global and local time models. The results

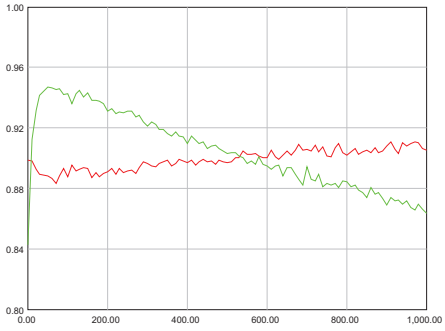

Fig. 7. $I_{NB}^{*(1)}(t)$ (red) and $I_{NB}^{*(2)}(t)$ (green) - Local time model

for $I_{NB}^{(i)}(t)$ are shown in Figure 4 and Figure 5, while the results for $I_{NB}^{*(i)}(t)$ are shown in Figure 6 and Figure 7. For series like systems like the one we are considering, it is typically the worst component that is the most important one. With the current probability models, Component 1 is better than Component 2 in the beginning, and worse than Component 2 when the aging effect starts to dominate. Hence, Component 2 is more important than Component 1 in the beginning, while the ranking is reversed when the aging of component 1 kicks in.

We observe that all the four plots are fairly similar in this particular case. The plots for the measure $I_{NB}^{*(i)}(t)$ tend to have a larger values since this measure is defined relative to the expected impact of component state changes, while the measure $I_{NB}^{(i)}(t)$ only takes into account the probability of an impact of component state changes. The other main difference between the plots is the importance of Component 2 decreases somewhat faster when the global time model is used, compared to the local time model. This is exactly the same effect as we saw in Figure 2 and Figure 3.

5. Conclusions

In the present paper we have introduced a multistate generalization of the trend-renewal model. By using this model and Monte Carlo simulation we can analyze expected performance and importance in multistate systems in non-homogenous cases. Two different time models have been suggested: the global time model and the local time model. The global time model is suitable for modeling component state processes in cases where the components are aging all the time irrespective of the component states. This type of model can also be used to represent seasonal variations in component behavior. The local time model is suitable when changes in transition rates are tied to specific component states.

The multistate trend-renewal model has been applied to importance measure estimation. Contrary to homogenous models the trend-renewal model typically results in component ranking which may change over time. When decisions regarding e.g., maintenance and improvements are based on importance ranking, this lack of stability may pose a challenge. In such cases there is a need for unifying time-independent measures. We will return to this issue in a future project.

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