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Nonlinear model for waves on water covered by ice

Resonant interaction equations and analytical solution of zeroth harmonic “group line” measured in experiments

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

Understanding of hydroelastic waves is essential since it interferes in several domains of study. This thesis presents an analytical investigation on wave-ice interaction, where ice is modelled as an elastic material. Mainly, in this thesis the focus is on the three wave interactions in the case of resonance for beam and plate structural elements. In addition, the non-resonant particular solution of the second order problem is also derived, where the analytical solution of the group line is found. The analysis starts by finding dispersion relations of different structural elements. After that, the governing equations of the coupled system between elastic sheet and fluid are derived. Then, the solution of the first order problem is found after applying regular perturbation. Three wave resonance investigation is done for beam and plate, the plate case is compared with the capillary-gravity waves. It is concluded that the hydro-elastic waves have a different behaviour than that of the capillary-gravity waves. The interaction equations in case of resonance were derived by means of multiple scales perturbation method and Green's theorem. It is proven that the stiffness of the elastic sheet covering fluid affects the group velocity and the amplitude of the wave field. To add, elastic parameters present in the interaction coefficients can influence the nonlinear interaction happening. The analytical solution of the group line is found by solving the second order problem of the coupled system, for the non-resonant case. The structure of the group line is probably a cloud looking like a straight line.

Dedication

I choose to dedicate this research to everyone who offered me a silent prayer which led to my success, to everyone who helped and encouraged me. Special thanks dedicated to my family, professors and colleagues.

Acknowledgements

Foremost, I would like to thank my supervisor, Professor Karsten Trulsen, for his guidance and advice since the first day I started my study. He assisted me in solving several problems I faced during those years. His willingness to motivate, encourage and guide me helped tremendously in writing this master thesis. Professor Karsten inspired me greatly with his constant support and ideas.

Likewise, I would like to thank Professor José Carlos Nieto Borge for sharing his thoughts and experience throughout the completion of this thesis.

Special thanks goes to my husband Hamad and son Mostafa, thanks for all the support and inspiration. You always make me smile even when I am stressed. My mom and dad, brothers and sister thank you for everything you did for me, I am extremely grateful. Family in law, I will never forget your help and support, huge thanks. My brother-in-law, thanks a lot for the last moment help.

An honourable thanks goes to my friends for their understanding and support. Moreover, special thanks and gratitude for everyone who helped me and gave me recommendations.

Notations

(x, y, z): Distance in meter

(t): Time in seconds

η(x, t): Surface elevation/displacement of surface in form of waves

φ(x, z, t): Velocity potential of surface elevation

k: Wavenumber

λ: Wavelength

ω: Angular frequency

T: Wave period

A : Complex amplitude

$\boldsymbol{r}$: Position vector

χ: Phase function

$\boldsymbol{c}_g$: Group velocity

ϵ: Characteristic steepness

E : Young's modulus.

I : Second moment of area

$f(x, t)$: Load per unit length

ρ_f : Mass density of fluid

D : Flexural rigidity of the plate

I_m : Moment of inertia per unit area

$q(x, y, t)$: Distributed load on the plate

Further notations are presented in the text where it is applied.

Contents

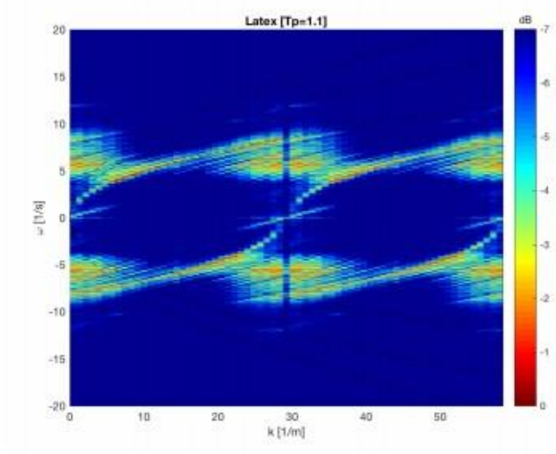
Chapter 1

Introduction

Waves can be defined as a propagating dynamic disturbance of one or more quantities. One needs to understand a large set of possible waves and the interaction between them [1]. In fact, wave interaction with each other and with the surrounding material have a huge impact on the environment and humanity.

1.1 Motivation

Ocean waves affect the way sea-ice forms, contribute to how the ice edge moves, penetrate further into the sea-ice, have more destructive power to break up the ice and to change the distribution of floe sizes because the ice is weaker, and assist in lateral melting. These interactions collectively identify a parametrization currently absent from Earth system models, as well as shortcomings in wave forecasts arising from limited understanding of the impact of sea-ice on ocean waves and vice versa [2]. Ice can be modelled as an elastic material and exploring ice-water interaction will enhance the understanding of wave behaviour in elastic materials. Furthermore, pollution released close to the ice, for example oil contamination due to accidental blow-up during oil extraction near the arctic regions, can be brought under the ice and transported rapidly for long distances. A possible contribution to rapid spreading of pollution is the existence of the group line associated with induced current for the wave motion in ice. The group line was detected in the experiments of *Olsen* (2019) where irregular surface waves on water were sent into a region where the water was covered by an elastic sheet that was supposed to resemble ice [7]. In the figure below, the line coming from the origin is called the zeroth harmonic or the group line.



Plot 1. Logarithmic scaled spectrum plot for Latex sheet with 0.2 mm thickness, Olsen (2019)

Information about the group line streaming is still lacking, therefore, it is essential to investigate the group line phenomenon. In this thesis, analytical investigation is made which may help in comprehension of ice-wave interaction, or elastic material-wave interaction.

1.2 *Previous work*

During the last period, the study of water waves was an area of great interest. The importance of water waves behaviour made researchers more engaged in investigating it. To add, recently research became richer due to the technological evolution, which increased the ability of accurate measurements.

Furthermore, research about ice-wave interaction, and waves that propagate on elastic sheets or membranes can be found in several domains such as fluid mechanics, geophysics, and biophysics. Such interaction is affecting our daily lives and environment. One example is the detection of a series of waves with approximately one-meter amplitude and 18 seconds period 560 km away from the ice edge in the Weddell sea, which lead to the breakup of the ice pack. The latter was encountered by a scientist on the *R/V Polarstern* [6]. The Weddell sea extreme waves have been explained and analysed by *Lui & Mollo Christensen* in their article *Wave Propagation in a Solid Ice Pack* [6]. In their article they derived the dispersion relation for waves under pack compression and compared group velocity to critical mean compressive stress, in addition to carrying out a stability analysis using the non-linear cubic Schrödinger equation and providing a non-linear model to describe waves in ice [7].

Regarding the resonant interaction, Harrison (1909) studied the three-wave resonance of two unidirectional capillary-gravity waves [19]. These waves are commonly called Wilton's ripples, after Wilton (1915) [20], although they were previously described by Harrison (1909). Another study was done by Phillips (1960) where he proved the nonexistence of the three-wave resonance of deep-water gravity waves [21]. To add, the quartet resonance of gravity waves was investigated by Phillips (1960). After five years, McGoldrick (1965) studied the configuration of resonant triads of gravity-capillary waves on infinite depth [15].

1.3 *Research Questions*

Wave-ice interaction theory is a wide field and includes various cases. In this Master thesis an investigation of the wave-ice interaction, where the ice is modelled as a thin elastic sheet, is performed. In the nonlinear problem, the risk of resonant growth is investigated and found to be present for certain wave numbers in three wave interaction. This analysis made us able to comprehend the difference between the behaviour of elastic wave resonance triads and that of capillary gravity waves triads. The possibility of resonance blow up leads to the need of slow modulations on time and space and a solvability condition that arrests that growth. The solvability condition, which is the resonance interaction equations, is found by means of Green's theorem. Those equations gave more understanding of how the interaction coefficients depend on the elastic parameters. Besides, how the group velocity and amplitudes were affected by the elasticity of the structural element. Furthermore, the experiments showed the presence of a strong zeroth harmonic ‘‘group line’’. The existence of the group line implies an induced streaming that can possibly carry pollution under the ice. For that the analytical solution of this group line was derived. The latter will lead to computing the structure and predicting the distribution of energy intensity of the group line.

1.4 Outline

Chapter 2 includes mathematical background of some concepts used during fulfilment of this thesis. It presents a mathematical description of ocean waves, short explanation of the method of multiple scales, and Green’s second identity.

In Chapter 3 a theoretical derivation of the linear dispersion relation of string, Euler-Bernoulli beam and plate is done. The plots of the dispersion relations are included and discussed shortly.

Chapter 4 contains the second order problem by regular perturbation and multiple scales method for the Euler-Bernoulli beam. It also presents resonance investigation of three wave resonance and the resonant interaction equations.

Chapter 5 presents same approach made in chapter 4 for a different structural element which is the plate. The resonant interaction equations for plate over fluid are derived. This chapter also includes the particular solution for the second order problem, where the analytical solution of the group line is derived.

In Chapter 6, short discussion of the dispersion relations is stated. Then, deeper consideration of the resonance analysis is presented. Furthermore, the interaction equations are discussed in section 6.3. Lastly, the nonlinearly forced response is discussed.

Chapter 7 presents a conclusion of this thesis and suggestions for further work are stated.

Chapter 2

Mathematical background

Richard Feynman wrote about water waves in his book *The Feynman lectures on physics* [4] ‘‘the worst possible example [of waves], because they are in no respect like sound and light; have all the complications that waves can have’’ [5]. In general, ocean waves can be described as unpredictable and random. In this chapter, we will present a mathematical description of ocean waves. We will also state some definitions that will be used in the coming chapters. Besides, we will present the multiple scales method which is one of the most important methods in the applied mathematics domain. This method will be applied while solving the nonlinear system in chapter 4. Finally, from vector calculus, Green’s second identity is expressed.

2.1 Description of waves

In the very beginning, it is important to describe a monochromatic wave, which is also known as simple harmonic wave. By definition, it is a sinusoidal wave with a unique period T and a unique wave length λ [9]. The monochromatic wave is represented as follows

$$\eta(r,t) = Re\{Ae^{i(\boldsymbol{k}.\boldsymbol{r}-\omega t)}\} = |A|\cos(\boldsymbol{k}.\boldsymbol{r} - \omega t + \arg A) \qquad (2.1)$$

Where,
A is a complex amplitude
$\boldsymbol{k}$ is the wavenumber vector
$\boldsymbol{r}$ is the position vector

The phase function is denoted by

$$\chi = (\boldsymbol{k}.\boldsymbol{r} - \omega t + \arg A)$$

The phase surface is the equiscalar surface of the phase function $\chi =$ constant.
One property of the wave is the phase speed, which is the speed that the phase surface moves in its normal direction. [9]

Waves can be interpreted as free waves if they can exist without being forced, if not it will be a forced wave. Free waves satisfy what is called the dispersion relation, which is a relation between the wavenumber vector $\boldsymbol{k}$ and the angular frequency ω.
Linear dispersion relation is the relation that is independent of the amplitude A. On the other hand, if the dispersion relation is depending on A , then it is called nonlinear. The dispersion relation gives information about the behaviour of the wave.

If the dispersion relation specifies a proportionality between the wavenumber k and the angular frequency ω, then the waves are said to be non-dispersive, otherwise we say they are dispersive. For non-dispersive waves, the phase speed does not depend on the frequency, while for dispersive waves it depends on the frequency. If the dispersion relation depends only on the wavenumber k and not on the direction of the wavenumber vector **k** the waves are said to be isotropic, otherwise we say they are anisotropic. Isotropic waves have the same properties in all directions, anisotropic waves have different properties in different directions. [9]

By definition, the velocity of propagation through space of the whole envelope of the wave is called the group velocity. Mathematically, the group velocity is the gradient $\boldsymbol{c}_g = \frac{\partial \omega}{\partial \boldsymbol{k}}$.

Last but not least, if the system studied, meaning the governing equations was linear and unforced, then the waves will be free. This means that the solution will be of the form stated in equation (2.1). The principle of superposition can be used to combine irregular wave solutions.

$$\eta(\boldsymbol{r},t) = \sum_n A_n e^{i(\boldsymbol{k}_n.\boldsymbol{r}-\omega_n t)} \tag{2.2}$$

Each term in the above equation is considered a free monochromatic wave with a complex amplitude A_n.

In case the amplitudes A_n are independent stochastic variables, then the Central Limit theorem predicts that the statistical distribution of the resulting wave field should be Gaussian. [9]

For nonlinear governing equations, the linear superposition principle will be not valid. Nevertheless, for weakly nonlinear equations, with nonlinear contribution characterized by small parameter ϵ <<1, then the irregular wave solution can be expressed as follows

$$\eta(\boldsymbol{r},t) = \eta_L(\boldsymbol{r},t) + \epsilon\eta_{NL}(\boldsymbol{r},t) \tag{2.3}$$

2.2 The Multiple Scales Perturbation Method

Equations arising from mathematical models usually cannot be solved in exact form. Therefore, we often resort to approximation and numerical methods. Foremost among approximation techniques are perturbation methods [10]. For a model equation that includes small terms, perturbation methods can be applied in order to find an approximation solution for the problem. If the effect of the physical process is small, then small terms will appear in the equations. To illustrate, the viscosity could be small contrasted to the advection in a fluid flow problem. Another example is in the motion of a projectile, where the force caused by air resistance may be small compared to gravity. These low-order effects are represented by terms in the model equations, and when compared to the other terms, are negligible. When scaled properly, the order of the magnitude of these terms is represented by small coefficient parameter called ϵ. Perturbation methods can be used to all types of equations considered in applied mathematics. [10]

The method of multiple scales is one of the most important perturbation methods. This method contains techniques that lead to constructing uniformly valid approximations to the solutions of perturbation problems for several values of independent variables. The headstone of the method of multiple scales is introducing fast scale and slow scale variables for an independent variable, then dealing with the introduced scales as if they are independent. As a consequence of the latter step, the additional freedom that came from the new independent variables will be utilized in removing the secular terms. The latter puts constraints on the approximate solution, which are called solvability conditions. [11]

The starting point in the procedure of the method of multiple scales is letting $\tau = \epsilon t$ (τ defines a long-time scale because τ is not small when t is if order 1/ ϵ or larger [12]) Then, assuming a perturbation expansion as follows,

$$y(t) = Y_0(t,\tau) + \epsilon Y_1(t,\tau) + \cdots \tag{2.4}$$

Subsequently, for finding the derivatives of $y(t)$,the chain rule is used for partial differentiation.

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \cdots \tag{2.5}$$

And

$$\frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \cdots\right)^2 = \frac{\partial^2}{\partial {t_0}^2} + 2\epsilon \frac{\partial^2}{\partial t_0 \partial t_1} + \cdots \tag{1.6}$$

The next step will be substituting in the equations we need to solve, then using the accomplished freedom in removing the secular terms.

The multiple scales method was studied by many researchers and it is used in several physical and mathematical problems. For instance, analysis of a damped oscillator, studying the Rayleigh oscillator and solving boundary layer problems. The multiple scales method is used in the solution process of our system, where the plate or beam is placed over fluid.

2.3 Green's second identity

Generally, Green's theorem in vector calculus links a line integral around a simple closed curve C to a double integral over the plane region D bounded by C. One special case of Green's theorem is Stokes' Theorem for two dimensional systems. Using Green's theorem, a set of identities was derived and named after the mathematician *George Green*, the discoverer of the theorem [13].
In this section, we will briefly shine light on Green's second identity, which is a vector calculus identity that relates the bulk with the boundary of a region on which differential operators act.

Green's second identity is derived from Green's first identity for the pair of functions (u, v)over the domain D. The first identity in three dimensional is:

$$\iiint v\Delta u dx = \oint\!\!\oint \left(v\frac{\partial u}{\partial n}\right) dS - \iiint \nabla u . \nabla v dx \tag{2.7}$$

When interchanging *u* and *v*, Green's first identity will be then,

$$\iiint u\Delta v dx = \oint\!\!\oint \left(u\frac{\partial v}{\partial n}\right) dS - \iiint \nabla u . \nabla v dx \tag{2.8}$$

The second term of the right-hand side in equations (2.7) and (2.8) are exactly the same, then if we subtract the above equations, we will get,

$$\iiint (u\Delta v - u\Delta v) dx = \oint\!\!\oint \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \tag{2.9}$$

The above equation is known as Green's second identity for a pair of functions (u, v).
Later during the chapters, Green's second identity will be used to find a solvability condition that prevents the unbounded resonant growth of the nonlinear problem. This is accomplished by relating the bulk with the boundary of a region.

Chapter 3

Linear dispersion relations for transverse waves in different structural elements

Structural elements are the elements used in structural analysis making the study of complex systems simpler. Those elements are designed to withstand forces and moments. In the following sections different kinds of elements will be studied such as line elements (e.g., string and beam), surface elements (e.g., plate, free surface).

Transversal waves are waves in which particles vibrate at right angles to the direction of propagation of wave. This type of wave is used in this chapter as a disturbance of the structural elements.

3.1 Transverse waves in a string:

In this section we will derive the wave equation of transverse waves in a string, in order to derive the dispersion relation. To start the analysis, it is beneficial to define the system studied. Consider a string under small transverse displacement, the x-coordinate is along the string and the displacement ψ is applied in the z-direction (tangent to the string).
Consider a segment of the string in the x-direction that separates two points x and $x + dx$ as shown in the figure below

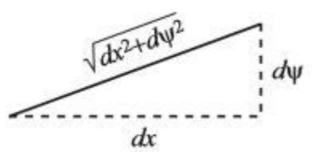


Fig 1. Segment of the string in the x-direction. Available at: https://www.people.fas.harvard.edu/~djmorin/waves/transverse.pdf

Now using Newton's second law of motion $F = ma$, the wave equation is to be found.

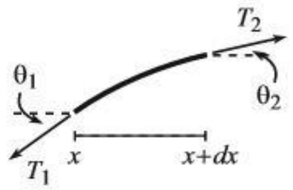


Fig 2. Forces acting on the string segment. Available at: https://www.people.fas.harvard.edu/~djmorin/waves/transverse.pdf

Starting from calculating the tension forces in the string where T is string tension magnitude.

$$\vec{T}_1 = -T \frac{\overrightarrow{\iota_x} + \psi_x \overrightarrow{\iota_z}}{\sqrt{(1 + {\psi_x}^2)}} \tag{3.1}$$

$$\vec{T}_2 = T \frac{\overrightarrow{\iota_x} + \psi_x \overrightarrow{\iota_z}}{\sqrt{(1 + {\psi_x}^2)}} \tag{3.2}$$

Taylor expansion is applied on T_2 around point x to get

$$\vec{T}_2 = T \left(\frac{\overrightarrow{\iota_x} + \psi_x \overrightarrow{\iota_z}}{\sqrt{(1 + {\psi_x}^2)}} + dx \frac{\partial}{\partial x} \frac{\overrightarrow{\iota_x} + \psi_x \overrightarrow{\iota_z}}{\sqrt{(1 + {\psi_x}^2)}} \right) \tag{3.3}$$

The total force will be as follows

$$\vec{F}_{net} = T dx \left(\frac{\psi_{xx} \sqrt{1 + {\psi_x}^2}\, \overrightarrow{\iota_z} - (\overrightarrow{\iota_x} + \psi_x \overrightarrow{\iota_z}) \frac{\psi_x \psi_{xx}}{\sqrt{(1 + {\psi_x}^2)}}}{1 + {\psi_x}^2} \right) \tag{3.4}$$

Knowing that the slope ψ_x is considered small, only the terms that are linear in ψ are included. Then the total force is

$$\vec{F}_{net} = T dx \psi_{xx} \overrightarrow{\iota_z} \tag{3.5}$$

Now applying Newton’s law knowing that the mass of small segment of string is μdx where µ is the mass per unit length

$$F = T dx \psi_{xx} = \mu dx \psi_{tt} \tag{3.6}$$

The wave equation of a string under transverse displacement is then

$$\psi_{tt} = \frac{T}{\mu} \psi_{xx} \tag{3.7}$$

Assuming that the solution is of the form $\psi(x,t) = A sin(kx - \omega t)$, knowing that ω is the frequency and k is the wave number. Then the dispersion relation is $\omega = \sqrt{\frac{T}{\mu}}k$.

3.2 Transverse waves in a beam:

By definition, beams are structural elements that resist loads applied laterally to the beam's axis. In this section, Euler-Lagrange equation and Newton's second law approaches are used to find the general form of Euler-Bernoulli beam equation.
For simpler analysis, consider a homogeneous beam with constant rectangular cross section and it endures only linear elastic deformation. Also, it is slender with small deflections to be taken into consideration.
Starting with the static beam equation; that is the relation between the deflection in the beam and the applied load.
The Euler- Bernoulli equation

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w(x,t)}{\partial x^2}\right) = f(x,t) \tag{3.8}$$

Where,
$w(x)$ is the deflection of the beam in the z-direction at a position x.
E is Young's modulus.
$f(x,t)$ is the load per unit length
I is the second moment of area, with respect to the y-axis, of the beam's cross-section which is given by

$$I = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} z^2 dz dy = \frac{ba^3}{12} \tag{3.9}$$

Where, b is the width, and a is the thickness of the beam.
Here, and due to forthcoming use, parallel axis theorem is applied for the second moment of area, then $I = \frac{ba^3}{12} + \frac{ba^3}{4}$
For such geometry and composition, EI will be considered constant (flexural rigidity) and the beam's deflection (w) can be written as follows

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} = f(x,t) \tag{3.10}$$

Derivation of dynamic beam equation using the Euler-Lagrange equation

Starting by the functional S which is given by

$$S = \int_{t_1}^{t_2} \int_0^L \frac{1}{2} \mu \left(\frac{\partial w}{\partial t}\right)^2 - \frac{1}{2} EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + f(x,t) w(x,t)\, dxdt \tag{3.11}$$

The first term is the kinetic energy knowing that μ is the mass per unit length, the second term is the potential energy due to internal forces which is considered with negative sign (sign convention).
Now, Euler-Lagrange equation is used to determine the function that minimizes the functional S.
Lagrangian is in this case

$$\mathcal{L} = \frac{1}{2} \mu \left(\frac{\partial w}{\partial t}\right)^2 - \frac{1}{2} EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + f(x,t) w(x,t) = \mathcal{L}(x, t, w, \dot{w}, w_{xx}) \tag{3.12}$$

The general form of the Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial w} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{w}}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right) = 0 \tag{3.13}$$

Then,

$$\frac{\partial \mathcal{L}}{\partial w} = f\ ; \quad \left(\frac{\partial \mathcal{L}}{\partial \dot{w}}\right) = \mu \dot{w}; \quad \left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right) = -EI w_{xx} \tag{3.14}$$

Replace into the general form in (3.13) to get

$$f - \mu \ddot{w} - (EI w_{xx})_{xx} = 0 \tag{3.15}$$

And when we consider the beam homogenous E and I will be constants, ending up with a simpler equation which is

$$EI \frac{d^4 w}{dx^4} = -\mu \frac{\partial^2 w}{\partial t^2} + f \tag{3.16}$$

Equation (3.16) is the governing equation for the dynamics of the Euler-Bernoulli beam.
As in the previous section, assume that the solution of equation (3.16) is of the form $w = Ae^{i(kx - wt)}$ and $f(x,t)$ equals zero, then the dispersion relation is $\omega = \pm \sqrt{\frac{EI}{\mu}} k^2$.

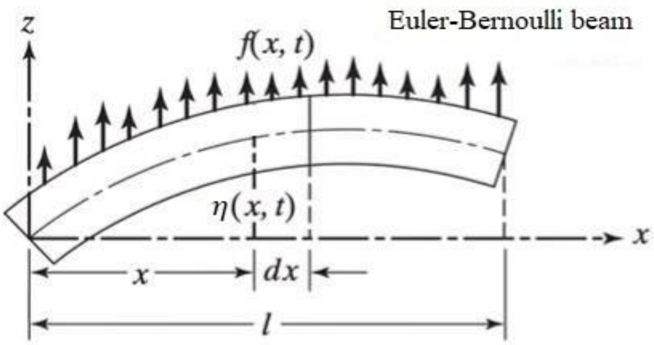


Fig 3. Euler/Bernoulli beam. Available at:
https://www.youtube.com/watch?v=3R8q3becNvg

Derivation of dynamic beam equation using the Newton's second law

Consider Euler- Bernoulli beam as presented in figure 3,
$\eta(x,t)$ is the transverse displacement
$f(x,t)$ is the load per unit length

Take a differential slice dx

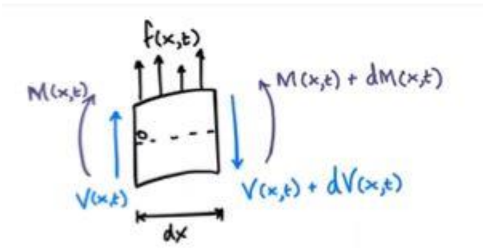


Fig 4. Differential slice of the beam. Available at:
https://www.youtube.com/watch?v=3R8q3becNvg

In the figure, $V(x,t)$ stands for the shear force, and $M(x,t)$ stands for the moment. The dotted line is the neutral axis of the beam. It is an axis in the cross section of a beam, along which there are no longitudinal stresses or strains.
Firstly, Newton's second law is applied to the beam element
The force equation is

$$\sum F = ma$$

$$V - (V + dV) + f(x,t)dx = \rho A(x)dx\frac{\partial^2 \eta(x,t)}{\partial t^2} \tag{3.17}$$

A(x) is the cross-sectional area of the beam, ρ is the mass density.

The moment equation at point O is (considering counter clockwise direction as positive)

$$(M + dM) - M - (V + dV)dx + \frac{f(x,t)dxdx}{2} = 0 \tag{3.18}$$

Note that the zero on the right-hand side is due to the Euler-Bernoulli assumption, that is there is no rotation at that element.

We know from calculus that

$$dV = \frac{\partial V}{\partial x}dx; \tag{3.19-1}$$

$$dM = \frac{\partial M}{\partial x}dx \tag{3.19-2}$$

Substitute equation (3.19-1) into equation (3.17) to get

$$\rho A(x)\frac{\partial^2 \eta(x,t)}{\partial t^2}dx = -\frac{\partial V}{\partial x}dx + f(x,t)dx \tag{3.20}$$

Now, substitute equation (3.19-2) into equation (3.18) to get

$$\frac{\partial M}{\partial x}dx - Vdx - \frac{\partial V}{\partial x}dx^2 + f(x,t)\frac{(dx)^2}{2} = 0 \tag{3.21}$$

It should be noted that dx is infinitesimal, then dx^2 is negligible.
Consequently, equation (3.21) can be written as

$$\frac{\partial M}{\partial x} = V \tag{3.22}$$

Substitute (3.22) into (3.20)

$$\rho A(x)\frac{\partial^2 \eta(x,t)}{\partial t^2} = -\frac{\partial^2 M(x,t)}{\partial x^2} + f(x,t) \tag{3.23}$$

With Euler-Bernoulli beam assumption in mind, the moment $M(x,t)$ can be written as

$M(x,t) = EI\frac{\partial^2 \eta(x,t)}{\partial x^2}$ where EI is the flexural rigidity of the beam.

Altogether

We can write the Euler-Bernoulli beam equation as

$$\rho A(x)\frac{\partial^2\eta(x,t)}{\partial t^2} + EI\frac{\partial^4\eta(x,t)}{\partial x^4} = f(x,t) \qquad (3.24)$$

Both approaches used in this section led to the general form of Euler-Bernoulli equation.

Suppose now that the beam is situated on a fluid such that the force $f(x,t)$ is the net force between the beam and the fluid. Here, the contact area between the fluid and the beam $A_c = bdx$, knowing that $f(x,t)$ is the force per unit length dx.

For incompressible fluid and inviscid flow, the continuity equation and Navier Stokes equations can be written as

$$\nabla.\vec{v} = 0 \qquad (3.25\text{-}1)$$

$$\rho_f\frac{D\vec{v}}{Dt} \cong \rho_f\vec{g} - \nabla \mathrm{p} \qquad (3.25\text{-}2)$$

Equation (3.25-2) is known also as Euler's equation, where $\vec{v} = (u,v,w)$is the fluid velocity. Assume a potential flow ($\vec{v} = \nabla\varphi$), where $\varphi(\boldsymbol{r},t)$ denotes the velocity potential.

Integrate with respect to all space coordinates to get Euler's equation, which is also known as Bernoulli's pressure equation

$$p_f(\mathrm{x,z,t}) = p_b + p_a - \rho_f\left\{\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + gz\right\}$$

where p_f is the fluid pressure, p_b is the pressure from the beam and p_a is the atmospheric pressure.

z is the vertical coordinate.

ρ_f is the mass density of fluid.

The dynamic surface condition at the interface between water and the beam is

$$p_f - p_b - p_a = \rho a\frac{\partial^2\eta(x,t)}{\partial t^2} + \frac{EI}{b}\frac{\partial^4\eta(x,t)}{\partial x^4} = -\rho_f\left\{\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + g\eta\right\} \ at\ z = \eta \qquad (3.26)$$

where a is the thickness, b is the width of the beam.

The kinematic surface condition at the interface between water and the beam is

$$\frac{\partial \eta}{\partial t} + \nabla\varphi . \nabla\eta = \frac{\partial \varphi}{\partial z} \quad at\ z = \eta \tag{3.27}$$

For simplification, only linear factors are used in this derivation. Apply Taylor expansion around $z = 0$,then leading order problem will be

$$\frac{\partial \eta}{\partial t} = \frac{\partial \varphi}{\partial z} \tag{3.28}$$

$$\rho_f \left\{ \frac{\partial \varphi}{\partial t} + g\eta \right\} + \left(\rho a \frac{\partial^2 \eta(x,t)}{\partial t^2} \right) + \frac{EI}{b} \frac{\partial^4 \eta(x,t)}{\partial x^4} = 0 \tag{3.29}$$

Equations (1.28) and (1.29) can be solved by assuming a monochromatic elementary wave solution

$$\begin{pmatrix} \eta(\boldsymbol{x},t) \\ \varphi(\boldsymbol{r},t) \end{pmatrix} = \begin{pmatrix} \hat{\eta} \\ \hat{\varphi}(z) \end{pmatrix} e^{i(\boldsymbol{k}.\boldsymbol{x} - \omega t)} \tag{3.30}$$

$\boldsymbol{k} = k_x i_x + k_y i_y$ is the wave vector

$\boldsymbol{x}$ is the horizontal position vector

$\boldsymbol{r}$ is the three-dimensional position vector

ω is the angular frequency

In order to find the dispersion relation for the wave equation in (3.29), the solution of $\hat{\varphi}(z)$ is found first from the continuity equation and the bottom boundary condition.

$$\nabla^2 \varphi = 0 \ \text{ at } -\text{h} < \text{z} < 0 \tag{3.31}$$

$$\frac{\partial \varphi}{\partial z} = 0 \ \ at\ \ z = -h \tag{3.32}$$

Where, h is the depth of the fluid

Assume that $\hat{\varphi}(z) = e^{\lambda z}$

Then the characteristic polynomial from equation (3.31) is

$$-k^2 + \lambda^2 = 0 \tag{3.33}$$

As a result, $\lambda = \pm k$

Take the solution of $\hat{\varphi}(z) = \frac{Ccosh\ k(z+h)}{\sinh(kh)} + \frac{Dsinh\ k(z+h)}{\sinh(kh)}$, it is preferred to divide by $\sinh(kh)$ to avoid reaching infinity as h tends to infinity.

Substituting $\hat{\varphi}(z)$ in to the boundary condition (3.32) will lead to

$\hat{\varphi}(z) = C\frac{\cosh k(z+h)}{\sinh(kh)}$, where $k = \sqrt{k_x^2 + k_y^2}$ is the wave number.

The equations (3.29) and (3.31) then give the following linear system

$$\begin{pmatrix} -i\omega & -k \\ \frac{EI}{b}k^4 + \rho_f g - \rho a\omega^2 & -\rho_f i\omega \coth(kh) \end{pmatrix} \begin{pmatrix} \hat{\eta} \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.34}$$

Finally, to avoid trivial solution, the determinant of coefficient matrix must be zero, which will therefor give the dispersion relation for a beam over fluid,

$$\omega^2 = \frac{\frac{EI}{\rho_f b}k^5 + gk}{\frac{\rho a}{\rho_f}k + \coth(kh)}$$

3.3 *Transverse waves in a plate:*

This section presents a discussion of transvers waves in plates under no load, and plates placed over fluid. Plates are plane structural elements that have small thickness compared to the planer dimensions.
Finding the equations of motion in order to model the dynamics of a plate, some simplifying assumptions are used.

- The plate is not subjected to any in-plane forces when it is under transverse vibration, which means that the neutral fibres in the plate remains unstrained
- There is no shear deformation in the z-direction (Kirchhoff hypothesis)
- During the transverse deflection of the plate the slopes remain small
- Plate thickness is constant

Consider a plate of thickness a laying in the x-y plane, and it deflects in the z-direction. Take a differential element from this plate, the stresses acting on this element are the normal stresses σ_{xx}, σ_{yy}, and shear stresses $\sigma_{xy}, \sigma_{xz}, \sigma_{yz}$.

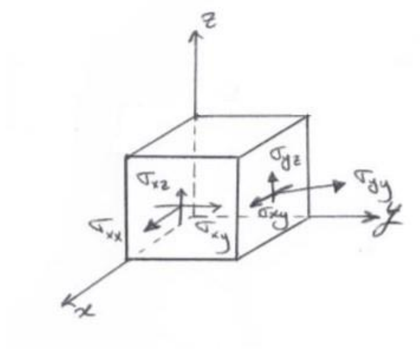


Fig 5. Stresses on infinitesimal element

In order to have a two-dimensional model, integrate over the thickness.

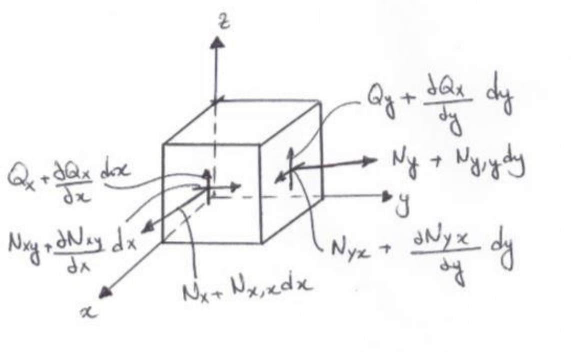


Fig 6. *Force resultants on infinitesimal element*

The stress resultants per unit length due to the normal stresses are

$$N_x = \int_{-a/2}^{a/2} \sigma_{xx}\, dz\ ;\ \ N_y = \int_{-a/2}^{a/2} \sigma_{yy}\, dz\ ; \qquad (3.35)$$

In plane normal stress resultant per unit length due to shear stress is

$$N_{xy} = \int_{-a/2}^{a/2} \sigma_{xy}\, dz \qquad (3.36)$$

Out of plane shear stress resultants per unit length are

$$Q_x = \int_{-a/2}^{a/2} \sigma_{xz}\, dz\ ;\ \ Q_y = \int_{-a/2}^{a/2} \sigma_{yz}\, dz\ ; \qquad (3.37)$$

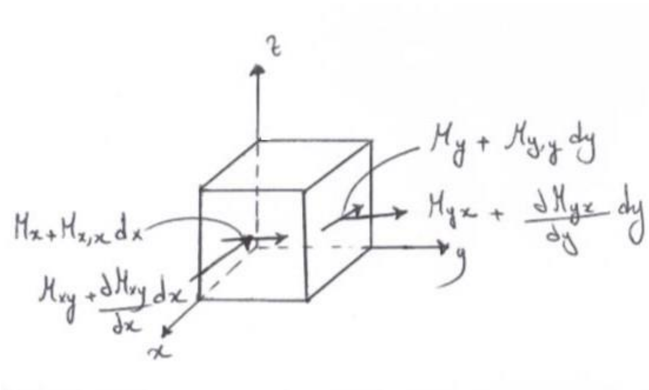


Fig 7. Moment resultants on infinitesimal element

Moment resultants per unit length are

$$M_x = \int_{-a/2}^{a/2} z\sigma_{xx}\, dz\,; \quad M_y = \int_{-a/2}^{a/2} z\sigma_{yy}\, dz\,; \quad M_{xy} = \int_{-a/2}^{a/2} z\sigma_{xy}\, dz \qquad (3.38)$$

Hooke's law is to be used next in order to find a relation between the stresses and strains. Hooke's law is also known as the law of elasticity, it states that for relatively small deformations of an object, the displacement or size of the deformation is directly proportional to the deforming force or load [22].

The relation between stresses and strains is then

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right]; \quad \sigma_{yy} = \frac{E}{1-\nu^2}\left[\nu\varepsilon_{xx} + \varepsilon_{yy}\right]; \qquad (3.39)$$

$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}$$

$\varepsilon_{xz} = \varepsilon_{yz} = 0$ there is no shear deformation of the element along the z-direction (Kirchhoff hypothesis).

E is Young's modulus and ν is Poisson ratio.
Next, the strain in terms of the deflection of the plate is calculated from the geometry of deformation of the plate as shown in figure 8.
$tan\theta = \frac{\partial \eta}{\partial x} \cong sin\theta \cong \theta$(for small angle$\theta$) the location of a point from the neutral surface.
The deflection in the x-direction and the y-direction can approximately written as
$u(x, y, z, t) = -z\frac{\partial \eta(x,y,t)}{\partial x}\,; v(x, y, z, t) = -z\frac{\partial \eta(x,y,t)}{\partial y}$

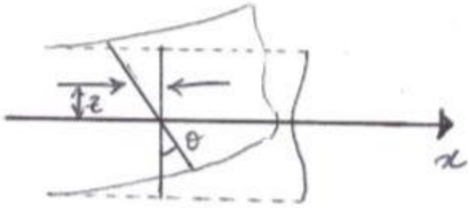


Fig 8. Geometry of deformation

Then, the strain field will be

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} = -z\frac{\partial^2\eta}{\partial x^2} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} = -z\frac{\partial^2\eta}{\partial y^2} \\ \varepsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\frac{\partial^2\eta}{\partial x\partial y} \end{aligned} \tag{3.40}$$

Clearly, the strains are linear in the z-direction, then the stresses in equation (3.39) are also linear in z- direction. Consequently, the normal stress resultants N_x, N_y, N_{xy} in equations (3.35) and (3.36) will go to zero, and the non-zero remaining resultants are the shear and moment.

The moment resultants can be calculated by substituting equations (3.39) and (3.40) in the moment resultants expressions in equation (3.38) and doing the integration

$$M_x = \int_{-a/2}^{a/2} z\sigma_{xx}\, dz$$

$$= \int_{-a/2}^{a/2} z\left(\frac{E}{1-\nu^2}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right]\right) dz$$

$$= \int_{-a/2}^{a/2} z^2\left(-\frac{E}{1-\nu^2}\left[\frac{\partial^2\eta}{\partial x^2} + \nu\frac{\partial^2\eta}{\partial y^2}\right]\right) dz$$

$$= -\frac{Ea^3}{12(1-\nu^2)}\left[\frac{\partial^2\eta}{\partial x^2} + \nu\frac{\partial^2\eta}{\partial y^2}\right]$$

Then,

$$\begin{aligned} M_x &= -D\left[\frac{\partial^2\eta}{\partial x^2} + \nu\frac{\partial^2\eta}{\partial y^2}\right] \\ M_y &= -D\left[\frac{\partial^2\eta}{\partial y^2} + \nu\frac{\partial^2\eta}{\partial x^2}\right] \\ M_{xy} &= -D(1-\nu)\frac{\partial^2\eta}{\partial x\partial y} \end{aligned} \tag{3.41}$$

D is the flexural rigidity of the plate $D = \frac{Ea^3}{12(1-\nu^2)}$

Altogether, the equations of motion can be written as

Transverse dynamics

$$\rho a dx dy \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial Q_y}{\partial y} dy dx + \frac{\partial Q_x}{\partial x} dx dy \qquad (3.42)$$

Rotational dynamics

Rotation about the y- axis

$$I_m \frac{\partial^2}{\partial t^2}\left(\frac{\partial \eta}{\partial x}\right) = -\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} + Q_x \qquad (3.43)$$

Rotation about the x- axis

$$I_m \frac{\partial^2}{\partial t^2}\left(\frac{\partial \eta}{\partial x}\right) = -\frac{\partial M_y}{\partial x} - \frac{\partial M_{xy}}{\partial x} + Q_y \qquad (3.44)$$

Where, I_m is the moment of inertia per unit area $I_m = \frac{\rho a^3}{12} + \frac{\rho a^3}{4}$

Using equations (3.43) and (3.44),Q_x and Q_y in equation (3.42) can be written as function of deflection of plate and moments.
Finally, the equation of motion of a plate can be written as

$$\rho a \frac{\partial^2 \eta}{\partial t^2} - I_m \nabla^2 \frac{\partial^2 \eta}{\partial t^2} + D\nabla^4 \eta = 0 \qquad (3.45)$$

If the moment of inertia per unit area I_m was small, equation (3.45) will reduce to

$$\rho a \frac{\partial^2 \eta}{\partial t^2} + D\nabla^4 \eta = 0 \qquad (3.46)$$

Assume that the solution of equation (3.45) is of the form $\eta(x, y, t) = Ae^{i(\boldsymbol{k}.\boldsymbol{r} - \omega t)}$

$\boldsymbol{k}$ is the wave vector

$\boldsymbol{r}$ is the position vector

ω is the angular frequency

Then the dispersion relation is $\omega^2 = \frac{Dk^4}{\rho a + Ik^2}$

In case of presence of distributed load on the plate such as lateral pressure $q(x, y, t)$, that is force per unit area, the equation of motion (1.45) will be of the form

$$\rho a \frac{\partial^2 \eta}{\partial t^2} - I_m \nabla^2 \frac{\partial^2 \eta}{\partial t^2} + D\nabla^4 \eta + q(x, y, t) = 0 \tag{3.47}$$

Just like the process done to the beam, suppose that the plate is situated on a fluid such that the force $q(x, y, t)$ is the fluid pressure on the plate. In order to find the dispersion relation for a plate over fluid, the only difference will be in the dynamic condition.

The dynamic condition on the interface between the fluid surface and the plate is

$$\rho a \frac{\partial^2 \eta}{\partial t^2} - I_m \nabla^2 \frac{\partial^2 \eta}{\partial t^2} + D\nabla^4 \eta + \rho_f \frac{\partial \varphi}{\partial t} + \rho_f g \eta = 0 \; at \; z = 0 \tag{3.48}$$

After finding the solution for $\varphi(z)$, equations (3.27) and (3.48) give the following linear system

$$\begin{pmatrix} -i\omega & -k \\ \rho_f g - \rho a \omega^2 - I_m \omega^2 k^2 + Dk^4 & -\rho_f i\omega \coth(kh) \end{pmatrix} \begin{pmatrix} \hat{\eta} \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.49}$$

Finally, to avoid trivial solution, the determinant of coefficient matrix must be zero, which will therefor give the dispersion relation of a plate over fluid

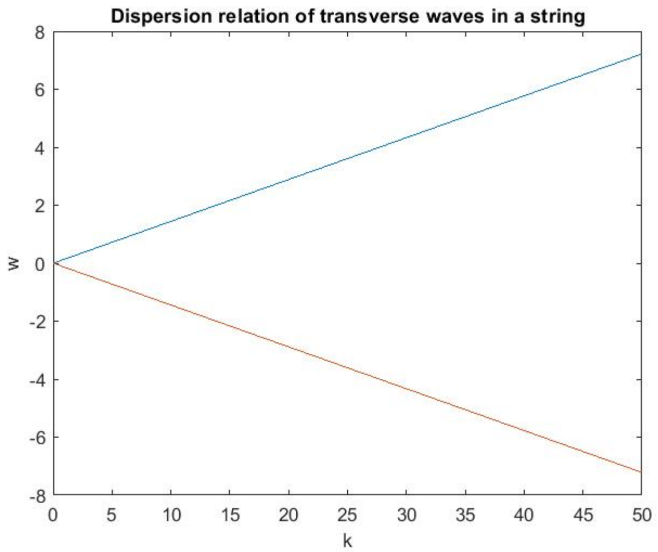
$$\omega^2 = \frac{\frac{D}{\rho_f} k^5 + gk}{\frac{\rho a}{\rho_f} k + \frac{I_m}{\rho_f} k^3 + \coth(kh)}$$

3.4 *Dispersion relations plots:*

In the previous three sections, dispersion relations for string, Euler-Bernoulli beam, beam over fluid, plate and plate over fluid were found. Now, in order to discuss those dispersion relations further, plots for the angular frequency versus the wavenumber for the structural elements studied are represented below. The plots are created using MATLAB codes, the vertical axis shows the angular frequency$(1/s)$ and the horizontal axis represents the wavenumber$(1/m)$.

To start by the dispersion relation of the string which is given by

$$\omega = \sqrt{\frac{T}{\mu}} k$$

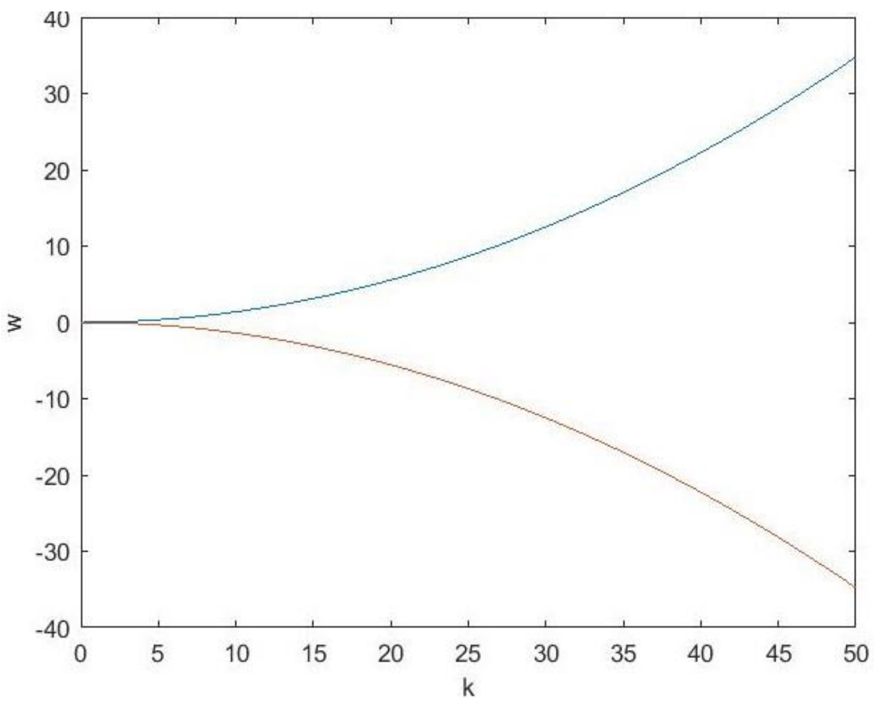


plot 2. Dispersion relation of transverse waves in a string

In the graph above, the value of the angular frequency increases proportionally to the value of the wave number. The dispersion relation above shows that the transverse waves in string are non-dispersive, since the wave number and angular frequency are proportional. As well as string dispersion relation is isotropic because waves behave equally in all directions, meaning that the dispersion relation depends on the value of wavenumber and not the wavenumber vector.

Secondly, the Euler-Bernoulli beam dispersion relation

$$\omega = \pm\sqrt{\frac{EI}{\mu}}k^2$$



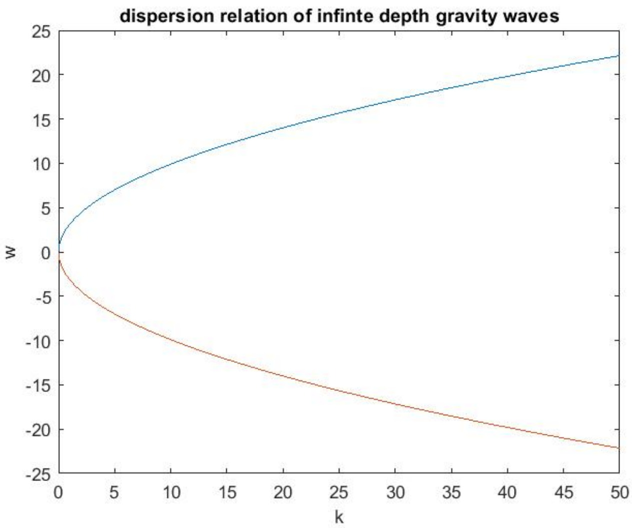
plot 3. Dispersion relation of transverse waves in Euler-Bernoulli beam

For the Euler-Bernoulli beam, the graph presents the positive part of a parabola since the value of wavenumber is positive. Obviously, the wavenumber and the angular frequency are not proportional, then the dispersion relation is dispersive, and the speed of wave propagation is not constant.

Before getting into the Euler Bernoulli beam over fluid it is important to show the dispersion relation of infinite depth gravity waves to illustrate the competition between the two behaviours; gravity waves and the beam placed over fluid.

Dispersion relation of infinite depth gravity waves

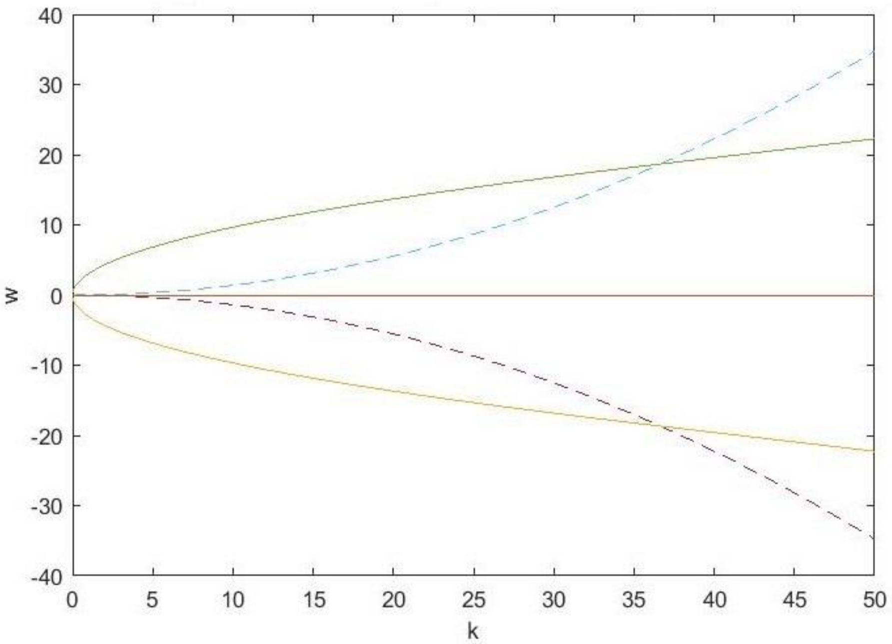
$$\omega^2 = gk$$



plot 4. Dispersion relation of infinite depth gravity waves

Further, Euler Bernoulli beam situated over fluid dispersion relation

$$\omega^2 = \frac{\frac{EI}{\rho_f b} k^5 + gk}{\frac{\rho a}{\rho_f} k + \coth(kh)}$$

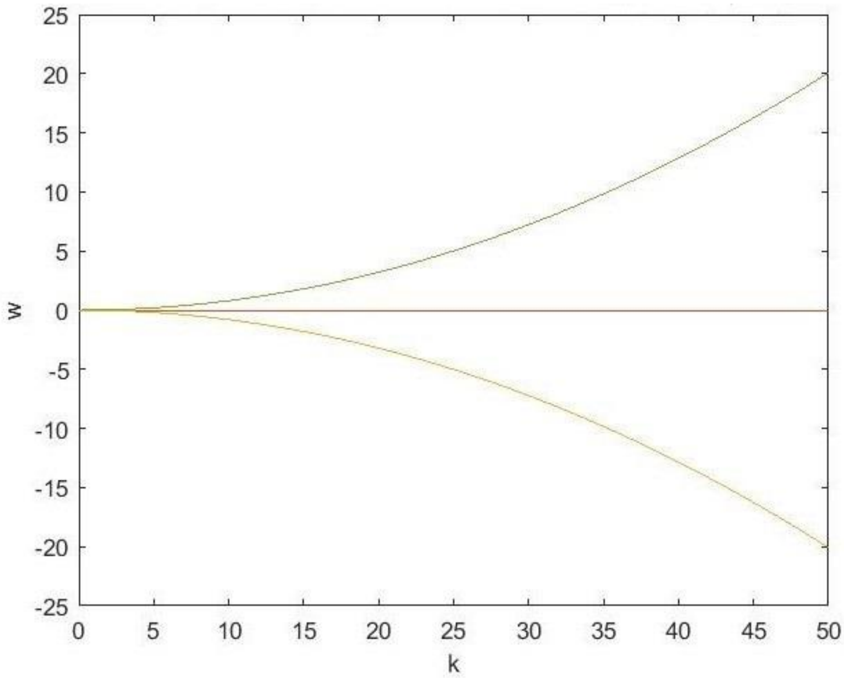


plot 5. Dispersion relation of transverse waves in free Euler-Bernoulli beam and Euler-Bernoulli beam over water

In the above graph, both dispersion relations for the free Euler-Bernoulli beam (dashed line) and Euler-Bernoulli beam over fluid are presented in order to make it more obvious how the Euler-Bernoulli beam behaves when placed over fluid. At low wavenumbers, that is high wave length; the behaviour of gravity waves leads, and the presence of the beam has less effect. However, as the wavenumber rises the behaviour of beam leads, after an inflection point is observed at k approximately equals $36.5(1/m)$. After this point, as the wavenumber increases the angular frequency increases similar to the free beam.

Last but not least, the dispersion relation of the plate

$$\omega^2 = \frac{Dk^4}{\rho a + Ik^2}$$

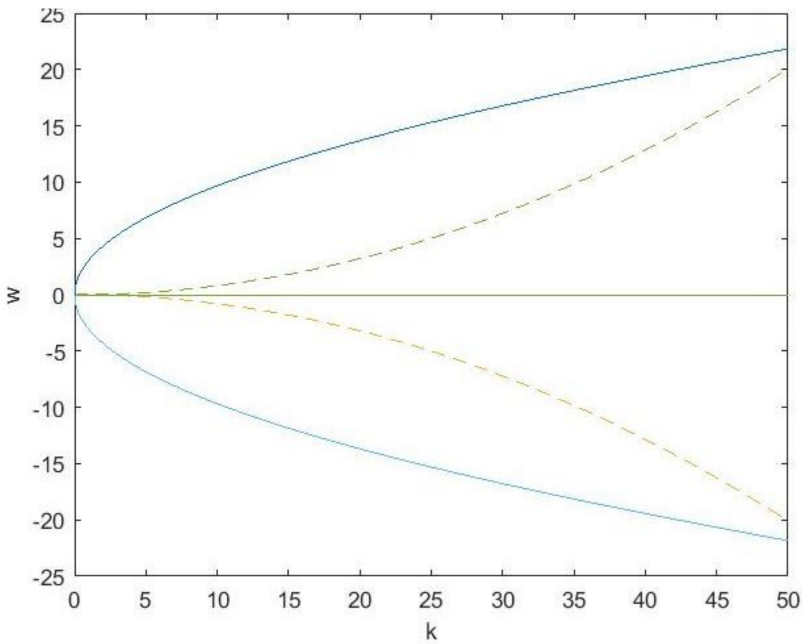


plot 6. *Dispersion relation of transverse waves in plate*

The behaviour of free plate is similar to that of the Euler-Bernoulli beam. Also, here the dispersion relation is dispersive, and the speed of wave propagation is not constant.

Finally, the dispersion relation of plate over fluid

$$\omega^2 = \frac{\frac{D}{\rho_f}k^5 + gk}{\frac{\rho a}{\rho_f}k + \frac{I_m}{\rho_f}k^3 + \coth(kh)}$$



plot 7. Dispersion relation of transverse waves in free plates and plates over fluid

For the plate, a competition between two behaviours is observed just like what was observed in the behaviour of the beam. At low wave numbers, that is high wave length; the behaviour of gravity waves leads, and the presence of the plate has less effect. However, as the wave number increases the behaviour of the plate leads, after an inflection point is observed after $k = 50(1/m)$.
The inflection point in the beam was detected at smaller wave number (at k approximately equals 36.5$(1/m)$) than that of the plate.
It should be noted that the material properties have an obvious impact on the curvature of the dispersion relation graph.

Chapter 4

Nonlinear dispersion relation for transverse waves in Euler-Bernoulli beam

In the previous chapter, the wave equations for string, Euler-Bernoulli beam and plate were found. Then, we assumed that the plate and the beam were placed over a fluid and the dispersion relations for those wave equations were discussed. Up to this point, we were concentrating on the linear parts of equations in order to find linear dispersion relations for the structural elements. In this chapter, Euler-Bernoulli beam will be studied, where nonlinear terms will be included which will influence the dispersion relation found previously.

4.1 Governing equations, normalization

It is useful to start by recalling the equations that describe the fluid motion (when the beam is placed over it) for potential flow used in the previous chapter.

The continuity equation for incompressible fluid

$$\nabla^2\varphi = 0 \quad at \ -\mathrm{h} < \mathrm{z} < \eta \tag{4.1}$$

The kinematic bottom condition

$$\frac{\partial\varphi}{\partial z} = 0 \ \ at \ \ z = -h \tag{4.2}$$

The kinematic surface condition

$$\frac{\partial\eta}{\partial t} + \nabla\varphi.\nabla\eta = \frac{\partial\varphi}{\partial z} \quad at \ z = \eta \tag{4.3}$$

The dynamic surface condition

$$\rho_f\left\{\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + g\eta\right\} + \left(\rho a\frac{\partial^2\eta(x,t)}{\partial t^2}\right) + \frac{EI}{b}\frac{\partial^4\eta(x,t)}{\partial x^4} = 0 \ at \ z = \eta \tag{4.4}$$

Now, we need to scale (normalize) with respect to a characteristic wave field that have the following features

k_c is a characteristic wave number

ω_c is the characteristic angular frequency

a_c is the characteristic amplitude for the surface elevation

Thus, we can define a characteristic steepness which is given by $\epsilon = k_c a_c$.

At this time, normalization can be carried out as follows

$(x', y', z') = k_c(x, y, z)$; $t' = \omega_c t$; $\eta = a_c \eta'$; $\varphi = \frac{\omega_c a_c}{k_c} \varphi'$; $g = \frac{{\omega_c}^2}{k_c} g'$; $h' = k_c h$; $\frac{\rho a}{\rho_f} = \frac{1}{k_c} \frac{\rho' a'}{{\rho'}_f}$; $\frac{EI}{\rho_f b} = \frac{{\omega_c}^2}{{k_c}^5} \frac{E' I'}{{\rho'}_f b'}$

Substitute the normalized quantities in equations (4.1-4.4), then the normalized equations will be as follows

The continuity equation for incompressible fluid

$$\nabla'^2 \varphi' = 0 \quad at \ -\mathrm{h}' < \mathrm{z}' < \epsilon\eta' \tag{4.5}$$

The kinematic bottom condition

$$\frac{\partial \varphi'}{\partial z'} = 0 \ \ at \ \ z' = -h' \tag{4.6}$$

The kinematic surface condition

$$\frac{\partial \eta'}{\partial t'} + \epsilon \nabla' \varphi' . \nabla' \eta' = \frac{\partial \varphi'}{\partial z'} \quad at \ z' = \epsilon \eta' \tag{4.7}$$

The dynamic surface condition

$$\frac{\partial \varphi'}{\partial t'} + \frac{1}{2} \epsilon (\nabla' \varphi')^2 + g' \eta' + \frac{\rho' a'}{{\rho'}_f} \frac{\partial^2 \eta'}{\partial {t'}^2} + \frac{E' I'}{{\rho'}_f b'} \frac{\partial^4 {\eta'}^{(x,t)}}{\partial {x'}^4} = 0 \ at \ z = \eta \tag{4.8}$$

For simplicity, we will drop the primes in the equations keeping the steepness ϵ. Assuming that the steepness is considerably small, $\epsilon \ll 1$, then, the next step is to carry out Taylor- expansion about $z = 0$ such that for any function $f(z)$

$$f(z) = f(0) + \epsilon \, \eta \frac{\partial f}{\partial z}(0) + \frac{1}{2} \epsilon^2 \eta^2 \frac{\partial^2 f}{\partial z^2}(0) + \cdots \tag{4.9}$$

And we get within the first three orders

The continuity equation for incompressible fluid

$$\nabla^2 \varphi = 0 \quad at \ -\mathrm{h} < \mathrm{z} < 0 \tag{4.10}$$

The kinematic bottom condition

$$\frac{\partial\varphi}{\partial z} = 0 \ \ at \ \ z = -h \tag{4.11}$$

The kinematic surface condition

$$\frac{\partial\eta}{\partial t} + \epsilon\, \nabla\varphi.\nabla\eta + \epsilon^2\eta\nabla\frac{\partial\varphi}{\partial z}.\nabla\eta = \frac{\partial\varphi}{\partial z} + \epsilon\,\eta\,\frac{\partial^2\varphi}{\partial z^2} + \frac{1}{2}\epsilon^2\eta^2\,\frac{\partial^3\varphi}{\partial z^3} \quad at\ z = 0 \tag{4.12}$$

The dynamic surface condition

$$\left\{\frac{\partial\varphi}{\partial t} + \epsilon\,\eta\,\frac{\partial^2\varphi}{\partial z\partial t} + \frac{1}{2}\epsilon^2\eta^2\frac{\partial^3\varphi}{\partial z^2\partial t} + \frac{1}{2}\epsilon(\nabla\varphi)^2 + \epsilon^2\eta\nabla\varphi.\nabla\frac{\partial\varphi}{\partial z} + g\eta\right\} + \frac{\rho a}{\rho_f}\frac{\partial^2\eta}{\partial t^2} + \frac{EI}{\rho_f b}\frac{\partial^4\eta(x,t)}{\partial x^4} = 0\ at\ z = 0 \tag{4.13}$$

4.2 Regular perturbation

After normalizing the governing equations of the system, we can now apply regular perturbation expansions in order to solve the above equations

$$\eta = \eta_1 + \epsilon\eta_2 + \epsilon^2\eta_3 + \cdots \tag{4.14}$$

$$\varphi = \varphi_1 + \epsilon\varphi_2 + \epsilon^2\varphi_3 + \cdots \tag{4.15}$$

Then the equations will be of the form (perturbing to the third order)

The continuity equation for incompressible fluid

$$\nabla^2\varphi_n = 0 \quad at\ -\mathrm{h} < \mathrm{z} < 0 \tag{4.16}$$

The kinematic bottom condition

$$\frac{\partial\varphi_n}{\partial z} = 0 \ \ at \ \ z = -h \tag{4.17}$$

The kinematic surface condition

$$\frac{\partial\eta_1}{\partial t}+\epsilon\frac{\partial\eta_2}{\partial t}+\epsilon^2\frac{\partial\eta_3}{\partial t}+\epsilon\nabla\varphi_1.\nabla\eta_1+\epsilon^2\nabla\varphi_2.\nabla\eta_1+\epsilon^2\nabla\varphi_1.\nabla\eta_2 \\ +\epsilon^2\eta_1\nabla\frac{\partial\varphi_1}{\partial z}.\nabla\eta_1=\frac{\partial\varphi_1}{\partial z}+\epsilon\frac{\partial\varphi_2}{\partial z}+\epsilon^2\frac{\partial\varphi_3}{\partial z} \\ +\epsilon\eta_1\frac{\partial^2\varphi_1}{\partial z^2}+\epsilon^2\eta_2\frac{\partial^2\varphi_1}{\partial z^2}+\epsilon^2\eta_1\frac{\partial^2\varphi_2}{\partial z^2}+\frac{1}{2}\epsilon^2{\eta_1}^2\frac{\partial^3\varphi_1}{\partial z^3}\quad at\ z=0 \tag{4.18}$$

The dynamic surface condition

$$\left\{\frac{\partial\varphi_1}{\partial t}+\epsilon\frac{\partial\varphi_2}{\partial t}+\epsilon^2\frac{\partial\varphi_3}{\partial t}+\epsilon\eta_1\frac{\partial^2\varphi_1}{\partial z\partial t}+\epsilon^2\eta_2\frac{\partial^2\varphi_1}{\partial z\partial t}+\epsilon^2\eta_1\frac{\partial^2\varphi_2}{\partial z\partial t}\right. \\ +\frac{1}{2}\epsilon^2{\eta_1}^2\frac{\partial^3\varphi_1}{\partial z^2\partial t}+\frac{1}{2}\epsilon(\nabla\varphi_1)^2+\epsilon^2\nabla\varphi_2.\nabla\varphi_1 \\ \left.+\epsilon^2\eta_1\nabla\varphi_1.\nabla\frac{\partial\varphi_1}{\partial z}+g\eta_1+\epsilon g\eta_2+\epsilon^2 g\eta_3\right\}+ \\ \frac{\rho a}{\rho_f}\left\{\frac{\partial^2\eta_1}{\partial t^2}+\epsilon\frac{\partial^2\eta_2}{\partial t^2}+\epsilon^2\frac{\partial^2\eta_3}{\partial t^2}\right\}+ \\ \frac{EI}{\rho_f b}\left\{\frac{\partial^4\eta_1}{\partial x^4}+\epsilon\frac{\partial^4\eta_2}{\partial x^4}+\epsilon^2\frac{\partial^4\eta_3}{\partial x^4}\right\}=0\quad at\ z=0 \tag{4.19}$$

4.3 Second order problem and resonance analysis

In chapter three, we assumed a monochromatic elementary wave solution given in equation (3.30). Depending on the matrix (3.34), a nontrivial solution will be as follows

$$\eta_1=\frac{d}{2}\ \ and\ \ C=-i\frac{\omega}{k}\frac{d}{2} \tag{4.20}$$

Note that d is a complex amplitude. The matrix (3.34) has one zero eigen value, then the nontrivial solution (4.20) depends on precisely one free complex amplitude d. Due to the linearity of the first order problem, the principle of superposition can be used in order to form a general solution for an irregular sea as a sum of monochromatic waves.

For a discrete superposition we have

$$\eta_1(\boldsymbol{x},t)=\frac{1}{2}\sum_j d_j e^{i(\boldsymbol{k_j}.\boldsymbol{x}-\omega_j t)} \tag{4.21}$$

$$\varphi_1(\boldsymbol{r},t) = \frac{1}{2}\sum_j -i\frac{\omega_j \cosh\left(k_j(z+h)\right)}{k_j \sinh\left(k_j h\right)} d_j e^{i(\boldsymbol{k_j}.\boldsymbol{x}-\omega_j t)} \tag{4.22}$$

d_j are complex amplitudes, and each pair of wave vector k_j and ω_j satisfies the dispersion relation found for the Euler-Bernoulli beam over fluid. The sum over *j* should include complex conjugates in order to have real surface elevation.

Second order problem

The leading order problem (linear) was solved in the previous chapter.

Now regarding the second order problem(ϵ^1), the equations will be

The continuity equation for incompressible fluid

$$\nabla^2\varphi_2 = 0 \quad at\ -\mathrm{h}\ < \mathrm{z} < 0 \tag{4.23}$$

The kinematic bottom condition

$$\frac{\partial \varphi_2}{\partial z} = 0\ \ at\ \ z = -h \tag{4.21}$$

The kinematic surface condition

$$\frac{\partial \eta_2}{\partial t} + \nabla\varphi_1.\nabla\eta_1 = \frac{\partial \varphi_2}{\partial z} + \eta_1\ \frac{\partial^2\varphi_1}{\partial z^2} \quad at\ z = 0 \tag{4.22}$$

The dynamic surface condition

$$\left\{\frac{\partial \varphi_2}{\partial t} + \eta_1\ \frac{\partial^2\varphi_1}{\partial z \partial t} + \frac{1}{2}(\nabla\varphi_1)^2 + g\eta_2\right\} + \frac{\rho a}{\rho_f}\left\{\frac{\partial^2\eta_2}{\partial t^2}\right\} + \frac{EI}{b}\left\{\frac{\partial^4\eta_2}{\partial x^4}\right\} = 0 \quad at\ z = 0 \tag{4.23}$$

Rearranging the equations above and substituting the solutions in (4.21) and (4.22) we will get the following

The continuity equation for incompressible fluid.

$$\nabla^2\varphi_2 = 0 \quad at\ -\mathrm{h} < \mathrm{z} < 0 \tag{4.24}$$

The kinematic bottom condition

$$\frac{\partial\varphi_2}{\partial z} = 0\ \ at\ \ z = -h \tag{4.25}$$

The kinematic surface condition

$$\frac{\partial\eta_2}{\partial t} - \frac{\partial\varphi_2}{\partial z} = \eta_1\ \frac{\partial^2\varphi_1}{\partial z^2} - \nabla\varphi_1.\nabla\eta_1 = \frac{1}{4}\sum_{j,l} -i\omega_j\left(\frac{\boldsymbol{k}_j.\boldsymbol{k}_l}{k_j} + k_j\right)\coth\left(k_j h\right) d_j d_l e^{i(\boldsymbol{K}_{j,l}.\boldsymbol{x}-\Omega_{j,l}t)} \quad at\ z = 0 \tag{4.26}$$

The dynamic surface condition

$$\frac{\partial\varphi_2}{\partial t} + g\eta_2 + \frac{\rho a}{\rho_f}\frac{\partial^2\eta_2}{\partial t^2} + \frac{EI}{b}\frac{\partial^4\eta_2}{\partial x^4} = -\eta_1\ \frac{\partial^2\varphi_1}{\partial z\partial t} - \frac{1}{2}(\nabla\varphi_1)^2 = \frac{1}{4}\sum_{j,l}\omega_j{}^2 - \frac{\omega_j\omega_l}{2}\left(\frac{\boldsymbol{k}_j.\boldsymbol{k}_l}{k_j k_l}\coth\left(k_j h\right)\coth(k_l h) - 1\right) d_j d_l e^{i(\boldsymbol{K}_{j,l}.\boldsymbol{x}-\Omega_{j,l}t)} \quad at\ z = 0 \tag{4.27}$$

Where, $\boldsymbol{K}_{j,l} = \boldsymbol{k}_j + \boldsymbol{k}_l$ and $\Omega_{j,l} = \omega_j + \omega_l$.

Nonlinear Resonance analysis

Indeed, at this time we should check if resonance could occur at the second order. Namely if two waves with wave numbers $\boldsymbol{k}_1\ and\ \boldsymbol{k}_2$ are present at a certain time instant, then the nonlinear interaction between them will produce a wave with wave number $\boldsymbol{k}_3$ such that $\boldsymbol{k}_3 = \boldsymbol{k}_1 \pm \boldsymbol{k}_2$. The produced wave frequency ω_3 will be equivalent to the sum or difference frequency $\omega_3 = \omega_1 \pm \omega_2$ where the signs $\pm$ occur together. If the latter is satisfied, meaning the sum or difference of frequency equals to ω_3 which is the natural frequency of the $\boldsymbol{k}_3$ wave, then $\boldsymbol{k}_3$ wave is excited at its natural frequency and the resonance can occur.

Now, we can write the resonance conditions as follows

$$\begin{cases} \pm\boldsymbol{k}_3 = \boldsymbol{k}_1 \pm \boldsymbol{k}_2 \\ \pm\omega_3 = \omega_1 \pm \omega_2 \end{cases} \tag{4.28}$$

In case of presence of solutions for the above equations, resonance may take place. In order to make the calculation easier we will use plus sign instead of $\pm$ signs in equation (4.28).

Special case: Three wave resonance of two unidirectional waves

One special case for the Euler-Bernoulli beam situated over fluid is assuming that $\boldsymbol{k}_1 = \boldsymbol{k}_2$ and the depth is infinite. This configuration was made by Wilton for capillary-gravity waves, where waves were commonly called Wilton's ripples, after Wilton (1915), although they were previously described by Harrison (1909).
We have

$$\begin{gathered} \boldsymbol{k}_3 = 2\boldsymbol{k}_1 \\ \omega_1 + \omega_2 = 2\omega_1 = \omega_3 \end{gathered} \tag{4.29}$$

Substituting the dispersion relation for Euler-Bernoulli beam situateded over fluid in to the resonance condition, hence the condition will be as follows

$$2\sqrt{\frac{\frac{EI}{\rho_f b}{k_1}^5 + gk_1}{\frac{\rho a}{\rho_f}k_1 + 1}} = \sqrt{\frac{32\frac{EI}{\rho_f b}{k_1}^5 + 2gk_1}{2\frac{\rho a}{\rho_f}k_1 + 1}} \tag{4.30}$$

The table below represents the properties of the Latex sheet

Property of sheet	Latex
Thickness $[\boldsymbol{mm}]$	0.2
Width $[\boldsymbol{mm}]$	1
Density $\left[\frac{\boldsymbol{kg}}{\boldsymbol{m}^3}\right]$	960
E-module $\left[\frac{\boldsymbol{N}}{\boldsymbol{m}^2}\right]$	0.0015

Table 1 *Properties of Latex sheet*

Assume that the fluid is water with density $\rho_f = 1000\,\frac{kg}{m^3}$ and acceleration of gravity $g = 9.81\,\frac{m}{s^2}$.

Subsituting the values presented in table 1 into equation (4.30) yields to a fifth order polynomial. After solving for k_1 five roots were achived, one positive real root, two negative real roots and two complex roots.

The positive real root found means that resonance condition is satisfied and resonance could occur. Consequently, slow modulation scales should be considered in order to have a non sacular solution.

4.4 Resonant interaction equations

Resonant interactions occur when a simple set of criteria coupling wave-vectors and the dispersion relation are met [15]. In the previous section we proved that resonance could occur, then we are forcing the linear system by its own natural frequency. The investigation of resonant interaction equations is popular in many fields, and it has numerous applications in engineering, medicine, etc. The theoretical analysis that leads to finding the resonant interaction equations will be presented in this section. Here, the multiple scales perturbation expansion will be used in order to arrest the unbounded growth in the solution. As mentioned in chapter 2, the technique is based on letting the amplitude $\hat{\eta}$ and $\hat{\varphi}$ vary slowly, much slower than the natural oscillation of the harmonic oscillator. To achieve the latter, the complex amplitudes of the leading order solution should be modulated using slow scales as follows $t_1 = \epsilon t$ and $\boldsymbol{x}_1 = \epsilon \boldsymbol{x}_0$.

Then, η and φ will be as a function of ($t_0, t_1, \boldsymbol{x}_0, \boldsymbol{x}_1$). Time and derivation is made by the chain rule as follows

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \cdots \tag{4.31}$$

And

$$\frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \cdots\right)^2 = \frac{\partial^2}{\partial {t_0}^2} + 2\epsilon \frac{\partial^2}{\partial t_0 \partial t_1} + \cdots \tag{4.32}$$

Similarly, the position vector derivation is performed. After applying time and space modulation to the kinematic and dynamic surface conditions (4.12) and (4.13) respectively we will get the following

The kinematic surface condition

$$\frac{\partial \eta}{\partial t_0} + \epsilon \frac{\partial \eta}{\partial t_1} + \epsilon \nabla \varphi . \nabla \eta + \epsilon^2 \eta \nabla \frac{\partial \varphi_1}{\partial z} . \nabla \eta = \frac{\partial \varphi}{\partial z} + \epsilon \eta \frac{\partial^2 \varphi}{\partial z^2} + \frac{1}{2} \epsilon^2 \eta^2 \frac{\partial^3 \varphi}{\partial z^3} \quad at\ z = 0 \tag{4.33}$$

The dynamic surface condition

$$\left\{\frac{\partial\varphi}{\partial t_0}+\epsilon\frac{\partial\varphi}{\partial t_1}+\epsilon\eta\ \frac{\partial^2\varphi}{\partial z\partial t_0}+\epsilon^2\eta\ \frac{\partial^2\varphi}{\partial z\partial t_1}+\frac{1}{2}\epsilon^2\eta^2\frac{\partial^3\varphi}{\partial z^2\partial t_0}+\frac{1}{2}\epsilon(\nabla\varphi)^2 + \epsilon^2\eta\nabla\varphi.\nabla\frac{\partial\varphi}{\partial z}+g\eta\right\}+ \frac{\rho a}{\rho_f}\left\{\frac{\partial^2\eta}{\partial {t_0}^2}+2\epsilon\frac{\partial^2\eta}{\partial t_0\partial t_1}+\epsilon^2\frac{\partial^2\eta}{\partial {t_1}^2}\right\}+ \frac{EI}{\rho_f b}\left\{\frac{\partial^4\eta}{\partial {x_0}^4}+4\epsilon\frac{\partial^4\eta}{\partial {x_0}^3\partial x_1}\right\}=0 \quad at\ z=0 \tag{4.34}$$

The suitable way to solve the above equations is to apply a regular perturbation expansion as shown in equations (4.14-15) . Equations (4.33) and (4.34) are then

The kinematic surface condition

$$\left(\frac{\partial\eta_1}{\partial t_0}+\epsilon\frac{\partial\eta_1}{\partial t_1}\right)+\epsilon\left(\frac{\partial\eta_2}{\partial t_0}+\epsilon\frac{\partial\eta_2}{\partial t_1}\right)+\epsilon^2\left(\frac{\partial\eta_3}{\partial t_0}+\epsilon\frac{\partial\eta_3}{\partial t_1}\right)+\epsilon\nabla\varphi_1.\nabla\eta_1 + \epsilon^2\nabla\varphi_2.\nabla\eta_1+\epsilon^2\nabla\varphi_1.\nabla\eta_2+\epsilon^2\eta_1\nabla\frac{\partial\varphi_1}{\partial z}.\nabla\eta_1 = \frac{\partial\varphi_1}{\partial z}+\epsilon\frac{\partial\varphi_2}{\partial z}+\epsilon^2\frac{\partial\varphi_3}{\partial z} +\epsilon\eta_1\ \frac{\partial^2\varphi_1}{\partial z^2}+\epsilon^2\eta_2\ \frac{\partial^2\varphi_1}{\partial z^2}+\epsilon^2\eta_1\ \frac{\partial^2\varphi_2}{\partial z^2}+\frac{1}{2}\epsilon^2{\eta_1}^2\ \frac{\partial^3\varphi_1}{\partial z^3}\quad at\ z=0 \tag{4.35}$$

The dynamic surface condition

$$\left\{\frac{\partial\varphi_1}{\partial t_0}+\epsilon\frac{\partial\varphi_2}{\partial t_0}+\epsilon^2\frac{\partial\varphi_3}{\partial t_0}+\epsilon\left(\frac{\partial\varphi_1}{\partial t_1}+\epsilon\frac{\partial\varphi_2}{\partial t_1}\right)+\epsilon\eta_1\ \frac{\partial^2\varphi_1}{\partial z\partial t_0} +\epsilon^2\eta_2\ \frac{\partial^2\varphi_1}{\partial z\partial t_0}+\epsilon^2\eta_1\ \frac{\partial^2\varphi_2}{\partial z\partial t_0}+\epsilon^2\eta_1\ \frac{\partial^2\varphi_2}{\partial z\partial t_1} +\frac{1}{2}\epsilon^2{\eta_1}^2\frac{\partial^3\varphi_1}{\partial z^2\partial t_0}+\frac{1}{2}\epsilon(\nabla\varphi_1)^2+\epsilon^2\nabla\varphi_2.\nabla\varphi_1 +\epsilon^2\eta_1\nabla\varphi_1.\nabla\frac{\partial\varphi_1}{\partial z}+g\eta_1+\epsilon g\eta_2+\epsilon^2 g\eta_3\right\}+ \frac{\rho a}{\rho_f}\left\{\frac{\partial^2\eta_1}{\partial {t_0}^2}+\epsilon\frac{\partial^2\eta_2}{\partial {t_0}^2}+\epsilon^2\frac{\partial^2\eta_3}{\partial {t_0}^2}+2\epsilon\frac{\partial^2\eta_1}{\partial t_0\partial t_1}+2\epsilon^2\frac{\partial^2\eta_1}{\partial t_0\partial t_1}+\epsilon^2\frac{\partial^2\eta_1}{\partial {t_1}^2}\right\}+ \frac{EI}{\rho_f b}\left\{\frac{\partial^4\eta_1}{\partial {x_0}^4}+\epsilon\frac{\partial^4\eta_2}{\partial {x_0}^4}+\epsilon^2\frac{\partial^4\eta_3}{\partial {x_0}^4}+4\epsilon\frac{\partial^4\eta_1}{\partial {x_0}^3\partial x_1}+4\epsilon^2\frac{\partial^4\eta_2}{\partial {x_0}^3\partial x_1}\right\}=0\quad at\ z=0 \tag{4.36}$$

The modulated first order problem will be exactly the same as the one solved in chapter 3, for that we are not going to re-mention it here. But the fundamental difference now is the assumption that complex amplitudes are modulated. Assume now that the solution of the above equations is of the following form

$$\begin{pmatrix} \eta(\boldsymbol{x},t) \\ \varphi(\boldsymbol{r},t) \end{pmatrix} = \begin{pmatrix} \hat{\eta}(\boldsymbol{x}_1,t_1) \\ \hat{\varphi}(\boldsymbol{x}_1,z,t_1) \end{pmatrix} e^{i(\boldsymbol{k}.\boldsymbol{x}_0-\omega t_0)} \tag{4.37}$$

then, the nontrivial solution will be as follows

$$\eta_1 = \frac{d(\boldsymbol{x}_1,t_1)}{2} \;\; and \;\; C = -i\frac{\omega}{k}\frac{d(\boldsymbol{x}_1,t_1)}{2} \tag{4.38}$$

Note that $d(\boldsymbol{x}_1,t_1)$ is a complex amplitude that depends on $\boldsymbol{x}_1$and t_1. Linearity of the first order problem, permits the employment of the principle of superposition forming a general solution for an irregular sea as a sum of monochromatic waves.

For a discrete superposition we have

$$\eta_1(\boldsymbol{x},t) = \frac{1}{2}\sum_j d_j(\boldsymbol{x}_1,t_1)e^{i\left(\boldsymbol{k_j}.\boldsymbol{x}_0-\omega_j t_0\right)} \tag{4.39}$$

$$\varphi_1(\boldsymbol{r},t) = \frac{1}{2}\sum_j -i\frac{\omega_j \cosh\left(k_j(z+h)\right)}{k_j \sinh\left(k_j h\right)} d_j(\boldsymbol{x}_1,t_1)e^{i\left(\boldsymbol{k_j}.\boldsymbol{x}_0-\omega_j t_0\right)} \tag{4.40}$$

$d_j(\boldsymbol{x}_1,t_1)$ are complex amplitudes, and each pair of wave vector k_j and ω_j satisfies the dispersion relation found for the Euler-Bernoulli beam over fluid. The sum over *j* should include complex conjugates in order to have real surface elevation.

Modulated second order problem ($\boldsymbol{\epsilon^1}$)

The second order problem(ϵ^1) after time modulation and regular perturbation, the equations will be

The continuity equation for incompressible fluid

$$\nabla^2\varphi_2 = H_2 = -2\frac{\partial^2\varphi_1}{\partial x_0 \partial x_1} - 2\frac{\partial^2\varphi_1}{\partial y_0 \partial y_1} \;\; at \;\; -\mathrm{h} < \mathrm{z} < 0 \tag{4.41}$$

The kinematic bottom condition

$$\frac{\partial\varphi_2}{\partial z} = 0 \;\; at \;\; z = -h \tag{4.42}$$

The kinematic surface condition

$$\left(\frac{\partial \eta_1}{\partial t_1}\right) + \left(\frac{\partial \eta_2}{\partial t_0}\right) + \nabla\varphi_1.\nabla\eta_1 = \frac{\partial \varphi_2}{\partial z} + \eta_1 \frac{\partial^2 \varphi_1}{\partial z^2} \quad at\ z = 0 \tag{4.43}$$

After rearranging we will get,

$$\frac{\partial \eta_2}{\partial t_0} - \frac{\partial \varphi_2}{\partial z} = F_2 = -\frac{\partial \eta_1}{\partial t_1} - \nabla\varphi_1.\nabla\eta_1 + \eta_1 \frac{\partial^2 \varphi_1}{\partial z^2} \quad at\ z = 0$$

The dynamic surface condition

$$\left\{\frac{\partial \varphi_2}{\partial t_0} + \frac{\partial \varphi_1}{\partial t_1} + \eta_1 \frac{\partial^2 \varphi_1}{\partial z \partial t_0} + \frac{1}{2}(\nabla\varphi_1)^2 + g\eta_2\right\} + \frac{\rho a}{\rho_f}\left\{\frac{\partial^2 \eta_2}{\partial {t_0}^2} + 2\frac{\partial^2 \eta_1}{\partial t_0 \partial t_1}\right\} + \frac{EI}{\rho_f b}\left\{\frac{\partial^4 \eta_2}{\partial {x_0}^4} + 4\frac{\partial^4 \eta_1}{\partial {x_0}^3 \partial x_1}\right\} = 0 \quad at\ z = 0 \tag{4.44}$$

After rearranging we will get,

$$\left\{\frac{\partial \varphi_2}{\partial t_0} + g\eta_2\right\} + \frac{\rho a}{\rho_f}\left\{\frac{\partial^2 \eta_2}{\partial {t_0}^2}\right\} + \frac{EI}{\rho_f b}\left\{\frac{\partial^4 \eta_2}{\partial {x_0}^4}\right\} =$$
$$G_2 = -\frac{\partial \varphi_1}{\partial t_1} - \eta_1 \frac{\partial^2 \varphi_1}{\partial z \partial t_0} - \frac{1}{2}(\nabla\varphi_1)^2 - 2\frac{\rho a}{\rho_f}\frac{\partial^2 \eta_1}{\partial t_0 \partial t_1} - 4\frac{EI}{\rho_f b}\frac{\partial^4 \eta_1}{\partial {x_0}^3 \partial x_1}$$
$$at\ z = 0$$

We arranged the equations in a way that all the second order unknows are on the left-hand side, and the forcing is at the right-hand side.

In order to solve this problem, let η_1 and φ_1 be of the form

$$\eta_1(\boldsymbol{x}, t) = \frac{1}{2}\sum_{j=1}^{3} A_{1,j}(\boldsymbol{x}_1, t_1) e^{i(\theta_j)} + c.c \tag{4.45}$$

$$\varphi_1(\boldsymbol{r}, t) = \frac{1}{2}\sum_{j=1}^{3} -i\frac{\omega_j \cosh\left(k_j(z+h)\right)}{k_j \sinh\left(k_j h\right)} A_{1,j}(\boldsymbol{x}_1, t_1) e^{i(\theta_j)} + c.c \tag{4.46}$$

Where, $\theta_j = \left(\boldsymbol{k_j}.\boldsymbol{x}_0 - \omega_j t_0\right)\ for\ j = 1,2,3$

Then,

$$H_2 = -\sum_{j=1}^{3} \frac{\omega_j \cosh\left(k_j(z+h)\right)}{k_j \sinh\left(k_j h\right)} \boldsymbol{k_j} \frac{\partial A_{1,j}}{\partial \boldsymbol{x}_1} e^{i(\theta_j)} + c.c \tag{4.47}$$

Taking out just one harmonic

$$H_2 = -\frac{\omega_1 \cosh\left(k_1(z+h)\right)}{k_1 \sinh(k_1 h)} \boldsymbol{k_1} \frac{\partial A_{1,1}}{\partial \boldsymbol{x}_1} e^{i(\theta_1)} + c.c$$

$$F_2 = -\frac{\partial \eta_1}{\partial t_1} - \nabla\varphi_1 . \nabla\eta_1 + \eta_1 \frac{\partial^2 \varphi_1}{\partial z^2} \tag{4.48}$$

$$= -\frac{1}{2}\sum_{j=1}^{3} \frac{\partial A_{1,j}}{\partial t_1} e^{i(\boldsymbol{\theta_j})}$$

$$-\frac{1}{4}\sum_{m,n=1}^{3} i\boldsymbol{k_m}.\boldsymbol{k_n} A_{1,m} A_{1,n} \frac{\omega_m}{k_m} \coth\,(k_m h) e^{i(\theta_m+\theta_n)}$$

$$-\frac{1}{4}\sum_{m,n=1}^{3} -i\boldsymbol{k_m}.\boldsymbol{k_n} A_{1,m} A^*_{1,n} \frac{\omega_m}{k_m} \coth\,(k_m h) e^{i(\theta_m-\theta_n)}$$

$$+\frac{1}{4}\sum_{m,n=1}^{3} -i\omega_n k_n A_{1,m} A_{1,n} \coth\,(k_n h) e^{i(\theta_m+\theta_n)}$$

$$+\frac{1}{4}\sum_{m,n=1}^{3} i\omega_n k_n A_{1,m} A^*_{1,n} \coth\,(k_n h) e^{i(\theta_m-\theta_n)} + c.c$$

Taking out just one harmonic

$$F_2 = -\frac{1}{2}\frac{\partial A_{1,1}}{\partial t_1} e^{i(\theta_1)} + \frac{1}{4} i\boldsymbol{k_3}.\boldsymbol{k_2} A_{1,3} A^*_{1,2} \frac{\omega_3}{k_3} \coth\,(k_3 h) e^{i(\theta_3-\theta_2)}$$
$$+\frac{1}{4} i\omega_2 k_2 A_{1,3} A^*_{1,2} \coth\,(k_2 h) e^{i(\theta_3-\theta_2)} + c.c$$

$$G_2 = -\frac{\partial\varphi_1}{\partial t_1} - \eta_1\ \frac{\partial^2\varphi_1}{\partial z\partial t_0} - \frac{1}{2}(\nabla\varphi_1)^2 - 2\frac{\rho a}{\rho_f}\frac{\partial^2\eta_1}{\partial t_0\partial t_1} - 4\frac{EI}{\rho_f b}\frac{\partial^4\eta_1}{\partial {x_0}^3\partial x_1}$$

$$= -\frac{1}{2}\sum_{j=1}^{3} -i\frac{\omega_j}{k_j}\frac{\partial A_{1,j}}{\partial t_1}\coth(k_j h)e^{i(\theta_j)} \tag{4.49}$$

$$-\frac{1}{4}\sum_{m,n=1}^{3} -\omega_n^2 A_{1,m}A_{1,n}e^{i(\theta_m+\theta_n)} - \frac{1}{4}\sum_{m,n=1}^{3} -\omega_n^2 A_{1,m}A_{1,n}^* e^{i(\theta_m-\theta_n)}$$

$$-\ \frac{1}{8}\sum_{m,n=1}^{3} \boldsymbol{k_m}.\boldsymbol{k_n}A_{1,m}A_{1,n}\frac{1}{k_m k_n}\coth(k_m h)\coth(k_n h)e^{i(\theta_m+\theta_n)}$$

$$-\frac{1}{8}\sum_{m,n=1}^{3} \boldsymbol{k_m}.\boldsymbol{k_n}A_{1,m}A_{1,n}^*\frac{1}{k_m k_n}\coth(k_m h)\coth(k_n h)e^{i(\theta_m-\theta_n)}$$

$$-\frac{1}{8}\sum_{m,n=1}^{3} -\omega_m\omega_n A_{1,m}A_{1,n}e^{i(\theta_m+\theta_n)} - \frac{1}{8}\sum_{m,n=1}^{3} \omega_m\omega_n A_{1,m}A_{1,n}^* e^{i(\theta_m-\theta_n)}$$

$$-\frac{\rho a}{\rho_f}\sum_{j=1}^{3} -i\omega_j\frac{\partial A_{1,j}}{\partial t_1}e^{i(\theta_j)} - 2\frac{EI}{\rho_f b}\sum_{j=1}^{3} -i\,k_j\boldsymbol{k_j}\frac{\partial A_{1,j}}{\partial \boldsymbol{x_1}}e^{i(\theta_j)} + c.c$$

Taking out just one harmonic

$$G_2 = \frac{1}{2}i\frac{\omega_1}{k_1}\frac{\partial A_{1,1}}{\partial t_1}\coth(k_1 h)e^{i(\theta_1)} + \frac{1}{4}\omega_2^2 A_{1,3}A_{1,2}^* e^{i(\theta_3-\theta_2)}$$
$$-\frac{1}{8}\boldsymbol{k_3}.\boldsymbol{k_2}A_{1,3}A_{1,2}^*\frac{1}{k_3k_2}\coth(k_3 h)\coth(k_2 h)e^{i(\theta_3-\theta_2)}$$
$$-\frac{1}{8}\omega_3\omega_2 A_{1,3}A_{1,2}^* e^{i(\theta_3-\theta_2)} + \frac{\rho a}{\rho_f}i\omega_1\frac{\partial A_{1,1}}{\partial t_1}e^{i(\theta_1)}$$
$$+\ 2\frac{EI}{\rho_f b}ik_1\boldsymbol{k_1}\frac{\partial A_{1,1}}{\partial \boldsymbol{x_1}}e^{i(\theta_1)} + c.c$$

If we are only interested with η_2 *and* φ_2 of the following forms (first harmonic)

$$\eta_2 = \frac{1}{2}A_{2,1}(\boldsymbol{x_1}, t_1)e^{i(\theta_1)} + c.c$$

$$\varphi_2(\boldsymbol{r}, t) = -\frac{1}{2}i\frac{\omega_1\cosh\big(k_1(z+h)\big)}{k_1\sinh(k_1 h)}A_{2,1}(\boldsymbol{x_1}, t_1)e^{i(\theta_1)} + c.c$$

Then, the equations above will be

Continuity equation

$$\frac{1}{2}i\frac{\omega_1\cosh\big(k_1(z+h)\big)}{\sinh(k_1 h)}(k_1^2 - k_1^2)A_{2,1}e^{i(\theta_1)} + c.c \tag{4.50}$$
$$= -\frac{\omega_1\cosh\big(k_1(z+h)\big)}{k_1\sinh(k_1 h)}\boldsymbol{k_1}.\frac{\partial A_{1,1}}{\partial \boldsymbol{x_1}}e^{i(\theta_1)} + c.c$$

The kinematic surface condition

$$-\frac{1}{2}i\omega_1 A_{2,1}e^{i(\theta_1)}+\frac{1}{2}i\frac{\omega_1\sinh\big(k_1(z+h)\big)}{\sinh(k_1h)}A_{2,1}e^{i(\theta_1)}+c.c$$
$$=-\frac{1}{2}\frac{\partial A_{1,1}}{\partial t_1}e^{i(\theta_1)}+\frac{1}{4}i\boldsymbol{k_3}.\boldsymbol{k_2}A_{1,3}A_{1,2}^{*}\frac{\omega_3}{k_3}e^{i(\theta_3-\theta_2)}$$
$$+\frac{1}{4}i\omega_2k_2A_{1,3}A_{1,2}^{*}e^{i(\theta_3-\theta_2)}+c.c=0\ at\ z=0 \tag{4.51}$$

The Dynamic surface condition

$$\frac{1}{2}\left(-i\frac{\omega_1^2}{k_1}+g-\frac{\rho a}{\rho_f}\omega_1^2+\frac{EI}{\rho_f b}k_1^4\right)A_{2,1}e^{i(\theta_1)}+c.c$$
$$=\frac{1}{2}i\frac{\omega_1}{k_1}\frac{\partial A_{1,1}}{\partial t_1}\coth\,(k_1h)e^{i(\theta_1)}+\frac{1}{4}\omega_2^2A_{1,3}A_{1,2}^{*}e^{i(\theta_3-\theta_2)}$$
$$-\frac{1}{8}\boldsymbol{k_3}.\boldsymbol{k_2}A_{1,3}A_{1,2}^{*}\frac{1}{k_3k_2}\coth(k_3h)\coth(k_2h)\,e^{i(\theta_3-\theta_2)}$$
$$-\frac{1}{8}\omega_3\omega_2A_{1,3}A_{1,2}^{*}e^{i(\theta_3-\theta_2)}+i\frac{\rho a}{\rho_f}\omega_1\frac{\partial A_{1,1}}{\partial t_1}e^{i(\theta_1)}$$
$$+2\frac{EI}{\rho_f b}ik_1\boldsymbol{k_1}.\frac{\partial A_{1,1}}{\partial \boldsymbol{x}_1}e^{i(\theta_1)}+c.c \tag{4.52}$$

Obviously, the assumed solution for the second order problem is not adequate for this problem, for that we should find a solvability condition that prevents the unbounded resonant growth. The homogeneous version of the boundary value problem has $\eta_1(\boldsymbol{x},t)$ and $\varphi_1(\boldsymbol{r},t)$ as a nontrivial solution. Then, the inhomogeneous problem must satisfy a solvability condition, which follows by applying Green's theorem, to $\eta_1(\boldsymbol{x},t)$, $\varphi_1(\boldsymbol{r},t)$, $\eta_2(\boldsymbol{x},t)$ and $\varphi_2(\boldsymbol{r},t)$[3]. The identity used here is Green's second identity.

Green's second identity

Let f and g be twice continuously differentiable functions in the control volume V, then

$$\iiint f\nabla^2 g-g\nabla^2 f dV=\oint(f\nabla g-g\nabla f).\boldsymbol{n}ds \tag{4.53}$$

In the case we are studying, let $f=\varphi_1(\boldsymbol{r},t)$ and $g=\varphi_2(\boldsymbol{r},t)$

Notice that from the continuity equation of the first two orders of the problem, we know that

$$\nabla^2 f = \nabla^2 \varphi_1 = 0$$

And

$$\nabla^2 g = \nabla^2 \varphi_2 = H_2(\varphi_1)$$

Then the second factor of the right-hand side of the identity will cancel out. Regarding the left-hand side of the identity, it will only have contribution at the free surface, that is $at\ z = 0$. Also, for the right-hand side we will focus only on the z-direction

Then the whole equation will be of the following form

$$\int_{-h}^{0} \varphi_1 H_2(\varphi_1) dz = \left[\varphi_1 \frac{\partial \varphi_2}{\partial z} - \varphi_2 \frac{\partial \varphi_1}{\partial z}\right]_{z=0} \tag{4.54}$$

We know from the kinematic surface condition of the modulated second order problem that

$$\frac{\partial \varphi_2}{\partial z} = \frac{\partial \eta_2}{\partial t_0} - F_2 = \frac{\partial \eta_2}{\partial t_0} + \frac{\partial \eta_1}{\partial t_1} + \nabla\varphi_1 . \nabla\eta_1 - \eta_1 \frac{\partial^2 \varphi_1}{\partial z^2} \tag{4.55}$$

From the dynamic condition we can find that

$$\begin{aligned}\frac{\partial \varphi_2}{\partial t_0} &= G_2 - g\eta_2 - \frac{\rho a}{\rho_f}\frac{\partial^2 \eta_2}{\partial {t_0}^2} - \frac{EI}{\rho_f b}\frac{\partial^4 \eta_2}{\partial {x_0}^4} \\ &= -\frac{\partial \varphi_1}{\partial t_1} - \eta_1 \frac{\partial^2 \varphi_1}{\partial z \partial t_0} - \frac{1}{2}(\nabla\varphi_1)^2 - 2\frac{\rho a}{\rho_f}\frac{\partial^2 \eta_1}{\partial t_0 \partial t_1} \\ &\quad - 4\frac{EI}{\rho_f b}\frac{\partial^4 \eta_1}{\partial {x_0}^3 \partial x_1} - g\eta_2 - \frac{\rho a}{\rho_f}\frac{\partial^2 \eta_2}{\partial {t_0}^2} - \frac{EI}{\rho_f b}\frac{\partial^4 \eta_2}{\partial {x_0}^4}\end{aligned} \tag{4.56}$$

From the assumed solution for the second order problem, we have

$$\frac{\partial \varphi_2}{\partial t_0} = -i\omega_1 \varphi_2 \ then\ \varphi_2 = \frac{i}{\omega_1}\frac{\partial \varphi_2}{\partial t_0}$$

As well as, from the kinematic condition in first order problem we have

$$\frac{\partial \varphi_1}{\partial z} = \frac{\partial \eta_1}{\partial t_0}$$

After substituting the above expressions into the right-hand side of Green's formula (4.54), the factors including $\eta_2(\boldsymbol{x},t)$ and $\varphi_2(\boldsymbol{r},t)$ will cancel out, and consequently we will then get the following

Concerning the left-hand side of equation (4.54)

$$\begin{aligned}\int_{-h}^{0} \varphi_1 H_2(\varphi_1) dz \\ = & \left\{\frac{i}{\omega_1}\left(\frac{\partial \eta_1}{\partial t_1}\right)\left(-g\eta_1 + \frac{\rho a}{\rho_f}\frac{\partial^2 \eta_1}{\partial {t_0}^2} - \frac{EI}{\rho_f b}\frac{\partial^4 \eta_2}{\partial {x_0}^4}\right)\right\} \\ & + \left\{\frac{i}{\omega_1}(\nabla\varphi_1 . \nabla\eta_1)\left(-g\eta_1 - \frac{\rho a}{\rho_f}\frac{\partial^2 \eta_1}{\partial {t_0}^2} - \frac{EI}{\rho_f b}\frac{\partial^4 \eta_2}{\partial {x_0}^4}\right)\right\} \\ & + \left\{\frac{i}{\omega_1}\left(\eta_1 \frac{\partial^2 \varphi_1}{\partial z^2}\right)\left(g\eta_1 + \frac{\rho a}{\rho_f}\frac{\partial^2 \eta_1}{\partial {t_0}^2} + \frac{EI}{\rho_f b}\frac{\partial^4 \eta_2}{\partial {x_0}^4}\right)\right\} \\ & - \left\{\frac{i}{\omega_1}\left(-\frac{\partial \eta_1}{\partial t_0}\right)\left(\frac{\partial \varphi_1}{\partial t_1} + \eta_1 \frac{\partial^2 \varphi_1}{\partial z \partial t_0} + \frac{1}{2}(\nabla\varphi_1)^2 \right.\right. \\ & \left.\left. + 4\frac{EI}{\rho_f b}\frac{\partial^4 \eta_1}{\partial {x_0}^3 \partial x_1}\right)\right\} \; at\; z = 0\end{aligned} \tag{4.57}$$

$$\begin{aligned}\int_{-h}^{0} \varphi_1 H_2(\varphi_1) dz \\ = \frac{i}{2}\frac{\omega_1^2}{k_1^2}\boldsymbol{k}_1 . \nabla_1 A_{1,1} A_{1,1}\left(\frac{2k_1 h + \sinh(2k_1 h)}{4k_1 sinh^2(k_1 h)}\right)\left[e^{2i\theta_1} + e^0\right] + c.c\end{aligned} \tag{4.58}$$

As $k_1 h$ tends to infinity, and by using L'Hôpital's rule [25], we can find that

$$\left(\frac{2k_1 h + \sinh(2k_1 h)}{4k_1 sinh^2(k_1 h)}\right) = \frac{1}{2k_1}$$

After finding the derivatives in the right hand-side of equation (4.57), picking out just the second harmonic (all the factors that include $e^{2i\theta_1}$), the overall equation is then

$$\frac{i}{4}\frac{\omega_1^2}{k_1^3}\boldsymbol{k}_1.\nabla_1 A_{1,1}+\frac{i}{\omega_1}\frac{\partial A_{1,1}}{\partial t_1}\left(\frac{1}{4}g+\frac{1}{4}\frac{\rho a}{\rho_f}\omega_1^2+\frac{1}{4}\frac{EI}{\rho_f b}k_1^4+\frac{1}{4}\frac{\omega_1^2}{k_1}\right)$$
$$+\frac{i}{\omega_1}\left(\frac{EI}{\rho_f b}\omega_1 k_1^2\boldsymbol{k}_1.\nabla_1 A_{1,1}\right)$$
$$=\frac{i}{\omega_1}\left[\frac{i}{8}\boldsymbol{k}_3.\boldsymbol{k}_2\frac{\omega_3}{k_3}\left(-g-\frac{\rho a}{\rho_f}\omega_1^2+\frac{EI}{\rho_f b}k_1^4\right)A_{1,3}A_{1,2}^*\right]$$
$$+\frac{i}{\omega_1}\left[\frac{i}{8}\omega_2 k_2\left(g-\frac{\rho a}{\rho_f}\omega_1^2+\frac{EI}{\rho_f b}k_1^4\right)A_{1,3}A_{1,2}^*\right]$$
$$-\frac{i}{\omega_1}\left[-\frac{i}{8}A_{1,3}A_{1,2}^*\omega_2^2\omega_1+\frac{i}{16}A_{1,3}A_{1,2}^*\omega_1\omega_2\omega_3\frac{\boldsymbol{k}_3.\boldsymbol{k}_2}{k_3k_2}\right.$$
$$\left.+\frac{i}{16}A_{1,3}A_{1,2}^*\omega_1\omega_2\omega_3\right] \tag{4.59}$$

Rearranging the above equation will lead to following, and assuming $\coth(k_1h)=1$

$$\frac{\partial A_{1,1}}{\partial t_1}+\boldsymbol{c}g_1.\nabla_1 A_{1,1}=\alpha_1 A_{1,3}A_{1,2}^* \tag{4.60}$$

Similarly, we will get the two other interaction equations and the set of equations will be as follows

$$\begin{aligned}\frac{\partial A_{1,1}}{\partial t_1}+\boldsymbol{c}g_1.\nabla_1 A_{1,1}&=\alpha_1 A_{1,3}A_{1,2}^*\\ \frac{\partial A_{1,2}}{\partial t_1}+\boldsymbol{c}g_2.\nabla_1 A_{1,2}&=\alpha_1 A_{1,3}A_{1,1}^*\\ \frac{\partial A_{1,3}}{\partial t_1}+\boldsymbol{c}g_3.\nabla_1 A_{1,3}&=\alpha_1 A_{1,1}A_{1,3}^*\end{aligned} \tag{4.61}$$

Where,

$$\boldsymbol{c}g_j=\frac{2\frac{EI}{\rho_f b}k_j^3\omega_j}{\frac{EI}{\rho_f b}k_j^4+g}+\frac{\frac{EI}{\rho_f b}k_j^2\omega_j}{2\left(\frac{EI}{\rho_f b}k_j^4+g\right)\left(\frac{\rho a}{\rho_f}+\frac{1}{k_j}\right)}+\frac{g\omega_j}{2k_j^2\left(\frac{EI}{\rho_f b}k_j^4+g\right)\left(\frac{\rho a}{\rho_f}+\frac{1}{k_j}\right)} \tag{4.62}$$

The group velocity can also be expressed in a factorized form

$$\boldsymbol{c}g_j=\frac{\frac{EI}{\rho_f b}k_j^4\left(4\frac{\rho a}{\rho_f}k_j+5\right)}{2\left(\frac{\rho a}{\rho_f}k_j+1\right)^{3/2}\sqrt{k_j\left(\frac{EI}{\rho_f b}k_j^4+g\right)}}$$

Knowing that

$$\omega_j = \sqrt{\frac{\frac{EI}{\rho_f b} k_j^4 + g}{\frac{\rho a}{\rho_f} + \frac{1}{k_j}}}$$

And

$$\alpha_j = \left\{ \frac{1}{g + \frac{\rho a}{\rho_f}\omega_j^2 + \frac{EI}{\rho_f b}k_j^4 + \frac{\omega_1^2}{k_j}} \right\} \left\{ \left(\frac{i}{2}\boldsymbol{k}_m.\boldsymbol{k}_n \frac{\omega_m}{k_m}\left(-g - \frac{\rho a}{\rho_f}\omega_j^2 + \frac{EI}{\rho_f b}k_j^4 \right) \right) + \left(\frac{i}{2}\omega_n^2 k_n \left(g - \frac{\rho a}{\rho_f}\omega_j^2 + \frac{EI}{\rho_f b}k_j^4 \right) \right) - \left(-\frac{i}{2}\omega_n^2\omega_j + \frac{i}{4}\omega_j \frac{\boldsymbol{k}_m.\boldsymbol{k}_n}{k_m k_n} + \frac{i}{4}\omega_j\omega_n\omega_m \right) \right\}$$

Where, j=1,2,3; m=1,2,3 and n=1,2,3.

Chapter 5

Nonlinear dispersion relations for transverse waves in a plate

In this chapter, we will go through same process as the previous one, but here the studied structural element will be a plate. In fact, the plate case is more interesting since ice sheets can be modelled as plates.

5.1 Plate governing equations, normalization

To begin with, the governing equations

The continuity equation for incompressible fluid

$$\nabla^2\varphi = 0 \quad at\ -\mathrm{h} < \mathrm{z} < \eta \tag{5.1}$$

The kinematic bottom condition

$$\frac{\partial\varphi}{\partial z} = 0\ \ at\ \ z = -h \tag{5.2}$$

The kinematic surface condition

$$\frac{\partial\eta}{\partial t} + \nabla\varphi.\nabla\eta = \frac{\partial\varphi}{\partial z} \quad at\ z = \eta \tag{5.3}$$

The dynamic surface condition

$$\rho_f\left\{\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + g\eta\right\} + \rho a\frac{\partial^2\eta}{\partial t^2} - I_m\nabla^2\frac{\partial^2\eta}{\partial t^2} + D\nabla^4\eta = 0\ at\ z = \eta \tag{5.4}$$

The normalization process is similar to that of the beam, here we will add the flexural rigidity and the moment of inertia per unit area factors

$\frac{I_m}{\rho_f} = \frac{1}{{k_c}^3}\frac{{I_m}'}{{\rho'}_f}$; $\frac{D}{\rho_f} = \frac{{\omega_c}^2}{{k_c}^5}\frac{D'}{{\rho'}_f}$

Introducing the normalized quantities in equations (5.1-5.4), then the normalized equations will be as follows

The continuity equation for incompressible fluid

$$\nabla'^2\varphi' = 0 \quad at\ -\mathrm{h}' < \mathrm{z}' < \epsilon\eta' \tag{5.5}$$

The kinematic bottom condition

$$\frac{\partial\varphi'}{\partial z'} = 0\ \ at\ \ z' = -h' \tag{5.6}$$

The kinematic surface condition

$$\frac{\partial\eta'}{\partial t'} + \epsilon\nabla'\varphi'.\nabla'\eta' = \frac{\partial\varphi'}{\partial z'} \quad at\ z' = \epsilon\eta' \tag{5.7}$$

The dynamic surface condition

$$\frac{\partial\varphi'}{\partial t'} + \frac{1}{2}\epsilon(\nabla'\varphi')^2 + g'\eta' + \frac{\rho' a'}{\rho'_f}\frac{\partial^2\eta'}{\partial t'^2} - \frac{I_m{}'}{\rho'_f}\frac{\partial^2\eta'}{\partial t'^2} + \frac{D'}{\rho'_f}\nabla'^4\eta' = 0\ at\ z = \eta \tag{5.8}$$

After dropping the prime for simplicity and keeping the steepness ϵ, applying Taylor's-expansion about $z = 0$ similar to equation (4.9) in the previous chapter, the first three orders of the governing equations will be

The continuity equation for incompressible fluid

$$\nabla^2\varphi = 0 \quad at\ -\mathrm{h} < \mathrm{z} < 0 \tag{5.9}$$

The kinematic bottom condition

$$\frac{\partial\varphi}{\partial z} = 0\ \ at\ \ z = -h \tag{5.10}$$

The kinematic surface condition

$$\frac{\partial\eta}{\partial t} + \epsilon\,\nabla\varphi.\nabla\eta + \epsilon^2\eta\nabla\frac{\partial\varphi}{\partial z}.\nabla\eta = \frac{\partial\varphi}{\partial z} + \epsilon\,\eta\,\frac{\partial^2\varphi}{\partial z^2} + \frac{1}{2}\epsilon^2\eta^2\,\frac{\partial^3\varphi}{\partial z^3} \quad at\ z = 0 \tag{5.11}$$

The dynamic surface condition

$$\left\{\frac{\partial\varphi}{\partial t}+\epsilon\,\eta\,\frac{\partial^2\varphi}{\partial z\partial t}+\frac{1}{2}\epsilon^2\eta^2\frac{\partial^3\varphi}{\partial z^2\partial t}+\frac{1}{2}\epsilon(\nabla\varphi)^2+\epsilon^2\eta\nabla\varphi.\nabla\frac{\partial\varphi}{\partial z}+g\eta\right\}+\frac{\rho a}{\rho_f}\frac{\partial^2\eta}{\partial t^2} -\frac{I_m}{\rho_f}\nabla^2\frac{\partial^2\eta}{\partial t^2}+\frac{D}{\rho_f}\nabla^4\eta=0\ at\ z=0 \tag{5.12}$$

For significance, we will not state the regular perturbed problem and the second order problem here, but resonance analysis for the plate will be stated. The regular perturbation of the above equations will be stated in section 5.4 since it will be utilized there.

5.2 Resonance analysis for Plate over fluid

In this section, we will study the existence of unbounded resonant growth if three waves can resonate in a triad.

The resonance conditions are

$$\begin{cases}\pm\boldsymbol{k}_3=\boldsymbol{k}_1\pm\boldsymbol{k}_2\\ \pm\omega_3=\omega_1\pm\omega_2\end{cases} \tag{5.13}$$

The fulfilment of the above conditions means that resonant growth is expected. Also, here the positive sign will be considered to simplify the calculation.

General case for nonlinear resonance condition
Consider the following confugration

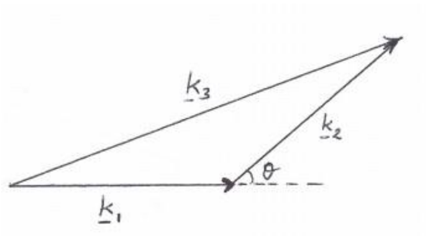
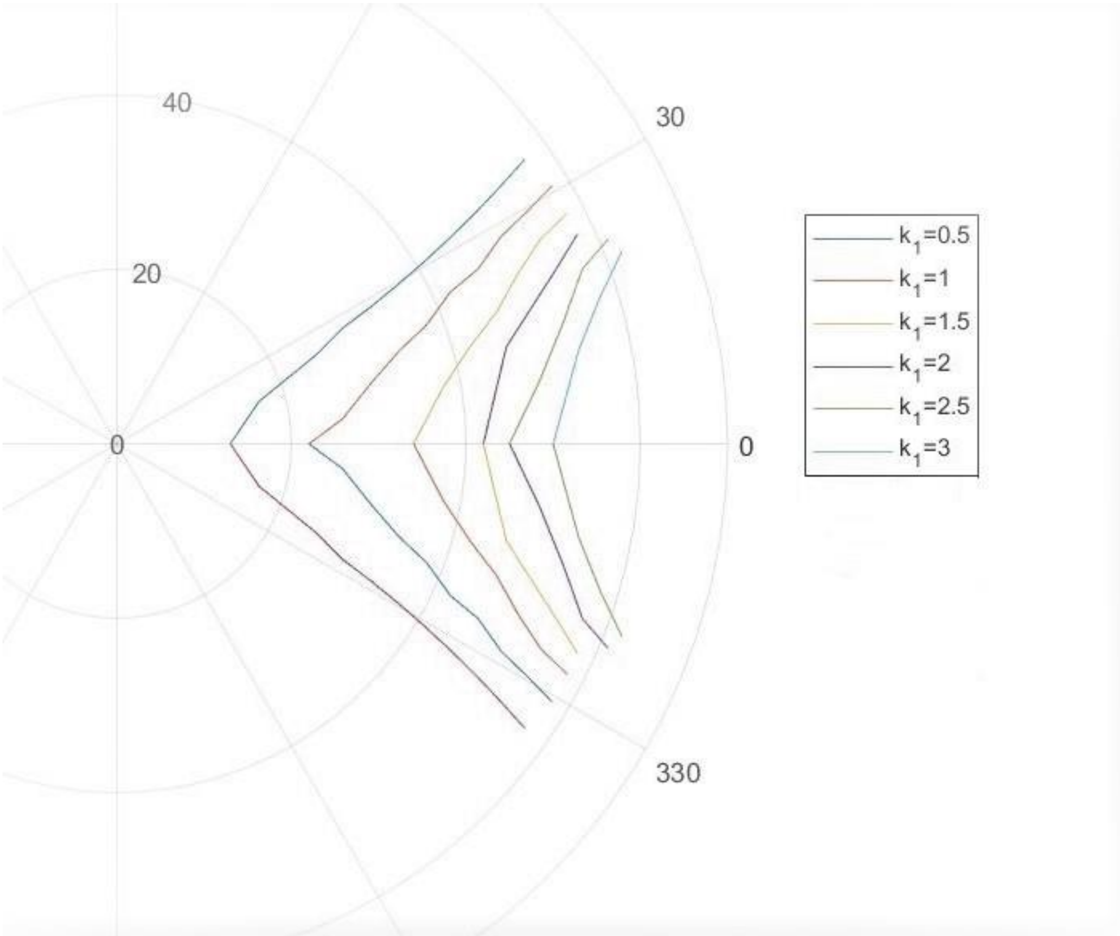


Fig 9. Resonance triad configuration

Let θ be the angle between $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, then the wave vectors will be represented as follows
$\boldsymbol{k}_1 = k(1,0)$, $\boldsymbol{k}_2 = (\alpha cos\ \theta, \alpha sin\ \theta)$, $\boldsymbol{k}_3 = (k + \alpha cos\ \theta, \alpha sin\ \theta)$
Substituting in the resonance condition for infinite depth, will give

$$\sqrt{\frac{\frac{D}{\rho_f}{k_3}^5 + gk_3}{\frac{\rho a}{\rho_f}k_3 + \frac{I_m}{\rho_f}{k_3}^3 + 1}} = \sqrt{\frac{\frac{D}{\rho_f}{k_1}^5 + gk_1}{\frac{\rho a}{\rho_f}k_1 + \frac{I_m}{\rho_f}{k_1}^3 + 1}} + \sqrt{\frac{\frac{D}{\rho_f}{k_2}^5 + gk_2}{\frac{\rho a}{\rho_f}k_2 + \frac{I_m}{\rho_f}{k_2}^3 + 1}} \qquad (5.14)$$

Rearranging the equation and using the properties of Latex sheet found in Table 1. Note that the Possion's ratio for the plate $\nu = 0.5$. Assuming that the fluid is water with density $\rho_f = 1000\frac{kg}{m^3}$, acceleration of gravity $g = 9.81\frac{m}{s^2}$.
The resonance condition will then be a polynomial function of k, α and $\cos\theta$. Solving for the value of $\cos\theta$ for fixed values of k and α , will lead to at least one real root. Resonance will occurr if $\cos\theta$ is smaller than 1.
$\boldsymbol{k}_3$ is the wave vector that will be produced due to the interaction of waves one and two having the wave numbers $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ respectively.



plot 8. *Solutions for resonance condition*

It is important to note that an extended comprehension of the results will be done in chapter 6, the solutions for resonance in elastic plate triad will be compared with the outcomes found by McGoldrick (1964) for capillary-gravity waves.

5.3 *Resonance interaction equations*

The risk of occurrence of resonance shown in the previous section, means that the linear system is forced by its own natural frequency. The importance of the resonance interaction equations in the elastic plate waves and the popularity of the technique required the analysis in this field.
Also, for the plate, we will use the multiple scales perturbation expansion in order to arrest the unbounded growth in the solution. The kinematic and dynamic conditions will then be

The kinematic surface condition

$$\frac{\partial\eta}{\partial t_0}+\epsilon\frac{\partial\eta}{\partial t_1}+\epsilon\nabla\varphi.\nabla\eta+\epsilon^2\eta\nabla\frac{\partial\varphi_1}{\partial z}.\nabla\eta=\frac{\partial\varphi}{\partial z} +\epsilon\eta\,\frac{\partial^2\varphi}{\partial z^2}+\frac{1}{2}\epsilon^2\eta^2\,\frac{\partial^3\varphi}{\partial z^3}\quad at\ z=0 \tag{5.15}$$

The dynamic surface condition

$$\begin{aligned}&\left\{\frac{\partial\varphi}{\partial t_0}+\epsilon\frac{\partial\varphi}{\partial t_1}+\epsilon\eta\,\frac{\partial^2\varphi}{\partial z\partial t_0}+\epsilon^2\eta\,\frac{\partial^2\varphi}{\partial z\partial t_1}+\frac{1}{2}\epsilon^2\eta^2\frac{\partial^3\varphi}{\partial z^2\partial t_0}+\frac{1}{2}\epsilon(\nabla\varphi)^2 +\epsilon^2\eta\nabla\varphi.\nabla\frac{\partial\varphi}{\partial z}+g\eta\right\}+\\ &\frac{\rho a}{\rho_f}\left\{\frac{\partial^2\eta}{\partial {t_0}^2}+2\epsilon\frac{\partial^2\eta}{\partial t_0\partial t_1}+\epsilon^2\frac{\partial^2\eta}{\partial {t_1}^2}\right\}\\ &-\frac{I_m}{\rho_f}\left\{{\nabla_0}^2\frac{\partial^2\eta}{\partial {t_0}^2}+2\epsilon{\nabla_0}^2\frac{\partial^2\eta}{\partial t_0\partial t_1}+2\epsilon\nabla_0.\nabla_{\mathbf{1}}\frac{\partial^2\eta}{\partial {t_0}^2}\right\}\\ &+\frac{D}{\rho_f}\left\{{\nabla_0}^4\eta+4\epsilon{\nabla_0}^2\nabla_0.\nabla_{\mathbf{1}}\eta\right\}=0\quad at\ z=0\end{aligned} \tag{5.16}$$

Subsequently, apply now regular perturbation expansion as shown in equations (4.14-15). The two surface conditions will be

The kinematic surface condition

$$\left(\frac{\partial\eta_1}{\partial t_0}+\epsilon\frac{\partial\eta_1}{\partial t_1}\right)+\epsilon\left(\frac{\partial\eta_2}{\partial t_0}+\epsilon\frac{\partial\eta_2}{\partial t_1}\right)+\epsilon^2\left(\frac{\partial\eta_3}{\partial t_0}+\epsilon\frac{\partial\eta_3}{\partial t_1}\right)+\epsilon\nabla\varphi_1.\nabla\eta_1 \tag{5.17}$$
$$+\epsilon^2\nabla\varphi_2.\nabla\eta_1+\epsilon^2\nabla\varphi_1.\nabla\eta_2+\epsilon^2\eta_1\nabla\frac{\partial\varphi_1}{\partial z}.\nabla\eta_1$$
$$=\frac{\partial\varphi_1}{\partial z}+\epsilon\frac{\partial\varphi_2}{\partial z}+\epsilon^2\frac{\partial\varphi_3}{\partial z}$$
$$+\epsilon\eta_1\frac{\partial^2\varphi_1}{\partial z^2}+\epsilon^2\eta_2\frac{\partial^2\varphi_1}{\partial z^2}+\epsilon^2\eta_1\frac{\partial^2\varphi_2}{\partial z^2}+\frac{1}{2}\epsilon^2{\eta_1}^2\frac{\partial^3\varphi_1}{\partial z^3}\quad at\ z=0$$

The dynamic surface condition

$$\left\{\frac{\partial\varphi_1}{\partial t_0}+\epsilon\frac{\partial\varphi_2}{\partial t_0}+\epsilon^2\frac{\partial\varphi_3}{\partial t_0}+\epsilon\left(\frac{\partial\varphi_1}{\partial t_1}+\epsilon\frac{\partial\varphi_2}{\partial t_1}\right)+\epsilon\eta_1\frac{\partial^2\varphi_1}{\partial z\partial t_0}+\epsilon^2\eta_2\frac{\partial^2\varphi_1}{\partial z\partial t_0}\right.$$
$$+\epsilon^2\eta_1\frac{\partial^2\varphi_2}{\partial z\partial t_0}+\epsilon^2\eta_1\frac{\partial^2\varphi_2}{\partial z\partial t_1}+\frac{1}{2}\epsilon^2{\eta_1}^2\frac{\partial^3\varphi_1}{\partial z^2\partial t_0}+\frac{1}{2}\epsilon(\nabla\varphi_1)^2 \tag{5.18}$$
$$\left.+\epsilon^2\nabla\varphi_2.\nabla\varphi_1+\epsilon^2\eta_1\nabla\varphi_1.\nabla\frac{\partial\varphi_1}{\partial z}+g\eta_1+\epsilon g\eta_2+\epsilon^2 g\eta_3\right\}+$$
$$\frac{\rho a}{\rho_f}\left\{\frac{\partial^2\eta_1}{\partial {t_0}^2}+\epsilon\frac{\partial^2\eta_2}{\partial {t_0}^2}+\epsilon^2\frac{\partial^2\eta_3}{\partial {t_0}^2}+2\epsilon\frac{\partial^2\eta_1}{\partial t_0\partial t_1}+2\epsilon^2\frac{\partial^2\eta_1}{\partial t_0\partial t_1}+\epsilon^2\frac{\partial^2\eta_1}{\partial {t_1}^2}\right\}$$
$$-\frac{I_m}{\rho_f}\left\{{\nabla_0}^2\frac{\partial^2\eta_1}{\partial {t_0}^2}+\epsilon{\nabla_0}^2\frac{\partial^2\eta_2}{\partial {t_0}^2}+2\epsilon{\nabla_0}^2\frac{\partial^2\eta_1}{\partial t_0\partial t_1}+2\epsilon\nabla_0.\nabla_{\mathbf{1}}\frac{\partial^2\eta_1}{\partial {t_0}^2}\right\}$$
$$+\frac{D}{\rho_f}\left\{{\nabla_0}^4\eta_1+\epsilon{\nabla_0}^4\eta_2+4\epsilon{\nabla_0}^2\nabla_0.\nabla_{\mathbf{1}}\eta_1\right\}=0\quad at\ z=0$$

Now, we have the modulated perturbed problem, we can extract the modulated second order directly since the first order is the linear problem solved in chapter 3. Using the assumed solution stated in equations (4.37-40).

Modulated second order problem (ϵ^{1})

The continuity equation for incompressible fluid

$$\nabla^2\varphi_2=H_2=-2\frac{\partial^2\varphi_1}{\partial x_0\partial x_1}-2\frac{\partial^2\varphi_1}{\partial y_0\partial y_1}\ at\ -\mathrm{h}<\mathrm{z}<0 \tag{5.19}$$

The kinematic bottom condition

$$\frac{\partial \varphi_2}{\partial z} = 0 \;\; at \;\; z = -h \tag{5.20}$$

The kinematic surface condition

$$\left(\frac{\partial \eta_1}{\partial t_1}\right) + \left(\frac{\partial \eta_2}{\partial t_0}\right) + \nabla\varphi_1 . \nabla\eta_1 = \frac{\partial \varphi_2}{\partial z} + \eta_1 \, \frac{\partial^2 \varphi_1}{\partial z^2} \quad at \; z = 0 \tag{5.21}$$

After rearranging we will get,

$$\frac{\partial \eta_2}{\partial t_0} - \frac{\partial \varphi_2}{\partial z} = F_2 = -\frac{\partial \eta_1}{\partial t_1} - \nabla\varphi_1 . \nabla\eta_1 + \eta_1 \, \frac{\partial^2 \varphi_1}{\partial z^2} \quad at \; z = 0$$

The dynamic surface condition

$$\left\{\frac{\partial \varphi_2}{\partial t_0} + \frac{\partial \varphi_1}{\partial t_1} + \eta_1 \, \frac{\partial^2 \varphi_1}{\partial z \partial t_0} + \frac{1}{2}(\nabla\varphi_1)^2 + g\eta_2\right\} + \frac{\rho a}{\rho_f}\left\{\frac{\partial^2 \eta_2}{\partial {t_0}^2} + 2\frac{\partial^2 \eta_1}{\partial t_0 \partial t_1}\right\} - \frac{I_m}{\rho_f}\left\{{\nabla_0}^2 \frac{\partial^2 \eta_2}{\partial {t_0}^2} + 2{\nabla_0}^2 \frac{\partial^2 \eta_1}{\partial t_0 \partial t_1} + 2\nabla_0 . \nabla_{\mathbf{1}} \frac{\partial^2 \eta_1}{\partial {t_0}^2}\right\} + \frac{D}{\rho_f}\left\{{\nabla_0}^4 \eta_2 + 4{\nabla_0}^2 \nabla_0 . \nabla_{\mathbf{1}} \eta_1\right\} = 0 \quad at \; z = 0 \tag{5.22}$$

After rearranging we will get,

$$\left\{\frac{\partial \varphi_2}{\partial t_0} + g\eta_2\right\} + \frac{\rho a}{\rho_f}\left\{\frac{\partial^2 \eta_2}{\partial {t_0}^2}\right\} - \frac{I_m}{\rho_f}\left\{{\nabla_0}^2 \frac{\partial^2 \eta_2}{\partial {t_0}^2}\right\} + \frac{D}{\rho_f}\left\{{\nabla_0}^4 \eta_2\right\} = G_2 = -\frac{\partial \varphi_1}{\partial t_1} - \eta_1 \, \frac{\partial^2 \varphi_1}{\partial z \partial t_0} - \frac{1}{2}(\nabla\varphi_1)^2 - 2\frac{\rho a}{\rho_f}\frac{\partial^2 \eta_1}{\partial t_0 \partial t_1} + 2\frac{I_m}{\rho_f}{\nabla_0}^2 \frac{\partial^2 \eta_1}{\partial t_0 \partial t_1} + 2\frac{I_m}{\rho_f}\nabla_0 . \nabla_{\mathbf{1}} \frac{\partial^2 \eta_1}{\partial {t_0}^2} - 4\frac{D}{\rho_f}{\nabla_0}^2 \nabla_0 . \nabla_{\mathbf{1}} \eta_1 \quad at \; z = 0$$

Green's second identity

Following, we will be applying Green's second identity in order to find the interaction equations. The application of the identity will be same as that used for the beam. So, we will use equations (4.53-55). Despite, the dynamic surface condition is different from that of the beam.

Here we have

$$
\begin{aligned}
\frac{\partial \varphi_2}{\partial t_0} = G_2 - g\eta_2 - \frac{\rho a}{\rho_f}\frac{\partial^2 \eta_2}{\partial {t_0}^2} + \frac{I_m}{\rho_f}{\nabla_0}^2\frac{\partial^2 \eta_2}{\partial {t_0}^2} - \frac{D}{\rho_f}{\nabla_0}^4\eta_2 \\
= -\frac{\partial \varphi_1}{\partial t_1} - \eta_1\,\frac{\partial^2 \varphi_1}{\partial z \partial t_0} - \frac{1}{2}(\nabla\varphi_1)^2 - 2\frac{\rho a}{\rho_f}\frac{\partial^2 \eta_1}{\partial t_0 \partial t_1} \\
+ 2\frac{I_m}{\rho_f}{\nabla_0}^2\frac{\partial^2 \eta_1}{\partial t_0 \partial t_1} + 2\frac{I_m}{\rho_f}\nabla_0.\nabla_{\mathbf{1}}\frac{\partial^2 \eta_1}{\partial {t_0}^2} - 4\frac{D}{\rho_f}{\nabla_0}^2\nabla_0.\nabla_{\mathbf{1}}\eta_1 \\
- g\eta_2 - \frac{\rho a}{\rho_f}\frac{\partial^2 \eta_2}{\partial {t_0}^2} + \frac{I_m}{\rho_f}{\nabla_0}^2\frac{\partial^2 \eta_2}{\partial {t_0}^2} - \frac{D}{\rho_f}{\nabla_0}^4\eta_2 \qquad (5.23)
\end{aligned}
$$

From the assumed solution for the second order problem, we have

$$\frac{\partial \varphi_2}{\partial t_0} = -i\omega_1\varphi_2 \; then \; \varphi_2 = \frac{i}{\omega_1}\frac{\partial \varphi_2}{\partial t_0}$$

As well as, from the kinematic condition in first order problem we have

$$\frac{\partial \varphi_1}{\partial z} = \frac{\partial \eta_1}{\partial t_0}$$

The substitution of the above expressions into the right-hand side of Green's formula (4.54), factors that include $\eta_2(\boldsymbol{x}, t)$ and $\varphi_2(\boldsymbol{r}, t)$ will cancel out, then Green's identity will be as follows

$$
\begin{aligned}
&\int_{-h}^{0} \varphi_1 H_2(\varphi_1) dz \\
&= \left\{ \frac{i}{\omega_1} \left(\frac{\partial \eta_1}{\partial t_1} \right) \left(-g\eta_1 + \frac{\rho a}{\rho_f} \frac{\partial^2 \eta_1}{\partial t_0{}^2} - \frac{I_m}{\rho_f} \nabla_0{}^2 \frac{\partial^2 \eta_1}{\partial t_0{}^2} - \frac{D}{\rho_f} \nabla_0{}^4 \eta_1 \right) \right\} \\
&+ \left\{ \frac{i}{\omega_1} (\nabla \varphi_1 . \nabla \eta_1) \left(-g\eta_1 - \frac{\rho a}{\rho_f} \frac{\partial^2 \eta_1}{\partial t_0{}^2} + \frac{I_m}{\rho_f} \nabla_0{}^2 \frac{\partial^2 \eta_1}{\partial t_0{}^2} \right. \right. \\
&\left. \left. - \frac{D}{\rho_f} \nabla_0{}^4 \eta_1 \right) \right\} \\
&+ \left\{ \frac{i}{\omega_1} \left(\eta_1 \ \frac{\partial^2 \varphi_1}{\partial z^2} \right) \left(g\eta_1 + \frac{\rho a}{\rho_f} \frac{\partial^2 \eta_1}{\partial t_0{}^2} - \frac{I_m}{\rho_f} \nabla_0{}^2 \frac{\partial^2 \eta_1}{\partial t_0{}^2} + \frac{D}{\rho_f} \nabla_0{}^4 \eta_1 \right) \right\} \\
&- \left\{ \frac{i}{\omega_1} \left(-\frac{\partial \eta_1}{\partial t_0} \right) \left(\frac{\partial \varphi_1}{\partial t_1} + \eta_1 \ \frac{\partial^2 \varphi_1}{\partial z \partial t_0} + \frac{1}{2} (\nabla \varphi_1)^2 \right. \right. \\
&\left. \left. - 2 \frac{I_m}{\rho_f} \nabla_0 . \nabla_{\mathbf{1}} \frac{\partial^2 \eta_1}{\partial t_0{}^2} + 4 \frac{D}{\rho_f} \nabla_0{}^2 \nabla_0 . \nabla_{\mathbf{1}} \eta_1 \right) \right\} \ at \ z = 0
\end{aligned}
\tag{5.24}
$$

The left-hand side of equation (5.24) is same as that for the beam. After finding the derivatives in the right hand-side of equation (5.24), picking out just the second harmonic (all the factors that include $e^{2i\theta_1}$), the overall equation is then

$$
\begin{aligned}
&\frac{i}{4} \frac{\omega_1^2}{k_1^3} \boldsymbol{k}_1 . \nabla_1 A_{1,1} + \frac{i}{\omega_1} \left(\frac{1}{4} g + \frac{1}{4} \frac{\rho a}{\rho_f} \omega_1^2 + \frac{1}{4} \frac{I_m}{\rho_f} k_1^2 \omega_1^2 + \frac{1}{4} \frac{D}{\rho_f} k_1^4 + \frac{1}{4} \frac{\omega_1^2}{k_1} \right) \frac{\partial A_{1,1}}{\partial t_1} \\
&\quad + \frac{i}{\omega_1} \left(-\frac{1}{2} \frac{I_m}{\rho_f} \omega_1^3 + \frac{D}{\rho_f} \omega_1 k_1^2 \right) \boldsymbol{k}_1 . \nabla_1 A_{1,1} \\
&\quad = \frac{i}{\omega_1} \left[\frac{i}{8} \boldsymbol{k}_3 . \boldsymbol{k}_2 \frac{\omega_3}{k_3} \left(-g - \frac{\rho a}{\rho_f} \omega_1^2 - \frac{I_m}{\rho_f} k_1^2 \omega_1^2 + \frac{D}{\rho_f} k_1^4 \right) A_{1,3} A_{1,2}^* \right] \\
&\quad + \frac{i}{\omega_1} \left[\frac{i}{8} \omega_2 k_2 \left(g - \frac{\rho a}{\rho_f} \omega_1^2 - \frac{I_m}{\rho_f} k_1^2 \omega_1^2 + \frac{D}{\rho_f} k_1^4 \right) A_{1,3} A_{1,2}^* \right] \\
&\quad - \frac{i}{\omega_1} \left[-\frac{i}{8} A_{1,3} A_{1,2}^* \omega_2^2 \omega_1 + \frac{i}{16} A_{1,3} A_{1,2}^* \omega_1 \omega_2 \omega_3 \frac{\boldsymbol{k}_3 . \boldsymbol{k}_2}{k_3 k_2} \right. \\
&\quad \left. + \frac{i}{16} A_{1,3} A_{1,2}^* \omega_1 \omega_2 \omega_3 \right]
\end{aligned}
\tag{5.25}
$$

Rearranging the above equation will lead to following, and assuming $\coth(k_1 h) = 1$

$$
\frac{\partial A_{1,1}}{\partial t_1} + \boldsymbol{c}g_1 . \nabla_1 A_{1,1} = \alpha_1 A_{1,3} A_{1,2}^* \tag{5.26}
$$

Similarly, we will get the two other interaction equations, then set of equations will be as follows

$$\frac{\partial A_{1,1}}{\partial t_1} + \boldsymbol{c}g_1.\nabla_1 A_{1,1} = \alpha_1 A_{1,3} A_{1,2}^*$$
$$\frac{\partial A_{1,2}}{\partial t_1} + \boldsymbol{c}g_2.\nabla_1 A_{1,2} = \alpha_1 A_{1,3} A_{1,1}^* \tag{5.27}$$
$$\frac{\partial A_{1,3}}{\partial t_1} + \boldsymbol{c}g_3.\nabla_1 A_{1,3} = \alpha_1 A_{1,1} A_{1,3}^*$$

Where,

the expanded form of $\boldsymbol{c}g_j$ is

$$\begin{aligned}\boldsymbol{c}g_j = {} & \frac{2\frac{D}{\rho_f}k_j^3\omega_j}{\frac{D}{\rho_f}k_j^4 + g} + \frac{\frac{D}{\rho_f}k_j^2\omega_j}{2\left(\frac{D}{\rho_f}k_j^4 + g\right)\left(\frac{\rho a}{\rho_f} + \frac{I_m}{\rho_f}k_j^2 + \frac{1}{k_j}\right)} \\ & + \frac{g\omega_j}{2k_j^2\left(\frac{D}{\rho_f}k_j^4 + g\right)\left(\frac{\rho a}{\rho_f} + \frac{I_m}{\rho_f}k_j^2 + \frac{1}{k_j}\right)} \\ & - \frac{\frac{I_m}{\rho_f}gk_j\omega_j}{\left(\frac{D}{\rho_f}k_j^4 + g\right)\left(\frac{\rho a}{\rho_f} + \frac{I_m}{\rho_f}k_j^2 + \frac{1}{k_j}\right)} \\ & - \frac{\frac{D}{\rho_f}\frac{I_m}{\rho_f}k_j^5\omega_j}{\left(\frac{D}{\rho_f}k_j^4 + g\right)\left(\frac{\rho a}{\rho_f} + \frac{I_m}{\rho_f}k_j^2 + \frac{1}{k_j}\right)}\end{aligned} \tag{5.28}$$

Knowing that

$$\omega_j = \sqrt{\frac{\frac{D}{\rho_f}k_j^4 + g}{\frac{\rho a}{\rho_f} + \frac{I_m}{\rho_f}k_j^2 + \frac{1}{k_j}}}$$

$\boldsymbol{c}g_j$ can also be expressed as follows

$$\boldsymbol{c}g_j = \frac{\frac{D}{\rho_f}k_j^4\left(4\frac{\rho a}{\rho_f}k_j + 2\frac{I_m}{\rho_f}k_j^3 + 5\right) - 2\frac{I_m}{\rho_f}gk_j^3 + g}{2\left(\frac{\rho a}{\rho_f}k_j + \frac{I_m}{\rho_f}k_j^3 + 1\right)^{3/2}\sqrt{k_j\left(\frac{D}{\rho_f}k_j^4 + g\right)}}$$

The interaction coefficients α_j are,

$$\alpha_j = \left\{\frac{1}{\left(g + \frac{\rho a}{\rho_f}\omega_j^2 + \frac{I_m}{\rho_f}k_j^2\omega_j^2 + \frac{D}{\rho_f}k_j^4 + \frac{\omega_j^2}{k_j}\right)}\right\}\left\{\left(\frac{i}{2}\boldsymbol{k}_m.\boldsymbol{k}_n\frac{\omega_m}{k_m}\left(-g - \frac{\rho a}{\rho_f}\omega_j^2 - \frac{I_m}{\rho_f}k_j^2\omega_j^2 + \frac{D}{\rho_f}k_j^4\right)\right) + \left(\frac{i}{2}\omega_n k_n\left(g - \frac{\rho a}{\rho_f}\omega_j^2 - \frac{I_m}{\rho_f}k_j^2\omega_j^2 + \frac{D}{\rho_f}k_j^4\right)\right) - \left(-\frac{i}{2}\omega_n^2\omega_j + \frac{i}{4}\omega_j\omega_n\omega_m\frac{\boldsymbol{k}_m.\boldsymbol{k}_n}{k_m k_n} + \frac{i}{4}\omega_j\omega_n\omega_m\right)\right\} \quad (5.29)$$

Where, j=1,2,3; m=1,2,3 and n=1,2,3.

5.4 *Nonlinearly forced response*

So far, we have discussed the case where $\boldsymbol{K_{j,l}}$ and $\Omega_{j,l}$ are non-zero and satisfy the dispersion relation. The system is then forced with its natural wave solution, which will eventually lead to resonance. To avoid this problem, we added slow scales $x_1 = \epsilon x$ and $t_1 = \epsilon t$ and found the resonance interaction equations.

In this section we will take into consideration the case where $\boldsymbol{K_{j,l}}$ and $\Omega_{j,l}$ are non-zero and does not satisfy the dispersion relation.

At this time, and for its importance we will state the regular perturbation of the governing equations (5.9-12)

The continuity equation for incompressible fluid

$$\nabla^2\varphi_n = 0 \quad at \ -\mathrm{h} < \mathrm{z} < 0 \quad (5.30)$$

The kinematic bottom condition

$$\frac{\partial\varphi_n}{\partial z} = 0 \ \ at \ \ z = -h \quad (5.31)$$

The kinematic surface condition

$$\frac{\partial\eta_1}{\partial t} + \epsilon\frac{\partial\eta_2}{\partial t} + \epsilon^2\frac{\partial\eta_3}{\partial t} + \epsilon\nabla\varphi_1.\nabla\eta_1 + \epsilon^2\nabla\varphi_2.\nabla\eta_1 + \epsilon^2\nabla\varphi_1.\nabla\eta_2 + \epsilon^2\eta_1\nabla\frac{\partial\varphi_1}{\partial z}.\nabla\eta_1 = \frac{\partial\varphi_1}{\partial z} + \epsilon\frac{\partial\varphi_2}{\partial z} + \epsilon^2\frac{\partial\varphi_3}{\partial z} \quad (5.32)$$

$$+\epsilon\eta_1 \frac{\partial^2\varphi_1}{\partial z^2} + \epsilon^2\eta_2 \frac{\partial^2\varphi_1}{\partial z^2} + \epsilon^2\eta_1 \frac{\partial^2\varphi_2}{\partial z^2} + \frac{1}{2}\epsilon^2{\eta_1}^2 \frac{\partial^3\varphi_1}{\partial z^3} \quad at\ z = 0$$

The dynamic surface condition

$$\left\{\frac{\partial\varphi_1}{\partial t} + \epsilon\frac{\partial\varphi_2}{\partial t} + \epsilon^2\frac{\partial\varphi_3}{\partial t} + \epsilon\eta_1 \frac{\partial^2\varphi_1}{\partial z\partial t} + \epsilon^2\eta_2 \frac{\partial^2\varphi_1}{\partial z\partial t} + \epsilon^2\eta_1 \frac{\partial^2\varphi_2}{\partial z\partial t} + \frac{1}{2}\epsilon^2{\eta_1}^2 \frac{\partial^3\varphi_1}{\partial z^2\partial t} + \frac{1}{2}\epsilon(\nabla\varphi_1)^2 + \epsilon^2\nabla\varphi_2.\nabla\varphi_1 + \epsilon^2\eta_1\nabla\varphi_1.\nabla\frac{\partial\varphi_1}{\partial z} + g\eta_1 + \epsilon g\eta_2 + \epsilon^2 g\eta_3\right\} + \frac{\rho a}{\rho_f}\left\{\frac{\partial^2\eta_1}{\partial t^2} + \epsilon\frac{\partial^2\eta_2}{\partial t^2} + \epsilon^2\frac{\partial^2\eta_3}{\partial t^2}\right\} - \frac{I_m}{\rho_f}\left\{\nabla^2\frac{\partial^2\eta_1}{\partial t^2} + \epsilon\nabla^2\frac{\partial^2\eta_2}{\partial t^2}\right\} \frac{D}{\rho_f}\{\nabla^4\eta_1 + \epsilon\nabla^4\eta_2\} = 0 \quad at\ z = 0 \tag{5.33}$$

And the second order kinematic and dynamic equations will be

The kinematic surface condition

$$\frac{\partial\eta_2}{\partial t} - \frac{\partial\varphi_2}{\partial z} = -\nabla\varphi_1.\nabla\eta_1 + \eta_1 \frac{\partial^2\varphi_1}{\partial z^2} \quad at\ z = 0 \tag{5.32}$$

The dynamic surface condition

$$\frac{\partial\varphi_2}{\partial t} + g\eta_2 + \frac{\rho a}{\rho_f}\frac{\partial^2\eta_2}{\partial t^2} - \frac{I_m}{\rho_f}\nabla^2\frac{\partial^2\eta_2}{\partial t^2} + \frac{D}{\rho_f}\nabla^4\eta_2 = -\eta_1 \frac{\partial^2\varphi_1}{\partial z\partial t} - \frac{1}{2}\epsilon(\nabla\varphi_1)^2 \quad at\ z = 0 \tag{5.33}$$

We assume that the particular solution due to each product term $\{j, l\}$ above will be bound or forced non-free simple harmonic waves,

$$\begin{pmatrix}\eta_{2j,l}(\boldsymbol{x},t)\\ \varphi_{2j,l}(\boldsymbol{r},t)\end{pmatrix} = \begin{pmatrix}\hat{\eta}_{2j,l}\\ \widehat{\varphi}_{2j,l}(z)\end{pmatrix} e^{i(\boldsymbol{K_{j,l}}.\boldsymbol{x} - \Omega_{j,l}t)} \tag{5.34}$$

We can find the solution of $\widehat{\varphi}_{2j,l}(z)$ from equations (5.30) and (5.31).

$$\widehat{\varphi}_{2j,l}(z) = C_{2j,l}\frac{\cosh K_{j,l}(\mathrm{z}+\mathrm{h})}{\sinh\,(K_{j,l}h)} \tag{5.35}$$

Where $K_{j,l}$ is $|\boldsymbol{K_{j,l}}|$, for deep water $(h \to \infty)$ the limiting behaviour of $\widehat{\varphi}_{2j,l} = C_{2j,l}e^{K_{j,l}z}$. At this time, after substituting the assumed solution in the left-hand side of the kinematic and the dynamic boundary conditions (5.32-5.33) respectively. Also substituting the first order solution in equations (4.21-22) to the right-hand side of (5.32-5.33) respectively. We will have the following,

$$\begin{pmatrix} -i\Omega_{j,l} & -K_{j,l} \\ g - \frac{\rho a}{\rho_f}{\Omega_{j,l}}^2 + \frac{I_m}{\rho_f}{K_{j,l}}^2{\Omega_{j,l}}^2 + \frac{D}{\rho_f}{K_{j,l}}^4 & -i\Omega_{j,l}\coth K_{j,l}h \end{pmatrix}\begin{pmatrix}\hat{\eta}_{2j,l} \\ C_{2j,l}\end{pmatrix} = \begin{pmatrix} -i\omega_j\left(\frac{\boldsymbol{k}_j.\boldsymbol{k}_l}{k_j} + k_j\right)F_j \\ {\omega_j}^2 - \frac{\omega_j\omega_l}{2}\left(\frac{\boldsymbol{k}_j.\boldsymbol{k}_l}{k_jk_l}F_jF_l - 1\right)\end{pmatrix}\frac{d_jd_l}{4} \tag{5.32}$$

Where, F_j is denotes $coth(k_jh)$ similarly for F_l. The pair $\boldsymbol{K_{j,l}}$ and $\Omega_{j,l}$ does not satisfy the dispersion relation, then the determinant of the coefficient matrix is non-zero (the matrix is not singular). Consequently, the matrix (5.32) can be solved straight forward.

Then $\hat{\eta}_{2j,l}$and $C_{2j,l}$ will have the following forms

$$\hat{\eta}_{2j,l} = \frac{\left\{-i\Omega_{j,l}\coth K_{j,l}h\left(-i\omega_j\left(\frac{\boldsymbol{k}_j.\boldsymbol{k}_l}{k_j}+k_j\right)F_j\right) + K_{j,l}\left({\omega_j}^2 - \frac{\omega_j\omega_l}{2}\left(\frac{\boldsymbol{k}_j.\boldsymbol{k}_l}{k_jk_l}F_jF_l - 1\right)\right)\right\}\frac{d_jd_l}{4}}{-{\Omega_{j,l}}^2\coth K_{j,l}h - K_{j,l}\left(g - \frac{\rho a}{\rho_f}{\Omega_{j,l}}^2 + \frac{I_m}{\rho_f}{K_{j,l}}^2{\Omega_{j,l}}^2 + \frac{D}{\rho_f}{K_{j,l}}^4\right)}$$

$$C_{2j,l} = \frac{\left\{\beta\left(-i\omega_j\left(\frac{\boldsymbol{k}_j.\boldsymbol{k}_l}{k_j}+k_j\right)F_j\right) - i\Omega_{j,l}\left({\omega_j}^2 - \frac{\omega_j\omega_l}{2}\left(\frac{\boldsymbol{k}_j.\boldsymbol{k}_l}{k_jk_l}F_jF_l - 1\right)\right)\right\}\frac{d_jd_l}{4}}{-{\Omega_{j,l}}^2\coth K_{j,l}h - K_{j,l}\left(g - \frac{\rho a}{\rho_f}{\Omega_{j,l}}^2 + \frac{I_m}{\rho_f}{K_{j,l}}^2{\Omega_{j,l}}^2 + \frac{D}{\rho_f}{K_{j,l}}^4\right)}$$

Noting that

$\beta=\left(-g + \frac{\rho a}{\rho_f}{\Omega_{j,l}}^2 - \frac{I_m}{\rho_f}{K_{j,l}}^2{\Omega_{j,l}}^2 - \frac{D}{\rho_f}{K_{j,l}}^4\right)$

It is essential to note here that $\hat{\eta}_{2j,l}$ is the analytical solution for the group line, that is the zeroth harmonic in the plot 1, in section 1.1. One more case exists, where $\boldsymbol{K_{j,l}}$ or $\Omega_{j,l}$ is zero, the second order solution will not be a propagating wave. We are not going to discuss this case in this thesis.

Chapter 6

Discussion

In this chapter, we will discuss the theoretical results attained during this study. First, we will take a brief look at the linear dispersion relations found in chapter 3. Secondly, the resonance analysis made for the beam and the plate is to be interpreted. Thereafter, a discussion of the beam and plate interaction equations found in chapter 4 and 5 is to be held. In the last section we will discuss the non-resonant interactions.

6.1 Linear dispersion relations

In chapter 3, the linear dispersion relations for the string, beam and plate were derived. For small thickness of the beam and the plate, the plots in chapter 1 showed that at low wavenumbers, the behaviour of gravity waves wins, then as the wave number increases, the elastic contribution become stronger and dominates after an inflection point. The wave length where the change of behaviour occurs is highly affected with the elasticity of the structural element. As the stiffness of the structural element increases the point of inflection will be detected at higher wavenumber. The results found where compatible with theoretical predictions, where a structural element is coupled with the hydrodynamics of inviscid incompressible flows [14].

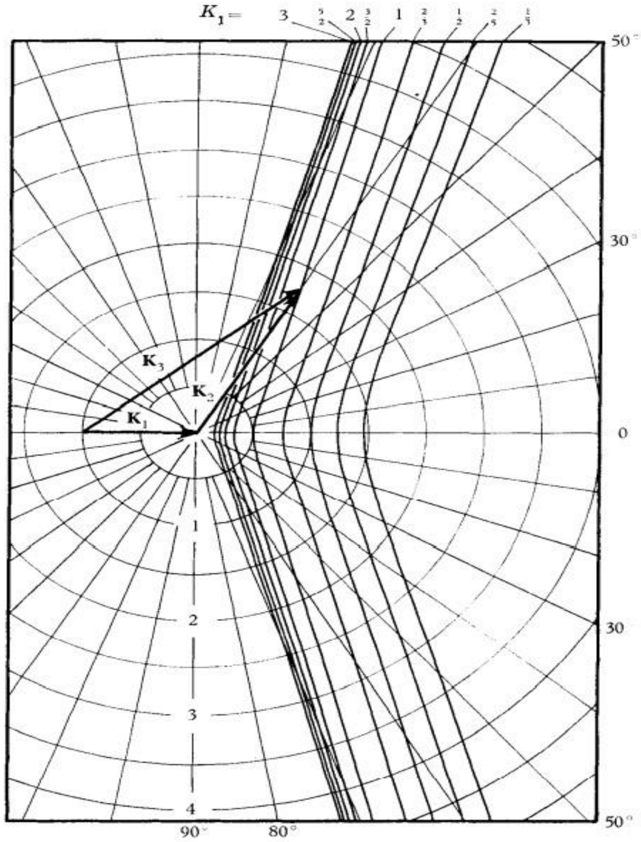
6.2 Resonance

Through the resonance analysis held in chapters 4 and 5, it has been proven that resonance could occur due to three wave interactions in case of fluid covered by elastic sheet, and deep-water assumption. The existence of non-colinear triads for the plate case, means that short-crested wave field could be detected.

The resonance investigation showed that there exists a minimum value for k_2 where the resonance could occur. The latter agrees with other studies such as *Resonant interactions among capillary-gravity waves*, which was held by McGoldrick (1964), and the study made by Phillips (1960) [15].

Following, a comparison is made between solutions of resonant triads for hydroelastic plate waves (plot 8) and capillary-gravity waves. The values of k_1 studied range between $0.5m^{-1}$ and $3m^{-1}$, where the minimum values for k_2 , ($k_{2\,min}$) range between $13m^{-1}$ and 50 m^{-1}, respectively. That is to say the value of $k_{2\,min}$ increases with the increasing value of k_1. Unlike, the behaviour of resonant triads in case of capillary-gravity waves (McGoldrick -1964), where the minimum values for k_2 decreased with the increase of the values of k_1. In the capillary gravity waves, $k_{2\,min}$ ranged between $1.5m^{-1}$ and $0.1m^{-1}$, for k_1 equals 1/3 and 3, respectively.

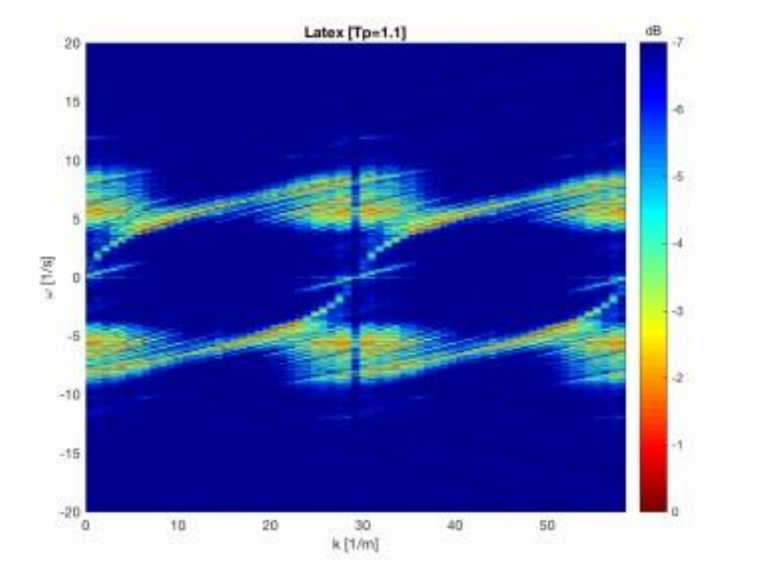
Another difference between capillary-gravity waves and hydroelastic waves is that the values of $k_{2\,min}$ in hydroelastic waves is higher than that of capillary gravity waves. Higher values of $k_{2\,min}$ in hydroelastic waves could be due to the presence of the elastic modulus E and the plate density ρ, which have considerably bigger values than the surface tension coefficient present in the resonance condition for capillary-gravity waves. The presented contrasts demonstrate an important difference between capillary gravity waves and elastic plate waves. The polar plot below presents solutions of resonant triads of capillary-gravity waves.



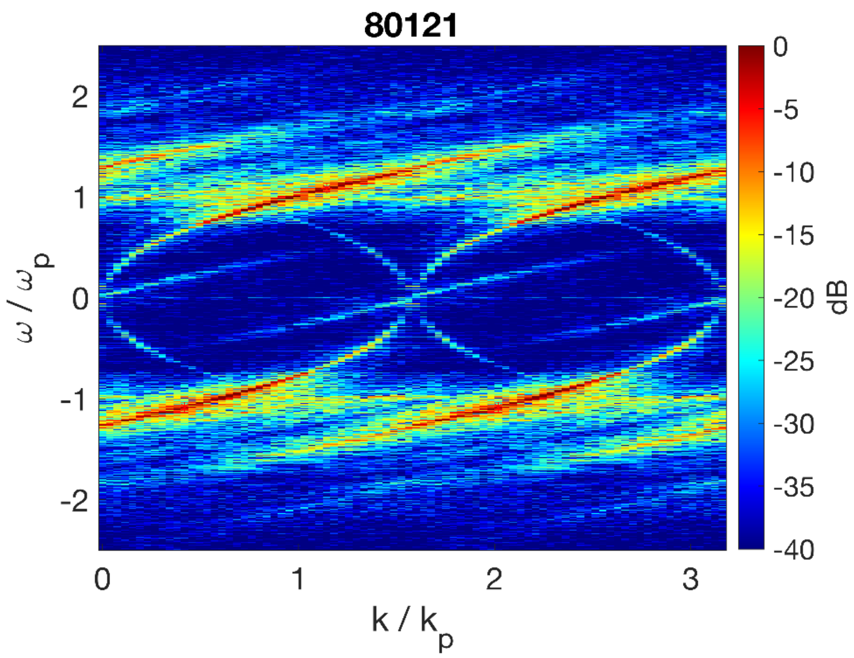
plot 9. Resonance solutions for capillary-gravity waves (McGoldrick-1964).

6.3 Resonant interaction equations

By definition, the resonant interaction is the interaction of three or more waves, which takes place when a certain configuration of wave vectors and dispersion equation are met. Resonant interactions can lead to wave instability, which in its turn will create turbulence, and high-dimensional chaos [15]. In the last decades, the interaction equations was popular due to their importance and the simplicity of the criteria. *A. V. Marchenko* studied the stability of flexural-gravity waves and quadratic interactions, where he used a numerical solution of the interaction equations to prove that in case of presence of resonant triad, the resonating waves may be strongly amplified [23]. The significance of the resonant case made interesting to be investigated in this thesis. The presented investigation is theoretical, but in fact it is remarkably close to experiments made in the laboratory. In (2019), experiments was held by *Ingrid Olsen* where irregular surface waves on water were sent into a region where the water was covered by an elastic sheet that was supposed to resemble ice [17]. Measurements were made of the surface elevation with good spatial resolution such that 2D Fourier analysis in time and space was possible. To illustrate the theoretical analysis held in this thesis, we will present figures resulting from the latter described experiment and similar experiment held by *Yiyi Whitchelo* (2021).



plot 10. Logarithmic scaled spectrum plot for Latex sheet with 0.2mm thickness *Olsen* (2019).



plot 11. Logarithmic scales spectrum plot for Latex sheet with 0.25mm thickness *Whitchelo* (2021)

In plot 10, the black dashed lines show linear dispersion relation and its harmonics up to the third order, the innermost is the first harmonic, which represents the linear dispersion relation. The curve above it is the second harmonic, and the straight line coming out of the origin is the zeroth harmonic, it is also known as the group line. Note that this plot is a reproduction from Olsen's thesis, the black dashed line will be more obvious on screen than on paper.

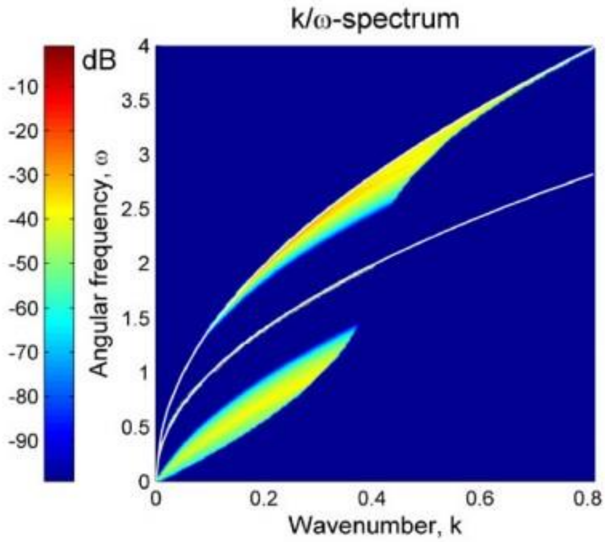
In this research, the need of Green's second identity is to the extent that the waves that occur on the dispersion relation (first harmonic) are forced by their own natural frequency. This will lead to resonant growth. To arrest the growth, a solvability condition is imposed on the forcing that we have on the right-hand side of the governing equations in the nonlinear system. The solvability condition is the set of interaction equations derived in chapter 4 and 5 for beam and plate, respectively. Using Green's formula, we get a relationship between the three waves of the resonant triad, combined per wise in quadratic products.

Now we will move into a closer look at the interaction equations. Regarding the group velocity term, one of the observations in the experiments held by *Olsen* (2019) is that the group velocity increases gradually as wave propagates into the elastic sheet. The latter can be proven theoretically, using the interaction equations. We can see that the group velocity is highly affected by the elastic material properties present in its own form and in the interaction coefficients. Since the elastic material properties such as elastic modulus E and density ρ substantially have bigger effect than the surface tension coefficient present in the group velocity form of free surface waves, then the group velocity of elastic waves will be higher than that for free waves. To add, from the equation of cg, it is intuitive to say that for lower wavenumbers, then longer waves, the group velocity is higher in elastic sheet.
During wave propagation the energy flux is conserved, then if the group velocity increases, this means that the wave amplitude will decrease. This is also consistent with the experimental results showing attenuation of amplitudes in the hydroelastic waves.
In case of three wave resonance, the interaction coefficients α_j present in the interaction equations will identify the direction of energy transfer between waves in a resonating triad. The energy sharing between three wave modes is complicated, it can be periodic back and forth, or it can go asymptotically in one direction [17]. In reality, the presence of a whole continuum of different resonant triads makes the situation more challenging. The study *A criterion for nonlinear wave stability* by *K. Hasselmann* suggests that shorter waves have tendency to share their energy with longer waves [24].

6.4 *Nonlinearly forced response*

In the non-resonant case, the second order problem can be solved precisely, since $\boldsymbol{K}_{j,l}$ and $\Omega_{j,l}$ are non-zero and does not satisfy the dispersion relation. In section 5.4, the second order problem was solved analytically and the equations of $\hat{\eta}_{2j,l}$and $C_{2j,l}$ where stated. In fact, $\hat{\eta}_{2j,l}$ is the analytical solution for the group line, that is the zeroth harmonic in the plots 11 and 12. Although the zeroth harmonic may look like a straight line, it is probably a cloud of a certain structure that looks like a straight line, having the slope of the group velocity of the principal wave component. The intensity of the cloud differs depending on the products of the two complex amplitudes interacting with each other. This means that the intensity of the cloud is not the same

in all locations. The analytical formulas of $\hat{\eta}_{2j,l}$and $C_{2j,l}$ calculated in section 5.4 predicts the distribution of intensity within the group line.
In fact, the non-resonant equations calculated in section 5.4 corresponds to the equations (12-14) in the study *Interpretation and observations of ocean wave spectra* by *Krogstad* and *Trulsen* (2010) [26]. In the latter study, the nonlinear Schrödinger equation and its generalizations approach was used. Below is a plot presented by *Krogstad* and *Trulsen* (2010) for an example of the first and second order (k, ω)-spectrum for unidirectional waves. This plot is actually a solution of equations (12-14) in the preceding paper, and it is similar to the experimental observations. The analytical solution derived in section 5.4 can be used to produce a similar plot and to investigate the structure of the group line.



plot 12. First and second order spectra for unidirectional waves *(Krogstad and Trulsen (2010))*

Chapter 7

Conclusion

7.1 Conclusion

'*Since a general solution must be judged impossible from want of analysis, we must be content with the knowledge of some special cases, and that all the more, since the development of various cases seems to be the only way of bringing us at last to a more perfect knowledge'* [18]. This thesis is a theoretical investigation for transverse waves in fluid covered by an elastic sheet. The governing equations of a plate and a beam placed over fluid systems was derived and studied for the linear and second order nonlinearity. Resonance investigation was performed and the resonant interaction equations for beam and plate was derived. For the non-resonant case, the particular solution was obtained for the plate over fluid. Resonance analysis showed that resonance could occur due to three wave interactions in hydroelastic waves, for deep water assumption. The resonance behaviour of waves in elastic sheet covering fluid is different from that for capillary-gravity waves, in regards of increasing value of $k_{2\,min}$ with the increase of k_1. To add, the presence of elastic properties of the sheet leads to higher values of k_2 than that for capillary gravity waves. The group velocity of elastic waves is higher than that of free waves and it increases with the increase of the wave length. Effects of elasticity leads to lower amplitudes of elastic waves than that of free waves. The zeroth, first and second harmonic presented in by *Olsen* (2019) were detected during the investigation. The presence of elastic parameters in the interaction coefficients means that the value of the bending stiffness influences the nonlinear interactions happening. Studying the non-resonant case leads to finding the analytical solution of the group line. It is concluded that the group line probably have a cloud structure looking like a straight line. Last but not least, the hydro-elastic waves are highly affected by the stiffness of the covering sheet. Finally, the investigation of fluid covered by elastic sheet case made us more knowledgeable about the resonant and non-resonant interactions of hydro-elastic waves.

7.2 Further work

Future investigation is the quantitative comparison between the predicted formulas of $\hat{\eta}_{2j,l}$and $C_{2j,l}$ derived in chapter 5, with the structure of the group line in the experimental plots (plots 10- 11). In the experiments, the spectrum used to simulate the data file of surface elevation is the JONSWAP-spectrum. The structure of the group line can be computed by inserting the frequencies and the wavenumbers from the latter spectrum into the analytical solution $\hat{\eta}_{2j,l}$. A plot similar to that found in the study *Interpretation and observations of ocean wave spectra* by *Krogstad* and *Trulsen* (plot- 12 in this thesis) is to be computed. In this thesis, the concern was about how the group line manifests in surface elevation, the next step can be to investigate how it manifests in velocity. In particular, if it implies enhanced mass transport into ice-covered water faster than we would have preferred due to environmental concerns. Besides, a similar analytical investigation could be done for sheets with different properties in order to emphasize the effect of elasticity on the waves.

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