Ocean Modelling 166 (2021) 101847

Contents lists available at ScienceDirect

Ocean Modelling

journal homepage: www.elsevier.com/locate/ocemod

On the Stokes drift in traveling surface pulses

Jan Erik H. Weber

Department of Geosciences, University of Oslo, P.O. Box 1022, Blindern NO-0313, Oslo, Norway

ARTICLE INFO

Keywords:
Shallow water equations
Surface pulses
Stokes drift
Solitary waves

ABSTRACT

Similar to the Stokes drift in periodic gravity waves, we introduce a second-order Stokes effect on the particle drift in isolated linear surface disturbances (pulses) in shallow water. For a linear disturbance with the shape and length scale of a KdV solitary wave, the model results agree surprisingly well with the observed drift in moderately steep solitary waves in the laboratory, as well as with more comprehensive theory for surface drift in solitary waves.

1. Introduction

Wave-induced particle drift in shallow water is a phenomenon that is of considerable interest for coastal engineers and environmentalists. It has an impact on sediment transport as well as the spread of effluents and pollution in the near-shore zone. For periodic irrotational waves, the story goes back to the seminal paper by Stokes (1847). Since then the number of papers on the mean drift in periodic surface waves has grown enormously. The next major step forward was due to Longuet-Higgins (1953), introducing the effect of viscosity into the wave drift problem. For a recent overview of direct Lagrangian calculations of the wave-induced drift in a viscous ocean, the reader is referred to Weber (2019), where also relevant references to inviscid studies are listed.

For non-periodic flows such as the motion of isolated disturbances, or pulses, along the sea surface, there is also a mass transport. Unlike the spiraling forward particle motion in periodic irrotational waves, this drift is just a finite forward particle displacement caused the passage of single pulse. The most celebrated disturbance of this kind is the solitary wave observed by Russell (1838, 1845), which led to theoretical investigations by Boussinesq (1871) and Rayleigh (1876). The lowest order solution is obtained by the KdV equation (Korteweg and de Vries, 1895), valid asymptotically in the limit of small amplitude and long wavelength. The solitary wave has been studied to third order in the parameter $\varepsilon = A/H$ by Grimshaw (1971), where A is the maximum surface amplitude and H is the undisturbed fluid depth, and to ninth order by Fenton (1972); see the review by Miles (1980). Existence theory for solitary waves has been developed by, amongst others, Amick and Toland (1981). More recently, Constantin and Escher (2007) analyzed formally particle paths associated with solitary-wave solutions; see also Constantin (2010) and Constantin et al. (2011). In Borluk and Kalisch (2012) velocity fields related to exact solutions of the KdV equation are reported, and particle trajectories are computed numerically.

In the present study, we follow the approach by Eames and McIntyre (1999) for the Lagrangian displacement due to periodic wave motion, but apply it to isolated surface pulses. The rest of the paper is organized as follows: In Section 2 we derive the exact equations for the particle drift in irrotational waves, and Section 3 gives the well-known results for periodic waves. Section 4 considers non-periodic flows in the form of isolated disturbances or pulses in shallow water, and discusses the Stokes drift in the general case. The pulses are solutions of the small-amplitude shallow water equation. In Section 5, we study a linear pulse with the spatial shape of solitary wave in the KdV approximation. However, we do not derive asymptotically valid results for the KdV solitary wave. When the Stokes drift is added to the drift in the linear pulse, we obtain surprisingly good fit with laboratory experiments on moderately steep solitary waves, as well as with higher order theory. Finally, Section 6 contains some concluding remarks.

2. Mathematical derivation

We consider two-dimensional motion of an incompressible and irrotational fluid in a horizontal layer. When undisturbed, the depth is constant and equal to H. The effect of the earth's rotation is not taken into account. The x-axis is along the undisturbed surface, and the z-axis is vertically upward. The bottom at $z = -H$ is impermeable. When we have a disturbance, the velocity components are (u, w), and free surface is given by $z = \eta(x, t)$, where t denotes time. Since the fluid is irrotational, we can write the velocity components $u = \partial\varphi/\partial x, w = \partial\varphi/\partial z$, where φ is the velocity potential. Hence

$$u = DX/Dt = \partial\varphi/\partial x, \tag{1}$$

where X is a horizontal particle coordinate, and D/Dt is the rate of change following a fluid particle. We consider waves of permanent form

E-mail address: j.e.weber@geo.uio.no.

https://doi.org/10.1016/j.ocemod.2021.101847
Received 19 April 2021; Received in revised form 21 June 2021; Accepted 18 July 2021
Available online 21 July 2021

that propagates with constant phase speed C. In that case $C\partial/\partial x = -\partial/\partial t$ and we can write (1) (Eames and McIntyre, 1999):

$$DX/Dt = -C^{-1}\partial\varphi/\partial t = -C^{-1}[D\varphi/Dt - \nabla\varphi\cdot\nabla\varphi]. \tag{2}$$

This expression has been derived earlier by Eames et al. (1994) in connection with the Darwin drift caused by a single body moving through an inviscid fluid.

In the same way, we can use that the fluid is incompressible, and write $w = \partial\psi/\partial x$, where ψ is the stream function. Hence

$$DZ/Dt = -C^{-1}\partial\psi/\partial t = -C^{-1}[D\psi/Dt - \nabla\varphi\cdot\nabla\psi] = -C^{-1}D\psi/Dt, \tag{3}$$

where Z is a vertical particle coordinate (Weber et al., 2014). We realize that although (2) and (3) are exact equations, the particle displacements are given in an implicit way. In integrating these equations in time, we must follow a fluid particle which at time $t = 0$ has a position $X_0(a,c), Z_0(a,c)$, where (a,c) are the independent Lagrangian variables. The gradient operator in (2) must also be evaluated in Lagrangian terms. Formally, from (2) and (3) we obtain for the Lagrangian horizontal and vertical displacements, skipping constants of integration:

$$X(a,c,t) = -C^{-1}\varphi(a,c,t) + X_S, \tag{4}$$

$$Z(a,c,t) = -C^{-1}\psi(a,c,t), \tag{5}$$

where

$$X_S(a,c,t) = C^{-1}\int[(\partial X/\partial t)^2 + (\partial Z/\partial t)^2]dt. \tag{6}$$

Here X_S is the Stokes particle displacement.

3. Periodic motion

For a periodic disturbance due to an infinite train of waves, we have that φ and ψ are purely periodic. Integrating (2) over the time period τ, we find

$$X(t+\tau) - X(t) = -C^{-1}[\varphi(t+\tau) - \varphi(t)] + X_S(t+\tau) - X_S(t). \tag{7}$$

Since the bracket on the right-hand side vanishes, one obtains the Lagrangian drift velocity u_L by dividing by the period (Eames and McIntyre, 1999):

$$u_L = \tau^{-1}[X(t+\tau) - X(t)] = \tau^{-1}\left[X_S(t+\tau) - X_S(t)\right] = u_S, \tag{8}$$

where u_S is the Stokes drift. This again demonstrates that the Lagrangian mean drift and the Stokes drift is equal in irrotational, inviscid periodic wave motion, i.e. the Eulerian mean motion vanishes identically (Longuet-Higgins, 1953).

Likewise, from (3) we obtain the trivial result for the vertical Lagrangian drift w_L:

$$w_L = \tau^{-1}[Z(t+\tau) - Z(t)] = -(\tau C)^{-1}[\psi(t+\tau) - \psi(t)] = 0. \tag{9}$$

The result (9) is also valid for horizontally propagating internal waves, which possess vorticity (Weber et al., 2014).

For periodic waves, the Stokes drift can be written from (6)

$$u_S = \tau^{-1}\left[X_S(t+\tau) - X_S(t)\right] = (\tau C)^{-1}\int_t^{t+\tau}\nabla\varphi\cdot\nabla\varphi dt \tag{10}$$

To lowest order, the Lagrangian and Eulerian solutions for periodic waves are equal. Therefore, one can evaluate the Stokes drift to second order from (10) by inserting for the linear Eulerian velocities in the integral on the right-hand side; see Longuet-Higgins (1953).

4. Non-periodic flows

The aim of this paper is to study the effect of the Stokes drift in small amplitude non-periodic waves. In the nonrotating, shallow-water approximation the lowest order surface elevation is governed by

$$\partial^2\eta/\partial t^2 + C_0^2\partial^2\eta/\partial x^2 = 0, \tag{11}$$

where $C_0^2 = gH$. A general solution for isolated disturbances, or pulses, that propagates to the right can be written

$$\eta = F(\xi). \tag{12}$$

Here

$$\xi = (x - C_0 t)/L \tag{13}$$

is the phase, where L is the typical lateral extent of the disturbance. The corresponding velocity components (u,w) become

$$u = gF(\xi)/C_0, \tag{14}$$

$$w = -C_0(z+H)F'(\xi)/(LH), \tag{15}$$

where the prime denotes differentiation with respect to ξ. Hence, since $u = \partial\varphi/\partial x$, $w = \partial\psi/\partial x$, we obtain from (4) and (5)

$$X = -(L/H)\left(\int Fd\xi + H^{-1}\int F^2 d\xi\right), \tag{16}$$

$$Z = (1 + z/H)F(\xi), \tag{17}$$

where the last term in (16) is the Stokes displacement in a linear pulse. Here we have used the shallow-water approximation $u^2 \gg w^2$ in (16).

Assuming that the pulse is symmetrical, and that the peak is situated at the origin at $t = 0$, we can write the maximum horizontal and vertical displacements $\{X\}$ and $\{Z\}$ as

$$\{X\} = 2LH^{-1}\left(\int_0^\infty F(\xi)d\xi + H^{-1}\int_0^\infty F^2(\xi)d\xi\right), \tag{18}$$

$$\{Z\} = H^{-1}(z+H)F(\xi=0). \tag{19}$$

This is a general result for linear symmetric pulses in shallow water where the effect of the Stokes drift has been taken into account.

5. A solitary wave-like pulse

The results in the last section can be applied to linear pulses of any shape. Of particular interest, is the surface shape that resembles a solitary wave in the KdV approximation. In this approximation the surface displacement is given by

$$\eta = A\,\mathrm{sech}^2[\alpha(x - Ct)/H], \tag{20}$$

where

$$\alpha^2 = 3\varepsilon/4, \tag{21}$$

and

$$C^2 = C_0^2(1+\varepsilon). \tag{22}$$

We consider a linear pulse with the shape of a KdV solitary wave, i.e. we take that

$$\eta = F(\xi) = A\mathrm{sech}^2(\xi), \tag{23}$$

where ξ is given by (13). The maximum horizontal and vertical drift of particles during the passage of this wave becomes from (18) and (19)

$$\{X\} = 2L\varepsilon\left(\int_0^\infty \mathrm{sech}^2(\xi)d\xi + \varepsilon\int_0^\infty \mathrm{sech}^4(\xi)d\xi\right) = 2L\varepsilon(1+2\varepsilon/3), \tag{24}$$

$$\{Z\} = (z+H)\varepsilon. \tag{25}$$

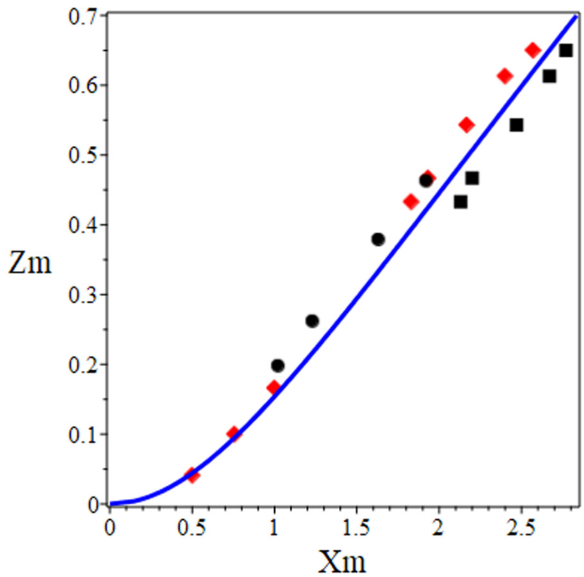

Fig. 1. Maximum horizontal drift versus maximum vertical drift induced by solitary waves. Filled black boxes: data for surface drift from Longuet-Higgins (1981). Filled black circles: data for surface drift from Hsu et al. (2012). Filled red diamonds: selected theoretical values from the ninth order theory by Fenton (1972). Blue curve: present theory from (27)–(28).

It is interesting to compare our results with the surface data in the laboratory experiments on solitary waves by Longuet-Higgins (1981) and Hsu et al. (2012), considering cases of not too steep waves. In linear theory the ratio L/H in (18) is arbitrary, as long as $L \gg H$. From (21) we have for long waves in the KdV approximation that

$$L = 2H/(3\varepsilon)^{1/2}. \tag{26}$$

This appears to be the relevant length scale we should use in our theory for comparison with laboratory experiments. Inserting into (24), the maximum particle displacements at the surface become

$$\{X\}/H = 2(L/H)\varepsilon(1 + 2\varepsilon/3) = (4/3)(3\varepsilon)^{1/2}\,[1 + 2\varepsilon/3], \tag{27}$$

$$\{Z\}/H = \varepsilon. \tag{28}$$

The results for the surface drift are depicted in Fig. 1, where the non-dimensional maximum displacements have been defined as Xm = $\{X\}/H$ and Zm = $\{Z\}/H$.

We note from the figure that our simple theoretical model, with the inclusion of the Stokes drift and the KdV length scale, is able to reproduce fairly well the surface drift in laboratory experiments, as well as the ninth order theoretical results by Fenton (1972). Formally, from a small-amplitude point of view, we have obviously depicted the blue curve in Fig. 1 for too large values of the parameter $\varepsilon = A/H$, where A is the maximum surface amplitude and H is the undisturbed fluid depth. It is therefore quite surprising that the fit with experimental data and highly nonlinear theory are so good for larger ε.

It is an interesting fact that formal power series expansions in the parameter ε for solitary waves breaks down rather quickly. As pointed out by Longuet-Higgins (1981), the Zm *vs.* Xm curve from the KdV solution very soon deviates from the experimental data when ε increases. It is also found that third order theory (Grimshaw, 1971) does not improve that result very much. Even Fenton's ninth order solution loses accuracy for $\varepsilon > 0.7$ (Longuet-Higgins and Fenton, 1974).

The present approach does not constitute a formal power series expansion, although it contains higher powers in ε. We start out with a linear pulse, but this is not a linear drift model. More correctly, it is a hybrid model that includes a nonlinear Stokes drift and a KdV length scale. Judging from Fig. 1, this combination apparently captures some of the essence of this problem.

6. Concluding remarks

Particle drift in isolated surface pulses in shallow water is a quantity that are of considerable interest for coastal engineers and environmentalists. A considerable amount of work has been devoted to the study of steep solitary waves; see Jonsson et al. (2000). However, isolated disturbances traveling along the sea surface are usually not very steep. An example here is a tsunami in the open ocean. Then a simple theoretical approach, like the present one, may yield reasonable results for the particle drift in small and moderately steep surface pulses.

CRediT authorship contribution statement

Jan Erik H. Weber: Ensuring that the descriptions are accurate.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Amick, C.J., Toland, J.F., 1981. On solitary waves of finite amplitude. Arch. Ration. Mech. Anal. 76, 9–95.
Borluk, H., Kalisch, H., 2012. Particle dynamics in the KdV approximation. Wave Motion 49, 691–709.
Boussinesq, J., 1871. Théorie de l'intumescence liquid appelée onde solitaire ou de translation, se propageant dans un chanal rectangulaire. C. R. Acad. Sci., Paris 72, 755–759.
Constantin, A., 2010. On the particle paths in solitary water waves. Q. Appl. Math. 68, 81–90.
Constantin, A., Escher, J., 2007. Particle trajectories in solitary water waves. Bull. Amer. Math. Soc. 44, 423–431.
Constantin, A., Escher, J., Hsu, H.C., 2011. Pressure beneath a solitary water wave: mathematical theory and experiments. Arch. Ration. Mech. Anal. 201, 251–269.
Eames, I., Belcher, S.E., Hunt, J.C.R., 1994. Drift, partial drift and Darwin's proposition. J. Fluid Mech. 254, 201–233.
Eames, I., McIntyre, M.E., 1999. On the connection between Stokes drift and Darwin drift. Math. Proc. Cambridge Philos. Soc. 126, 171–174.
Fenton, J.D., 1972. A ninth-order solution for the solitary wave. J. Fluid Mech. 53, 257–271.
Grimshaw, R., 1971. The solitary wave in water of variable depth. Part 2. J. Fluid Mech. 46, 611–622.
Hsu, H.-C., Chen, Y.-Y., Hwung, H.-H., 2012. Experimental study of the particle paths in solitary water waves. Phil. Trans. R. Soc. A 370, 1629–1637.
Jonsson, I.G., Thorup, P., Tamašauskas, H.T., 2000. Accuracy of asymptotic series for the kinematics of high solitary waves. Ocean Eng. 27, 511–529.
Korteweg, D.J., de Vries, G., 1895. On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves. Phil. Mag. 39, 422–443.
Longuet-Higgins, M.S., 1953. Mass Transport in Water Waves. Philos. Trans. R. Soc. Lond. A Math. Phys. Eng. Sci. 245, 535–581.
Longuet-Higgins, M.S., 1981. Trajectories of particles at the surface of steep solitary waves. J. Fluid Mech. 110, 239–247.
Longuet-Higgins, M.S., Fenton, J.D., 1974. On the mass, momentum, energy and circulation of a solitary wave. II. Proc. R. Soc. A 340, 471–493.
Miles, J.W., 1980. Solitary waves. Ann. Rev. Fluid Mech. 12, 11–43.
Rayleigh, Lord, 1876. On waves. Phil. Mag. 1 (5), 257–279.
Russell, J.S., 1838. Report of the committee on waves. In: Proc. 7th Meeting, Brit. Ass. Adv..
Russell, J.S., 1845. Report on waves. In: Proc. 14th Meeting, Brit. Ass. Adv. Sci. York. p. 1844.
Stokes, G.G., 1847. On the theory of oscillatory waves. Trans. Camb. Phil. Soc. 8, 441–455.
Weber, J.E., 2019. Lagrangian studies of wave-induced flows in a viscous ocean. Deep-Sea Res. II 160, 68–81.
Weber, J.E., Christensen, K.H., Broström, G., 2014. Stokes drift in internal equatorial Kelvin waves; continuous stratification versus two-layer models. J. Phys. Oceanogr. 44, 591–599.