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Abstract

We study the computational complexity of three representations of irrational numbers: standard Baire sequences, dual Baire sequences and contractors. Our main results: Irrationals whose standard Baire sequences are of low computational complexity might have dual Baire sequences of arbitrarily high computational complexity, and vice versa, irrationals whose dual Baire sequences are of low complexity might have standard Baire sequences of arbitrarily high complexity. Furthermore, for any subrecursive class $\mathcal{S}$ closed under primitive recursive operations, the class of irrationals that have a contractor in $\mathcal{S}$ is exactly the class of irrationals that have both a standard and a dual Baire sequence in $\mathcal{S}$. Our results implies that a subrecursive class closed under primitive recursive operations contains the continued fraction of an irrational number α if and only if there is a contractor for α in the class.

Keywords (separated by '-'): Computable analysis - Representation of irrationals - Subrecursion - Computational complexity - Baire sequences - Contraction maps

On Subrecursive Representation of Irrational Numbers: Contractors and Baire Sequences

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Abstract. We study the computational complexity of three representations of irrational numbers: standard Baire sequences, dual Baire sequences and contractors. Our main results: Irrationals whose standard Baire sequences are of low computational complexity might have dual Baire sequences of arbitrarily high computational complexity, and vice versa, irrationals whose dual Baire sequences are of low complexity might have standard Baire sequences of arbitrarily high complexity. Furthermore, for any subrecursive class $\mathcal{S}$ closed under primitive recursive operations, the class of irrationals that have a contractor in $\mathcal{S}$ is exactly the class of irrationals that have both a standard and a dual Baire sequence in $\mathcal{S}$. Our results implies that a subrecursive class closed under primitive recursive operations contains the continued fraction of an irrational number α if and only if there is a contractor for α in the class.

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1 Introduction

The theorems proved below complement the picture drawn in Kristiansen [4,5] and, particularly, Georgiev et al. [1]. Our investigations are motivated by the question: *Do we need, or do we not need, unbounded search in order to convert one representation of an irrational number into another representation?* A computation that does not apply unbounded search is called a *subrecursive* computation. Primitive recursive computations and (Kalmar) elementary computations are typical examples of subrecursive computations. A representation R_1 (of irrational numbers) is *subrecursive in* a representation R_2 if the R_1-representation of α can be subrecursively computed in the R_2-representation of α.

The reader that wants to know more about our motivations, or want further explanations, should consult the first few sections of Georgiev et al. [1]. This is a

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technical paper where our main concern is to give reasonably full proofs of some new theorems.[1]

What we will call a *Baire sequence* is an infinite sequence of natural numbers. Such a sequence $a_0, a_1, a_2, \ldots$ represents an irrational number α in the interval $(0, 1)$. We split the interval $(0, 1)$ into infinitely many open subintervals with rational endpoints. We may, e.g., use the splitting

$$(0/1 \ , \ 1/2 \ ) \ (\ 1/2 \ , \ 2/3 \ ) \ (\ 2/3 \ , \ 3/4 \ ) \ \ldots \ (\ n/(n+1) \ , \ (n+1)/(n+2) \ ) \ldots \ .$$

The first number of the sequence a_0 tells us in which of these intervals we find α. Thus if $a_0 = 17$, we find α in the interval $(17/18, 18/19)$. Then we split the interval $(17/18, 18/19)$ in a similar way. The second number of the sequence a_1 tells us in which of these intervals we find α, and thus we proceed.

In general, in order to split the interval (q, r), we need a strictly increasing sequence of rationals $s_0, s_1, s_2 \ldots$ such that $s_0 = q$ and $\lim_i s_i = r$. We will use the splitting $s_i = (a + ic)/(b + id)$ where a, b are (the unique) relatively prime natural numbers such that $q = a/b$ and c, d are (the unique) relatively prime natural numbers such that $r = c/d$ (let $0 = 0/1$ and $1 = 1/1$). This particular splitting makes our proof smooth and transparent, but our main results are invariant over all natural splittings.

We will say that the Baire sequences explained above are *standard*. The standard Baire sequence of the irrational number α will lexicographically precede standard Baire sequence of the irrational number β iff $\alpha < \beta$. We will also work with what we will call *dual* Baire sequences. The dual sequence of α will lexicographically precede the dual sequence of β iff $\alpha > \beta$. We get the dual sequences by using decreasing sequences of rationals to split intervals, e.g., the interval $(0, 1)$ may be split into the intervals

$$(\ 1/1 \ , \ 1/2 \ ) \ (\ 1/2 \ , \ 1/3 \ ) \ (\ 1/3 \ , \ 1/4 \ ) \ \ldots \ (\ 1/n \ , \ 1/(n+1) \ ) \ \ldots \quad .$$

Definition 1. *Let $f : \mathbb{N} \to \mathbb{N}$ be any function, and let $n \in \mathbb{N}$. We define the interval I_f^n by $I_f^0 = (0/1, 1/1)$ and*

$$I_f^{n+1} \quad = \quad \left(\frac{a + f(n)c}{b + f(n)d} \, , \, \frac{a + f(n)c + c}{b + f(n)d + d} \right)$$

if $I_f^n = (a/b, c/d)$. We define the interval J_f^n by $J_f^0 = (0/1, 1/1)$ and

$$J_f^{n+1} \quad = \quad \left(\frac{a + f(n)a + c}{b + f(n)b + d} \, , \, \frac{f(n)a + c}{f(n)b + d} \right)$$

if $J_f^n = (a/b, c/d)$. The function $B : \mathbb{N} \to \mathbb{N}$ is the standard Baire representation *of the irrational number $\alpha \in (0, 1)$ if we have $\alpha \in I_B^n$ for every n. The function $A : \mathbb{N} \to \mathbb{N}$ is the* dual Baire representation *of the irrational number $\alpha \in (0, 1)$ if we have $\alpha \in J_A^n$ for every n.*

[1] The author wants to thank Dag Normann for enlightening discussions which lead up to this paper. The author wants to thank Eyvind Briseid for helpful advice and for pinpointing weaknesses in an early version of this paper.

Before we discuss contractors, we will recall the *trace functions* introduced in Kristiansen [4]. A trace function for α is a function that move any rational number closer to α. The formal definition, which follows, is straightforward.

Definition 2. *A function $T : [0,1] \cap \mathbb{Q} \rightarrow (0,1) \cap \mathbb{Q}$ is a* trace function *for the irrational number α if we have $|\alpha - q| > |\alpha - T(q)|$ for any rational q.*

We will say that a trace function T *moves q to the right (left)* if $q < T(q)$ ($T(q) < q$). The easiest way to realize that a trace function indeed defines a unique real number, is probably to observe that a trace function T for α yields the Dedekind cut of α: if T moves q the right, then we know that q lies below α; if T moves q the left, then we know that q lies above α. Obviously, T cannot yield the Dedekind cut for any other number than α. It is proved in [4] that trace functions are subrecursively equivalent to continued fractions.

Intuitively, a *contractor* is a function that moves two (rational) numbers closer to each other. It turns out that also contractors can be used to represent irrational numbers.

Definition 3. *A function $F : [0,1] \cap \mathbb{Q} \rightarrow (0,1) \cap \mathbb{Q}$ is a* contractor *if we have*

$$F(q) \neq q \quad \text{and} \quad |F(q_1) - F(q_2)| \; < \; |q_1 - q_2|$$

for any rationals q, q_1, q_2 where $q_1 \neq q_2$.

Theorem 4. *Any contractor is a trace function for some irrational number.*

Proof. Let F be a contractor. If F moves q to the right (left), then F also move any rational less (greater) than q to the right (left); otherwise F would not be a contractor. We define two sequences $q_0, q_1, q_2 \ldots$ and $p_0, p_1, p_2 \ldots$ of rationals. Let $q_0 = 0$ and $p_0 = 1$. Let $q_{i+1} = (q_i + p_i)/2$ if F moves $(q_i + p_i)/2$ to the right; otherwise, let $q_{i+1} = q_i$. Let $p_{i+1} = (q_i + p_i)/2$ if F moves $(q_i + p_i)/2$ to the left; otherwise, let $p_{i+1} = p_i$ (Definition 3 requires that a contractor moves any rational number). Obviously, we have $\lim_i q_i = \lim_i p_i$, and obviously, this limit is an irrational number α. It is easy to see that F is a trace function for α. □

Definition 5. *A contractor F is a* contractor for *the irrational number α if F is a trace function for α (Theorem 4 shows that this definition makes sense).*

Contractors, also known as contraction maps, come in a number of variants. The variant given by Definition 3 is tailored for our purposes. Computational aspects of contractors have also been studied in proof mining, see Kohlenbach and Olivia [3] and Gerhardy and Kohlebach [2].

2 Technical Preliminaries

Definition 6. *For any string $\tau \in \{L, R\}^*$, we define the* interval addressed by *τ inductively over the structure of τ: The empty sequence addresses the interval $(0/1, 1/1)$. Furthermore*

$$\tau L \text{ addresses } \left(\frac{a}{b}, \frac{a+c}{b+d} \right) \text{ and } \tau R \text{ addresses } \left(\frac{a+c}{b+d}, \frac{c}{d} \right)$$

if τ addresses $(a/b, c/d)$. We will use $I[\tau]$ to denote the interval addressed by τ.

Definition 7. *Let α be an irrational number in the interval $(0, 1)$. Let a and b be relatively prime natural numbers with $b > 0$. The fraction a/b is a* left best approximant *of α if we have $c/d \leq a/b < \alpha$ or $\alpha < c/d$ for any natural numbers c, d with $0 < d \leq b$. The fraction a/b is a* right best approximant *of α if we have $\alpha < a/b \leq c/d$ or $c/d < \alpha$ for any natural numbers c, d with $0 < d \leq b$.*

Lemma 8. *Assume that the interval $(a/b, c/d)$ is addressed by some $\tau \in \{L, R\}^*$. Then, (i) a/b and c/d are, respectively, left and right best approximants of any irrational number in the interval $(a/b, c/d)$, and (ii) we have $c/d - a/b = 1/(db)$.*

Proof. If an interval $(a/b, c/d)$ is addressed by some $\tau \in \{L, R\}^*$, then a/b and c/d will be a Farey pair, that is, neighbors in the Farey series of order $\max(b, d)$. It is well know that the *mediant* of the pair, that is, $(a + c)/(b + d)$ will be in its lowest terms and lie in the interval, moreover, for any other vulgar fraction m/n that lie in the interval, we have $n > b + d$, see Richards [8]. It follows that (i) holds. Moreover, it well know that we have $cb - ad = 1$, or equivalently $c/d - a/b = 1/(db)$, for any Farey pair $(a/b, c/d)$, and thus (ii) also holds. □

The next lemma is the key to the proof of one of our main theorems.

Lemma 9. *(i) The string $R^{f(0)}LR^{f(1)}L \ldots R^{f(n)}L$ addresses the interval I_f^{n+1}. (ii) The string $L^{f(0)}RL^{f(1)}R \ldots L^{f(n)}R$ addresses the interval J_f^{n+1}.*

Proof. We prove (i). The proof of (ii) is symmetric.

Let $\tau = R^{f(0)}LR^{f(1)}L \ldots R^{f(n-1)}L$. Observe that we have $I[\tau] = (0/1, 1/1) = I_f^0$ when τ is the empty sequence.

Assume that $I[\tau] = I_f^n = (a/b, c/d)$. We need to prove that

$$I[\tau R^{f(n)}L] \;=\; I_f^{n+1} \; . \tag{1}$$

Let $k = f(n)$. We prove (1) by a secondary induction on k.

Assume $k = 0$. By Definition 6, we have

$$I[\tau R^{f(n)}L] \;=\; I[\tau R^0 L] \;=\; I[\tau L] \;=\; (\; a/b \, , \, (a + c)/(b + d) \;) \, .$$

By Definition 1, we have

$$I_f^{n+1} \;=\; (\; (a{+}kc)/(b{+}kd) \, , \, (a{+}kc{+}c)/(b{+}kd{+}d) \;) \;=\; (\; a/b \, , \, (a{+}c)/(b{+}d) \;) \, .$$

Thus (1) holds when $f(n) = 0$. Now, assume by induction hypothesis that

$$I[\tau R^k L] \;=\; \left(\frac{a + kc}{b + kd} \, , \, \frac{a + kc + c}{b + kd + d} \right) \, . \tag{2}$$

Observe that the right hand side of (2) is the definition of I_f^{n+1} with k for $f(n)$. Now, by (2) and Definition 6, we have

$$I[\tau R^k] \;=\; \left(\frac{a + kc}{b + kd} \, , \, \frac{c}{d} \right) \, . \tag{3}$$

Furthermore, by (3) and Definition 6, we have

$$I[\tau R^{k+1}] \;=\; \left(\frac{a+kc+c}{b+kd+d}\,,\, \frac{c}{d} \right) \;=\; \left(\frac{a+(k+1)c}{b+(k+1)d}\,,\, \frac{c}{d} \right) \tag{4}$$

and by (4) and Definition 6, we have

$$I[\tau R^{k+1}L] \;=\; \left(\frac{a+(k+1)c}{b+(k+1)d}\,,\, \frac{a+(k+1)c+c}{b+(k+1)d+d} \right) . \tag{5}$$

Observe that the right hand side of (5) is the definition of I_f^{n+1} with $k+1$ for $f(n)$. This proves that (1) holds. □

Note that it follows from the two lemmas above that the endpoints of the interval I_f^n (for any n and any f) will be best approximants of every irrational in the interval. The same goes for and J_f^n.

Lemma 10. *For any n and any f, let r_n denote the right endpoint of the interval I_f^n, and let ℓ_n denote the left endpoint of the J_f^n. Then, we have (i) $r_n - r_{n+1} > r_{n+1} - r_{n+2}$ and (ii) $\ell_{n+1} - \ell_n > \ell_{n+2} - \ell_{n+1}$.*

Proof. We prove (i). Assume $I_f^n = (a/b, c/d) = I[\tau]$. By Definition 1 and Lemma 9, we have

$$I_f^{n+1} \;=\; \left(\frac{a+f(n)c}{b+f(n)d}\,,\, \frac{a+f(n)c+c}{b+f(n)d+d} \right) \;=\; I[\tau R^{f(n)}L]\,. \tag{6}$$

Let $\mathbf{a} = a + f(n)c$, let $\mathbf{b} = b + f(n)d$ and let $k = f(n)$. We can now rewrite (6) as

$$I_f^{n+1} \;=\; \left(\frac{\mathbf{a}}{\mathbf{b}}\,,\, \frac{\mathbf{a}+c}{\mathbf{b}+d} \right) \;=\; I[\tau R^k L]\,. \tag{7}$$

By (7) and Definition 6, we have

$$I[\tau R^k] = (\mathbf{a}/\mathbf{b}, c/d) \quad \text{and} \quad I[\tau R^k R] = ((\mathbf{a}+c)/(\mathbf{b}+d), c/d)\,.$$

This shows that $((\mathbf{a}+c)/(\mathbf{b}+d), c/d)$ is addressed by some string in $\{L,R\}^*$. Thus, by Lemma 8 (ii), we have

$$\frac{c}{d} - \frac{\mathbf{a}+c}{\mathbf{b}+d} \;=\; \frac{1}{d(\mathbf{b}+d)}\,. \tag{8}$$

By Lemma 9, we have $I_f^{n+2} = I[\tau R^k L R^m L]$ where $m = f(n+1)$. We can assume that $m = 0$ since $m = 0$ yields the maximal distance between r_{n+1} and r_{n+2}. Thus, by Definition 6, $I_f^{n+2} = I[\tau R^k LL] = (\mathbf{a}/\mathbf{b}, (2\mathbf{a}+c)/(2\mathbf{b}+d))$. Moreover, again by Definition 6, we have

$$I[\tau R^k L] \;=\; \left(\frac{\mathbf{a}}{\mathbf{b}}\,,\, \frac{\mathbf{a}+c}{\mathbf{b}+d} \right) \quad \text{and} \quad I[\tau R^k LR] \;=\; \left(\frac{2\mathbf{a}+c}{2\mathbf{b}+d}\,,\, \frac{\mathbf{a}+c}{\mathbf{b}+d} \right) .$$

This shows that $((2\mathbf{a}+c)/(2\mathbf{b}+d), (\mathbf{a}+c)/(\mathbf{b}+d)$ is addressed by a string in $\{L, R\}^*$, and thus, by Lemma 8 (ii), we have

$$\frac{\mathbf{a}+c}{\mathbf{b}+d} - \frac{2\mathbf{a}+c}{2\mathbf{b}+d} = \frac{1}{(\mathbf{b}+d)(2\mathbf{b}+d)} . \tag{9}$$

Now we can conclude our proof of (i) with

$$r_n - r_{n+1} = \frac{c}{d} - \frac{\mathbf{a}+c}{\mathbf{b}+d} \overset{(8)}{=} \frac{1}{d(\mathbf{b}+d)} > \frac{1}{(\mathbf{b}+d)(2\mathbf{b}+d)} \overset{(9)}{=} \frac{\mathbf{a}+c}{\mathbf{b}+d} - \frac{2\mathbf{a}+c}{2\mathbf{b}+d} = r_{n+1} - r_{n+2} .$$

The proof of (ii) is symmetric. □

The *Hurwitz characteristic* of an irrational $\alpha \in (0, 1)$ is the (unique) infinite sequence Σ over the alphabet $\{L, R\}$ such that we have $\alpha \in I[\sigma]$ for any finite prefix σ of Σ. Hurwitz characteristics, which are subrecursively equivalent to Dedekind cuts, have been studied by Lehman [7] and, more recently, by Kristiansen and Simonsen [6].

3 Main Results

Theorem 11. *Let B and A be, respectively, the standard and the dual Baire sequence of α, and let F be any contractor for α. (i) We can compute B primitive recursively in F. (ii) We can compute A primitive recursively in F.*

Proof. We will show that the interval I_B^{n+1} and the value of $B(n)$ can be computed primitive recursively in F. It is trivial to compute the interval I_B^0. Assume that we have computed the interval $I_B^n = (a/b, c/d)$. First, we compute $c'/d' = F(c/d)$. Since F is a contractor for α, we have $a/b < \alpha < c'/d' < c/d$. Next, we find j such that

$$\frac{a+jc}{b+jd} < \frac{c'}{d'} \leq \frac{a+jc+c}{b+jd+d} .$$

Observe that $(\frac{a+jc}{b+jd}, \frac{a+jc+c}{b+jd+d})$ is an addressable interval and that c'/d' either lies inside, or is the right endpoint of, the interval. Thus, by Lemma 8, we have $d' \geq b+jd+d$. No unbounded search is needed to determine j. Indeed, j has to be less than d'. Thus we can primitive recursively compute j such that

$$\frac{a}{b} < \alpha < \frac{a+(j+1)c}{b+(j+1)d} .$$

Finally, we search for the least i less than or equal to $j+1$ such that F moves $(a+ic+c)/(b+id+d)$ to the left, and then we let $B(n)$ equal that i. This shows that we can compute $B(n)$ primitive recursively in F, and thus (i) holds. The proof of (ii) is symmetric. Use the contractor at the left endpoint of intervals in place of the right endpoint. □

Theorem 12. *Let B and A be, respectively, the standard and the dual Baire representation of α. We can compute a contractor for α primitive recursively in B and A (and we will need both oracles).*

Proof. Let r_i denote the right endpoint of the interval I_B^i, and let ℓ_i denote the left endpoint of the interval J_A^i. For every rational number $x \in [0,1]$, we define

$$F(x) \;=\; \begin{cases} r_{i+1} - (r_i - x)\dfrac{r_{i+1} - r_{i+2}}{r_i - r_{i+1}} & \text{if } r_{i+1} < x \leq r_i \\[2ex] \ell_{i+1} + (x - \ell_i)\dfrac{\ell_{i+2} - \ell_{i+1}}{\ell_{i+1} - \ell_i} & \text{if } \ell_i \leq x < \ell_{i+1}. \end{cases}$$

First we will prove that F is contractor, that is, we will prove that we have

$$|F(x) - F(y)| \;<\; |x - y| \tag{10}$$

for any rationals x, y where $x \neq y$. Once we have established that F is a contractor, it will be clear that F is a contractor for α.

Assume that one of the rationals x and y lies below α and that the other lies above. It is easy to see that F will move one of the numbers to the right and the other one to the left, and thus, (10) holds. Assume that both x and y lie at same side of α. We can w.l.o.g. assume that both lie below and that we have $x < y < \alpha$. The proof splits into two cases: (i) $\ell_i \leq x < y < \ell_{i+1}$ for some i, and (ii) $\ell_i \leq x < \ell_{i+1} \leq \ell_j \leq y < \ell_{j+1}$ for some i, j where $j \geq i+1$.

Case (i). Let $k = (\ell_{i+2} - \ell_{i+1})/(\ell_{i+1} - \ell_i)$. By Lemma 10, we have $k < 1$, and then by the definition of F, we have

$$F(y) - F(x) \;=\; \ell_{i+1} + (y - \ell_i)k - (\ell_{i+1} + (x - \ell_i)k) \;=\; (y - x)k \;<\; y - x$$

and thus (10) holds.

Case (ii). This case is slightly more involved, but in the end everything is straightforward. We omit the details.

This proves that F is a contractor for α. It remains to argue that F can be computed primitive recursively in B and A. Let q be an arbitrary rational in the interval $[0,1]$, and let m/n be q written in lowest terms.

(Claim) There exists $j < n$ such that $\ell_j \leq q < \ell_{j+1}$ or $r_{j+1} < q \leq r_j$.

In order to see that the claim holds, assume that $\alpha < q = m/n$. It follows from the lemmas in Sect. 2 that each $r_j = c_j/d_j$ is a right best approximant to α. Thus we have $n \geq d_j$ whenever $m/n \leq c_j/d_j$. Moreover, as $d_j > j$, we have $j < n$ such that $r_{j+1} < q = m/n \leq r_j$ if $\alpha < q$. If $q = m/n < \alpha$, a symmetric argument yields $j < n$ such that $\ell_j \leq q < \ell_{j+1}$. This proves the claim.

The sequence $r_0, r_1, r_2, \ldots$ can be computed primitive recursively in B, and the sequence $\ell_0, \ell_1, \ell_2, \ldots$ can be computed primitive recursively in A. Thus, it follows from the claim that F can be computed primitive recursively in B and A. □

It follows from the next theorem that we cannot compute the standard Baire sequence of an irrational α subrecursively in the dual Baire sequence of α. That requires unbounded search.

Theorem 13. *Let $\mathcal{S}$ be any subrecursive class. There exists an irrational number α such that (i) the standard Baire sequence of α is not in $\mathcal{S}$, and (ii) the dual Baire sequence of α is (Kalmar) elementary.*

Proof. A function f is *honest*, by definition, if $2^x \leq f(x)$, $f(x) \leq f(x+1)$ and the relation $f(x) = y$ is elementary. Let B be the an honest function which is not in $\mathcal{S}$. Such a B exists (a proof can be found in Georgiev et al. [1]). Now, B is the standard Baire sequence of some irrational number α, and since an irrational number only has one standard Baire sequence, the standard Baire sequence of α is not in $\mathcal{S}$. It remains to prove that the dual Baire sequence of α is elementary.

Let $a_n = B(0)+(\sum_{i=1}^{n} B(i)+1)$. Let $A(x) = 1$ if $x = a_n$ for some n; otherwise, let $A(x) = 0$. Since B is an honest function, we can check by elementary means if there exists n such that $x = a_n$. Hence A is an elementary function. We will prove that A is the dual Baire sequence of α.

For any natural number n, we define the strings σ_n and τ_n by

$$\sigma_n = L^{A(0)}RL^{A(1)}R\ldots L^{A(a_n-1)}RL^{A(a_n)} \text{ and } \tau_n = R^{B(0)}LR^{B(1)}L\ldots R^{B(n)}L\,.$$

We will prove the following claim by induction on n: $\sigma_n = \tau_n$ (claim).

Let $n = 0$. We have $a_0 = B(0)$ and thus, by the definition of A, we have

$$\sigma_0 \;=\; L^{A(0)}RL^{A(1)}R\ldots L^{A(a_0-1)}RL^{A(a_0)} \;=\; R^{a_0}L \;=\; R^{B(0)}L \;=\; \tau_0\,.$$

Let $n > 0$. By the definition of a_n, we have $a_n = a_{n-1} + B(n) + 1$, and thus $B(n) = a_n - (a_{n-1} + 1)$. Furthermore, we have

$$\sigma_n \overset{(1)}{=} \sigma_{n-1}RL^{A(a_{n-1}+1)}RL^{A(a_{n-1}+2)}\ldots RL^{A(a_n-1)}RL^{A(a_n)} \overset{(2)}{=} \sigma_{n-1}R^{a_n-(a_{n-1}+1)}L \overset{(3)}{=} \sigma_{n-1}R^{B(n)}L \overset{(4)}{=} \tau_{n-1}R^{B(n)}L \overset{(5)}{=} \tau_n$$

where (1) holds by the definition of σ_n; (2) holds by the definition of A; (3) holds by the definition of a_n; (4) holds by the induction hypothesis; and (5) holds by the definition of τ_n. This proves (claim).

It follows from (claim) and Lemma 9 that the inclusion $J_A^{a_n} \subseteq I_B^n$ holds for all n. This proves that A is the dual Baire sequence of α. □

Just for the record, the proof of the next theorem is symmetric to the proof of the preceding theorem.

Theorem 14. *Let $\mathcal{S}$ be any subrecursive class. There exists an irrational number α such that (i) the dual Baire sequence of α is not in $\mathcal{S}$, and (ii) the standard Baire sequence of α is (Kalmar) elementary.*

4 The Big Picture

Definition 15. *For any subrecursive class $\mathcal{S}$, let $\mathcal{S}_F$ denote the class of irrational numbers that have a* contractor *in $\mathcal{S}$, let $\mathcal{S}_{sB}$ denote the class of irrational numbers that have a* standard Baire sequence *in $\mathcal{S}$, and $\mathcal{S}_{dB}$ denote the class of irrational numbers that have a* dual Baire sequence *in $\mathcal{S}$.*

Corollary 16. *Let $\mathcal{S}$ be any subrecursive class closed under primitive recursive operations. Then, (i) $\mathcal{S}_{sB} \not\subseteq \mathcal{S}_{dB}$, (ii) $\mathcal{S}_{dB} \not\subseteq \mathcal{S}_{sB}$ and (iii) $\mathcal{S}_F = \mathcal{S}_{dB} \cap \mathcal{S}_{sB}$.*

Proof. Theorem 13 entails (i). Theorem 14 entails (ii). Theorem 11 entails $\mathcal{S}_F \subseteq \mathcal{S}_{dB} \cap \mathcal{S}_{sB}$. Theorem 12 entails $\mathcal{S}_{dB} \cap \mathcal{S}_{sB} \subseteq \mathcal{S}_F$. Thus, (iii) holds. □

Definition 17. *Let α be an irrational number in the interval $(0,1)$. A* left best approximation *of α is a sequence of fractions $\{a_i/b_i\}_{i\in\mathbb{N}}$ such that $(0/1) = (a_0/b_0) < (a_1/b_1) < (a_2/b_2) < \ldots$ and each a_i/b_i is a left best approximant of α (see Definition 7). A* right best approximation *of α is a sequence of fractions $\{a_i/b_i\}_{i\in\mathbb{N}}$ such that $(1/1) = (a_0/b_0) > (a_1/b_1) > (a_2/b_2) > \ldots$ and each a_i/b_i is a right best approximant of α. Clearly, both sequences converge to α.*

Let $\mathcal{S}_<$ denote the class of irrational numbers that have a left best approximation in the subrecursive class $\mathcal{S}$, and let $\mathcal{S}_>$ denote the class of irrational numbers that have a right best approximation in $\mathcal{S}$.

Theorem 18. *For any subrecursive class $\mathcal{S}$ closed under primitive recursion, we have $\mathcal{S}_< = \mathcal{S}_{dB}$ and $\mathcal{S}_> = \mathcal{S}_{sB}$.*

Proof. We say that a right best approximation of α is *complete* if every right best approximant occurs in the approximation. Note that the complete best approximation of an irrational α in the interval $(0,1)$ is unique.

Let B be the standard Baire sequence of α. Consider the interval I addressed by $R^{f(0)}LR^{f(1)}L\ldots R^{f(n)}L$. By Lemma 9, we have $I = I_B^{n+1}$. The right endpoint of I will be the n'th best approximant in the complete right best approximation of α. These considerations make it easy to see that the inclusion $\mathcal{S}_{sB} \subseteq \mathcal{S}_>$ holds.

Let $\{a_i/b_i\}_{i\in\mathbb{N}}$ be a right best approximation of α. We can w.l.o.g. assume that $\{a_i/b_i\}_{i\in\mathbb{N}}$ is complete since a complete right best approximation can be computed primitive recursively in an arbitrary right best approximation. We can primitive recursively in $\{a_i/b_i\}_{i\in\mathbb{N}}$ compute a (unique) string of the form $R^{k_0}LR^{k_1}L\ldots R^{k_n}L$ such that the right endpoint of the interval addressed by $R^{k_0}L\ldots R^{k_i}L$ equals a_{i+1}/b_{i+1} (for all $i \leq n$). Let B be the standard Baire sequence of α. By Lemma 9, we have $B(i) = k_i$ (for all $i \leq n$). These considerations make it easy to see that the inclusion $\mathcal{S}_> \subseteq \mathcal{S}_{sB}$ holds. This proves $\mathcal{S}_> = \mathcal{S}_{sB}$. The proof of $\mathcal{S}_< = \mathcal{S}_{dB}$ is of course symmetric. □

For any subrecursive class $\mathcal{S}$, let $\mathcal{S}_{g\uparrow}$ denote the class of irrational numbers that have a *general sum approximation from below* in $\mathcal{S}$, let $\mathcal{S}_{g\downarrow}$ denote the class of irrational numbers that have a *general sum approximation from above* in $\mathcal{S}$, furthermore, let $\mathcal{S}_{[]}$ denote the class of irrational numbers that have a *continued* *fraction* in $\mathcal{S}$. Definitions of general sum approximations from below and above can be found in [4] and [1]. It is proved in [4] that we have $\mathcal{S}_{[]} = \mathcal{S}_{g\uparrow} \cap \mathcal{S}_{g\downarrow}$ for any $\mathcal{S}$ closed under primitive recursive operations. It is proved in [1] that we have $\mathcal{S}_{<} = \mathcal{S}_{g\uparrow}$ and $\mathcal{S}_{>} = \mathcal{S}_{g\downarrow}$ for any $\mathcal{S}$ closed under primitive recursive operations. Thus, we have the following corollary.

Corollary 19. *For any subrecursive class $\mathcal{S}$ closed under primitive recursive operations, we have $\mathcal{S}_{[]} = \mathcal{S}_F$ and $\mathcal{S}_{<} = \mathcal{S}_{g\uparrow} = \mathcal{S}_{dB}$ and $\mathcal{S}_{>} = \mathcal{S}_{g\downarrow} = \mathcal{S}_{sB}$.*

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Chapter 28

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