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Importance Measures in Multistate Systems

Applications of generalisations of the Birnbaum-measure

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

Several examples of use of four importance measures for repairable multistate systems. The measures are based on the classical Birnbaum reliability importance measure, and can be characterised with respect to two features. First, they are either forward-looking or backward-looking. Second, they are either measuring criticality by the probability of criticality, or by the expected impact on the system state. We consider both periodic life cycles and more stochastic life cycles, modelled by semi-Markov processes. We look at several examples and see that the rankings of the components may differ between the measures. We find a result that shows that a component that is in parallel with the rest of the system is always with importance equal to 1 and thus most important in two of the measures. We also discover a new criterion to make two and two measures equal in pairs. Also, two generalisations of the Birnbaum structural importance measure to the case of multistate systems are proposed, where the two are based on the feature of being either forward-looking or backward-looking, and we prove that the parallel theorem still holds.

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Similarly to some of the calculations that are included in this master thesis, the completion of this project and degree has surely been a bumpy road. By finishing it and thereby adding my pebble to the stone wall of reliability theory, I need to thank the people around me who have helped me along the way.

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Larvik, October 2020
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CHAPTER 1

Introduction

Say you have a machine consisting of many components, and one day it stops functioning. Which component should you troubleshoot first?

Now, say that you have just fixed the failed component, and the machine is functioning again. Great! However, as you have recently discovered that troubleshooting is a tedious job, you want to start a preventive checking up on the components. What component should you prioritize first?

These two questions hint at the two main reasons for calculating the importance of components. The first is the troubleshooting reason, *after* system failures, to be able to analyse the system and find what component that most likely caused the failure. The second is the preventive reason, to be able to analyse and optimise a system *before* system failures.

The first approach to solve these kinds of problems statistically was done by Birnbaum in [3], with the introduction of importance measures. This is a thorough introduction to binary reliability theory, and Birnbaum here introduced both structural importance measures and reliability importance measures.

One weakness with binary reliability theory is that it describes the components and the systems as either functioning or failed. While this can be true in some cases, it is often an oversimplification of real-life situations. To let the components and the system be able to assume a range of states, from perfect functioning to complete failure, is more flexible. This is why multistate systems were introduced in the 1980s, with one of the first papers being Griffith [7], which was also the first to introduce *reliability importance vectors*. The ith element in such a vector represents the impact on the system reliability, given that the component was improved from state $i-1$ to state i. This approach can be considered a natural extension of the Birnbaum measure to the multistate case.

The methodology on multistate systems has been developed continuously since then, and for a thorough introduction to multistate reliability theory, we refer to [14].

There are several established ways of calculating importance measures, classical approaches include Birnbaum [3], Fussel and Vesely [6], Barlow and Proschan [2], and Natvig [13]. These four differ in character, but can all be described as *generic*, meaning that they depend only on the probabilistic properties of the components and the structure of the system.

A summary paper that reviews papers on importance measures is [1]. In general it is important to note that no importance measure will be universally best, irrespective of usage purpose, as they all show different aspects of the situations.

Reliability importance vectors can identify the state in a given component that is most important. However, as pointed out in [16], the most critical system component state may not always correspond to the most critical system component. Thus, in order to simplify the comparison of components, *composite* importance measures were introduced in [15]. A composite measure combines the elements in the reliability importance vector into a unified quantity. [16] presents several different composite importance measures, with the common idea that importance is quantified with respect to the impact on the system followed by improving a component's state. Moreover, system performance is evaluated relative to an external demand, and considering both constant and variable demand models.

This has been further developed into the decision oriented approach for evaluating importance measures. Here, the measures are used as tools for optimising performance and minimise cost. These models are thereby not generic, as they depend on an external function in addition to the probabilistic properties of the components and the structure of the system. This external function can be an objective function, cost function, or demand models. The approaches include [4], [19], [20] and [18].

In this thesis we will perform a closer examination of four generic and composite importance measures, introduced by Huseby in [10] and in [12], that are generalisations of the Birnbaum reliability importance measure. They are true generalisations in the way that they reduce to the Birnbaum measure when calculated on the binary case. In the multistate case, however, they are able to describe different aspects of the system under consideration.

Our measures can be categorised along two axes; *forward-looking* versus *backward-looking*, and *probability-based* versus *impact-based*.

- The forward-looking measures focus on the *next* component states, so the most important component will be the one with highest probability or impact of *changing* the system state.
- The backward-looking measures focus on the *previous* component states, so the most important component will be the one with highest probability or impact of *having changed* the system state.
- The probability-based measures calculate the probability for a component's state change to cause a system state change. The most important component is the one with the highest probability.
- The impact-based measures calculate the expected absolute difference in system state upon a component's state change. The most important component is the one with the highest expected difference, often referred to as impact.

Our main aim in this thesis is to understand these four importance measures better, and demonstrate them as a method to answer the questions raised initially in this introduction. We will try to use the parameters at hand to investigate several cases and discover results that are interesting and improves our understanding of the measures. Furthermore, we will present a few new results discovered while simulating examples. Additionally, we will take a look at Birnbaum's structural importance measure from [3], generalise it to multistate systems, and use it as a tool to counter the results of the four reliability importance measures.

In Chapter 2, we will introduce the main theory and framework concerning criticality, reliability importance, and structural importance. We will begin with the binary case and extend to the multistate case.

The main part of this thesis is in Chapter 3 and Chapter 4. In each of these chapters, we will look at two systems consisting of three components through several examples. In Chapter 3, our examples are based on deterministic life cycles, while in Chapter 4, our examples are based on stochastic life cycles. Chapter 4 also compares some of its results with some of the results from Chapter 3.

Chapter 5 is a more practical—and maybe also a bit whimsical—multistate system example analysis. Here we aim to describe, if not *the* reality, well at least *a* reality, within our multistate framework.

CHAPTER 2

Criticality and importance in binary and multistate monotone systems

In order to introduce multistate systems properly, we will look at binary state systems first. This chapter is based on and will look similar to several other articles and books within the same field, such as [10] and [12].

2.1 Binary monotone systems

We define a binary monotone system as an ordered pair (C, ϕ), where $C = 1, ..., n$ is the set of components and ϕ is the structure function. We let $\boldsymbol{X}(t) = (X_1(t), ..., X_n(t))$, with $X_i(t)$ the state variable of component i at time t, where $i \in C$. We will also use the standard notation from reliability theory, $\phi(x_i, \boldsymbol{X})$, which is defined as $\phi(x_i, \boldsymbol{X}) = \phi(X_1, \ldots, X_{i-1}, x_i, X_{i+1}, \ldots, X_n)$, and if the index i of x_i is obvious from context, we will write $\phi(u, \boldsymbol{X})$.

We define $X_i(t) = 1$ when component i is functioning at time t, and $X_i(t) = 0$ if it is not functioning. Component i is said to be *critical* if

$$\phi(0_i, \boldsymbol{X}(t)) = 0, \text{and } \phi(1_i, \boldsymbol{X}(t)) = 1. \tag{2.1}$$

That is, if a component's functioning (failing) makes the system function (fail), it is critical.

As (C, ϕ) is a monotone system, the structure function ϕ is binary and non-decreasing. We therefore always have $0 \leq \phi(0_i, \boldsymbol{X}(t)) \leq \phi(1_i, \boldsymbol{X}(t)) \leq 1$. If we use that property on (2.1), we have that if component i is critical , then

$$\phi(0_i, \boldsymbol{X}(t)) < \phi(1_i, \boldsymbol{X}(t)). \tag{2.2}$$

Equivalently, we can say that component i is critical at time t if

$$\phi(0_i, \boldsymbol{X}(t)) \neq \phi(1_i, \boldsymbol{X}(t)). \tag{2.3}$$

We introduce notation for the *upcoming* state $X_i^+(t)$ of repairable components, in order to be able to extend the framework further later on.

$$X_i^+(t) = \begin{cases} 0 & \text{for } X_i(t) = 1 \\ 1 & \text{for } X_i(t) = 0. \end{cases}$$

So we can now rewrite (2.3) as

$$\phi(X_i^+(t), \boldsymbol{X}(t)) \neq \phi(X_i(t), \boldsymbol{X}(t)). \tag{2.4}$$

Hence, a component i is critical at time t if changing the state of component i at time t also results in a system state change.

The reliability importance measure introduced by Birnbaum in [3], $I_B^{(i)}(t)$, is defined for component $i \in C$ at time t as the probability that component i is critical at time t. Using (2.4) we get that

$$I_B^{(i)}(t) = P[\phi(X_i^+(t), \boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t))]. \tag{2.5}$$

The *asymptotic* Birnbaum importance measure of component $i \in C$ is defined as:

$$I_B^{(i)} = \lim_{t\to\infty} I_B^{(i)}(t) = \lim_{t\to\infty} P[\phi(X_i^+(t), \boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t))]. \tag{2.6}$$

If we assume that the component state processes $X_1(t), \ldots, X_n(t)$ are independent and that the limiting distributions for these processes exists, we can introduce $\boldsymbol{p} = (p_1, \ldots, p_n)$ such that

$$p_i = \lim_{t\to\infty} P(X_i(t) = 1), \quad i \in C.$$

and the reliability of the system

$$h = h(\boldsymbol{p}) = \lim_{t\to\infty} P(\phi(\boldsymbol{X}(t)) = 1).$$

It is then very well-known that we have (for instance in [9]):

$$I_B^{(i)} = \frac{\partial h}{\partial p_i}(\boldsymbol{p}), \quad i \in C, \tag{2.7}$$

which in words means that the asymptotic importance of component i can be interpreted as the change rate of the asymptotic system availability, with respect to a small change in the asymptotic component availability.

2.2 Multistate monotone systems

For a thorough introduction to multistate monotone systems, we refer to [14].

We define a multistate monotone system rather similar to the binary case, as an ordered pair (C, ϕ) with $C = 1, ..., n$ as the set of components and ϕ is the structure function. We let $\boldsymbol{X}(t) = (X_1(t), ..., X_n(t))$, with $X_i(t)$ the state variable of component i at time t, where $i \in C$. So far it is all the same, but here both the components and the system may be in multiple states. If $i \in C$, we let $S_i = \{0, 1, \ldots, r_i\}$ denote the set of states for component i. Furthermore, we assume that each component i starts at its top-level state r_i and then at random points of time traverses downwards through the set until it reaches state 0. At this stage the component is replaced by a new component, and a new life cycle begins.

We also introduce for each component i a function $f_i : S_i \to \mathbb{R}$, which represents the *physical state* of the component as a function of the state. Thus, if $X_i(t) = x_i \in S_i$, then the physical state of component i at time t is $f_i(X_i(t))$.

A practical example of this is that component i is a pipeline, where the physical state of the component at a given point of time may be the capacity of the pipeline at this point of time. Being a physical property of the pipeline, this may be any arbitrary non-negative number depending on the state of the component, and the function f_i provides a convenient way of encoding this directly into the model.

Note that the functions $f_1, \ldots, f_n$ do not necessarily need to be monotone. This introduces additional flexibility to the modelling of component states within the predefined life cycle of the component. It permits performing minimal repairs on the components, which here could be maintenance or improvement of the component before it reaches its failure state. Also, in many real-life situations one may think of several possibilities of how the component states should be ordered. In particular, consider some kind of machine part or an engine that becomes more efficient after some time functioning, that is so-called burn-in cases, where a new component starts its life cycle at some intermediate state before reaching its perfect functioning state.

The structure function ϕ represents the state of the system expressed as a function of the states of the components. It is common in multistate reliability theory to assume that ϕ also assumes values in a set of non-negative integers. In this context, however, we let the structure function represent the physical state of the system. Moreover, we assume that ϕ can be written as:

$$\phi(\boldsymbol{X}(t)) = \phi(f_1(X_1(t)), \ldots, f_n(X_n(t))).$$

This assumption implies that the physical state of the system is a function of the physical states of the components. Furthermore, it seems reasonable to allow the physical state of the system to be expressed as a non-decreasing function of the physical states of the components, reflecting the physical monotonicity of the system. It should be noted that ϕ does not necessarily need to be non-decreasing in component states x_i, $i \in C$. Hence, assume that ϕ is a non-decreasing function of the physical state functions $f_1, \ldots, f_n$. The advantage with this approach is that the system state is expressed in terms of physical quantities rather than being encoded more abstractly as non-negative integers.

2.3 Importance measures based on probability of criticality

In multistate systems, we will have use for a different definition for the *next* and the *previous* state. As we defined in Chapter 2.1, $X_i^+(t)$ does still denote the upcoming state, but the definition will be, for a component i with states $S_i = \{0, 1, \ldots, r_i\}$ and with a deterministic, downward, one-step life cycle in a repairable system:

$$X_i^+(t) = \begin{cases} X_i(t) - 1 & \text{for } X_i(t) > 0 \\ r_i & \text{for } X_i(t) = 0. \end{cases} \tag{2.8}$$

And the definition for the previous state $X_i^-(t)$ is the other way around,

$$X_i^-(t) = \begin{cases} X_i(t) + 1 & \text{for } X_i(t) < r_i \\ 0 & \text{for } X_i(t) = r_i. \end{cases} \tag{2.9}$$

A more general model with stochastic transitions will be considered in Chapter 4. We now also introduce two notions of criticality. Component i is defined to be *n-critical* at time t if:

$$\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t)). \tag{2.10}$$

Hence, a component i is n-critical at time t if changing it to its next state would result in a system change. Similarly, we say that component i is *p-critical* at time t if:

$$\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^-(t), \boldsymbol{X}(t)). \tag{2.11}$$

Hence, a component i is p-critical at time t if changing it to its *previous* state would result in a system change.

Now, to the first two importance measures we will look closer at in this thesis, introduced in [10]. They are generalisations of the earlier mentioned Birnbaum measure $I_B^{(i)}$ we looked at in (2.5). We define the *n-Birnbaum measure* of importance of component i at time t, denoted $I_{NB}^{(i)}(t)$, as the probability that the component is n-critical at time t. Similarly, we define the *p-Birnbaum measure* of importance of component i at time t, denoted $I_{PB}^{(i)}(t)$, as the probability that the component is p-critical at time t. By use of equations (2.10) and (2.11), we get:

$$I_{NB}^{(i)}(t) = P[\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t))] \tag{2.12}$$

and

$$I_{PB}^{(i)}(t) = P[\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^-(t), \boldsymbol{X}(t))]. \tag{2.13}$$

Hence, $I_{NB}^{(i)}(t)$ is the probability that component i will change the system state by entering its next state, while $I_{PB}^{(i)}(t)$ is the probability that component i has changed the system in its previous change of state. In other words, we can say that $I_{NB}^{(i)}(t)$ is forward-looking and $I_{PB}^{(i)}(t)$ is backward-looking. When we refer to both these measures together, we will call them *probability-based measures*.

It is easy to verify that in the binary case $I_{NB}^{(i)}(t) = I_{PB}^{(i)}(t) = I_B^{(i)}(t)$. In the multistate case, however, we may have that $I_{NB}^{(i)}(t) \neq I_{PB}^{(i)}(t)$. Let us take a closer look at the difference between the two importance measures by rewriting the expressions (2.12) and (2.13), respectively, by conditioning on the state $u \in S_i$ of component $i \in C$.

$$\begin{aligned} I_{NB}^{(i)}(t) = &\sum_{u=1}^{r_i} P[\phi(u, \boldsymbol{X}(t)) \neq \phi(u^+, \boldsymbol{X}(t))] \cdot P[X_i(t) = u] \\ &+ P[\phi(0, \boldsymbol{X}(t)) \neq \phi(r_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = 0] \end{aligned} \tag{2.14}$$

$$\begin{aligned} I_{PB}^{(i)}(t) = &\sum_{u=0}^{r_i-1} P[\phi(u^-, \boldsymbol{X}(t)) \neq \phi(u, \boldsymbol{X}(t))] \cdot P[X_i(t) = u] \\ &+ P[\phi(0, \boldsymbol{X}(t)) \neq \phi(r_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = r_i]. \end{aligned} \tag{2.15}$$

And by changing the summation index in (2.15) comparison becomes even easier:

$$
\begin{aligned}
I_{PB}^{(i)}(t) = & \sum_{u=1}^{r_i} P[\phi(u, \boldsymbol{X}(t)) \neq \phi(u^+, \boldsymbol{X}(t))] \cdot P[X_i(t) = u^+] \\
& + P[\phi(0, \boldsymbol{X}(t)) \neq \phi(r_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = r_i].
\end{aligned} \tag{2.16}
$$

We compare (2.14) and (2.16), and see that $P[X_i(t) = u]$ in (2.14) is "replaced" by $P[X_i(t) = u^+]$ in (2.16), $u = 1, 2, \ldots, r_i$. Further, $P[X_i(t) = 0]$ in (2.14) is "replaced" by $P[X_i(t) = r_i]$ in (2.16). By these observations it is easy to see that if $P[X_i(t) = 0] = P[X_i(t) = 1] = \ldots = P[X_i(t) = r_i]$, we will also have that $I_{NB}^{(i)}(t) = I_{PB}^{(i)}(t)$. In general, the two importance measures will still be different.

2.4 Importance measures based on expected physical criticality

For a given multistate system C, ϕ, we introduce $\Delta_N^{(i)}(t)$ and $\Delta_P^{(i)}$ defined for $t > 0$ and $i \in C$ as:

$$\Delta_N^{(i)}(t) = |\phi(X_i(t), \boldsymbol{X}(t)) - \phi(X_i^+(t), \boldsymbol{X}(t))| \tag{2.17}$$

$$\Delta_P^{(i)}(t) = |\phi(X_i^-(t), \boldsymbol{X}(t)) - \phi(X_i(t), \boldsymbol{X}(t))| \tag{2.18}$$

Hence, $\Delta_N^{(i)}(t)$ denotes the absolute value of the change in system state, as a result of component i changing from its present state $X_i(t)$ to its next state $X_i^+(t)$. Similarly, $\Delta_P^{(i)}(t)$ denotes the absolute value of the change in system state, as a result of component i changing from its previous state $X_i^-(t)$ to its present state $X_i(t)$.

The *n*-Birnbaum measure* of importance of component i at time t, denoted $I_{NB}^{*(i)}$, is defined as the expected value of $\Delta_N^(i)(t)$:

$$I_{NB}^{*(i)}(t) = E|\phi(X_i(t), \boldsymbol{X}(t)) - \phi(X_i^+(t), \boldsymbol{X}(t))| \tag{2.19}$$

Similarly, the *p*-Birnbaum measure* of importance of component i at time t, denoted $I_{PB}^{*(i)}(t)$, is defined as the expected value of $\Delta_P^{(i)}(t)$

$$I_{PB}^{*(i)}(t) = E|\phi(X_i^-(t), \boldsymbol{X}(t)) - \phi(X_i(t), \boldsymbol{X}(t))|. \tag{2.20}$$

Thus, $I_{NB}^{*(i)}(t)$ is the expected physical impact on the system, given that component i enters its next state, while $I_{PB}^{*(i)}(t)$ is the expected physical impact of the most recent state change for component i. As with the probability-based measures from Chapter 2.3, we also here have one forward-looking importance measure, $I_{NB}^{*(i)}$, and one backward-looking, $I_{PB}^{*(i)}(t)$. When we refer to both these measures together, we will call them *impact-based measures*. The main difference between them and the probability-based measures, is in the way we quantify criticality. Expected physical impact can potentially capture more detailed information about the system than probability.

Note that in the binary case all the different measures are the same:

$$I_{NB}^{(i)}(t) = I_{PB}^{(i)}(t) = I_{NB}^{*(i)}(t) = I_{PB}^{*(i)}(t) = I_{B}^{(i)}(t).$$

In order to show how these measures can be computed, we expand expression (2.19) and (2.20) like we did in Chapter 2.3, by conditioning on the state $u \in S_i$ of component $i \in C$, and obtain:

$$\begin{aligned} I_{NB}^{*(i)}(t) =& \sum_{u=1}^{r_i} E|\phi(u, \boldsymbol{X}(t)) - \phi(u^+, \boldsymbol{X}(t))| \cdot P[X_i(t) = u] \\ &+ E|\phi(0, \boldsymbol{X}(t)) - \phi(r_i, \boldsymbol{X}(t))] \cdot P[X_i(t) = 0| \end{aligned} \tag{2.21}$$

$$\begin{aligned} I_{PB}^{*(i)}(t) =& \sum_{u=0}^{r_i-1} E|\phi(u^-, \boldsymbol{X}(t)) - \phi(u, \boldsymbol{X}(t))| \cdot P[X_i(t) = u] \\ &+ E|\phi(0, \boldsymbol{X}(t)) - \phi(r_i, \boldsymbol{X}(t))| \cdot P[X_i(t) = r_i]. \end{aligned} \tag{2.22}$$

We change the summation index in (2.22), and get:

$$\begin{aligned} I_{PB}^{*(i)}(t) =& \sum_{u=1}^{r_i} E|\phi(u, \boldsymbol{X}(t)) - \phi(u^+, \boldsymbol{X}(t))| \cdot P[X_i(t) = u^+] \\ &+ E|\phi(0, \boldsymbol{X}(t)) - \phi(r_i, \boldsymbol{X}(t))| \cdot P[X_i(t) = r_i]. \end{aligned} \tag{2.23}$$

Using the same arguments as the ones that follow (2.16), we have that $I_{NB}^{*(i)}(t) = I_{PB}^{*(i)}(t)$ if $P[X_i(t) = 0] = P[X_i(t) = 1] = \ldots = P[X_i(t) = r_i]$. In general, the two measures will still be different.

2.5 Structural importance

In addition to the reliability importance measure $I_B^{(i)}(t)$, see (2.5), Birnbaum also produced a structural importance measure $J_B^{(i)}$ in [3]. We will now look at this measure, as well as a make a generalisation into multistate systems, with use of our definitions of n-critical and p-critical.

Let (C, ϕ) be a binary monotone system with $C = 1, \ldots, n$, and let $i \in C$. From [3] we have the Birnbaum measure for *structural* importance of component i, denoted $J_B^{(i)}$, is defined as:

$$J_B^{(i)} = \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{X})} [\phi(1, \boldsymbol{X}) - \phi(0, \boldsymbol{X})]. \tag{2.24}$$

It is then very well-known that we have these results:

Theorem 2.5.1. *Let (C, ϕ) be a binary monotone system with $C = 1, \ldots, n$, and let $i, j \in C$.*

- *Assume that component i is in* series *with the rest of the system, while j is not. Then $J_B^{(i)} > J_B^{(j)}$.*

- *Assume that component i is in* parallel *with the rest of the system, while j is not. Then $J_B^{(i)} > J_B^{(j)}$.*

We can adapt the definition in (2.24) to the multistate case and to our forward- and backward-looking framework, by using our definitions of n-critical and p-critical in (2.10) and (2.11), respectively. Because structural importance is intuitively independent of time, we omit time t from these definitions. We will also make use of the indicator function $\mathbf{1}(y)$:

$$\mathbf{1}(y) = \begin{cases} 1 & \text{if } y = \text{True} \\ 0 & \text{if } y = \text{False} \end{cases}$$

First, we rephrase (2.24):

$$\begin{aligned} J_B^{(i)} &= \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{X})} [\phi(1, \boldsymbol{X}) - \phi(0, \boldsymbol{X})] \\ &= \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{X})} \sum_{u=0}^{1} \mathbf{1}[\phi(u, \boldsymbol{X}) \neq \phi(u^+, \boldsymbol{X})] \frac{1}{2} \\ &= \frac{1}{2^n} \sum_{(\cdot_i, \boldsymbol{X})} \sum_{u=0}^{1} \mathbf{1}[\phi(u, \boldsymbol{X}) \neq \phi((u^+, \boldsymbol{X})]. \end{aligned}$$

This measure has similarities with the $I_{NB}^{(i)}(t)$-measure in (2.14). In fact, we observe that $J_B^{(i)}$ corresponds to a binary case of $I_{NB}^{(i)}(t)$ where a) $P(X_i = x_i) = \frac{1}{2}$ and b) $P[(\cdot_i, \boldsymbol{X}) = (\cdot_i, x)] = 1/[2^{n-1}]$. Here a) means that the probability for functioning (or not) is 1/2 (1/2), and b) means that all possible combinations of the n components' states are equal in probability, i.e. $1/[2^{n-1}]$.[1]

We can now move on to the multistate case.

Let (C, ϕ) be a multistate monotone system with $C = 1, \dots, n$, and let $i \in C$. Also let $S_i = \{0, 1, \dots, r_i\}$ denote the set of states for component i. The n-Birnbaum measure for structural importance of component i, denoted $J_{NB}^{(i)}$, is defined as:

$$\begin{aligned} J_{NB}^{(i)} &= \frac{1}{\prod_{j \neq i}(r_j + 1)} \sum_{(\cdot_i, \boldsymbol{X})} \sum_{u=0}^{r_i} \mathbf{1}[\phi(u, \boldsymbol{X}) \neq \phi(u^+, \boldsymbol{X})] \frac{1}{r_i + 1} \\ &= \frac{1}{\prod_{j \in C}(r_j + 1)} \sum_{(\cdot_i, \boldsymbol{X})} \sum_{u=0}^{r_i} \mathbf{1}[\phi(u, \boldsymbol{X}) \neq \phi(u^+, \boldsymbol{X})]. \end{aligned} \tag{2.25}$$

Recall also our definitions of the next and previous state from (2.8) and (2.9), which will come to use in the innermost sum.

Likewise, the p-Birnbaum measure for structural importance of component $i \in C$ and $u \in S_i$, denoted $J_{PB}^{(i)}$, is defined as:

$$\begin{aligned} J_{PB}^{(i)} &= \frac{1}{\prod_{j \neq i}(r_j + 1)} \sum_{(\cdot_i, \boldsymbol{X})} \sum_{u=0}^{r_i} \mathbf{1}[\phi(u, \boldsymbol{X}) \neq \phi(u^-, \boldsymbol{X})] \frac{1}{r_i + 1} \\ &= \frac{1}{\prod_{j \in C}(r_j + 1)} \sum_{(\cdot_i, \boldsymbol{X})} \sum_{u=0}^{r_i} \mathbf{1}[\phi(u, \boldsymbol{X}) \neq \phi(u^-, \boldsymbol{X})]. \end{aligned} \tag{2.26}$$

[1] The same holds for $I_{PB}^{(i)}(t)$.

These two measures are true generalisations, as when measuring a binary system it reduces to the original Birnbaum structural importance measure, i.e. $J_{NB}^{(i)} = J_{PB}^{(i)} = J_{B}^{(i)}$.

Furthermore, as this clearly is true:

$$\mathbf{1}[\phi(u, \boldsymbol{X}) \neq \phi(u^+, \boldsymbol{X})] = \mathbf{1}[\phi(u^+, \boldsymbol{X}) \neq \phi(u, \boldsymbol{X})],$$

we see also that every possible pair $[u, u^+]$ will give equal values as $[u^+, u]$ inside the indicator function in (2.25). Then for all $u \in S_i$ there is a corresponding $v = u^+ \in S_i$ such that $[u^+, u] = [v, v^-]$, and we can again rephrase this equation such that:

$$\begin{aligned}
J_{NB}^{(i)} &= \frac{1}{\prod_{j \in C}(r_j + 1)} \sum_{(\cdot_i, \boldsymbol{X})} \sum_{u=0}^{r_i} \mathbf{1}[\phi(u, \boldsymbol{X}) \neq \phi(u^+, \boldsymbol{X})] \\
&= \frac{1}{\prod_{j \in C}(r_j + 1)} \sum_{(\cdot_i, \boldsymbol{X})} \sum_{u=0}^{r_i} \mathbf{1}[\phi(u^+, \boldsymbol{X}) \neq \phi(u, \boldsymbol{X})] \\
&= \frac{1}{\prod_{j \in C}(r_j + 1)} \sum_{(\cdot_i, \boldsymbol{X})} \sum_{v=0}^{r_i} \mathbf{1}[\phi(v, \boldsymbol{X}) \neq \phi(v^-, \boldsymbol{X})] \\
&= J_{PB}^{(i)}.
\end{aligned}$$

Thus, in a system where the components follow life cycles as given in (2.8) and (2.9), we have that $J_{NB}^{(i)} = J_{PB}^{(i)}$.

CHAPTER 3

Examples of importance measures for multistate systems with deterministic life cycles

3.1 System A, System B and flow networks

We will now try to use the four importance measures introduced in Chapter 2 on different multistate systems. We will focus on two rather simple systems, System A and System B, both consisting of three components. The reason for this is to be able to say something more general about how the different importance measures responds to both structure and the other parameters it will depend on, included in the following list. The two systems are shown in Figure 3.1. It should be mentioned that we are not the first to look at these two three-component systems, as they are somewhat classical within system analysis, one example is in Chapter 7.2 in [14]. However, we examine our four more recent importance measures in this chapter and the next, so the similarities stops with the systems.

We will have the following parameters to work with, in order to investigate how they influence the importance measures:

- How many states each component has, i.e. $r_i + 1$.
- The function $f_i(u)$ which gives the level component i is functioning at in state u.
- The waiting time before next state change in a component, which in our examples will be drawn from the exponential distribution with expected waiting time $\mu_u^{(i)}$, hence it is defined within each state in each component.

(a) System A. (b) System B.

Figure 3.1: The two main systems we will look at, throughout this paper.

The choice of the exponential distribution implies that this is actually a Markov process, a different choice of distribution would lead to a semi-Markov process. A waiting time distribution must return values on the positive half axis, we chose the exponential distribution as it makes the stationary distribution converge rather quickly. However, any choice of waiting time distributions would be covered by our framework. Thus, the choice of exponential distribution is not essential for this thesis.

We are modelling our systems as *directed network flow systems*, where the system state is the amount of flow that can be transported through the network. $f_i(u)$ is to be considered as the capacity of component i when in state u.

A directed network flow system consists of a set of points, called nodes, and a set of lines between the nodes, often referred to as edges - but for continuity we will refer to them as components. The direction of the system is indicated by the arrows, see Figure 3.1 for examples. We only consider *source-to-terminal* flow networks where one of the nodes is the source node, marked in our figures with "Start", while one of the nodes is the terminal node, marked in our figures with "End".

Let us shortly describe the systems in Figure 3.1, both in order to understand the terms *series* and *parallel* properly, and make this information accessible for those who can not visually see the figures. In both the systems, we have three nodes put up horizontally after each other, where the leftmost is the source node, and the rightmost is the terminal node. The difference between the systems lies in how these nodes are connected through the components.

In System A, component 1 connects the source and the middle node and both component 2 and 3 connects the middle and the terminal node, i.e. there are two paths between these nodes. We say that component 1 is in series with the rest of the system, and that component 2 and 3 are in parallel with each other.

In System B, component 1 connects the source and the terminal node, component 2 connects the source and the middle node, and component 3 connects the middle node and the terminal node. We say that component 1 is in parallel with the rest of the system, and that component 2 and 3 are in series with each other.

A minimal cut set is a minimal set of components which, upon failure, will break the connection between the endpoints of the network, and thereby make the system fail. Let K_l, where $l = 1, \ldots, m$, be the minimal cut sets of the network. By applying the max-flow min-cut theorem [5] we get that the system state is given by:

$$\phi(\boldsymbol{X}(t)) = \min_{1 \le l \le m} \sum_{i \in K_l} X_i(t). \tag{3.1}$$

In System A, we have minimal cut sets $K_1 = \{1\}$ and $K_2 = \{2, 3\}$, hence $\phi(\boldsymbol{X}(t)) = \min(X_1(t), [X_2(t) + X_3(t)])$. In System B, we have minimal cut sets $K_1 = \{1, 2\}$ and $K_2 = \{1, 3\}$, hence $\phi(\boldsymbol{X}(t)) = \min([X_1(t) + X_2(t)], [X_1(t) + X_3(t)])$.

For convenience we will choose $\mu_u^{(i)}$'s such that $\sum_{v=0}^{r_i} \mu_v^{(i)} = 10$ for each component i. This makes it easier to compute what the ratio of time component i spends in state u will be converging towards, i.e. the asymptotic probability for component i to be in state u. For convenience in calculations we will sometimes use the notation $p_{i,u}$ for the asymptotic probabilities $p_{i,u} = \lim_{t\to\infty} P(X_i(t) = u) = \mu_u^{(i)}/[\sum_{v=0}^{r_i} \mu_v^{(i)}] = \mu_u^{(i)}/10$. This is a result based on basic renewal theory, see [17].

3.2 Structural importance in our systems

Using the structural importance measure $J_{NB}^{(i)} = J_{PB}^{(i)}$ from (2.25) and (2.26) on System A, and also that in Chapter 3.4, we will work only with examples where $r_1 = 3$, $r_2 = 2$ and $r_3 = 3$, we find the structural importance measure of the components to be:

$$J_{NB}^{(1)} = 37/48 \qquad J_{NB}^{(2)} = 15/48 \qquad J_{NB}^{(3)} = 16/48.$$

Using the $J_{NB}^{(i)} = J_{PB}^{(i)}$ measure on System B, and that we in Chapter 3.5 will work only with examples where $r_1 = 3$, $r_2 = 2$ and $r_3 = 2$, we find the structural importance measure of the components to be:

$$J_{NB}^{(1)} = 36/36 = 1 \qquad J_{NB}^{(2)} = 20/36 = 5/9 \qquad J_{NB}^{(3)} = 20/36 = 5/9.$$

We now also obtain the following theorem, as an extension of Theorem 2.5.1.

Theorem 3.2.1. *Let (C, ϕ) be a multistate monotone system modelled as a flow network with $C = 1, \ldots, n$, and let $i, j \in C$. Let $u \neq u^+$ for $i \in C$ and $u \in S_i$. Assume that component i is in* parallel *with the rest of the system, while k is not. Then $J_{NB}^{(i)} = J_{PB}^{(i)} = 1 > J_{NB}^{(j)} = J_{PB}^{(j)}$.*

Proof. We have

$$J_{NB}^{(i)} = J_{PB}^{(i)} = \frac{1}{\prod_{j\in C}(r_j+1)} \sum_{(\cdot_i, \boldsymbol{X})} \sum_{u=0}^{r_i} \mathbf{1}[\phi(u, \boldsymbol{X}) \neq \phi(u^+, \boldsymbol{X})].$$

If we consider the inner sum above, we know that the indicator function here by assumption will be 1 for all $u = 0, \ldots, r_i$, as component i is part of all minimal cut sets. Hence, we have:

$$\begin{aligned} J_{NB}^{(i)} = J_{PB}^{(i)} &= \frac{1}{\prod_{j\in C}(r_j+1)} \sum_{(\cdot_i, \boldsymbol{X})} (r_i + 1) \\ &= \frac{1}{\prod_{j\in C}(r_j+1)} [\prod_{j\neq i}(r_j+1)] \cdot (r_i+1) \\ &= \frac{\prod_{j\in C}(r_j+1)}{\prod_{j\in C}(r_j+1)} = 1 \end{aligned}$$

Where the second line is a result of $(r_i + 1)$ not depending on $\boldsymbol{X}$. Furthermore, as component k is *not* in parallel with the rest of the system, there exists at least one minimal cut set K_{not} such that $k \notin K_{not}$, hence the result that $1 > J_{NB}^{(k)} = J_{PB}^{(k)}$ follows. ■

It is well-known that the classical Birnbaum reliability importance measure $I_B^{(i)}(t)$ from (2.5) at page 6 does not have similar properties as $J_B^{(i)}$ has in Theorem 2.5.1.

But what about our four new measures? In the examples that will follow in Chapter 3.4 we will see demonstrated the following findings.

Assume that (C, ϕ) is a multistate monotone system modelled as a flow network with $C = 1, \ldots, n$, where $X_i^+(t)$ is given by (2.8) and $X_i^-(t)$ is given by (2.9). Let $i, j \in C$, and assume that component i is in *series* with the rest of the system, while component j is not. Then we will see, as in the Birnbaum importance measure, that:

- $I_{NB}^{(i)}(t)$ is not necessarily larger than $I_{NB}^{(j)}(t)$
- $I_{PB}^{(i)}(t)$ is not necessarily larger than $I_{PB}^{(j)}(t)$
- $I_{NB}^{*(i)}(t)$ is not necessarily larger than $I_{NB}^{*(j)}(t)$
- $I_{PB}^{*(i)}(t)$ is not necessarily larger than $I_{PB}^{*(j)}(t)$

We use the phrase "not necessarily" in order to hint that it is still often the case. However, we will in Chapter 3.5 see demonstrated the following findings.

Assume that (C, ϕ) is a multistate monotone system modelled as a flow network with $C = 1, \ldots, n$, where $X_i^+(t)$ is given by (2.8) and $X_i^-(t)$ is given by (2.9). Let $i, j \in C$, and assume that component i is in *parallel* with the rest of the system, while component j is not. Then we will see:

- $I_{NB}^{(i)}(t) > I_{NB}^{(j)}(t)$
- $I_{PB}^{(i)}(t) > I_{PB}^{(j)}(t)$
- $I_{NB}^{*(i)}(t)$ is not necessarily larger than $I_{NB}^{*(j)}(t)$
- $I_{PB}^{*(i)}(t)$ is not necessarily larger than $I_{PB}^{*(j)}(t)$

These two first bullet points will now be formalised.

Theorem 3.2.2 (Halle's parallel theorem)**.** *Assume that (C, ϕ) is a multistate monotone system modelled as a flow network with $C = 1, \ldots, n$, where $X_i^+(t) \neq X_i(t)$ and $X_i^-(t) \neq X_i(t)$). Assume also that $f_i(X_i(t)) \neq f_i(X_i^+(t))$ and $f_i(X_i(t)) \neq f_i(X_i^-(t))$, i.e. no two states following each other have the same capacity.*

Assume that component i is in parallel with the rest of the system, while component j is not. Then:

- $I_{NB}^{(i)}(t) = 1 > I_{NB}^{(j)}(t)$
- $I_{PB}^{(i)}(t) = 1 > I_{PB}^{(j)}(t)$

Proof. We prove the first statement, and the other will follow from an equivalent line of arguments. We omit the physical state functions f_i from the proof under the assumption that $f_i(X_i(t)) = X_i(t)$, however the proof would follow the exact same line of arguments if the functions were included.

We observe that if component i is in parallel with the rest of the system, $i \in K_l$ for all $l = 1, ..., m$. We use the max-flow-min-cut theorem (3.1), and obtain:

$$\begin{aligned}
\phi(X_i(t), \boldsymbol{X}(t)) &= \min([X_i(t) + \sum_{k \in K_1, k \neq i} X_k(t)], \ldots, [X_i(t) + \sum_{k \in K_m, k \neq i} X_k(t)]) \\
&= X_i(t) + \min([\sum_{k \in K_1, k \neq i} X_k(t)], \ldots, [\sum_{k \in K_m, k \neq i} X_k(t)]) \\
\phi(X_i^+(t), \boldsymbol{X}(t)) &= \min([X_i^+(t) + \sum_{k \in K_1, k \neq i} X_k(t)], \ldots, [X_i^+(t) + \sum_{k \in K_m, k \neq i} X_k(t)] \\
&= X_i^+(t) + \min([\sum_{k \in K_1, k \neq i} X_k(t)], \ldots, [\sum_{k \in K_m, k \neq i} X_k(t)])
\end{aligned}$$

Which gives this as a result from initial assumptions:

$$\phi(X_i(t), \boldsymbol{X}(t)) - \phi(X_i^+(t), \boldsymbol{X}(t)) = X_i(t) - X_i^+(t) \neq 0$$

and we use the above to see that

$$\begin{aligned}
I_{NB}^{(i)}(t) &= P[\phi(X_i(t), \boldsymbol{X}(t)) \neq \phi(X_i^+(t), \boldsymbol{X}(t))] \\
&= 1.
\end{aligned}$$

Furthermore, as component j is *not* in parallel with the rest of the system, there exists at least one minimal cut set K_{not} such that $j \notin K_{not}$, hence the result that $1 > I_{NB}^{(j)}(t)$ follows. ■

3.3 Multicue

We will be using the software Multicue to do our simulations in this chapter. Multicue is developed by Tobias Abrahamsen and Arne Bang Huseby, with support from the Department of Mathematics at the University of Oslo[8]. It is a further development of Huseby's program Eventcue. One thing that makes Multicue different from Eventcue is, as the name hints towards, that Multicue handles *multi*state systems. In Multicue there exists functions for computing the different importance measures $I_{NB}^{(i)}(t)$, $I_{PB}^{(i)}(t)$, $I_{NB}^{*(i)}(t)$ and $I_{PB}^{*(i)}(t)$, which is what we mostly will use.

Multicue and Eventcue are programs built on *discrete event simulation*, which is further described in Chapter 3 in [11], but we quote a quick intro on this from the given source:

> In order to simulate such a system, we use an object oriented approach where the components as well as the system are represented as objects. The component objects are equipped with methods for generating failure and repair events according to their respective life- and repair time distributions. The system object determines the state of the system as a function of the component states. To keep track of the events and process them in the correct order, they are organised in a dynamic queue sorted with respect to the points of time of the events. The component processes place their upcoming events into the queue where they stay until they are processed.

i	1 ■				2 ■			3 ■			
u	0	1	2	3	0	1	2	0	1	2	3
$f_i(u)$	0	1	2	3	0	1	2	0	1	2	3
$\mu_u^{(i)}$	2	1	3	4	2	3	5	2	1	2	5

Table 3.1: Parameters used in Example 3.4.1.

In the examples found in Chapter 3.4 and 3.5, to make the graphs easier to read and compare with calculations based on asymptotic probabilities, we have taken the cumulative mean of the simulations.

Figures that will follow from simulations are all produced in Multicue[1]. A standard simulation, unless otherwise mentioned, will consist of 1000 simulations over a simulation time of 1000. This seems to be numbers that ensure convergence at a satisfying level.

On the other hand, in some cases it could be more interesting to investigate the initial phases of the importance measures. The ratio between the simulation time and the sum of expected waiting times, $\sum_{v=0}^{r_i} \mu_v^{(i)}$, would then be important. In the following examples this ratio will be $1000/10 = 100$, hence we expect to go through the life cycles in each component 100 times in each simulation, and we will mainly look at the asymptotic values. A ratio closer to 1 could tell us more about the initial phase and less about the asymptotic values.

3.4 Examples on System A

Example 3.4.1

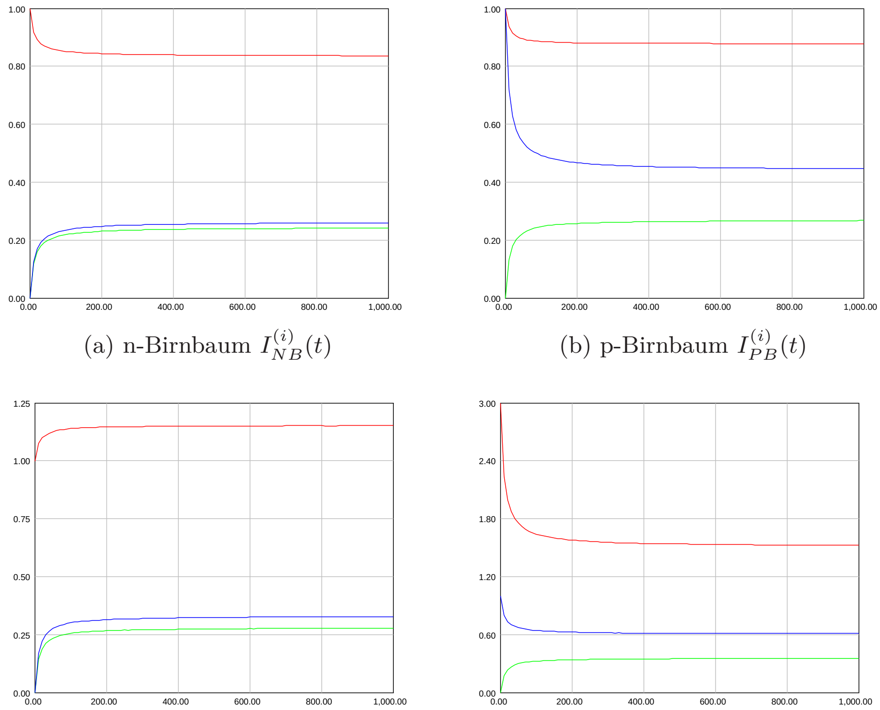
First, let us ease into the examples with a rather simple system. The main aim with this example is both to show how we establish our examples, and to demonstrate calculations of two of the measures.

The parameters are defined in Table 3.1. We simulate and obtain the plots of the importance measures shown in Figure 3.2. To explain the figure, and later figures, we need to specify that component 1 will always have the colour red, component 2 will always have the colour green and component 3 will always have the colour blue[2]. This is also emphasised in the top row of Table 3.1.

We observe that we have the same ranking between the components throughout the different importance measures, with component 1 being the most important, followed by component 3 and finally component 2, which matches their respective structural importance. This makes sense also as they are rather similar when we consider their respective waiting times $\mu_u^{(i)}$. We also observe a larger space between component 2 and 3 in the backward-looking measures. This space is because they both spend the most time, about 50% asymptotically, in their respective top state, $r_2 = 2$ and $r_3 = 3$. From these top states a look at the previous state, i.e. not functioning, would have higher probability for changing the system in component 3's case than component 2. It would also have stronger impact on the system state.

[1] The figures shown in Figure 3.1, though, is to be considered as illustrations of the systems, rather than exact replicas of how the figures actually were put up in Multicue.

[2] This is the case also later, when we look at System B.



(a) n-Birnbaum $I_{NB}^{(i)}(t)$

(b) p-Birnbaum $I_{PB}^{(i)}(t)$

(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$

(d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

Figure 3.2: The cumulative mean for simulations of the four different importance measures, measuring System A in Example 3.4.1.

As mentioned initially, we will now take a closer look at the calculations. We will calculate $I_{NB}^{(2)}$ first. Recall that we have the asymptotic probabilities where $p_{i,u} = P(X_i(t) = u) = \mu_u^{(i)}/[\sum_{v=0}^{r_i} \mu_v^{(i)}]$, thus $p_{2,0} = 0.2$, $p_{2,1} = 0.3$ and $p_{2,2} = 0.5$, and similarly calculated for the component 1 and 3. If we make use of (2.14), and that we now look at the asymptotic probabilities, thus we omit t from the equations, we have

$$
\begin{aligned}
I_{NB}^{(2)} &= \sum_{u=1}^{2} P[\phi(u, \boldsymbol{X}) \neq \phi(u^+, \boldsymbol{X})] \cdot P[X_2 = u] \\
&+ P[\phi(0, \boldsymbol{X}) \neq \phi(2, \boldsymbol{X})] \cdot P[X_2 = 0] \\
&= P[\phi(1, \boldsymbol{X}) \neq \phi(0, \boldsymbol{X})] \cdot P[X_2 = 1] \\
&+ P[\phi(2, \boldsymbol{X}) \neq \phi(1, \boldsymbol{X})] \cdot P[X_2 = 2] \\
&+ P[\phi(0, \boldsymbol{X}) \neq \phi(2, \boldsymbol{X})] \cdot P[X_2 = 0].
\end{aligned}
$$

As we see we have three parts, one for each state of component 2 and the probability of it causing a system change by moving to its next state. We put up calculation tables to cover all possible values of $\boldsymbol{X}$, i.e., in what cases of $\boldsymbol{X}$ for $u = 1, 2$, is $\phi(u, \boldsymbol{X}) \neq \phi(u^+, \boldsymbol{X})$ and in what cases of $\boldsymbol{X}$ is $\phi(0, \boldsymbol{X}) \neq \phi(2, \boldsymbol{X})$. The three parts of the calculation are likewise split into three tables, namely Table 3.2, 3.3 and 3.4. Recall that $\boldsymbol{X} = [X_1, X_2, X_3]$ and that in this case where

X_1	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
True?	0	0	0	0	1	0	0	0	1	1	0	0	1	1	1	0

Table 3.2: Calculation table for $\phi(1, \boldsymbol{X}) \neq \phi(0, \boldsymbol{X})$

X_1	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
True?	0	0	0	0	0	0	0	0	1	0	0	0	1	1	0	0

Table 3.3: Calculation table for $\phi(2, \boldsymbol{X}) \neq \phi(1, \boldsymbol{X})$

X_1	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
True?	0	0	0	0	1	0	0	0	1	1	0	0	1	1	1	0

Table 3.4: Calculation table for $\phi(0, \boldsymbol{X}) \neq \phi(2, \boldsymbol{X})$

X_2 is the component we want to investigate the effects on the system on, we must consider all combinations of X_1 and X_3. This is ensured in the two top rows of the tables. The bottom rows in the tables get value 1 if the expression given in the caption is true, and 0 if it is false. If it is false, the probability of the expression will also be 0, hence it is not included in the further calculations.

I.e., the first entry on the bottom row of Table 3.2 is 0. Here we consider[3] the case of whether $\phi(0, 1, 0) \neq \phi(0, 0, 0)$. As this case has component 1 in state 0, hence not functioning, any change in component 2 will not affect the system. Thus, $\phi(0, 1, 0) \neq \phi(0, 0, 0)$ is *not* true and the entry in the bottom row is 0.

$$
\begin{aligned}
I_{NB}^{(2)}(t) &= \{p_{1,1}p_{3,0} + p_{1,2}p_{3,0} + p_{1,2}p_{3,1} + p_{1,3}p_{3,0} + p_{1,3}p_{3,1} + p_{1,3}p_{3,2}\} \cdot p_{2,1} \\
&\quad + \{p_{1,2}p_{3,0} + p_{1,3}p_{3,0} + p_{1,3}p_{3,1}\} \cdot p_{2,2} \\
&\quad + \{p_{1,1}p_{3,0} + p_{1,2}p_{3,0} + p_{1,2}p_{3,1} + p_{1,3}p_{3,0} + p_{1,3}p_{3,1} + p_{1,3}p_{3,2}\} \cdot p_{2,0} \\
&= \{0.1 \cdot 0.2 + 0.3 \cdot 0.2 + 0.3 \cdot 0.1 + 0.4 \cdot 0.2 + 0.4 \cdot 0.1 + 0.4 \cdot 0.2\} \cdot 0.3 \\
&\quad + \{0.3 \cdot 0.2 + 0.4 \cdot 0.2 + 0.4 \cdot 0.1\} \cdot 0.5 \\
&\quad + \{0.1 \cdot 0.2 + 0.3 \cdot 0.2 + 0.3 \cdot 0.1 + 0.4 \cdot 0.2 + 0.4 \cdot 0.1 + 0.4 \cdot 0.2\} \cdot 0.2 \\
&= 0.31 \cdot 0.3 + 0.18 \cdot 0.5 + 0.31 \cdot 0.2 = \underline{0.245}.
\end{aligned}
$$

To explain the above calculations, we continue with splitting the calculations into three parts, one for each line. The first term on the first line, $p_{1,1}p_{3,0}$, is found by looking for the first 1 in the bottom row of Table 3.2, which we find located under $X_1 = 1$ and $X_3 = 0$. These two events have the probabilities $p_{1,1} = 0.1$ and $p_{3,0} = 0.2$, respectively, and the probability that they both occur at the same time is therefore $p_{1,1}p_{3,0} = 0.02$. The second term on the first line is found by looking for the second 1 in the bottom row of Table 3.2 and likewise calculated, and so on.

[3] In this example $f_i(u) = u$, so we omit these capacity functions from the system state function, for easier reading.

X_1	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
Diff	0	0	0	0	1	0	0	0	1	1	0	0	1	1	1	0

Table 3.5: Calculation table for $|\phi(1,\boldsymbol{X})-\phi(0,\boldsymbol{X})|$

X_1	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
Diff	0	0	0	0	0	0	0	0	1	0	0	0	1	1	0	0

Table 3.6: Calculation table for $|\phi(2,\boldsymbol{X})-\phi(1,\boldsymbol{X})|$

X_1	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
Diff	0	0	0	0	1	0	0	0	2	1	0	0	2	2	1	0

Table 3.7: Calculation table for $|\phi(0,\boldsymbol{X})-\phi(2,\boldsymbol{X})|$

We get the answer $I_{NB}^{(2)} = 0.245$ which resembles the end point value of the green graph in Figure 3.2a. Similar calculations can be done for $I_{NB}^{(1)}$ and $I_{NB}^{(3)}$ resulting in, respectively, 0.836 and 0.263, which also resemble their respective end point values.

Now, for the $I_{PB}^{*(2)}(t)$ we look at the *impact* of *having* changed states. If we make use of (2.22), and again use that we now look at the asymptotic probabilities, so that we omit t from the equations, we have

$$\begin{aligned}
I_{PB}^{*(2)} &= \sum_{u=0}^{1} E|\phi(u^-,\boldsymbol{X})-\phi(u,\boldsymbol{X})|\cdot P[X_2=u] \\
&\quad + E|\phi(0,\boldsymbol{X})-\phi(2,\boldsymbol{X})|\cdot P[X_2=2] \\
&= E|\phi(1,\boldsymbol{X})-\phi(0,\boldsymbol{X})|\cdot P[X_2=0] \\
&\quad + E|\phi(2,\boldsymbol{X})-\phi(1,\boldsymbol{X})|\cdot P[X_2=1] \\
&\quad + E|\phi(0,\boldsymbol{X})-\phi(2,\boldsymbol{X})|\cdot P[X_2=2].
\end{aligned}$$

We observe that the main differences with the above expression and the previous one for $I_{NB}^{(2)}$ are these two: (i) we look at *expectations* over differences, not probabilities for system states to differ, and (ii) that they are backward-looking. We put up three similar tables as earlier for simplifying calculations, Table 3.5, 3.6 and 3.7. Note that we now look at the absolute values of the differences in the system states, when changing states of component 2. This is also shown in the numbers in the table, as they in this example can be both 0, 1 or 2. With help from the calculation tables we obtain the following calculations:

$$
\begin{aligned}
&I_{PB}^{*(2)} = \\
&= \{p_{1,1}p_{3,0} + p_{1,2}p_{3,0} + p_{1,2}p_{3,1} + p_{1,3}p_{3,0} + p_{1,3}p_{3,1} + p_{1,3}p_{3,2}\} \cdot p_{2,0} \\
&+ \{p_{1,2}p_{3,0} + p_{1,3}p_{3,0} + p_{1,3}p_{3,1}\} \cdot p_{2,1} \\
&+ \{p_{1,1}p_{3,0} + 2 \cdot p_{1,2}p_{3,0} + p_{1,2}p_{3,1} + 2 \cdot p_{1,3}p_{3,0} + 2 \cdot p_{1,3}p_{3,1} + p_{1,3}p_{3,2}\} \cdot p_{2,2} \\
&= \{0.1 \cdot 0.2 + 0.3 \cdot 0.2 + 0.3 \cdot 0.1 + 0.4 \cdot 0.2 + 0.4 \cdot 0.1 + 0.4 \cdot 0.2\} \cdot 0.2 \\
&+ \{0.3 \cdot 0.2 + 0.4 \cdot 0.2 + 0.4 \cdot 0.1\} \cdot 0.3 \\
&+ \{0.1 \cdot 0.2 + 2 \cdot 0.3 \cdot 0.2 + 0.3 \cdot 0.1 + 2 \cdot 0.4 \cdot 0.2 + 2 \cdot 0.4 \cdot 0.1 + 0.4 \cdot 0.2\} \cdot 0.5 \\
&= 0.31 \cdot 0.2 + 0.18 \cdot 0.3 + 0.49 \cdot 0.5 = \underline{0.361}.
\end{aligned}
$$

The calculations are performed very similarly to when we calculated $I_{NB}^{(2)}$ above, but now, some of the elements are multiplied by 2, in the third line. This is corresponding with where we find the number 2 in the bottom line in Table 3.7. We get the answer $I_{PB}^{*(2)}(t) = 0.361$ which resembles the end point value of the green graph in Figure 3.2d. Similar calculations can be done for $I_{PB}^{*(1)}$ and $I_{PB}^{*(3)}$ resulting in, respectively, 1.513 and 0.611, which also resembles their respective end point values.

We chose to go through these two importance measures, one forward-looking and one backward-looking, as well as one probability-based and one impact-based. The two remaining importance measures $I_{PB}^{(i)}$ and $I_{NB}^{*(i)}$ can be calculated in similar fashions. In this example the numbers of components and states are small, and also the $p_{i,u}$-values are rather easy to work with, so that the calculation hopefully did not get too messy to read through. However, we maybe also demonstrated why these measures preferably are calculated through simulation, and not by hand. Even in this small, three-component system, the calculation gets rather messy.

On the other hand, this first system's parameters are not very realistic, given that it assumes a mean downtime for each component at 20%. Let us move onto something perhaps a bit more realistic, at least when considering the downtime.

To explain the motivation for the next examples, let us walk through the path from the previous one to them. Rather than an aimless walk with varying the different parameters at hand, we tried to investigate the combinatorics of them.

Say a component roughly can be categorised to be in one of three different levels of reliability, given their chosen $\mu_u^{(i)}$'s and the relation between them.

The component can be either (H) for high reliability, with low downtime and high probability to be in its top state, or (M) for medium reliability, with a more even distribution between states, or it can be (L) for low reliability, with high downtime and low probability to be in its top state.

We can put them together in an array, where the first entry corresponds to component 1, the second to component 2 and the third to component 3. Example 3.4.1 can then be considered as a case of [M,M,M]. The number of different combinations we can choose our component arrays from will then be $3^3 = 27$. Keeping the same r_i's and $f_i(u)$'s as in the previous example, we investigated the asymptotic importance in all of the different combinations.

i	1				2			3			
u	0	1	2	3	0	1	2	0	1	2	3
$f_i(u)$	0	1	2	3	0	1	2	0	1	2	3
$\mu_u^{(i)}$	0.1	0.1	0.1	9.7	9.8	0.1	0.1	0.1	0.1	0.1	9.7

Table 3.8: Parameters used in Example 3.4.2.

These more general observations were made, concerning the asymptotic values of the measures:

- In the cases where component 1 was least reliable out of the three, i.e. [L,M,M], [L,M,H], [L,H,H], [L,H,M] or [M,H,H], component 1 was always most important throughout the different importance measures.
- If all the components were equally reliable, i.e. [L,L,L], [M,M,M] or [H,H,H], the rankings were as we observed in Example 3.4.1, i.e. following their structural importance.
- If component 1 was most reliable, i.e. [H,L,L], [H,L,M], [H,M,M], [H,M,L] or [M,L,L], it was least important.
- In the cases where component 2 and 3 were about equally reliable, i.e. [*,L,L], [*,M,M], [*,H,H] where * is H, M or L, component 3 with its higher capacity would also be more important than component 2, hence also following their structural importance.

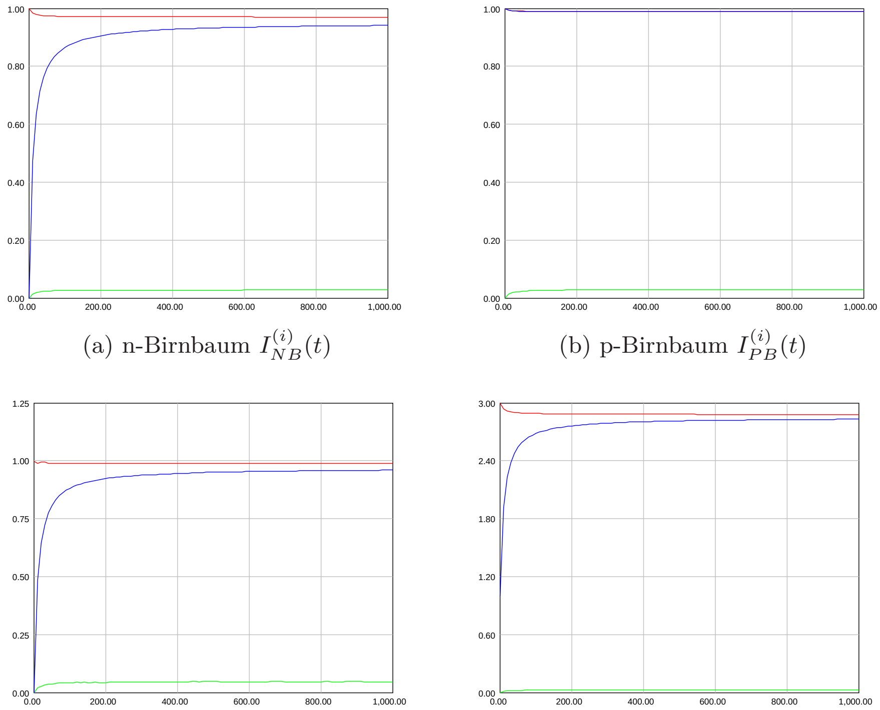
One general pattern here is that the more reliable a component is, the less important it is in the four measures we consider in this thesis. Furthermore, the higher capacity a component has, the more important.

Example 3.4.2

Let us look at a case that is not generalised above. Here component 1 and 3 have high reliability, while component 2 has low reliability, i.e. [H, L, H]. The parameters are given in Table 3.8. This is a rather extreme case regarding the choices of $\mu_u^{(i)}$, which is a choice we made to show the trends more clearly.

We understand that this case is almost to be considered as something close to a series system consisting only of component 1 and 3. As they also share the exact same waiting time distributions such that $\mu_u^{(1)} = \mu_u^{(3)}$, they should be closely linked.

We see the resulting plots in Figure 3.3, and observe what could be expected. If we consider the main system state here, where component 1 is in state 3, component 2 in state 0, and component 3 in state 3. This happens about 92% of the time, since $p_{1,3} \cdot p_{2,0} \cdot p_{3,3} = 0.97 \cdot 0.98 \cdot 0.97 \approx 0.92$. Moving forward or backwards from this system state, it is very much likely that both component 1 and 3 can change the system state, hence explaining the high values of importance for component 1 and 3 in all measures. Furthermore, component 2 cannot change the system state neither forwards nor backwards from its state 0, it will need component 3 to change state from its state 3 first in order for component 2 to be able to change the system state.



(a) n-Birnbaum $I_{NB}^{(i)}(t)$

(b) p-Birnbaum $I_{PB}^{(i)}(t)$

(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$

(d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

Figure 3.3: The four different importance measures, measuring System A in Example 3.4.2. Note that in (b) the red and the blue graph are very close.

i	1 ■				2 ■			3 ■			
u	0	1	2	3	0	1	2	0	1	2	3
$f_i(u)$	0	1	2	3	0	1	2	0	1	2	3
$\mu_u^{(i)}$	0.1	0.1	0.1	9.7	0.1	0.1	9.8	9.7	0.1	0.1	0.1

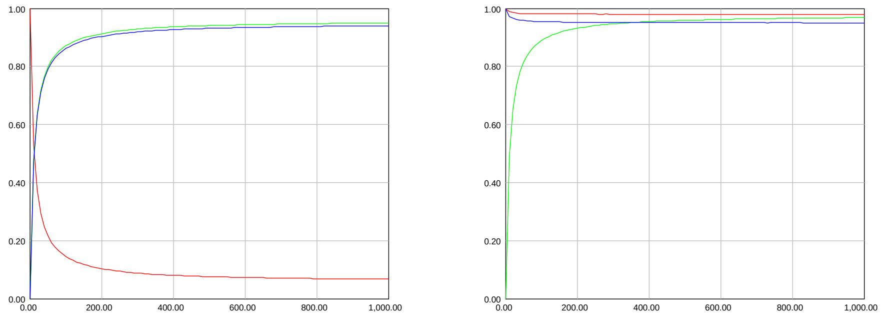
Table 3.9: Parameters used in Example 3.4.3.

Say that component 2 is functioning at state 2 while component 3 is in state 3, which is rarely the case, but still: if so, the probability for component 3 to change the system when going backwards from state 3 to state 0 is a bit higher than the probability for component 3 to change the system when going forwards from state 3 to state 2. This explains why the graphs for component 1 and 3 are closer in Figure 3.3b than in Figure 3.3a.

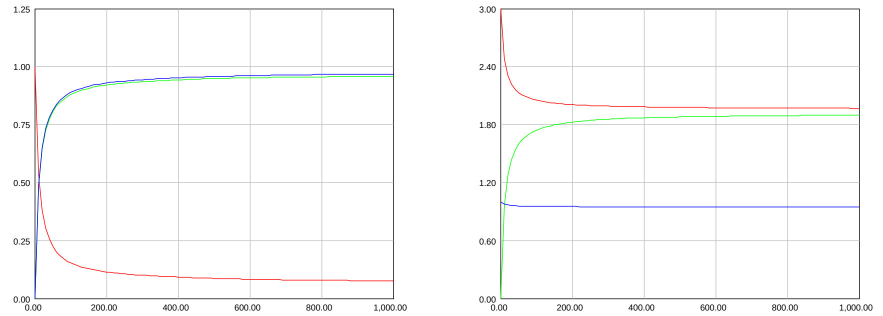
Example 3.4.3

Next, let us look at another example which at first glance looks rather similar to the previous one. Component 1 is the same as before, but now component 2 is the reliable one while component 3 is very unreliable, i.e. [H, H, L].

The parameters are given in Table 3.9, and the plots follow in Figure 3.4. Here we observe something very different from Example 3.4.2. The system is



(a) n-Birnbaum $I_{NB}^{(i)}(t)$ (b) p-Birnbaum $I_{PB}^{(i)}(t)$



(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$ (d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

Figure 3.4: The four different importance measures, measuring System A in Example 3.4.3.

still to be considered as something close to a series system, with component 3 rarely functioning, but we see that the system is more dependent on component 3 than it was on component 2 in Example 3.4.2. This is because component 3 has the same capacity as component 1 with $r_1 = r_3 = 3$, while $r_2 = 2$. This example is thus more of a "bottleneck situation" than Example 3.4.2.

Let us look at the forward-looking measures first. We observe that in both measures, component 2 and 3 are closely linked. This makes sense, because the most dominating system state is when $X_1(t) = 3$, $X_2(t) = 2$ and $X_3(t) = 0$, hence $p_{1,3} \cdot p_{2,2} \cdot p_{3,0} = 0.97 \cdot 0.98 \cdot 0.97 = 0.922$. This would result in a system state $\phi(\boldsymbol{X}(t)) = \min(3, [2 + 0]) = 2$, recall the max-flow-min-cut theorem (3.1). 92,2% of the time this is the system state, and at this point a change forwards in state either of component 2 and 3 would also change the system state.

We also observe that in this dominating system state, a state change in component 1 would not change the system state, which results in low importance values. Component 1 thus shows in the forward-looking measures cases where it is not at all most important. This shows what we discussed in Chapter 3.2. Here is a case where the structural importance is not dominant in our importance measures, as other factors "power through". Even though component 1 is in series with the rest of the system, we have $I_{NB}^{(1)} < I_{NB}^{(i)}$ and $I_{NB}^{*(1)} < I_{NB}^{*(i)}$ for $i = 2, 3$.

Even though $I_{NB}^{(2)}(t) > I_{NB}^{(3)}(t)$, barely, we also note that $I_{NB}^{*(2)}(t) < I_{NB}^{*(3)}(t)$, also barely. This is because component 3 has a higher potential capacity than component 2, i.e. $r_2 = 2 > r_3 = 3$, which makes greater impact on the system upon state change.

The backward-looking measures show something rather different. We observe the same rankings at the end points between the two measures, although the graph of $I_{PB}^{*(3)}(t)$ is at a totally different level than $I_{PB}^{(3)}(t)$, compared with the others. First; the components are all close to being equally important in Figure 3.4b for some of the same reasoning as above. The most dominating system state is still the same, i.e. $X_1(t) = 3$, $X_2(t) = 2$ and $X_3(t) = 0$, but the previous states of each of these will here be $X_1^-(t) = 0$, $X_2^-(t) = 0$ and $X_3^-(t) = 1$. This gives these respective changes in system states, recall (2.18);

$$\Delta_P^{(1)}(t) = |\phi(0, \boldsymbol{X}(t)) - \phi(3, \boldsymbol{X}(t))| = |\min(0, [2+0]) - \min(3, [2+0])| = 2$$

$$\Delta_P^{(2)}(t) = |\phi(0, \boldsymbol{X}(t)) - \phi(2, \boldsymbol{X}(t))| = |\min(3, [0+0]) - \min(3, [2+0])| = 2$$

$$\Delta_P^{(3)}(t) = |\phi(1, \boldsymbol{X}(t)) - \phi(0, \boldsymbol{X}(t))| = |\min(3, [2+1]) - \min(3, [2+0])| = 1$$

which also explains Figure 3.4d quite straightforwardly.

About the **initial phases of the plots**, we have now observed a total of 12 different plots over the cumulative mean of the importance measures, in Figure 3.2, 3.3 and 3.4. All picture System A with $r_1 = 3$, $r_2 = 2$ and $r_3 = 3$, and recall that all components always start in their top state. There are some tendencies we can see throughout these examples about their initial states in the different measures. The graphs of the probability-based measures always start at 1 or 0. So, take Figure 3.4a at page 25. Keeping the other components in their top state but changing component 1 to its next state *will* cause a system state change. Keeping the other components in their top state but changing component 2 to its next state will *not* cause a system state change. Keeping the other components in their top state but changing component 3 to its next state will *not* cause a system state change. After a short amount of time, in this example, component 3 will have traversed its way down to its failure state, and stay there for most of the time, while component 1 and 2 will stay most of their time in its top state. Hence a change in component 1 will rarely change the system state, while a change in component 2 or 3 will often change the system state. This explains the end points of the graph.

The slopes of the graphs in the initial phases tells us most about how far from their asymptotic values the graphs are at time $t = 0$. I.e. in Figure 3.4b we see that the red and blue graphs, respectively belonging to component 1 and 3, start off very close to their endpoints, while the green graph of component 2 starts off further away from its endpoint. This clearly influences their slope. However, it should be mentioned that if the ratio we introduced at the end of Chapter 3.3, considering the fraction of the simulation time over the sum of expected waiting times, were closer to 1, the slope of the graphs would also be less steep.

When we look at Figure 3.4b we see that both component 1 and 3 start in 1 on the y-axis. The previous states is based on our definitions in (2.9) for all the components are the bottom state 0. It is important to note that this is a choice made for the simulation, as the previous state at time $t = 0$ does not

i	1 ■				2 ■			3 ■			
u	0	1	2	3	0	1	2	0	1	2	3
$f_i(u)$	0	1	2	3	0	1	2	0	1	2	3
$\mu_u^{(i)}$	0.1	1.9	3	5	0.1	3.9	6	9.7	0.1	0.1	0.1

Table 3.10: Parameters used in Example 3.4.4.

actually exist. Keeping the other components in their top state but changing component 1 to its previous state will cause a system state change. Keeping the other components in their top state but changing component 2 to its previous state will *not* cause a system state change. Keeping the other components in their top state but changing component 3 to its previous state will cause a system state change. Hence the starting points at time $t = 0$ in Figure 3.4b are explained.

The plots in Figure 3.4c and 3.4d show as we know the expected physical criticality, respectively forward- and backward-looking. In the initial phase in Figure 3.4c, we see that any state change will not cause a system change higher than 1. In 3.4d, we see that component 1's graph begins at 3, as the previous state is set to be state 0, and a change for component 1 from state 3 to state 0 would change the system state from 3 to 0.

Example 3.4.4

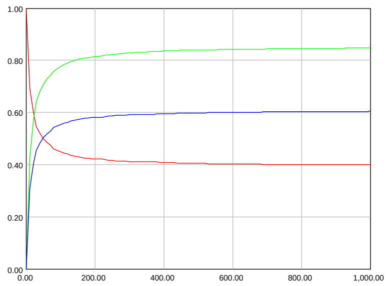
As seen in the previous example, component 2 is already close to most important for all the importance measures. Can we make it most important for them all? Let us see.

The parameters are given in Table 3.10 and the plots are found in Figure 3.5. This can be considered as a case of [M, M, L], so it has some similarities with the previous example. However, we observe that by reducing the reliability of component 1 and 2, which again will reduce the mean system state, component 2 increases in importance and is most important throughout the different measures.

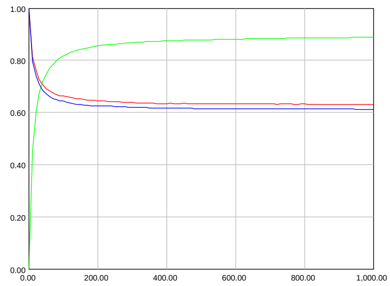
The reason for this is that for a larger portion of the time, component 2 will have the same or less capacity than component 1, and thereby be able both to *change* the system or recently *have changed* the system state. Hence, by reducing the mean capacity of component 1, component 2 with its lower capacity than component 1 has more chance of fulfilling the flow from component 1, and grows in importance.

If we return to look at the most dominating system state, where $X_1(t) = 3$, $X_2(t) = 2$ and $X_3(t) = 0$, which now only is the case asymptotically about 29% of the time, because $p_{1,3} \cdot p_{2,2} \cdot p_{3,0} = 0.5 \cdot 0.6 \cdot 0.97 = 0.291$. Here $\phi(\boldsymbol{X}(t)) = \min(X_1(t), (X_2(t) + X_3(t))) = \min(3, (2+0)) = 2$, the *next* state for the three components would be respectively $X_1^+(t) = 2$, $X_2^+(t) = 1$ and $X_3^+(t) = 3$. This again would only result in a system state change for component 2 and 3, and hence we have explained the forward-looking measures.

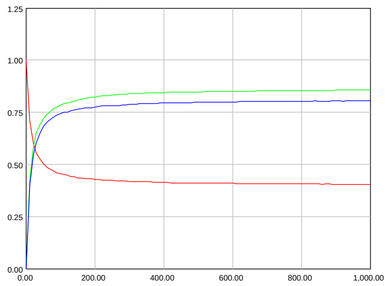
Let us consider the *previous* states, we have to go to the third most dominating system state, before we see a situation where not all the components would cause a change in system state by changing to its respective previous



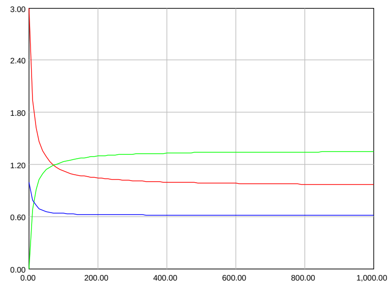
(a) n-Birnbaum $I_{NB}^{(i)}(t)$



(b) p-Birnbaum $I_{PB}^{(i)}(t)$



(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$



(d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

Figure 3.5: The four different importance measures, measuring System A in Example 3.4.4.

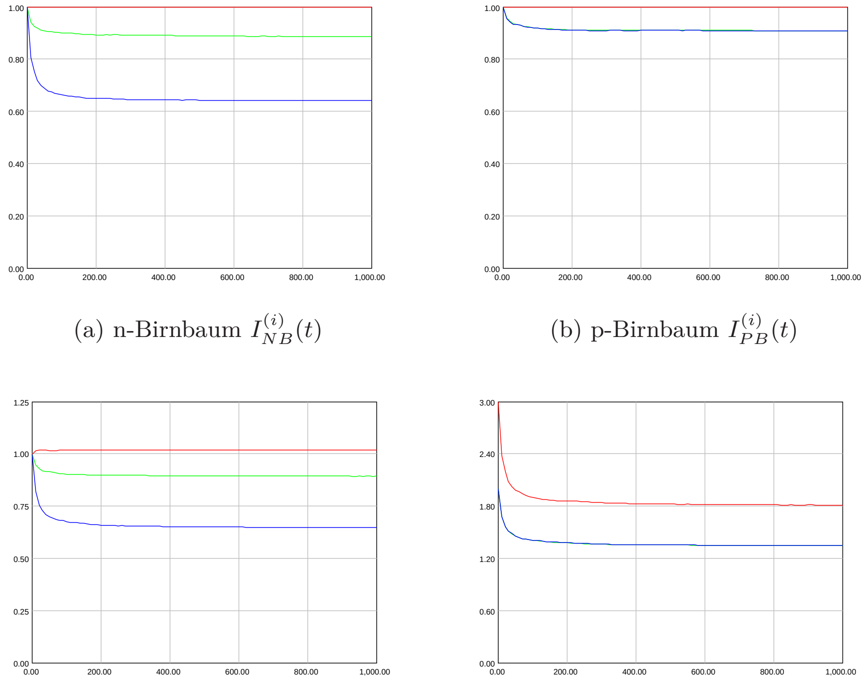
state. This third most dominating system state is when $X_1(t) = 2$, $X_2(t) = 2$ and $X_3(t) = 0$, which is the case about 17% of the time, with $\phi(\boldsymbol{X}(t)) = \min(2, (2+0)) = 2$. The respective previous states $X_1^-(t) = 3$, $X_2^-(t) = 0$ and $X_3^-(t) = 1$ would result in these respective system states $\phi(\boldsymbol{X}(t)) = 2$, $\phi(\boldsymbol{X}(t)) = 0$ and $\phi(\boldsymbol{X}(t)) = 2$. So here component 2 is the only one that would have caused system change, hence it is more important. If we consider the fourth and fifth most dominating system states as well, we find further explanation of Figure 3.5b.

This example is a pure demonstration of what we discussed in Chapter 3.2. Even though component 1 is the most structurally important component, it is here least important in the forward-looking measures, and close to least important in the backward-looking measures. Likewise with component 2, but the complete opposite.

About the **capacity functions** $f_i(u)$ given in all the tables, this far we have chosen $f_i(u) = u$ for all i, u throughout the examples. The reason for this is that when we investigated different functions $f_i(u)$, given the same structure and values of r_i as in the previous examples, we found that the main findings were the same. I.e, if we let $f_2(2) = 3$ and kept $f_i(u) = u$ in all the other cases, what we basically got was that component 2 was functioning very much like component 3, as their maximum capacity were the same. The only

i	1				2			3		
u	0	1	2	3	0	1	2	0	1	2
$f_i(u)$	0	1	2	3	0	1	2	0	1	2
$\mu_u^{(i)}$	0.1	2	3.9	4	0.1	4.4	5.5	0.1	1.9	8

Table 3.11: Parameters used in Example 3.5.1.



(a) n-Birnbaum $I_{NB}^{(i)}(t)$

(b) p-Birnbaum $I_{PB}^{(i)}(t)$

(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$

(d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

Figure 3.6: The four different importance measures, measuring System B in Example 3.5.1. Note that in (b) and (d) the green and the blue graph are very close.

difference was that there was only one state between maximum capacity and not functioning in component 2, and two states between maximum capacity and not functioning in component 3. In the importance measures we are investigating in this thesis, to change $f_i(u) = u$ to something else is an important option for the models to be as flexible as possible, but this far in Chapter 3 we found no use for demonstrating it.

3.5 Examples on System B

Example 3.5.1

Let us start with a rather straightforward example. Recall System B from Figure 3.1b at page 13. Here component 1 is in parallel with component 2 and

X_1	0	0	0	1	1	1	2	2	2	3	3	3
X_k	0	1	2	0	1	2	0	1	2	0	1	2
$\phi(1, \boldsymbol{X}) \neq \phi(0, \boldsymbol{X})$	0	1	1	0	1	1	0	1	1	0	1	1
$\phi(2, \boldsymbol{X}) \neq \phi(1, \boldsymbol{X})$	0	0	1	0	0	1	0	0	1	0	0	1
$\phi(0, \boldsymbol{X}) \neq \phi(2, \boldsymbol{X})$	0	1	1	0	1	1	0	1	1	0	1	1

Table 3.12: Calculation table used for finding both $I_{PB}^{(2)}$ and $I_{PB}^{(3)}$, with respectively $k = 3$ or $k = 2$ in row number 2.

3, where the latter two are in series with each other. We put up Table 3.11, note that now $r_3 = 2$.

The resulting plots are in Figure 3.6. What first strikes us is that $I_{NB}^{(1)}(t) = I_{PB}^{(1)}(t) = 1$ for all t. This is the result from Theorem 3.2.2, recall from Chapter 3.1 that System B has two minimal cut sets, where component 1 is member of both sets. Thus, $\phi(\boldsymbol{X}(t)) = \min([X_1(t) + X_2(t)], [X_1(t) + X_3(t)])$, so the probability that a change of state in component 1 will change the system state has to be 1.

We see that component 3 has a higher probability to be in its top state than component 2, although the probabilities to be in their bottom states are equal. This difference shows in the forward-looking measures, as we observe that component 2 is much more important in both of them, hence showing again that the less reliable, the more important.

For a larger portion of the time, asymptotically about 89% since $P(X_2 <= X_3) = (p_{2,0} + p_{2,1} + p_{2,2}) \cdot p_{3,2} + (p_{2,0} + p_{2,1}) \cdot p_{3,1} + p_{2,0} \cdot p_{3,0} = 1 \cdot 0.8 + 0.45 \cdot 0.19 + 0.01 \cdot 0.01 \approx 0.89$, component 2 will function as a bottleneck compared with component 3, and thus be able to change the system with a change to its next state. The equivalent opposite occurs only about 64% of the time.

In the backward-looking measures we see that component 2 and 3 follow each other very closely, and hence also follows their structural importance. However, this actually seems to be the case whenever $p_{2,0} = p_{3,0}$, for System B where $r_i = 2$ and $f_i(u) = u$ for $i = 2, 3$. Let us look into that. For $i = 2, 3$ we have

$$
\begin{aligned}
I_{PB}^{(i)} &= \sum_{u=1}^{2} P[\phi(u, \boldsymbol{X}) \neq \phi(u^+, \boldsymbol{X})] \cdot P[X_i = u+] \\
&+ P[\phi(0, \boldsymbol{X}) \neq \phi(2, \boldsymbol{X})] \cdot P[X_i = 2] \\
&= P[\phi(1, \boldsymbol{X}) \neq \phi(0, \boldsymbol{X})] \cdot P[X_i = 0] \\
&+ P[\phi(2, \boldsymbol{X}) \neq \phi(1, \boldsymbol{X})] \cdot P[X_i = 1] \\
&+ P[\phi(0, \boldsymbol{X}) \neq \phi(2, \boldsymbol{X})] \cdot P[X_i = 2].
\end{aligned}
$$

Further, as a result of equal structural importance and capacity, we obtain the same calculation tables for both of them, see Table 3.12. We begin with component 2.

$$\begin{aligned}
I_{PB}^{(2)} &= \{p_{1,0}p_{3,1} + p_{1,0}p_{3,2} + p_{1,1}p_{3,1} + p_{1,1}p_{3,2} \\
&\quad + p_{1,2}p_{3,1} + p_{1,2}p_{3,2} + p_{1,3}p_{3,1} + p_{1,3}p_{3,2}\} \cdot p_{2,0} \\
&+ \{p_{1,0}p_{3,2} + p_{1,1}p_{3,2} + p_{1,2}p_{3,2} + p_{1,3}p_{3,2}\} \cdot p_{2,1} \\
&+ \{p_{1,0}p_{3,1} + p_{1,0}p_{3,2} + p_{1,1}p_{3,1} + p_{1,1}p_{3,2} \\
&\quad + p_{1,2}p_{3,1} + p_{1,2}p_{3,2} + p_{1,3}p_{3,1} + p_{1,3}p_{3,2}\} \cdot p_{2,2} \\
&= \{p_{3,1} + p_{3,2}\} \cdot p_{2,0} + p_{3,2} \cdot p_{2,1} + \{p_{3,1} + p_{3,2}\} \cdot p_{2,2} \\
&= p_{3,2} + p_{3,1} \cdot \{p_{2,0} + p_{2,2}\}.
\end{aligned}$$

Here we used that $\sum_{u=0}^{r_i} p_{i,u} = 1$ for $i = 1, 2, 3$ and $u = 0, 1, \ldots, r_i$. We observe that component 1 gets irrelevant for the calculations, which is reasonable from a quick view at the system. The calculation of $I_{PB}^{(3)}$ is exactly the same, only exchanged component 2 with component 3 and vice versa, hence $I_{PB}^{(3)} = p_{2,2} + p_{2,1} \cdot \{p_{3,0} + p_{3,2}\}$. In order to show that $I_{PB}^{(2)} = I_{PB}^{(3)}$, the difference between them has to be 0. Let us look into that. In the following calculations we use that in this case $p_{2,0} = p_{3,0}$, and we have marked where we have used it with bold font. We have also used again that $\sum_{u=0}^{r_i} p_{i,u} = 1$, this is also marked with bold font.

$$\begin{aligned}
I_{PB}^{(2)} - I_{PB}^{(3)} &= p_{3,2} + p_{3,1} \cdot \{p_{2,0} + p_{2,2}\} - p_{2,2} - p_{2,1} \cdot \{p_{3,0} + p_{3,2}\} \\
&= p_{3,2} + p_{3,1} \cdot \{p_{2,0} + p_{2,2}\} - p_{2,2} - p_{2,1} \cdot \{\boldsymbol{p_{2,0}} + p_{3,2}\} \\
&= p_{3,2} + p_{3,1} \cdot \{p_{2,0} + p_{2,2}\} - p_{2,2} - p_{2,0}p_{2,1} - p_{2,1}p_{3,2} \\
&= p_{3,2} \cdot \{1 - p_{2,1}\} + p_{3,1} \cdot \{p_{2,0} + p_{2,2}\} - p_{2,2} - p_{2,0}p_{2,1} \\
&= p_{3,2} \cdot \{1 - p_{2,1}\} + p_{3,1} \cdot \{\mathbf{1} - \boldsymbol{p_{2,1}}\} - p_{2,2} - p_{2,0}p_{2,1} \\
&= \{p_{3,1} + p_{3,2}\} \cdot \{1 - p_{2,1}\} - p_{2,2} - p_{2,0}p_{2,1} \\
&= \{\mathbf{1} - \boldsymbol{p_{3,0}}\} \cdot \{1 - p_{2,1}\} - p_{2,2} - p_{2,0}p_{2,1} \\
&= \{1 - \boldsymbol{p_{2,0}}\} \cdot \{1 - p_{2,1}\} - p_{2,2} - p_{2,0}p_{2,1} \\
&= 1 - p_{2,1} - p_{2,0} + p_{2,0}p_{2,1} - p_{2,2} - p_{2,0}p_{2,1} \\
&= 0
\end{aligned}$$

So in System B, when $p_{2,0} = p_{3,0}$, $r_i = 2$, and $f_i(u) = u$ for $i = 2, 3$ and $u \in S_2 = S_3$, we have that $I_{PB}^{(2)} = I_{PB}^{(3)}$. How come?

If we flip the definition of $I_{PB}^{(2)}$, we have:

$$\begin{aligned}
I_{PB}^{(2)} &= 1 - P(\text{component } i \text{ is not critical}) \\
&= 1 - p_{3,0} - p_{2,1} \cdot p_{3,1}.
\end{aligned}$$

Likewise we have $I_{PB}^{(3)} = 1 - p_{2,0} - p_{2,1} \cdot p_{3,1}$. Hence, the criterion $p_{2,0} = p_{3,0}$ makes them equal. A small result like this opens up for several new questions. Let us briefly address some of them.

- **What about a similar case where $r_2 = r_3 > 2$?** If we put up the same definition as above to calculate this, when $r_2 = r_3 = 3$, we end up with another criterion for $I_{PB}^{(2)} = I_{PB}^{(3)}$, namely that $p_{2,2}p_{3,1} = p_{2,1}p_{3,2}$.

i	1 ■				2 ■			3 ■		
u	0	1	2	3	0	1	2	0	1	2
$f_i(u)$	0	1	2	3	0	1	4	0	1	4
$\mu_u^{(i)}$	0.1	2	3.9	4	0.1	4.4	5.5	0.1	1.9	8

Table 3.13: Parameters used in Example 3.5.2.

- **What about a similar case with *three* components in series with each other?** We have counterexamples such that with three components in series, i.e. the system had component 1 in parallel with component 2, 3 and 4, which again were in series with each other, $I_{PB}^{(2)} \neq I_{PB}^{(3)} \neq I_{PB}^{(3)}$.
- **Does this result hold for the non-asymptotic case, i.e. for all values of t?** Although $I_{PB}^{(2)}(t)$ is closely linked to $I_{PB}^{(3)}(t)$ throughout the simulation time, there is no equality. This is easily seen by examining the initial phase. Here component 2 will have expected waiting time 5.5 before state change, while component 3 will have expected waiting time 8 before state change, which quickly will change their backward-looking importance.

The criterion that $f_i(u) = u$ may in these calculations seem a bit redundant. There *are* cases where $f_i(u) \neq u$, for all $u \in S_i$ and $i = 2, 3$, that would lead to the same result. I.e., if $f_2(1) = f_3(1) = 1.5$ when $f_i(u) = u$ for $i = 2, 3$ and $u = 0, 2$, we would still have that $I_{PB}^{(2)} = I_{PB}^{(3)}$. However, if we put $f_2(1) = f_3(1) = 2.5$ and $f_i(u) = u$ for $i = 2, 3$ and $u = 0, 2$, the resulting $I_{PB}^{(2)} \neq I_{PB}^{(3)}$. So for convenience and simplicity, we chose to add $f_i(u) = u$ as a criterion.

Similar calculations as above can be performed to see that, under the same criteria, $I_{PB}^{*(2)} = I_{PB}^{*(3)}$. Note that here the criterion that $f_i(u) = u$ for $i = 2, 3$ comes more into play. This we will look at in the following example.

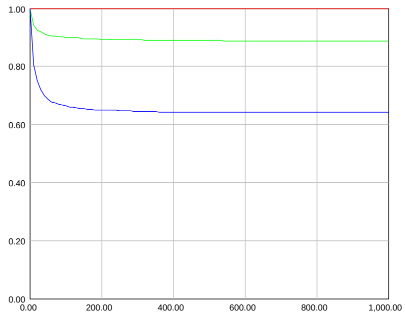
Example 3.5.2

Finally, we will make some adjustments in the $f_i(u)$ functions. In this example, see Table 3.13, we have the same $\mu_u^{(i)}$-values as in Example 3.5.1, but here we put $f_2(2) = f_3(2) = 4$, thus the combined capacity of component 2 and 3 is larger than component 1. The resulting plots are found in Figure 3.7.

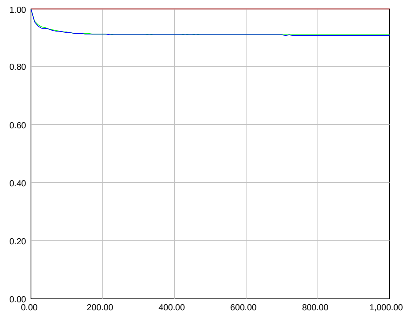
As we maybe expected, this adjustment makes no changes compared to Example 3.5.1 in the probability-based measures. We note that all component 1's importance measures is not influenced in any way from the adjustments, compared to the previous example[4]. This is also expected, as we found out in the calculations in the previous example, that component 1 was irrelevant in the calculations of component 2 and 3's importance. This is the vice versa of that, i.e. component 2 and 3 is irrelevant in the calculation of component 1's importance.

When it comes to the impact-based measures in Figure 3.7c and 3.7d, we observe that this higher capacity naturally leads to higher impact, hence

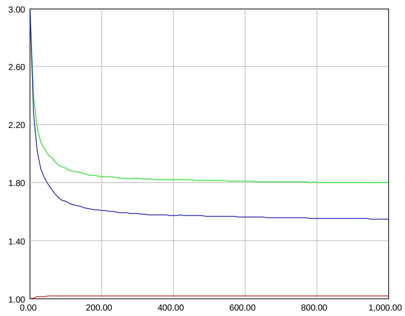
[4]Note changes in the y-axis from Example 3.5.1.



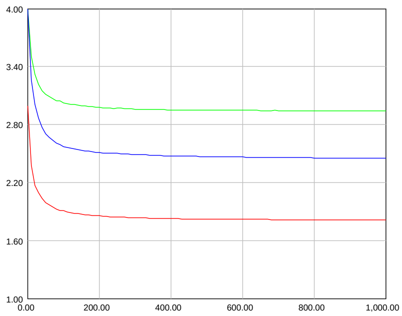
(a) n-Birnbaum $I_{NB}^{(i)}(t)$



(b) p-Birnbaum $I_{PB}^{(i)}(t)$



(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$



(d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

Figure 3.7: The four different importance measures, measuring System B in Example 3.5.2. Note that in (b) the green and the blue graph are very close.

increasing the importance. In the previous example we looked at a case where component 1 was most important throughout the different measures, due to it being part of both minimal cut sets. Here we see that even though this still is true, when component 2 and 3 together offer higher capacity than component 1, they have greater impact on the system. Hence their impact-based measures are higher. This also demonstrates how the impact-based measures and the probability-based measures can tell different stories about the system, and the components' importance.

In Figure 3.7d we see that the graphs of component 2 and component 3 are no longer equal, as we observed in Figure 3.6d. The reason for this is that within the calculations of $I_{PB}^{*(2)}$ and $I_{PB}^{*(3)}$, we end up with respectively weighting the probabilities $p_{2,2}$ and $p_{3,2}$ more, because of the higher capacity. Therefore, as $p_{2,2}$ and $p_{3,2}$ are not equal, neither are $I_{PB}^{*(2)}$ and $I_{PB}^{*(3)}$.

CHAPTER 4

Importance measures for multistate systems with non-deterministic life cycles

4.1 Definitions of importance measures for multistate systems with non-deterministic life cycles

First, we must mention that the theory in this chapter is from [10], and this chapter will also follow section 4 in that article closely.

In the previous chapters we assumed that each life cycle of a component was deterministic, with respect to the states the component transitioned through. As a result the *next* and *previous* states at a given point in time, denoted respectively $X_i^+(t)$ and $X_i^-(t)$, were both determined with probability one by the current state $X_i(t)$. Now, we relax this assumption and allow the components to have state transitions that follow a Markov chain, referred to as the *built-in Markov chain*[1]. Thus, each time component i enters a state $u \in S_i$, it remains there for a stochastic amount of time given by the waiting time distribution, before it makes a transition into state $v \in S_i$ with probability $P_{uv}^{(i)}$. The full matrix of transition probabilities for the built-in Markov chain for component i is denoted $\boldsymbol{P}^{(i)}$, $i \in C$. Given this matrix we have that:

$$P(X_i^+(t) = v | X_i(t) = u) = P_{uv}^{(i)}, \quad u, v \in S_i. \tag{4.1}$$

In order to find a similar expression for the conditional distribution of $X_i^-(t)$, we need the transition matrix for the backwards version of the built-in Markov chain, which we denote by $\boldsymbol{Q}^{(i)}$. It then follows that:

$$P(X_i^-(t) = v | X_i(t) = u) = Q_{uv}^{(i)}, \quad u, v \in S_i. \tag{4.2}$$

Within this more general context the definitions of $I_{NB}^{(i)}(t)$ and $I_{PB}^{(i)}(t)$ given in (2.12) and (2.13) at page 8 are still valid. However, the equations (2.14) and (2.15) need to be modified as follows:

$$I_{NB}^{(i)}(t) = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}(t)) \neq \phi(v, \boldsymbol{X}(t))] \cdot P[X_i(t) = u] \cdot P_{uv}^{(i)} \tag{4.3}$$

[1] The examples in Chapter 3 can be considered as special cases of this, with deterministic transition matrices. More about this in Chapter 4.2.

and

$$I_{PB}^{(i)}(t) = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}(t)) \neq \phi(v, \boldsymbol{X}(t))] \cdot P[X_i(t) = u] \cdot Q_{uv}^{(i)}. \tag{4.4}$$

The difference between the two above equations and (2.14) and (2.15) is that the mathematical expressions now must take into account that a more general probability model is used for the component processes. Where the transitions before were deterministic, we now need a sum that handles all combinations of u and v, including the cases where $u = v$.

Now, we focus on the asymptotic properties of the processes and ommit the time t from the notation. For component $i \in C$ we denote the stationary probabilities of the built-in Markov chain by $\pi_u^{(i)}$, $u \in S_i$. We then have, the following well-known relation between the transition matrices $\boldsymbol{P}^{(i)}$ and $\boldsymbol{Q}^{(i)}$, see [17]:

$$Q_{uv}^{(i)} = \frac{\pi_v^{(i)}}{\pi_u^{(i)}} P_{vu}^{(i)}, \quad u, v \in S_i. \tag{4.5}$$

Note that if the stationary distribution of the built-in Markov chain is *uniform*, i.e., if $\pi_u^{(i)} = 1/(r_i + 1)$ for all $u \in S_i$, we have:

$$\boldsymbol{Q}^{(i)} = (\boldsymbol{P}^{(i)})^T, \quad i \in C. \tag{4.6}$$

It is well-known that an irreducible aperiodic finite Markov chain has a uniform stationary distribution if and only if $\boldsymbol{P}^{(i)}$ is a *doubly stochastic matrix*, i.e., all rows and columns sums up to 1.

Let us introduce the times spent in each state between the transitions. More specifically, we define $W_{ku}^{(i)}$ as the kth waiting time in state u for component i. Furthermore, we assume that all the waiting times are independent, and that for all components $i \in C$ and states $u \in S_i$, the waiting times $W_{1u}^{(i)}, W_{2u}^{(i)}, \ldots$ are identically distributed with finite mean $\mu_u^{(i)}$. Then it follows from standard renewal theory, see [17], that the stationary distribution of X_i is given by:

$$P[X_i = u] = \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}}, \quad u \in S_i, i \in C. \tag{4.7}$$

Combining (4.7) with (4.3) and (4.4), we get the following expressions for the stationary importance measures:

$$I_{NB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot P_{uv}^{(i)} \tag{4.8}$$

$$I_{PB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot Q_{uv}^{(i)} \tag{4.9}$$

Using these formulas we have two theorems from [10], followed by a fresh theorem from this thesis:

Theorem 4.1.1 (From [10].)**.** *Assume that* $\mu_0^{(i)} = \cdots = \mu_{r_i}^{(i)}$. *Then* $I_{NB}^{(i)} = I_{PB}^{(i)}$.

Another special case occurs when the transition matrix $\boldsymbol{P}^{(i)}$ is doubly stochastic.

Theorem 4.1.2 (From [10].)**.** *Assume that the transition matrix $P^{(i)}$ is doubly stochastic. Then we have:*

$$I_{NB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{uv}^{(i)}$$

$$I_{PB}^{(i)} = \sum_{u,v \in S_i} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{vu}^{(i)}$$

Based on the above theorem, we now obtain this new result:

Theorem 4.1.3. *Assume that the transition matrix $\boldsymbol{P}^{(i)}$ is both doubly stochastic and symmetric, i.e. $\boldsymbol{P}^{(i)} = (\boldsymbol{P}^{(i)})^T$. Then*

$$I_{NB}^{(i)} = I_{PB}^{(i)}$$

Proof. From Theorem 4.1.2, we observe that for $I_{NB}^{(i)} = I_{PB}^{(i)}$ to be true we need to have $P_{uv}^{(i)} = P_{vu}^{(i)}$. This is given by the assumption that $\boldsymbol{P}^{(i)}$ is symmetric. ■

Finally, we look at how to expand the importance measures based on expected physical criticality, introduced in Chapter 2.4 for non-deterministic processes. We observe that the equations (2.19) and (2.20) are valid also in the general case. In order to calculate the *n*-Birnbaum measure* and the *p*-Birnbaum measure*, we again expand (2.19) and (2.20) by conditioning on the state of component i and get formulas similar to (4.3) and (4.4):

$$I_{NB}^{*(i)}(t) = \sum_{u,v \in S_i} E|\phi(u, \boldsymbol{X}(t)) - \phi(v, \boldsymbol{X}(t))| \cdot P[X_i(t) = u] \cdot P_{uv}^{(i)} \tag{4.10}$$

$$I_{PB}^{*(i)}(t) = \sum_{u,v \in S_i} E|\phi(u, \boldsymbol{X}(t)) - \phi(v, \boldsymbol{X}(t))| \cdot P[X_i(t) = u] \cdot Q_{uv}^{(i)} \tag{4.11}$$

Focusing on the asymptotic properties and using the same arguments as we did for $I_{NB}^{(i)}$ and $I_{PB}^{(i)}$, we obtain the following analogues to equations (4.8) and (4.9):

$$I_{NB}^{*(i)} = \sum_{u,v \in S_i} E|\phi(u, \boldsymbol{X}) - \phi(v, \boldsymbol{X})| \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot P_{uv}^{(i)} \tag{4.12}$$

$$I_{PB}^{*(i)} = \sum_{u,v \in S_i} E|\phi(u, \boldsymbol{X}) - \phi(v, \boldsymbol{X})| \cdot \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}} \cdot Q_{uv}^{(i)}. \tag{4.13}$$

Using these formulas we again have two theorems from [10], followed again by a fresh theorem from this thesis:

Theorem 4.1.4 (From [10].)**.** *Assume that* $\mu_0^{(i)} = \cdots = \mu_{r_i}^{(i)}$*. Then* $I_{NB}^{*(i)} = I_{PB}^{*(i)}$*.*

Theorem 4.1.5 (From [10].)**.** *Assume that the transition matrix* $\boldsymbol{P}^{(i)}$ *is doubly stochastic. Then we have:*

$$I_{NB}^{*(i)} = \sum_{u,v \in S_i} E|\phi(u, \boldsymbol{X}) - \phi(v, \boldsymbol{X})| \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{uv}^{(i)}$$

$$I_{PB}^{*(i)} = \sum_{u,v \in S_i} E|\phi(u, \boldsymbol{X}) - \phi(v, \boldsymbol{X})| \cdot \frac{\mu_u^{(i)}}{\sum_{v \in S_i} \mu_v^{(i)}} \cdot P_{vu}^{(i)}$$

Based on the above theorem, we now obtain this new result:

Theorem 4.1.6. *Assume that the transition matrix* $\boldsymbol{P}^{(i)}$ *is both doubly stochastic and symmetric, i.e.* $\boldsymbol{P}^{(i)} = (\boldsymbol{P}^{(i)})^T$*. Then*

$$I_{NB}^{*(i)} = I_{PB}^{*(i)}.$$

Proof. From Theorem 4.1.5, we observe that for $I_{NB}^{*(i)} = I_{PB}^{*(i)}$ to be true we need to have $P_{uv}^{(i)} = P_{vu}^{(i)}$. This is given by the assumption that $\boldsymbol{P}^{(i)}$ is symmetric. ■

4.2 Examples on System A

In Chapter 3 we only used specific deterministic versions of the transition matrices. I.e. $\boldsymbol{P}^{(i)}$ would be, depending on the value of r_i, a $(r_i+1) \times (r_i+1)$-matrix. We will give this deterministic downward transition matrix the "name" $\boldsymbol{D}_{r_i+1}$, and $\boldsymbol{D}_4$ looks like this:

$$\boldsymbol{D}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and more general we have the $(r_i+1) \times (r_i+1)$-matrix

$$\boldsymbol{D}_{r_i+1} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Before we continue we will also define another transition matrix by the value of r_i, namely $\boldsymbol{U}_{r_i+1}$. This transition matrix models uniform transitions between states, and $\boldsymbol{U}_4$ will look like this:

$$\boldsymbol{U}_4 = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

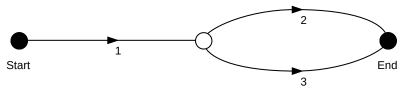


Figure 4.1: System A.

Thus, we have the $(r_i+1)\times(r_i+1)$-matrix

$$\boldsymbol{U}_{r_i+1} = \begin{bmatrix} 0 & \frac{1}{r_i} & \dots & & \frac{1}{r_i} \\ \frac{1}{r_i} & 0 & \frac{1}{r_i} & \dots & \frac{1}{r_i} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \frac{1}{r_i} & \dots & & \frac{1}{r_i} & 0 \end{bmatrix}. \tag{4.14}$$

Since both $\boldsymbol{U}_{r_i+1}$ and $\boldsymbol{D}_{r_i+1}$ are doubly stochastic, their stationary distributions are uniform with $\pi_u^{(i)} = 1/(r_i+1)$.

As observed here in the definition of $\boldsymbol{D}_{r_i+1}$ and $\boldsymbol{U}_{r_i+1}$ we have let the diagonals consist of zeroes. This is a common assumption, but our framework is not limited to this. In our first examples the diagonal, i.e. the probability that a component will go from its present state to the same state in the next step, will be zero. Still the theory defined in this chapter and earlier will surely handle transition matrices with non-zero diagonals. We will take a closer look at this in Example 4.3.1 and 4.3.2.

We use a more recent program, also developed by Arne Bang Huseby, to model the following examples. This is because Multicue does not handle stochastic life cycles. However, the program we use produces files of the simulations that can be further processed and analysed in Multicue, see Chapter 3.3.

In this chapter we will to some extent look at the calculation of importance measures, to see especially how the chosen $\boldsymbol{P}^{(i)}$ affects the measures. As we have walked through calculations of both impact-based and probability-based importance measures in Chapter 3, we will now focus on the calculations of $I_{NB}^{(i)}$. The reason for this choice is that there are no large differences between calculating the different measures, and the effect of the newly introduced $\boldsymbol{P}^{(i)}$ will be of the same kind. Furthermore, by Theorem 4.1.3 we have that $I_{NB}^{(i)} = I_{PB}^{(i)}$ when the transition matrix is symmetric, which is the case in some of the examples. Again, when we perform the calculations it is still at the asymptotic level, hence independent of time.

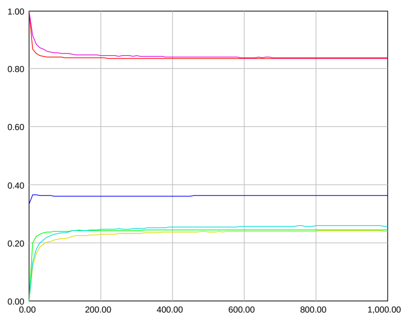
Example 4.2.1

This example can be considered as a continuation of Example 3.4.1. We are back in System A, see Figure 4.1, and with the same parameters as in Example 3.4.1, given again in the top four rows of Table 4.1.

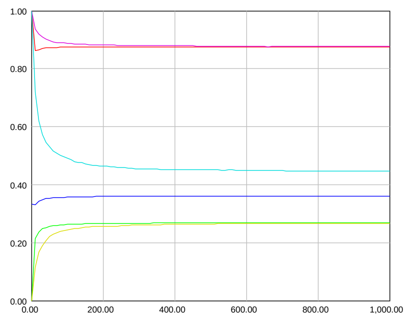
In this example we have to aims. The first is to demonstrate calculations in this framework, and the second is to observe how a non-deterministic simulation

i	1				2			3			
u	0	1	2	3	0	1	2	0	1	2	3
$f_i(u)$	0	1	2	3	0	1	2	0	1	2	3
$\mu_u^{(i)}$	2	1	3	4	2	3	5	2	1	2	5
$\boldsymbol{P}^{(i)}$	$\boldsymbol{D}_4$				$\boldsymbol{D}_3$			$\boldsymbol{U}_4$			

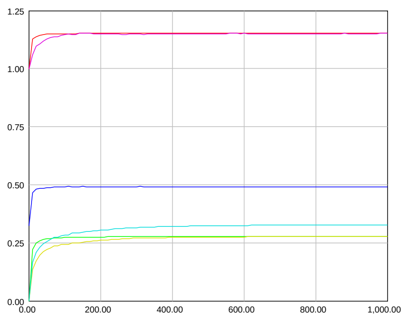
Table 4.1: Parameters used in Example 4.2.1.



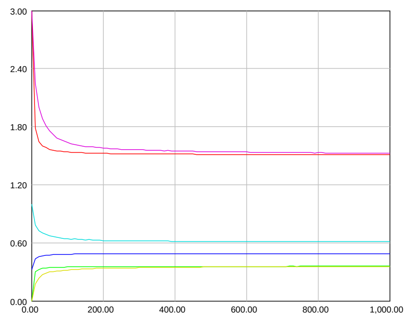
(a) n-Birnbaum $I_{NB}^{(i)}(t)$



(b) p-Birnbaum $I_{PB}^{(i)}(t)$



(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$



(d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

Figure 4.2: The cumulative mean for simulations of the four different importance measures, measuring System A in Example 4.2.1, and comparing with results from Example 3.4.1.

of one of the components turns out, by comparing it with the earlier example. Thus, we begin with a small adjustment where component 3 has a different transition matrix than the deterministic case in Example 3.4.1, namely the uniform transition matrix, see bottom row in Table 4.1, and (4.14) where the $\boldsymbol{U}_{r_i+1}$-matrix is defined. Component 1 and 2 still follow the deterministic life cycle from Example 3.4.1.

In order to visualise how this adjustment in component 3 changes the importance measures, we look at Figure 4.2. As before, the red, green, and blue graph represent component 1, 2, and 3 respectively, but there are also three more graphs in the plots. The purple, yellow, and turquoise graph *also* represent component 1, 2, and 3 respectively, but are based on Example 3.4.1. Thus, these last three graphs show the same as in Figure 3.2 at page 19. The four plots thereby together shows a "before and after"-picture of how the introduction of a uniform transition matrix in component 3 affects the different importance measures.

As we maybe could expect we see that there are not any significant changes[2] in the importance measures for component 1 and 2. They still approach the same asymptotic values as they did in Example 3.4.1.

However, we see that the importance of component 3 has increased in the forward-looking measures, and decreased in the backward-looking cases. This is a natural result, considering that the most dominating state for component 3 is the top state $u = 3$, where it is about 50% of the time. From this state the situation is random, contrary to the case in Example 3.4.1, in both the forward-looking measures and the backward-looking measures.

In Example 3.4.1 the *next* state after state 3 was state 2 with probability 1. Here it is any of the states 0, 1, or 2, all with probability $\frac{1}{3}$. Hence state 3 is a less safe state to be in than before, with potentially bigger impact on the system, which is increasing its importance.

In Example 3.4.1 the *previous* state from state 3 was state 0 with probability 1. Here it is any of the states 0, 1, or 2, all with probability $\frac{1}{3}$. Hence state 3 is a safer state to be in than before, with potentially smaller impact on the system, which is decreasing its importance.

Since $\boldsymbol{P}^{(3)}$ is symmetric, we see Theorem 4.1.3 and 4.1.6 visualised here, as we can see that $I_{NB}^{(3)} = I_{PB}^{(3)}$, and when we take the different y-axis into account we can also see that $I_{NB}^{*(3)} = I_{PB}^{*(3)}$.

In fact it may seem like $I_{NB}^{(3)} = I_{PB}^{(3)}$ is precisely in the middle of the interval of $[I_{NB}^{(3)}, I_{PB}^{(3)}]$ from Example 3.4.1, and the same for the corresponding impact-based measures. We checked it out, and even though it is close to being true, it is not. Let us first do the calculations, and come back to this later.

The calculations of the asymptotic importance measures can be done manually in this example, just as in Example 3.4.1. The calculations though get more messy, as there are $(r_i + 1)^2$ different combinations of u and v that must be considered, and even if we exclude where $u = v$,[3] we have a remaining

[2] The small dip we observe here in the initial phase of the backward-looking measures will also be observed in later figures in this chapter. Our program has used a shortcut solution for selecting the previous state in the initialization of the components, which is now stochastic and not deterministic as in Chapter 3. This shortcut only interferes with the first point and not the asymptotic values. Still, this first point makes a larger impact on our plots as we plot the cumulative mean of the importance measures.

[3] As in these cases the contributions to the sum will be zero as $\phi(u, \boldsymbol{X}) = \phi(v, \boldsymbol{X})$.

X_1		0	0	0	1	1	1	2	2	2	3	3	3
X_2		0	1	2	0	1	2	0	1	2	0	1	2
u	v												
0	1	0	0	0	1	0	0	1	1	0	1	1	1
0	2	0	0	0	1	0	0	1	1	0	1	1	1
0	3	0	0	0	1	0	0	1	1	0	1	1	1
1	0	0	0	0	1	0	0	1	1	0	1	1	1
1	2	0	0	0	0	0	0	1	0	0	1	1	0
1	3	0	0	0	0	0	0	1	0	0	1	1	0
2	0	0	0	0	1	0	0	1	1	0	1	1	1
2	1	0	0	0	0	0	0	1	0	0	1	1	0
2	3	0	0	0	0	0	0	0	0	0	1	0	0
3	0	0	0	0	1	0	0	1	1	0	1	1	1
3	1	0	0	0	0	0	0	1	0	0	1	1	0
3	2	0	0	0	0	0	0	0	0	0	1	0	0

Table 4.2: Calculation table for $\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})$ in Example 4.2.1.

X_1	0	0	0	1	1	1	2	2	2	3	3	3
X_2	0	1	2	0	1	2	0	1	2	0	1	2
$\min(u, v)$												
0	0	0	0	1	0	0	1	1	0	1	1	1
1	0	0	0	0	0	0	1	0	0	1	1	0
2	0	0	0	0	0	0	0	0	0	1	0	0

Table 4.3: Simplified calculation table for $\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})$ in Example 4.2.1.

amount of $(r_i + 1)^2 - (r_i + 1)$ contributions to the sum in all the importance measures.

Still, let us embark on this more bumpy journey and try to look at $I_{NB}^{(3)}$. As $\boldsymbol{P}^{(3)} = \boldsymbol{U}_4$ is doubly stochastic, we will base our calculations on the formulas given in Theorem 4.1.2.

In order to compute $P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})]$, we put up a calculation table, Table 4.2. It looks a bit different than in the previous chapter, but the main understanding will be somewhat similar as in Example 3.4.1. We note in the table that there are three "groups" of situations; the (u, v)-pairs $(0, 1), (0, 2), (0, 3), (1, 0), (2, 0), (3, 0)$ all have the exact same arrays of 0s and 1s following them, as do $(1, 2), (1, 3), (2, 1), (3, 1)$ and lastly $(3, 2), (2, 3)$. We marked the groups with different colors, and observe that the groups have the same $\min(u, v)$. So Table 4.2 can be shortened into Table 4.3. Let us compute $P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})]$ for the three different groups:

$$
\begin{aligned}
P[\phi(u,\boldsymbol{X}) \neq \phi(v,\boldsymbol{X})] &= 0.1\cdot 0.2 + 0.3\cdot 0.2 + 0.3\cdot 0.3 \\
&\quad + 0.4\cdot 0.2 + 0.4\cdot 0.3 + 0.4\cdot 0.5 = \underline{0.57} \\
&\quad \text{for } u, v \text{ where } \min(u,v) = 0. \\
P[\phi(u,\boldsymbol{X}) \neq \phi(v,\boldsymbol{X})] &= 0.3\cdot 0.2 + 0.4\cdot 0.2 + 0.4\cdot 0.3 = \underline{0.26} \\
&\quad \text{for } u, v \text{ where } \min(u,v) = 1. \\
P[\phi(u,\boldsymbol{X}) \neq \phi(v,\boldsymbol{X})] &= 0.4\cdot 0.2 = \underline{0.08} \\
&\quad \text{for } u, v \text{ where } \min(u,v) = 2.
\end{aligned}
$$

Here the first term on the first line, $0.1 \cdot 0.2$, is referring to $P(X_1 = 1) \cdot P(X_2 = 0)$, as the first 1 in the third row of Table 4.3 is found right below where $X_1 = 1$ and $X_2 = 0$. The values for these probabilities are derived from the values of $\mu_u^{(3)}$ in Table 4.1. The second term $0.3 \cdot 0.2$ is referring to $P(X_1 = 2) \cdot P(X_2 = 0)$, as the second 1 in the third row of Table 4.3 is found right below where $X_1 = 2$ and $X_2 = 0$. And so on.

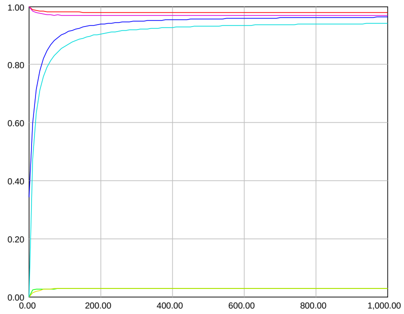
Recall that $P_{uv}^{(3)} = \frac{1}{3}$ for all $u \neq v$ in this example (and $P_{uv}^{(3)} = 0$ for $u = v$). In the following calculations we have made further use of the three groups. In the fraction multiplied with 0.57 the first value in the numerator, $\mu_0^{(3)}$, is found by locating the u-value in the first column of the first -colored row in Table 4.2 ($u = 0$). Since $u = 0$ we then look for the value of $\mu_0^{(3)}$ in Table 4.1, ($\mu_0^{(3)} = 2$). Finally we divide by $\sum_{v \in S_3} \mu_v^{(3)} = 10$, and we obtain $\frac{2}{10}$ as the first term in the fraction after 0.57. The second term is found by looking for the value of u in the second -colored row, and so on. Similarly we find the values following 0.26 by considering the -colored rows, and the values following 0.08 by considering the -colored rows.

$$
\begin{aligned}
I_{NB}^{(3)} &= \sum_{u,v \in S_3} P[\phi(u,\boldsymbol{X}) \neq \phi(v,\boldsymbol{X})] \cdot \frac{\mu_u^{(3)}}{\sum_{v \in S_3} \mu_v^{(3)}} \cdot P_{uv}^{(3)} \\
&= \frac{1}{3} \sum_{u,v \in S_3, u \neq v} P[\phi(u,\boldsymbol{X}) \neq \phi(v,\boldsymbol{X})] \cdot \frac{\mu_u^{(3)}}{\sum_{v \in S_3} \mu_v^{(3)}} \\
&= \frac{1}{3} \cdot [0.57 \cdot \frac{\mu_0^{(3)} + \mu_0^{(3)} + \mu_0^{(3)} + \mu_1^{(3)} + \mu_2^{(3)} + \mu_3^{(3)}}{\sum_{v \in S_i} \mu_v^{(3)}} \\
&\quad + 0.26 \cdot \frac{\mu_1^{(3)} + \mu_1^{(3)} + \mu_2^{(3)} + \mu_3^{(3)}}{\sum_{v \in S_i} \mu_v^{(3)}} + 0.08 \cdot \frac{\mu_2^{(3)} + \mu_3^{(3)}}{\sum_{v \in S_i} \mu_v^{(3)}}] \\
&= \frac{1}{3} \cdot [0.57 \cdot \frac{2+2+2+1+2+5}{10} \\
&\quad + 0.26 \cdot \frac{1+1+2+5}{10} + 0.08 \cdot \frac{2+5}{10}] \\
&= \frac{1}{3} \cdot [0.57 \cdot 1.4 + 0.26 \cdot 0.9 + 0.08 \cdot 0.7] \approx \underline{0.36}
\end{aligned}
$$

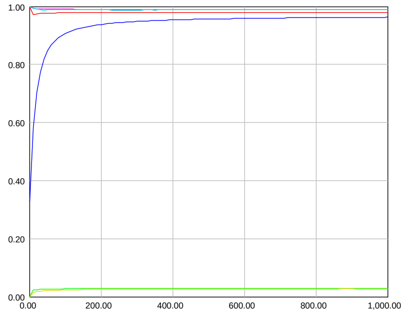
Now, we warned about the bumpy journey. Note that by Theorem 4.1.3, we also have that $I_{PB}^{(3)} = I_{NB}^{(3)} \approx 0.36$, since $\boldsymbol{P}^{(3)} = \boldsymbol{U}_4$ is symmetric. This also resembles the endpoints in Figure 4.2a and 4.2b.

i	1 ■				2 ■			3 ■			
u	0	1	2	3	0	1	2	0	1	2	3
$f_i(u)$	0	1	2	3	0	1	2	0	1	2	3
$\mu_u^{(i)}$	0.1	0.1	0.1	9.7	9.8	0.1	0.1	0.1	0.1	0.1	9.7
$\boldsymbol{P}^{(i)}$	$\boldsymbol{U}_4$				$\boldsymbol{U}_3$			$\boldsymbol{U}_4$			

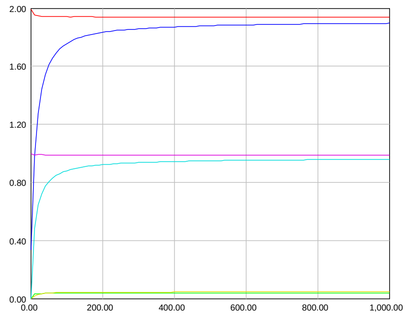
Table 4.4: Parameters used in Example 4.2.2.



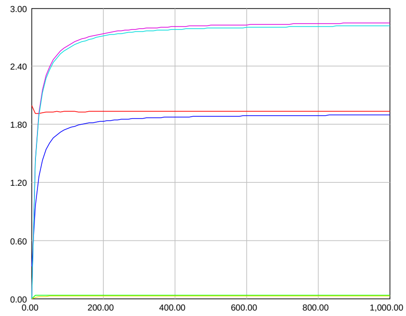
(a) n-Birnbaum $I_{NB}^{(i)}(t)$



(b) p-Birnbaum $I_{PB}^{(i)}(t)$



(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$



(d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

Figure 4.3: The cumulative mean for simulations of the four different importance measures, measuring System A in Example 4.2.2, and comparing with results from Example 3.4.2.

Short about the initial phase in the example; we also observe a change from the before case in the initial phase of component 3's graph for $I_{NB}^{(3)}$ in Figure 4.2a. Here it begins at 0.33, which is reasonable as it still starts in state 3, but now there is a 33% probability for the next state to be state 0, and hence reducing the system state. The same can be found in the plots for the other importance measures as well.

Example 4.2.2

In this example, which can be considered as a continuation of Example 3.4.2, we will investigate what happens when we choose somewhat extreme values of $\mu_u^{(i)}$ and expand the use of the uniform transition matrix $\boldsymbol{U}_{r_i+1}$. Short reminder of Example 3.4.2, this was the one where System A was modelled close to a series system consisting of only component 1 and 3.

As the previous example showed some interesting effects of introducing a uniform transition matrix in component 3, let us look at what happens when we use $\boldsymbol{P}^{(i)} = \boldsymbol{U}_{r_i+1}$ in all the three components. The parameters are given in Table 4.1, and the resulting plots are found in Figure 4.3. Again, the red, green, and blue graph correspond to component 1, 2, and 3 in this example, while the purple, yellow, and turquoise correspond to component 1, 2, and 3 in Example 3.4.2.

We observe some of the same effects as we looked at in Example 4.2.1, especially with component 1 and 3. Putting $\boldsymbol{P}^{(1)} = \boldsymbol{P}^{(3)} = \boldsymbol{U}_4$ results in an increase in importance in the forward-looking measures and a decrease in importance in the backward-looking measure. This is more visible in the impact-based measures, but the effect is there in the probability-based measures as well, but with the extreme waiting times, the relative effect is less. However component 2 seems rather unchanged by changing its transition matrix.

Let us compute $I_{NB}^{(3)}$ manually in this example too. We obtain the same calculation table as Table 4.2, as we are in the same system with the same r_i and $f_i(u)$ for all $i \in C, u \in S_i$. The differences are found in these probabilities:

$$
\begin{aligned}
P[\phi(u,\boldsymbol{X}) \neq \phi(v,\boldsymbol{X})] &= 0.01 \cdot 0.98 + 0.01 \cdot 0.98 + 0.01 \cdot 0.01 \\
&\quad + 0.97 \cdot 0.98 + 0.97 \cdot 0.01 + 0.97 \cdot 0.01 = \underline{0.9897} \\
&\quad \text{for } u, v \text{ where } \min(u,v) = 0. \\
P[\phi(u,\boldsymbol{X}) \neq \phi(v,\boldsymbol{X})] &= 0.01 \cdot 0.98 + 0.97 \cdot 0.98 + 0.97 \cdot 0.01 = \underline{0.9701} \\
&\quad \text{for } u, v \text{ where } \min(u,v) = 1. \\
P[\phi(u,\boldsymbol{X}) \neq \phi(v,\boldsymbol{X})] &= 0.97 \cdot 0.98 = \underline{0.9506} \\
&\quad \text{for } u, v \text{ where } \min(u,v) = 2.
\end{aligned}
$$

And further;

$$
\begin{aligned}
I_{NB}^{(3)} = I_{PB}^{(3)} &= \sum_{u,v \in S_3} P[\phi(u,\boldsymbol{X}) \neq \phi(v,\boldsymbol{X})] \cdot \frac{\mu_u^{(3)}}{\sum_{v \in S_3} \mu_v^{(3)}} \cdot P_{uv}^{(3)} \\
&= \frac{1}{3} \sum_{u,v \in S_3, u \neq v} P[\phi(u,\boldsymbol{X}) \neq \phi(v,\boldsymbol{X})] \cdot \frac{\mu_u^{(3)}}{\sum_{v \in S_3} \mu_v^{(3)}} \\
&= \frac{1}{3} \cdot [0.9897 \cdot \frac{\mu_0^{(3)} + \mu_0^{(3)} + \mu_0^{(3)} + \mu_1^{(3)} + \mu_2^{(3)} + \mu_3^{(3)}}{\sum_{v \in S_i} \mu_v^{(3)}} \\
&\quad + 0.9701 \cdot \frac{\mu_1^{(3)} + \mu_1^{(3)} + \mu_2^{(3)} + \mu_3^{(3)}}{\sum_{v \in S_i} \mu_v^{(3)}} + 0.9506 \cdot \frac{\mu_2^{(3)} + \mu_3^{(3)}}{\sum_{v \in S_i} \mu_v^{(3)}}] \\
&= \frac{1}{3} \cdot [0.9897 \cdot \frac{0.1 + 0.1 + 0.1 + 0.1 + 0.1 + 9.7}{10} \\
&\quad + 0.9701 \cdot \frac{0.1 + 0.1 + 0.1 + 9.7}{10} + 0.9506 \cdot \frac{0.1 + 9.7}{10}] \\
&= \frac{1}{3} \cdot [0.9897 \cdot 1.02 + 0.9701 \cdot 1.00 + 0.9506 \cdot 0.98] \approx \underline{0.97}.
\end{aligned}
$$

i	1				2			3			
u	0	1	2	3	0	1	2	0	1	2	3
$f_i(u)$	0	1	2	3	0	1	2	0	1	2	3
$\mu_u^{(i)}$	0.1	1.9	3	5	0.1	3.9	6	9.7	0.1	0.1	0.1
$\boldsymbol{P}^{(i)}$	$\boldsymbol{D}_4$				$\boldsymbol{D}_3$			$\begin{bmatrix} 0 & 0.05 & 0.05 & 0.90 \\ 0.05 & 0 & 0.05 & 0.90 \\ 0.05 & 0.05 & 0 & 0.90 \\ 0.05 & 0.05 & 0.90 & 0 \end{bmatrix}$			

Table 4.5: Parameters used in Example 4.2.3.

When we compare the above calculations with the ones we looked at in Example 4.2.1, we observe that the computed $P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})]$-values in the present example are way closer to 1 than in Example 4.2.1. This is where the different result turns out.

What were the same values when $\boldsymbol{P}^{(i)} = \boldsymbol{D}_{r_i+1}$, as we looked at in Example 3.4.2? We did the calculations and found that $I_{NB}^{(3)} \approx 0.92$ was the case in Example 3.4.2.

So we observe that due to extreme choices of waiting times, the change of transition matrix from $\boldsymbol{D}_{r_i+1}$ to $\boldsymbol{U}_{r_i+1}$ for all $i \in C$, does not interfere too much with the probability-based importance measures $I_{NB}^{(i)}$ and $I_{PB}^{(i)}$. The ratio of before and after in this case is $0.92/0.97 \approx 0.95$, while in Example 4.2.1 the same ratio[4] was $0.26/0.36 \approx 0.73$.

Example 4.2.3

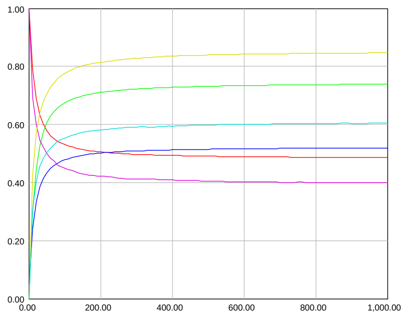
This example is a continuation of Example 3.4.4, and a result of questioning ourselves "but what if we do this", just to check it out. The parameters are found in Table 4.5.

Here component 1 and 2 are kept as they were in Example 3.4.4, rather reliable considering their respective $\mu_u^{(i)}$-s, while component 3 is not very reliable. At the same time, component 3 is now equipped with a transition matrix that should imply high reliability. How do these two traits influence the importance measure, while dragging in opposite directions? The results are found in Figure 4.4, once again the purple, yellow and turquoise graph represents respectively component 1, 2 and 3 from the previous case Example 3.4.4.

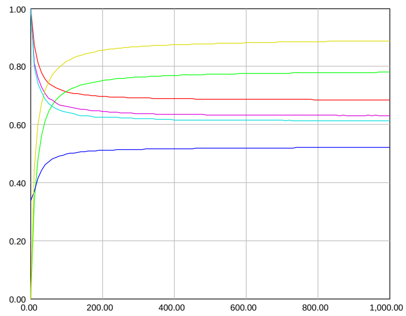
The first surprise here, is perhaps that component 1 and 2 show different results than before, even though we did not change anything about them. So this change in component 3 influenced component 1 and 2. More specific, we see that component 1 has increased while component 2 has decreased in importance throughout the measures. Component 3 has decreased in the forward-looking measures and $I_{PB}^{(3)}$, while in Figure 4.4d it is almost the same as before, only a small increase.

First things first: how can this change in component 3 affect the other components this much? Let us look at $I_{NB}^{(1)}$, and as $\boldsymbol{P}^{(1)}$ is doubly stochastic, we use Theorem 4.1.2.

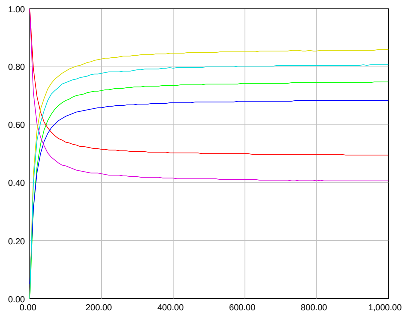
[4]The number 0.263 for $I_{NB}^{(3)}$ is found in Example 3.4.1.



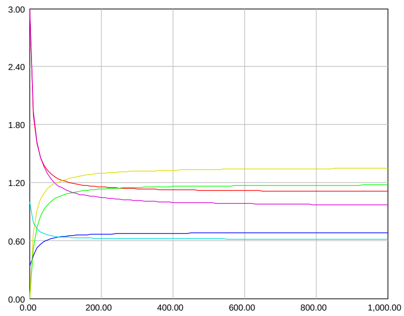
(a) n-Birnbaum $I_{NB}^{(i)}(t)$



(b) p-Birnbaum $I_{PB}^{(i)}(t)$



(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$



(d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

Figure 4.4: The cumulative mean for simulations of the four different importance measures, measuring System A in Example 4.2.3, and comparing with results from Example 3.4.4.

$$I_{NB}^{(1)} = \sum_{u,v \in S_1} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\mu_u^{(1)}}{\sum_{v \in S_1} \mu_v^{(3)}} \cdot P_{uv}^{(1)}$$

Considering the three factors within the sum, we see that both $\mu_u^{(1)} / \sum_{v \in S_1} \mu_v^{(1)}$ and $P_{uv}^{(1)}$ clearly does not depend on component 3. So the answer must be within $P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})]$. The probabilities $P(X_2 = u)$ for $u \in S_2$ have not changed, they are still $\mu_u^{(2)} / \sum_{v \in S_2} \mu_v^{(2)}$ as $\boldsymbol{P}^{(2)} = \boldsymbol{D}_3$. However, the probabilities $P(X_3 = u)$ for $u \in S_3$ *have* changed, they are now $\pi_u^{(3)} \mu_u^{(3)} / \sum_{v \in S_3} \pi_u^{(3)} \mu_v^{(3)}$. We calculated $\boldsymbol{\pi}^{(3)}$ by its definition $\boldsymbol{\pi}^{(3)} \boldsymbol{P}^{(3)} = \boldsymbol{\pi}^{(3)}$, and found $\boldsymbol{\pi}^{(3)} \approx [0.048, 0.048, 0.430, 0.474]$, and further

$$\begin{aligned} &P(X_3 = 0) \approx 0.829 \qquad P(X_3 = 1) \approx 0.009 \\ &P(X_3 = 2) \approx 0.077 \qquad P(X_3 = 3) \approx 0.085. \end{aligned}$$

The explanation is thus that since we looked at cases in Example 4.2.1 where the transition matrix of the changed component was doubly stochastic, and therefore its stationary distribution uniform, the other unchanged components did not change in importance compared with the similar examples in Chapter 3.

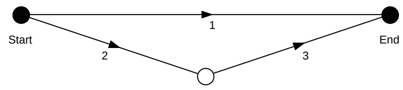


Figure 4.5: System B.

i	1				2			3		
u	0	1	2	3	0	1	2	0	1	2
$f_i(u)$	0	1	2	3	0	1	4	0	1	4
a) $\mu_u^{(i)}$	0.1	0.1	0.1	9.7	0.1	1.9	8.0	0.1	1.9	8.0
b) $\mu_u^{(i)}$	0.1	0.1	0.1	9.7	8.0	1.9	0.1	8.0	1.9	0.1
$\boldsymbol{P}^{(i)}$	$\boldsymbol{D}_4$				$\begin{bmatrix} 0.0 & 0.2 & 0.8 \\ 0.8 & 0.0 & 0.2 \\ 0.2 & 0.8 & 0.0 \end{bmatrix}$			$\begin{bmatrix} 0.2 & 0.0 & 0.8 \\ 0.8 & 0.2 & 0.0 \\ 0.0 & 0.8 & 0.2 \end{bmatrix}$		

Table 4.6: Parameters used in Example 4.3.1.

In this example, $\boldsymbol{P}^{(3)}$ is *not* doubly stochastic, and hence this fact influenced the importance measures of the other components. It is also rather intuitive that by introducing this transition matrix in component 3, we also make it more reliable than it was, hence increasing its mean capacity. The consequence of this is that component 1 increases in importance, and component 2 decreases.

Now, what happens in Figure 4.4d with component 3? It seems to be rather straightforward, as in this example both component 1 and 2 are rather reliable, the impact component 3 can cause for the system is not too big. Due to the close starting points, the before[5] and after graph crosses paths in their initial phases.

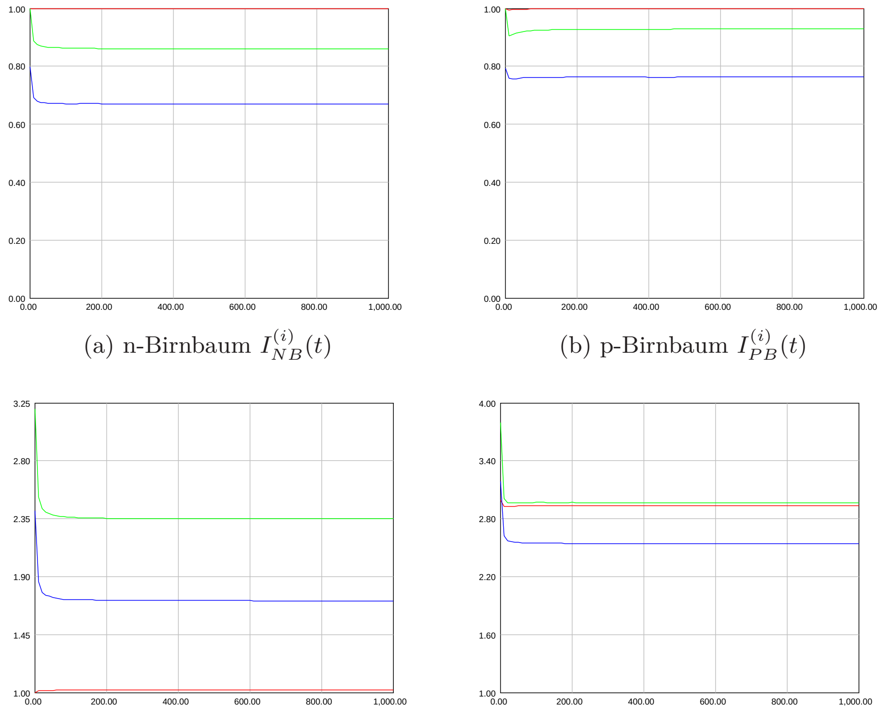
4.3 Examples on System B

Example 4.3.1

In this example we will look at the diagonal of the transition matrix, i.e. the probability for a component to stay in its current state upon state change. We will now look at System B, see Figure 4.5. We split this example in two cases; part a) where the two components we look at (component 2 and 3) are rather reliable according to their waiting times, and part b) where the components are not so reliable. The parameters for both parts are given in Table 4.6. We observe some similarities with Example 3.5.2, although we will now not compare with it. Note also the $f_i(u)$-values.

Before we look at the resulting figures and part a) and b), let us just briefly look at the cases at hand. Component 2 and 3 have equivalent placements in the system, and we made them equal in two other ways such that $\mu_u^{(2)} = \mu_u^{(3)}$ and $f_2(u) = f_3(u)$, in both part a) and b). The only difference between them is their transition matrices. Component 2 has zeroes on the diagonal of its

[5]from Example 3.4.4



(a) n-Birnbaum $I^{(i)}_{NB}(t)$ (b) p-Birnbaum $I^{(i)}_{PB}(t)$

(c) n*-Birnbaum $I^{*(i)}_{NB}(t)$ (d) p*-Birnbaum $I^{*(i)}_{PB}(t)$

Figure 4.6: The cumulative mean for simulations of the four different importance measures, measuring System B in Example 4.3.1, part a).

transition matrix, while component 3 has the value 0.2 in all the entries on the diagonal. The transition matrices has the value 0.8 in the exact same positions, and 0.0 where the other component have 0.2 and vice versa.

Part a) As their given $\mu^{(i)}_u$ implies rather reliable components, the diagonal in component 3 could be expected to amplify this trait, and be more reliable than component 2. Let us take a look at the plots, found in Figure 4.6.

We see the expected result throughout the plots, as component 3 is more reliable, its importance decreases. Let us look into it.

First, we can easily see that this holds, as a result of that $\mu^{(2)}_u = \mu^{(3)}_u$ and that both $\boldsymbol{P}^{(2)}$ and $\boldsymbol{P}^{(3)}$ are doubly stochastic:

$$\begin{aligned}
P(X_2 = 0) = P(X_3 = 0) = 0.01\\
P(X_2 = 1) = P(X_3 = 1) = 0.19\\
P(X_2 = 2) = P(X_3 = 2) = 0.80.
\end{aligned}$$

So this factor within the sum in the computation of the importance measures will be the same for component 2 and 3. The factor $P^{(2)}_{uv} \neq P^{(3)}_{uv}$ for all pairs of $[u, v]$, except these three pairs $[u, v]$: $[0, 2], [1, 0], [2, 1]$, where $P^{(2)}_{uv} = P^{(3)}_{uv} = 0.8$. The factor $P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})]$ is equal for component 2 and 3, due to equivalent placing in the system, equal capacity and that $P[X_2 = u] = P[X_3 = u]$. Recall also that $S_2 = S_3$, and that when $u = v$ we have $P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] = 0$. Thus, when we now look at the difference between $I_{NB}^{(2)}$ and $I_{NB}^{(3)}$, we are left with only these three pairs of $[u, v]$ not being "zeroed out": $[0, 1], [1, 2]$ and $[2, 0]$:

$$\begin{aligned}
I_{NB}^{(2)} - I_{NB}^{(3)} = & \sum_{u,v \in S_2} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\mu_u^{(2)}}{\sum_{v \in S_2} \mu_v^{(2)}} \cdot P_{uv}^{(2)} \\
& - \sum_{u,v \in S_3} P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\mu_u^{(3)}}{\sum_{v \in S_3} \mu_v^{(3)}} \cdot P_{uv}^{(3)} \\
& = \sum_{u,v \in S_2} [P[\phi(u, \boldsymbol{X}) \neq \phi(v, \boldsymbol{X})] \cdot \frac{\mu_u^{(2)}}{\sum_{v \in S_2} \mu_v^{(2)}}] \cdot [P_{uv}^{(2)} - P_{uv}^{(3)}] \\
& = [P[\phi(0, \boldsymbol{X}) \neq \phi(1, \boldsymbol{X})] \cdot \frac{\mu_0^{(2)}}{\sum_{v \in S_2} \mu_v^{(2)}}] \cdot [P_{01}^{(2)} - P_{01}^{(3)}] \\
& + [P[\phi(1, \boldsymbol{X}) \neq \phi(2, \boldsymbol{X})] \cdot \frac{\mu_1^{(2)}}{\sum_{v \in S_2} \mu_v^{(2)}}] \cdot [P_{12}^{(2)} - P_{12}^{(3)}] \\
& + [P[\phi(2, \boldsymbol{X}) \neq \phi(0, \boldsymbol{X})] \cdot \frac{\mu_2^{(2)}}{\sum_{v \in S_2} \mu_v^{(2)}}] \cdot [P_{20}^{(2)} - P_{20}^{(3)}] \\
& = 0.99 \cdot 0.01 \cdot [0.2 - 0.0] \\
& + 0.80 \cdot 0.19 \cdot [0.2 - 0.0] \\
& + 0.99 \cdot 0.80 \cdot [0.2 - 0.0] \\
& = \underline{0.19078.}
\end{aligned}$$

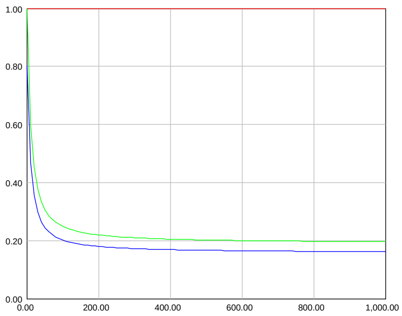
Part b) Now, does the same line of arguments hold for a case where both the components are somewhat unreliable? Does the non-zero uniform diagonal still amplify the traits given by chosen waiting times? Is component 3 here more important, as a consequence of the diagonal amplifying its traits of low reliability?

If we look at similar calculations as in part a), we find that:

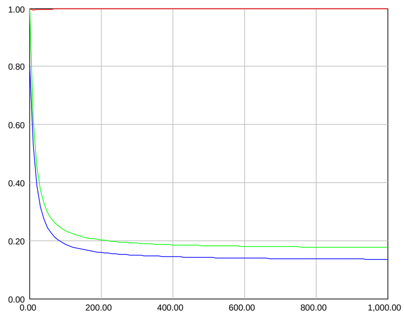
$$\begin{aligned}
I_{NB}^{(2)} - I_{NB}^{(3)} = & (\cdots) \\
& = 0.20 \cdot 0.80 \cdot [0.2 - 0.0] \\
& + 0.01 \cdot 0.19 \cdot [0.2 - 0.0] \\
& + 0.20 \cdot 0.01 \cdot [0.2 - 0.0] \\
& = \underline{0.03278.}
\end{aligned}$$

So we see that component 2 is still more important, but the difference is much smaller. This is also clearly visualized in Figure 4.7, especially when we compare it with Figure 4.6.

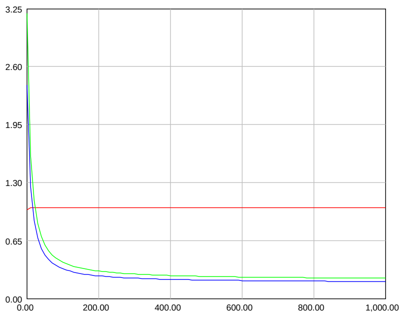
The reason is that component 2 changes states more often than component 3, as a consequence of the zeroes on the diagonal. It is rather intuitive, that the more a component changes state, the more important it will be, at least when considering the probability-based measures.



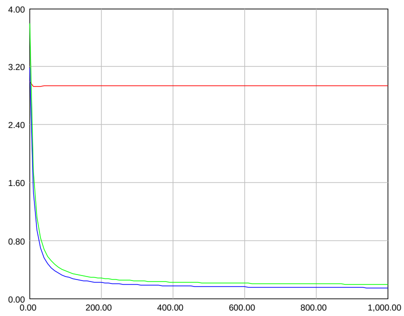
(a) n-Birnbaum $I_{NB}^{(i)}(t)$



(b) p-Birnbaum $I_{PB}^{(i)}(t)$



(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$



(d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

Figure 4.7: The cumulative mean for simulations of the four different importance measures, measuring System B in Example 4.3.1, part b).

This shows kind of neatly that the introduction of a non-zero uniform diagonal in the transition matrix of a component increases its reliability, hence its importance decreases. This happens regardless of the $\mu_u^{(i)}$-values.

Note also that in this case, compared to part a), the components have way less probability to change the system state. This naturally leads to that their importance has generally decreased throughout the measures.

We also see that Theorem 3.2.2 still holds here for component 1.

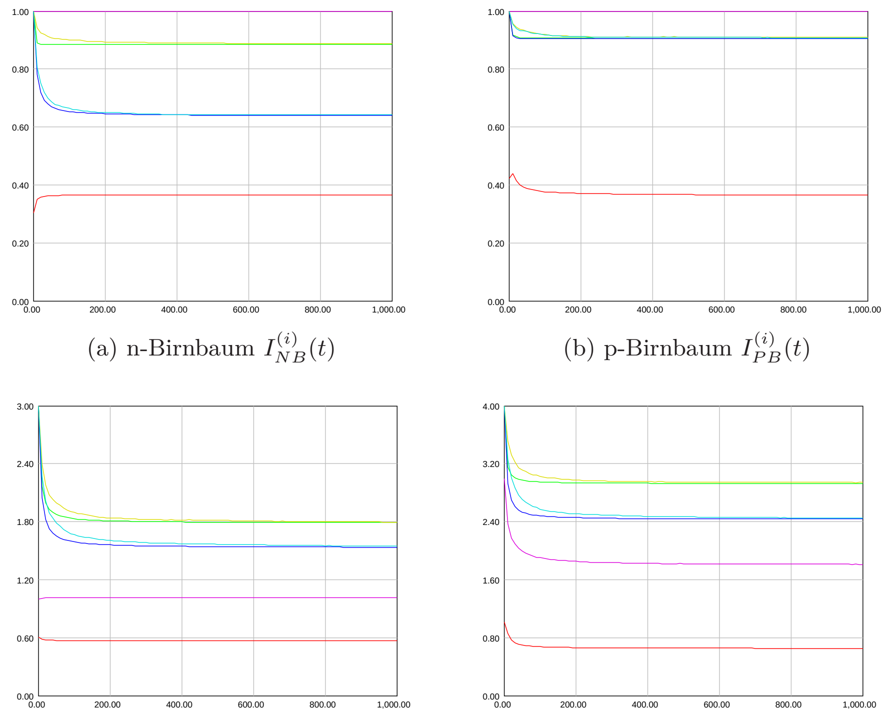
Example 4.3.2

In the previous example we looked closer at what happens when we no longer have zeroes on the diagonal in the transition matrix, but we only looked at cases where the values on the diagonal was uniform. Now we will return to and compare with Example 3.5.2, the one where component 1 was such that $I_{NB}^{(1)}(t) = I_{PB}^{(1)}(t) = 1$ at all time t. Still, because we adjusted $f_i(u)$ a bit we made it so that component 1 was *not* most important in the impact-based measures.

If we now add values larger than zero on the diagonal of component 1 in System B, then component 1 will no longer have the property from Theorem 3.2.2 that $I_{NB}^{(1)}(t) = I_{PB}^{(1)}(t) = 1$ at all time t, as it upon state change is no longer necessarily leaving its current state and hence not changing the system

i	1 ■				2 ■			3 ■		
u	0	1	2	3	0	1	2	0	1	2
$f_i(u)$	0	1	2	3	0	1	4	0	1	4
$\mu_u^{(i)}$	0.1	2	3.9	4	0.1	4.4	5.5	0.1	1.9	8
$\boldsymbol{P}^{(i)}$	$\begin{bmatrix} 0.4 & 0.1 & 0.2 & 0.3 \\ 0.3 & 0.5 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.7 \end{bmatrix}$				$\boldsymbol{D}_3$			$\boldsymbol{D}_3$		

Table 4.7: Parameters used in Example 4.3.2



(a) n-Birnbaum $I_{NB}^{(i)}(t)$ (b) p-Birnbaum $I_{PB}^{(i)}(t)$

(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$ (d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

Figure 4.8: The cumulative mean for simulations of the four different importance measures, measuring System A in Example 4.3.2, and comparing with results from Example 3.5.2.

state. This also demonstrates the need for the assumption that $X_i^+ \neq X_i$ in Theorem 3.2.2, and shows that the theorem can hold also with stochastic state transitions, as long as this assumption in the theorem is fulfilled.

The table of parameters is given in Table 4.7, and the resulting figures of the importance measures is found in Figure 4.8. Recall that red, green and blue are the graphs simulated for component 1, 2 and 3 respectively in this example, while purple, yellow and turquoise are respectively component 1, 2 and 3 in Example 3.5.2. So we now mainly will comment the red graph, compared with the purple, as this is where the changes were made.

Here we observe large changes in component 1's importance throughout the measures. As the diagonal of $\boldsymbol{P}^{(1)}$ makes the the transition from state u to state u possible, its importance drops a substantial amount. As this result is rather intuitive, we will not dive more into it, but it was an interesting follow-up from results in Chapter 3.5.

However, we will take a quick look at the values for $P(X_1 = u)$, just to see how the diagonal makes an effect here. First, we calculated $\boldsymbol{\pi}^{(1)} \approx [0.202, 0.207, 0.240, 0.351]$, and then:

$$\begin{aligned}
&P(X_1 = u) = \frac{\pi_u^{(1)} \mu_u^{(1)}}{\sum_{v \in S_1} \pi_v^{(1)} \mu_v^{(1)}} \quad \text{s.t.} \\
&P(X_1 = 0) \approx 0.007 \\
&P(X_1 = 1) \approx 0.149 \\
&P(X_1 = 2) \approx 0.338 \\
&P(X_1 = 3) \approx 0.506
\end{aligned}$$

Here we get a direct view of how the transition matrix influences the stationary probabilities. If we had a case of doubly stochastic transition matrix, hence only looked at $P(X_1 = u) = \mu_u^{(1)} / \sum_{v \in S_1} \mu_v^{(1)}$, the values for $P(X_1 = u)$ would be 0.01, 0.20, 0.39 and 0.40, for $u = 0, 1, 2, 3$ respectively. We note that for $u = 0, 1, 2$ the stationary probabilities values decreases with the introduction of transition matrix $\boldsymbol{P}^{(1)}$, while for $u = 3$ it has increased. This is also to be expected as a consequence of that P_{uu} is increasing in u.

CHAPTER 5

A more specific example

Up until now we have looked at system A and B, both constructed by the idea to make simple structures so that the number of parameters that can affect the system and thereby the importance measures, has an upper limit. Hence also making general findings easier to achieve.

However, during the spring of 2020 something else came to our attention, a machine that is not an actual machine. Still, when this machine is not working, neither is anyone else.

Assume you have a family consisting of two parents and j children, $j = 0, 1, \ldots$ under the age of 10 and above 1, i.e. they are not able to take care of themselves, but still big enough to be in daycare or school. Assume also that the level of sickness that is allowed in daycare and school, has recently decreased due to a pandemic. For both the parents to be able to maintain a full workday, they will need that all the j children are in their respective care facility.

However, as this is not always the case, let us see if we can model it as a multistate system, where each child is a component[1]. The system state $\phi(\boldsymbol{X})$ tells us at what level the parents are able to work at that time, such that when $\phi(\boldsymbol{X})$ is at its maximum capacity, both parents are able to work 100%. Each child has these three states[2]:

- State 0 - Quarantine
- State 1 - Sick
- State 2 - Healthy

In our example we now choose $j = 3$, and the parameters are shown in Table 5.1. The capacity function $f_i(u)$ here will differ based on the age of the children, as the older they are, they can entertain themselves more, but also if they are younger they probably sleep more. Both cases may make the parent watching them able to do a bit work. Any experienced parent will also agree that when a child is not sick, but in quarantine for some reason, it is a larger

[1]The term "child" and "component" will both be used, depending on whether the context is based on human or more theoretical concepts.

[2]Quarantine can be one of two cases; either externally imposed, based on some event in the surroundings putting the child in quarantine, or internally imposed as a consequence of healing from illness, in order to limit contagiousness. This is reflected later in the choice of $\boldsymbol{P}^{(i)}$.

i	1			2			3		
u	0	1	2	0	1	2	0	1	2
$f_i(u)$	0.4	0.7	1.0	0.0	0.3	1.0	0.2	0.4	1.0
$\mu_u^{(i)}$	1.2	1.3	7.5	1.4	1.6	7.0	1.6	1.7	6.7
$\boldsymbol{P}^{(i)}$	$\boldsymbol{P}^{(1)}$			$\boldsymbol{P}^{(2)}$			$\boldsymbol{P}^{(3)}$		

Table 5.1: Parameters used in the pandemic example.

burden for the parent, than when a child is sick. The capacity function takes this into account as well. In Table 5.1 we find the capacity functions, and it can also be mentioned that child 1, 2 and 3 is aged 6, 4 and 2 years old, respectively.

It is a reasonable assumption that all components have the same transition matrix, i.e. for $i = 1, 2, 3$ we have:

$$\boldsymbol{P}^{(i)} = \begin{bmatrix} 0.05 & 0.05 & 0.90 \\ 0.50 & 0.50 & 0.00 \\ 0.05 & 0.05 & 0.90 \end{bmatrix}$$

With resulting stationary distribution:

$$\boldsymbol{\pi}^{(i)} = [\pi_0^{(i)}, \pi_1^{(i)}, \pi_2^{(i)}] = \left[\frac{1}{11}, \frac{1}{11}, \frac{9}{11}\right].$$

Recall that

$$P(X_i = u) = \frac{\pi_u^{(i)} \mu_u^{(i)}}{\sum_{v \in S_i} \pi_v^{(i)} \mu_v^{(i)}}$$

so the asymptotic probabilities for the components can be calculated:

$$\begin{array}{lll} P(X_1 = 0) = 0.017 & P(X_1 = 1) = 0.019 & P(X_1 = 2) = 0.964 \\ P(X_2 = 0) = 0.021 & P(X_2 = 1) = 0.024 & P(X_2 = 2) = 0.955 \\ P(X_3 = 0) = 0.025 & P(X_3 = 1) = 0.027 & P(X_3 = 2) = 0.948 \end{array}$$

This corresponds to that child 3 is sick or in quarantine about every 19th day, while child 1 is sick or in quarantine about every 28th day, which seems about right for the given children.

We have a case where the diagonal in $\boldsymbol{P}^{(i)}$ influences how long a child stays in one state. Our interpretation is that the waiting time $\mu_2^{(i)}$ as "how long until the child comes across a new source of infection". Then the value $P_{20}^{(i)}$ and $P_{21}^{(i)}$ gives the probability that this source leads to quarantine or sickness, respectively, while $P_{22}^{(i)}$ will in this understanding be the probability that the source of infection is fought by the childs immune system.

The waiting times are also arranged to reflect that the older a child gets, the more its immune system is working, hence it will—more likely—take longer until the child is sick again. The other values can also be interpreted in a similar fashion.

In order to put the components in a system, we first note that the components do not have different roles in the system. One is not more or less structurally important for the system to function properly. Hence a symmetric system where they all have the same structural importance, is the obvious choice. We then

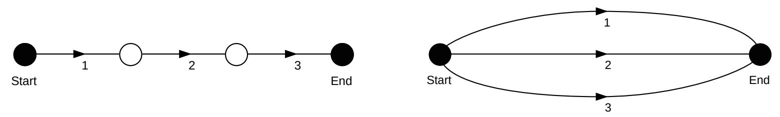


Figure 5.1: The two systems we consider in this chapter, the series system on the left and the parallel system on the right.

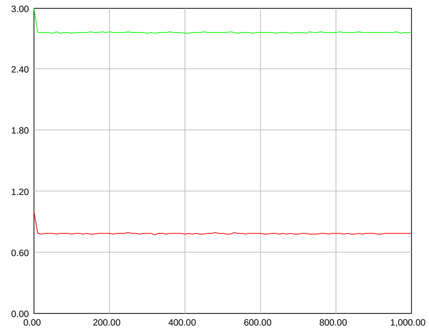


Figure 5.2: The mean system states for the series case in red ■, and the parallel case in green ■.

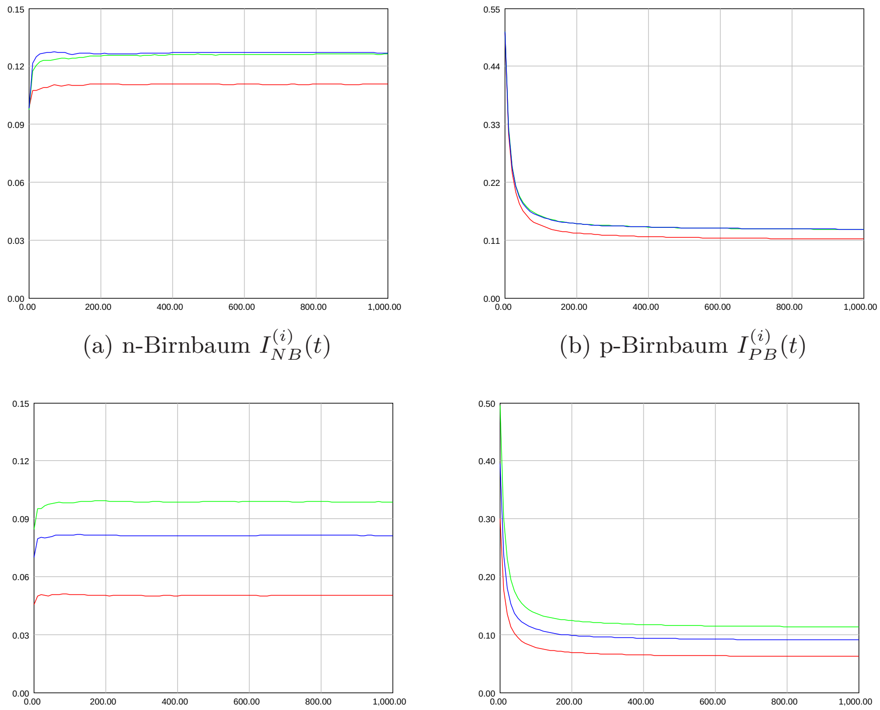
have to choose between a series system and a parallel system, they are both illustrated in Figure 5.1.

If we choose the series system, this will have its obvious weaknesses as one component can choke the system with not functioning at its top state. This may, however, *feel* like a good description of the reality for most parents. If we choose the system up as a parallel system, this will in a good way show the nuance of differences in system state based on what child is sick. What seems reasonable is to model the system both ways, and look at both cases, before a brief discussion and conclusion in the end. But first, let us take a quick look at the resulting mean system states[3], found in Figure 5.2.

In order compare the two systems by their mean system states, we can look at the ratio between the mean system state and top system state. In the series case this is $\approx 0.78/1.00 = 0.78$, while in the parallel case this is $\approx 2.76/3.00 = 0.92$. So the series system—as earlier suggested—has a mean system state further away from its top state than the parallel system. This is due to the earlier mentioned possibility of one component choking the system. This translates to that in the series case, a set of parents can expect to be working full about 78% of the time, while in the parallel case this percentage is at a level of 92%.

We consider both systems to be good approximations of the real life we are trying to describe here, so we will look further into both of them. Now, what child is most important?

[3]where the mean is taken over the 1000 simulations, not cumulatively



(a) n-Birnbaum $I_{NB}^{(i)}(t)$

(b) p-Birnbaum $I_{PB}^{(i)}(t)$

(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$

(d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

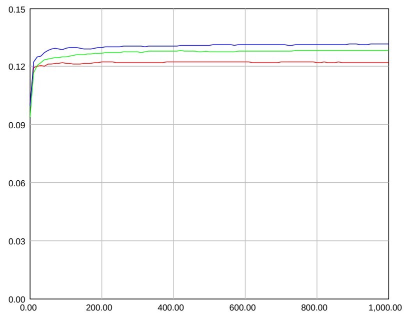
Figure 5.3: The cumulative mean for simulations of the four different importance measures, measuring the series case.

5.1 Simulation of the series case and the parallel case

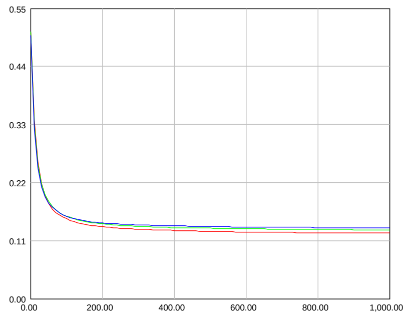
In the series case, we find the resulting plots in Figure 5.3. As the difference between the components in this example are rather small, note that we have made different choices of sections on the y-axis, compared to the examples in previous chapters. This is in order to zoom in on the graphs and show where the differences between them are. What is interesting is still not the level of importance, but how the different components' importance relate to each other.

We observe that child 1 is least important throughout the measures, which is to be expected as it was the most reliable one. Child 2 and 3 are almost equal in the probability-based measures, with a slightly higher importance of child 3. Still, in the impact-based measures, the relation between the three children are very clear: child 2 is most important, followed by child 3 and then child 1. This is because while $f_3(0) = 0.2$, $f_2(0) = 0$, so child 2 is the only child able to totally choke the system. This shows here, even though child 3 is slightly less reliable than child 2.

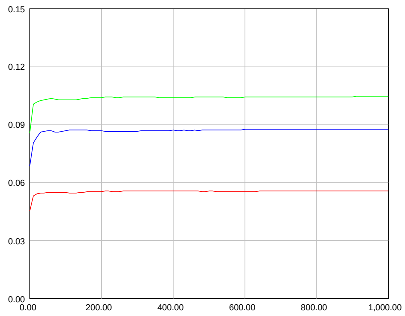
When it comes to the parallel case, the plots are in Figure 5.4. Note that also here the y-axis is not consistent with the examples in previous chapters. We observe quickly that the results here are very much similar to what we found in the series case. The same relation between the different children's importance is found in these plots, although the levels they end up at are slightly different.



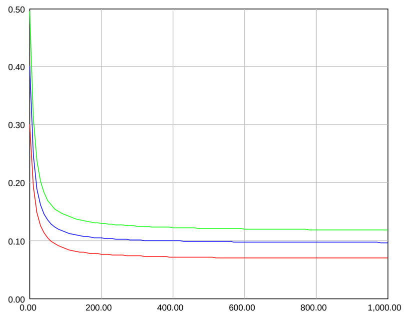
(a) n-Birnbaum $I_{NB}^{(i)}(t)$



(b) p-Birnbaum $I_{PB}^{(i)}(t)$



(c) n*-Birnbaum $I_{NB}^{*(i)}(t)$



(d) p*-Birnbaum $I_{PB}^{*(i)}(t)$

Figure 5.4: The cumulative mean for simulations of the four different importance measures, measuring the parallel case.

Child 2's middle level in the probability-based measures is more visible here. However, child 1's bottom level throughout the measures is more clear in the series case, i.e. the difference from the two other children is larger.

The forward-looking measures calculate the risk ahead of us, which can be translated to "what child is most likely to stay home the next day?", or in the impact-based measure, "at what level can we expect to be able to work the next day?".

However, the backward-looking measures are not as easily translated. One translation, at least for the probability-based measure, could be at the occasion where there suddenly is sickness inside the four walls of the family's home, and no one knows what the source of infection was. Well, according to our plots, the main suspect for bringing sickness would be child 3, even though the margins between the children are very small. The impact-based backward-looking measure could then be interpreted to be a tool for the parents' employers, telling them what child is most likely sick or in quarantine if they observe a drop in one of the parents working hours.

5.2 What child is more important?

Luckily, this example only found slight margins between the children's importance, hence no parent with more than one child should think that any child is *much* more important than the other. However, we can with large confidence say that the child that has the ability to make it impossible for the parent who is looking after him/her to work at all, in our example child 2, makes more impact on the system's function. However, the least reliable child, in our example child 3, has a slightly higher probability for changing the system.

The in our example should probably wash the children's hands in this order; child 2, child 3, and finally child 1, as a preventive method for keeping the family functioning.

CHAPTER 6

Concluding remarks

Throughout this thesis we have tried to wrap our head around the four importance measures. We have tried to understand what influences them—or not—and if so, in what way.

Remember the machine we talked about in the introduction? Let us return to the initial questions, and see if we can answer them with a toolbox full of importance measures.

Our first question was, upon machine failure, which component should you troubleshoot first? With the $I_{PB}^{(i)}(t)$-measure, you can get to know what component that most likely made the system state change - and in this case fail. If the machine was brand new and suddenly broke down, you should probably consider the $I_{PB}^{*(i)}$-measure. This would tell you what component that most likely made the greatest impact on the system.

After fixing the broken component, you want to start a preventive checking up on the components. What component should you prioritize first? Well, with the $I_{NB}^{(i)}(t)$-measure you can find what component is most likely to cause a system state change next, and you can start with this one. Or, if you only want to consider the component that will make the greatest impact on the system state, you should check the $I_{NB}^{*(i)}$-measure, of course.

This little story with the imaginary machine is of course a bit naive and simplified in its assumptions. Still, it tells how our measures could be used, in a hopefully understandable fashion.

6.1 Main results

The theoretical addition to multistate reliability theory, made in this thesis, includes:

- Two extensions of the Birnbaum binary structural importance measure into the multistate case, (2.25) and (2.26).
- Theorem 3.2.1 that takes these two above extensions and proves that a component in parallel with the rest of the system always have structural importance equal 1.
- Theorem 3.2.2 which proves that a component that is in parallel with the rest of the system, and that has transitions such that it cannot enter the same state or capacity, always have $I_{NB}^{(i)}(t) = I_{PB}^{(i)}(t) = 1$.

- Theorem 4.1.3 proves that if a transition matrix is both doubly stochastic and symmetric, then $I_{NB}^{(i)} = I_{PB}^{(i)}$.
- Theorem 4.1.6 proves that if a transition matrix is both doubly stochastic and symmetric, then $I_{NB}^{*(i)} = I_{PB}^{*(i)}$.

We have learned through some of the examples that our measures may rank the components the same way as the structural importance measures. However, we also have examples proving otherwise for all the four measures.

If we consider component 1 in System A first, where it was structurally most important. If it was equipped with high reliability—which would be a reasonable choice for a component that essential—it was often less important than the other components, given that at least one of the other components had lower reliability. We have cases like Example 3.4.3 where component 1 was absolutely least important in the forward-looking measures, but most important in the backward-looking measures. If we translate this to the machine story above, we should never prioritise to do preventive maintaining of this component, however, it would be the first we would check if the machine broke down.

Now, take component 2 in System A. It is structurally least important as it has lower capacity than component 3, which it is in parallel with. However, we observed in Example 3.4.4 that it almost blossomed in importance. As component 3 had low reliability, and component 1 often delivered lower than its max capacity, suddenly component 2 was a force to be reckoned with, and should no longer be ignored.

When we look at System B and Theorem 3.2.2, it would be tempting to stop examination and conclude that component 1 is most important. Still, Example 3.5.2 told us a different story when we considered the impact the component was able to make on the system state. This is an important notion.

Then, we introduced stochastic life cycles in the components. The main result was that when we allowed transitions to any state except the one the component was in before, through our uniform transition matrix, this increased importance in the forward-looking measures and decreased importance in the backward-looking measures.

We discovered in Example 4.2.3 on System A, that if we only added a transition matrix that was not doubly stochastic in one/some of the components, this would influence the other components' importance as well.

Another result that was useful—even though it is rather intuitive—is the consequence of having a non-zero diagonal in the transition matrix of component 1 in System B. This will lead to it not being included in Theorem 3.2.2, thus no longer most important, in fact in Example 4.3.2, it is absolutely least important throughout the measures.

6.2 Further work

During the production of this thesis, we had to put aside until later some of the things that caught our attention. Now, upon finishing the writing, we see some loose threads that could be interesting to investigate further. These includes:

- Non-stationary waiting times was originally a part of the project, but had to be cut as the thesis developed. This is a topic introduced in an upcoming paper by Huseby and Halle, that we did not have time to look at.

- We did not have time to prove an equivalent theorem to Theorem 3.2.1 for when a component is in series with the rest of the system. The structural measures strongly hint to that this actually is the case in multistate systems just as in binary systems.
- Moreover, Theorem 3.2.1 could most likely be made stronger by reducing the assumptions, more specific we would like to omit the assumption of a flow network, if possible.
- The reliability measures we have looked at considers *all* changes in system state. One could argue that only the negative—or positive—changes in system state is of interest. A further extension of the measures that takes this into account could be interesting.
- What cases, if any, make $J_{NB}^{(i)} \neq J_{PB}^{(i)}$?

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