Potentiality and Indeterminacy in Mathematics

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Abstract

The purpose of this article is to explore the use of modal logic and/or intuitionistic logic to explicate potentiality and incomplete or indeterminate domains in mathematics. Our primary applications are the traditional notion of potential infinity, predicativity, a version of real analysis based on Brouwerian choice sequences, and a potentialist account of the iterative hierarchy in set theory.

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Section 1 is a brief presentation of the history and philosophy behind potentialism, with a focus on mathematics. We argue that modality provides the best (or, at any rate, a very good) framework to explicate potentialism. Section 2 develops the proper modal logics for various kinds of potentiality. One key issue is the proper background logic for this. Should it be classical or intuitionistic? We argue that this distinction turns on a central philosophical thesis that the potentialist might (or might not) adopt, concerning modal propositions. Section 3 provides sketches of different applications.

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1 Potential infinity

1.1 A modal analysis

Aristotle famously rejected the notion of the actual infinite—a complete, existing entity with infinitely many members. He argued that the only sensible notion is that of potential infinity. In *Physics* 3.6 (206a27-29), he wrote:

> For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different.

As Richard Sorabji [41] (322-3) once put it, for Aristotle, "infinity is an extended finitude"(see also [24], [25]).

The attitude toward the infinite was echoed by the vast majority of mathematicians and philosophers at least until late in the nineteenth century. In 1831, for example, Gauss [10] wrote:

> I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking.

Aristotle did accept what is sometimes called *potential infinity*, against the ancient atomists (see [30]). He argued that this makes sense of the mathematics of his day.[1] Subsequent mathematicians followed this, and, indeed, made brilliant use of potential infinity. But what is potential infinity?

The notion can be motivated by considering *procedures* that can be repeated indefinitely.[2] A nice example is provided by Aristotle's claim, against the atomists, that matter is infinitely divisible. Consider a stick. However many times one has divided the stick, it is always possible to divide it again (or so it is assumed).

[1] Aristotle wrote (*Physics 207b27*):

> Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the infinite and do not use it. They postulate only that a finite straight line may be produced as far as they wish. It is possible to have divided into the same ratio as the largest quantity another magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them to have such an infinite instead, while its existence will be in the sphere of real magnitudes.

See [21] for a detailed analysis of ancient Greek mathematics on these issues.

[2] There is some controversy over whether Aristotle took procedures like these to be central to the view, or whether he was more concerned with the structure of matter and space. See, for example, [19],[24], [25], and [5].

It is natural to explicate this in a modal way.[3] This yields the following analysis of the infinite divisibility of a stick s:

$$\Box\forall x(Pxs \to \Diamond\exists y\, Pyx), \tag{1}$$

where Pxy means that x is a *proper* part of y. If, by contrast, the parts of the stick formed an actual infinity, the following would hold:

$$\forall x(Pxs \to \exists y\, Pyx). \tag{2}$$

According to Aristotle, the stick does not have, and cannot have, infinitely many parts:

$$\neg\Diamond\forall x(Pxs \to \exists y\, Pyx). \tag{3}$$

By endorsing (1) and (3), one is asserting that the divisions of the stick are *merely potentially infinite*. We thus see that a potential infinity is not the same as the possibility of an actual infinity. This contrasts with many uses of the word 'potential'; e.g. to say that someone is a potential champion is to say that possibly he or she is a champion. On the Aristotelian view, there is no totality or collection that could become infinite (whatever that might mean).

As noted above, our present concern is with mathematics. According to Aristotle, the natural numbers are merely potentially infinite. We can represent this view as the conjunction of the following theses:

$$\Box\forall m\Diamond\exists n\,\textsc{Succ}(m,n) \tag{4}$$

$$\neg\Diamond\forall m\exists n\,\textsc{Succ}(m,n), \tag{5}$$

where $\textsc{Succ}(m,n)$ states that n comes right after m. The modal language thus provides a nice way to distinguish the merely potential infinite from the actual infinite.[4]

Like David Hilbert ([18]), we are not looking to leave Cantor's paradise. Following contemporary practice, we accept the existence of actually infinite collections. We suggest, however, that there is room for potentiality in contemporary mathematics, and in the philosophy of contemporary mathematics. The modal analysis helps explicate that. It provides a framework in which actual and potential infinity can live side by side, sometimes in the very same context (see [28]). In other words, we

[3] We make use here of contemporary modal notions. We make no attempt to recapitulate what Aristotle himself says about modality.

[4] For Aristotle, there *cannot* be an actual infinity. So, here, one might say $\Box\neg\Diamond\forall m\exists n\,\textsc{Succ}(m,n)$.

envisage cases where there are actual infinities, but other “collections” or “totalities” are merely potential.[5] We describe some examples below.

1.2 Three orientations towards the infinite

It is useful to distinguish different orientations towards a given infinite totality. *Actualism* unreservedly accepts actual infinities, of a given kind, and thus finds no use for modal notions—or at least no use that is specific to the analysis of the infinity in question. Actualists maintain that the non-modal language of ordinary mathematics is already fully explicit and thus deny that we need a translation into some modal language. Furthermore, actualists accept classical logic when reasoning about the infinite (or the infinite in question).

Potentialism is the orientation that stands opposed to actualism. According to it, the objects with which mathematics is concerned—or some of the objects with which mathematics is concerned—are generated successively, and at least some of these generative processes cannot be completed. So there is an inherent potentiality to (at least some) mathematical objects.

There is room for disagreement about *which* processes can be completed. As noted from the above passages, the traditional Aristotelian form of potentialism takes a very restrictive view, insisting that at any stage, there are never more than finitely many objects, but that we always (i.e., necessarily) have the ability to go on and generate more. Recall Sorabji’s suggestion that, for Aristotle, infinity is “extended finitude”. Generalized forms of potentialism take a more relaxed attitude. Potentialism about set theory provides an extreme example. According to this view, any generative process that is indexed by a set-theoretic ordinal can be completed, but it is impossible to complete the the entire process of forming sets. The so-called “relative predicativism” lies in between the traditional Aristotelian orientation and potentialized set theory. The view accepts the natural numbers as a complete infinity, but insists that sets of natural numbers are defined in stages, and there is no stage at which all sets of natural numbers exist together, so to speak. Other sorts of relative predicativity are possible as well. Brouwerian choice sequences can also be taken to fit the mold of potentialism. See §3 below for a bit more detail on these cases.

Potentialists also differ from each other with respect to a *qualitative* matter. As characterized above, potentialism is the view some or all of the *objects* with which mathematics is concerned are successively generated and that some of these

[5]Strictly speaking, there are no sets, collections, or totalities that are potentially infinite. But it is useful to use a count noun to talk about the kinds of “things” said to be potentially infinite. We will use “collections” for this, sometimes in scare quotes.

generative processes cannot be completed. What about *the truths* of mathematics? Of course, on any form of potentialism, these are modal truths concerned with certain generative processes. But how should these modal truths be understood?

Liberal potentialists regard the modal truths as unproblematic, adopting bivalence for the modal language. Consider Goldbach's conjecture. As potentialists interpret it, the conjecture says that necessarily any even natural number that is generated can be written as a sum of two primes. Liberal potentialists maintain that this modal statement has an unproblematic truth-value—it is either true or false. Their approach to modal theorizing in mathematics is thus much like a realist approach to modal theorizing in general: there are objective truths about the relevant modal aspects of reality, and this objectivity warrants the use of some classical form of modal logic.

Strict potentialists differ from their liberal cousins by requiring, not only that every object be generated at some stage of a process, but also that every truth be "made true" at some stage. Consider, again, the Goldbach conjecture. If there are counterexamples to the conjecture, then its negation will presumably be "made true" at the stage where the first counterexample is generated. But suppose there never will be any counterexamples. Since the conjecture is concerned with *all* the natural numbers, it is hard to see how it could be "made true" without completing the generation of natural numbers. This completion would, however, violate the strict potentialists' requirement that any truth be made true at some stage of the process.

We suggest that strict potentialists should adopt a modal logic whose underlying logic is intuitionistic (or intermediate between classical and intuitionistic logic); this allows them to adopt a conception of universal generality which does not presuppose that all the instances are available, thus overcoming the problem just identified. In particular, strict potentialists should not accept every instance of the law of excluded middle in the background modal language (see [28] for more details). This dovetails with a view that Solomon Feferman and others adopt towards predicative mathematics, and it has ramifications for the articulation of predicativism and the extent of the mathematics that it captures.

1.3 The modality

Here, as elsewhere, it is often useful to invoke the contemporary heuristic of possible worlds when discussing the modality in question. Here we insist that this is *only* heuristic, as a manner-of-speaking. Our official theory is formulated in the modal language, with (one or both of) the modal operators as primitive. The modal language is rock bottom, not explained or defined in terms of anything else.[6]

The potentialist does, of course, reject the now common thesis that mathematical objects exist of necessity (if they exist at all). To invoke the heuristic, the now common thesis is that all mathematical objects exist in all worlds. The potentialist gives that up. There is no world with all of the objects in question—all natural numbers for the Aristotelian, all sets of numbers for the relative predicativist, all sets for the set-theoretic potentialist, etc.

The potentialist does, however, maintain that once a mathematical object comes into existence—by being constructed—it continues to exist, of necessity. To paraphrase Aristotle (from another context), the potentialist accepts generation, but not corruption.

What about the philosophical nature of the modality invoked in the analysis of potentiality? For the Aristotelian, it can perhaps be the ordinary metaphysical modality invoked in contemporary philosophy (or perhaps defined from that notion). The idea is that mathematical objects are generated successively, in time. At any stage—in any world—there are finitely many natural numbers, but each such world has access to another where some more numbers have been generated. Given enough time, any given natural number can be generated.[7]

Charles Parsons [32] once argued that this sort of modality does not make sense for the richer potentialisms, where the "procedure" of generating mathematical objects extends into the transfinite. Intuitively, generation takes place in time, and the richer potentialisms stretch the the notion of time too far. Of course, the potentialist is not going to presuppose a totality of ordinals (or anything else) by which to make sense of the generation.

So perhaps the non-Aristotelian potentialist should simply sever any link between metaphysical modality and the modality invoked in explicating potential infinity. Instead, one might regard the latter as an altogether distinct kind of modality, say the logico-mathematical modality of [34] or [11] (though see also [32]), or the interpretational modality of [9], [26], or [42].

Here we remain neutral on the exact interpretation of the modal operators. What

[6]If a potentialist did make use of the explication of modality in terms of possible worlds, she would, presumably, think of the collection of worlds as itself potentially infinite. So it is not clear that there is much of a gain in understanding, analysis, or the like. Here we make use of the usual possible-worlds semantics to obtain results about what is, and is not, derivable in the formal systems. We do not directly address the interesting question concerning the extent to which a potentialist can accept our results. Thanks to two referees for pressing this matter.

[7]In terms of possible worlds, the relevant modality is the one that results from restricting the accessibility associated with metaphysical modality by imposing the additional requirement that domains don't ever decrease along the accessibility relation. This restriction can be captured proof-theoretically, using the resources of plural logic (see [28], p. 188, n. 15).

matters for us are the structural features of any plausible interpretation. That is, we are concerned to develop the right modal *logic*.

2 The logic of potentiality

2.1 The modal logic

For the time being, we will be neutral on the liberal vs. strict divide and thus also on whether the non-modal part of the logic should be classical or intuitionistic. To invoke the heuristic, the idea is that a "possible world" has access to other possible worlds that contain objects that have been constructed or generated from those in the first world. From the perspective of the earlier world, the "new" objects in the second exist only potentially.

Geometry provides a good illustration, if we take seriously the constructive language in, say, Euclid's *Elements*. One world might contain a line segment, and a "later" (or accessible) world might contain a bisect of that line segment. Another later world might contain an extension of that line segment. Other sorts of constructions are arithmetic: the later world might contain more natural numbers than those of the first, say the successor of the largest natural number in the first world. Or, for a third kind of example, the later world may contain a set whose members are all in the first world.

An Aristotelian (or Gauss, etc.) assumes that every possible world is finite, in the sense that it contains only finitely many objects. This, of course, just is the rejection of the actually infinite. As noted, we make no such assumption here. Our goal is to *contrast* the actually infinite and the potentially infinite, so we need a framework where both can occur (to speak loosely). An actual infinity—or, to be precise, the possibility of an actual infinity—is realized at a possible world if it contains infinitely many objects.

As noted, we also assume that objects are not destroyed in the process of construction or generation. So, to continue the heuristic, it follows from the foregoing that the domains of the possible worlds grow (or, better, are non-decreasing) along the accessibility relation. So we assume:

$$w_1 \leq w_2 \rightarrow D(w_1) \subseteq D(w_2) \tag{6}$$

where '$w_1 \leq w_2$' says that w_2 is accessible from w_1, and for each world w, $D(w)$ is the domain of w. As is well-known, the conditional (6) entails that the converse Barcan formula is valid. That is,

$$\exists x \Diamond \phi(x) \rightarrow \Diamond \exists x \varphi(x). \tag{CBF}$$

For present purposes, we can think of a possible world as determined completely by the mathematical objects—regions, numbers, sets, etc.—it contains. In other words, we assume the converse of (6). We will talk neutrally about the extra mathematical objects existing at a world w_2 but not at an "earlier" world w_1 which accesses w_2, as having been "constructed" or "generated". This motivates the following principle:

> **Partial ordering**: The accessibility relation $\leq$ is a partial order. That is, it is reflexive, transitive, and anti-symmetric.

So the underlying logic is at least S4. So far, then, we have S4 plus (CBF).[8]

At any stage in the process of construction, we generally have a choice of which objects to generate. For some types of construction, but not all, it makes sense to require that a license to generate objects is not revoked at accessible worlds. Intuitively geometric construction is like this. For example, we might have, at some stage, two intervals that don't yet have bisections. We can choose to bisect one or the other of them, or perhaps to bisect both simultaneously. Assume we are at a world w_0 where we can choose to generate objects, in different ways, so as to arrive at either w_1 or w_2. Say at w_1 we bisect an interval i and at w_2 we bisect another interval j. It seems plausible to require that the licence to bisect i can be executed at w_2 or any later world. In other words, nothing we do can prevent us from being able to bisect the other interval.

This corresponds to a requirement that any two worlds w_1 and w_2 accessible from a common world have a common extension w_3. This is a directedness property known as *convergence* and formalized as follows:

$$\forall w_0 \forall w_1 \forall w_2 (w_0 \leq w_1 \wedge w_0 \leq w_2 \rightarrow \exists w_3 (w_1 \leq w_3 \ \wedge \ w_2 \leq w_3))$$

For constructions that have this property, then, we adopt the following principle:

> **Convergence**: The accessibility relation $\leq$ is convergent.

This principle ensures that, whenever we have a choice of mathematical objects to generate, the order in which we choose to proceed is irrelevant. Whichever object(s) we choose to generate first, the other(s) can always be generated later. Unless $\leq$ is convergent, our choice whether to extend the ontology of w_0 to that of w_1 or that of w_2 might have an enduring effect.

[8] Recall that S4 and (non-free) first-order logic entails (CBF). We can also require the accessibility relation to be well-founded, on the grounds that all mathematical construction has to start somewhere. However, nothing of substance turns on this here.

It is well known that the convergence of $\leq$ ensures the soundness of the following principle:

$$\Diamond\Box p \rightarrow \Box\Diamond p. \tag{G}$$

The modal propositional logic that results from adding this principle to a complete axiomatization of S4 is known as S4.2. As noted, not all construction principles sanction this principle. We give an example below.

2.2 The logic of potential infinity

What is the correct logic when reasoning about the potentially infinite? Informal glosses aside, the language of contemporary mathematics is strictly non-modal. We thus need a translation to serve as a bridge connecting the non-modal language in which mathematics is ordinarily formulated with the modal language in which our analysis of potentiality is developed. Suppose we adopt a translation $*$ from a non-modal language $\mathcal{L}$ to a corresponding modal language $\mathcal{L}^{\Diamond}$. The question of the right logic of potential infinity is the question of which entailment relations obtain in $\mathcal{L}$.

To determine whether $\varphi_1, \ldots, \varphi_n$ entail ψ, in the non-modal system, we need to (i) apply the translation and (ii) ask whether $\varphi_1^*, \ldots, \varphi_n^*$ entail ψ^* in the modal system. This means that the right logic of potential infinity depends on several factors. First, the logic depends on the bridge that we choose to connect the non-modal language of ordinary mathematics with the modal language in which our analysis of potential infinity is given. Second, the logic obviously depends on our modal analysis of potential infinity; in particular, on the modal logic that is used in this analysis—including the underlying logic of the modal language, whether it is classical or intuitionistic. Let us now turn to the first factor.

The heart of potentialism, as we see it, is the idea that the existential quantifier of ordinary non-modal mathematics has an implicit modal aspect. Consider the statement that a given number has a successor. For the Aristotelian, this is a proposition that each number *potentially* has a successor—that it is *possible* to generate a successor. This suggests that the right translation of $\exists$ is $\Diamond\exists$.

Since we consider both classical and intuitionistic backgrounds, we treat the universal quantifier separately. But it is understood in a dual way. When a potentialist says that a given property holds of all objects (of a certain sort), he means that the property holds of all objects (of that sort) *whenever they are generated.* This suggests that $\forall$ be translated as $\Box\forall$.

Thus understood, the quantifiers of ordinary non-modal mathematics are understood as devices for generalizing over absolutely all objects, not only the ones available at some stage, but also any that we may go on to generate. In our modal language, these generalizations are effected by the strings $\Box\forall$ and $\Diamond\exists$. Although these strings are strictly speaking composites of a modal operator and a quantifier proper, they behave logically just like quantifiers ranging over all entities at all (future) worlds. We will therefore refer to the strings as *modalized quantifiers*.

Our proposal is thus that each quantifier of the non-modal language is translated as the corresponding modalized quantifier.Each connective is translated as itself. Let us call this the *potentialist translation*, and let $\varphi^\Diamond$ represent the translation of φ.[9] We say that a formula is *fully modalized* just in case all of its quantifiers are modalized. Clearly, the potentialist translation of any non-modal formula is fully modalized.

Say that a formula φ is *stable* if the necessitations of the universal closures of the following two conditionals hold:

$$\varphi \to \Box\varphi \qquad\qquad \neg\varphi \to \Box\neg\varphi$$

Intuitively, a formula is stable just in case it never "changes its mind", in the sense that, if the formula is true (or false) of certain objects at some world, it remains true (or false) of these objects at all "later" worlds as well.

We are now ready to state two key results, which answer the question about the correct logic for those kinds of potentiality that enjoy the above convergence property. Let $\vdash$ be the relation of classical deducibility in a non-modal first-order language $\mathcal{L}$. Let $\mathcal{L}^\Diamond$ be the corresponding modal language, and let $\vdash^\Diamond$ be deducibility, in this corresponding language, by $\vdash$, S4.2, and axioms asserting the stability of all atomic predicates of $\mathcal{L}$.

Theorem 1 (Classical potentialist mirroring)**.** *For any formulas $\varphi_1, \ldots, \varphi_n$, and ψ of $\mathcal{L}$, we have:*

$$\varphi_1, \ldots, \varphi_n \vdash \psi \quad \textit{iff} \quad \varphi_1^\Diamond, \ldots, \varphi_n^\Diamond \vdash^\Diamond \psi^\Diamond.$$

(See [26] for a proof.)

The theorem has a simple moral. Suppose we are interested in logical relations between formulas in the range of the potentialist translation, in a classical (first-order) modal theory that includes S4.2 and the stability axioms. Then we may delete all the modal operators and proceed by the ordinary non-modal logic underlying $\vdash$.[10] In particular, under the stated assumptions, the modalized quantifiers $\Box\forall$ and $\Diamond\exists$

[9] This is an alternative to the more familiar Gödel translation, which translates '$\forall$' as '$\Box\forall$' (as we do), '$\exists$' as itself, and also adopts a non-homophonic translation of negation and the conditional. This translation is poorly suited to explicating potentialism. For example, the translation of $\forall m \exists n \textsc{Succ}(m,n)$ is $\Box\forall m \exists n \Box\textsc{Succ}(m,n)$, which, as discussed in Section 1.1, a potentialist would reject. For more detail, see [28].

[10] There are interesting issues concerning comprehension axioms in higher-order frameworks. See [28], §7.

behave logically just as ordinary quantifiers, except that they generalize across all (accessible) possible worlds rather than a single world. This buttresses our choice of the potentialist translation as the bridge connecting actualist and potentialist theories. We will observe, as we go along, that the stability axioms on which the mirroring theorem relies are acceptable.

It is important to note, however, that our interest won't always be limited to formulas in the range of the potentialist translation. One can often use the extra expressive resources afforded by the modal language to engage in reasoning that takes us outside of this range. The modal language allows us to look at the subject matter under a finer resolution, which can be turned on or off, according to our needs.[11]

An important upshot of the theorem is that ordinary classical first-order logic is validated via this bridge. However, this response depends on the robustness of our grasp on the modality. We noted that our *liberal* potentialist accepts classical logic when it comes to the modality. Our first mirroring theorem fits in nicely with that perspective. As noted above, however, Linnebo and Shapiro [28] argue that a stricter form of potentialism pushes in the direction of intuitionistic logic. What to do then?

The answer is given by a second mirroring theorem, which we now explain. As usual, we say that a formula φ is *decidable* in a given (intuitionistic) theory if the universal closure of $\varphi \vee \neg\varphi$ is deducible in that theory. Let $\vdash_{\text{int}}$ be the relation of intuitionistic deducibility in a first-order language $\mathcal{L}$, and let $\vdash^{\Diamond}_{\text{int}}$ be deducibility in the modal language corresponding to $\mathcal{L}$, by $\vdash_{\text{int}}$, S4.2, the stability axioms for all atomic predicates of $\mathcal{L}$, and the decidability of all atomic formulas of $\mathcal{L}$.[12]

Theorem 2 (Intuitionistic potentialist mirroring)**.** *For any formulas $\varphi_1, \ldots, \varphi_n$, and ψ of $\mathcal{L}$, we have:*

$$\varphi_1, \ldots, \varphi_n \vdash_{int} \psi \quad \textit{iff} \quad \varphi_1^{\Diamond}, \ldots, \varphi_n^{\Diamond} \vdash^{\Diamond}_{int} \psi^{\Diamond}.$$

(See [28] for a proof.)

Together, the two mirroring theorems show how our analysis of quantification over a potentially infinite domain can be separated from the question of whether the

[11] A salient example is the Aristotelian statement, above, rejecting the actual infinity of the natural numbers:

$$\neg\Diamond\forall m \exists n \textsc{Succ}(m, n) \tag{5}$$

This is not in the range of the potentialist translation, and so has no counterpart in the non-modal framework. Moreover, the formula, $\neg\Box\forall m\Diamond\exists n\textsc{Succ}(m, n)$, which is in the range of the translation, is the contradictory opposite of (4)

[12] The intuitionistic modal predicate system must be formulated with some care, since the two modal operators are not inter-definable. See [40] for the details.

appropriate logic is classical or intuitionistic—at least for those kinds of potentiality that have the convergence property (and for which the underlying logic is first-order). Hold fixed our modal analysis of potential infinity, the propositional modal logic S4.2, and the potentialist bridge. Then the appropriate logic of potential infinity depends entirely on the (first-order) logic used in the modal system. Whichever logic we plug in on the modal end—classical or intuitionistic—we also get out on the non-modal end. Since liberal potentialists see no reason to plug in anything other than *classical* first-order logic, they can reasonably regard this as the correct logic for potential infinity, for the cases in question.

3 Applications

We will now describe some applications of the framework presented above.

3.1 Aristotelian potentialism

Let us begin with *Aristotelian potentialism*, that is, the view that even the natural numbers do not form a completed "collection", only a potential one. The view has two parts. First, there is the positive thesis that necessarily, given any natural number, it is possible for there to exist a successor of it. As before, let '$\textsc{Succ}(m, n)$' express that the immediate successor of m is n. The mentioned view can then be formalized as:

$$\Box\forall m\Diamond\exists n\textsc{Succ}(m, n) \tag{4}$$

Next, there is the negative thesis that it is impossible for all of the natural numbers to exist simultaneously:

$$\neg\Diamond\forall m\exists n\textsc{Succ}(m, n) \tag{5}$$

Once again, we can answer the vexed question of the correct logic for ordinary non-modal reasoning about the natural numbers when these are understood as merely potential. Provided our view of the modality is sufficiently robust to warrant the use of classical logic combined with a modal logic at least as strong as S4.2, the mentioned kind of reasoning is governed by classical first-order logic. This is the upshot of our first mirroring theorem. The second such theorem ensures that, if only intuitionistic logic can be assumed in the modal language, then only intuitionistic logic is warranted in the ordinary non-modal language.

It is also instructive to use our framework to locate some kindred views. Geoffrey Hellman's modal structuralism [11] provides an example. The subtle details of the view don't matter for present purposes. Hellman avoids asserting the existence of infinitely many objects. Instead, he asserts the *possible* existence of a model of second-order Dedekind-Peano arithmetic. In effect, this is to assert the contradictory opposite of (5), i.e. $\Diamond \forall m \exists n \textsc{Succ}(m, n)$. In present terms, this is to assert the *possibility of an actual infinity.* This is a strictly stronger modal commitment than that of the Aristotelian potentialist, though still a weaker one than the claim famously disputed by Hilbert [18], namely that there actually exists a completed infinity of objects.

3.2 Set-theoretic potentialism

Cantor famously rejected the Aristotelian ban on actual infinities, which had been the dominant view in mathematics and philosophy up until his time. At times, he appears to endorse the diametrically opposite view that for every potential infinity, there is a corresponding actual infinity:

> ... every potential infinite, if it is to be applicable in a rigorous mathematical way, presupposes an actual infinite ([3], 410–411).

At least at times of his career, however, Cantor retained traces of the old potentialist view, only now applied to the "multiplicity" of all sets rather than the "multiplicity" of natural numbers. In a much quoted letter to Dedekind, in 1899, he wrote:

> [I]t is necessary ... to distinguish two kinds of multiplicities (by this I always mean definite multiplicities). For a multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities* ... If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into 'one thing', I call it a *consistent multiplicity* or a 'set'. ([8], 931-932)

In other words, although it is possible for all the natural numbers to "exist together", or to form a completed totality, this cannot be said about the "collection" of all sets or the "collection" all ordinals. These are "inconsistent multiplicities" whose members cannot all "coexist". In present terms, the sets and the ordinals are potential collections.

Several more recent thinkers have been inspired by these ideas of Cantor's, inchoate though they may be. There are two main traditions. One is nicely encapsulated in the following passage by Charles Parsons:

> A multiplicity of objects that exist together *can* constitute a set, but it is not necessary that they *do*. ... However, the converse does hold and is expressed by the principle that the existence of a set implies that of all its elements. (pp. 293-4)

This requires some explanation. First, there is the idea that a set exists *potentially* relative to its elements. When the elements of some would-be set exist, we have all that it takes to define or specify the set in question: it is the set of precisely *these things*. Then, there is the related idea that the elements are ontologically prior to their set. The elements can exist although the set does not—much like a floor of a building can exist without the higher floors that it supports. But a higher floor cannot exist without the lower floors that support it. Likewise, a set cannot exist without its elements, which are prior to it and on which the set is therefore ontologically dependent.

This view suggests that any objects potentially form a set. In the language of plural logic:[13]

$$\Box\forall xx\Diamond\exists y\textsc{Set}(xx, y) \tag{7}$$

However, on pain of paradox, we cannot admit a corresponding completed totality; that is, we have:

$$\neg\Diamond\forall xx\exists y\textsc{Set}(xx, y) \tag{8}$$

Notice the parallel with the two theses, (4) and (5) of Aristotelian potentialism.

This "potentialist" view of set theory is interesting, philosophically as well as technically. Potentialist set theories have been developed with (7) at their heart ([42], [26], inspired by [32]). Moreover, by applying the potentialist translation described in Section 2.2, these theories validate either classical or intuitionistic logic, depending on whether the modal logic employed is classical or intuitionistic.

A second tradition takes its departure from Ernst Zermelo's famous 1930 article [46]. Studying models of second-order ZF set theory—henceforth ZF2—which includes a standard replacement axiom, Zermelo comes to the conclusion that the distinction between sets and proper classes is only a relative one: what is a proper class in one model is merely a set from the point of view of a larger model.

> But [the set-theoretic paradoxes] are only apparent 'contradictions', and depend solely on confusing set theory itself, which is not categorically determined by its axioms, with individual models representing it. What appears as an 'ultrafinite non- or super-set' in one model is, in the succeeding model, a perfectly good, valid set with both a cardinal number

[13] See Boolos [2] for the seminal contribution and [27] for a recent survey.

> and an ordinal type, and is itself a foundation stone for the construction of a new domain. ([46], 1233)

Let us spell things out. Consider a model of ZF2 based on a domain M and a membership relation $R \subseteq M \times M$, in terms of which the membership predicate $\in$ is interpreted. The model is said to be *standard* if (i) the membership relation R is well founded, (ii) the model has a maximality property akin to the axiom of separation:

> Consider any a in M. Let X be the collection of objects that bear R to a. Then, for any subcollection $Y \subseteq X$, there must be some b in M such that Y is the collection of objects that bear R to b,

and (iii) a similar clause for replacement holds. Letting M and M^+ range over standard models, Zermelo's idea can be formalized as the following extendability principle:[14]

$$\Box \forall M \Diamond \exists M^+ (M^+ \text{ properly extends } M) \tag{EP}$$

This approach to set theory has been developed further by Putnam [34] and Hellman [11]. In particular, they show how this approach too enables us to interpret ordinary first-order set theoretic discourse. To do so, we need a translation from the language of ordinary set theory into the language that talks about possible models and their extensions. A simple example suffices to convey the idea, which is quite intuitive. Consider the claim that for every ordinal there is a greater ordinal: $\forall \alpha \exists \beta (\alpha < \beta)$. This claim is translated as:

> Necessarily, for every standard model and every object α that plays the role of an ordinal in this model, possibly there is an extended standard model containing an object β that also plays the role of an ordinal, and according to which α is smaller than β.

How much set theory does this validate? Sam Roberts [35] provides an answer by formulating a modal structuralist set theory in which a slight strengthening of Zermelo set theory is faithfully interpretable.

3.3 Predicativism

A third view in the foundations of mathematics where potentialist ideas naturally come up is predicativism. This may be surprising, given that predicativism is often

[14] Admittedly, Zermelo's language isn't consistently modal or potentialist. He does write, however, that "every categorically determined domain can also be interpreted as a set" (1232) and describes this step as "a creative advance" (1233).

seen as encapsulated in Russell's Vicious Circle principle, which instructs us that no entity can be defined in a way that quantifies over a totality to which this entity belongs, on the grounds that any such definition would be unacceptably circular. It is not immediately obvious what this non-circularity requirement has to do with potentialist ideas.

However, potentialist ideas figure centrally in other characterizations of predicativism. Some authors connect predicativism closely with the view that some totalities are inherently *potential*. Consider Solomon Feferman:[15]

> ... we can never speak sensibly (in the predicative conception) of the "totality"of all sets as a "completed totality" but only as a *potential totality* whose full content is never fully grasped but only *realized in stages*. ([12], p. 2)

The potentialist framework described above is useful for explicating these ideas.

To see how, consider predicativism relative to the natural numbers. This is the view that takes the natural numbers to be a completed infinity and then proceeds to generate sets of natural numbers in a predicative manner. We thus start with a base world containing all of the natural numbers. We now consider a system of possible worlds which add more and more sets of natural numbers. The essential constraint on this generative process is that the sets we add be given a stable definition, that is, a definition that isn't disrupted as more entities are generated and the domain thus expands.[16] To ensure this definitional stability, it suffices to restrict all quantifiers to sets available at the relevant world.

How might this restriction be effected? The crux is to observe that a little bit of "coding" enables us to use a single set of natural numbers to represent a countable collection of such sets. If X is a set of numbers and n is a number, we define the *n-section of X*, denoted X_n, as $\{x|\langle n,x\rangle \in X\}$. If X is a set-variable and φ is a formula without any occurrences of X, let $\varphi^{<X}$ be the result of restricting the set-quantifiers in φ to the sections of X. That is, we translate $\forall Y\psi(Y)$ as

$$\forall Y\forall z(Y = X_z \to \psi^{<X}(Y)),$$

where z is a new first-order variable. And $\exists Y\, \psi(Y)$ is translated in the obvious dual manner.

We contend that all of the sets that exist at any given world can be "coded up" as the sections of a single set X that exists at some other world. This means that all sets that are predicatively definable, relative to a certain world, are definable by

[15] Other examples can be found Poincaré [33], p. 463.

[16] In fact, this emphasis on stability of definitions goes back to [33].

a formula of the form $\varphi^{<X}$. The desired predicative set formation principle can thus be formulated as:

$$\Diamond\exists Y\Box\forall x(x \in Y \leftrightarrow \varphi^{<X}(x)) \tag{9}$$

where the formula φ does not contain X free, but may contain parameters. It is possible to formulate stronger principles which assert not only the possibility of generating a single, predicatively defined set but of simultaneously generating all sets that are predicatively definable relative to a certain possible world. Call this step a *predicative jump*.

Eventually, we lay down that, for any relation R which by predicatively acceptable means can be proved to be a well-order, it is possible to iterate the predicative jump along R. The exact analysis of this idea is subtle and somewhat controversial, so cannot be discussed here (see [29]).

3.4 Free choice sequenecs

L.E.J. Brouwer's approach to intuitionistic real analysis made crucial use of *free choice sequences*. Each such sequence can be thought of as generated by an ideal mathematician. At any one time, the mathematician has specified some finite initial segment of the sequence, but she always has the ability to go on and specify a larger initial segment. However, it is not in the mathematician's power to complete the specification of the entire sequence. Each choice sequence is thus a potentially infinite object: at each moment, it consists of some finite initial segment, and there is always a possibility of going on.

As realized by Saul Kripke [23] and Joan Moschovakis [31], the idea of choice sequences naturally admits of a modal explication. For instance, while there is no upper bound to how long a sequence α can be, it cannot be infinitely long:

$$\Box\forall n\Diamond\, l(\alpha) \geq n \tag{10}$$

$$\neg\Diamond\Box\forall n(l(\alpha) \geq n) \tag{11}$$

Consider a free choice sequence α and suppose a is an initial segment. Then, for any natural number n, it is possible that α should have n as its next entry:

$$\Box\forall a\forall n\Diamond\exists x(x = a^\frown n)$$

where '$a^\frown n$' is the result of appending n to the end of a.

This brings out a novel phenomenon not encountered in the forms of potentialism discussed above: the generation in question is *indeterministic*. Suppose α has length 10. While it is possible that the 11th entry should be 0, there are many other, incompatible possibilities: if the 11th entry turns out to be 0, it will always remain 0, which means that the possibility of this entry being 1—which existed when the sequence had only 10 entries—has been shut down forever.

This indeterminacy has some important consequences. Most immediately, it means that the convergence property discussed in Section 2.1 fails. And this failure has important knock-on effects. Without convergence, we lose the justification for the axiom G, $\Diamond\Box\phi \rightarrow \Box\Diamond\phi$, and the mirroring theorem is no longer available. There may, however, be other translations from the non-modal language of intuitionistic analysis into our classical modal language. A natural contender is the Gödel translation—although as explained in footnote 9, this is poorly suited to explicate potentiality. But the translation would at least have the effect of rendering the logic of choice sequences intuitionistic.

Much work still remains to be done. First, we need to provide a more complete theory of choice sequences in a classical modal language based on S4 or some related system. Second, it would be good to provide a translation from the non-modal language into the modal one that better captures potentialist ideas.

4 Conclusion

We have outlined a powerful and very general framework for analyzing a wide variety of potentialist ideas. We have made good progress applying this framework to various such ideas, although much work still remains.

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