Infinite Dimensional Analysis, Quantum Probability and Related Topics

Regularity Properties of the Stochastic Flow of a Skew Fractional Brownian Motion

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In this paper we prove, for small Hurst parameters, the higher order differentiability of a stochastic flow associated with a stochastic differential equation driven by an additive multi-dimensional fractional Brownian noise, where the bounded variation part is given by the local time of the unknown solution process. The proof of this result relies on Fourier analysis based variational calculus techniques and on intrinsic properties of the fractional Brownian motion.

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1. Introduction

Consider a $d-$dimensional fractional Brownian motion (fBm)

$$B_t^H = (B_t^{1,H}, \dots, B_t^{d,H}),\ 0 \leq t \leq T$$

with Hurst parameter $H \in (0,1)$ constructed on some complete probability space $(\Omega, \mathcal{F}, \mu)$. Here $B_\cdot^{1,H}, \dots, B_\cdot^{d,H}$ are independent $1-$dimensional fractional Brownian motions, that is centered Gaussian processes with a covariance structure given by

$$R_H(t,s) = E[B_t^{i,H} B_s^{i,H}] = \frac{1}{2}(s^{2H} + t^{2H} + |t-s|^{2H}).$$

We mention that for $H = \frac{1}{2}$ the fBm is a standard Wiener process. If $H \neq \frac{1}{2}$, the fBm is neither a semimartingale nor a Markov process. See e.g. [19] for more information about fBm.

In this paper we want to study the regularity of solutions X_t^x, $0 \le t \le T$ to the stochastic differential equation (SDE)

$$X_t^x = x + \alpha L_t(X^x) \cdot \mathbf{1}_d + B_t^H, \; 0 \le t \le T \tag{1.1}$$

with respect to their initial condition $x \in \mathbb{R}^d$. Here the Hurst parameter H of the fBm is small, that is $H \in (0, 1/2)$, $\alpha \in \mathbb{R}$, $\mathbf{1}_d$ is the vector with entries 1 and $L_t(X^x)$, $0 \le t \le T$ is the local time of the unknown solution process, which one can define as

$$L_t(X^x) = \lim_{\varepsilon \searrow 0} \int_0^t \varphi_\varepsilon(X_s^x) ds$$

where the limit is in probability and φ_ε approximates, in distribution, the Dirac delta function δ_0 in zero. Here a commonly used approximation φ_ε is given by

$$\varphi_\varepsilon(x) = \varepsilon^{-\frac{d}{2}} \varphi(\varepsilon^{-\frac{1}{2}} x), \; \varepsilon > 0, \tag{1.2}$$

where φ is a $d-$dimensional Gaussian probability density.

In the Wiener case, that is $H = \frac{1}{2}$, and $d = 1$ solutions to equations of this type are referred to as Skew Brownian motion in the literature and were first studied by [13] and [23] in the weak and strong sense. See also the related articles [12], [24], [21], [5], [10] and [15]. In the sequel, we may therefore also call solutions to (1.1) Skew Fractional Brownian motions.

In the case of $H \in (0, 1/2)$ the authors in [4] recently constructed strong solutions to (1.1) by using techniques from Malliavin calculus. In fact, the authors prove the following result:

Theorem 1.1. *Let $H < \frac{1}{2(d+2)}$. Then for all $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$, there exists a strong solution to X_t^x, $0 \le t \le T$ to (1.1). Moreover, X_t^x is Malliavin differentiable for all $0 \le t \le T$.*

For general $H \in (0,1)$ we shall mention the striking work in [6], where the authors analyze *path by path* solutions to (1.1) in the framework of Besov spaces

$B^{\alpha+1}_{\infty,\infty}$ by employing techniques based e.g. on the Leray-Schauder-Tychonoff fixed point theorem and a comparison principle with respect to an averaging operator. For $H < \frac{1}{2(d+1)}$, which is a slightly looser upper bound for H than the one in Theorem 1.1, the authors obtain existence of strong solutions. In the case $H < \frac{1}{2(d+2)}$,which corresponds to the condition in Theorem 1.1 they even prove path by path uniqueness, but not Malliavin differentiability of such solutions. Further, for $H < \frac{1}{2(d+3)}$ the authors are able to construct unique Lipschitz flows. However, a disadvantage of the latter approach is that it, in contrast to the method in [4], cannot be used for the construction of strong solutions to SDE's with additive fractional noise, where the drift vector field belongs to $L^\infty(\mathbb{R}^d)$.

The objective of this paper is to significantly improve the result in [6] with respect to the regularity of the stochastic flow in (1.1). For example in the case of $H < \frac{1}{2(d+3)}$, under which the authors obtain a Lipschitz flow, we can show that the flow must be twice locally Sobolev differentiable. Moreover, we will prove for $H < \frac{1}{2(d-1+2k)}$ that the flow belongs to the Sobolev space $W^{k,p}_{loc}$ for all $p \geq 2$.

The method used in this article is based on "local time variational calculus" techniques developed in the papers [4], [2], [3]. See also [17], [18], [9], [11] in the case of a (cylindrical) Wiener process or a Lévy process.

2. Main Result

Using a "local time variational calculus" technique for fractional Brownian motion developed in [4], [2], [3] we aim at proving in this section higher order differentiability of the stochastic flow associated with the SDE (1.1).

The main result of our paper is the following:

Theorem 2.1.
Let $H < \frac{1}{2(d-1+2k)}$ for $k \in \mathbb{N}$ and $\mathcal{U} \subset \mathbb{R}^d$ be a bounded and open set. Further, let X^x_t, $0 \leq t \leq T$ be the strong solution to (1.1) as constructed in Theorem 1.1. Then the associated stochastic flow with respect to (1.1) is $k-$times Sobolev differentiable on $\mathcal{U}$ $\mu-$a.e.. More precisely, for all $0 \leq t \leq T$

$$(x \longmapsto X^x_t) \in \bigcap_{p\geq 2} L^2(\Omega; W^{k,p}(\mathcal{U})).$$

Remark 2.1. Let us mention here that the regularity result in Theorem 2.1 is a significant improvement of that obtained in [6] in the case of (distributional) drift vector fields in Besov spaces $B^{\alpha+1}_{\infty,\infty}$ for $\alpha > 2 - 1/(2H)$, where the authors prove Lipschitzianity of the associated stochastic flow.

In order to prove Theorem 2.1 we need a some definition and an auxiliary result:

Definition 2.1. Let $H < \frac{1}{2(d+2)}$. We then denote by $\mathbb{L}$ the class of sequences of vector fields $\varphi_n : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}$, $n\geq 1$ such that the SDE

$$Y_t^x = y + \int_0^t \varphi_n(u, Y_u^y)\cdot \mathbf{1}_d du + B_t^H,\ 0\leq t\leq T \tag{2.1}$$

admits a unique strong solution for all $y\in\mathbb{R}^d$, $n\geq 1$ and such that

$$\int_0^\cdot \varphi_n(u, B_u^H + y)du \in I_{0^+}^{H+\frac{1}{2}}(L^2)$$

for all $y\in\mathbb{R}^d$, $n\geq 1$ as well as

$$\sup_{n\geq 1} E\left[\exp(k\int_0^T (K_H^{-1}(\int_0^\cdot \varphi_n(u, B_u^H + y)du)(s))^2 ds)\right] < \infty$$

for all $k\in\mathbb{R}$. See the Appendix for the definition of the space $I_{0^+}^{H+\frac{1}{2}}(L^2)$ and the operator K_H^{-1}.

Remark 2.2. It follows from Lemma 3.1 in the Appendix that the approximation sequence $\varphi_{x,\varepsilon}$ for $\varepsilon = 1/n, n\geq 1$ with respect to the Dirac delta function in x in (3.4) belongs to the class $\mathbb{L}$.

The proof of Theorem 2.1 mainly relies on the following estimate (compare Lemma 7 in [2] and Theorem 5.1 in [3]):

Lemma 2.1. *Assume that $H < \frac{1}{2(d-1+2k)}$. Let $\varphi_n : \mathbb{R}^d \longrightarrow \mathbb{R}$, $n\geq 1$ belong to $\mathbb{L}$ in Definition 2.1 and $\varphi_n \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz function space) for all $n\geq 1$. Denote by $X_t^{x,n}$, $0\leq t\leq T$ the strong solution to the SDE (2.1) with respect to the drift φ_n for each $n\geq 1$. Fix integers $p\geq 2$. Then*

$$\sup_{x\in\mathbb{R}^d} E\left[\left\|\frac{\partial^k}{\partial x^k} X_t^{x,n}\right\|^p\right] \leq M\cdot C_{p,k,H,d,T}(\|\varphi_n\|_{L^1(\mathbb{R}^d)}) < \infty$$

for all $n\geq 1$ for some continuous function $C_{p,k,H,d,T} : [0,\infty)^2 \longrightarrow [0,\infty)$ and a constant M depending only on φ_n, $n\geq 1$ and p.

Proof. To simplify notation we set $b = \varphi_n \cdot \mathbf{1}_d$ for a fixed $n\geq 1$ and denote the corresponding solution by $X_t^x = X_t^{x,n}$, $0\leq t\leq T$. Since the stochastic flow associated with the smooth vector field b is also smooth (compare to e.g. [14]), we obtain that

$$\frac{\partial}{\partial x} X_t^x = I_{d\times d} + \int_s^t Db(X_u^x)\cdot \frac{\partial}{\partial x} X_u^x du, \tag{2.2}$$

where $Db : \mathbb{R}^d \longrightarrow L(\mathbb{R}^d, \mathbb{R}^d)$ is the derivative of b with respect to the space variable.

Using Picard iteration, we see that

$$\frac{\partial}{\partial x} X_t^x = I_{d\times d} + \sum_{m\geq 1} \int_{\Delta_{0,t}^m} Db(X_{u_1}^x)\ldots Db(X_{u_m}^x) du_m \ldots du_1, \tag{2.3}$$

where

$$\Delta_{s,t}^m = \{(u_m, \ldots u_1) \in [0,T]^m : \theta < u_m < \ldots < u_1 < t\}.$$

By differentiating both sides with respect to x in connection with dominated convergence, we also get that

$$\frac{\partial^2}{\partial x^2} X_t^x = \sum_{m\geq 1} \int_{\Delta_{0,t}^m} \frac{\partial}{\partial x} [Db(X_{u_1}^x)\ldots Db(X_{u_m}^x)] du_m \ldots du_1.$$

Using the Leibniz and chain rule, we have that

$$\frac{\partial}{\partial x}[Db(X_{u_1}^x)\ldots Db(X_{u_m}^x)] = \sum_{r=1}^{m} Db(X_{u_1}^x)\ldots D^2 b(X_{u_r}^x) \frac{\partial}{\partial x} X_{u_r}^x \ldots Db(X_{u_m}^x),$$

where $D^2 b = D(Db) : \mathbb{R}^d \longrightarrow L(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^d))$.

So it follows from (2.3) that

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} X_t^x &= \sum_{m_1\geq 1} \int_{\Delta_{0,t}^{m_1}} \sum_{r=1}^{m_1} Db(X_{u_1}^x)\ldots D^2 b(X_{u_r}^x) \\
&\quad \times \left(I_{d\times d} + \sum_{m_2\geq 1} \int_{\Delta_{0,u_r}^{m_2}} Db(X_{v_1}^x)\ldots Db(X_{v_{m_2}}^x) dv_{m_2} \ldots dv_1 \right) \\
&\quad \times Db(X_{u_{r+1}}^x)\ldots Db(X_{u_{m_1}}^x) du_{m_1} \ldots du_1 \\
&= \sum_{m_1\geq 1} \sum_{r=1}^{m_1} \int_{\Delta_{0,t}^{m_1}} Db(X_{u_1}^x)\ldots D^2 b(X_{u_r}^x)\ldots Db(X_{u_{m_1}}^x) du_{m_1} \ldots du_1 \\
&\quad + \sum_{m_1\geq 1} \sum_{r=1}^{m_1} \sum_{m_2\geq 1} \int_{\Delta_{0,t}^{m_1}} \int_{\Delta_{0,u_r}^{m_2}} Db(X_{u_1}^x)\ldots D^2 b(X_{u_r}^x) \\
&\quad \times Db(X_{v_1}^x)\ldots Db(X_{v_{m_2}}^x) Db(X_{u_{r+1}}^x)\ldots Db(X_{u_{m_1}}^x) \\
&\quad dv_{m_2} \ldots dv_1 du_{m_1} \ldots du_1 \\
&=: I_1 + I_2.
\end{aligned} \tag{2.4}$$

We now aim at applying Lemma 3.3 (in connection with Lemma 3.2) to the term I_2 in (2.4) and find that

$$I_2 = \sum_{m_1 \geq 1} \sum_{r=1}^{m_1} \sum_{m_2 \geq 1} \int_{\Delta_{0,t}^{m_1+m_2}} \mathcal{H}_{m_1+m_2}^X(u) du_{m_1+m_2} \dots du_1 \tag{2.5}$$

for $u = (u_1, \dots, u_{m_1+m_2})$, where the integrand $\mathcal{H}_{m_1+m_2}^X(u) \in \mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d$ has entries given by sums of at most $C(d)^{m_1+m_2}$ terms, which are products of length $m_1 + m_2$ of functions belonging to the set

$$\left\{ \frac{\partial^{\gamma^{(1)}+\dots+\gamma^{(d)}}}{\partial^{\gamma^{(1)}} x_1 \dots \partial^{\gamma^{(d)}} x_d} b^{(r)}(X_u^x),\ r = 1, \dots, d,\ \gamma^{(1)} + \dots + \gamma^{(d)} \leq 2,\ \gamma^{(l)} \in \mathbb{N}_0,\ l = 1, \dots, d \right\}.$$

Here it is important note that the terms in these products for which we have equality in

$$\gamma^{(1)} + \dots + \gamma^{(d)} \leq 2$$

appear only once in (2.5). So the total order of derivatives $|\alpha|$ (in the sense of Lemma 3.3 in the Appendix) of those products of functions is given by

$$|\alpha| = m_1 + m_2 + 1. \tag{2.6}$$

Let us choose $p, c, r \in [1, \infty)$ such that $cp = 2^q$ for some integer q and $\frac{1}{r} + \frac{1}{c} = 1$. Then we can use Hölder's inequality and Girsanov's theorem (see Theorem 3.1) in connection with Definition 2.1 and obtain that

$$E[\|I_2\|^p] \leq M \left(\sum_{m_1 \geq 1} \sum_{r=1}^{m_1} \sum_{m_2 \geq 1} \sum_{i \in I} \left\| \int_{\Delta_{0,t}^{m_1+m_2}} \mathcal{H}_i^{B^H}(u) du_{m_1+m_2} \dots du_1 \right\|_{L^{2^q}(\Omega;\mathbb{R})} \right)^p,$$

where $M < \infty$ is a constant depending only on $\varphi_n, n \geq 1$ and p. Here $\#I \leq K^{m_1+m_2}$ for a constant $K = K(d)$ and the integrands $\mathcal{H}_i^{B^H}(u)$ are of the form

$$\mathcal{H}_i^{B^H}(u) = \prod_{l=1}^{m_1+m_2} h_l(u_l), h_l \in \Lambda, l = 1, \dots, m_1 + m_2$$

where

$$\Lambda := \left\{ \begin{array}{l} \frac{\partial^{\gamma^{(1)}+\dots+\gamma^{(d)}}}{\partial^{\gamma^{(1)}} x_1 \dots \partial^{\gamma^{(d)}} x_d} b^{(r)}(x + B_u^H),\ r = 1, \dots, d, \\ \gamma^{(1)} + \dots + \gamma^{(d)} \leq 2,\ \gamma^{(l)} \in \mathbb{N}_0,\ l = 1, \dots, d \end{array} \right\}.$$

Also here functions with second order derivatives only appear once in those products.

Set

$$J = \left(\int_{\Delta_{0,t}^{m_1+m_2}} \mathcal{H}_i^{B^H}(u) du_{m_1+m_2} \dots du_1 \right)^{2^q}.$$

Using Lemma 3.2 once more, successively q-times, we find that J can be written as a sum of length at most $K(q)^{m_1+m_2}$ with summands of the form

$$\int_{\Delta_{0,t}^{2^q(m_1+m_2)}} \prod_{l=1}^{2^q(m_1+m_2)} f_l(u_l) du_{2^q(m_1+m_2)} \dots du_1, \tag{2.7}$$

where $f_l \in \Lambda$ for all l.

Here the number of factors f_l in the above product having a second order derivative is exactly 2^q. So the total order of the derivatives involved in (2.7) in the sense of Lemma 3.3 (where one in that lemma formally replaces X_u^x by $x + B_u^H$ in the corresponding expressions) is given by

$$|\alpha| = 2^q(m_1 + m_2 + 1). \tag{2.8}$$

Now we can apply Theorem 3.2 for $m = 2^q(m_1 + m_2)$ and $\varepsilon_j = 0$ and get that

$$\begin{aligned} &\left| E\left[\int_{\Delta_{0,t}^{2^q(m_1+m_2)}} \prod_{l=1}^{2^q(m_1+m_2)} f_l(u_l) du_{2^q(m_1+m_2)} \dots du_1 \right] \right| \\ &\leq C^{m_1+m_2} (\|b\|_{L^1(\mathbb{R}^d)})^{2^q(m_1+m_2)} \\ &\times \frac{((2(2^q(m_1+m_2+1))!)^{1/4}}{\Gamma(-H(2d2^q(m_1+m_2) + 2^2 2^q(m_1+m_2+1)) + 22^q(m_1+m_2))^{1/2}} \end{aligned}$$

for a constant C depending on H, T, d and q.

So the latter combined with (2) shows that

$$\begin{aligned} &E[\|I_2\|^p] \\ &\leq M \left(\sum_{m_1 \geq 1} \sum_{m_2 \geq 1} K^{m_1+m_2} ((\|b\|_{L^1(\mathbb{R}^d)})^{2^q(m_1+m_2)} \right. \\ &\left. \times \frac{((2(2^q(m_1+m_2+1))!)^{1/4}}{\Gamma(-H(2d2^q(m_1+m_2) + 2^2 2^q(m_1+m_2+1)) + 22^q(m_1+m_2))^{1/2}})^{1/2^q} \right)^p \end{aligned}$$

for a constant K depending on H, T, d, p and q.

Since $\frac{1}{2(d+3)} \leq \frac{1}{2(d+2\frac{m_1+m_2+1}{m_1+m_2})}$ for $m_1, m_2 \geq 1$, the above sum converges if $H < \frac{1}{2(d+3)}$.

On the other hand one derives in the same way a similar estimate for $E[\|I_1\|^p]$. Altogether the proof follows for $k = 2$.

We now explain the generalization of the latter reasoning to the case $k \geq 2$: In this case, we find that

$$\frac{\partial^k}{\partial x^k} X_t^x = I_1 + \ldots + I_{2^{k-1}}, \tag{2.9}$$

where each I_i, $i = 1, \ldots, 2^{k-1}$ is a sum of iterated integrals over simplices of the form $\Delta_{0,u}^{m_j}$, $0 < u < t$, $j = 1, \ldots, k$ with integrands having at most one product factor $D^k b$, whereas the other factors are of the form $D^j b, j \leq k-1$.

For convenience, we introduce the following notation: For given multi-indices $m. = (m_1, \ldots, m_k)$ and $r := (r_1, \ldots, r_{k-1})$ we define

$$m_j^- := \sum_{i=1}^{j} m_i$$

and

$$\sum_{\substack{m \geq 1 \\ r_l \leq m_l^- \\ l=1,\ldots,k-1}} := \sum_{m_1 \geq 1} \sum_{r_1=1}^{m_1} \sum_{m_2 \geq 1} \sum_{r_2=1}^{m_2^-} \cdots \sum_{r_{k-1}=1}^{m_{k-1}^-} \sum_{m_k \geq 1} .$$

In what follows, we confine ourselves without loss of generality to the estimation of the term $I_{2^{k-1}}$ in (2.9). Just as in the case $k = 2$, we obtain by employing Lemma 3.3 (in connection with Lemma 3.2) that

$$I_{2^{k-1}} = \sum_{\substack{m \geq 1 \\ r_l \leq m_l^- \\ l=1,\ldots,k-1}} \int_{\Delta_{0,t}^{m_1+\ldots+m_k}} \mathcal{H}_{m_1+\ldots+m_k}^X(u) du_{m_1+m_2} \ldots du_1 \tag{2.10}$$

for $u = (u_{m_1+\ldots+m_k}, \ldots, u_1)$, where the integrand $\mathcal{H}_{m_1+\ldots+m_k}^X(u) \in \otimes_{j=1}^{k+1} \mathbb{R}^d$ has entries given by sums of at most $C(d)^{m_1+\ldots+m_k}$ terms, which are products of length $m_1 + \ldots + m_k$ of functions, which are elements in

$$\left\{ \begin{array}{c} \frac{\partial^{\gamma^{(1)}+\ldots+\gamma^{(d)}}}{\partial^{\gamma^{(1)}} x_1 \ldots \partial^{\gamma^{(d)}} x_d} b^{(r)}(X_u^x), r = 1, \ldots, d, \\ \gamma^{(1)} + \ldots + \gamma^{(d)} \leq k, \gamma^{(l)} \in \mathbb{N}_0, l = 1, \ldots, d \end{array} \right\}.$$

As in the case $k = 2$ we can apply Lemma 3.3 in the Appendix and find that the total order of derivatives $|\alpha|$ of those products of functions is

$$|\alpha| = m_1 + \ldots + m_k + k - 1. \tag{2.11}$$

Then we proceed as before and choose $p, c, r \in [1, \infty)$ such that $cp = 2^q$ for some integer q and $\frac{1}{r} + \frac{1}{c} = 1$ and get by using Hölder's inequality and Girsanov's theorem (see Theorem 3.1) in connection with Definition 2.1 that

$$\begin{aligned} &E[\|I_{2^{k-1}}\|^p] \\ &\leq M \left(\sum_{\substack{m \geq 1 \\ r_l \leq m_l^- \\ l=1,\ldots,k-1}} \sum_{i \in I} \left\| \int_{\Delta_{0,t}^{m_1+m_2}} \mathcal{H}_i^{B^H}(u) du_{m_1+\ldots+m_k} \ldots du_1 \right\|_{L^{2^q}(\Omega;\mathbb{R})} \right)^p \end{aligned} \tag{2.12}$$

where $M < \infty$ is a constant depending only on $\varphi_n, n \geq 1$ and p. Here $\#I \leq K^{m_1+\ldots+m_k}$ for a constant $K = K(d)$ and the integrands $\mathcal{H}_i^{B^H}(u)$ take the form

$$\mathcal{H}_i^{B^H}(u) = \prod_{l=1}^{m_1+\ldots+m_k} h_l(u_l), \; h_l \in \Lambda, \; l = 1, \ldots, m_1 + \ldots + m_k,$$

where

$$\Lambda := \left\{ \begin{array}{l} \frac{\partial^{\gamma^{(1)}+\ldots+\gamma^{(d)}}}{\partial^{\gamma^{(1)}} x_1 \ldots \partial^{\gamma^{(d)}} x_d} b^{(r)}(x + B_u^H), \; r = 1, \ldots, d, \\ \gamma^{(1)} + \ldots + \gamma^{(d)} \leq k, \; \gamma^{(l)} \in \mathbb{N}_0, \; l = 1, \ldots, d \end{array} \right\}.$$

Let

$$J = \left(\int_{\Delta_{0,t}^{m_1+\ldots+m_k}} \mathcal{H}_i^{B^H}(u) du_{m_1+\ldots+m_k} \ldots du_1 \right)^{2^q}.$$

Again repeated use of Lemma 3.2 in the Appendix shows that J can be represented as a sum of, at most of length $K(q)^{m_1+\ldots m_k}$ with summands of the form

$$\int_{\Delta_{0,t}^{2^q(m_1+\ldots+m_k)}} \prod_{l=1}^{2^q(m_1+\ldots+m_k)} f_l(u_l) du_{2^q(m_1+\ldots+m_k)} \ldots du_1, \tag{2.13}$$

where $f_l \in \Lambda$ for all l.

Using once more Lemma 3.3 (where one in that Lemma formally replaces X_u^x by $x + B_u^H$ in the corresponding terms) it follows that the total order of the derivatives in the products of functions in (2.13) is given by

$$|\alpha| = 2^q(m_1 + \ldots + m_k + k - 1). \tag{2.14}$$

Then Proposition 3.2 for $m = 2^q(m_1 + \ldots + m_k)$ and $\varepsilon_j = 0$ yields

$$\left| E\left[\int_{\Delta_{0,t}^{2^q(m_1+\ldots+m_k)}} \prod_{l=1}^{2^q(m_1+\ldots+m_k)} f_l(u_l) du_{2^q(m_1+\ldots+m_k)} \ldots du_1 \right] \right|$$
$$\leq C^{m_1+\ldots+m_k} (\|b\|_{L^1(\mathbb{R}^d)})^{2^q(m_1+\ldots+m_k)}$$
$$\times \frac{((2(2^q(m_1+\ldots+m_k+k-1))!)^{1/4}}{\Gamma(-H(2d2^q(m_1+\ldots+m_k)+2^2 2^q(m_1+\ldots+m_k+k-1))+22^q(m_1+\ldots+m_k))^{1/2}}$$

for a constant C depending on H, T, d and q.

Hence (2.12) implies that

$$E[\|I_{2^{k-1}}\|^p]$$
$$\leq M\left(\sum_{m_1\geq 1} \ldots \sum_{m_k \geq 1} K^{m_1+\ldots+m_k} ((\|b\|_{L^1(\mathbb{R}^d)})^{2^q(m_1+\ldots+m_k)} \right.$$
$$\left. \times \frac{((2(2^q(m_1+\ldots+m_k+k-1))!)^{1/4}}{\Gamma(-H(2d2^q(m_1+\ldots+m_k)+2^2 2^q(m_1+\ldots+m_k+k-1))+22^q(m_1+\ldots+m_k))^{1/2}})^{1/2^q} \right)^p$$
$$\leq M\left(\sum_{m\geq 1} \sum_{\substack{l_1,\ldots,l_k\geq 0:\\ l_1+\ldots+l_k=m}} K^m ((\|b\|_{L^1(\mathbb{R}^d)})^{2^q m} \right.$$
$$\left. \times \frac{((2(2^q(m+k-1))!)^{1/4}}{\Gamma(-H(2d2^q m+2^2 2^q(m+k-1))+22^q m)^{1/2}})^{1/2^q} \right)^p$$

for a constant K depending on H, T, d, p and q.

Since we assumed that $H < \frac{1}{2(d-1+2k)}$ the above sum converges. So the proof follows. □

Using Lemma 2.1 we can now prove the main result.

Proof. [Proof of Theorem 2.1] Following the ideas in Theorem 5.2 in [3] or Proposition 4.2 in [17], we approximate the Dirac distribution δ_0 in zero by $\varphi_{1/n}$, where $\varphi_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$, $\varepsilon > 0$ is given as in (1.2). Set $b_n = \varphi_{1/n} \cdot \mathbf{1}_d$. Denote by $X_t^{n,x}$, $0 \leq t \leq T$ the solution to (2.1) associated with the vector field b_n, starting in x. Let $\phi \in C_c^\infty(\mathcal{U};\mathbb{R}^d)$ and define for fixed $t \in [0,T]$ the sequence of random variables

$$\left\langle X_t^{n,\cdot}, \phi \right\rangle := \int_{\mathcal{U}} \langle X_t^{n,x}, \phi \rangle_{\mathbb{R}^d}\, dx,\ n \geq 1$$

Using the same reasoning as in the proof of Theorem 5.2 in [3], which is based on a compactness criterion for square integrable functionals of Wiener processes (see

[7]), combined with the estimates of Lemma 5.6 in [4] one shows that there exists a subsequence n_j, $j \geq 1$ such that

$$\left\langle X_t^{n_j,\cdot}, \phi \right\rangle \underset{j \longrightarrow \infty}{\longrightarrow} \langle X_t^{\cdot}, \phi \rangle \tag{2.15}$$

in $L^2(\Omega)$ strongly for all $\phi \in C_c^\infty(\mathcal{U}; \mathbb{R}^d)$, where X_s^x, $0 \leq s \leq T$ is the strong solution of Theorem 1.1. Note that we also have that

$$X_t^{n,x} \underset{n \longrightarrow \infty}{\longrightarrow} X_t^x$$

in $L^2(\Omega)$ strongly. See Corollary 5.7 in [4].

On the other hand, it follows from Lemma 2.1 that

$$\begin{aligned} &\sup_{n \geq 1} \left\| X_t^{n,\cdot} \right\|^2_{L^2(\Omega; W^{k,p}(\mathcal{U}))} \\ &\leq \sum_{i=0}^{k} \left(\int_{\mathcal{U}} \sup_{n \geq 1} E\left[\left\| D^i X_t^{n,x} \right\|^p \right] dx \right)^{\frac{2}{p}} < \infty \end{aligned}$$

for $H < \frac{1}{2(d-1+2k)}$.

Since $L^2(\Omega; W^{k,p}(\mathcal{U}))$ is reflexive for $p > 1$, there exists a subsequence $n_j, j \geq 1$ such that

$$X_t^{n,\cdot} \underset{j \longrightarrow \infty}{\longrightarrow} Y$$

in $L^2(\Omega; W^{k,p}(\mathcal{U}))$ weakly. For convenience, let $n_j, j \geq 1$ be the same subsequence as in (2.15). Further, we have for all $A \in \mathcal{F}$, $\phi \in C_c^\infty(\mathcal{U}; \mathbb{R}^d)$, $\alpha^{(1)} + \ldots + \alpha^{(d)} \leq k$ with $\alpha^{(i)} \in \mathbb{N}_0$, $i = 1, \ldots, d$ that

$$\begin{aligned} &E\left[1_A \left\langle X_t^{n_j,\cdot}, \frac{\partial^{\alpha^{(1)} + \ldots + \alpha^{(d)}}}{\partial^{\alpha^{(1)}} x_1 \ldots \partial^{\alpha^{(d)}} x_d} \phi \right\rangle \right] \\ &= (-1)^{\alpha^{(1)} + \ldots + \alpha^{(d)}} E\left[1_A \left\langle \frac{\partial^{\alpha^{(1)} + \ldots + \alpha^{(d)}}}{\partial^{\alpha^{(1)}} x_1 \ldots \partial^{\alpha^{(d)}} x_d} X_t^{n_j,\cdot}, \phi \right\rangle \right] \\ &\underset{j \longrightarrow \infty}{\longrightarrow} (-1)^{\alpha^{(1)} + \ldots + \alpha^{(d)}} E\left[1_A \left\langle \frac{\partial^{\alpha^{(1)} + \ldots + \alpha^{(d)}}}{\partial^{\alpha^{(1)}} x_1 \ldots \partial^{\alpha^{(d)}} x_d} Y, \phi \right\rangle \right]. \end{aligned}$$

We also know from (2.15) that

$$E\left[1_A \left\langle X_t^{n_j,\cdot}, \frac{\partial^{\alpha^{(1)} + \ldots + \alpha^{(d)}}}{\partial^{\alpha^{(1)}} x_1 \ldots \partial^{\alpha^{(d)}} x_d} \phi \right\rangle \right] \underset{j \longrightarrow \infty}{\longrightarrow} E\left[1_A \left\langle X_t^{\cdot}, \frac{\partial^{\alpha^{(1)} + \ldots + \alpha^{(d)}}}{\partial^{\alpha^{(1)}} x_1 \ldots \partial^{\alpha^{(d)}} x_d} \phi \right\rangle \right].$$

So $X_t^{\cdot} \in L^2(\Omega; W^{k,p}(\mathcal{U}))$ for all $p \geq 2$. □

3. Appendix

In view of the need for a version of Girsanov's theorem for fractional Brownian motion, which we use in connection with the proof of Lemma 2.1, we recall some basic concepts from fractional calculus (see [22] and [16]).

Let $a,\, b \in \mathbb{R}$ with $a < b$. Let $f \in L^p([a,b])$ with $p \geq 1$ and $\alpha > 0$. Define the *left-* and *right-sided Riemann-Liouville fractional integrals* by

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy$$

and

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy$$

for almost all $x \in [a,b]$, where Γ denotes the Gamma function.

For a given integer $p \geq 1$, let $I_{a^+}^{\alpha}(L^p)$ (resp. $I_{b^-}^{\alpha}(L^p)$) be the image of $L^p([a,b])$ of the operator $I_{a^+}^{\alpha}$ (resp. $I_{b^-}^{\alpha}$). If $f \in I_{a^+}^{\alpha}(L^p)$ (resp. $f \in I_{b^-}^{\alpha}(L^p)$) and $0 < \alpha < 1$ then we can introduce the *left-* and *right-sided Riemann-Liouville fractional derivatives* by

$$D_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^{\alpha}} dy$$

and

$$D_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(y)}{(y-x)^{\alpha}} dy.$$

The left- and right-sided derivatives of f also have the following representations

$$D_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right)$$

and

$$D_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}} dy \right).$$

Using the above definitions, one finds that

$$I_{a^+}^{\alpha}(D_{a^+}^{\alpha} f) = f$$

for all $f \in I_{a^+}^{\alpha}(L^p)$ and

$$D_{a^+}^{\alpha}(I_{a^+}^{\alpha} f) = f$$

for all $f \in L^p([a,b])$ and similarly for $I_{b^-}^{\alpha}$ and $D_{b^-}^{\alpha}$.

Consider now a d-dimensional *fractional Brownian motion* $B^H = \{B_t^H, t \in [0,T]\}$ with Hurst parameter $H \in (0, 1/2)$, that is B^H is a centered Gaussian process with a covariance function given by

$$(R_H(t,s))_{i,j} := E[B_t^{H,(i)} B_s^{H,(j)}] = \delta_{ij} \frac{1}{2} \left(t^{2H} + s^{2H} - |t-s|^{2H} \right), \quad i,j = 1, \ldots, d,$$

where δ_{ij} is one, if $i = j$, or zero else.

Next, we want to briefly pass in review a construction of the fractional Brownian motion, which can be found in [19]. For convenience let $d = 1$.

Denote by $\mathcal{E}$ the set of step functions on $[0,T]$ and by $\mathcal{H}$ the Hilbert space given by the completion of $\mathcal{E}$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

From that we obtain an extension of the mapping $1_{[0,t]} \mapsto B_t$ to an isometry between $\mathcal{H}$ and a Gaussian subspace of $L^2(\Omega)$ associated with B^H. Denote by $\varphi \mapsto B^H(\varphi)$ this isometry.

It turns out that for $H < 1/2$ the covariance function $R_H(t,s)$ can be represented as

$$R_H(t,s) = \int_0^{t \wedge s} K_H(t,u) K_H(s,u) du, \tag{3.1}$$

where

$$K_H(t,s) = c_H \left[\left(\frac{t}{s} \right)^{H - \frac{1}{2}} (t-s)^{H - \frac{1}{2}} + \left(\frac{1}{2} - H \right) s^{\frac{1}{2} - H} \int_s^t u^{H - \frac{3}{2}} (u-s)^{H - \frac{1}{2}} du \right].$$

Here $c_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+1/2)}}$ and β is the Beta function. See [19, Proposition 5.1.3].

Using the kernel K_H, one can define by means (3.1) an isometry K_H^* between $\mathcal{E}$ and $L^2([0,T])$ such that $(K_H^* 1_{[0,t]})(s) = K_H(t,s) 1_{[0,t]}(s)$. This isometry extends to the Hilbert space $\mathcal{H}$ and has the following representations in terms of fractional derivatives

$$(K_H^* \varphi)(s) = c_H \Gamma \left(H + \frac{1}{2} \right) s^{\frac{1}{2} - H} \left(D_{T^-}^{\frac{1}{2} - H} u^{H - \frac{1}{2}} \varphi(u) \right) (s)$$

and

$$(K_H^*\varphi)(s) = c_H\Gamma\left(H+\frac{1}{2}\right)\left(D_{T^-}^{\frac{1}{2}-H}\varphi(s)\right)(s) + c_H\left(\frac{1}{2}-H\right)\int_s^T \varphi(t)(t-s)^{H-\frac{3}{2}}\left(1-\left(\frac{t}{s}\right)^{H-\frac{1}{2}}\right)dt.$$

for $\varphi \in \mathcal{H}$ One also has that $\mathcal{H} = I_{T^-}^{\frac{1}{2}-H}(L^2)$. See [8] and [1, Proposition 6].

Using the fact that K_H^* is an isometry from $\mathcal{H}$ into $L^2([0,T])$ the d-dimensional process $W = \{W_t, t \in [0,T]\}$ defined by

$$W_t := B^H((K_H^*)^{-1}(1_{[0,t]})) \tag{3.2}$$

is a Wiener process and the process B^H has the representation

$$B_t^H = \int_0^t K_H(t,s)dW_s. \tag{3.3}$$

See [1].

In the sequel we also need the concept of fractional Brownian motion with respect to a filtration.

Definition 3.1. Let $\mathcal{G} = \{\mathcal{G}_t\}_{t\in[0,T]}$ be a filtration on $(\Omega, \mathcal{F}, P)$ satisfy the usual conditions. A fractional Brownian motion B^H is called a $\mathcal{G}$-fractional Brownian motion if the process W defined by (3.2) is a $\mathcal{G}$-Brownian motion.

In what follows, let W be a standard Wiener process on a probability space $(\Omega, \mathfrak{A}, P)$ endowed with the natural filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ which is generated by W and augmented by all P-null sets. We denote by $B := B^H$ the fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$ given by the representation (3.3).

We want to employ a version of Girsanov's theorem for fractional Brownian motion which goes back to [8, Theorem 4.9]. The version we recall here is that given in [20, Theorem 2]. In doing so, we have to introduce the definition of an isomorphism K_H from $L^2([0,T])$ onto $I_{0+}^{H+\frac{1}{2}}(L^2)$ associated with the kernel $K_H(t,s)$ in terms of the fractional integrals as follows (see [8, Theorem 2.1]):

$$(K_H\varphi)(s) = I_{0^+}^{2H} s^{\frac{1}{2}-H} I_{0^+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}}\varphi, \quad \varphi \in L^2([0,T]).$$

From that and the properties of the Riemann-Liouville fractional integrals and derivatives one can see that the inverse of K_H has the representation

$$(K_H^{-1}\varphi)(s) = s^{\frac{1}{2}-H} D_{0^+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_{0^+}^{2H}\varphi(s), \quad \varphi \in I_{0+}^{H+\frac{1}{2}}(L^2).$$

The latter shows that if φ is absolutely continuous, see [20], one gets that

$$(K_H^{-1}\varphi)(s) = s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H}\varphi'(s).$$

Theorem 3.1 (Girsanov's theorem for fBm). *Let $u = \{u_t, t \in [0,T]\}$ be an $\mathcal{F}$-adapted process with integrable trajectories and set $\widetilde{B}_t^H = B_t^H + \int_0^t u_s ds, \quad t \in [0,T]$. Assume that*

(i) $\int_0^{\cdot} u_s ds \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T]))$, P-a.s.
(ii) $E[\xi_T] = 1$ where

$$\xi_T := \exp\left\{ -\int_0^T K_H^{-1}\left(\int_0^{\cdot} u_r dr\right)(s) dW_s - \frac{1}{2}\int_0^T K_H^{-1}\left(\int_0^{\cdot} u_r dr\right)^2 (s) ds \right\}.$$

Then the shifted process $\widetilde{B}^H$ is an $\mathcal{F}$-fractional Brownian motion with Hurst parameter H under the new probability $\widetilde{P}$ defined by $\frac{d\widetilde{P}}{dP} = \xi_T$.

Remark 3.1. As for the multi-dimensional case, define

$$(K_H\varphi)(s) := ((K_H\varphi^{(1)})(s), \dots, (K_H\varphi^{(d)})(s))^*, \quad \varphi \in L^2([0,T];\mathbb{R}^d),$$

where $*$ denotes transposition. Similarly for K_H^{-1} and K_H^*.

In this paper we also make use of the following technical lemma, whose proof can be found in [4, Lemma 5.3]

Lemma 3.1. *Let $x \in \mathbb{R}^d$. If $H < \frac{1}{2(1+d)}$ then*

$$\sup_{\varepsilon>0} E\left[\exp(k\int_0^T (K_H^{-1}(\int_0^{\cdot} \varphi_{x,\varepsilon}(B_u^H)du)(t))^2 dt)\right] < \infty$$

for all $k \in \mathbb{R}$, where

$$\varphi_{x,\varepsilon}(B_u^H) = \frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}} \exp(-\frac{\left|B_u^H - x\right|_{\mathbb{R}^d}^2}{2\varepsilon}). \tag{3.4}$$

In the following we also need an integration by parts formula for iterated integrals based on *shuffle permutations*. For this purpose, let m and n be integers. We define $S(m,n)$ as the set of shuffle permutations, i.e. the set of permutations $\sigma : \{1,\dots,m+n\} \to \{1,\dots,m+n\}$ such that $\sigma(1) < \dots < \sigma(m)$ and $\sigma(m+1) < \dots < \sigma(m+n)$.

Define the m-dimensional simplex for $0 \le \theta < t \le T$,

$$\Delta_{\theta,t}^m := \{(s_m,\dots,s_1) \in [0,T]^m : \theta < s_m < \dots < s_1 < t\}.$$

The product of two simplices can be written as the following union

$$\Delta_{\theta,t}^m \times \Delta_{\theta,t}^n = \bigcup_{\sigma\in S(m,n)} \{(w_{m+n},\dots,w_1) \in [0,T]^{m+n} : \theta < w_{\sigma(m+n)} < \dots < w_{\sigma(1)} < t\} \cup \mathcal{N}\ ,$$

where the set $\mathcal{N}$ has null Lebesgue measure. Hence, if $f_i : [0,T] \to \mathbb{R}$, $i = 1, \dots, m+n$ are integrable functions we have

$$\int_{\Delta^m_{\theta,t}} \prod_{j=1}^{m} f_j(s_j) ds_m \dots ds_1 \int_{\Delta^n_{\theta,t}} \prod_{j=m+1}^{m+n} f_j(s_j) ds_{m+n} \dots ds_{m+1}$$
$$= \sum_{\sigma \in S(m,n)} \int_{\Delta^{m+n}_{\theta,t}} \prod_{j=1}^{m+n} f_{\sigma(j)}(w_j) dw_{m+n} \cdots dw_1. \quad (3.5)$$

The latter relation can be generalized as follows (see [3]):

Lemma 3.2. *Let n, p and k be non-negative integers, $k \leq n$. Assume we have integrable functions $f_j : [0,T] \to \mathbb{R}$, $j = 1, \dots, n$ and $g_i : [0,T] \to \mathbb{R}$, $i = 1, \dots, p$. We may then write*

$$\int_{\Delta^n_{\theta,t}} f_1(s_1) \dots f_k(s_k) \int_{\Delta^p_{\theta,s_k}} g_1(r_1) \dots g_p(r_p) dr_p \dots dr_1 f_{k+1}(s_{k+1}) \dots f_n(s_n) ds_n \dots ds_1$$
$$= \sum_{\sigma \in A_{n,p}} \int_{\Delta^{n+p}_{\theta,t}} h^\sigma_1(w_1) \dots h^\sigma_{n+p}(w_{n+p}) dw_{n+p} \dots dw_1,$$

where $h^\sigma_l \in \{f_j, g_i : 1 \leq j \leq n, 1 \leq i \leq p\}$. Above $A_{n,p}$ denotes a subset of permutations of $\{1, \dots, n+p\}$ such that $\#A_{n,p} \leq C^{n+p}$ for an appropriate constant $C \geq 1$. Here we defined $s_0 = \theta$.

The proof of Lemma 2.1 requires an important estimate (see e.g. Proposition 3.3 in [4] for a new proof). To this end, let m be an integer and let $f : [0,T]^m \times (\mathbb{R}^d)^m \to \mathbb{R}$ be a function of the form

$$f(s,z) = \prod_{j=1}^{m} f_j(s_j, z_j), \quad s = (s_1, \dots, s_m) \in [0,T]^m, \quad z = (z_1, \dots, z_m) \in (\mathbb{R}^d)^m, \quad (3.6)$$

where $f_j : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $j = 1, \dots, m$ are smooth functions with compact support. Further, let $\varkappa : [0,T]^m \to \mathbb{R}$ be a function of the form

$$\varkappa(s) = \prod_{j=1}^{m} \varkappa_j(s_j), \quad s \in [0,T]^m, \quad (3.7)$$

where $\varkappa_j : [0,T] \to \mathbb{R}$, $j = 1, \dots, m$ are integrable functions.

Next, denote by α_j a multi-index and D^{α_j} its corresponding differential operator. For $\alpha = (\alpha_1, \dots, \alpha_m)$ considered an element of $\mathbb{N}_0^{d \times m}$ so that $|\alpha| := \sum_{j=1}^{m} \sum_{l=1}^{d} \alpha_j^{(l)}$, we write

$$D^\alpha f(s,z) = \prod_{j=1}^{m} D^{\alpha_j} f_j(s_j, z_j).$$

Theorem 3.2. *Let $B^H, H \in (0, 1/2)$ be a standard $d-$dimensional fractional Brownian motion and functions f and $\varkappa$ as in (3.6), respectively as in (3.7). Let $\theta, t \in [0, T]$ with $\theta < t$ and*

$$\varkappa_j(s) = (K_H(s, \theta))^{\varepsilon_j}, \theta < s < t$$

for every $j = 1, \ldots, m$ with $(\varepsilon_1, \ldots, \varepsilon_m) \in \{0, 1\}^m$. Let $\alpha \in (\mathbb{N}_0^d)^m$ be a multi-index. If

$$H < \frac{\frac{1}{2} - \gamma}{(d - 1 + 2\sum_{l=1}^{d} \alpha_j^{(l)})}$$

for all j, where $\gamma \in (0, H)$ is sufficiently small, then there exists a universal constant C (depending on H, T and d, but independent of m, $\{f_i\}_{i=1,\ldots,m}$ and α) such that for any $\theta, t \in [0, T]$ with $\theta < t$ we have

$$\begin{aligned} &\left| E \int_{\Delta_{\theta,t}^m} \left(\prod_{j=1}^{m} D^{\alpha_j} f_j(s_j, B_{s_j}^H) \varkappa_j(s_j) \right) ds \right| \\ &\leq C^{m+|\alpha|} \prod_{j=1}^{m} \|f_j(\cdot, z_j)\|_{L^1(\mathbb{R}^d; L^\infty([0,T]))} \theta^{(H-\frac{1}{2})\sum_{j=1}^{m} \varepsilon_j} \\ &\quad \times \frac{(\prod_{l=1}^{d} (2|\alpha^{(l)}|)!)^{1/4} (t-\theta)^{-H(md+2|\alpha|)-(H-\frac{1}{2}-\gamma)\sum_{j=1}^{m} \varepsilon_j + m}}{\Gamma(-H(2md + 4|\alpha|) + 2(H - \frac{1}{2} - \gamma)\sum_{j=1}^{m} \varepsilon_j + 2m)^{1/2}}. \end{aligned}$$

Remark 3.2. The above theorem also holds true for functions $\{f_i\}_{i=1,\ldots,m}$ in the Schwartz function space.

Finally, we also need the following auxiliary result in connection with the proof of Lemma 2.1:

Lemma 3.3. *Let n, p and k be non-negative integers, $k \leq n$. Assume we have functions $f_j : [0, T] \to \mathbb{R}$, $j = 1, \ldots, n$ and $g_i : [0, T] \to \mathbb{R}$, $i = 1, \ldots, p$ such that*

$$f_j \in \left\{ \frac{\partial^{\alpha_j^{(1)} + \ldots + \alpha_j^{(d)}}}{\partial^{\alpha_j^{(1)}} x_1 \ldots \partial^{\alpha_j^{(d)}} x_d} b^{(r)}(X_u^x),\ r = 1, \ldots, d \right\},\ j = 1, \ldots, n$$

and

$$g_i \in \left\{ \frac{\partial^{\beta_i^{(1)} + \ldots + \beta_i^{(d)}}}{\partial^{\beta_i^{(1)}} x_1 \ldots \partial^{\beta_i^{(d)}} x_d} b^{(r)}(X_u^x),\ r = 1, \ldots, d \right\},\ i = 1, \ldots, p$$

for $\alpha := (\alpha_j^{(l)}) \in \mathbb{N}_0^{d \times n}$ and $\beta := (\beta_i^{(l)}) \in \mathbb{N}_0^{d \times p}$, where $X_\cdot^x$ is the strong solution to

$$X_t^x = x + \int_0^t b(X_u^x)du + B_t^H,\ 0 \le t \le T$$

for $b = (b^{(1)}, \dots, b^{(d)})$ *with* $b^{(r)} \in \mathcal{S}(\mathbb{R}^d)$ *for all* $r = 1, \dots, d$*. So (as we shall say in the sequel) the product* $g_1(r_1) \cdot \dots \cdot g_p(r_p)$ *has a total order of derivatives* $|\beta| = \sum_{l=1}^d \sum_{i=1}^p \beta_i^{(l)}$*. We know from Lemma 3.2 that*

$$\int_{\Delta_{\theta,t}^n} f_1(s_1)\dots f_k(s_k) \int_{\Delta_{\theta,s_k}^p} g_1(r_1)\dots g_p(r_p)dr_p\dots dr_1 f_{k+1}(s_{k+1})\dots f_n(s_n)ds_n\dots ds_1$$
$$= \sum_{\sigma \in A_{n,p}} \int_{\Delta_{\theta,t}^{n+p}} h_1^\sigma(w_1)\dots h_{n+p}^\sigma(w_{n+p})dw_{n+p}\dots dw_1, \tag{3.8}$$

where $h_l^\sigma \in \{f_j, g_i : 1 \le j \le n,\ 1 \le i \le p\}$*,* $A_{n,p}$ *is a subset of permutations of* $\{1, \dots, n+p\}$ *such that* $\#A_{n,p} \le C^{n+p}$ *for an appropriate constant* $C \ge 1$*, and* $s_0 = \theta$*. Then the products*

$$h_1^\sigma(w_1) \cdot \dots \cdot h_{n+p}^\sigma(w_{n+p})$$

have a total order of derivatives given by $|\alpha| + |\beta|$*.*

Proof. The result is proved by induction on n. For $n = 1$ and $k = 0$ the result is trivial. For $k = 1$ we have

$$\int_\theta^t f_1(s_1) \int_{\Delta_{\theta,s_1}^p} g_1(r_1)\dots g_p(r_p) \quad dr_p\dots dr_1 ds_1$$
$$= \int_{\Delta_{\theta,t}^{p+1}} f_1(w_1)g_1(w_2)\dots g_p(w_{p+1})dw_{p+1}\dots dw_1,$$

where we have put $w_1 = s_1,\ w_2 = r_1, \dots, w_{p+1} = r_p$. Hence the total order of derivatives involved in the product of the last integral is given by $\sum_{l=1}^d \alpha_1^{(l)} + \sum_{l=1}^d \sum_{i=1}^p \beta_i^{(l)} = |\alpha| + |\beta|$.

Assume the result holds for n and let us show that this implies that the result is true for $n+1$. Either $k = 0, 1$ or $2 \le k \le n+1$. For $k = 0$ the result is trivial. For $k = 1$ we have

$$\int_{\Delta_{\theta,t}^{n+1}} f_1(s_1) \int_{\Delta_{\theta,s_1}^p} g_1(r_1)\dots g_p(r_p)dr_p\dots dr_1 f_2(s_2)\dots f_{n+1}(s_{n+1})ds_{n+1}\dots ds_1$$
$$= \int_\theta^t f_1(s_1) \left(\int_{\Delta_{\theta,s_1}^n} \int_{\Delta_{\theta,s_1}^p} g_1(r_1)\dots g_p(r_p)dr_p\dots dr_1 f_2(s_2)\dots f_{n+1}(s_{n+1})ds_{n+1}\dots ds_2 \right) ds_1.$$

From (3.5) we observe by using the shuffle permutations that the latter inner double integral on diagonals can be written as a sum of integrals on diagonals of length $p+n$ with products having a total order of derivatives given by $\sum_{l=1}\sum_{j=2}^{n+1}\alpha_j^{(l)}+\sum_{l=1}^{d}\sum_{i=1}^{p}\beta_i^{(l)}$. Hence we obtain a sum of products, whose total order of derivatives is $\sum_{l=1}^{d}\sum_{j=2}^{n+1}\alpha_j^{(l)}+\sum_{l=1}^{d}\sum_{i=1}^{p}\beta_i^{(l)}+\sum_{l=1}^{d}\alpha_1^{(l)}=|\alpha|+|\beta|$.

For $k\geq 2$ we have (in connection with Lemma 3.2) from the induction hypothesis that

$$\int_{\Delta_{\theta,t}^{n+1}} f_1(s_1)\dots f_k(s_k)\int_{\Delta_{\theta,s_k}^{p}} g_1(r_1)\dots g_p(r_p)dr_p\dots dr_1 f_{k+1}(s_{k+1})\dots f_{n+1}(s_{n+1})ds_{n+1}\dots ds_1$$
$$=\int_\theta^t f_1(s_1)\int_{\Delta_{\theta,s_1}^{n}} f_2(s_2)\dots f_k(s_k)\int_{\Delta_{\theta,s_k}^{p}} g_1(r_1)\dots g_p(r_p)dr_p\dots dr_1$$
$$\times f_{k+1}(s_{k+1})\dots f_{n+1}(s_{n+1})ds_{n+1}\dots ds_2 ds_1$$
$$=\sum_{\sigma\in A_{n,p}}\int_\theta^t f_1(s_1)\int_{\Delta_{\theta,s_1}^{n+p}} h_1^\sigma(w_1)\dots h_{n+p}^\sigma(w_{n+p})dw_{n+p}\dots dw_1 ds_1,$$

where each of the products $h_1^\sigma(w_1)\cdot\dots\cdot h_{n+p}^\sigma(w_{n+p})$ have a total order of derivatives given by $\sum_{l=1}\sum_{j=2}^{n+1}\alpha_j^{(l)}+\sum_{l=1}^{d}\sum_{i=1}^{p}\beta_i^{(l)}$. Thus we get a sum with respect to a set of permutations $A_{n+1,p}$ with products having a total order of derivatives which is

$$\sum_{l=1}^{d}\sum_{j=2}^{n+1}\alpha_j^{(l)}+\sum_{l=1}^{d}\sum_{i=1}^{p}\beta_i^{(l)}+\sum_{l=1}^{d}\alpha_1^{(l)}=|\alpha|+|\beta|. \qquad \square$$

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