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Multiplicative Tate Spectral Sequences

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To my mother,
for introducing me to the wonderful world of mathematics.

Acknowledgements

Newton famously said

> *"If I have seen further than others, it is by standing upon the shoulders of giants."*

and while it is true that we owe a huge mathematical debt to the "giants" before us, we sometimes forget the debt we owe to the equals that we stand beside. Contrary to how it is often portrayed in popular media, mathematics is not a solitary activity, and the biggest portion of mathematical research comes out of discussion and collaboration. With this is mind, I would like to express my sincerest gratitude to the homotopy theory community in general. The homotopy theory community is warm, inclusive, and generous in its mathematical advice, and this thesis would not be even half of what it is without the support this community offers. If you have ever organised a conference I have been to, held a talk I have seen, written an article I have read, offered advice big or small, or just listed to me rant or rave about maths, then know that I am grateful to you. That being said, there are a couple of people I want to thank by name.

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Contents

Chapter 1

Introduction

The thesis you are reading is the result of research conducted by the author during her employment as a PhD student supervised by John Rognes at the Department of Mathematics at the University of Oslo. The thesis consists of two papers:

Paper I A. Hedenlund and J. Rognes. *A multiplicative Tate spectral sequence for compact Lie group actions.* 2020.

Paper II A. Hedenlund. *Multiplicative spectral sequences via décalage.* 2020.

This first chapter is meant as both a historical background and a larger context for the two papers included in this thesis. We will try to keep things conceptual, focusing on *why* rather than the more technical *how*, which we leave to the two papers and the other references provided in the bibliography. My hope is that the reader, after finishing this introduction, will have an understanding for what results to expect in the two papers, and why the author saw the importance in pursuing them, even if the reader is not necessarily a member of the same field of mathematics as herself and might not care about the same type of questions. We will successively narrow down the main themes of this thesis, starting from a very broad context, and ending in the more specific topics to be covered.

Section 1 We introduce the context in which this thesis is written: homotopical algebra. We look at two frameworks that have historically been used to deal with homotopical phenomena: model categories and ∞-categories. We introduce the main mathematical objects that we study: spectra.

Section 2 We introduce the technical tools that we will use and study in this thesis: spectral sequences. We discuss what we mean by convergence of spectral sequences, and how to to think about multiplicative structures in spectral sequences.

Section 3 We introduce the specific topic in homotopical algebra we would like to study, namely Tate constructions. We start with an introduction to the classical concept of Tate cohomology, go on to discuss its generalisation to homotopical algebra, and introduce the Tate spectral sequence. We explain what technical difficulties one might expect when considering such a spectral sequence. We end with a short digression where the author explains her own personal reasons for studying the Tate construction, which comes from a background in algebraic K-theory and topological Hochschild homology.

1.1 Homotopical algebra

The broad context in which this thesis takes place is within the mathematical subject known as homotopical algebra. I could call the context algebraic topology, or homotopy theory, but I feel like that gives a wrong indication of the flavour of the results contained in this thesis, which are often more algebraic and less topological and/or geometrical. While an aspiring mathematician's first exposure to "homotopy" often comes packaged in a topology course, one could argue that this is mostly a historical feature, and that there is nothing intrinsically topological about the concept, at all. In this first section, I hope to convince the reader that homotopical algebra is so much more than just a subset of algebraic topology, with applications going far beyond topological questions. It is often more fruitful to think of homotopical algebra as a natural extension of algebra; what Waldhausen envisioned with his "brave new rings"-paradigm. From this point of view, homotopical algebra is like doing algebra over a "deeper" base ring; while $\mathbb{Z}$ is the initial commutative ring in classical algebra, the so-called sphere spectrum $\mathbb{S}$ is the initial "commutative" ring in homotopical algebra. We can think of homotopical algebra as what we get if we base-change classical algebra along the Hurewicz homomorphism $\mathbb{S} \to \mathbb{Z}$.

In this section, we start with a motivation for thinking "homotopically". We go on to discuss two of the (many) approaches to homotopical algebra that exist: model categories and ∞-categories. Lastly we discuss the main mathematical objects that we will study in this thesis: spectra, the abelian groups of homotopical algebra.

Homotopical algebra	Algebra
space/∞-groupoid/anima	set
spectrum	abelian group
$\mathbb{E}_1$-ring spectrum	associative and unital ring
$\mathbb{E}_\infty$-ring spectrum	commutative ring
$\mathrm{Fin}^{\simeq}$ - the underlying space/∞-groupoid/anima of the category of finite sets	$\mathbb{N}$ - the monoid of natural numbers
$\mathbb{S}$ - the sphere spectrum	$\mathbb{Z}$ - the ring of integers

1.1.1 Homotopical algebra: A motivation

Many important results in mathematics deal with the question of "figuring out what objects are the same". This can be something as rudimentary as stating that $1 + 1$ is the same thing as 2, or something as advanced as stating that:

> *A simply connected closed 3-manifold is "the same thing" as a 3-dimensional sphere.*

One standard way of rigorously dealing with the concept of "sameness" in mathematics is through the very useful and powerful language of category theory. In the sense of category theory, we talk about two mathematical objects x and y of the same category as "the same" (or: **isomorphic**) if we can find morphisms $f : x \to y$ and $g : y \to x$ whose two compositions gf and fg are equal to the identity morphisms on x and y, respectively [Mac98; Bor94]. For example, in the statement above, the category we are considering is that of topological spaces with continuous maps between them. Two spaces that are "the same" in the above sense are referred to as being **homeomorphic**, and a more mathematically rigorous statement of what we have written above becomes:

> *Every simply connected closed 3-manifold is homeomorphic to the 3-sphere.*

This statement is known as the Poincaré conjecture, and is to this day the only one of the seven Millennium Prize Problems to have been solved. The formulation of the Poincaré conjecture coincides with the conception of the mathematical subject known as algebraic topology, where the rough aim can be expressed as "finding algebraic invariants that classify topological spaces up to homeomorphism". One of the main branches of algebraic topology is homotopy theory, where the idea is to use a weaker notion of "sameness" called homotopy equivalences, in order to approach this aim. Explicitly, two topological spaces X and Y are said to be **homotopy equivalent** if there is a homotopy equivalence between them. This means that we can find continuous maps $f : X \to Y$ and $g : Y \to X$ such that gf and fg are homotopic to the identity maps. Here, two maps $\phi, \psi : A \to B$ are homotopic if there is **homotopy** between them; a continuous map

$$H : I \times A \to B \quad \text{such that} \quad \begin{cases} H(0,x) = \phi(x) \\ H(1,x) = \psi(x)\,. \end{cases}$$

One equivalent formulation of the Poincaré conjecture says, in layman's terms, that this weaker form of "sameness", is enough to guarantee the stronger form, when we are dealing with the class of topological spaces known as "3-manifolds":

> *A 3-manifold which is homotopy equivalent to the 3-sphere is also homeomorphic to the 3-sphere.*

At this point it is worth taking a step back and note that similar notions of this "weaker sameness" can be found in many other fields of mathematics, as the following examples show.

Example 1.1.1. Two chain complexes C_* and D_* are said to be chain homotopic if there is a chain homotopy equivalence between them. This means that we can find chain maps $f : C_* \to D_*$ and $g : D_* \to C_*$ such that gf and fg are chain homotopic to the identities on C_* and D_*, respectively. Here, two chain maps $\phi, \psi : A_* \to B_*$ are called chain homotopic if there is a **chain homotopy** between them; a collection of maps $s_n : A_n \to B_{n+1}$ satisfying

$$\phi_n - \psi_n = s_{n-1}\partial_n^A + \partial_{n+1}^B s_n\,.$$

Example 1.1.2. Even in category theory itself we do not care about categories up to isomorphism, but rather, only up to equivalence. Recall that two categories $\mathscr{C}$ and $\mathscr{D}$ are called equivalent if there are functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that GF and FG are naturally isomorphic to the identity functors. Here, a **natural transformation** between two functors $\Phi, \Psi : \mathscr{A} \to \mathscr{B}$ is a class of morphisms $\tau_a : \Phi a \to \Psi a$ in $\mathscr{B}$ indexed by the objects of $\mathscr{A}$ and such that for every morphism $f : a \to a'$ in $\mathscr{A}$, the diagram

$$\begin{array}{ccc} \Phi a & \xrightarrow{\tau_a} & \Psi a \\ \downarrow{\scriptstyle \Phi f} & & \downarrow{\scriptstyle \Psi f} \\ \Phi a' & \xrightarrow{\tau_{a'}} & \Psi a' \end{array}$$

commutes. We say that we have a **natural isomorphism** if all the morphisms $\tau_a : \Phi a \to \Psi a$ are isomorphisms.

The context of the above examples are different: the first example can be placed under homological algebra, while the second example belongs to category theory. However, we can note that all of these examples describe essentially the same concept of "sameness", a concept which involved some notion of morphisms between morphisms. We called these by different names (homotopy, chain homotopy, and natural transformation) depending on the context, but collectively these morphisms between morphisms can be referred to as 2-morphisms. It turns out that it is often useful to also think about morphisms between these 2-morphisms, what we would call 3-morphisms, and so on. The study of such structures can have many different names depending on the direction you approach them from, but in this thesis, we will use the terminology *higher category theory* and make the following distinctions:

Higher category theory The study of the structures involving an infinite hierarchy of morphisms. Here, the structures, the ∞-categories, are themselves the objects of interest.

Homotopy theory The classical study of spaces as living in the context of higher category theory, rather than in ordinary category theory, via continuous maps, homotopies between continuous maps, homotopies between homotopies between continuous maps, and so on.

Homotopical algebra The study of mathematical objects that live naturally in the context of higher category theory. In particular, it is the study of those "homotopical" objects that posess some extra structure, like a multiplicative structure of some kind, as for example spectra[1].

We will return to the mathematical objects mentioned in the last point, spectra, in Section 1.1.3.

[1] There is not a clear consensus on what the term *homotopical algebra* should mean. Quillen was arguably the first one to use the terminilogy and explains it as "the generalization of homological algebra to arbitrary categories which results by considering a simplicial object as being a generalization of a chain complex" [Qui67]. In the light of the discussion in Section 1.1.3.3, we feel confident in claiming that the study of spectra are at the heart of homotopical algebra,

1.1.2 Approaches to homotopical algebra

Dealing with homotopical phenomena can be quite complicated as it is often hard to get a concrete grasp on our objects. Indeed, as the above discussion shows, at its extreme end, homotopical algebra involves keeping track of an infinite hierarchy of morphisms, which is not always easy in practice. The hands-on approach usually does not cut it, so one of the main burning questions when working with some version of homotopical algebra becomes finding a framework that is rigid enough to grasp the objects we are working with, but flexible enough to actually prove things. In this section, we introduce two such prevalent frameworks, namely model categories and ∞-categories.

1.1.2.1 Model categories

Historically, one common way to deal with homotopical phenomena is to use model categories. This is especially true if one comes from the direction of homotopy theory, where the broad goal for a long time was, and still is to some degree, to "classify continuous maps between spaces up to homotopy". The set of homotopy classes of (based) continuous maps $Y \to X$ is typically denoted $[Y, X]$. The most relevant case is when Y is the n-sphere, in which case

$$\pi_n(X) := [S^n, X]$$

is referred to as the n**th homotopy group**[2] of X. The basic motivating question is this: We want to treat homotopy equivalences as if they are isomorphisms, so why not simply add formal inverses to them? The first big road-block with this line of thought is that what we get by inverting an arbitrary class of morphisms might not be a category. Model categories were introduced by Quillen and provide ways to deal with these sorts of set theoretical issues. Briefly, a **model category** $\mathcal{M}$ is a bicomplete category together with three distinguished classes of morphisms, weak equivalences, fibrations, and cofibrations, together with a bunch of axioms [Qui67; Hov99]. The **homotopy category** of $\mathcal{M}$ is the localisation of $\mathcal{M}$ with respect to the weak equivalences. Here, localisation simply means that we formally invert all the weak equivalences $\mathcal{W}$, forcing them to become isomorphisms:

$$\mathrm{Ho}(\mathcal{M}) = \mathcal{M}[\mathcal{W}^{-1}]$$

The model axioms make sure that this can be done without any set-theoretical problems. More specifically, they guarantee that

$$\mathrm{Ho}(\mathcal{M})(Y, X) := [Y, X]$$

as the term was originally indended to be used. Cisinski seems to use the term in a much wider sense in [Cis19] when he writes that it is "the study of the compatibility of localisations with (co)limits". This is certainly relevant to homotopical algebra, but one could argue that it describes homotopical behaviour in a larger generality. Our use of *homotopical algebra* is close to Lurie's use of *higher algebra*, with an affinity towards *homotopical* rather than *higher* as we feel that the former is more descriptive.

[2] When $n = 0$, this is just a set, so using the word "group" is perhaps not so appropriate. For $n \geq 1$ they are groups, though. For $n \geq 2$, they are additionally abelian by an Eckmann–Hilton argument.

is a set for all objects X and Y, so that $\mathrm{Ho}(\mathcal{M})$ is actually a category.

The employment of homotopical algebra in the framework of model categories has been shown to be quite powerful. Some notable examples are Quillen's work in rational homotopy theory [Qui69] and, more recently, Voevodsky's proof of the Milnor conjecture [Voe03b; Voe03a] and later the more general Bloch–Kato conjecture [Voe11], which heavily rely on model categorical methods. However, it is worth noting that a lot of possibly important information goes lost when we pass from a model category to its homotopy category. While you remember that two maps are homotopic, you lose the information on *how* they were homotopic in the first place. As in, you lose explicit information about the homotopy between the two homotopic maps.

Another problem is that homotopical phenomena interact quite badly with colimits and limits. To illustrate by an example, consider the category **CW** consisting of CW-complexes and continuous maps between them. Along the program described above, we can deal with homotopical questions in **CW** by endowing this category with a suitable model structure. We can then formally invert the homotopy equivalences (which are the weak equivalences in this model category) to obtain the category

$$\mathcal{H} = \mathrm{Ho}(\mathbf{CW})\,.$$

This is referred to as the **classical homotopy category**[3]; the objects are still CW-complexes, but the morphisms are now homotopy classes of continuous maps. For a simple example of how colimits interact badly with such constructions, consider the two diagrams

$$D = \left(\begin{array}{ccc} S^0 & \longrightarrow & * \\ \big\downarrow & & \\ * & & \end{array}\right) \quad \text{and} \quad D' = \left(\begin{array}{ccc} S^0 & \longrightarrow & I \\ \big\downarrow & & \\ I & & \end{array}\right),$$

where the two points of S^0 are sent to the endpoints of the intervals I in diagram D'. The two diagrams are levelwise homotopy equivalent, but note that their respective colimits, the pushouts, are not. The pushout of the left hand side is a point $*$, but the pushout of the right hand side is the circle S^1, which is certainly not contractible. Classically, the solution to this problem is to introduce homotopy colimits, but *a priori* this is just a method of constructing something that "behaves like a colimit" and is invariant under levelwise homotopy equivalences of diagrams. In particular, homotopy colimits do not have a similar universal property to that of ordinary colimits, neither in the category **CW**, nor in the category $\mathcal{H}$.

[3]Many different model categories can give rise to equivalent homotopy categories, and this is certainly true for the classical homotopy category. We can alternatively take the category of simplicial sets and give its standard Kan-Quillen model structure.

1.1.2.2 ∞-categories

To be able to use homotopical algebra to its best potential, you have to find a way to encode all the information on higher morphisms between the objects you want to study. One way of doing this, which seems to become more and more prevalent, is ∞-categories, as developed by the likes of Boardman–Vogt, Joyal, and Lurie [BV73; Joy02; Lur09; Lur17]. These are certain simplicial sets that can be said to behave like categories. We emphasize that the combinatorial behaviour of simplicial sets makes them extremely practical to work with in many situations.

Definition 1.1.3. An **∞-category**[4] is a simplicial set X in which every inner horn $\Lambda^n_k \to X$ can be extended to a simplex $\Delta^n \to X$.

To explain in which sense simplicial sets are category-like, we recall the nerve functor. The nerve functor is defined as evaluation at the fully faithful inclusion functor from the simplex category to the category of categories, in the sense that:

$$N : \mathrm{Cat} \longrightarrow \mathrm{sSet}\,, \quad \mathscr{C} \mapsto \mathrm{Hom}_{\mathrm{Cat}}(i(-), \mathscr{C})\,.$$

The reader can check for themself that the 0-simplices in the simplicial set $N\mathscr{C}$ are given by the objects in $\mathscr{C}$, the 1-simplices by the morphisms in $\mathscr{C}$, the 2-simplices by composable pairs of morphisms, and so on. We can let this serve as a paradigm when thinking of a simplicial set X as category-like:

- We can think of a 0-simplex $v : \Delta^0 \to X$ as an object of X.
- We can think of a 1-simplex $\phi : \Delta^1 \to X$ as a morphism[5] of X from the source $x = d_1(\phi)$ to the target $y = d_0(\phi)$.
- We can think of a 2-simplex $\sigma : \Delta^2 \to X$ as witnessing that $d_1(\sigma)$ is the composition of the map $\phi = d_2(\sigma)$ and $\psi = d_0(\sigma)$, informally visualised as the diagram:

$$\begin{array}{ccccc} & & d_0(\phi) = d_1(\psi) & & \\ & {\scriptstyle \phi = d_2(\sigma)}\nearrow & & \searrow{\scriptstyle \psi = d_0(\sigma)} & \\ d_1(\phi) & & \xrightarrow[\psi\phi = d_1(\sigma)]{} & & d_0(\psi) \end{array}\,.$$

Nerves of categories are simplicial sets exhibiting a certain specific property: every inner horn $\Lambda^n_k \to N\mathscr{C}$ can be *uniquely* extended to a simplex $\Delta^n \to N\mathscr{C}$; see [Lur09, Proposition 1.1.2.2]. In particular, note that nerves of categories are examples of ∞-categories. However, there are many examples of ∞-categories that cannot be written as the nerve of an ordinary category. Using the category-like description of the simplicial set X, the inner 2-horn condition of Definition 1.1.3 is supposed to tell us that any pair of composable morphisms in X

[4] Also called quasi-category or weak Kan complex.

[5] From this point of view, it makes sense to think of the degenerate 1-simplex $\mathrm{id}_v = s_0(v)$ as the identity morphism of the object $v : \Delta^0 \to X$.

has a composite. Indeed, any pair of composable morphisms $\phi, \psi : \Delta^1 \to X$ determines an inner 2-horn $\Lambda^2_1 \to X$, informally visualised as:

$$\begin{array}{ccccc} & & d_0(\phi) = d_1(\psi) & & \\ & \overset{\phi}{\nearrow} & & \overset{\psi}{\searrow} & \\ d_1(\phi) & & \cdots\cdots\cdots\cdots\cdots\rightarrow & & d_0(\psi) \end{array} .$$

By the inner 2-horn condition, the dotted map in the diagram exists. While this composite is not unique in a strict sense, the rest of the inner horn conditions guarantee that it is unique in the sense of higher category theory. Indeed, possible compositions of ϕ and ψ form a simplicial set, and the condition that we have lifts for *all* inner horns guarantees that this simplicial set is actually a Kan complex, and even more, that this Kan complex is contractible. If we want to emphasize this higher categorical view of uniqueness we usually speak of something "being unique up to contractible choice"[6].

The concept of ∞-categories solves the conundrum regarding homotopy and colimits by adding something lying in between the category of CW-complexes and the classical homotopy category:

$$\mathbf{CW} \longrightarrow \mathcal{S} \longrightarrow \mathcal{H}$$

This something is the ∞-category of spaces[7]. There are ∞-categorical interpretations of colimits and limits, with suitable universal properties [Lur09, Chapter 4]. The ∞-category $\mathcal{S}$ has all of these, and what is classically referred to as a homotopy (co)limit in $\mathbf{CW}$ is mapped to such a (co)limit in $\mathcal{S}$.

1.1.3 Spectra

One common description of higher category theory is that it is category theory not built on sets, but on spaces. From this point of view, we can understand

[6] A way to rigorously state this sort of uniqueness of compositions as the characterising feature of an ∞-category is given by the following result.

Theorem 1.1.4 (Joyal)**.** *A simplicial set X is an ∞-category if and only if the restriction map* $\mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Lambda^2_1, X)$ *is an acyclic Kan fibration.*

We can think of $\mathrm{Map}(\Lambda^2_1, X)$ as "the space composition problems in X" and of $\mathrm{Map}(\Delta^2, X)$ as the "the space of solutions to composition problems". The theorem above tells us that the characterizing property of an ∞-category is that these two spaces are the same, from a homotopical point of view.

[7] Whether it is reasonable to call the objects in this ∞-category "spaces" is up for debate. From this historical account it seems reasonable, but it is worth noting that both the ∞-category $\mathcal{S}$ and the category $\mathcal{H}$ can be constructed in many other ways, that do not necessarily make use of spaces, as we usually think of them. Perhaps a better term for the objects in $\mathcal{S}$ is "homotopy types" or "∞-groupoids". In [CS19], the authors argue for the terminology "anima", in the sense of the "soul" of a space. Indeed, the ∞-category $\mathcal{S}$ can be obtained from the category of sets in a process of freely adding sifted colimits, which the authors refer to as "animation". We leave it to the reader to make up their own mind on what they think is the best word, but stick with "spaces" in this thesis, as it is the most well-used terminology at the time of writing.

homotopical algebra as algebra not built on sets, but on spaces. It turns out that the world of classical algebra can be embedded faithfully into the world of homotopical algebra and this allows for several interesting generalisations of algebra into a homotopical setting. The abelian groups of homotopical algebra are referred to as spectra. In the next couple of sections we explain what spectra are from various points of view.

1.1.3.1 Spectra as stable spaces

The earliest motivation behind spectra is that they describe a relatively well-behaved part of homotopy theory, namely stable behaviours. This motivation can be traced back to Freudenthal's suspension theorem, which tells us that the sequence

$$[Y,X] \longrightarrow [\Sigma Y, \Sigma X] \longrightarrow \cdots \longrightarrow [\Sigma^n Y, \Sigma^n X] \longrightarrow \cdots$$

of homotopy classes of maps will eventually stabilise [Fre38]. In the case that Y is an n-sphere, we write

$$\pi_n^{\mathrm{st}}(X) = \operatorname*{colim}_k \pi_{n+k}(\Sigma^k X)$$

for this stabilised value and call it the **nth stable homotopy group** of X. The subfield of **stable homotopy theory** can roughly be understood as the study of stable homotopy groups. Freudenthal's suspension theorem suggests the idea of introducing a category with objects that reflect this stable phenomenon and in which it is natural to study these types of stable behaviours. This category is known as the category of spectra, and was first introduced by Lima [Lim60]. In his sense, a **spectrum** X is simply an infinite sequences of pointed spaces $\{X_n\}_{n=0}^{\infty}$ equipped with continuous maps $\Sigma X_n \to X_{n+1}$ from the suspension of the nth space to the $(n+1)$th space. Note that every space K naturally gives rise to a spectrum, the **(unreduced) suspension spectrum** $\Sigma_+^\infty K$, where the nth space is the nth suspension of K_+, with the obvious continuous maps between the different levels. Arguably, the most important example is the **sphere spectrum** which is simply defined as the suspension spectrum of the point:

$$\mathbb{S} = \Sigma_+^\infty * \ .$$

The sphere spectrum $\mathbb{S}$ plays the same role in homotopical algebra, as the integers $\mathbb{Z}$ play in classical algebra; it is the initial "commutative" ring of homotopical algebra. For a more algebraic example of spectra, recall that, given an abelian group A and non-negative integer n, it is always possible to construct a space $K(A,n)$, the nth Eilenberg–Mac Lane space of A, in such a way that

$$\pi_k K(A,n) = \begin{cases} A & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

It is known that there are homotopy equivalences $\Omega K(A, n+1) \simeq K(A,n)$, so that we have left adjoint maps $\Sigma K(A,n) \to K(A,n+1)$. These make sure that the Eilenberg–Mac Lane spaces assemble into a spectrum, called the **Eilenberg–Mac Lane spectrum** of A and denoted HA.

1.1.3.2 Spectra as cohomology theories

The example of Eilenberg–Mac Lane spectra at the end of the previous section shows how spectra can be algebraic in nature. However, *a priori* there is nothing intrinsically algebraic about them, in general. Possibly, a better explanation for in what way they are algebraic is through Atiyah–Whitehead work on generalised cohomology theories. Recall that a **generalised cohomology theory** h^* is sequence of functors $h^n : \mathcal{H}^{\mathrm{op}} \to \mathrm{Ab}$ together with natural isomorphisms $\sigma : h^n \to h^{n+1} \circ \Sigma$, satisfying exactness and additivity [Whi62]. These conditions guarantee, via Brown's representability theorem [Bro62], that the functors $h^n : \mathcal{H}^{\mathrm{op}} \to \mathrm{Set}$ are all representable. That is, we can find pointed CW-complexes E_n such that

$$h^n(X) \cong [X, E_n].$$

Using the suspension isomorphism for generalised cohomology theories we have bijections

$$[X, E_n] \cong [\Sigma X, E_{n+1}] \cong [X, \Omega E_{n+1}]\,,$$

and by full faithfulness of the Yoneda embedding we thus have homotopy equivalences $E_n \to \Omega E_{n+1}$. The left adjoint $\Sigma E_n \to E_{n+1}$ of this map is the wanted structure map in our spectrum $\{E_n\}_{n=0}^{\infty}$. Hence, the spaces E_n assemble into a spectrum. The converse is also true: every spectrum gives rise to a homology and cohomology theory. This suggests that we want to put a model structure on the category of spectra such that two spectra are equivalent if and only if they give rise to isomorphic (co)homology theories. This leads us to the definition of the **stable homotopy category** $\mathcal{SHC}$; what we get if we localise with respect to the weak equivalences in that model structure [BF78].

Ordinary cohomology with coefficients in the abelian group A gives us back the Eilenberg–Mac Lane spectrum via the above discussion. However, there are many examples of more exotic versions of cohomology theories, each of which gives rise to its own spectrum. Some examples are:

- Various flavours of topological K-theory: complex topological K-theory KU, real topological K-theory KO, and quaternionic topological K-theory KSp, ...
- Various flavours of cobordism[8]: complex cobordism MU, unoriented cobordism MO, oriented cobordism MSO,...
- Elliptic cohomology: elliptic curves gives rise to formal groups which in turn give rise to cohomology theories via Landweber's exact functor theorem [Lan76].

[8]The sphere spectrum also belongs here and corresponds to framed cobordism via the Pontryagin–Thom theorem.

1.1.3.3 Spectra as an analogy of unbounded chain complex

One alternative way of viewing spectra, which might be useful for people that care neither about stable behaviour of spaces nor generalised cohomology theories, is as follows:

> *Spectra are to spaces, what unbounded chain complexes are to non-negatively graded chain complexes*[9].

This can give a reason for working with spectra, at least if the reader sees the point in working with chain complexes and agrees that it is generally a stupid idea to restrict mathematics to the setting of non-negatively graded ones. Let us imagine a world where we only have access to non-negatively graded chain complexes. How would we construct the category of unbounded chain complexes from this? By thinking backwards, we could start by noting that every unbounded chain complex can be written as a colimit of bounded below ones simply by truncating:

$$C = \operatorname*{colim}_{n\to\infty} \tau_{\geq -n} C \,.$$

This allows us to describe unbounded chain complexes as colimits of bounded below chain complexes. A bounded below chain complex can be made into a non-negatively graded chain complex by suspending it enough times. Indeed, let us write

$$C^{(n)} = (\tau_{\geq -n} C)[n]\,, \quad n \geq 0\,,$$

and notice that this is always a non-negatively graded chain complex. In terms of non-negative chain complexes, suspensions, and colimits, our original chain complex can be written as the colimit of the system

$$C^{(0)} \longrightarrow C^{(1)}[-1] \longrightarrow C^{(2)}[-2] \longrightarrow \cdots \,.$$

This data could alternatively be phrased as:

1. A sequence $\{C^{(n)}\}_{n=0}^{\infty}$ of non-negative chain complexes.
2. A chain map $C^{(n)}[1] \to C^{(n+1)}$ for every non-negative integer n.

Compare this to the definition of a spectrum from Section 1.1.3.1. This might provide meaning to the concept of spectra, especially if the reader is already using simplicial methods to deal with questions concerning the derived category of chain complexes, via the Dold–Kan correspondence

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab}) \simeq \mathrm{Ch}(\mathrm{Ab})_{\geq 0}\,.$$

A simplicial abelian group is in particular a Kan complex, which is what is usually interpret as a “space” in the theory of ∞-categories. From this point of view, it seems reasonable to make the switch from simplicial methods to homotopical algebra when you want to better understand unbounded chain complexes and the derived category of such.

[9]Thank you to Fabian Hebestreit for calling me in the middle of the night to explain this point of view when he had his eureka moment on spectra. It was as enlightening to me.

1.2 Spectral sequences

If spectra are the mathematical objects that we are interested in, then spectral sequences are the tools that we will use to study them with[10]. Both of the papers included in this thesis deal with spectral sequences, in some way or another. We discuss in order: what spectral sequences are, different ways they arise, what we mean by convergence of spectral sequences, and how to deal with multiplicative structures. In particular, although it is of course an important part of the subject, we will *not* discuss how to work with, manipulate, and compute with spectral sequences. This often depends very heavily on the spectral sequence in question, and it is very hard to say something in general. Instead, we focus on the aim of giving the reader a feel for what they are and how to think about them, and refer the reader who is hungry for more to [McC01].

1.2.1 Spectral sequences

Since their conception by Leray [Ler46], spectral sequences have proven to be incredibly useful tools in various subjects of mathematics. One can view them as a generalisation of the concept of an exact sequence, and they are primarily used for the same purpose, namely for computations of homotopy and/or homology groups. Let us start with the most basic definition. In what follows, we will consider the category of abelian groups, although similar definitions can be made in any abelian category. A **spectral sequence** (of abelian groups) consists of the following data:

1. for every integer $r \geq 1$, a bigraded abelian group $E^r = E^r_{p,q}$;
2. for every integer $r \geq 1$, a map $d^r : E^r \to E^r$ of bidegree $(-r, r-1)$ such that $d^r \circ d^r = 0$;
3. for every integer $r \geq 1$, an isomorphism $E^{r+1} \cong H(E^r, d^r)$ of bigraded abelian groups, where H refers to taking homology.

A morphism of spectral sequences is a collection of morphisms of bigraded abelian groups compatible with the differentials and with the isomorphisms $E^{r+1} \cong H(E^r, d^r)$, in the obvious way. This makes spectral sequences into a category which we denote as SSEQ. It is common to refer to the bigraded abelian group $E^r_{*,*}$ as the rth page of the spectral sequence, and to visualise it as a page in an imagined book, where we pass from one page to the next by taking homology.

Remark 1.2.1. There are many other grading conventions for spectral sequences. The one described above is called homological Serre grading. Another grading convention that is used is homological Adams grading $E^r_{n,s}$. To go between these two grading conventions we can use the linear transformations

$$(n, s) \mapsto (-s, n+s) \quad \text{and} \quad (p, q) \mapsto (p+q, -p)\,.$$

[10]Disclaimer: there is really no etymological connection between *spectra* and *spectral sequences* other than derivatives of the word *spectrum* being overused in mathematics.

In homological Adams grading the d^r-differentials would go

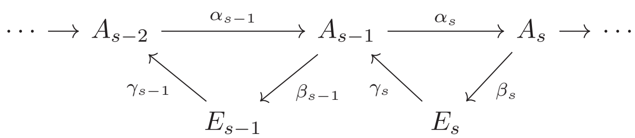
$$d^r_{n,s} : E^r_{n,s} \to E^r_{n-1,s+r} \,.$$

Note that the grading conventions are not consistent between the papers contained in this thesis!

A common situation where spectral sequences arise is when considering filtrations of mathematical objects. In this way, spectral sequences provide means to translate homotopical information into algebraic information, that can then be processed in the standard fashion of homological algebra. This will be our main point of view on spectral sequences in this thesis; that they are convenient ways to store and process large amounts of mathematically information. One could argue that this is the source of both their their usefulness and difficulty. There are a number of convenient stepping-stones when passing from a filtration to a spectral sequence, which we now cover briefly. In all the sections below, $\mathscr{A}$ denotes the graded abelian category of abelian groups.

1.2.1.1 Exact couples

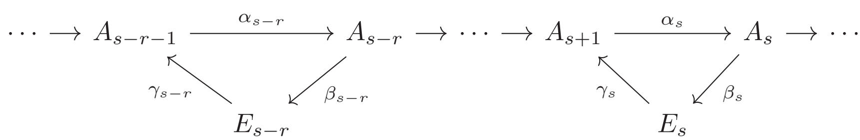
After Massey [Mas52, Section 1.4], we define an **(unrolled) exact couple** as a diagram



in $\mathscr{A}$, in which every triangle is exact. Here, the internal degrees of the maps α_s, β_s, and γ_s are 0, 0, and -1, respectively. Such an object gives rise to a spectral sequence by setting the E^1-page and d^1-differential to be

$$E^1_{s,*} = E_s \quad \text{and} \quad d^1 = \beta_{s-1} \circ \gamma_s \,.$$

The higher pages are given by considering the part



of the exact couple; see [HR19, Lemma 3.4]. While exact couples are useful for building additive spectral sequences, they have the disadvantage in that there is no useful notion of a pairing of exact couples; we will return to this point in the section dealing with multiplicative structures on spectral sequences.

1.2.1.2 Cartan–Eilenberg systems

For multiplicative considerations it is often convenient to work with Cartan–Eilenberg systems [CE56, Section XV.7]. Consider the following two categories:

- The category $\mathbb{Z}^{[1]}$ whose objects are pairs (i, j) of integers with $i \leq j$, and where have a single morphism $(i, j) \to (i', j')$ precisely when $i \leq i'$ and $j \leq j'$.
- The category $\mathbb{Z}^{[2]}$ whose objects are triples (i, j, k) of integers with $i \leq j \leq k$, and where have a single morphism $(i, j, k) \to (i', j', k')$ precisely when $i \leq i'$, $j \leq j'$, and $k \leq k'$.

Note that we have three obvious functors $d_0, d_1, d_2 \colon \mathbb{Z}^{[2]} \longrightarrow \mathbb{Z}^{[1]}$ and two natural transformations $\iota \colon d_2 \longrightarrow d_1$ and $\pi \colon d_1 \longrightarrow d_0$. We define an **finite Cartan–Eilenberg system** as a pair (H, ∂) where $H \colon \mathbb{Z}^{[1]} \to \mathscr{A}$ is a functor and $\partial \colon Hd_0 \to Hd_2$ is a natural transformation, such that the triangle

$$\begin{array}{ccc} Hd_2 & \xrightarrow{H\iota} & Hd_1 \\ & {\scriptstyle\partial}\nwarrow \quad \swarrow{\scriptstyle H\pi} & \\ & Hd_0 & \end{array}$$

is exact. Adding an initial object $-\infty$ and terminal object ∞ to the poset $\mathbb{Z}$ gives us the notion of an **extended Cartan–Eilenberg** system. An extended Cartan–Eilenberg system thus associates to each pair (i, j) with $-\infty \leq i \leq j \leq \infty$ a graded abelian group $H(i, j)$, in a functorial way. Furthermore, it associates to each triple (i, j, k) with $-\infty \leq i \leq j \leq k \leq \infty$ a long exact sequence

$$\ldots \longrightarrow H(i, j) \longrightarrow H(i, k) \longrightarrow H(j, k) \xrightarrow{\partial} H(i, j) \longrightarrow \ldots,$$

where ∂ is a natural transformation of total degree -1.

Remark 1.2.2. The notion of a finite Cartan–Eilenberg system is connected to the notion of a $\mathbb{Z}$-complex in the ∞-category of spectra, as in [Lur17, Definition 1.2.2.2]. Indeed, given a $\mathbb{Z}$-complex $X : \mathbb{Z}^{[1]} \to \mathrm{Sp}$ the composition

$$\mathbb{Z}^{[1]} \xrightarrow{X} \mathrm{Sp} \xrightarrow{\pi_*} \mathscr{A}$$

forms a finite Cartan–Eilenberg system.

An extended Cartan–Eilenberg system gives rise to an exact couple by setting

$$A_s = H(-\infty, s) \quad \text{and} \quad E_s = H(s-1, s)$$

and

$$\begin{aligned} \alpha_s &: H(-\infty, s-1) \longrightarrow H(-\infty, s) \\ \beta_s &: H(-\infty, s) \longrightarrow H(s-1, s) \\ \gamma_s &: H(s-1, s) \longrightarrow H(-\infty, s-1) \end{aligned}$$

where the first two maps are induced by the maps $s-1 \leq s$ and $-\infty \leq s-1$ in the poset $\mathbb{Z}$, while the last map is induced by the natural transformation ∂ in the Cartan–Eilenberg system [HR19, Section 7].

1.2.1.3 Décalage

Informally, décalage is a way to make sense of “turning a page in the spectral sequence” on the level of filtrations. This was first introduced by Deligne in relation to his studies on Hodge structures [Del71]. Without going into too much detail: given a filtered chain complex (K, F), the associated decalée $\mathrm{Déc}(K)$ is a new filtered chain complex $(K, \mathrm{Déc}(F))$. This new filtered chain complex is constructed in such a way that the spectral sequence associated to $(K, \mathrm{Déc}(F))$ is isomorphic, after reindexing, to the spectral sequence associated to (K, F), but shifted forward one page:

$$E^r_{n,s}(\mathrm{Déc}(K)) \cong E^{r+1}_{n,s-n}(K)$$

in homological Adams grading. Although not originally phrased in this language, we can make sense of décalage using a t-structure on the derived filtered category, called the Beilinson t-structure [Beĭ87; BMS19]. We will study this approach to spectral sequences in Paper II.

1.2.2 Convergence

One of the main questions when working with spectral sequences is:

Is the spectral sequence computing what we want it to compute?

This is the question of convergence of spectral sequences. At the inception of the subject of spectral sequences, dealing with this question usually involved imposing quite severe finiteness conditions on the objects, but as the subject developed it became apparent that better considerations were needed. One groundbreaking article is [Boa99] and its introduction of the notion of conditional convergence.

In order to talk about convergence, we need to first establish some terminology regarding filtrations of abelian groups. For us, a **filtration** is simply a sequence of injective homomorphisms of abelian groups

$$\cdots \longrightarrow F^{q+1} \longrightarrow F^q \longrightarrow F^{q-1} \longrightarrow \cdots .$$

We consider this as an abstract filtration, and not as a filtration of a specific group, though we could of course say that is a filtration of $\mathrm{colim}_q F^q$. In this sense, all filtrations we work with are exhaustive, in Boardman's terminology. A filtration is called **derived complete** if the total derived inverse limit

$$\operatorname*{Rlim}_q F^q \simeq 0$$

vanishes.

Remark 1.2.3. We warn the reader that we use the symbol Rlim differently from Boardman here. What we mean by Rlim is the total right derived functor of the limit, and not just its first right derived functor. To clarify, for us, the derived inverse limit $\mathrm{Rlim}_q F^q$ is an object of $\mathcal{D}\mathbb{Z}_{[-1,0]}$, since the sequential limit functor only has two non-vanishing right derived functors. What Boardman writes as $\mathrm{Rlim}_q F^q$, we would instead write as

$$H_{-1}(\operatorname*{Rlim}_q F^q) = \lim_q{}^1 F^q \,.$$

Hence, being derived complete is equivalent to being complete and Hausdorff, in Boardman's terminology.

Given a spectral sequence $E^{\star}_{*,*}$, the E^2-page has a filtration

$$0 = B^2_{p,q} \subset B^3_{p,q} \subset \cdots \subset B^r_{p,q} \subset \cdots \subset Z^r_{p,q} \subset \cdots \subset Z^3_{p,q} \subset Z^2_{p,q} = E^2_{p,q}$$

of abelian groups in such a way that

$$E^r_{p,q} \cong Z^r_{p,q}/B^r_{p,q} \,.$$

We write

$$Z^{\infty}_{p,q} = \lim_r Z^r_{p,q} \quad \text{and} \quad B^{\infty}_{p,q} = \operatorname*{colim}_r B^r_{p,q}$$

and call these the **infinite cycles** and the **infinite boundaries**, respectively. The abelian groups

$$E^{\infty}_{p,q} = Z^{\infty}_{p,q}/B^{\infty}_{p,q} \quad \text{and} \quad RE^{\infty}_{p,q} = \lim_r{}^1 Z^r_{p,q}$$

are referred to as the **limit page** and the **derived limit page** of the spectral sequence, respectively. The point of convergence is to connect these objects, which are internal to the spectral sequence, to some filtration on the wanted target of the spectral sequence. We start with arguably the most useful notion of convergence. **Strong convergence** of a spectral sequence $E^{\star}_{*,*}$ to a graded group G_* consists of:

1. A derived complete filtration $F^{\star}_{*}$ for every integer $*$.

2. An isomorphism
$$E^{\infty}_{p,q} \cong F^q_{p+q}/F^{q+1}_{p+q}$$
for every pair of integers p and q.

3. An isomorphism
$$G_* \cong \operatorname*{colim}_q F^q_*$$
for every integer $*$.

We will often abusively say "the spectral sequence converges strongly to G_*" even though strong convergence is technically not a property of a spectral sequence, but rather extra structure. If our spectral sequence is strongly convergent[11], then G_* can be recovered from the spectral sequence in question in the strongest possible sense via [Boa99, Proposition 2.5]. However, there are a lot of spectral sequences that are not *a priori* strongly convergent. For those, Boardman introduced the notion of conditional convergence. If we are given an exact couple such that

$$\operatorname{Rlim}_s A_s \simeq 0\,,$$

we say that the associated spectral sequence **converges conditionally**. Note that conditional convergence is slightly unsatisfactory from a structural point of view in the sense that conditional convergence technically is a property of an exact couple, and not a property or structure on the spectral sequence itself. In other words, given a spectral sequence, with no information on how it arose, the question "Does the spectral sequence converge conditionally?" does not even makes sense. Nevertheless, the concept of conditional convergence is very useful in that it allows one to deduce strong convergence from conditions that are entirely internal to the spectral sequence in question, and which in many cases are easy to check. Indeed, depending on what sort of spectral sequence you have, the following table summarises sufficient criteria for strong convergence [Boa99, Theorem 6.1, Theorem 7.3, Theorem 8.2]:

half-plane with exiting differentials	conditional convergence
half-plane with entering differentials	conditional convergence + vanishing of the derived limit page RE^∞
whole-plane	conditional convergence + vanishing of the derived limit page RE^∞ + vanishing of Boardman's whole-plane obstruction W

Let us end this section by discussing the term W appearing in the last entry, which we have yet to explain. Instead of introducing the necessary terminology for introducing it in Boardman's language we refer to [HR19] where the authors give a simplified description of Boardman's obstruction group in terms of Cartan–Eilenberg systems. They show that it can be expressed as the kernel

$$W \cong \ker(\kappa)$$

of the canonical colimit-limit exchange map

$$\kappa : \operatorname*{colim}_i \lim_j H(i,j) \longrightarrow \lim_j \operatorname*{colim}_i H(i,j)\,.$$

[11]Or more correctly: "is endowed with the structure of strong convergence".

We note that although Boardman's obstruction group depends on the exact couple and/or the Cartan–Eilenberg system, there are criteria internal to the spectral sequence itself that guarantee the vanishing of Boardman's obstruction group [Boa99, Lemma 8.1].

Remark 1.2.4. As mentioned in the previous section, the notion of a spectral sequence makes sense more generally in an abelian category $\mathscr{A}$. To make sense of convergence for such spectral sequences, we need some assumptions on the abelian category, though. Assuming that sequential colimits and limits in $\mathscr{A}$ behaves as in the category of abelian groups will do the trick. What "behaving as" should mean is subtle, though. One might expect that we want sequential colimits and infinite countable products to be exact. In the terminology of Grothendieck's Tohoku paper [Gro57], we should require the abelian category $\mathscr{A}$ to satisfy AB5 and AB4*. However, in light of Neeman's counterexample to AB4* being sufficient to guarantee that $\lim^1$ vanishes on Mittag–Leffler sequences [Nee02], and Boardman heavily making use of Mittag–Leffler short exact sequences in his paper, we refrain from making any definite claims on this matter.

1.2.3 Multiplicative structures

One of the main foci of this thesis is multiplicative structures on spectral sequences. When the object we want to study has some extra structure, like some sort of pairing, it is useful, if not often essential, to incorporate this structure in the spectral sequence used to study the object. Such a structure can then be heavily exploited in computations.

The category of spectral sequences is not a symmetric monoidal category, so it does not make sense to talk about the tensor product of two spectral sequences. However, it does make sense to talk about multilinear maps of spectral sequences. This makes the category of spectral sequences into a multicategory, or, as it is also called, a coloured operad. Given spectral sequences (C^r, d^r), (D^r, d^r), and (E^r, d^r), a **bilinear map** (or **pairing**)

$$\phi : (C^\star_{*,*}, D^\star_{*,*}) \longrightarrow E^\star_{*,*}$$

is a collection of morphisms

$$\phi^r : C^r_{p,q} \otimes D^r_{p',q'} \longrightarrow E^r_{p+p',q+q'}$$

such that the following conditions hold:

1. We have
$$d^r \phi^r = \phi^r(d^r \otimes 1 + 1 \otimes d^r)$$
as morphisms $C^r_{p,q} \otimes D^r_{p',q'} \to E^r_{p+p'-r,q+q'+r-1}$.

2. The diagram

$$\begin{array}{ccc} C^{r+1}_{p,q} \otimes D^{r+1}_{p',q'} & \xrightarrow{\phi^{r+1}} & E^{r+1}_{p+p',q+q'} \\ \downarrow & & \downarrow \cong \\ H_{p+p',q+q'}(C^r_{*,*} \otimes D^r_{*,*}) & \xrightarrow{H(\phi^r)} & H(E^r_{n+n',s+s'}) \end{array}$$

commutes.

The three ways of passing from filtrations to spectral sequences that we covered in Section 1.2.1 are more and less suitable for dealing with multiplicative structures on the associated spectral sequences. As already mentioned, exact couples are at a disadvantage in that there is no useful notion of a pairing of exact couples[12]. To deal with multiplicative questions it is therefore better to use one of the two other constructions: Cartan–Eilenberg systems or décalage. There is a suitable definition for a pairing of a finite Cartan–Eilenberg system. Indeed, given finite Cartan–Eilenberg systems (H', ∂), (H'', ∂) and (H, ∂), a **pairing**

$$\phi : (H', H'') \to H$$

is a collection of homomorphisms

$$\phi_r \colon H'(i-r, i) \otimes H''(j-r, j) \longrightarrow H(i+j-r, i+j)$$

of total degree 0, for all integers i and j and $r \geq 1$. These are required to satisfy the following two conditions:

1. Each square

$$\begin{array}{ccc} H'(i-r, i) \otimes H''(j-r, j) & \xrightarrow{\phi_r} & H(i+j-r, i+j) \\ \downarrow & & \downarrow \\ H'(i'-r', i') \otimes H''(j'-r', j') & \xrightarrow{\phi_{r'}} & H(i'+j'-r', i'+j') \end{array}$$

commutes, for all integers i, j, i', j' and $r, r' \geq 1$ with $i \leq i'$, $i - r \leq i' - r'$, $j \leq j'$ and $j - r \leq j' - r'$.

[12] The reader might disagree by referring to the paper [Mas54]. However, the structure and properties involved in the notion of a "pairing of exact couples" according to Massey are abundant enough to the point that the claim that a pairing of exact couples leads to a pairing of spectral sequences is essentially a tautology. In practice, the conditions one would need to check in order to show that one has a pairing of exact couples are essentially the same ones one would need to check in order to show that one has a pairing of the associated spectral sequences, rendering the use of exact couples as a stepping stone between filtrations and spectral sequences pointless when dealing with multiplicative questions. Hence our phrasing "no useful notion".

2. In the (non-commutative) diagram

$$\begin{array}{ccccc} H'(i-r,i)\otimes H''(j-r,j) & & \xrightarrow{\eta\otimes\partial} & & H'(i-1,i)\otimes H''(j-r-1,j-r) \\ & \searrow^{\phi_r} & & & \\ \Big\downarrow{\scriptstyle \partial\otimes\eta} & & H(i+j-r,i+j) & & \Big\downarrow{\scriptstyle \phi_1} \\ & & & \searrow^{\partial} & \\ H'(i-r-1,i-r)\otimes H''(j-1,j) & & \xrightarrow{\phi_1} & & H(i+j-r-1,i+j-r) \end{array}$$

   the inner composition is the sum of the two outer ones:

$$\partial\phi_r = \phi_1(\partial\otimes\eta) + \phi_1(\eta\otimes\partial)\,.$$

   In terms of elements, this identity in $H(i+j-r-1, i+j-r)$ can be written

$$\partial\phi_r(x\otimes y) = \phi_1(\partial x\otimes \eta y) + (-1)^{\|x\|}\phi_1(\eta x\otimes \partial y)$$

   for $x \in H'(i-r,i)$ of total degree $\|x\|$ and $y \in H''(j-r,j)$.

This definition was exploited by Douady to show that such a pairing of Cartan–Eilenberg systems gives rise to a pairing of the associated spectral sequences [Dou59a; Dou59b].

1.3 The Tate construction

The specific topic in homotopical algebra that this thesis is concerned about is the Tate construction. This construction was first introduced by Greenlees and should be seen as a generalisation of Tate cohomology to the setting of spectra. In particular, the research in this thesis concerns the Tate spectral sequence, which is a spectral sequence that computes the homotopy groups of the Tate construction on some spectrum with group action. We start by discussing Tate cohomology, to explain the classical context, and go on to define the Tate construction in the setting of G-spectra. Next, we explore the Tate spectral sequence and what issues one might expect to pop up when studying this spectral sequence. Lastly, I explain my own personal reason for studying the Tate spectral sequence, which is connected to my interest in topological Hochschild homology and algebraic K-theory.

1.3.1 Tate cohomology

Tate cohomology was first introduced by Tate in his study of class field theory [Tat52]. We give a very brief introduction to the subject following [Bro82, Chapter VI] and [CE56, Chapter XII]. In what follows, G will be a finite group and M a G-module. The basic idea is that Tate cohomology is a way to splice together group homology and group cohomology of G with coefficients in M into a single cohomology theory. We usually do this via the so-called norm map: the map from the G-orbits to the G-fixed points of M defined as

$$\mathrm{Nm}_G : M_G \to M^G\,, \quad m \mapsto \sum_{g\in G} g\cdot m\,.$$

The **Tate cohomology groups** of G with coefficients in M can then be defined as

$$\hat{H}^i(G;M) = \begin{cases} H^i(G;M) & i \geq 1, \\ \mathrm{coker}(\mathrm{Nm}_G) & i = 0, \\ \ker(\mathrm{Nm}_G) & i = -1, \\ H_{-i-1}(G;M) & i \leq -2\,. \end{cases}$$

It turns out that the Tate cohomology groups can also be phrased as the (co)homology groups of a (co)chain complex of G-modules. Indeed, let P_* denote a projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[G]$-module, and note that the dual resolution $\mathrm{Hom}(P,\mathbb{Z})_*$ is a 'coresolution' of projective modules since $\mathbb{Z}[G]$ is a quasi-Frobenius algebra: projective and injective modules over $\mathbb{Z}[G]$ coincide. The spliced resolution $\hat{P}_*$, informally visualised as

$$\begin{array}{ccccccccccc} \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow & P_0 & \xrightarrow{\quad\mathrm{Nm}_G\quad} & \mathrm{Hom}(P_0,\mathbb{Z}) & \longrightarrow \mathrm{Hom}(P_1,\mathbb{Z}) \longrightarrow \cdots\,, \\ & {\scriptstyle\epsilon}\searrow & & \nearrow{\scriptstyle\epsilon^*} & \\ & & \mathbb{Z} & & \end{array}$$

is referred to as a **complete resolution**. We can then define Tate cohomology as the cohomology groups

$$\hat{H}^i(G;M) = H^i\,\mathrm{Hom}_{\mathbb{Z}[G]}(\hat{P}_*, M)\,.$$

One thing that is worth noting is that Tate cohomology is indeed a multiplicative cohomology theory in the sense that we can define a cup product

$$\smile : \hat{H}^i(G;M) \otimes \hat{H}^j(G;N) \longrightarrow \hat{H}^{i+j}(G; M\otimes N)$$

that extends the cup product on group cohomology, in a suitable sense. Although this can be done using complete resolutions and completed tensor products, as in [Bro82, Section VI.6], we here take the opportunity to introduce another perspective on Tate cohomology that is useful when thinking of the cup product. Given a projective resolution P_*, consider the mapping cone $\widetilde{P}_* = \mathrm{cone}(\epsilon : P_* \to \mathbb{Z})$ of the augmentation. After [Gre95], we define the **Tate complex** via the tensor product

$$T_*(M) = \widetilde{P}_* \otimes \mathrm{Hom}(P_*,\mathbb{Z})\,;$$

this chain complex is quasi-isomorphic to $\mathrm{Hom}(\hat{P}_*, M)$ after taking G-fixed points, by a zig-zag

$$\Big(\widetilde{P}_* \otimes \mathrm{Hom}(P_*,M)\Big)^G \xrightarrow{\ \sim\ } \Big(\widetilde{P}_* \otimes \mathrm{Hom}(\hat{P}_*,M)\Big)^G \xleftarrow[\ \sim\ ]{} \mathrm{Hom}_{\mathbb{Z}[G]}(\hat{P}_*, M)\,.$$

This shows that Tate cohomology can alternatively be described as the homology of the G-fixed points of the Tate complex:

$$\hat{H}^i(G;M) = H_{-i}\left(\widetilde{P}_* \otimes \mathrm{Hom}(P,M)_*\right)^G$$

The advantage with this perspective is that the cup product on Tate cohomology can be described by using G-chain maps

$$\Psi : P_* \to P_* \otimes P_* \quad \text{and} \quad \Phi : \widetilde{P}_* \otimes \widetilde{P}_* \to \widetilde{P}_*$$

lifting the identity map $\mathrm{id} : \mathbb{Z} \to \mathbb{Z} \otimes \mathbb{Z}$ and extending the fold map $\widetilde{P}_* \oplus_{\mathbb{Z}} \widetilde{P}_* \to \widetilde{P}_*$, respectively. Indeed, for G-modules M and N, the composite pairing

$$\begin{aligned}
\widetilde{P}_* \otimes \mathrm{Hom}(P_*, M) \otimes \widetilde{P}_* \otimes \mathrm{Hom}(P_*, N) & \\
&\longrightarrow \widetilde{P}_* \otimes \widetilde{P}_* \otimes \mathrm{Hom}(P_*, M) \otimes \mathrm{Hom}(P_*, N) \\
&\longrightarrow \widetilde{P}_* \otimes \widetilde{P}_* \otimes \mathrm{Hom}(P_* \otimes P_*, M \otimes N) \\
&\xrightarrow{\Phi\otimes\Psi^*} \widetilde{P}_* \otimes \mathrm{Hom}(P_*, M \otimes N)
\end{aligned}$$

is G-equivariant, and so induces an associative, unital, and graded commutative pairing

$$\smile : \hat{H}^i(G;M) \otimes \hat{H}^j(G;N) \longrightarrow \hat{H}^{i+j}(G;M \otimes N)$$

after passing to homology, which we refer to as the cup product on Tate cohomology. This extends the cup product on ordinary group cohomology in the obvious way.

1.3.2 The Tate construction

The Tate construction on a G-spectrum is the incarnation of Tate cohomology in homotopical algebra. Although it was first defined in the setting of genuine equivariant homotopy theory, it turns out that the Tate construction only depends on the naïve equivariant homotopy type of our spectrum, so to simplify the discussion, we here give an account in a much more naïve setting. We return to the classical point of view at the end of the section, in Remark 1.3.2.

Let G be a topological group, and let BG denote a fixed classifying space for it. We usually refer to the stable ∞-category

$$\mathrm{Sp}^{BG} = \mathrm{Fun}(BG, \mathrm{Sp})$$

as the ∞-category of **G-spectra**[13]. Here Sp denotes the ∞-category of spectra and the reader is encouraged to think of G-spectra as analogues to G-modules. We will follow ordinary ∞-categorical notation; in particular, all colimits and

[13] Again, we point out that these are not genuine G-spectra in the sense of equivariantly homotopy theory, but rather a naïve version. However, all constructions we consider in this thesis depend only on the naïve homotopy type of our spectra, so this is a point that we will sweep under the rug for now.

limits are implicitly derived, and we denote the smash product of spectra by $\otimes$. Given a G-spectrum X, there a two obvious things we can do with it, namely take the (homotopy) colimit or limit of our functor:

$$X_{hG} = \operatorname*{colim}_{BG} X \quad \text{and} \quad X^{hG} = \lim_{BG} X\,.$$

These spectra are referred to as the **homotopy orbits** and **homotopy fixed points** of X, respectively. One can view these as spectrum level versions of group homology and group cohomology, respectively. Indeed, if M is a G-module for some finite group G, then the homotopy orbits and homotopy fixed points of the Eilenberg–Mac Lane spectrum HM recover group homology and group cohomology on homotopy groups:

$$\pi_*(HM_{hG}) \cong H_*(G; M) \quad \text{and} \quad \pi_*(HM^{hG}) \cong H^{-*}(G; M)\,.$$

As mentioned, the Tate construction is the homotopical algebra version of Tate cohomology and, as in the classical case, we can access it by defining a suitable norm map between orbits and fixed points. To this end, we follow the discussion in [Rog08, Section 5.2], and consider $G\times G$-spectra, that is, functors $BG\times BG \to \mathrm{Sp}$. Note that a G-spectrum X can always be considered as a $G \times G$-spectrum by adding an extra trivial G-action. Another important $G \times G$-spectrum is the spherical group ring

$$\mathbb{S}[G] = \Sigma^\infty_+ G$$

with its obvious left and right G-actions coming from left and right multiplication of G on itself. Note that we have a canonical colimit-limit exchange map

$$\kappa : \operatorname*{colim}_{BG} \lim_{1\times BG} (X \otimes \mathbb{S}[G]) \to \lim_{BG} \operatorname*{colim}_{BG\times 1} (X \otimes \mathbb{S}[G])\,.$$

Simply unravelling the source and the target, we see that this identifies to a map

$$\mathrm{Nm}_G : (X \otimes \mathbb{S}[G]^{h(1\times G)})_{hG} \to X^{hG}$$

which we refer to as the **norm map**. The G-spectrum

$$D_G = \mathbb{S}[G]^{h(1\times G)}$$

appearing in the source is referred to as the **dualising spectrum** of G. The **Tate construction** on the G-spectrum X can be defined as the cofiber of the norm map

$$X^{tG} = \mathrm{cofib}(\mathrm{Nm}_G : (X \otimes D_{BG})_{hG} \longrightarrow X^{hG})\,,$$

and recovers Tate cohomology on homotopy groups, in the sense that

$$\pi_*(HM^{tG}) \cong \hat{H}^{-*}(G; M)$$

whenever M is a G-module for some finite group G.

Remark 1.3.1. In general, the dualising spectrum can be quite hard to understand. However, there is a very nice description of the dualising spectrum when G is a compact Lie group, due to Klein. In [Kle01], he identifies the dualising spectrum of a compact Lie group G with the representation sphere on the adjoint representation of G:

$$D_G = \mathbb{S}^{\mathrm{Ad}(G)}\,.$$

In particular, note that when G is a finite discrete group, which is a compact Lie group of dimension 0, the dualising module is just the sphere spectrum with trivial action, which explains why the dualising spectrum does not appear in the classical situation.

The homotopy fixed point functor $(-)^{hG} : \mathrm{Sp}^{BG} \to \mathrm{Sp}$ is lax symmetric monoidal, in the same way as group cohomology can be endowed with the graded commutative cup product. In the previous section, we saw that the cup product could be extended to Tate cohomology, and the same is true in homotopical algebra: the Tate construction functor can be endowed with the structure of a lax symmetric monoidal functor in such a way that the natural transformation

$$(-)^{hG} \longrightarrow (-)^{tG}$$

is symmetric monoidal, at least when G is a compact Lie group. In fact, this lax symmetric monoidal structure is unique up to contractible choice [NS18, Theorem I.3.1, Theorem I.4.1].

So far, we have not motivated why we should be interested in the Tate construction, so let us now connect it to something quite fundamental in algebraic topology. Firstly, note that we are not restricted to working with the classifying space of a group. Indeed, we may replace BG with some general space B all the way through[14]. This provides us with a perfectly good definition of Tate cohomology of the space B. For example, if $H\mathbb{Z}$ denotes the Eilenberg–Mac Lane spectrum of the integers, let us write $\underline{H\mathbb{Z}}$ for the same spectrum viewed as trivially parametrised over B. Then it seems reasonable to define Tate cohomology of B with coefficients in $\mathbb{Z}$ as the homotopy groups

$$\hat{H}^{-*}(B;\mathbb{Z}) = \pi_*\left(\mathrm{cofib}\left(\mathrm{Nm}_B : \operatorname*{colim}_B(\underline{H\mathbb{Z}} \otimes D_B) \to \lim_B \underline{H\mathbb{Z}}\right)\right)\,.$$

One could speculate that the main reason this has not been studied in the past is that it is trivial on a very big and important class of spaces, namely finite dimensional manifolds. If $B = M$ is an n-manifold, then the dualising spectrum is fibre-wise a sphere shifted into degree $-n$, and the norm map

$$\mathrm{Nm}_M : (\underline{H\mathbb{Z}} \otimes D_M)_{hM} \to \underline{H\mathbb{Z}}^{hM}$$

[14] As is common when working with ∞-categories, we mean “space” as “Kan complex” here. Note that $\delta_!(\mathbb{S}) = \mathbb{S}[G]$ where $\delta_!$ denotes induction along the diagonal map $\delta : BG \to BG \times BG$, so the obvious replacement for the spherical group ring will be the parametrised spectrum we get when inducing up the trivial sphere spectrum along the diagonal map $\delta : B \to B \times B$.

precisely induces the map

$$H_{*+n}(M;\mathbb{Z}_\omega) \to H^{-*}(M;\mathbb{Z})$$

appearing in the statement of generalised Poincaré duality, where $\mathbb{Z}_\omega$ is the orientation module of M. In this case, the norm map is an equivalence; this is quite literally the statement of the generalised Poincaré duality. Of course, this is a very special situation, and for general B, for example for classifying spaces of groups, we cannot expect the norm map to be an equivalence. The above discussion tells us that we could think of the dualising spectrum D_B as a spectrum level analogue of an orientation module for the space B, and of the Tate construction as a measure for the failure of a generalised version of Poincaré duality to hold [Kle07].

Remark 1.3.2. The Tate construction was first introduced by Greenless in the setting of genuine equivariant stable homotopy theory [Gre87; GM95]. This relies on explicit point-set models of spectra. We have already seen one such model; the one given by Lima that we recalled in Section 1.1.3.1. The issue with this model is that, although the stable homotopy category $\mathcal{SHC}$ is symmetric monoidal via the smash product [Ada74, Section III.4], we cannot endow the category that Lima sets up with a reasonable symmetric monoidal structure. This is less than stellar; if you want to do any type of algebra involving spectra, you better have access to some sort of tensor product on them. In the 90's, there was a boom of modified symmetric monoidal models of spectra, including, but not limited to: S-modules [EKMM97], symmetric spectra [HSS00], and orthogonal spectra [MMSS01]. We focus on orthogonal spectra, as it is the most well-used when passing to the equivariant setting. Roughly, an **orthogonal spectrum** is a spectrum X where the nth space X_n is endowed with an action of the orthogonal group $O(n)$. The category of orthogonal spectra is indeed closed symmetric monoidal and can be endowed with a model structure in such a way that its homotopy category is equivalent to the stable homotopy category.

We pass to equivariant stable homotopy theory when we add a group action to the picture. We will try to stay informal in this discussion and refer the reader to [Sch18, Section 3.1] for a more thorough discussion. Roughly, an **orthogonal G-spectrum** is an orthogonal spectrum with an action of a compact Lie group G. This is a closed symmetric monoidal category via the underlying closed symmetric monoidal structure on orthogonal spectra, equipped with diagonal and conjugate G-action. The distinction between "genuine" and "naïve" equivariant homotopy theory comes in when we define the weak equivalences; the notion of a weak equivalence in genuine equivariant homotopy theory is significantly stronger than simply asking for a G-equivariant map that is a weak equivalence in the underlying model category of non-equivariant orthogonal spectra. This inevitably leads to some homotopy theoretical difficulties when one studies fixed points. There is an obvious fixed point functor that takes an orthogonal G-spectrum X to the orthogonal spectrum whose nth level is given by the set-theoretic fixed points X_n^G, but this does not necessarily preserve (genuine) weak equivalences of G-spectra. Instead, one needs to derive this functor, and we so obtain a homotopically meaningful functor $(-)^G$ referred to as the **genuine fixed points**.

In genuine stable homotopy theory, a point-set model for the homotopy fixed points is given as

$$X^{hG} = F(EG_+, X)^G \,,$$

where EG denotes a free (non-equivariantly) contractible G-space and $F(-,-)$ refers to the function objects in the closed monoidal structure on orthogonal G-spectra. Similarly, a point-set model for the Tate construction is

$$X^{tG} = \left(\widetilde{EG} \wedge F(EG_+, X)\right)^G$$

where $\widetilde{EG}$ is the mapping cone of the collapse map $EG_+ \to S^0$. In this setting, the multiplicative structure on homotopy fixed points and the Tate construction relies on the existence of G-equivariant maps

$$EG_+ \longrightarrow EG_+ \wedge EG_+ \quad \text{and} \quad \widetilde{EG} \wedge \widetilde{EG} \longrightarrow \widetilde{EG}\,.$$

Such maps exist due to obstruction theory and are unique up to homotopy.

1.3.3 The Tate spectral sequence

The main common thread through this thesis is the Tate spectral sequence. This is a spectral sequence which is supposed to compute the homotopy groups of the Tate construction on a G-spectrum[15] for some topological group G. We will sketchily refer to this spectral sequence, to be constructed in various different ways, as

$$\hat{H}^{-p}(G, \pi_q(X)) \Longrightarrow \pi_{p+q}(X^{tG})\,.$$

There are essentially three questions to consider here:

1. How do we algebraically make sense of the left hand side?
2. How do we make sure that the spectral sequence is multiplicative?
3. How do we make sure that the spectral sequence converges?

We note that none of these questions are particularly straight-forward. Regarding the first question: we do have a good algebraic understanding for Tate cohomology when G is a finite group. However, what "Tate cohomology of a compact Lie group G" should mean is less clear, for example. Multiplicativity of the Tate spectral sequence is a technical question that involves homotopical control of the maps of our potential filtrations. Finally, the third question is made extra tricky by the fact that the Tate spectral sequence is generally a whole-plane spectral sequence, so we need to take Boardman's whole-plane obstruction into account.

[15] Or more generally, a parametrised spectrum.

Tate spectral sequences for actions of finite groups are relatively well understood and can be constructed in a number of different ways. Below, we outline three approaches and point out some advantages and disadvantages with each method, with indications where the methods used for finite groups are insufficient for more general topological groups.

1.3.3.1 The Greenlees–May construction

Arguably the first construction of the Tate spectral sequence is due to Greenlees–May and works in analogy to the complete resolutions view of Tate cohomology [Gre87; GM95]. Here, we work in the context of genuine equivariant stable homotopy theory, and so take our model for the Tate construction as

$$X^{tG} = \left(\widetilde{EG} \wedge F(EG_+, X)\right)^G .$$

We construct a filtration of $\widetilde{EG}$ by using its G-CW structure, dualise this filtration by taking Spanier–Whitehead duals, and splice the two filtration together to obtain a bi-infinite filtration of $\widetilde{EG}$. In turn, this induces a bi-infinite filtration on the Tate construction which we refer to as the **Greenlees filtration**. The first page of the spectral sequences can be expressed as

$$E^1_{*,*} \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\hat{P}_*, \pi_*(X)) ,$$

where $\hat{P}_*$ is a complete resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[G]$-module. We conclude that the second page of this spectral sequence is given by the Tate cohomology groups:

$$E^2_{*,*} \cong \hat{H}^{-*}(G; \pi_*(X)) .$$

It is straight-forward to show that the Tate spectral sequence constructed in this way is conditionally convergent. The problem with this filtration is that it is not clear how to endow the resulting spectral sequence with a multiplicative structure. Indeed, you tend to run into difficult technical problems regarding the homotopies once you start mixing negative and positive indices in the Greenlees-filtration. One way of avoiding this problem is to construct a filtration on the Tate construction that does not involve dualising.

1.3.3.2 The Hesselholt–Madsen construction

The Hesselholt–Madsen construction of the Tate spectral sequence works in analogy to the Tate complex view of Tate cohomology, and so circumvents the need to dualise [HM03, Section 4]. Here, we work with the filtrations EG and $\widetilde{EG}$ coming from the G-CW structure on the two spaces, simultaneously. These filtrations induces a filtration on the Tate construction by smashing them together and totalising the obtained bigraded filtration. The first page of the Tate spectral sequence obtained in this way is given by

$$E^1_{*,*} = \left(\widetilde{P}_* \otimes \mathrm{Hom}(P_*, \pi_*(X))\right)^G .$$

Since this chain complex is quasi-isomorphic to $\mathrm{Hom}_{\mathbb{Z}[G]}(\hat{P}_*, \pi_*(X))$ (see the discussion in Section 1.3.1), we know that the second page of this spectral sequence is also given by the Tate cohomology groups:

$$E^2_{*,*} \cong \hat{H}^{-*}(G; \pi_*(X))\,.$$

The Hesselholt–Madsen construction of the Tate spectral sequence can be endowed with a multiplicative structure. To deal with homotopical issues Hesselholt and Madsen employ functorial G-CW replacements to convert the G-spectra to G-CW spectra. One can also find functorial G-CW replacements when G is a non-finite compact Lie group, but the monoidal properties of such are less clear, which is an issue when moving from the setting of finite groups to compact Lie groups. We saw before that the multiplicative structure on the Tate construction relied on the existence of G-equivariant maps $EG_+ \to EG_+ \wedge EG_+$ and $\widetilde{EG} \wedge \widetilde{EG} \to \widetilde{EG}$. For finite groups these can be chosen to be cellular, so that they preserve the chosen filtration. On the first page of the Hesselholt–Madsen Tate spectral sequence, the pairings of filtrations correspond to the maps

$$P_* \longrightarrow P_* \otimes P_* \quad \text{and} \quad \widetilde{P}_* \otimes \widetilde{P}_* \longrightarrow \widetilde{P}_*$$

which precisely induce the cup product on Tate cohomology.

1.3.3.3 The Postnikov tower construction

Given a space or a spectrum there is always a way to kill off the homotopy groups over or under a certain degree. The filtrations so obtained are referred to as the Postnikov and Whitehead towers, respectively, and it is possible to construct the Tate spectral sequence also using these constructions. We focus on the Whitehead tower construction, since this has better multiplicative properties than the Postnikov one, see for example [Dug03]. Roughly, the construction of the Tate spectral sequence constructed in this way proceeds by taking the G-equivariant Whitehead (or Postnikov) tower of a G-spectrum X and then taking the G-Tate construction on each level. This construction is also covered in [GM95]. The fundamental difficulty with this construction, when moving from finite groups to compact Lie groups, is that is not straight-forward to access the effect of the action of G on X on the level of the homotopy groups of X.

Note that the construction illustrated here is a fundamentally different view on the Tate spectral sequence; for both the Greenlees–May and the Hesselholt–Madsen filtration we start with a cellular filtration related to the group G, while in the Postnikov/Whitehead filtration construction we consider filtrations of the spectrum X. This is reminiscent to various constructions of the Atiyah–Hirzebruch spectral sequence.

1.3.4 Digression: Topological periodic homology

The author came to study the Tate construction via an interest in topological Hochschild homology, which can be viewed as a lift of Hochschild homology to the setting of homotopical algebra. Topological Hochschild homology was first introduced by Bökstedt in the unpublished manuscript [Bök85b] as a tool to study algebraic K-theory. He was motivated by the 'brave new rings' paradigm of Waldhausen. One of the main advantages of topological Hochschild homology is that it is a lot more amenable to computations than algebraic K-theory, yet still allows access to a significant portion of information on the former via trace methods. Some of the first computations on topological Hochschild homology are due to Bökstedt and coauthors; most notable are the articles [Bök85a] and [BHM93]. In the first of these articles Bökstedt computed topological Hochschild homology of the prime fields $\mathbb{F}_p$, as well as of the integers $\mathbb{Z}$. The computation of $\mathrm{THH}(\mathbb{Z})$ allowed mathematicians to stretch the boundaries for what was know about algebraic K-theory of $\mathbb{Z}$, which is in turn intimately connected to the Kummer–Vandiver conjecture in algebraic number theory.

As a cyclic object, topological Hochschild homology has the structure of a $\mathbb{T}$-spectrum, where $\mathbb{T}$ denotes the circle group. The Tate construction on topological Hochschild homology with respect to the entire $\mathbb{T}$-action is referred to as topological periodic homology, and denoted

$$\mathrm{TP}(R) = \mathrm{THH}(R)^{t\mathbb{T}}\,.$$

While this construction had been studied before, it was put in the spotlight by Hesselholt who showed that it has important connections to Hasse–Weil zeta functions [Hes18]. Work pioneered by Hesselholt and coauthors, had previously understood the importance of topological Hochschild homology, and its various refinements, in arithmetic contexts (see for example [Hes96; GH99; HM03; Hes06]). The research field is still a very active one, especially after the recent simplified reformulation of cyclotomic spectra in the ∞-categorical framework by Nikolaus–Scholze [NS18] motivated by research of Bhatt–Morrow–Scholze on integral p-adic Hodge theory [BMS18; BMS19].

Chapter 2

Summary of papers

In this chapter, we give summary of the papers found in this thesis. As the relevant background and history can be found in the previous chapter, we focus on the main results.

2.1 A multiplicative Tate spectral sequence for compact Lie group actions

The aim of this paper, which is joint with John Rognes, is to construct a multiplicative Tate spectral sequence when we are dealing with actions of compact Lie groups. Here, we work in the setting of genuine equivariant stable homotopy theory and so use equivariant orthogonal spectra as our model for G-spectra, as described in Remark 1.3.2. Given a commutative (non-equivariant) orthogonal ring spectrum R we consider the group ring

$$R[G] = R \wedge G_+ \,.$$

If the homotopy groups $R[G]_* = \pi_*(R[G])$ are flat over $R_* = \pi_*(R)$, then the group structure on G makes sure that $R[G]_*$ is a Hopf algebra over R_*. If we moreover assume that $R[G]_*$ is finitely generated projective over R_*, we show that we have access to a multiplicative G-Tate spectral sequence

$$E^2_{p,*} = \widehat{\mathrm{Ext}}^{-p}_{R[G]_*}(R_*, \pi_*(X)) \Longrightarrow \pi_{p+*}(X^{tG})$$

where the E^2-page is given by the complete Ext groups of R_* over $R[G]_*$ with coefficients in the $R[G]_*$-module $\pi_*(X)$. This spectral sequence will be strongly convergent under mild hypotheses, such as for instance in the case when the RE^∞-page vanishes and the spectrum X is bounded below. The paper can be said to consist of two parts: an algebraic and a topological one. The algebraic part consists of giving an algebraic formulation for the E^2-page of the spectral sequence described above, while the topological part consists of actual constructions of the spectral sequences from sequences of orthogonal G-spectra.

Note that the topological framework allows for Hopf algebras over very complicated rings, like $\mathbb{S}_*$, so we do not want to restrict ourselves to an oversimplified algebraic setting. Given a Hopf algebra Γ over a (possibly graded) commutative ring k, we study the Tate complex

$$T_*(M) = \widetilde{P}_* \otimes \mathrm{Hom}(P_*, M)$$

for a Γ-module M, where $P_* \to k$ is a projective resolution of k as a trivial Γ-module and $\widetilde{P}_*$ is its mapping cone. In particular, if Γ is finitely generated projective over k, then we show that the homology of the Γ-invariants of the Tate complex is isomorphic to what is referred to as complete Ext [CK97; Mis94] of Γ with coefficients in M:

$$\widehat{\operatorname{Ext}}^n_\Gamma(k, M) \cong H_{-n} \operatorname{Hom}_\Gamma(k, T_*(M))$$

The key ingredient is a result by Pareigis [Par71] which identifies the k-dual of a finitely generated Hopf algebra with the induced Γ-module

$$\operatorname{Hom}_k(\Gamma, k) \cong \operatorname{Ind}_{\mathrm{k}}^\Gamma P(\operatorname{Hom}_k(\Gamma, k)),$$

where $P(\operatorname{Hom}_k(\Gamma, k))$ is the primitives for the Γ-coaction on $\operatorname{Hom}_k(\Gamma, k)$, which is a finitely generated projective k-module of constant rank 1. In particular, this result implies that induced modules over Γ are coinduced, and vice versa.

The main motivation behind working with Tate complexes, as opposed to complete resolutions, has to do with multiplicative structures on the Tate spectral sequence. In the algebraic setting, we show that there is an associative, unital, and graded commutative pairing on complete Ext

$$\smile \colon \widehat{\operatorname{Ext}}_\Gamma(k, M) \otimes \widehat{\operatorname{Ext}}_\Gamma(k, N) \longrightarrow \operatorname{Ext}_\Gamma(k, M \otimes N)$$

which we refer to simply as the cup product. This extends the ordinary cup product on Ext, in a suitable sense.

The topological part of the paper can be said to contain two sub-parts: a more general consideration of spectral sequences coming from filtrations of orthogonal G-spectra via Cartan–Eilenberg systems, and the construction of the Tate spectral sequence, more specifically. In the more general part, we show that a pairing of sequences of orthogonal G-spectra gives rise of a pairing of the corresponding spectral sequences, via the use of Cartan–Eilenberg systems. This can be regarded as folklore, but we felt that an explicit reference for this fact was not available at the time of writing, so we decided to give a complete proof. Here, we use hands-on methods to handle homotopy theoretical issues. In particular, we use the classical mapping telescope construction, which has convenient monoidal properties, to deal with sequential homotopy colimits of spectra. As mentioned in Section 1.3.3.2, in the case of finite groups Hesselholt and Madsen [HM03, Section 4.3] instead use a functorial G-CW replacement deal with these sorts of issues in the Tate spectral sequence. There exists a functorial G-CW replacement also for compact Lie groups [Sey83], but its monoidal properties are less clear.

To construct the G-Tate spectral sequence, we use the point-set model

$$X^{tG} = \left(\widetilde{EG} \wedge F(EG_+, X)\right)^G \simeq \left((R \wedge \widetilde{EG}) \wedge_R F_R(R \wedge EG_+, X)\right)^G .$$

The idea is to start by giving the free G-space EG the simplicial skeletal filtration coming from the construction of EG using the simplicial bar construction [May72]. This induces filtrations on $F_R(R \wedge EG_+, X)$ and $R \wedge \widetilde{EG}$, which can be combined into a filtration called the Hesselholt–Madsen filtration:

$$HM_c(X) = \operatorname*{hocolim}_{a+b \leq c} \widetilde{E}_a \wedge F(E/E_{-b-1}, X).$$

As we have mentioned before, the existence of a multiplicative structure on the Hesselholt–Madsen G-Tate spectral sequence relies on the existence of filtration-preserving maps $EG_+ \to EG_+ \wedge EG_+$ and $\widetilde{EG} \wedge \widetilde{EG} \to \widetilde{EG}$. The first is known to exist, and we prove by obstruction theory that the second one exists under the assumption that $R[G]_*$ is projective over R_*. The work we have done in the more general setting of Cartan–Eilenberg systems from sequences of orthogonal G-spectra then guarantees that the Tate spectral sequence is multiplicative. However, convergence is much less clear.

To settle questions about convergence, we compare the Hesselholt–Madsen filtration to another possible filtration of the Tate construction, that we dub the Greenlees–May filtration and refer to as $GM_\star(X)$. It is not hard to show that the G-Tate spectral sequence associated to this filtration is conditionally convergent. By showing that there is a map of filtrations

$$\alpha : GM_\star(X) \longrightarrow HM_\star(X)$$

which induces an isomorphism of spectral sequences from the E^2-page and on, we can then deduce convergence results for the Hesselholt–Madsen G-Tate spectral sequence under mild conditions, such as when the spectrum X is bounded below and the derived limit term RE^∞ vanishes.

2.2 Multiplicative spectral sequences via décalage

The aim of this article is to give an clear account of the subject of multiplicative spectral sequences using the modern language of ∞-categories, and from this access highly structured results regarding the passage from filtrations to spectral sequences. I am the sole author of this paper, but it is worth noting that it builds on joint work together with Achim Krause and Thomas Nikolaus. I hope to eventually publish the results of the article in a joint paper that also includes (yet unfinished) considerations of conditional convergence from the décalage point of view.

As we explained in Section 1.2.1.3, décalage is a way to make sense of "turning the page in a spectral sequence" on the level of filtrations. Although it was initially defined quite hands-on in the context of filtered chain complexes, one can make sense of décalage in the language of the Beilinson t-structure. This allows one to generalise the construction also to spectra. Indeed, the stable ∞-category $\mathrm{Tow}(\mathrm{Sp}) = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Sp})$, whose objects we refer to as filtrations, can be equipped with a t-structure by declaring the Beilinson n-connective filtrations to be the objects in the subcategory

$$\mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei}}_{\geq n} = \{X \in \mathrm{Tow}(\mathrm{Sp}) \mid \mathrm{Gr}^q(X) \in \mathrm{Sp}_{\geq n-q} \text{ for all } n\}\,,$$

where

$$\mathrm{Gr}^q(X) = X(q)/X(q+1) = \mathrm{cofib}(X(q+1) \to X(q))\,,$$

and the cofibre is meant in the ∞-categorical sense, i.e. as a homotopy colimit. The heart of this t-structure is the abelian category of chain complexes of abelian groups. This is shown in [BMS19, Section 5], and we take care to show that this equivalence of categories is also compatible with all the multiplicative structures involved. It is an observation by Antieau that décalage can be phrased using the cover functors in the Beilinson t-structure, and that this extends the notion of décalage to the setting of filtered spectra. In the paper, we define the décalée of the filtration X as the filtration $\mathrm{Déc}(X)$ given by

$$\cdots \longrightarrow \operatorname*{colim}_q(\tau^{\mathrm{Bei}}_{\geq n+1}X)(q) \longrightarrow \operatorname*{colim}_q(\tau^{\mathrm{Bei}}_{\geq n}X)(q) \longrightarrow \operatorname*{colim}_q(\tau^{\mathrm{Bei}}_{\geq n-1}X)(q) \longrightarrow \cdots,$$

where the middle term is placed in filtration degree n. The above construction gives us a functor

$$\mathrm{Déc} : \mathrm{Tow}(\mathrm{Sp}) \longrightarrow \mathrm{Tow}(\mathrm{Sp})$$

that we refer to as décalage. If X is a complete filtration of A, then $\mathrm{Déc}(X)$ is a complete filtration of A, as well, and we can use iterated décalage to build a spectral sequence associated to X. Explicitly, we show that the assignment

$$E^r_{n,s}(X) = \pi_n(\mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)))$$

determines a spectral sequence (in homological Adams grading). Here, the differential $d^r_{n,s} : E^r_{n,s} \to E^r_{n-1,s+r}$ is induced by the connecting homomorphism in the pushout square

$$\begin{array}{ccc} \mathrm{Gr}^{(r-1)n+s+1}(\mathrm{Déc}^{r-1}(X)) & \longrightarrow & \dfrac{\mathrm{Déc}^{r-1}(X)((r-1)n+s)}{\mathrm{Déc}^{r-1}(X)((r-1)n+s+2)} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)). \end{array}$$

This is essentially trivial once we prove that the nth associated graded of the décalée of X can be expressed as the Eilenberg–Mac Lane spectrum of the nth Beilinson homotopy groups of X:

$$\mathrm{Gr}^n\,\mathrm{Déc}(X) \simeq H\pi^{\mathrm{Bei}}_n(X)[n].$$

Indeed, the isomorphism between the $r+1$th page of the associated spectral sequence and the homology of the rth page is precisely induced by this equivalence. Careful considerations shows that this equivalence is symmetric monoidal in a suitable sense, and this allows us to prove that the functor

$$E^{\star}_{*,*} : \mathrm{Tow}(\mathrm{Sp}) \longrightarrow \mathrm{SSEQ}$$

from filtrations to spectral sequences admits the structure of a map of ∞-operads. Let us elaborate on this statement. The ∞-category $\mathrm{Tow}(\mathrm{Sp})$ can be endowed with a symmetric monoidal structure via Day convolution. The category of spectral sequences is not symmetric monoidal, however. But while cannot technically speak of the tensor product of two spectral sequences, it is possible to speak of multilinear maps of such, and this makes spectral sequences into a coloured operad. Hence Tow(Sp) and SSEQ both fit into the framework of ∞-operads, and the above statement has meaning.

In an appendix, we describe how to prove the Tate spectral sequence is multiplicative, using the machinery developed in the rest of the paper. In particular, we look closer at the Tate spectral sequence for topological Hochschild homology and topological restriction homology at the prime p, and describe in what sense topological periodic homology is a version of 2-periodic crystalline cohomology.

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