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Geometric and arithmetic properties of motivic cohomology theories

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Preface

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List of papers

Paper I

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Druzhinin, A. and Kolderup, H. “Cohomological correspondence categories”. In: *Algebr. Geom. Topol.* vol. 20-3 (2020), pp. 1487–1541.
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Paper IV

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Kolderup, H. *Remarks on classical number theoretic aspects of Milnor–Witt* K-*theory*. 2019. Preprint. arXiv: 1906.07506.

Contents

Chapter 1

Introduction

The theme of this thesis lies in the field of motivic homotopy theory. More precisely, we study various motivic cohomology theories—or rather *categories of motives*, which are the homes of motivic cohomology theories—from two different points of view: on the one hand we aim to provide a few foundational results on various categories of motives via the study of the underlying geometric properties of motivic cohomology theories; on the other hand, we study some manifestations of motivic cohomology theories in number theory. We focus in particular on *Milnor–Witt* K-*theory*, which blends quadratic forms into the number theoretic information contained in Milnor K-theory, and which can be thought of as the initial motivic cohomology theory. In summary, this thesis investigates interactions between geometry, number theory, cohomology and quadratic forms that take place in motivic homotopy theory. In this introduction we aim to make this statement more precise by presenting some of these interactions via examples that ultimately lead to the definitions of the main characters of motivic homotopy theory. We will also shed light on what we mean by "motivic cohomology theories" and explain the title of the thesis in more detail. Along the way we mention how the results of this thesis fit into the picture.

1.1 Pythagorean triples

An ancient problem bearing Pythagoras' name provides a first glimpse at how geometry, arithmetic, cohomology and quadratic forms complement each other, interact with each other and are, in some sense, unified in the motivic world. Thus, before we move on to defining the main objects of motivic homotopy theory, let us take a look at the problem of finding all *Pythagorean triples*—that is, finding all integer solutions to the equation

$$x^2 + y^2 = z^2. \tag{1.1}$$

Asking for solutions of an equation over the integers is intrinsically a number theoretic problem. On the other hand, an equivalent formulation of the problem is to ask for rational points on the unit circle $x^2 + y^2 = 1$, which is a geometric question. This point of view can be further translated into finding the kernel of the norm homomorphism $\mathrm{N}\colon \mathbf{Q}(i)^\times \to \mathbf{Q}^\times$ given by $a + ib \mapsto a^2 + b^2$. But this is now a cohomological question: indeed, if we let σ denote the generator of the Galois group $G := \mathrm{Gal}(\mathbf{Q}(i)/\mathbf{Q})$, then $\ker(\mathrm{N})/(\sigma - 1)\mathbf{Q}(i)^\times$ is isomorphic to the first Galois cohomology group $\mathrm{H}^1(G, \mathbf{Q}(i)^\times)$ [Mil13, II Example 1.20]. Now Hilbert's Theorem 90 asserts that this cohomology group is trivial, which implies that an element $\alpha \in \mathbf{Q}(i)^\times$ lies in the kernel of the norm map if and only if

$\alpha = \sigma\beta/\beta$ for some $\beta \in \mathbf{Q}(i)^\times$. Writing $\beta = a + ib$, we then have

$$\frac{\sigma\beta}{\beta} = \frac{a - ib}{a + ib} = \frac{a^2 - b^2}{a^2 + b^2} - i\frac{2ab}{a^2 + b^2}.$$

This computation shows that the Pythagorean triples are parametrized by

$$(x, y, z) = (a^2 - b^2, 2ab, a^2 + b^2);$$

furthermore, it illustrates that it can often be fruitful to translate a mathematical problem into a question concerning cohomology groups.

Yet another take on the problem of finding Pythagorean triples is to view the equation (1.1) as asking whether the quadratic form $x^2 + y^2 - z^2$ is isotropic over $\mathbf{Q}$, i.e., whether it has a nontrivial rational zero. This point of view allows for a local-global principle to enter the picture: the form $x^2 + y^2 - z^2$ defines an element in the Witt ring $\mathrm{W}(\mathbf{Q})$ of equivalence classes of quadratic forms over $\mathbf{Q}$, and we can then apply the celebrated Hasse–Minkowski theorem which states that the natural homomorphism

$$\mathrm{W}(\mathbf{Q}) \to \bigoplus_{2 \leq p \leq \infty} \mathrm{W}(\mathbf{Q}_p) \tag{1.2}$$

into the Witt rings of all completions of $\mathbf{Q}$ is injective. In particular, this means that a quadratic form over $\mathbf{Q}$ is isotropic if and only if it is isotropic over $\mathbf{Q}_\infty = \mathbf{R}$ and over $\mathbf{Q}_p$ for all primes p. This principle gives another way to solve quadratic Diophantine equations similar to (1.1): if we can find a solution in $\mathbf{R}$ as well as a solution modulo a power of any prime number, then there is a solution over $\mathbf{Q}$.

We have now seen that the problem of finding Pythagorean triples can be tackled from a geometric, arithmetic, cohomological, and quadratic forms point of view. In 1970 Milnor published the paper [Mil70] in which he conjectured that all the different perspectives on the equation (1.1) above can in general be neatly linked together via the object $\mathrm{K}^{\mathrm{M}}_*(\mathbf{Q})$ now known as Milnor K-theory. More precisely, Milnor's conjecture states that for any field k there are natural isomorphisms

$$\begin{tikzcd}[column sep=small] & \mathrm{K}^{\mathrm{M}}_n(k)/2 \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ \mathrm{I}^n(k)/\mathrm{I}^{n+1}(k) & & \mathrm{H}^n_{\text{ét}}(k, \mu_2^{\otimes n}) \end{tikzcd} \tag{1.3}$$

for each $n \geq 0$. Here, $\mathrm{I}(k)$ denotes the ideal of even dimensional quadratic forms in the Witt ring of k, and $\mathrm{H}^*_{\text{ét}}$ denotes the étale cohomology groups generalizing the Galois cohomology group we encountered above.

Several years after Milnor stated the conjectural relationship between cohomology, quadratic forms and Milnor K-theory, Morel and Voevodsky provided a natural environment to study these objects via the introduction of *motivic homotopy theory* and the *motivic stable homotopy ∞-category*[1] $\mathbf{SH}(k)$ [MV99; VSF00]. In fact, both the Milnor K-groups and the étale cohomology groups above are examples of *motivic cohomology groups*, which are represented in $\mathbf{SH}(k)$ by the motivic Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbf{Z}$. Milnor's conjecture was eventually settled by Voevodsky [Voe03; Voe11; Voe96b] and Orlov–Vishik–Voevodsky [OVV07] (see also [RØ16] for an alternative proof) by means of, among other things, constructing power operations on $\mathrm{H}\mathbf{Z}$. This success was the first major achievement of motivic homotopy theory.

Below we will explain some of the basic constructions in the theory of motives and motivic homotopy theory as well as revisit the groups encountered above. For simplicity we work over a field k, although many of the constructions are valid in a more general setting. We let Sm_k denote the category of smooth, separated schemes of finite type over k.

1.2 Correspondences, transfers and motives

Above we have seen several examples of cohomology theories arising from a geometric or arithmetic origin; in particular, we saw the étale cohomology groups and their relation to Milnor K-theory and Witt groups. Étale cohomology, introduced by Grothendieck in the 1960's [Mil80], has proved to be an enormously powerful tool in the study of algebraic varieties—notably via the vital role these cohomology groups played in connection with the Weil conjectures. The structural properties of the étale cohomology groups further fuelled the advent of other cohomology theories, such as ℓ-adic cohomology used by Deligne to settle the final part of the Weil conjectures. In the wake of this outburst of various cohomology theories in algebraic geometry, Grothendieck observed several similarities between the different theories: they are all contravariant functors satisfying a version of Poincaré duality, Künneth- and Lefschetz theorems, and they come equipped with cycle class maps from Chow groups [MNP13, Chapter 1]. Cohomology theories satisfying these properties are nowadays referred to as *Weil cohomology theories*. In light of these observations, Grothendieck crystallized in a letter to Serre from 1964 the idea that there should be an underlying structure giving rise to all Weil cohomology theories. More precisely,

[1] In this thesis we have chosen to work in the language of ∞-categories [Lur09]. We do so in order to streamline the notation with Paper II, where Lurie's ∞-categorical version of the Barr–Beck theorem [Lur17, Theorem 4.7.3.5] is needed. This choice moreover allows us exploit the universal property of $\mathbf{SH}(-)$ [Rob15, Corollary 2.39], and furthermore allows for the discussion in Section 1.6 on the distinctions between discrete and nondiscrete ∞-categories of correspondences. Note however that we will mainly work with *presentable* ∞-categories [Lur09, Definition 5.5.0.1]. By [Lur09, Proposition A.3.7.6], any presentable ∞-category is equivalent to the underlying ∞-category of a model category, and any adjoint pair of such ∞-categories comes from a Quillen adjunction. By [Hoy17, §1.3], the motivic unstable and stable homotopy ∞-categories we will consider are the underlying ∞-categories of the corresponding model categories of motivic spaces or spectra as constructed by Morel and Voevodsky [MV99]. Thus, whenever we discuss motivic categories and adjunctions between such, the reader who wishes may safely think in terms of model categories rather than ∞-categories.

Grothendieck speculated upon the existence of the *motif* of a variety—an object which should contain the essence of all cohomological information of the variety [Gro86; Mil14]. The name *motif*, or *motive*, is inspired from art and iconography in which a motif is an element that in a sense constitutes the essence of the image and is often seen repeated in other works. Carrying this notion to the world of algebraic varieties, the motive of a variety should thus be an object representing the common cohomological information revealed each time a Weil cohomology theory is applied to it. In other words, a motive should be a universal cohomology theory on algebraic varieties.

The actual construction of a category of motives was also initiated by Grothendieck. More precisely, he constructed the category of *pure motives* from the category of smooth projective varieties over k by means of enlarging the mapping sets, allowing certain types of *correspondences* between varieties, and then performing an idempotent completion. See [Mil14] for details. Whether or not this category has all the desired properties, however, relies on several existence conjectures on algebraic cycles now known as the *standard conjectures* [Kle94]. Furthermore, Grothendieck envisioned the existence of another category, that of *mixed motives*, which should define the motive of any variety. The construction of the category of mixed motives is still an open problem [MNP13].

In 1987 Beilinson offered through a series of conjectures an alternative approach to the theory of mixed motives [Bei87]. Instead of trying to construct the category of mixed motives directly, one could try to construct a derived category of *motivic complexes* equipped with a hypothetical t-structure whose heart should consist of mixed motives. This point of view was later taken up by Voevodsky in [Voe96a] and ultimately led to Suslin and Voevodsky's construction of the *derived category of motives* $\mathbf{DM}(k)$ [VSF00]. Although there is in general no desired t-structure on $\mathbf{DM}(k)$ giving rise to mixed motives [Voe00b, Proposition 4.3.8], the category $\mathbf{DM}(k)$ nevertheless allows for a definition of the sought-after universal cohomology theory on smooth varieties over k. This cohomology theory is now referred to as *motivic cohomology*. Suslin and Voevodsky's construction of $\mathbf{DM}(k)$ relies in particular on the notion of *homotopy invariant presheaves with transfers*, which may in some sense be thought of as an abstraction of some of the properties satisfied by Chow groups and other Weil cohomology theories, and which constitutes a central tool in this thesis. Below we take a closer look at the notion of homotopy invariant presheaves with transfers before we explain Suslin and Voevodsky's construction of the derived category of motives $\mathbf{DM}(k)$ in more detail.

1.2.1 Why homotopy invariant presheaves with transfers?

In the spirit of Section 1.1 let us return to questions concerning sums of two squares, this time in a more geometric setting. Take a smooth affine curve X over k equipped with a finite surjective morphism $\pi\colon X \to \mathbf{A}^1$. Suppose we are given an invertible regular function f on X which we know to be a sum of two squares away from a closed point of X, say $f|_{X\setminus x} = g_1^2 + g_2^2 \in k[X \setminus x]$. Then one can ask, is f a global sum of two squares?

When dealing with sums of squares in Section 1.1 we saw that the norm homomorphism $\mathrm{N}\colon \mathbf{Q}(i)^\times \to \mathbf{Q}^\times$ was particularly useful, suggesting that it might be worthwhile to try a similar approach in order to answer the question above. In fact, since the scheme X is finite over $\mathbf{A}^1$ we do have a norm homomorphism also in this geometric setting, namely the map

$$\mathrm{N}\colon k[X]^\times \to k[\mathbf{A}^1]^\times$$

given by fiber integration, i.e.,

$$\mathrm{N}(u)(t) := \prod_{y\in\pi^{-1}(t)} u(y).$$

Now, the invertible regular functions on $\mathbf{A}^1$ are just the constant functions, i.e., $k[\mathbf{A}^1]^\times = k^\times$. In particular, $\mathrm{N}(f)$ is constant. Say $\pi(x) = 0$, and let $\{x_2, \ldots, x_n\}$ be the remaining points in the fiber over 0. Let also $\pi^{-1}(1) = \{y_1, \ldots, y_m\}$ be the points in the fiber over 1. Then, since $\mathrm{N}(f)$ is constant, it follows that

$$f(x)f(x_2)\cdots f(x_n) = \mathrm{N}(f)(0) = \mathrm{N}(f)(1) = f(y_1)\cdots f(y_m).$$

In other words,

$$f(x) = f(y_1)\cdots f(y_m)f(x_2)^{-1}\cdots f(x_n)^{-1}.$$

The right hand side being a product of sums of two squares, it follows that $f(x)$ is the sum of two squares as well.

The above computation concerns sections of the sheaf of units $\mathcal{O}^\times$; it essentially demonstrates that the restriction homomorphism $\mathcal{O}^\times(X) \to \mathcal{O}^\times(X \setminus x)$ is injective. The argument exploits two special properties of this sheaf:

(i) It is *homotopy invariant*—or $\mathbf{A}^1$*-invariant*—on smooth k-schemes, i.e., $\mathcal{O}^\times(Y \times \mathbf{A}^1) \cong \mathcal{O}^\times(Y)$ for all $Y \in \mathrm{Sm}_k$.

(ii) It comes equipped with norm maps, or “wrong way maps”

$$\mathrm{N}\colon \mathcal{O}^\times(Y) \to \mathcal{O}^\times(Y')$$

for any finite and surjective morphism $Y \to Y'$ in Sm_k.

A presheaf of abelian groups on smooth k-schemes possessing the two properties above[2] is called a homotopy invariant presheaf with transfers. More succinctly, a presheaf with transfers can be defined as a presheaf on a certain enlargement Cor_k of the category Sm_k that allows more morphisms. Following [MVW06], let Cor_k denote the additive category whose objects are the same as those of Sm_k, and whose morphisms are given as

$$\mathrm{Cor}_k(X, Y) := \bigoplus_T \mathbf{Z},$$

[2]In addition, we should assume that the presheaf carries finite coproducts to finite products.

where the direct sum is taken over all closed irreducible subsets T of $X \times Y$ which become finite and surjective over a component of X when equipped with the reduced scheme structure. The abelian group $\mathrm{Cor}_k(X, Y)$ is called the group of *finite correspondences from X to Y*. We can think of finite correspondences as multivalued maps taking only finitely many values. Composition in Cor_k can be defined by pulling back finite correspondences to a triple Cartesian product, intersecting and then pushing forward; see [MVW06, Lecture 1] for more details. The resulting category Cor_k is additive and symmetric monoidal with respect to the Cartesian product. We refer to this category as the category of *finite correspondences*, and the homotopy invariant presheaves on this category satisfy several remarkable properties generalizing the injectivity result for the map $\mathcal{O}^\times(X) \to \mathcal{O}^\times(X \setminus x)$ we considered above:

Theorem 1.2.1 ([Voe00a]). *Suppose that k is a perfect field and let $\mathscr{F}$ be a homotopy invariant presheaf with transfers over k. Then the following hold:*

1. *For any smooth k-scheme X and any dense open subscheme U of X, the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$ is injective.*
2. *The associated Zariski sheaf $\mathscr{F}_{\mathrm{Zar}}$ can be extended to a presheaf with transfers in a unique way.*
3. *The Zariski- and the Nisnevich sheafification of $\mathscr{F}$ coincide, i.e., $\mathscr{F}_{\mathrm{Zar}} \cong \mathscr{F}_{\mathrm{Nis}}$.*
4. *For each $n \geq 0$, the cohomology sheaves $\mathrm{H}^n_{\mathrm{Nis}}(-, \mathscr{F}_{\mathrm{Nis}})$ are homotopy invariant presheaves with transfers.*

The main point of the above theorem is that being a homotopy invariant presheaf with transfers is preserved under Nisnevich sheafification. This is one of the two fundamental properties shared by homotopy invariant presheaves with transfers, the other one being a cancellation property with respect to smashing with the group scheme $\mathbf{G}_m$. Although the formulation of the cancellation theorem requires some notation that will be explained in the next subsection, we state it here for future reference:

Theorem 1.2.2 ([Voe10]). *Let k be a perfect field. Then for any two effective motives $K, L \in \mathbf{DM}^{\mathrm{eff}}(k)$, the natural map*

$$[K, L]_{\mathbf{DM}^{\mathrm{eff}}(k)} \to [K(1), L(1)]_{\mathbf{DM}^{\mathrm{eff}}(k)}$$

is an isomorphism.

As we will see below, the homotopy invariant presheaves with transfers constitute the main ingredient in Suslin and Voevodsky's construction of the ∞-category $\mathbf{DM}(k)$ [VSF00]; Theorems 1.2.1 and 1.2.2 above ensure that their construction results in a well behaved stable ∞-category in which we can define motivic cohomology.

1.2.2 Motives

We will now provide some details on the construction of Suslin and Voevodsky's derived category of motives $\mathbf{DM}(k)$. As mentioned above, the category $\mathbf{DM}(k)$ should contain the (co)homological information of any smooth k-scheme X. In particular, there should be a functor from Sm_k mapping a scheme $X \in \mathrm{Sm}_k$ to a complex $\mathrm{M}(X)$ which computes the motivic cohomology of X. In order to motivate the construction of this complex, let us draw some parallels from topology.

Recall that the singular homology groups of a topological space X are computed as the homology of the singular chain complex

$$\mathrm{Sing}_\bullet(X) = (\cdots \to \mathbf{Z}[\mathrm{Map}_{\mathrm{Top}}(\Delta^n, X)] \to \mathbf{Z}[\mathrm{Map}_{\mathrm{Top}}(\Delta^{n-1}, X)] \to \cdots),$$

where Δ^n denotes the standard n-simplex. In algebraic geometry there is an analog of the topological n-simplex, namely $\Delta^n_k := \mathrm{Spec}(k[t_0, \dots, t_n]/(\sum_i t_i - 1))$. However, the analogous complex

$$\cdots \to \mathbf{Z}[\mathrm{Map}_{\mathrm{Sm}_k}(\Delta^n_k, X)] \to \mathbf{Z}[\mathrm{Map}_{\mathrm{Sm}_k}(\Delta^{n-1}_k, X)] \to \cdots$$

does not define any interesting homology groups of a scheme X. This is because there are, in general, too few morphisms in the category Sm_k; for example, if X is a smooth projective curve of positive genus then X receives only constant maps from Δ^n_k.

In order to define a singular homology theory on schemes, we can instead draw inspiration from the Dold–Thom theorem. Let X and Y be topological spaces, and define the set $\mathrm{Map}_{\mathrm{mult}}(Y, X)$ of *multivalued maps from Y to X* as the set of those subsets $T \subseteq Y \times X$ for which the projection to Y is a covering. Then there is a bijection

$$\mathrm{Map}_{\mathrm{mult}}(Y, X) \cong \mathrm{Map}(Y, \mathrm{Sym}^\infty(X))$$

between multivalued maps from Y to X and the set of maps from Y to the infinite symmetric product of X [Gór06, VII §78]. By the Dold–Thom theorem, the homotopy groups $\pi_i(\mathrm{Sym}^\infty(X))$ of $\mathrm{Sym}^\infty(X)$ are precisely the reduced homology groups $\widetilde{\mathrm{H}}_i(X)$ of the space X. This suggests another approach to the definition of singular homology of schemes: instead of using the free abelian group on $\mathrm{Map}_{\mathrm{Sm}_k}(\Delta^n_k, X)$, we should consider the free abelian group on *multivalued maps* from Δ^n_k to X. In other words, we should use the group $\mathrm{Cor}_k(\Delta^n_k, X)$ of finite correspondences. The resulting complex

$$\mathrm{M}(X) := (\cdots \to \mathrm{Cor}_k(\Delta^n_k \times (-), X) \to \mathrm{Cor}_k(\Delta^{n-1}_k \times (-), X) \to \cdots)$$

is referred to as the *motive* of X, or the *Suslin complex* of X. The motive of X is a complex of presheaves with transfers that contains homological information about the scheme X. In fact more is true: each presheaf $\mathrm{Cor}_k(\Delta^n_k \times (-), X)$ is a sheaf in the Nisnevich topology [MVW06, Lemma 6.2], and the cohomology presheaves of $\mathrm{M}(X)$ are homotopy invariant [MVW06, Corollary 2.19]. Any complex satisfying these properties is referred to as an *effective motivic complex*, and these complexes constitute the objects of an ∞-category $\mathbf{DM}^{\mathrm{eff}}(k)$ [Voe00b], [BH18, §14]. The symmetric monoidal structure on finite correspondences extends to $\mathbf{DM}^{\mathrm{eff}}(k)$, whose unit object is the motive of a point, $\mathrm{M}(\mathrm{Spec}(k))$. For this reason, we write $\mathbf{Z}$ for the unit motive $\mathrm{M}(\mathrm{Spec}(k))$. Moreover, in $\mathbf{DM}^{\mathrm{eff}}(k)$ the motive of the projective line splits into a copy of $\mathbf{Z}$ and the shifted *Tate motive* $\mathbf{Z}(1)$, i.e.,

$$\mathrm{M}(\mathbf{P}^1) \simeq \mathbf{Z} \oplus \mathbf{Z}(1)[2].$$

This splitting reflects the fact that in classical topology, the singular cohomology of the projective line is concentrated in degrees 0 and 2. By [MVW06, Theorem 4.1], the motivic complex $\mathbf{Z}(1)$ may alternatively be described as $\mathbf{Z}(1) \simeq \mathcal{O}^{\times}[-1]$. Using the Tate motive $\mathbf{Z}(1)$ we can define, for any $n \geq 0$, the *Tate twist* $\mathscr{F}^{\bullet}(n)$ of a motivic complex $\mathscr{F}^{\bullet}$ as $\mathscr{F}^{\bullet}(n) := \mathscr{F}^{\bullet} \otimes \mathbf{Z}(1)^{\otimes n}$. We can think of the Tate twist as analogous to smashing with a sphere in the unstable homotopy category. Hence, similar to how the stable homotopy category is constructed from the unstable one by inverting the smash product with the circle, we can invert the Tate twist and obtain the *derived category of motives*

$$\mathbf{DM}(k) := \mathbf{DM}^{\mathrm{eff}}(k)[\mathbf{Z}(1)^{\otimes -1}].$$

This procedure yields a presentable, symmetric monoidal, stable ∞-category $\mathbf{DM}(k)$ equipped with a functor $\mathrm{M}\colon \mathrm{Sm}_k \to \mathbf{DM}(k)$ that sends a smooth k-scheme X to its motive $\mathrm{M}(X) \in \mathbf{DM}(k)$. By the cancellation theorem, the natural functor $\mathbf{DM}^{\mathrm{eff}}(k) \to \mathbf{DM}(k)$ is fully faithful. The key point is now that in $\mathbf{DM}(k)$ we can define, for any pair of integers p and q, the *motivic cohomology groups of X in bidegree* (p, q) as

$$\mathrm{H}^{p,q}(X, \mathbf{Z}) := [\mathrm{M}(X), \mathbf{Z}(q)[p]]_{\mathbf{DM}(k)} := \pi_0 \operatorname{Map}_{\mathbf{DM}(k)}(\mathrm{M}(X), \mathbf{Z}(q)[p]).$$

Suslin and Voevodsky's definition of motivic cohomology is believed to be a satisfactory construction of a universal cohomology theory as envisioned by Grothendieck. Furthermore, several well known invariants appear as motivic cohomology groups. For example, the diagonal motivic cohomology groups of the base field recover the Milnor K-groups, i.e., $\mathrm{H}^{n,n}(k, \mathbf{Z}) \cong \mathrm{K}^{\mathrm{M}}_n(k)$ [NS89; Tot92]. On the other hand, there is an isomorphism from $\mathrm{H}^{2n,n}(X, \mathbf{Z})$ to the Chow group $\mathrm{CH}^n(X)$ for any $X \in \mathrm{Sm}_k$ [MVW06, Lecture 17].

1.3 Motivic homotopy theory

After Suslin and Voevodsky's construction of the derived category of motives $\mathbf{DM}(k)$, the foundations of a "nonlinear" version of the theory were laid by Morel and Voevodsky in the works [MV99; Voe98]. More precisely, Morel and Voevodsky introduced the *motivic unstable and stable homotopy ∞-categories* $\mathbf{H}(k)$ and $\mathbf{SH}(k)$ which serve as homotopical counterparts to the homological $\mathbf{DM}^{\mathrm{eff}}(k)$ and $\mathbf{DM}(k)$. Below we briefly review the construction of these ∞-categories.

1.3.1 Homotopy invariant theories

As we have seen above, the sheaf of units $\mathcal{O}^\times$ is *homotopy invariant* on smooth k-schemes, that is, $\mathcal{O}^\times(X \times \mathbf{A}^1) \cong \mathcal{O}^\times(X)$ for any $X \in \mathrm{Sm}_k$. In fact, we have seen that $\mathcal{O}^\times$ is a homotopy invariant presheaf with transfers. As a sheaf on Sm_k we can identify $\mathcal{O}^\times$ with the sheaf of algebraic K-groups K_1 and note that, by [Wei13, V Theorem 6.3], the phenomenon of homotopy invariance remains true for all higher K-groups as well: for $X \in \mathrm{Sm}_k$ we have $\mathrm{K}_n(X \times \mathbf{A}^1) \cong \mathrm{K}_n(X)$ for any $n \geq 0$. However, the sheaf K_n does not possess the structure of presheaf with transfers in general [MVW06, Example 2.7]. Therefore algebraic K-theory is not represented in $\mathbf{DM}(k)$. On the other hand, in topology we have the stable homotopy category and the Brown representability theorem which provide a convenient setting to study topological K-theory. Thus we can ask if there is a more general construction than that of $\mathbf{DM}(k)$ providing a homotopy theoretic framework to study sheaves like algebraic K-theory. This is accomplished in the work [MV99]. Indeed, one of the features of Morel and Voevodsky's motivic homotopy theory is that algebraic K-theory, as well as many other interesting invariants of smooth k-schemes, becomes representable in the motivic stable homotopy ∞-category $\mathbf{SH}(k)$.

1.3.2 Motivic unstable and stable homotopy theory

In order to construct $\mathbf{SH}(k)$ we start with the category Sm_k consisting of smooth, separated schemes of finite type over k. This category is however poorly suited to do homotopy theory. For example, as taking colimits is a ubiquitous maneuver in homotopy theory we need to enlarge the category Sm_k by freely adjoining all small colimits. By [Lur09, Corollary 5.1.5.8], the operation of freely adjoining small colimits to Sm_k is equivalent to instead considering presheaves of spaces—or ∞-groupoids—on Sm_k. Let Spc denote the ∞-category of spaces, for which we can use Kan complexes as a suitable model. We can then consider the ∞-category $\mathrm{PSh}(\mathrm{Sm}_k) := \mathrm{Fun}(\mathrm{Sm}_k^{\mathrm{op}}, \mathrm{Spc})$ of presheaves of spaces on Sm_k, and note that we are now in a position to impose $\mathbf{A}^1$-invariance as mentioned above. We call a presheaf $\mathscr{F} \in \mathrm{PSh}(\mathrm{Sm}_k)$ *homotopy invariant*, or $\mathbf{A}^1$*-invariant*, if the projection $X \times \mathbf{A}^1 \to X$ induces an equivalence $\mathscr{F}(X) \xrightarrow{\simeq} \mathscr{F}(X \times \mathbf{A}^1)$ for any $X \in \mathrm{Sm}_k$. The inclusion $\mathrm{PSh}_{\mathbf{A}^1}(\mathrm{Sm}_k) \hookrightarrow \mathrm{PSh}(\mathrm{Sm}_k)$ of the full subcategory[3] of homotopy invariant presheaves admits a left adjoint $\mathrm{L}_{\mathbf{A}^1} : \mathrm{PSh}(\mathrm{Sm}_k) \to \mathrm{PSh}_{\mathbf{A}^1}(\mathrm{Sm}_k)$, which we refer to as $\mathbf{A}^1$-localization.

As a result of the above procedure we have forced the affine line $\mathbf{A}^1$ to play a role analogous to that of the unit interval in topology. However, in topology we also have coverings by open sets. In order to capture local phenomena we should specify a topology also in the motivic setting. There is a lot of room for choice here, but for the purposes of this thesis the Nisnevich topology [MVW06, Lecture 12] is the most suitable. Thus we consider the inclusion $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) \hookrightarrow \mathrm{PSh}(\mathrm{Sm}_k)$ of the full subcategory of Nisnevich sheaves on Sm_k.

[3]Following the terminology of Lurie [Lur09, Remark 1.2.11.1] we use the term "subcategory" rather than "sub-∞-category".

This inclusion admits a left adjoint $\mathrm{L}_{\mathrm{Nis}} \colon \mathrm{PSh}(\mathrm{Sm}_k) \to \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ referred to as Nisnevich localization. The *motivic unstable homotopy ∞-category* $\mathbf{H}(k)$ is then the full subcategory of $\mathrm{PSh}(\mathrm{Sm}_k)$ spanned by homotopy invariant Nisnevich sheaves. The objects of $\mathbf{H}(k)$ are referred to as *motivic spaces*, and the localization $\mathrm{L}_{\mathrm{mot}} \colon \mathrm{PSh}(\mathrm{Sm}_k) \to \mathbf{H}(k)$ is called the *motivic localization functor*. For $X \in \mathrm{Sm}_k$ we write X also for the motivic space represented by the scheme X.

Parallelling the role of pointed spaces in topology, we obtain a pointed variant $\mathbf{H}_\bullet(k)$ of the above construction by defining $\mathbf{H}_\bullet(k)$ to be the undercategory $\mathbf{H}(k)_{*/}$, where $* := \mathrm{Spec}(k)$ is the final object of $\mathbf{H}(k)$. There is a functor $(-)_+ \colon \mathbf{H}(k) \to \mathbf{H}_\bullet(k)$ given by adding a base point, i.e., $X_+ := X \amalg \mathrm{Spec}(k)$. In the ∞-category $\mathbf{H}_\bullet(k)$ we have the object $S^1_s := \Delta^1/\partial\Delta^1$ pointed by the class of $\partial\Delta^1$, and the object $\mathbf{G}_m^{\wedge 1} := (\mathbf{G}_m, 1)$ pointed by 1. Both play a role resembling that of the circle S^1 in topology. The smash powers of these objects form a bigraded family of motivic spheres $S^{p,q} := S_s^{p-q} \wedge \mathbf{G}_m^{\wedge q}$, and smashing with the sphere $S^{p,q}$ defines an endofunctor on $\mathbf{H}_\bullet(k)$ denoted by $\Sigma^{p,q}$. By considering the colimit of the diagram $\mathbf{A}^1 \leftarrow \mathbf{G}_m \to \mathbf{A}^1$ along with the fact that $\mathbf{A}^1$ is contractible in $\mathbf{H}_\bullet(k)$, we find that there is a string of equivalences

$$S^{2,1} := S^1_s \wedge \mathbf{G}_m^{\wedge 1} \simeq (\mathbf{P}^1, \infty) \simeq \mathbf{A}^1/\mathbf{A}^1 \setminus 0 =: \mathrm{T}$$

in $\mathbf{H}_\bullet(k)$. The object T above, known as the *Tate object*, can be thought of as analogous to the Tate motive $\mathbf{Z}(1)[2]$ of $\mathbf{DM}(k)$; stabilizing with respect to it yields the *motivic stable homotopy ∞-category* $\mathbf{SH}(k)$, i.e.,

$$\mathbf{SH}(k) := \mathbf{H}_\bullet(k)[\mathrm{T}^{\wedge -1}].$$

We refer to [Rob15, Corollary 2.22] for more details on this stabilization process. We have an adjunction

$$\Sigma^\infty_{\mathrm{T}} : \mathbf{H}_\bullet(k) \rightleftarrows \mathbf{SH}(k) : \Omega^\infty_{\mathrm{T}}$$

in which the left adjoint sends a scheme $X \in \mathrm{Sm}_k$ to its suspension spectrum $\Sigma^\infty_{\mathrm{T}} X_+$ whose n-th space is $X_+ \wedge \mathrm{T}^{\wedge n}$. In particular, the suspension spectrum of $\mathrm{Spec}(k)$ gives the unit for the symmetric monoidal structure on $\mathbf{SH}(k)$, that is, the *motivic sphere spectrum*

$$\mathbf{1} := \Sigma^\infty_{\mathrm{T}}(\mathrm{Spec}(k)_+) \in \mathbf{SH}(k).$$

In addition to the sphere spectrum, the ∞-category $\mathbf{SH}(k)$ contains an abundance of interesting objects; rather than listing those here let us only note that, as promised in Section 1.3.1, algebraic K-theory is representable by a spectrum $\mathrm{KGL} \in \mathbf{SH}(k)$ [PPR09]. Furthermore, the motivic stable homotopy category enjoys a fully fledged formalism of six functors [Ayo07; Hoy17]. In fact, $\mathbf{SH}(k)$ is in a certain sense initial among the categories possessing Grothendieck's six operations: by [Rob15, Corollary 2.39], the ∞-category $\mathbf{SH}(k)$ is universal among the presentable, symmetric monoidal ∞-categories $\mathscr{D}$ that are equipped with a functor $F \colon \mathrm{Sm}_k \to \mathscr{D}$ for which the following is satisfied:

- The functor F is homotopy invariant and satisfies Nisnevich descent.
- The cofiber of the map $F(\mathrm{Spec}(k)) \to F(\mathbf{P}^1)$ induced by the rational point ∞ acts invertibly on $\mathscr{D}$.

In particular, there is a functor γ^* from $\mathbf{SH}(k)$ to the derived category of motives $\mathbf{DM}(k)$. In fact, the functor γ^* constitutes the left adjoint of a free-forgetful adjunction [CD19, §10.1]

$$\gamma^* : \mathbf{SH}(k) \rightleftarrows \mathbf{DM}(k) : \gamma_*. \tag{1.4}$$

Applying the sphere spectrum to the unit $\mathrm{id} \to \gamma_*\gamma^*$ of this adjunction we obtain the *motivic Eilenberg–Mac Lane spectrum*

$$\mathrm{H}\mathbf{Z} := \gamma_*\gamma^*(\mathbf{1}) \in \mathbf{SH}(k)$$

we encountered in Section 1.1. The spectrum $\mathrm{H}\mathbf{Z}$ represents motivic cohomology groups, that is,

$$\mathrm{H}^{p,q}(X, \mathbf{Z}) \cong [\Sigma^\infty_{\mathrm{T}} X_+, \Sigma^{p,q}\mathrm{H}\mathbf{Z}]_{\mathbf{SH}(k)}$$

for any $X \in \mathrm{Sm}_k$. Parallelling the fact that the derived category of abelian groups is equivalent to the category of modules over the classical Eilenberg–Mac Lane spectrum [SS03], the main theorem of [RØ08] asserts that there is an equivalence

$$\mathbf{DM}(k) \simeq \mathrm{Mod}_{\mathrm{H}\mathbf{Z}[1/e]}(\mathbf{SH}(k)).$$

Here e denotes the exponential characteristic of the field k.

1.4 Milnor–Witt K-theory and cohomology theories with quadratic forms

One of the most basic questions in stable homotopy theory is, what are the stable homotopy classes of endomorphisms of the sphere? More generally, what are the stable homotopy groups of the sphere spectrum? In the motivic world, the analogous question means to compute the groups

$$\pi_{p,q}(\mathbf{1}) := [\Sigma^{p,q}\mathbf{1}, \mathbf{1}]_{\mathbf{SH}(k)}.$$

More generally, given any spectrum $E \in \mathbf{SH}(k)$ we can ask for its homotopy groups $\pi_{p,q}(E) := [\Sigma^{p,q}\mathbf{1}, E]_{\mathbf{SH}(k)}$ for any $p, q \in \mathbf{Z}$. However, one key property on which motivic homotopy theory differs from classical homotopy theory is that a spectrum in $\mathbf{SH}(k)$ need not be cellular [DI05]. For example, if X is the spectrum of a nontrivial field extension of k then $\Sigma^\infty_{\mathrm{T}} X_+ \in \mathbf{SH}(k)$ is not cellular [IØ19, §1]. As a consequence, the motivic homotopy groups do not in general detect equivalences. The notion of *motivic stable homotopy sheaves* has been introduced in order to remedy this problem. These objects are Nisnevich sheaves of abelian groups on Sm_k which generalize the homotopy groups we encountered above in the sense that they allow maps out of any smooth k-scheme, not only

Spec(k). In more detail, the *sheaf* $\underline{\pi}_p(E)_q$ *of motivic homotopy groups* of a spectrum $E \in \mathbf{SH}(k)$ is defined as the Nisnevich sheaf associated to the presheaf $\pi_p(E)_q$ on Sm_k given by

$$\pi_p(E)_q(U) := [\Sigma^{p,0}\Sigma^\infty_{\mathrm{T}} U_+, \Sigma^{q,q} E]_{\mathbf{SH}(k)}$$

for $U \in \mathrm{Sm}_k$. Thus the homotopy groups we encountered above are given as global sections of the sheaves of homotopy groups, i.e., $\underline{\pi}_p(E)_q(\mathrm{Spec}(k)) = \pi_{p-q,-q}(E)$.

Having defined motivic homotopy groups and sheaves of such, the main question is how to compute them. The first main result in this direction is Morel's calculation [Mor04a, Theorem 6.4.1] which shows that

$$\pi_{n,n}(\mathbf{1}) \cong \mathrm{K}^{\mathrm{MW}}_{-n}(k)$$

for any integer n. In terms of sheaves of homotopy groups, the result reads

$$\underline{\pi}_0(\mathbf{1})_{-n} \cong \mathbf{K}^{\mathrm{MW}}_{-n}.$$

The groups $\mathrm{K}^{\mathrm{MW}}_*(k)$ are referred to as the *Milnor–Witt* K*-groups* of the field k, and can be defined as follows [Mor12, Chapter 3]. Take the free graded associative $\mathbf{Z}$-algebra on one generator $[a]$ of degree $+1$ for each unit $a \in k^\times$, as well as one generator η of degree -1, and impose the following relations:

(1) $[a][1-a] = 0$.

(2) $[ab] = [a] + [b] + \eta[a][b]$.

(3) $\eta[a] = [a]\eta$.

(4) $(2 + \eta[-1])\eta = 0$.

The resulting graded ring is the Milnor–Witt K-theory of k, denoted $\mathrm{K}^{\mathrm{MW}}_*(k)$. By [Mor12, Lemma 3.6 (1)], the symbols $[a_1, \ldots, a_n] := [a_1] \cdots [a_n] \in \mathrm{K}^{\mathrm{MW}}_n(k)$ generate $\mathrm{K}^{\mathrm{MW}}_n(k)$. As the name suggests, the Milnor–Witt K-theory of k blends quadratic forms coming from the Witt ring of k with the Milnor K-theory of k. More precisely, by [Mor04b], the group $\mathrm{K}^{\mathrm{MW}}_n(k)$ fits in a pullback square

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_n(k) \arrow[r, "p"] \arrow[d] & \mathrm{K}^{\mathrm{M}}_n(k) \arrow[d] \\
\mathrm{I}^n(k) \arrow[r] & \mathrm{I}^n(k)/\mathrm{I}^{n+1}(k),
\end{tikzcd}$$

where $\mathrm{I}^n(k)$ is the n-th power of the fundamental ideal in the Witt ring of k. The map p is given by killing η, while the left hand vertical homomorphism is given by mapping $[a_1, \ldots, a_n]$ to the Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle \in \mathrm{I}^n(k)$. In particular, the negative Milnor–Witt K-groups of k are all isomorphic to the Witt group of k, while $\mathrm{K}^{\mathrm{MW}}_0(k)$ recovers the Grothendieck–Witt group $\mathrm{GW}(k)$ of k [Mor12, Lemma 3.10]. In fact, the Milnor–Witt K-groups define an unramified Nisnevich sheaf $\mathbf{K}^{\mathrm{MW}}_n$ of abelian groups on the category Sm_k [Mor12, Chapter 2].

Keeping in mind that the Chow groups $\mathrm{CH}^n(X)$ of a smooth k-scheme X can be defined in terms of the cohomology of the Zariski sheaf on Sm_k associated to the presheaf of Milnor K-groups $U \mapsto \mathbf{K}_n^{\mathrm{M}}(U)$ [Gil05], taking the cohomology of the sheaf $\mathbf{K}_n^{\mathrm{MW}}$ allows us to define a version of Chow groups that take into account quadratic form theory. However, in order to obtain proper pushforward maps we need to twist by local coefficient systems [Mor12, Remark 3.21]. For any line bundle $\mathscr{L}$ on $X \in \mathrm{Sm}_k$, the *n-th Milnor–Witt sheaf on X twisted by $\mathscr{L}$* is defined as

$$\mathbf{K}_n^{\mathrm{MW}}(\mathscr{L}) := \mathbf{K}_n^{\mathrm{MW}} \otimes_{\mathbf{Z}[\mathbf{G}_m]} \mathbf{Z}[\mathscr{L}^\times].$$

Here $\mathbf{Z}[\mathbf{G}_m]$ and $\mathbf{Z}[\mathscr{L}^\times]$ denotes the Nisnevich sheaves associated to $U \mapsto \mathbf{Z}[\mathcal{O}^\times(U)]$ and $U \mapsto \mathbf{Z}[\mathscr{L}(U) \setminus 0]$, respectively. We refer to [CF17b, §1.2] for more details. We can now define the *Chow–Witt groups* $\widetilde{\mathrm{CH}}^n(X, \mathscr{L})$ of a scheme $X \in \mathrm{Sm}_k$, twisted by $\mathscr{L}$, as

$$\widetilde{\mathrm{CH}}^n(X, \mathscr{L}) := \mathrm{H}^n_{\mathrm{Nis}}(X, \mathbf{K}_n^{\mathrm{MW}}(\mathscr{L})).$$

Similarly we can define Chow–Witt groups with support $\widetilde{\mathrm{CH}}^n_Z(X, \mathscr{L})$ on closed subsets $Z \subseteq X$ by taking cohomology with support. The upshot is that we obtain pushforward maps

$$f_*\colon \widetilde{\mathrm{CH}}^n_Z(X, \omega_f \otimes f^*\mathscr{L}) \to \widetilde{\mathrm{CH}}^{n+\dim Y - \dim X}_{f(Z)}(Y, \mathscr{L})$$

for any morphism $f\colon X \to Y$ such that $f|_Z$ is proper [Mor12, Corollary 5.30]. This construction results in a theory similar to that of Chow groups whose geometric significance lies in the fact that the Chow–Witt groups detect splittings of vector bundles. Indeed, for any oriented rank n vector bundle ξ on X there is a corresponding Euler class $e(\xi)$ in $\widetilde{\mathrm{CH}}^n(X)$ with the property that ξ splits off a trivial line bundle if and only if $e(\xi) = 0$. We refer to [Mor12, §8.2] and [AF16] for more details on the construction and basic properties of the Euler class.

The introduction of Milnor–Witt K-groups and Chow–Witt groups has sparked off several branches of research in motivic homotopy theory, around many of which the topics of this thesis revolve. The basic idea is to blend the theory of quadratic forms into classical topics using Milnor–Witt K-groups. In this direction there is a substantial program, initiated by Barge–Morel, Fasel, Hoyois, Kass–Wickelgren, Levine, Wendt and others [BM00; DJK18; Fas07; Hor+19; Hoy14; KW17; Lev18; SW19; Wen18], which aims to incorporate the theory of quadratic forms into intersection theory and enumerative geometry. To give a few examples, Hoyois obtained in [Hoy14] a quadratic version of the Grothendieck–Lefschetz fixed point formula, while Kass and Wickelgren counted in [KW17] the number of lines on a smooth cubic surface by using quadratic forms instead of the integers. The result of Kass and Wickelgren contains information on the number of lines of hyperbolic and elliptic type, and gives back the classical well known answer 27 by taking ranks of the quadratic forms involved. In a similar direction, Levine's work [Lev18] sets up a framework for enumerative geometry with quadratic forms. Although this is not precisely the path we will tread in this thesis, it is very much related. Indeed, it is possible to attach quadratic forms to finite correspondences and then try to parallel the whole construction of Suslin and Voevodsky's derived category of motives. This was initiated and completed by Calmès–Déglise–Fasel in [CF17b; DF17a; DF17b] and has been the starting point as well as a central theme of this thesis. We will therefore elaborate a bit further on this matter.

1.4.1 Milnor–Witt correspondences

Recall that a finite correspondence, in the sense of Suslin and Voevodsky, amounts to a closed subset of a Cartesian product of schemes, along with an integer attached to each irreducible component of the closed subset. In other words, the group $\mathrm{Cor}_k(X, Y)$ of finite correspondences from X to Y can be described as a colimit of Chow groups with support,

$$\mathrm{Cor}_k(X, Y) = \varinjlim_T \mathrm{CH}^{\dim Y}_T(X \times Y).$$

Here the colimit is taken over all closed subsets T of $X \times Y$ such that each irreducible component of the reduced subscheme associated to T is finite and surjective over a component of X. Assuming that k is a perfect field of characteristic different from 2, Calmès and Fasel [CF17b] replace Chow groups by Chow–Witt groups and show that this defines a new category of correspondences $\widetilde{\mathrm{Cor}}_k$, baptized the category of *finite Milnor–Witt correspondences*. The mapping sets of $\widetilde{\mathrm{Cor}}_k$ are given as

$$\widetilde{\mathrm{Cor}}_k(X, Y) := \varinjlim_T \widetilde{\mathrm{CH}}^{\dim Y}_T(X \times Y, p_Y^* \omega_{Y/k}),$$

where $p_Y : X \times Y \to Y$ is the projection, and the colimit runs over the closed subsets $T \subseteq X \times Y$ as above. A finite Milnor–Witt correspondence thus consists of an ordinary finite correspondence along with a quadratic form defined over the function field of each irreducible component of the support of the correspondence. By taking the rank of these quadratic forms we obtain an ordinary finite correspondence. In fact, this defines a forgetful functor $\widetilde{\mathrm{Cor}}_k \to \mathrm{Cor}_k$.

Given the category $\widetilde{\mathrm{Cor}}_k$ one can try to perform similar constructions as for the classical category Cor_k. In Paper I we prove that the analog of Theorem 1.2.1 is true for homotopy invariant presheaves of abelian groups on the category $\widetilde{\mathrm{Cor}}_k$. On the other hand, the cancellation theorem for Milnor–Witt correspondences was established by Fasel–Østvær in [FØ17]. As a result, Voevodsky's construction of the derived category of motives $\mathbf{DM}(k)$ can be carried out in the setting of Milnor–Witt correspondences; the resulting ∞-category $\widetilde{\mathbf{DM}}(k)$ of *Milnor–Witt motivic complexes* was constructed by Déglise and Fasel in [DF17a]. The ∞-category $\widetilde{\mathbf{DM}}(k)$ is in some sense closer to $\mathbf{SH}(k)$ than $\mathbf{DM}(k)$; for example, the group of endomorphisms of the unit object in $\widetilde{\mathbf{DM}}(k)$ is isomorphic to the endomorphisms of the sphere spectrum $\mathbf{1} \in \mathbf{SH}(k)$ [CF17a]. As is the case for

$\mathbf{DM}(k)$, there is an adjunction

$$\widetilde{\gamma}^* : \mathbf{SH}(k) \rightleftarrows \widetilde{\mathbf{DM}}(k) : \widetilde{\gamma}_* \tag{1.5}$$

to the motivic stable homotopy category, and the resulting motivic Eilenberg–Mac Lane spectrum $\mathrm{H}\widetilde{\mathbf{Z}} := \widetilde{\gamma}_*\widetilde{\gamma}^*(\mathbf{1})$ represents *Milnor–Witt motivic cohomology* $\mathrm{H}^{p,q}_{\mathrm{MW}}(X, \mathbf{Z})$. In particular, the diagonal motivic homotopy groups of $\mathrm{H}\widetilde{\mathbf{Z}}$ satisfy

$$\pi_{-n,-n}\mathrm{H}\widetilde{\mathbf{Z}} \cong \mathrm{H}^{n,n}_{\mathrm{MW}}(k, \mathbf{Z}) \cong \mathrm{K}^{\mathrm{MW}}_n(k)$$

for any $n \in \mathbf{Z}$ [CF17a]. Moreover, the results of Paper II show that the adjunction (1.5) realizes $\widetilde{\mathbf{DM}}(k)$ as the ∞-category of highly structured modules over the $\mathcal{E}_\infty$-ring spectrum $\mathrm{H}\widetilde{\mathbf{Z}}$ after inverting the exponential characteristic of k.

1.4.2 Milnor–Witt K-theory and number theory

Above we have discussed some examples on how Milnor–Witt K-theory allows us to blend quadratic form theory and geometry, in the setting of refined enumerative geometry and finite Milnor–Witt correspondences. In Paper V we aim to take a few steps in a similar direction in number theory, thus providing for the "arithmetic" part of the title of this thesis. We will now explain this in more detail. To do so, let us start by first recalling some classical results on the relations between number theory and K-theory.

One of the most fundamental objects of study in algebraic number theory is the *class group* Cl_k of a number field k. It measures the extent to which the ring of integers $\mathcal{O}_k$ in k fails to be a unique factorization domain. Furthermore, the exact sequence

$$1 \to \mathcal{O}_k^\times \to k^\times \to \bigoplus_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_k)^{(1)}} \mathbf{Z} \xrightarrow{\mathrm{div}} \mathrm{Cl}_k \to 1$$

shows that the class group is linked to another fundamental invariant in number theory, the *group of global units* $\mathcal{O}_k^\times$. Both of these invariants are manifestations of K-theoretic phenomena. Indeed, from the characterization of finitely generated projective modules over Dedekind domains it follows that the class group of k is precisely the torsion subgroup of the zeroth algebraic K-group $\mathrm{K}_0(\mathcal{O}_k)$ of $\mathcal{O}_k$ [Mil71, §1]. On the other hand, the group of units $\mathcal{O}_k^\times$ appears as the first algebraic K-group $\mathrm{K}_1(\mathcal{O}_k)$ [Ros94, §2.3]. The study of K_2 of a number field and its number theoretic properties was initiated by Tate [BT73] who computed that

$$\mathrm{K}_2(\mathbf{Q}) \cong \mathbf{Z}/2 \oplus \bigoplus_{p\geq 2} \mathbf{F}_p^\times. \tag{1.6}$$

The calculation involves an induction argument over the prime numbers which Tate remarks is lifted directly from the first proof of the quadratic reciprocity law given by Gauss [Mil71, p. 102]. To explain Tate's computation, it can be useful to keep the analogous case of $\mathrm{K}_1(\mathbf{Q}) \cong \mathbf{Q}^\times$ in mind: by extracting from a nonzero rational number its sign along with its valuation at each prime p we obtain an isomorphism

$$\mathrm{K}_1(\mathbf{Q}) \xrightarrow{\cong} \mathbf{Z}/2 \oplus \bigoplus_{p \geq 2} \mathbf{Z}.$$

For the group $\mathrm{K}_2(\mathbf{Q})$ we have Matsumoto's theorem asserting that $\mathrm{K}_2(\mathbf{Q})$ is additively generated by symbols $\{x, y\}$ for $x, y \in \mathbf{Q}^\times$, subject only to the bilinearity relation $\{xy, z\} = \{x, z\} + \{y, z\}$ and the Steinberg relation $\{x, 1-x\} = 0$. The local Hilbert symbols at the various primes provide "2-dimensional analogs" of the maps on $\mathrm{K}_1(\mathbf{Q})$ defined by the sign and by p-adic valuation: for p an odd prime, the local Hilbert symbol at p is given by

$$(x, y)_p \equiv (-1)^{v_p(x)v_p(y)} x^{v_p(y)} y^{-v_p(x)} \pmod{p},$$

while for the infinite place of $\mathbf{Q}$, the Hilbert symbol $(x, y)_\infty$ is given as -1 if both x and y are negative, and $+1$ otherwise [Gra03, II §7]. These symbols define a map from $\mathrm{K}_2(\mathbf{Q})$ to the right hand side of (1.6) which Tate proved to be an isomorphism. By the universality of K_2 with respect to Steinberg symbols, Tate's result gives rise to the product formula for Hilbert symbols on $\mathbf{Q}$,

$$\prod_{2 \leq p \leq \infty} (x, y)_p = 1,$$

which is an equivalent formulation of the law of quadratic reciprocity [Gra03, II Example 3.4.2, II Theorem 7.8.1.2]. The above discussion can be packaged into the assertion of Moore's theorem on uniqueness of reciprocity laws over the field $\mathbf{Q}$ [Gra03, II Theorem 7.6], which states that there is an exact sequence

$$0 \to \mathrm{K}_2(\mathbf{Q}) \xrightarrow{h} \mathbf{Z}/2 \oplus \bigoplus_{p \geq 2} \mu(\mathbf{Q}_p) \to \mathbf{Z}/2 \to 0. \tag{1.7}$$

The map h is the *global Hilbert symbol*, whose components are the local Hilbert symbols at the different places of $\mathbf{Q}$. Tate's computation and Moore's exact sequence shed light on the relationship between number theory and K_2 by showing that K_2 is intimately linked to reciprocity laws.

In Paper V we investigate similar properties for Milnor–Witt K-theory, so that we in a sense blend quadratic forms into the picture. In particular, we define Hilbert symbols on Milnor–Witt K-groups and show an analog of Moore's reciprocity sequence (1.7). Over the field $\mathbf{Q}$ we can perhaps think of this as a "quadratic" quadratic reciprocity law: namely, we have an exact sequence

$$0 \to \mathrm{K}_2^{\mathrm{MW}}(\mathbf{Q}) \xrightarrow{h^{\mathrm{MW}}} \mathbf{Z} \oplus \bigoplus_{p \geq 2} \mu(\mathbf{Q}_p) \to \mathbf{Z}/2 \to 0,$$

where h^{MW} is the global MW-Hilbert symbol.

The above exact sequences are examples of "local-global principles" parallelling the Hasse–Minkowski theorem (1.2) we encountered in Section 1.1.

Another example of a local-global principle is given by Hasse's norm theorem. This classical result states that if L/k is a finite cyclic extension of number fields, then an element of $k^\times$ is a norm from $L^\times$ if and only if it is a local norm at every place of k. We can think of this result as a norm theorem for the functor K_1. In [BR84], Bak and Rehmann extended Hasse's norm theorem to K_2. In fact, their result is valid for any finite extension of number fields L/k. It states that an element of $\mathrm{K}_2(k)$ lies in the image of the norm map $\mathrm{N}_{L/k}\colon \mathrm{K}_2(L) \to \mathrm{K}_2(k)$ if and only if its image in each $\mathrm{K}_2(k_v)$ lies in the image of the map

$$\bigoplus_{w|v} \mathrm{N}_{L_w/k_v}\colon \bigoplus_{w|v} \mathrm{K}_2(L_w) \to \mathrm{K}_2(k_v).$$

By [BR84, p. 4], this result can be reformulated as the exactness of the sequence

$$\mathrm{K}_2(L) \xrightarrow{\mathrm{N}_{L/k}} \mathrm{K}_2(k) \xrightarrow{\bigoplus_{v\in\Sigma_{L/k}} h_v} \bigoplus_{v\in\Sigma_{L/k}} \mathbf{Z}/2 \to 0. \tag{1.8}$$

Here $\Sigma_{L/k}$ denotes the set of infinite real places of k that are complexified in the extension L/k, and h_v denotes the local Hilbert symbol at v. The final result of Paper V aims to extend Bak and Rehmann's result to the setting of Milnor–Witt K-theory. More precisely, we show that there is an exact sequence

$$\mathrm{K}_2^{\mathrm{MW}}(L) \xrightarrow{\tau_{L/k}} \mathrm{K}_2^{\mathrm{MW}}(k) \xrightarrow{\bigoplus_{v\in\Sigma_{L/k}} h_v^{\mathrm{MW}}} \bigoplus_{v\in\Sigma_{L/k}} \mathbf{Z} \to 0,$$

where $\tau_{L/k}$ denotes the transfer map on Milnor–Witt K-theory, defined similarly as the norm maps on Milnor K-theory [Mor12, Chapter 3], and the right hand map is given by the local Milnor–Witt Hilbert symbols.

1.5 Correspondences arising from other cohomology theories

In Section 1.4 we encountered the notion of finite Milnor–Witt correspondences, which is a variant of Suslin and Voevodsky's finite correspondences. The derived category of motives $\widetilde{\mathbf{DM}}(k)$ associated to $\widetilde{\mathrm{Cor}}_k$ provides a better approximation to the motivic stable homotopy category and is well suited for computations. In fact, constructions of variants of Suslin and Voevodsky's $\mathbf{DM}(k)$ have proved so fruitful that it has resulted in a menagerie of different correspondence categories. To give a few examples, let us first mention the category of K_0-correspondences introduced by Walker in [Wal96] and further studied by Suslin [Sus03]. By proving that the category of K_0-correspondences satisfies properties similar to that of Cor_k, Suslin identified Grayson's motivic cohomology [Gra95] with ordinary motivic cohomology and established the motivic Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = \mathrm{H}^{p-q,-q}(X, \mathbf{Z}) \implies \mathrm{K}_{-p-q}(X) \tag{1.9}$$

which computes algebraic K-theory from motivic cohomology. One of the main features of $\widetilde{\mathbf{DM}}(k)$ is that it gives an analogous picture for Hermitian K-theory. To explain this, first recall that the ∞-category $\widetilde{\mathbf{DM}}(k)$ gives rise to the Milnor–Witt motivic cohomology groups $\mathrm{H}^{p,q}_{\mathrm{MW}}(X, \mathbf{Z})$. The work of Bachmann and Fasel [Bac17; BF18] shows that these cohomology groups occur at the E^1-page of the *very effective slice spectral sequence for the Hermitian* K*-theory spectrum* $\mathrm{KQ} \in \mathbf{SH}(k)$,

$$E^1_{p,w,q} = \pi_{p,w}\widetilde{s}_q(\mathrm{KQ}) \implies \pi_{p,w}(\mathrm{KQ}).$$

See for example [Hor05] for details on the spectrum KQ. The symbol $\widetilde{s}_q$ above denotes the *very effective slice functor* [SØ12]. This spectral sequence is a Hermitian analog of the above motivic Atiyah–Hirzebruch spectral sequence (1.9) [CF17b, Introduction].

As a final example of a correspondence category and its applications we offer the category of framed correspondences Fr_* introduced by Voevodsky and studied by Ananyevskiy, Garkusha and Panin [AGP18; GP18a; GP18b]. This results in an ∞-category of framed motives that classifies motivic infinite $\mathbf{P}^1$-loop spaces [Elm+19].

Now, for each new category of correspondences one needs to prove analogs of Theorems 1.2.1 and 1.2.2 on homotopy invariance and cancellation. The topic of Paper III is to provide an axiomatic approach to these results for a certain class of correspondence categories, namely those that arise from some cohomology theory on Sm_k. Examples of correspondences of this sort include the original Cor_k, which can be defined using Chow groups, and Milnor–Witt correspondences $\widetilde{\mathrm{Cor}}_k$, which are defined via Chow–Witt groups. In particular, Paper III generalizes the results of Paper I. For any appropriate cohomology theory A^* on smooth schemes, we define in Paper III a correspondence category Cor^A_k of *finite A-correspondences* and construct an associated derived category of motives $\mathbf{DM}_A(k)$ satisfying properties similar to that of $\mathbf{DM}(k)$ and $\widetilde{\mathbf{DM}}(k)$. We obtain an adjunction

$$\gamma_A^* : \mathbf{SH}(k) \rightleftarrows \mathbf{DM}_A(k) : \gamma_*^A$$

analogous to (1.4) and (1.5), as well as an associated motivic Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbf{Z}_A \in \mathbf{SH}(k)$ representing a variant of motivic cohomology. By the results of Paper II, the ∞-category $\mathbf{DM}_A(k)$ is equivalent to the ∞-category of modules over $\mathrm{H}\mathbf{Z}_A$ after inverting the exponential characteristic of k. In Paper IV we identify $\mathbf{DM}_A(k)$ for various choices of A and furthermore prove that the spectrum $\mathrm{H}\mathbf{Z}_A$ belongs to the heart of the effective homotopy t-structure on $\mathbf{SH}(k)^{\mathrm{eff}}$, i.e.,

$$\mathrm{H}\mathbf{Z}_A \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}.$$

We recall that $\mathbf{SH}(k)^{\mathrm{eff}}$ denotes the full subcategory of $\mathbf{SH}(k)$ generated under colimits by $\Sigma^{-n,0}\Sigma^\infty_{\mathrm{T}} X_+$ for $X \in \mathrm{Sm}_k$, $n \geq 0$ [BH18, §13], and that the *effective homotopy* t*-structure* on $\mathbf{SH}(k)^{\mathrm{eff}}$ is defined by

$$\mathbf{SH}(k)^{\mathrm{eff}}_{\geq 0} := \{E \in \mathbf{SH}(k)^{\mathrm{eff}} : \underline{\pi}_n(E)_0 = 0 \text{ for all } n < 0\},$$
$$\mathbf{SH}(k)^{\mathrm{eff}}_{\leq 0} := \{E \in \mathbf{SH}(k)^{\mathrm{eff}} : \underline{\pi}_n(E)_0 = 0 \text{ for all } n > 0\}.$$

The *heart of the effective homotopy* t*-structure* is then the abelian category

$$\mathbf{SH}(k)^{\mathrm{eff},\heartsuit} := \mathbf{SH}(k)^{\mathrm{eff}}_{\geq 0} \cap \mathbf{SH}(k)^{\mathrm{eff}}_{\leq 0}.$$

Thus the results of Paper III give rise to a parametrized family of $\mathcal{E}_\infty$-ring spectra in $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$. Conversely, we show in Paper III that we can start with an $\mathcal{E}_\infty$-ring spectrum E in $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ and define a cohomological correspondence category Cor_k^E from it. In other words, we can attach to each cohomological correspondence category a ring spectrum in $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$, and vice versa.

1.6 What is a motivic cohomology theory?

Having explored some of the main characters of motivic homotopy theory we can now return to the title of this thesis and explain in more detail what we mean by a "study of various motivic cohomology theories".

As envisioned by Grothendieck, the category of motives should be the home of a universal cohomology theory on Sm_k. Although the category of mixed motives is still out of reach, the *cohomological* properties of mixed motives depend only on its derived category for which Voevodsky's $\mathbf{DM}(k)$ provides a good candidate. As we have seen, the ∞-category $\mathbf{DM}(k)$ gives rise to the motivic cohomology theory $\mathrm{H}\mathbf{Z}$. However, as discussed in Section 1.5 there are also other variants of the derived category of motives; in fact, there is a parametrized family $\mathbf{DM}_A(k)$ of such. Each ∞-category $\mathbf{DM}_A(k)$ defines a spectrum $\mathrm{H}\mathbf{Z}_A$, not necessarily equivalent to $\mathrm{H}\mathbf{Z}$, but nevertheless sharing many of its basic properties. We therefore regard the various spectra $\mathrm{H}\mathbf{Z}_A \in \mathbf{SH}^{\mathrm{eff},\heartsuit}$ as *motivic cohomology theories* as well. Conversely, as mentioned in Section 1.5 any ring spectrum E in the heart of the effective homotopy t-structure on $\mathbf{SH}(k)$ gives rise to a derived category of motives $\mathbf{DM}_E(k)$. Hence, the ring spectra contained in $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ correspond to various motivic cohomology theories. But what about other motivic spectra? The work [Elm+20] constructs, for *any* ring spectrum $E \in \mathbf{SH}(k)$, a category $\mathrm{h}\mathbf{Corr}_k^E$ which is conjectured to be the homotopy category of an ∞-category $\mathbf{Corr}_k^E$ of finite E-correspondences. By [Elm+20, Lemma 4.1.21, Remark 4.1.22] the category $\mathbf{Corr}_k^E$ is a 1-category—i.e., discrete—if and only if $E \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$, in which case $\mathbf{Corr}_k^E$ coincides with the category Cor_k^E as defined in Paper III. What is the significance of the assertion that $\mathbf{Corr}_k^E$ is discrete? From topology, we know that for an $\mathcal{E}_\infty$-ring spectrum E in the classical stable homotopy category, the functoriality of the Becker–Gottlieb transfer on E may introduce higher homotopies; to obtain strict functoriality one needs to pass to modules over the classical Eilenberg–Mac Lane spectrum and hence to ordinary cohomology. In motivic homotopy theory we can think of the correspondences in the category $\mathbf{Corr}_k^E$ as encoding a sort of Becker–Gottlieb transfer. By the above discussion, the analogous procedure in motivic homotopy theory to strictifying the functoriality of the transfers would therefore be to pass to $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$.

To summarize, a ring spectrum $E \in \mathbf{SH}(k)$ should define a motivic cohomology theory if and only if $\mathbf{Corr}_k^E$ is discrete, and this latter condition is again equivalent to demanding $E \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$. We therefore propose to think of a motivic cohomology theory as an $\mathcal{E}_\infty$-ring spectrum in the category $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$. With this definition, we can make precise the statement made in the beginning of this introduction that Milnor–Witt K-theory is the initial motivic cohomology theory. Indeed, the truncation of the motivic sphere spectrum $\mathbf{1} \in \mathbf{SH}(k)$ to the heart of the effective homotopy t-structure is $\underline{\pi}_0^{\mathrm{eff}}(\mathbf{1}) \simeq f_0\mathbf{K}_*^{\mathrm{MW}}$, the effective cover of the homotopy module of Milnor–Witt K-theory [Bac17]. Hence the initial ring spectrum in $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ is given by Milnor–Witt K-theory. Furthermore, by [BF18] the spectrum $f_0\mathbf{K}_*^{\mathrm{MW}}$ coincides with the Milnor–Witt motivic cohomology spectrum $\mathrm{H}\widetilde{\mathbf{Z}}$ as defined via the ∞-category $\widetilde{\mathbf{DM}}(k)$ we encountered in Section 1.4. Thus, the category $\widetilde{\mathrm{Cor}}_k$ of finite Milnor–Witt correspondences constitutes the initial 1-category of correspondences [Elm+20, §1] and is in this sense the initial motivic cohomology theory.

1.7 Geometric and arithmetic properties of motivic cohomology theories

To conclude this introduction we may now summarize the above discussion as well as the title of the thesis as follows. Our main objects of study are motivic cohomology theories, i.e., $\mathcal{E}_\infty$-ring spectra in $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$. From a geometric point of view we study these ring spectra in terms of correspondence categories and their associated derived categories of motives, highlighting how each such category gives rise to a motivic cohomology theory. This is the main topic of Papers I, II, III and IV. From an arithmetic point of view, we study in Paper V some of the number theoretic information contained in the initial motivic cohomology theory, i.e., in Milnor–Witt motivic cohomology $\mathrm{H}\widetilde{\mathbf{Z}}$. We can consider the results of Paper V as an extension of the classical arithmetic properties of Milnor K-theory, and ordinary motivic cohomology $\mathrm{H}\mathbf{Z}$, to the quadratic setting of Milnor–Witt K-theory. Hence, this thesis investigates interactions between geometry, number theory, cohomology and quadratic forms, that take place in motivic homotopy theory.

1.8 Summary of papers

This thesis consists of five papers. The first four papers concern the geometric properties of motivic cohomology theories, while the last paper deals with the arithmetic properties. Each paper is summarized below.

Paper I concerns the basic properties of presheaves on the category $\widetilde{\mathrm{Cor}}_k$ of finite Milnor–Witt correspondences. Specifically, we show that for any homotopy invariant presheaf $\mathscr{F}$ on $\widetilde{\mathrm{Cor}}_k$, the associated Nisnevich sheaf $\mathscr{F}_{\mathrm{Nis}}$ is also homotopy invariant. For the proof we follow a similar path as Druzhinin's proof in the case of Witt- and Grothendieck–Witt correspondences [Dru14; Dru18] and as Garkusha–Panin's proof in the case of framed correspondences [GP18b]; it consists of showing various excision results as well as a moving lemma for the presheaf $\mathscr{F}$. Having these results at hand, the preservation of homotopy invariance under sheafification follows by a formal argument.

The excision results for the presheaf $\mathscr{F}$ roughly read as follows. Suppose that $i\colon V \hookrightarrow U$ is an inclusion of open subschemes of the affine line. If x is a closed point contained in V, then the map i induces an isomorphism

$$i^*\colon \frac{\mathscr{F}(U \setminus x)}{\mathscr{F}(U)} \xrightarrow{\cong} \frac{\mathscr{F}(V \setminus x)}{\mathscr{F}(V)}.$$

We also show a similar result for étale neighborhoods. Finally, the moving lemma for $\mathscr{F}$ can be formulated as follows. Pick a scheme $X \in \mathrm{Sm}_k$ along with a closed point $x \in X$ and a closed subscheme $Z \subseteq X$ containing x. If s is a global section of the homotopy invariant presheaf $\mathscr{F}$ on $\widetilde{\mathrm{Cor}}_k$ which vanishes at $X \setminus Z$, then $s_x = 0$ in the stalk of $\mathscr{F}$ at x.

We note that the result on homotopy invariance of $\mathscr{F}_{\mathrm{Nis}}$ was obtained in [DF17a] as well, by using the fact that there exists a functor from the category of framed correspondences to $\widetilde{\mathrm{Cor}}_k$ along with the fact that the result holds in the former category by the work of Garkusha and Panin [GP18b]. The point of Paper I is however to give a proof internal to the category of Milnor–Witt correspondences.

Paper II is joint work with E. Elmanto and concerns the following question: which stable ∞-categories can be realized as categories of modules over some motivic ring spectrum? We give an axiomatic approach using Lurie's ∞-categorical version of Barr–Beck's monadicity theorem [Lur17, Theorem 4.7.3.5]. Being equivalent to modules over the motivic Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbf{Z}$, Voevodsky's category $\mathbf{DM}(k)$ provides the first example of such a module category [RØ08]. Inspired by this result we axiomatize the properties of the category Cor_k of finite correspondences in order to produce a family of derived categories of motives, all of which will then, after inverting the residue characteristic, be a module category over a motivic spectrum. In particular, this result applies to the categories $\mathbf{DM}_A(k)$ constructed in Paper III, ensuring that each such category is a module category over the associated motivic Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbf{Z}_A$.

A similar axiomatization is obtained by Garkusha in [Gar19], however, using different methods. Finally, in the last part of the paper we adopt Cisinski and Déglise's techniques from [CD15] to extend the results to Noetherian base schemes that are regular over a field.

Paper III is joint work with A. Druzhinin in which we introduce a class of correspondence categories parametrized by suitable cohomology theories on smooth schemes. We generalize the results of Paper I by showing that homotopy invariance is preserved under Nisnevich sheafification for presheaves on these cohomological correspondence categories. We furthermore show that the analog of Voevodsky's cancellation theorem holds for these types of correspondences and use the results to construct their associated derived categories of motives. Finally, we note that each such cohomological correspondence category Cor_k^A defines an associated motivic Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbf{Z}_A \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$.

We contend that it should be possible to extend the results of this paper to more general cohomology theories—in particular, to cohomology theories for which one needs to twist by arbitrary virtual vector bundles in order to have a proper pushforward. This would allow one to replace the cohomology theory A^* with any $\mathcal{E}_\infty$-ring spectrum E in $\mathbf{SH}(k)$. However, according to [Elm+20, Lemma 4.1.21, Remark 4.1.22], the discrete correspondence category Cor_k^E depends only on the truncation $\underline{\pi}_0^{\mathrm{eff}}(E)$ of E to the effective heart $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$. In the case of the sphere spectrum, $\underline{\pi}_0^{\mathrm{eff}}(\mathbf{1})$ is the Milnor–Witt motivic cohomology spectrum $\mathrm{H}\widetilde{\mathbf{Z}}$, and hence, we reconstruct $\widetilde{\mathrm{Cor}}_k$ which is already a cohomological correspondence category in the sense of this paper. In this regard, the theory of Paper III is the most general theory we can obtain without passing to nondiscrete ∞-categories of correspondences.

Paper IV is a continuation of Paper III where we study the derived category of motives $\mathbf{DM}_A(k)$ and the motivic Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbf{Z}_A$ associated to a cohomological correspondence category Cor_k^A in more detail. In particular, we give a proof that $\mathrm{H}\mathbf{Z}_A$ is effective and compare the categories $\mathbf{DM}_A(k)$ for various choices of the cohomology theory A^*.

Paper V investigates number theoretic properties of Milnor–Witt K-theory, in the sense that we establish analogs of classical number theoretic results for Milnor–Witt K-groups. We focus in particular on the lower Milnor–Witt K-groups $\mathrm{K}_1^{\mathrm{MW}}$ and $\mathrm{K}_2^{\mathrm{MW}}$. For the functor $\mathrm{K}_1^{\mathrm{MW}}$ applied to a number field we define valuations, idèles and relate the associated idèle class group to the classical one. For the functor $\mathrm{K}_2^{\mathrm{MW}}$ on number fields we define Hilbert symbols and show an analog of Moore's reciprocity sequence. We also discuss $\mathrm{K}_2^{\mathrm{MW}}$ of rings of integers and a Hasse type norm theorem for $\mathrm{K}_2^{\mathrm{MW}}$.

There are many possible further questions in this direction. For example, we can ask if there is a variant of the Galois group that receives a reciprocity map from the Milnor–Witt idèles, giving an enhanced Artin reciprocity law. Another direction is to investigate whether one can connect the zeroth Milnor–Witt homology group $\mathrm{H}_0^{\mathrm{MW}}(X)$ [AN19] of a smooth curve X over a field of positive characteristic with variants of tame coverings of X [SS00]. One may also look for generalizations of the results to Milnor–Witt motivic cohomology groups or higher Hermitian K-groups.

1.8.1 Notational and editorial remarks

In order to fit the format used for Ph.D. theses at the University of Oslo, the papers have been lightly edited compared to their online arXiv versions. The notation used in the papers is for the most part consistent throughout.

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Papers

I

Paper I

Homotopy invariance of Nisnevich sheaves with Milnor–Witt transfers

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Abstract

The category of finite Milnor–Witt correspondences, introduced by Calmès and Fasel, provides a new type of correspondences closer to the motivic homotopy theoretic framework than Suslin–Voevodsky's finite correspondences. A fundamental result in the theory of ordinary correspondences concerns homotopy invariance of sheaves with transfers, and in the present paper we address this question in the setting of Milnor–Witt correspondences. Employing techniques due to Druzhinin, Fasel–Østvær and Garkusha–Panin, we show that homotopy invariance of presheaves with Milnor–Witt transfers is preserved under Nisnevich sheafification.

Contents

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I.1 Introduction

A stepping stone toward Voevodsky's construction of the derived category of motives $\mathbf{DM}(k)$ [Voe00b] is the notion of finite correspondences between smooth k-schemes. Such correspondences are in a certain sense multivalued functions taking only finitely many values. By considering finite correspondences instead of ordinary morphisms of schemes, one performs a linearization which allows for extra elbowroom and flexibility, and which in turn makes it possible to prove strong theorems. One of the "fundamental theorems" in the theory of correspondences concerns homotopy invariance, and is crucial for constructing the theory of motives.

Theorem I.1.1 ([Voe00a, Theorem 5.6]). *For any homotopy invariant presheaf $\mathscr{F}$ on the category Cor_k of finite correspondences, the associated Nisnevich sheaf $\mathscr{F}_{\mathrm{Nis}}$ is also homotopy invariant.*

In [CF17], Calmès and Fasel introduce a new type of correspondences called finite Milnor–Witt correspondences (or finite MW-correspondences for short). Milnor–Witt correspondences provide a setting that is closer to the motivic homotopy theoretic framework than Suslin–Voevodsky's correspondences; for example, the zero-line of sheaves of motivic homotopy groups of the sphere spectrum do not admit ordinary transfers, but they do admit MW-transfers [CF17]. Roughly speaking, a finite MW-correspondence amounts to an ordinary finite correspondence along with an unramified quadratic form defined on the function field of each irreducible component of the support of the correspondence. We briefly recall some results in the theory of MW-correspondences below. Our present goal is to prove a homotopy invariance result similar to Theorem I.1.1 for sheaves with MW-transfers:

Theorem I.1.2. *Let k be a field of characteristic*[1] *0. Then, for any homotopy invariant presheaf $\mathscr{F}$ on the category $\widetilde{\mathrm{Cor}}_k$ of finite MW-correspondences, the associated Nisnevich sheaf $\mathscr{F}_{\mathrm{Nis}}$ is also homotopy invariant.*

We note that this result is already known by work of Déglise and Fasel [DF17, Theorem 3.2.9]. Their proof uses the fact that there is a functor $\mathrm{Fr}_*(k) \to \widetilde{\mathrm{Cor}}_k$ from the category of framed correspondences to MW-correspondences. As the analog of Theorem I.1.2 is known for framed correspondences by work of Garkusha and Panin [GP18], it follows that the desired result also holds for $\widetilde{\mathrm{Cor}}_k$. The purpose of this paper is to give a more direct proof by using geometric input provided in [GP18, §13] to produce homotopies in $\widetilde{\mathrm{Cor}}_k$. Along the way we obtain results on MW-correspondences of independent interest. The proof strategy is due to Druzhinin [Dru16] and Garkusha–Panin [GP18], and uses techniques developed in [FØ17].

[1] The assumption on the characteristic is there because Milnor–Witt correspondences are currently not defined over nonperfect fields. The only place where this assumption is used is in Section I.8 where we need to consider Milnor–Witt correspondences defined over function fields of smooth k-schemes, which may in general be nonperfect. Otherwise, all excision results are valid for infinite perfect fields of characteristic different from 2.

Recollections on Milnor–Witt correspondences

The Milnor–Witt K-groups $\mathrm{K}_n^{\mathrm{MW}}(k)$ of a field k arose in the context of motivic stable homotopy groups of spheres. More precisely, in [Mor04, Theorem 6.4.1] Morel established isomorphisms

$$\pi_{n,n}\mathbf{1} \cong \mathrm{K}_{-n}^{\mathrm{MW}}(k) \tag{I.1}$$

for all $n \in \mathbf{Z}$, where $\mathbf{1} \in \mathbf{SH}(k)$ denotes the sphere spectrum. The groups $\mathrm{K}_n^{\mathrm{MW}}(k)$ admit a description in terms of generators and relations:

Definition I.1.3 (Hopkins–Morel)**.** Let k be a field. The *Milnor–Witt* K*-theory* $\mathrm{K}_*^{\mathrm{MW}}(k)$ of the field k is the graded associative $\mathbf{Z}$-algebra with one generator $[a]$ for each unit $a \in k^\times$, of degree $+1$, and one generator η of degree -1, subject to the following relations:

(1) $[a][1-a] = 0$ for any $a \in k^\times \setminus \{1\}$ (Steinberg relation).

(2) $\eta[a] = [a]\eta$ (η-commutativity).

(3) $[ab] = [a] + [b] + \eta[a][b]$ (twisted η-logarithmic relation).

(4) $(2 + \eta[-1])\eta = 0$ (hyperbolic relation).

We let $\mathrm{K}_n^{\mathrm{MW}}(k)$ denote the n-th graded piece of $\mathrm{K}_*^{\mathrm{MW}}(k)$. The product $[a_1]\cdots[a_n]$ in $\mathrm{K}_n^{\mathrm{MW}}(k)$ may also be denoted by $[a_1, \ldots, a_n]$.

Under the isomorphism (I.1) above, the element $[a] \in \mathrm{K}_1^{\mathrm{MW}}(k)$ corresponds to a class $[a] \in \pi_{-1,-1}\mathbf{1}$. A representative for $[a]$ is given by the pointed map

$$[a]\colon \operatorname{Spec}(k)_+ \to (\mathbf{G}_m, 1)$$

sending the non-basepoint to the point $a \in \mathbf{G}_m$. On the other hand, the element $\eta \in \mathrm{K}_{-1}^{\mathrm{MW}}(k)$ corresponds to the motivic Hopf map $\eta \in \pi_{1,1}\mathbf{1}$ represented by the natural projection [Mor04, §6]

$$\eta\colon \mathbf{A}^2 \setminus 0 \to \mathbf{P}^1.$$

As the sphere spectrum is initial in the category of motivic ring spectra, the homotopy groups $\pi_{p,q}(E)$ of a motivic ring spectrum E inherit the relations of $\pi_{p,q}\mathbf{1}$ via the unit map $\mathbf{1} \to E$. Thus Milnor–Witt K-theory is a fundamental object in motivic homotopy theory. In [CF17], Calmès and Fasel employ sheaves of Milnor–Witt K-theory to set up the theory of MW-correspondences. Based on the fact that the group $\mathrm{Cor}_k(X,Y)$ of finite correspondences from X to Y can be expressed as a colimit of Chow groups with support,

$$\begin{aligned} \mathrm{Cor}_k(X,Y) &= \varinjlim_{T \in \mathcal{A}(X,Y)} \mathrm{H}_T^{d_Y}(X \times Y, \mathbf{K}_{d_Y}^{\mathrm{M}}) \\ &= \varinjlim_{T \in \mathcal{A}(X,Y)} \mathrm{CH}_T^{d_Y}(X \times Y), \end{aligned}$$

Calmès and Fasel replace Milnor K-theory (and Chow groups) with (twisted) Milnor–Witt K-theory (and Chow–Witt groups), and define the group of finite MW-correspondences from X to Y as

$$\begin{aligned}\widetilde{\mathrm{Cor}}_k(X,Y) &:= \varinjlim_{T\in\mathcal{A}(X,Y)} \mathrm{H}^{d_Y}_T(X\times Y, \mathbf{K}^{\mathrm{MW}}_{d_Y}, p_Y^*\omega_{Y/k})\\ &= \varinjlim_{T\in\mathcal{A}(X,Y)} \widetilde{\mathrm{CH}}^{d_Y}_T(X\times Y, p_Y^*\omega_{Y/k}),\end{aligned}$$

where $p_Y\colon X\times Y\to Y$ is the projection. Here Y is assumed to be equidimensional of dimension d_Y, and $\mathcal{A}(X,Y)$ is the partially ordered set of closed subsets T of $X\times Y$ such that each irreducible component of T (with its reduced structure) is finite and surjective over X. Moreover, $\mathbf{K}^{\mathrm{MW}}_n$ is the n-th unramified Milnor–Witt K-theory sheaf, as defined in [Mor12, §5]. We note that the Nisnevich cohomology groups $\mathrm{H}^p(X,\mathbf{K}^{\mathrm{MW}}_q,\mathscr{L})$ of the Milnor–Witt sheaf $\mathbf{K}^{\mathrm{MW}}_q(\mathscr{L})$ twisted by a line bundle $\mathscr{L}$ can be computed using the Rost–Schmid complex [Mor12, Chapter 5], which provides a flabby resolution of $\mathbf{K}^{\mathrm{MW}}_q(\mathscr{L})$. Recall that the p-th term of the Rost–Schmid complex is given by

$$\mathrm{C}^p(X,\mathbf{K}^{\mathrm{MW}}_q,\mathscr{L}) := \bigoplus_{x\in X^{(p)}} \mathrm{K}^{\mathrm{MW}}_{q-p}(k(x), \wedge^p(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \otimes_{k(x)} \mathscr{L}_x),$$

where $X^{(p)}$ denotes the set of codimension p-points of X. We let $\widetilde{\mathrm{Cor}}_k$ denote the category of finite MW-correspondences. The category $\widetilde{\mathrm{Cor}}_k$ is symmetric monoidal, and comes equipped with an embedding $\mathrm{Sm}_k\to\widetilde{\mathrm{Cor}}_k$ from smooth k-schemes, as well as a forgetful functor $\widetilde{\mathrm{Cor}}_k\to\mathrm{Cor}_k$ to Suslin–Voevodsky's correspondences; see [CF17] for details.

Let $\widetilde{\mathrm{PSh}}(k)$ denote the category of presheaves with MW-transfers, i.e., additive presheaves of abelian groups $\mathscr{F}\colon\widetilde{\mathrm{Cor}}_k^{\mathrm{op}}\to\mathrm{Ab}$. As noted in [CF17], there are more presheaves on $\widetilde{\mathrm{Cor}}_k$ than on Cor_k. One example is of course provided by the sheaves $\mathbf{K}^{\mathrm{MW}}_*$, which admit MW-transfers but not ordinary transfers [CF17]. Among the various presheaves with MW-transfers, the homotopy invariant ones will be of most interest to us.

Definition I.1.4. A presheaf $\mathscr{F}\in\widetilde{\mathrm{PSh}}(k)$ with MW-transfers is *homotopy invariant* if for each $X\in\mathrm{Sm}_k$, the projection $p\colon X\times\mathbf{A}^1\to X$ induces an isomorphism $p^*\colon\mathscr{F}(X)\xrightarrow{\cong}\mathscr{F}(X\times\mathbf{A}^1)$. Equivalently, the zero section $i_0\colon X\to X\times\mathbf{A}^1$ induces an isomorphism $i_0^*\colon\mathscr{F}(X\times\mathbf{A}^1)\xrightarrow{\cong}\mathscr{F}(X)$.

Let us also mention that by [DF17, Lemma 1.2.10], the Nisnevich sheaf $\mathscr{F}_{\mathrm{Nis}}$ associated to a presheaf $\mathscr{F}\in\widetilde{\mathrm{PSh}}(k)$ comes equipped with a unique MW-transfer structure. This result follows essentially from [DF17, Lemma 1.2.6], which states that if $p\colon U\to X$ is a Nisnevich covering of a smooth k-scheme X, and if $\widetilde{\mathrm{c}}(X)$ denotes the representable presheaf $\widetilde{\mathrm{c}}(X)(Y):=\widetilde{\mathrm{Cor}}_k(Y,X)$, then the Čech-complex $\widetilde{\mathrm{c}}(U^\bullet_X)\to\widetilde{\mathrm{c}}(X)\to 0$ is exact on the associated Nisnevich sheaves.

Extending presheaves to essentially smooth schemes

In this paper we will consider two closely related ways to extend presheaves on $\widetilde{\mathrm{Cor}}_k$ to essentially smooth schemes over k. This allows us to formulate statements also about local schemes or henselian local schemes.

1. The first method is the standard way of defining the value of a presheaf on limits of schemes as a colimit of the presheaf values, and will be used in Sections I.9–I.13. We briefly recall some details on this matter, following [CF17, §5.1]. Let $\mathcal{P}$ be the category consisting of projective systems $((X_\lambda)_{\lambda\in I}, f_{\lambda\mu})$ such that $X_\lambda \in \mathrm{Sm}_k$ and such that the transition morphisms $f_{\lambda\mu}\colon X_\lambda \to X_\mu$ are affine and étale. By [CF17, §5.1], the limit of such a projective system exists in the category of schemes Sch. Moreover, the functor $\mathcal{P} \to \mathrm{Sch}$ sending a projective system to its limit defines an equivalence of categories between $\mathcal{P}$ and the full subcategory $\overline{\mathrm{Sm}}_k$ of Sch consisting of schemes over k that are limits of projective systems from $\mathcal{P}$ [CF17, §5.1].

   Now let $\mathscr{F}$ be a presheaf on Sm_k. We can extend $\mathscr{F}$ to a presheaf $\overline{\mathscr{F}}$ on $\overline{\mathrm{Sm}}_k$ by setting $\overline{\mathscr{F}}((X_\lambda)_{\lambda\in I}) := \varinjlim_{\lambda\in I} \mathscr{F}(X_\lambda)$. By [CF17, §5.1] this gives a well defined presheaf on $\overline{\mathrm{Sm}}_k$ that coincides with $\mathscr{F}$ when restricted to Sm_k. In particular, we can extend the presheaf $\widetilde{\mathrm{Cor}}_k(-, X)$ to $\overline{\mathrm{Sm}}_k$.

   The above construction can furthermore be carried out for Chow–Witt groups with support. Roughly speaking, we can define a category $\widetilde{\mathcal{P}}$ consisting of projective systems of triples $(X_\lambda, Z_\lambda, \mathscr{L}_\lambda)$ of a smooth k-scheme X_λ, a closed subscheme Z_λ of X_λ and a line bundle $\mathscr{L}_\lambda$ on X_λ. If the limit $(X, Z, \mathscr{L})$ of such a projective system is such that X is regular, then the pullback induces an isomorphism [CF17, Lemma 5.7]

   $$\varinjlim_{\lambda} \widetilde{\mathrm{CH}}^*_{Z_\lambda}(X_\lambda, \mathscr{L}_\lambda) \xrightarrow{\cong} \widetilde{\mathrm{CH}}^*_Z(X, \mathscr{L}).$$

   This allows us to pass to Chow–Witt groups of local schemes U in order to produce MW-correspondences on U, which will be needed in Sections I.9–I.12. However, in order to unburden our notation we may drop the bar both from $\overline{\mathscr{F}}$ and $\overline{\mathrm{Sm}}_k$ when evaluating presheaves on limits of schemes.

2. A second method of extending presheaves will be carried out in Section I.8 in order to show that certain results that hold for open subsets of $\mathbf{A}^1_k$ are also valid for open subsets of $\mathbf{A}^1_K$, where K is some finitely generated field extension K of the ground field k. This trick was suggested to the author by I. Panin, and involves extending a presheaf on $\widetilde{\mathrm{Cor}}_k$ to a certain full subcategory of $\widetilde{\mathrm{Cor}}_K$. See Section I.8 for details.

Outline

In Section I.2 we establish some notation and collect a few lemmas needed later on.

In Section I.3 we review how Cartier divisors give rise to finite Milnor–Witt correspondences, following [FØ17]. This gives a procedure to construct desired homotopies in the later sections.

In Section I.4 we prove the first main ingredient of the proof of Theorem I.1.2, which is a Zariski excision result for MW-presheaves. More precisely, in Theorem I.4.1 we show that if $V \subseteq U \subseteq \mathbf{A}^1$ are two Zariski open neighborhoods of a closed point $x \in \mathbf{A}^1$, then the inclusion $i\colon V \hookrightarrow U$ induces an isomorphism[2]

$$i^*\colon \frac{\mathscr{F}(U \setminus x)}{\mathscr{F}(U)} \xrightarrow{\cong} \frac{\mathscr{F}(V \setminus x)}{\mathscr{F}(V)}$$

for any homotopy invariant $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$. The proof of Zariski excision consists of producing left and right inverses in $\widetilde{\mathrm{Cor}}_k$ of i up to homotopy. This is done in Sections I.6 and I.7.

In Section I.8, we extend the results of Section I.4 to open subsets of $\mathbf{A}^1_K$, where K is a finitely generated field extension of the ground field k.

In Section I.9 we prove a "moving lemma" for MW-correspondences (see Theorem I.9.1), which can be informally stated as follows. Let $X \in \mathrm{Sm}_k$, and pick a closed point $x \in X$ along with a closed subscheme $Z \subsetneq X$ containing the point x. Then, up to $\mathbf{A}^1$-homotopy, we are able to "move the point x away from Z" using MW-correspondences. See Section I.9 for more details.

In Section I.10 we prove the last main ingredient of the proof of Theorem I.1.2, namely a Nisnevich excision result. The situation is as follows. Given an elementary distinguished Nisnevich square

$$\begin{tikzcd} V' \arrow[r] \arrow[d] & X' \arrow[d, "\Pi"] \\ V \arrow[r] & X \end{tikzcd}$$

with X and X' affine and k-smooth, let $S := (X \setminus V)_{\mathrm{red}}$ and $S' := (X' \setminus V')_{\mathrm{red}}$. Suppose that $x \in S$ and $x' \in S'$ are two points satisfying $\Pi(x') = x$, and put $U := \mathrm{Spec}(\mathcal{O}_{X,x})$ and $U' := \mathrm{Spec}(\mathcal{O}_{X',x'})$. Then the map Π induces an isomorphism[3]

$$\Pi^*\colon \frac{\mathscr{F}(U \setminus S)}{\mathscr{F}(U)} \xrightarrow{\cong} \frac{\mathscr{F}(U' \setminus S')}{\mathscr{F}(U')}$$

for any homotopy invariant $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$. Again the proof consists of producing left and right inverses to Π up to homotopy, which is done in Sections I.11 and I.12.

Finally, in Section I.13 we will see how homotopy invariance of the associated Nisnevich sheaf $\mathscr{F}_{\mathrm{Nis}}$ follows from the above results.

[2] We show in Section I.5 that the restriction maps $\mathscr{F}(U) \to \mathscr{F}(U \setminus x)$ and $\mathscr{F}(V) \to \mathscr{F}(V \setminus x)$ are injective, justifying the notation used in the formulation of Zariski excision.

[3] It follows from Theorem I.9.1 that the restriction maps $\mathscr{F}(U) \to \mathscr{F}(U \setminus S)$ and $\mathscr{F}(U') \to \mathscr{F}(U' \setminus S')$ are injective, justifying the notation used in the formulation of Nisnevich excision. See Section I.10 for details.

Conventions

Throughout we will assume that k is an infinite perfect field of characteristic different from 2. In Sections I.8 and I.13, k is furthermore assumed to be of characteristic 0. We let Sm_k denote the category of smooth separated schemes of finite type over k. All undecorated fiber products mean fiber product over k. Throughout, the symbols i_0 and i_1 will denote the rational points $i_0, i_1 \colon \mathrm{Spec}(k) \to \mathbf{A}^1$ given by 0 and 1, respectively.

We will frequently abuse notation and write simply $f \in \widetilde{\mathrm{Cor}}_k(X,Y)$ for $\widetilde{\gamma}_f$, where $\widetilde{\gamma}_f$ is the image of a morphism of schemes $f \colon X \to Y$ under the embedding $\widetilde{\gamma} \colon \mathrm{Sm}_k \to \widetilde{\mathrm{Cor}}_k$ of [CF17, §4.3]. We let $\sim_{\mathbf{A}^1}$ denote $\mathbf{A}^1$-homotopy equivalence. Following Calmès–Fasel [CF17], if $p_Y : X \times Y \to Y$ is the projection, we may write ω_Y as shorthand for $p_Y^* \omega_{Y/k}$ if no confusion is likely to arise. Note that ω_Y is then canonically isomorphic to $\omega_{X\times Y/X}$. In general, given a morphism of schemes $f \colon X \to Y$ we write $\omega_f := \omega_{X/k} \otimes f^*\omega_{Y/k}^\vee$.

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I.2 Pairs of Milnor–Witt correspondences

We will frequently encounter the situation of a pair $U \subseteq X$ of schemes, and we will be led to study the associated quotient $\mathscr{F}(U)/\operatorname{im}(\mathscr{F}(X) \to \mathscr{F}(U))$ for a given presheaf $\mathscr{F}$ on $\widetilde{\mathrm{Cor}}_k$. It is therefore notationally convenient to introduce a category $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ of pairs of MW-correspondences.

Following [GP18] we let SmOp_k denote the category whose objects are pairs (X,U) with $X \in \mathrm{Sm}_k$ and U a Zariski open subscheme of X, and whose morphisms are maps $f \colon (X,U) \to (Y,V)$, where $f \colon X \to Y$ is a morphism of schemes such that $f(U) \subseteq V$. Below we extend this notion of morphisms of pairs to MW-correspondences.

Definition I.2.1 ([GP18, Definition 2.3])**.** Let $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ denote the category whose objects are those of SmOp_k and whose morphisms are defined as follows. For $(X,U),(Y,V) \in \mathrm{SmOp}_k$, with open immersions $j_X : U \to X$ and $j_Y : V \to Y$, let

$$\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((X,U),(Y,V)) := \ker\left(\widetilde{\mathrm{Cor}}_k(X,Y) \oplus \widetilde{\mathrm{Cor}}_k(U,V) \xrightarrow{j_X^* - (j_Y)_*} \widetilde{\mathrm{Cor}}_k(U,Y)\right).$$

Thus a morphism in $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ is a pair (α, β), where $\alpha \in \widetilde{\mathrm{Cor}}_k(X, Y)$ and $\beta \in \widetilde{\mathrm{Cor}}_k(U, V)$, such that the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] & Y \\ U \arrow[u, "j_X"] \arrow[r, "\beta"] & V \arrow[u, "j_Y"'] \end{tikzcd}$$

commutes in $\widetilde{\mathrm{Cor}}_k$. Composition in $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ is defined by $(\alpha, \beta) \circ (\gamma, \delta) := (\alpha \circ \gamma, \beta \circ \delta)$.

The category SmOp_k contains Sm_k as a full subcategory, the embedding $\mathrm{Sm}_k \to \mathrm{SmOp}_k$ being defined by $X \mapsto (X, \varnothing)$. Moreover, the embedding $\mathrm{Sm}_k \to \mathrm{SmOp}_k$ induces a fully faithful embedding $\widetilde{\mathrm{Cor}}_k \to \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ which on morphisms is given by $\alpha \mapsto (\alpha, 0)$.

Proposition I.2.2 ([GP18, Construction 2.8])**.** *Suppose that $\mathscr{F}$ is a presheaf on $\widetilde{\mathrm{Cor}}_k$. For any $(X, U) \in \mathrm{SmOp}_k$, let $\mathscr{F}(X, U) := \mathscr{F}(U)/\operatorname{im}(\mathscr{F}(X) \to \mathscr{F}(U))$. Then, for any $(\alpha, \beta) \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((X, U), (Y, V))$, $\mathscr{F}$ induces a morphism*

$$(\alpha, \beta)^* \colon \mathscr{F}(Y, V) \to \mathscr{F}(X, U).$$

Definition I.2.3 ([GP18, Definition 2.3])**.** Define the homotopy category $\mathrm{h}\widetilde{\mathrm{Cor}}_k$ of $\widetilde{\mathrm{Cor}}_k$ as follows. The objects of $\mathrm{h}\widetilde{\mathrm{Cor}}_k$ are the same as those of $\widetilde{\mathrm{Cor}}_k$, and the morphisms are given by

$$\begin{aligned} \mathrm{h}\widetilde{\mathrm{Cor}}_k(X, Y) &:= \widetilde{\mathrm{Cor}}_k(X, Y)/\sim_{\mathbf{A}^1} \\ &= \operatorname{coker}\left(\widetilde{\mathrm{Cor}}_k(\mathbf{A}^1 \times X, Y) \xrightarrow{i_0^* - i_1^*} \widetilde{\mathrm{Cor}}_k(X, Y)\right). \end{aligned}$$

Similarly, let $\mathrm{h}\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ denote the category whose objects are those of $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$, and whose morphisms are given by

$$\begin{aligned} &\mathrm{h}\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((X, U), (Y, V)) := \\ &\operatorname{coker}\left(\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}(\mathbf{A}^1 \times (X, U), (Y, V)) \xrightarrow{i_0^* - i_1^*} \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((X, U), (Y, V))\right). \end{aligned}$$

Here $\mathbf{A}^1 \times (X, U)$ is shorthand for $(\mathbf{A}^1 \times X, \mathbf{A}^1 \times U)$. If $\alpha \in \widetilde{\mathrm{Cor}}_k(X, Y)$ is a finite MW-correspondence, we write $\overline{\alpha}$ for the image of α in $\mathrm{h}\widetilde{\mathrm{Cor}}_k(X, Y)$. Similarly, if (α, β) is a morphism in $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ from (X, U) to (Y, V), write $\overline{(\alpha, \beta)}$ for the image of α in $\mathrm{h}\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((X, U), (Y, V))$. Note that a presheaf on $\widetilde{\mathrm{Cor}}_k$ is homotopy invariant if and only if it factors through $\mathrm{h}\widetilde{\mathrm{Cor}}_k$. Moreover, the embedding $\widetilde{\mathrm{Cor}}_k \to \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ induces a fully faithful embedding $\mathrm{h}\widetilde{\mathrm{Cor}}_k \to \mathrm{h}\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$.

Next we record a few observations that will come in handy later on:

Lemma I.2.4. *Suppose that α is a finite* MW*-correspondence from X to Y. Let $T_1, \ldots, T_n$ be the connected components of the support T of α. Then, for each $i = 1, \ldots, n$ there are uniquely determined finite* MW*-correspondences α_i supported on T_i such that $\alpha = \sum_i \alpha_i$.*

Proof. Since $\alpha \in \bigoplus_{x \in (X \times Y)^{(d_Y)}} \mathrm{K}_0^{\mathrm{MW}}(k(x), \wedge^{d_Y}(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \otimes (\omega_Y)_x)$, we may write $\alpha = \sum_i \alpha_i$ where α_i is supported on T_i. To conclude we must show that $\alpha_i \in \widetilde{\mathrm{CH}}^{d_Y}_{T_i}(X \times Y, \omega_Y)$, i.e., that $\partial(\alpha_i) = 0$ for all i. Now $\partial_x(\alpha_i) = 0$ for all $x \in X \times Y$ except perhaps for $x \in T_i$. But since T_i is disjoint from the other T_j's and $\partial(\alpha) = 0$ by assumption, we must have $\partial_x(\alpha_i) = 0$ also for $x \in T_i$. ■

Lemma I.2.5. *Let X be a smooth scheme, let $q \in \mathbf{Z}$ be an integer, and let $\mathscr{L}$ be a line bundle over X. Let $j\colon U \to X$ be a Zariski open subscheme, and suppose that $T \subseteq U$ is a subset which is closed in both U and X. Then the map*

$$j^*\colon \mathrm{H}^p_T(X, \mathbf{K}^{\mathrm{MW}}_q, \mathscr{L}) \to \mathrm{H}^p_T(U, \mathbf{K}^{\mathrm{MW}}_q, j^*\mathscr{L})$$

is an isomorphism for each $p \in \mathbf{Z}$, with inverse j_*, the finite pushforward of [CF17, §3].

Proof. The map j^* is an isomorphism by étale excision [CF17, Lemma 3.7]. The composition j^*j_* is the identity map on the Rost–Schmid complex supported on T, which implies the claim. ■

Corollary I.2.6. *Let $X, Y \in \mathrm{Sm}_k$, and let $j\colon V \to Y$ be a Zariski open subscheme. Suppose that $\alpha \in \widetilde{\mathrm{Cor}}_k(X, Y)$ is a finite* MW*-correspondence such that* $\operatorname{supp}\alpha \subseteq$ *$X \times V$. Then there is a unique finite* MW*-correspondence $\beta \in \widetilde{\mathrm{Cor}}_k(X, V)$ such that $j \circ \beta = \alpha$. In fact, we have $\beta = (1 \times j)^*\alpha$.*

Proof. Let $T := \operatorname{supp}\alpha$, so that by Lemma I.2.5 we have mutually inverse isomorphisms

$$(1 \times j)^*\colon \widetilde{\mathrm{CH}}^{d_Y}_T(X \times Y, \omega_Y) \rightleftarrows \widetilde{\mathrm{CH}}^{d_Y}_T(X \times V, \omega_V) : (1 \times j)_*$$

with $\alpha \in \widetilde{\mathrm{CH}}^{d_Y}_T(X \times Y, \omega_Y)$. Thus, if $\beta := (1 \times j)^*(\alpha)$ then $(1 \times j)_*\beta = \alpha$. We conclude the equality $(1 \times j)_*\beta = j \circ \beta$ from [CF17, Example 4.18]. ■

Lemma I.2.7. *Suppose that $j_X\colon U \to X$ and $j_Y\colon V \to Y$ are open subschemes of smooth connected k-schemes X, Y. Assume further that $\alpha \in \widetilde{\mathrm{Cor}}_k(X, Y)$ is a finite* MW*-correspondence such that the support $T := \operatorname{supp}\alpha$ satisfies $T \cap (U \times Y) \subseteq U \times V$. Let $\alpha' := (j_X \times j_Y)^*(\alpha)$. Then we have*

$$(\alpha, \alpha') \in \widetilde{\mathrm{Cor}}^{\mathrm{pr}}_k((X, U), (Y, V)).$$

Proof. First we show that $\alpha' \in \widetilde{\mathrm{Cor}}_k(U,V)$. By contravariant functoriality of Chow–Witt groups we may write $\alpha' = (1 \times j_Y)^*(j_X \times 1)^*(\alpha)$. Now

$$(j_X \times 1)^*(\alpha) = \alpha \circ j_X \in \widetilde{\mathrm{Cor}}_k(U,Y)$$

by [CF17, Example 4.17]. By [CF17, Lemmas 4.8, 4.10], $\mathrm{supp}(j_X \times 1)^*(\alpha) = T \cap (U \times Y)$ is finite and surjective over U. Since $T \cap (U \times Y) \subseteq U \times V$, we have

$$\alpha' \in \widetilde{\mathrm{CH}}^{d_Y}_{T\cap(U\times Y)}(U \times V, (1 \times j_Y)^*\omega_Y),$$

where $d_Y := \dim Y$. As j_Y is an open embedding we have

$$(1 \times j_Y)^*\omega_Y \cong \omega_V;$$

hence α' is a finite MW-correspondence from U to V.

Next we show that the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] & Y \\ U \arrow[u, "j_X"] \arrow[r, "\alpha'"] & V \arrow[u, "j_Y"'] \end{tikzcd}$$

commutes in $\widetilde{\mathrm{Cor}}_k$. As $T \cap (U \times Y) = T \cap (U \times V)$, the morphism $(j_X \times 1)^*$ factors as

$$\begin{tikzcd} \widetilde{\mathrm{CH}}^{d_Y}_{T}(X \times Y, \omega_Y) \arrow[r, "(j_X \times j_Y)^*"] \arrow[dr, "(j_X\times 1)^*"'] & \widetilde{\mathrm{CH}}^{d_Y}_{T\cap(U\times V)}(U \times V, \omega_V) \arrow[d, "(1\times j_Y)_*"] \\ & \widetilde{\mathrm{CH}}^{d_Y}_{T\cap(U\times Y)}(U \times Y, \omega_Y). \end{tikzcd}$$

Hence

$$j_Y \circ \alpha' = (1 \times j_Y)_*(j_X \times j_Y)^*(\alpha) = (j_X \times 1)^*(\alpha) = \alpha \circ j_X$$

by [CF17, Examples 4.17, 4.18]. ■

Relative Milnor–Witt correspondences

For later reference, let us also briefly mention the notion of finite Milnor–Witt correspondences relative to a base scheme $S \in \mathrm{Sm}_k$.

Definition I.2.8. Let $S \in \mathrm{Sm}_k$ be a smooth k-scheme. For any $X, Y \in \mathrm{Sm}_S$, let $p\colon X \times_S Y \to X$ denote the projection, and let d denote the relative dimension of p. We define the group of *finite relative* MW-*correspondences* from X to Y as

$$\widetilde{\mathrm{Cor}}_S(X,Y) := \varinjlim_T \widetilde{\mathrm{CH}}^d_T(X \times_S Y, \omega_p),$$

where the colimit runs over all closed subsets T of $X \times_S Y$ such that each irreducible component of T_{red} is finite and surjective over X.

One can show that the groups $\widetilde{\mathrm{Cor}}_S(X,Y)$ define the mapping sets of a category $\widetilde{\mathrm{Cor}}_S$ of *finite relative* MW-*correspondences*. However, below we will only need the definition of the groups $\widetilde{\mathrm{Cor}}_S(X,Y)$, and so we will not pursue the study of the category $\widetilde{\mathrm{Cor}}_S$ in further detail here.

Lemma I.2.9. *Let $S \in \mathrm{Sm}_k$ be a smooth k-scheme, and let $X, Y \in \mathrm{Sm}_S$. Then the canonical morphism $f\colon X \times_S Y \to X \times Y$ induces a homomorphism*

$$f_*\colon \widetilde{\mathrm{Cor}}_S(X,Y) \to \widetilde{\mathrm{Cor}}_k(X,Y)$$

given as the pushforward on Chow–Witt groups.

Proof. Let $d_Y := \dim Y$ and $d_S := \dim S$. Then the projection $p\colon X \times_S Y \to X$ has relative dimension $d_Y - d_S$, and the pushforward map on Chow–Witt groups is given as

$$f_*\colon \widetilde{\mathrm{CH}}^{d_Y - d_S}_T(X \times_S Y, \omega_p) \to \widetilde{\mathrm{CH}}^{d_Y}_{f(T)}(X \times Y, \omega_Y),$$

for any admissible subset T. Since f is finite, $f(T)$ is also an admissible subset. Hence, composing with the canonical map to the colimit $\widetilde{\mathrm{Cor}}_k(X,Y)$ on the right hand side, we obtain the desired homomorphism. ■

I.3 Milnor–Witt correspondences from Cartier divisors

Let us recall from [FØ17, §2] how a Cartier divisor gives rise to a finite MW-correspondence. Suppose that $X \in \mathrm{Sm}_k$ is a smooth integral k-scheme, and let $D = \{(U_i, f_i)\}$ be a Cartier divisor on X, with support $|D|$. We can associate a cohomology class

$$\widetilde{\mathrm{div}}(D) \in \mathrm{H}^1_{|D|}(X, \mathbf{K}^{\mathrm{MW}}_1, \mathcal{O}_X(D)) = \widetilde{\mathrm{CH}}^1_{|D|}(X, \mathcal{O}_X(D))$$

to D as follows. If $x \in X^{(1)}$ is a codimension 1-point on X, choose i such that $x \in U_i$. Consider the element

$$[f_i] \otimes f_i^{-1} \in \mathrm{K}^{\mathrm{MW}}_1(k(X), \mathcal{O}_X(D) \otimes k(X)).$$

Definition I.3.1 ([FØ17, Definition 2.1.1])**.** In the above setting, define

$$\widetilde{\mathrm{ord}}_x(D) := \partial_x([f_i] \otimes f_i^{-1}) \in \mathrm{K}^{\mathrm{MW}}_0(k(x), (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \otimes_{k(x)} \mathcal{O}_X(D)_x),$$

and

$$\widetilde{\mathrm{ord}}(D) := \sum_{x \in X^{(1)} \cap |D|} \widetilde{\mathrm{ord}}_x(D) \in \mathrm{C}^1(X, \mathbf{K}^{\mathrm{MW}}_1, \mathcal{O}_X(D)).$$

By [FØ17, Lemma 2.1.2], the definition of $\widetilde{\mathrm{ord}}_x(D)$ does not depend on the choice of U_i, and by [FØ17, Lemma 2.1.3] we have $\partial(\widetilde{\mathrm{ord}}(D)) = 0$. Therefore the element $\widetilde{\mathrm{ord}}(D)$ defines a cohomology class in $\widetilde{\mathrm{CH}}^1_{|D|}(X, \mathcal{O}_X(D))$, which we denote by $\widetilde{\mathrm{div}}(D)$.

Lemma I.3.2. *Let $X \in \mathrm{Sm}_k$ be a smooth integral k-scheme and suppose that D and D' are two Cartier divisors on X such that*

- *the supports of D and D' are disjoint, and*
- *there are trivializations $\chi\colon \mathcal{O}_X \xrightarrow{\cong} \mathcal{O}(D)$ and $\chi'\colon \mathcal{O}_X \xrightarrow{\cong} \mathcal{O}(D')$.*

Then χ and χ' induce an isomorphism

$$\widetilde{\mathrm{CH}}^1_{|D+D'|}(X, \mathcal{O}(D+D')) \cong \widetilde{\mathrm{CH}}^1_{|D|}(X, \mathcal{O}(D)) \oplus \widetilde{\mathrm{CH}}^1_{|D'|}(X, \mathcal{O}(D')).$$

Under this isomorphism we have the identification

$$\widetilde{\mathrm{div}}(D+D') = \widetilde{\mathrm{div}}(D) + \widetilde{\mathrm{div}}(D').$$

Proof. Since $\mathcal{O}(D+D') \cong \mathcal{O}(D) \otimes \mathcal{O}(D')$, χ and χ' furnish a trivialization

$$\chi \otimes \chi'\colon \mathcal{O}(D+D') \cong \mathcal{O}_X.$$

As $|D+D'| = |D| \amalg |D'|$, we thus obtain isomorphisms

$$\begin{aligned}\widetilde{\mathrm{CH}}^1_{|D+D'|}(X, \mathcal{O}(D+D')) &\cong \widetilde{\mathrm{CH}}^1_{|D|}(X, \mathcal{O}(D+D')) \oplus \widetilde{\mathrm{CH}}^1_{|D'|}(X, \mathcal{O}(D+D')) \\ &\cong \widetilde{\mathrm{CH}}^1_{|D|}(X) \oplus \widetilde{\mathrm{CH}}^1_{|D'|}(X) \\ &\cong \widetilde{\mathrm{CH}}^1_{|D|}(X, \mathcal{O}(D)) \oplus \widetilde{\mathrm{CH}}^1_{|D'|}(X, \mathcal{O}(D')).\end{aligned}$$

To show the last claim, let D and D' be given by the data $\{(U_i, f_i)\}$ respectively $\{(U_i, f'_i)\}$, so that $D + D' = \{(U_i, f_i f'_i)\}$. Let $x \in X^{(1)} \cap |D|$, and choose an i such that $x \in U_i$. Since the vanishing loci of f_i and f'_i are disjoint we may assume that $f'_i \in \Gamma(U_i, \mathcal{O}_X^\times)$, shrinking U_i if necessary. Hence $\partial_x([f'_i]) = 0$, and we obtain

$$\begin{aligned}\partial_x([f_i f'_i] \otimes (f_i f'_i)^{-1}) &= \partial_x(([f'_i] + \langle f'_i\rangle [f_i]) \otimes (f_i f'_i)^{-1}) \\ &= \langle f'_i\rangle \langle (f'_i)^{-1}\rangle \partial_x([f_i] \otimes f_i^{-1}) \\ &= \partial_x([f_i] \otimes f_i^{-1}).\end{aligned}$$

Thus $\partial_x([f_i f'_i] \otimes (f_i f'_i)^{-1}) = \widetilde{\mathrm{ord}}_x(D)$. A similar argument shows that

$$\partial_x([f_i f'_i] \otimes (f_i f'_i)^{-1}) = \widetilde{\mathrm{ord}}_x(D')$$

for all $x \in X^{(1)} \cap |D'|$, and the result follows. ■

If we require a condition on the line bundle $\mathcal{O}(D)$ and on the support of D, the class $\widetilde{\mathrm{div}}(D)$ does indeed give rise to a finite MW-correspondence:

Lemma I.3.3. *Let X and Y be smooth connected k-schemes with $\dim Y = 1$. Let D be a Cartier divisor on $X \times Y$. Suppose that*

- *there is an isomorphism* $\chi\colon \mathcal{O}_{X\times Y}(D) \xrightarrow{\cong} \omega_Y$, *and*
- *each irreducible component of the support* $|D|$ *of* D *is finite and surjective over* X.

Then the image of $\widetilde{\operatorname{div}}(D)$ *under the isomorphism*

$$\widetilde{\mathrm{CH}}^1_{|D|}(X\times Y, \mathcal{O}_{X\times Y}(D)) \xrightarrow{\cong} \widetilde{\mathrm{CH}}^1_{|D|}(X\times Y, \omega_Y)$$

induced by χ *defines a finite* MW*-correspondence* $\widetilde{\operatorname{div}}(D,\chi) \in \widetilde{\mathrm{Cor}}_k(X,Y)$.

Proof. By assumption, $|D|$ is an admissible subset, hence the claim follows. ■

Lemma I.3.4. *Assume the hypotheses of Lemma I.3.3, and let* $f\colon X' \to X$ *be a morphism of smooth* k*-schemes. Then*

$$\widetilde{\operatorname{div}}(D,\chi)\circ f = \widetilde{\operatorname{div}}((f\times 1)^*D, (f\times 1)^*\chi) \in \widetilde{\mathrm{Cor}}_k(X',Y).$$

Proof. As $\widetilde{\operatorname{div}}(D,\chi)\circ f = (f\times 1)^*\widetilde{\operatorname{div}}(D,\chi)$, the claim follows from the fact that $(f\times 1)^*$ commutes with the boundary map ∂ in the Rost–Schmid complex. ■

For later reference, let us also state the version of Corollary I.2.6 for Cartier-divisors:

Lemma I.3.5. *Assume the hypotheses of Lemma I.3.3. Suppose moreover that* $j\colon V\to Y$ *is a Zariski open subscheme of* Y *such that support* $|D|$ *is contained in* $X\times V$. *Then there exists a unique finite* MW*-correspondence* $\beta\in\widetilde{\mathrm{Cor}}_k(X,V)$ *such that* $j\circ\beta = \widetilde{\operatorname{div}}(D,\chi)$. *In fact,* β *is given by*

$$\beta = \widetilde{\operatorname{div}}((1\times j)^*D, (1\times j)^*\chi).$$

Proof. By the same argument as in the proof of Lemma I.3.4 we have

$$(1\times j)^*\widetilde{\operatorname{div}}(D,\chi) = \widetilde{\operatorname{div}}((1\times j)^*D, (1\times j)^*\chi).$$

Hence the claim follows from Corollary I.2.6. ■

The above lemmas give a procedure to construct a morphism of pairs from a Cartier divisor:

Lemma I.3.6. *Assume the hypotheses of Lemma I.3.3, and let* $j_X\colon U\to X$ *and* $j_Y\colon V\to Y$ *be open subschemes. Let* $D' := D|_{U\times Y}$ *be the restriction of* D *to* $U\times Y$. *Suppose that* $|D'|\subseteq U\times V$. *Then*

$$(\widetilde{\operatorname{div}}(D,\chi), \widetilde{\operatorname{div}}((j_X\times j_Y)^*D, (j_X\times j_Y)^*\chi)) \in \widetilde{\mathrm{Cor}}^{\mathrm{pr}}_k((X,U),(Y,V)).$$

Proof. By Lemma I.3.4, $\widetilde{\operatorname{div}}((j_X\times j_Y)^*D) = (j_X\times j_Y)^*\widetilde{\operatorname{div}}(D)$, hence the claim follows from Lemma I.2.7. ■

We will frequently make use of the following well known fact in order to determine if the support of a given principal divisor satisfies the hypotheses of Lemma I.3.3:

Lemma I.3.7. *Let A be a ring, and suppose that P is a monic polynomial in $A[t]$. Then* $\mathrm{Spec}(A[t]/(P)) \to \mathrm{Spec}(A)$ *is finite, and every irreducible component of* $\mathrm{Spec}(A[t]/(P))$ *surjects onto* $\mathrm{Spec}(A)$.

Proof. Write $P(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$, and let $M := A[t]/(P)$. Then M is generated as an A-module by $1, t, \ldots, t^{n-1}$, hence $\mathrm{Spec}(A[t]/(P)) \to \mathrm{Spec}(A)$ is finite. As $A[t]/(P)$ is integral over A, it follows that the morphism is surjective as well. ∎

I.4 Zariski excision on the affine line

The aim of this section is to prove the following excision result:

Theorem I.4.1. *Let $x \in \mathbf{A}^1$ be a closed point and suppose that $V \subseteq U \subseteq \mathbf{A}^1$ are two Zariski open neighborhoods of x. Let $i\colon V \hookrightarrow U$ denote the inclusion, and let $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$ be a homotopy invariant presheaf with* MW*-transfers. Then the induced map*

$$i^*\colon \frac{\mathscr{F}(U \setminus x)}{\mathscr{F}(U)} \to \frac{\mathscr{F}(V \setminus x)}{\mathscr{F}(V)}$$

is an isomorphism.

The proof of Zariski excision proceeds in three steps. First we prove:

Theorem I.4.2 (Injectivity on the affine line)**.** *With the notation as in Theorem I.4.1, there exists a finite* MW*-correspondence $\Phi \in \widetilde{\mathrm{Cor}}_k(U, V)$ such that*

$$\bar{i} \circ \overline{\Phi} = \mathrm{id}_U$$

in $\mathrm{h}\widetilde{\mathrm{Cor}}_k$.

Theorem I.4.2 implies that $\Phi^* \circ i^* = \mathrm{id}_{\mathscr{F}(U)}$ for any homotopy invariant $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$, i.e., that i^* is injective. In particular, letting $V = U \setminus y$ for a closed point y of U, this means that $\mathscr{F}(U)$ is a subgroup of $\mathscr{F}(U \setminus y)$, justifying the notation of Theorem I.4.1.

The next step is then to extend Theorem I.4.2 to the category $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ of pairs. By abuse of notation, write i also for the inclusion $i\colon (V, V \setminus x) \hookrightarrow (U, U \setminus x)$ in SmOp_k. By Proposition I.2.2, i induces a map

$$i^*\colon \frac{\mathscr{F}(U \setminus x)}{\mathscr{F}(U)} \to \frac{\mathscr{F}(V \setminus x)}{\mathscr{F}(V)}$$

on the quotient, and the following theorem tells us that i^* is injective:

Theorem I.4.3 (Injectivity of Zariski excision)**.** *There exists a finite Milnor–Witt correspondence* $\Phi \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus x), (V, V \setminus x))$ *such that*

$$\bar{i} \circ \overline{\Phi} = \mathrm{id}_{(U, U\setminus x)}$$

in $\mathrm{h}\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$.

In the final step we establish surjectivity of i^*:

Theorem I.4.4 (Surjectivity of Zariski excision)**.** *With the notation as in Theorem I.4.1, there exist finite* MW*-correspondences* $\Psi \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus x), (V, V \setminus x))$ *and* $\Theta \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((V, V \setminus x), (V \setminus x, V \setminus x))$ *such that*

$$\overline{\Psi} \circ \bar{i} - \overline{j_V} \circ \overline{\Theta} = \mathrm{id}_{(V, V\setminus x)}$$

in $\mathrm{h}\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$, *where* $j_V \colon (V \setminus x, V \setminus x) \hookrightarrow (V, V \setminus x)$ *denotes the inclusion in* SmOp_k.

We note that Theorem I.4.1 is a consequence of Theorems I.4.3 and I.4.4:

Proof of Theorem I.4.1. As Φ is a morphism of pairs by Theorem I.4.3, Proposition I.2.2 tells us that Φ induces a morphism on the quotient

$$\Phi^* \colon \frac{\mathscr{F}(V \setminus x)}{\mathscr{F}(V)} \to \frac{\mathscr{F}(U \setminus x)}{\mathscr{F}(U)}.$$

Moreover, $\Phi^* \circ i^* = \mathrm{id}$ by Theorem I.4.3, hence i^* is injective.

On the other hand, as Θ maps to $(V \setminus x, V \setminus x)$ by Theorem I.4.4, it follows that $j_V \circ \Theta$ induces the trivial map on the quotient. Hence

$$i^* \circ \Psi^* = \mathrm{id} \colon \frac{\mathscr{F}(V \setminus x)}{\mathscr{F}(V)} \to \frac{\mathscr{F}(V \setminus x)}{\mathscr{F}(V)},$$

so that i^* is surjective. ■

It is therefore enough to prove Theorems I.4.2, I.4.3, and I.4.4.

I.5 Injectivity on the affine line

We continue with the same notation as in Theorem I.4.1. Thus $V \subseteq U \subseteq \mathbf{A}^1$ are two Zariski open neighborhoods of a closed point $x \in \mathbf{A}^1$, with inclusion $i \colon V \to U$. In order to produce the desired MW-correspondence $\Phi \in \widetilde{\mathrm{Cor}}_k(U, V)$ of Theorem I.4.2, we will need to consider certain "thick diagonals"

$$\Delta_m \in \widetilde{\mathrm{Cor}}_k(U, U),$$

constructed as follows.

Let $U \times U \subseteq \mathbf{A}^2$ have coordinates X and Y, respectively, and let $\Delta := \Delta(U) \subseteq U \times U$ denote the diagonal. For each $m \geq 1$, let f_m denote the polynomial

$$f_m(X, Y) := (Y - X)^m \in k[U \times U].$$

As f_m is monic in Y, it follows from Lemma I.3.7 that the support of the divisor

$$D_m := \mathcal{V}(f_m) := \{f_m = 0\} \subseteq U \times U$$

is finite and surjective over U. Moreover, as D_m is a principal Cartier divisor on $U \times U$, there is a trivialization $\mathcal{O}(D_m) \cong \mathcal{O}_{U\times U}$ given by $f_m^{-1} \mapsto 1$. We further obtain an isomorphism $\chi_m \colon \mathcal{O}(D_m) \cong \omega_U$ by $f_m^{-1} \mapsto dY$. By Lemma I.3.3, it follows that the divisor D_m gives rise to a finite MW-correspondence from U to U.

Definition I.5.1. For each $m \geq 1$, let

$$\Delta_m := \widetilde{\operatorname{div}}(D_m, \chi_m) \in \widetilde{\operatorname{Cor}}_k(U, U)$$

be the finite MW-correspondence defined by the data D_m and χ_m above.

Remark I.5.2. By the definition of $\widetilde{\operatorname{div}}(D_m, \chi_m)$, we see that Δ_m is given by the total residue

$$\Delta_m = \partial([f_m] \otimes dY) \in \widetilde{\mathrm{CH}}^1_\Delta(U \times U, \omega_U)$$

of the element $[f_m] \otimes dY \in \mathrm{K}_1^{\mathrm{MW}}(k(U \times U), \omega_U)$. Thus the support of the MW-correspondence Δ_m is the diagonal $\Delta = D_1 \subseteq U \times U$.

Lemma I.5.3. *For any $m \geq 0$ we have*

$$\Delta_{m+1} - \Delta_m = \langle -1 \rangle^m \cdot \Delta_1 \in \widetilde{\operatorname{Cor}}_k(U, U),$$

with $\Delta_1 = \mathrm{id}_U$.

Proof. Since Δ_m is supported on the diagonal $\Delta \subseteq U \times U$, it suffices to compute the residue $\partial_y([f_m] \otimes dY)$ at the codimension 1-point $y \in (U \times U)^{(1)}$ corresponding to the diagonal.

Recall from [Mor12, Lemma 3.14] that for any integer $n \geq 0$ we have $[a^n] = n_\epsilon [a]$ in $\mathrm{K}_1^{\mathrm{MW}}$, where $n_\epsilon = \sum_{i=1}^n \langle (-1)^{i-1} \rangle$. We thus get

$$\partial_y([f_m] \otimes dY) = m_\epsilon \otimes \overline{(Y - X)} dY \in \mathrm{K}_0^{\mathrm{MW}}(k(y), (\mathfrak{m}_y / \mathfrak{m}_y^2)^\vee \otimes (\omega_U)_y).$$

For $m = 1$, this reads $\Delta_1 = \langle 1 \rangle \otimes \overline{(Y - X)} dY = \mathrm{id}_U$. In the general case we obtain

$$\Delta_{m+1} - \Delta_m = ((m+1)_\epsilon - m_\epsilon) \otimes \overline{(Y - X)} dY = \langle (-1)^m \rangle \cdot \mathrm{id}_U,$$

using that $\Delta_1 = \mathrm{id}_U \in \widetilde{\operatorname{Cor}}_k(U, U)$. ∎

Our next objective is to prove the following:

Lemma I.5.4. *For $m \gg 0$ there exists a finite* MW*-correspondence $\Phi_m \colon U \to V$ such that $i \circ \Phi_m = \Delta_m$ in $\mathrm{h}\widetilde{\operatorname{Cor}}_k(U, U)$.*

Having established these properties of Δ_m and Φ_m, we will set $\Phi := \Phi_{m+1} - \Phi_m$ and show that we then have $i \circ \Phi \sim_{\mathbf{A}^1} \mathrm{id}_U$ provided m is an even integer $\gg 0$. To define Φ_m, we will need to ensure the existence of polynomials with certain specified properties.

Lemma I.5.5 ([GP18, §5])**.** *Let $A := \mathbf{A}^1 \setminus U$ and $B := U \setminus V$. For $m \gg 0$, there exists a polynomial $G_m \in k[U][Y] = k[U \times \mathbf{A}^1]$, monic and of degree m in Y, satisfying the following properties:*

(1) $G_m(Y)|_{U\times B} = 1$.

(2) $G_m(Y)|_{U\times A} = (Y - X)^m|_{U\times A}$.

(3) $G_m(Y)|_{U\times x} = (Y - X)^m|_{U\times x}$.

Remark I.5.6. The above polynomials, as well as those in Sections I.6 and I.7, are all constructed using variants of the Chinese remainder theorem, allowing us to find polynomials with specified behavior at given subschemes. The requirement that the desired polynomial be monic can be thought of as specifying its behavior at infinity. For example, the Chinese remainder theorem establishes a surjection $k[U \times \mathbf{A}^1] \to k[U \times A] \oplus k[U \times B]$, from which we can deduce Lemma I.5.5.

Lemma I.5.7. *Let D_{G_m} be the divisor on $U \times U$ defined by G_m, and let*

$$\phi_m \colon \mathcal{O}(D_{G_m}) \cong \omega_U$$

be the isomorphism determined by choosing the generator dY for ω_U. Then

$$\widetilde{\mathrm{div}}((1 \times i)^* D_{G_m}, (1 \times i)^* \phi_m) \in \widetilde{\mathrm{Cor}}_k(U, V).$$

Proof. Since G_m is monic in Y, the support $\mathcal{V}(G_m)$ of D_{G_m} is finite and surjective over U by Lemma I.3.7. Using the trivializations of $\mathcal{O}(D_{G_m})$ and of ω_U, Lemma I.3.3 implies that $\widetilde{\mathrm{div}}(D_{G_m}, \phi_m) \in \widetilde{\mathrm{Cor}}_k(U, U)$. Now, the polynomial G_m satisfies the following:

- $G_m|_{U\times A} \in k[U \times A]^\times$. This follows from the fact that $U \times A = U \times (\mathbf{A}^1 \setminus U)$ contains no diagonal points.
- $G_m|_{U\times B} \in k[U \times B]^\times$. This is obvious, as $G_m|_{U\times B} = 1$.

The above properties imply that $\mathcal{V}(G_m) \subseteq U \times V$. Hence the claim follows from Lemma I.3.5. ∎

Definition I.5.8. For $m \gg 0$, we define

$$\Phi_m := \widetilde{\mathrm{div}}((1 \times i)^* D_{G_m}, (1 \times i)^* \phi_m) \in \widetilde{\mathrm{Cor}}_k(U, V).$$

We now aim to define a homotopy $\mathscr{H}_m \colon i \circ \Phi_m \sim_{\mathbf{A}^1} \Delta_m$. Consider the product $\mathbf{A}^1 \times U \times \mathbf{A}^1$, where θ is the coordinate of the first copy of $\mathbf{A}^1$, U has coordinate X and the last $\mathbf{A}^1$ has coordinate Y. Let $H_\theta \in k[\mathbf{A}^1 \times U \times \mathbf{A}^1]$ be the polynomial

$$H_\theta(Y) := \theta G_m + (1-\theta)(Y-X)^m.$$

Since $U \times A$ contains no diagonal points, the restriction

$$G_m(Y)|_{U\times A} = (Y-X)^m|_{U\times A}$$

does not vanish on $U \times A$. It follows that

$$H_\theta(Y)|_{\mathbf{A}^1\times U\times A} = (Y-X)^m|_{\mathbf{A}^1\times U\times A} \in k[\mathbf{A}^1 \times U \times A]^\times.$$

Hence $\mathcal{V}(H_\theta) \subseteq \mathbf{A}^1 \times U \times U$. Let D_{H_θ} be the principal Cartier divisor on $\mathbf{A}^1 \times U \times U$ defined by H_θ, and let $\psi\colon \mathcal{O}(D_{H_\theta}) \cong \omega_U$ be the isomorphism given by choosing the generator dY for ω_U.

Lemma I.5.9. *Let $\mathscr{H}_m := \widetilde{\operatorname{div}}(D_{H_\theta}, \psi)$. Then $\mathscr{H}_m \in \widetilde{\operatorname{Cor}}_k(\mathbf{A}^1 \times U, U)$.*

Proof. As G_m is monic and of degree m in Y, it follows that the linear combination H_θ of G_m and $(Y-X)^m$ is also monic and of degree m in Y. Therefore the support $\mathcal{V}(H_\theta)$ of D_{H_θ} is finite and surjective over $\mathbf{A}^1 \times U$ by Lemma I.3.7. The result then follows from Lemma I.3.3. ∎

Lemma I.5.10. *Let $\mathscr{H}_m|_0 := \mathscr{H}_m \circ i_0, \mathscr{H}_m|_1 := \mathscr{H}_m \circ i_1 \in \widetilde{\operatorname{Cor}}_k(U,U)$ denote the respective precompositions of $\mathscr{H}_m \in \widetilde{\operatorname{Cor}}_k(\mathbf{A}^1 \times U, U)$ with the rational points $i_0, i_1\colon U \to \mathbf{A}^1 \times U$. Then $\mathscr{H}_m|_0 = \Delta_m$ and $\mathscr{H}_m|_1 = i \circ \Phi_m$.*

Proof. By Lemma I.3.4 we have

$$\mathscr{H}_0 = \widetilde{\operatorname{div}}((i_0 \times 1)^* D_{H_\theta}, (i_0 \times 1)^*\psi) = \widetilde{\operatorname{div}}(D_m, \chi_m) = \Delta_m.$$

On the other hand,

$$\mathscr{H}_1 = \widetilde{\operatorname{div}}((i_1 \times 1)^* D_{H_\theta}, (i_1 \times 1)^*\psi) = \widetilde{\operatorname{div}}(D_{G_m}, \phi_m) = i \circ \Phi_m$$

by Lemma I.3.5. ∎

We are now ready to prove the injectivity of the induced morphism

$$i^*\colon \mathscr{F}(U) \to \mathscr{F}(V),$$

for any homotopy invariant $\mathscr{F} \in \widetilde{\operatorname{PSh}}(k)$.

Proof of Theorem I.4.2. Let $m \gg 0$ be an integer large enough so that the polynomial G_m of Lemma I.5.5 exists. If $\Phi := \Phi_{2m+1} - \Phi_{2m}$, we then have

$$i \circ \Phi \sim_{\mathbf{A}^1} (\Delta_{2m+1} - \Delta_{2m}) = \langle(-1)^{2m}\rangle \mathrm{id}_U = \mathrm{id}_U$$

by Lemma I.5.3. As $\mathscr{F}$ is homotopy invariant, this yields $\Phi^* \circ i^* = \mathrm{id}_{\mathscr{F}(U)}$, hence i^* is injective. ∎

I.6 Injectivity of Zariski excision

We wish to extend Theorem I.4.2 to the category of pairs $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$—in other words to produce a morphism

$$(\Phi_m, \Phi_m^x) \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus x), (V, V \setminus x))$$

and a homotopy

$$(\mathscr{H}_m, \mathscr{H}_m^x) \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}(\mathbf{A}^1 \times (U, U \setminus x), (U, U \setminus x))$$

from Δ_m to $(i, i|_{V\setminus x}) \circ (\Phi_m, \Phi_m^x)$. This establishes Theorem I.4.3.

Let j_U and j_V denote the respective open immersions $j_U : U \setminus x \to U$ and $j_V : V \setminus x \to V$.

Lemma I.6.1. *Let*

$$\Phi_m^x := \widetilde{\mathrm{div}}((j_U \times j_V)^* D_{G_m}, (j_U \times j_V)^* \phi_m).$$

Then (Φ_m, Φ_m^x) *constitutes a morphism in* $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ *from* $(U, U \setminus x)$ *to* $(V, V \setminus x)$.

Proof. By Lemma I.3.6, it suffices to show that the support of $(j_U \times 1)^* D_{G_m}$ is contained in $(U \setminus x) \times (V \setminus x)$. As we already know that

$$\mathcal{V}(G_m) \cap ((U \setminus x) \times \mathbf{A}^1) \subseteq (U \setminus x) \times V,$$

it is enough to check that G_m does not vanish on $(U \setminus x) \times x$. By condition (3) of Lemma I.5.5, $G_m(Y)|_{U\times x} = (Y - X)^m|_{U\times x}$. As $(U \setminus x) \times x$ contains no diagonal points, it therefore follows that $G_m|_{(U\setminus x)\times x} \in k[(U \setminus x) \times x]^\times$. Hence $\mathcal{V}(G_m) \cap ((U \setminus x) \times \mathbf{A}^1) \subseteq (U \setminus x) \times (V \setminus x)$. ■

Lemma I.6.2. *Let* $\mathscr{H}_\theta^x := \widetilde{\mathrm{div}}(((1 \times j_U) \times j_U)^* D_{H_\theta}, ((1 \times j_U) \times j_U)^* \psi)$. *Then*

$$(\mathscr{H}_\theta, \mathscr{H}_\theta^x) \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}(\mathbf{A}^1 \times (U, U \setminus x), (U, U \setminus x)).$$

Proof. In light of Lemma I.3.6, it remains to check that

$$\mathcal{V}(H_\theta) \cap (\mathbf{A}^1 \times (U \setminus x) \times \mathbf{A}^1) \subseteq \mathbf{A}^1 \times (U \setminus x) \times (U \setminus x).$$

It is sufficient to show that H_θ does not vanish on $\mathbf{A}^1 \times (U \setminus x) \times x$. But

$$H_\theta(Y)|_{\mathbf{A}^1\times(U\setminus x)\times x} = \theta \cdot (Y - X)^m + (1-\theta) \cdot (Y - X)^m = (Y - X)^m|_{\mathbf{A}^1\times(U\setminus x)\times x},$$

and $(Y - X)^m|_{\mathbf{A}^1\times(U\setminus x)\times x} \in k[\mathbf{A}^1 \times (U \setminus x) \times x]^\times$ as $(U \setminus x) \times x$ contains no diagonal points. Whence the claim. ■

Proof of Theorem I.4.3. By a similar argument as in the proof of Lemma I.5.10, $(\mathscr{H}_\theta, \mathscr{H}_\theta^x)$ is a homotopy from Δ_m to $(i, i|_{V\setminus x}) \circ (\Phi_m, \Phi_m^x)$. Thus the same proof as that of Theorem I.4.2 applies. ■

I.7 Surjectivity of Zariski excision

We proceed to prove Theorem I.4.4. To begin with, we interpolate polynomials in a similar fashion as Lemma I.5.5:

Lemma I.7.1 ([GP18, §5]). *For $m \gg 0$ there exists a polynomial*

$$G_m(Y) \in k[U][Y] = k[U \times \mathbf{A}^1],$$

monic and of degree m in Y, satisfying the following properties:

(1′) $G_m(Y)|_{U \times B} = 1.$

(2′) $G_m(Y)|_{U \times A} = (Y - X)|_{U \times A}.$

(3′) $G_m(Y)|_{U \times x} = (Y - X)|_{U \times x}.$

Lemma I.7.2 ([GP18, §5]). *For $m \gg 0$ there exists a polynomial*

$$F_{m-1}(Y) \in k[V][Y] = k[V \times \mathbf{A}^1],$$

monic and of degree $m-1$ in Y, satisfying the following properties:

(1″) $F_{m-1}(Y)|_{V \times B} = (Y - X)^{-1} \in k[V \times B]^\times.$

(2″) $F_{m-1}(Y)|_{V \times A} = 1.$

(3″) $F_{m-1}(Y)|_{\Delta(V)} = 1.$

Remark I.7.3. As $B = U \setminus V$, the set $V \times B$ does not contain any diagonal points. Hence the function $Y - X$ is invertible on $V \times B$, so (1″) makes sense.

Definition I.7.4. Set

$$E_m := (Y - X) \cdot F_{m-1} \in k[V][Y]$$

and

$$H_\theta := \theta G_m + (1 - \theta) E_m \in k[\mathbf{A}^1 \times V][Y],$$

where θ is the coordinate of $\mathbf{A}^1$.

Observe that the divisor $\mathcal{V}(E_m)$ satisfies

$$\mathcal{V}(E_m) = \mathcal{V}(Y - X) \cup \mathcal{V}(F_{m-1}) = \Delta(V) \cup \mathcal{V}(F_{m-1}).$$

In fact, by (3″), this union is a disjoint union. Moreover, using the definition of F_{m-1} we see that E_m enjoys the following properties:

(1_E) $E_m(Y)|_{V \times B} = 1 = G_m(Y)|_{V \times B}.$

(2_E) $E_m(Y)|_{V \times A} = (Y - X)|_{V \times A} = G_m(Y)|_{V \times A}.$

(3_E) $E_m(Y)|_{V \times x} = (Y - X)|_{V \times x} = G_m(Y)|_{V \times x}.$

The last property (3_E) implies:

$(3'_E)$ $E_m(Y)|_{(V\setminus x)\times x} = G_m(Y)|_{(V\setminus x)\times x} \in k[(V\setminus x)\times x]^\times$.

Let us first construct the finite MW-correspondence $\Psi \in \widetilde{\mathrm{Cor}}_k(U,V)$ using the polynomial G_m of Lemma I.7.1 for $m \gg 0$. By Lemma I.7.1, $\mathcal{V}(G_m) \subseteq U\times V$, and we may consider the principal divisor D_{G_m} on $U\times V$ defined by G_m. Let $\psi\colon \mathcal{O}(D_{G_m}) \cong \omega_V$ be the isomorphism determined by choosing the generator dY for ω_V.

Lemma I.7.5. *Put*

$$\Psi := \widetilde{\mathrm{div}}(D_{G_m},\psi)$$

and

$$\Psi^x := \widetilde{\mathrm{div}}((j_U\times j_V)^* D_{G_m}, (j_U\times j_V)^*\psi).$$

Then

$$(\Psi,\Psi^x) \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U,U\setminus x),(V,V\setminus x)).$$

Proof. Since G_m is monic in Y, $\mathcal{V}(G_m)$ is finite and surjective over U by Lemma I.3.7. Thus Lemma I.3.3 ensures that Ψ is a finite MW-correspondence from U to V. Moreover, as $G_m(Y)|_{U\times x} = (Y-X)|_{U\times x}$, it follows that $G_m|_{(U\setminus x)\times x}$ is invertible on $(U\setminus x)\times x$. Hence there is an inclusion

$$\mathcal{V}(G_m)\cap((U\setminus x)\times V) \subseteq (U\setminus x)\times(V\setminus x).$$

By Lemma I.3.6 it follows that (Ψ,Ψ^x) is a morphism of pairs from $(U,U\setminus x)$ to $(V,V\setminus x)$. ■

In order to define the desired homotopy, we proceed in a familiar fashion. By (1_E) and (2_E), H_θ is invertible on $\mathbf{A}^1\times V\times B$ and $\mathbf{A}^1\times V\times A$. Hence $\mathcal{V}(H_\theta) \subseteq \mathbf{A}^1\times V\times V$, and we may consider the divisor D_{H_θ} on $\mathbf{A}^1\times V\times V$. We let $\chi\colon \mathcal{O}(D_{H_\theta}) \cong \omega_V$ be the isomorphism given by choosing the generator dY for ω_V.

Lemma I.7.6. *Let* $\mathscr{H}_\theta := \widetilde{\mathrm{div}}(D_{H_\theta},\chi)$ *and*

$$\mathscr{H}_\theta^x := \widetilde{\mathrm{div}}(((1\times j_V)\times j_V)^* D_{H_\theta}, ((1\times j_V)\times j_V)^*\chi).$$

Then

$$(\mathscr{H}_\theta,\mathscr{H}_\theta^x) \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}(\mathbf{A}^1\times(V,V\setminus x),(V,V\setminus x)).$$

Proof. To see that $\mathscr{H}_\theta$ is a finite MW-correspondence from $\mathbf{A}^1\times V$ to V, note that both G_m and E_m are monic and of the same degree in Y. Therefore the linear combination H_θ of G_m and E_m is also monic in Y, and it follows that the support $\mathcal{V}(H_\theta)$ of D_{H_θ} is finite and surjective over $\mathbf{A}^1\times V$ by Lemma I.3.7. Hence $\mathscr{H}_\theta \in \widetilde{\mathrm{Cor}}_k(\mathbf{A}^1\times V,V)$ by Lemma I.3.3.

Turning to $\mathscr{H}_\theta^x$, we must show that

$$\mathcal{V}(H_\theta)\cap(\mathbf{A}^1\times(V\setminus x)\times V) \subseteq \mathbf{A}^1\times(V\setminus x)\times(V\setminus x).$$

We already know that H_θ is invertible on $\mathbf{A}^1 \times (V \setminus x) \times A$ and on $\mathbf{A}^1 \times (V \setminus x) \times B$. It remains to check the set $\mathbf{A}^1 \times (V \setminus x) \times x$. But by ($3_E$) and ($3'_E$) we have

$$E_m(Y)|_{(V\setminus x)\times x} = G_m(Y)|_{(V\setminus x)\times x} = (Y - X)|_{(V\setminus x)\times x},$$

which is invertible as $(V \setminus x) \times x$ does not intersect the diagonal. Therefore the linear combination H_θ of E_m and G_m is also invertible on $(V \setminus x) \times x$, and the claim follows. Using Lemma I.3.6, this shows that $(\mathscr{H}_\theta, \mathscr{H}_\theta^x)$ constitutes a morphism of pairs from $\mathbf{A}^1 \times (V, V \setminus x)$ to $(V, V \setminus x)$. ∎

Let us compute the start- and endpoints $\mathscr{H}_0$, $\mathscr{H}_1$ of the homotopy $\mathscr{H}_\theta$—that is, the precomposition of $\mathscr{H}_\theta$ with the rational points $i_0, i_1 \colon V \to \mathbf{A}^1 \times V$.

Lemma I.7.7. *We have $\mathscr{H}_0 = \mathrm{id}_V + j_V \circ \Theta$ where $\Theta \in \widetilde{\mathrm{Cor}}_k(V, V \setminus x)$. On the other hand, $\mathscr{H}_1 = \Psi \circ i$, where $i \colon V \hookrightarrow U$ is the inclusion.*

Proof. By Lemma I.3.4 we have $\mathscr{H}_1 = \widetilde{\mathrm{div}}((i_1 \times 1)^* D_{H_\theta}, (i_1 \times 1)^*\chi) = \Psi \circ i$. As for $\mathscr{H}_0$, we have

$$\mathscr{H}_0 = \widetilde{\mathrm{div}}((i_0 \times 1)^* D_{H_\theta}, (i_0 \times 1)^*\chi) = \widetilde{\mathrm{div}}(D_{E_m}, (i_0 \times 1)^*\chi),$$

where D_{E_m} is the principal Cartier divisor on $V \times V$ defined by the polynomial E_m. Let $D_{F_{m-1}}$ be the principal divisor on $V \times V$ defined by F_{m-1}. As $\mathcal{V}(E_m) = \Delta(V) \amalg \mathcal{V}(F_{m-1})$, Lemma I.3.2 tells us that

$$\mathscr{H}_0 = \Delta_1 + \widetilde{\mathrm{div}}(D_{F_{m-1}}, (i_0 \times 1)^*\chi).$$

Here Δ_1 is the divisor defined in Definition I.5.1, satisfying $\Delta_1 = \mathrm{id}_V$. As $\mathcal{V}(F_{m-1}) \subseteq V \times (V \setminus x)$, Lemma I.3.5 ensures that there is a unique element

$$\Theta \in \widetilde{\mathrm{CH}}^1_{\mathcal{V}(F_{m-1})}(V \times (V \setminus x), \omega_{V\setminus x})$$

such that $j_V \circ \Theta = \widetilde{\mathrm{div}}(D_{E_m}, (i_0 \times 1)^*\chi)$. By Lemma I.7.2, $\mathcal{V}(F_{m-1})$ is finite and surjective over $V \setminus x$, and hence $\Theta \in \widetilde{\mathrm{Cor}}_k(V, V \setminus x)$. ∎

Proof of Theorem I.4.4. The result now follows directly from Lemma I.7.7. ∎

I.8 Zariski excision on $\mathbf{A}^1_K$

We now aim to extend the results of Section I.4 to open subsets of $\mathbf{A}^1_K$, where $K = k(X)$ is the function field of some integral k-scheme $X \in \mathrm{Sm}_k$. This can be achieved by the following trick, which was suggested to the author by I. Panin: given a presheaf $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$, we can extend $\mathscr{F}$ to a presheaf $\mathscr{F}^X$ on a certain full subcategory of $\widetilde{\mathrm{Cor}}_K$, and then use Zariski excision for presheaves on $\widetilde{\mathrm{Cor}}_K$.

In this section, the field k is assumed to be of characteristic 0.

Remark I.8.1. Notice that the results of Section I.4 show that Zariski excision on $\mathbf{A}^1_K$ holds for any homotopy invariant presheaf on $\widetilde{\mathrm{Cor}}_K$ by simply letting the ground field be K. The point of this section, however, is to show that we can obtain Zariski excision on $\mathbf{A}^1_K$ also for homotopy invariant presheaves on $\widetilde{\mathrm{Cor}}_k$.

Definition I.8.2. Let $X \in \mathrm{Sm}_k$ be a smooth integral k-scheme, and let $K := k(X)$ be the function field of X. We define the category $\widetilde{\mathrm{Cor}}_K^X$ as follows. Its objects are pairs $(Y, V \subseteq Y_K)$ consisting of a smooth k-scheme $Y \in \mathrm{Sm}_k$ along with an open subscheme V of $Y_K := Y \times_k \mathrm{Spec}(k(X))$. The morphisms of $\widetilde{\mathrm{Cor}}_K^X$ are given as

$$\mathrm{Hom}_{\widetilde{\mathrm{Cor}}_K^X}((Y, V), (Y', V')) := \widetilde{\mathrm{Cor}}_K(V, V').$$

Abusing notation, we may write simply V for an object (Y, V) of $\widetilde{\mathrm{Cor}}_K^X$.

Remark I.8.3. Since any open subscheme V of Y_K is K-smooth, $\widetilde{\mathrm{Cor}}_K^X$ is equivalent to the full subcategory of $\widetilde{\mathrm{Cor}}_K$ whose objects are those $V \in \widetilde{\mathrm{Cor}}_K$ for which there exists $Y \in \mathrm{Sm}_k$ along with an open embedding $V \hookrightarrow Y_K$.

Let us fix some notation:

Definition I.8.4. If $(Y, V) \in \widetilde{\mathrm{Cor}}_K^X$, we define the following subschemes of Y_K and $Y \times_k X$:

- $Z := Y_K \setminus V$;
- $\mathscr{Z} := \overline{Z}$, the Zariski closure of Z in $Y \times_k X$;
- $\mathscr{V} := (Y \times_k X) \setminus \mathscr{Z}$.

Let also $\mathscr{V}_K := \mathscr{V} \times_X \mathrm{Spec}(K)$ denote the generic fiber of the projection $p_X|_{\mathscr{V}} \colon \mathscr{V} \to X$. Note that we then have $\mathscr{V}_K = V$. Furthermore, for each open subscheme X_i of X, set

$$\mathscr{V}(X_i) := \mathscr{V} \cap (Y \times_k X_i),$$

the intersection being taken in $Y \times_k X$. Then we have $V = \mathscr{V}_K = \varprojlim_{X_i \subseteq X} \mathscr{V}(X_i)$, where the limit runs over all nonempty open subsets of X. In particular, $\mathscr{V} = \mathscr{V}(X)$.

Definition I.8.5. Let $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$ be a presheaf. For any $V \in \widetilde{\mathrm{Cor}}_K^X$, we set

$$\mathscr{F}^X(V) := \varinjlim_i \mathscr{F}(\mathscr{V}(X_i)).$$

In particular, if $(Y, V) = (\mathbf{A}^1_k, \mathbf{A}^1_K)$, then

$$\mathscr{F}^X(V) = \mathscr{F}(\mathbf{A}^1_K) = \varinjlim_i \mathscr{F}(\mathbf{A}^1_k \times_k X_i).$$

Remark I.8.6. Notice that if $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$ is homotopy invariant, then we have $\mathscr{F}^X(\mathbf{A}^1_K) \cong \mathscr{F}^X(K)$.

Our goal is now to promote $\mathscr{F}^X$ to a presheaf on $\widetilde{\mathrm{Cor}}^X_K$. For $U, V \in \widetilde{\mathrm{Cor}}^X_K$, this means that we need to define a natural restriction map $\alpha^*\colon \mathscr{F}^X(U) \to \mathscr{F}^X(V)$ for any $\alpha \in \widetilde{\mathrm{Cor}}^X_K(V, U)$. To do this we need some preparations. First, recall that we can write $U = \varprojlim_i \mathscr{U}(X_i)$, $V = \varprojlim_i \mathscr{V}(X_i)$, where $\mathscr{U}$, $\mathscr{V}$ and X_i are as in Definition I.8.4.

Lemma I.8.7. *With the notations as above, we have a natural isomorphism*

$$\varinjlim_i \widetilde{\mathrm{Cor}}_{X_i}(\mathscr{V}(X_i), \mathscr{U}(X_i)) \xrightarrow{\cong} \widetilde{\mathrm{Cor}}_K(V, U).$$

Proof. Rewriting U as $\mathscr{U} \times_X \mathrm{Spec}(K)$, we obtain the chain of natural isomorphisms

$$\begin{aligned}
\widetilde{\mathrm{Cor}}_K(V, U) &\cong \widetilde{\mathrm{Cor}}_K(V, \mathscr{U} \times_X \mathrm{Spec}(K)) \\
&\cong \widetilde{\mathrm{Cor}}_X(V, \mathscr{U}) \\
&\cong \varinjlim_i \widetilde{\mathrm{Cor}}_X(\mathscr{V}(X_i), \mathscr{U}) \\
&\cong \varinjlim_i \widetilde{\mathrm{Cor}}_{X_i}(\mathscr{V}(X_i), \mathscr{U}(X_i)).
\end{aligned}$$

Here the penultimate isomorphism follows from a similar argument as that of [CF17, Lemmas 4.6 and 5.10]. ■

For any $\alpha \in \widetilde{\mathrm{Cor}}^X_K(V, U)$, we can now define a natural map

$$\alpha^*\colon \mathscr{F}^X(U) \to \mathscr{F}^X(V)$$

as follows. Using Lemma I.8.7, we may choose a representative

$$\alpha_i \in \widetilde{\mathrm{Cor}}_{X_i}(\mathscr{V}(X_i), \mathscr{U}(X_i))$$

mapping to α. Let

$$f_i\colon \mathscr{V}(X_i) \times_{X_i} \mathscr{U}(X_i) \to \mathscr{V}(X_i) \times_k \mathscr{U}(X_i)$$

denote the canonical morphism. By Lemma I.2.9, f_i induces a homomorphism

$$(f_i)_*\colon \widetilde{\mathrm{Cor}}_{X_i}(\mathscr{V}(X_i), \mathscr{U}(X_i)) \to \widetilde{\mathrm{Cor}}_k(\mathscr{V}(X_i), \mathscr{U}(X_i)).$$

Definition I.8.8. With the notations as above, set

$$\alpha^* := \varinjlim_{j \geq i}((f_j)_*(\alpha_j))^*\colon \mathscr{F}^X(U) \to \mathscr{F}^X(V).$$

For any pair of indices $i \leq j$, the following commutative diagram in $\widetilde{\mathrm{Cor}}_k$,

$$\begin{tikzcd}
\mathscr{V}(X_j) \arrow[r, hook] \arrow[d, "(f_j)_*(\alpha_j)"'] & \mathscr{V}(X_i) \arrow[d, "(f_i)_*(\alpha_i)"] \\
\mathscr{U}(X_j) \arrow[r, hook] & \mathscr{U}(X_i),
\end{tikzcd}$$

shows that the definition of α^* does not depend on the lift α_i.

Lemma I.8.9 (Injectivity on $\mathbf{A}^1_K$)**.** *Let $(\mathbf{A}^1_k, U)$ and $(\mathbf{A}^1_k, V)$ be two objects of $\widetilde{\mathrm{Cor}}^X_K$ such that V is nonempty and $V \subseteq U$. Write $i\colon V \hookrightarrow U$ for the inclusion. Then the induced map*

$$i^*\colon \mathscr{F}^X(U) \to \mathscr{F}^X(V)$$

is injective for any homotopy invariant presheaf $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$.

Proof. Injectivity on the affine line gives a homotopy $\Phi \in \widetilde{\mathrm{Cor}}_K(U, V)$ such that $i \circ \Phi \sim_{\mathbf{A}^1} \mathrm{id}_U$. Since Φ is a morphism in $\widetilde{\mathrm{Cor}}^X_K$ and $\mathscr{F}^X$ is a presheaf on $\widetilde{\mathrm{Cor}}^X_K$, the result follows. ■

Lemma I.8.10 (Zariski excision on $\mathbf{A}^1_K$)**.** *Let $x \in \mathbf{A}^1_K$ be a closed point, and let $(\mathbf{A}^1_k, U) \in \widetilde{\mathrm{Cor}}^X_K$ be such that $x \in U$. Denote by $i\colon U \hookrightarrow \mathbf{A}^1_K$ the inclusion. Then i induces an isomorphism*

$$i^*\colon \frac{\mathscr{F}^X(\mathbf{A}^1_K \setminus x)}{\mathscr{F}^X(\mathbf{A}^1_K)} \xrightarrow{\cong} \frac{\mathscr{F}^X(U \setminus x)}{\mathscr{F}^X(U)}$$

for any homotopy invariant presheaf $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$.

Proof. This follows similarly as in Lemma I.8.9 above. ■

I.9 Injectivity for local schemes

The goal of this section is to prove the following theorem.

Theorem I.9.1. *Let X be a smooth k-scheme and $x \in X$ a closed point. Let $U := \mathrm{Spec}(\mathcal{O}_{X,x})$ and write $\mathrm{can}\colon U \to X$ for the canonical inclusion. Suppose that $i\colon Z \to X$ is a closed subscheme of codimension ≥ 1 in X satisfying $x \in Z$. Let $j\colon X \setminus Z \hookrightarrow X$ denote the open complement. Then there exists a finite* MW*-correspondence $\Phi \in \widetilde{\mathrm{Cor}}_k(U, X \setminus Z)$ such that the diagram*

$$\begin{tikzcd}
 & X \setminus Z \arrow[d, "j"] \\
U \arrow[ur, "\Phi"] \arrow[r, "\mathrm{can}"] & X
\end{tikzcd}$$

commutes in $\mathrm{h}\widetilde{\mathrm{Cor}}_k$.

For homotopy invariant presheaves on $\widetilde{\mathrm{Cor}}_k$ we immediately obtain:

Corollary I.9.2. *Suppose that $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$ is a homotopy invariant presheaf with* MW*-transfers. If $s \in \mathscr{F}(X)$ is a section such that $s|_{X \setminus Z} = 0$, then $s|_U = 0$.*

Let $X^\circ \subseteq X$ be a Zariski open neighborhood of the point x, and let $Z^\circ := Z \cap X^\circ$. As noted in [GP18, §8], it is enough to solve the problem for the triple U, X° and $X^\circ \setminus Z^\circ$. In particular, we may assume that X is irreducible and that the canonical sheaf $\omega_{X/k}$ is trivial. In fact, we will shrink X so that we are in the situation of a relative curve over a quasi-projective scheme. The advantage of this approach is that it turns problems regarding subschemes of high codimension into problems regarding divisors, which is a much more flexible setting. For the shrinking process we refer to the following theorem, which is originally due to M. Artin.

Theorem I.9.3 ([PSV09, Proposition 1]). *Let X, Z and $x \in Z$ be as in Theorem I.9.1. Then there is a Zariski open neighborhood $X^\circ \subseteq X$ of the point x, an open immersion $X^\circ \hookrightarrow \overline{X}^\circ$, a Zariski open subscheme B of $\mathbf{P}^{\dim X - 1}$ and a commutative diagram*

$$\begin{tikzcd} X^\circ \arrow[r, hook] \arrow[dr, "p"'] & \overline{X}^\circ \arrow[d, "\overline{p}"] & X^\circ_\infty \arrow[l, hook'] \arrow[dl, "p_\infty"] \\ & B & \end{tikzcd}$$

satisfying the following properties:

(1) *$\overline{p}$ is a smooth projective morphism, whose fibers are irreducible projective curves.*

(2) *$X^\circ_\infty = \overline{X}^\circ \setminus X^\circ$, and $p_\infty \colon X^\circ_\infty \to B$ is finite étale.*

(3) *The morphism $p|_{Z \cap X^\circ} \colon Z \cap X^\circ \to B$ is finite (where the intersection is taken in $\overline{X}^\circ$).*

The morphism $p \colon X^\circ \to B$ is called an almost elementary fibration.

Following [GP18, §8], we may shrink X such that there exists an almost elementary fibration $p \colon X \to B$ and such that $\omega_{X/k}$ and $\omega_{B/k}$ are trivial, i.e., $\omega_{X/k} \cong \mathcal{O}_X$ and $\omega_{B/k} \cong \mathcal{O}_B$. Let $\mathscr{X} := X \times_B U$ and $\mathscr{Z} := Z \times_B U$. Let also $p_X \colon \mathscr{X} \to X$ and $p_U \colon \mathscr{X} \to U$ be the projections onto X and U, respectively, and let d_X denote the dimension of X. Finally, let Δ denote the morphism $\Delta := (\mathrm{can}, \mathrm{id}) \colon U \to \mathscr{X}$.

Lemma I.9.4 ([GP18, Lemma 8.1]). *There exists a finite surjective morphism*

$$H_\theta = (h_\theta, p_U) \colon \mathscr{X} \to \mathbf{A}^1 \times U$$

over U, such that if we let $\mathscr{D}_1 := H_\theta^{-1}(1 \times U)$ and $\mathscr{D}_0 := H_\theta^{-1}(0 \times U)$ denote the scheme-theoretic preimages, then the following hold:

(1) $\mathscr{D}_1 \subseteq \mathscr{X} \setminus \mathscr{Z}$.

(2) $\mathscr{D}_0 = \Delta(U) \amalg \mathscr{D}'_0$ *with* $\mathscr{D}'_0 \subseteq \mathscr{X} \setminus \mathscr{Z}$.

We will use Lemma I.9.4 to produce the desired MW-correspondence Φ. The aim is to define Φ as the image $(H_\theta \times 1)_*(p_X)$ of the projection $p_X \in \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathscr{X} \times X, \omega_X)$ under the pushforward map

$$(H_\theta \times 1)_* \colon \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathscr{X} \times X, \omega_{H_\theta \times 1} \otimes \omega_X) \to \widetilde{\mathrm{CH}}^{d_X}_{(H_\theta \times 1)(\Gamma_{p_X})}(\mathbf{A}^1 \times U \times X, \omega_X).$$

To this end, we need a trivialization of $\omega_{H_\theta \times 1} = \omega_{\mathscr{X} \times X/k} \otimes (H_\theta \times 1)^* \omega^\vee_{\mathbf{A}^1 \times U \times X/k}$. Now, as U is local we have $\omega_{U/k} \cong \mathcal{O}_U$. Keeping in mind the discussion preceding Lemma I.9.4, it follows that the relative bundle $\omega_{H_\theta \times 1}$ is also trivial. Thus we may choose an isomorphism $\chi \colon \mathcal{O}_X \cong \omega_{H_\theta \times 1}$.

Definition I.9.5. Let $p_X \in \widetilde{\mathrm{Cor}}_k(\mathscr{X}, X)$ denote the projection. Using the trivialization χ above, we let $\mathscr{H}^\chi_\theta \in \widetilde{\mathrm{Cor}}_k(\mathbf{A}^1 \times U, X)$ denote the image of $p_X \in \widetilde{\mathrm{Cor}}_k(\mathscr{X}, X)$ under the composition

$$\widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathscr{X} \times X, \omega_X) \xrightarrow{\cong} \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathscr{X} \times X, \omega_{H_\theta \times 1} \otimes \omega_X)$$
$$\xrightarrow{(H_\theta \times 1)_*} \widetilde{\mathrm{CH}}^{d_X}_{(H_\theta \times 1)(\Gamma_{p_X})}(\mathbf{A}^1 \times U \times X, \omega_X).$$

Lemma I.9.6. *The morphism $H_\theta \times 1$ maps $\Gamma_{p_X} \cong \mathscr{X}$ isomorphically onto its image. Let $\mathscr{H}^\chi_0 := \mathscr{H}^\chi_\theta \circ i_0$ and $\mathscr{H}^\chi_1 := \mathscr{H}^\chi_\theta \circ i_1$. Identifying $\mathscr{X}$ with its image in $\mathbf{A}^1 \times U \times X$, we then have* $\operatorname{supp} \mathscr{H}^\chi_\theta = \mathscr{X}$, $\operatorname{supp} \mathscr{H}^\chi_0 = \mathscr{D}_0$, *and* $\operatorname{supp} \mathscr{H}^\chi_1 = \mathscr{D}_1$.

Proof. If $y = ((x,u),x)$, $y' = ((x',u'),x') \in \Gamma_{p_X}$ is such that

$$(H_\theta \times 1)(y) = (h_\theta(x,u),u,x) = (H_\theta \times 1)(y') = (h_\theta(x',u'),u',x'),$$

it follows that $x = x'$ and $u = u'$, hence $y = y'$. Thus we can consider $\mathscr{X}$ as a subscheme of $\mathbf{A}^1 \times U \times X$ by $(x,u) \mapsto (h_\theta(x,u),u,x)$. Now, the MW-correspondence p_X is supported on Γ_{p_X}, hence $\operatorname{supp} \mathscr{H}^\chi_\theta = (H_\theta \times 1)(\Gamma_{p_X}) \cong \mathscr{X}$. We turn to the restrictions $\mathscr{H}^\chi_0$ and $\mathscr{H}^\chi_1$ of the homotopy $\mathscr{H}^\chi_\theta$. By [CF17, Example 4.17] we have $\mathscr{H}^\chi_\theta \circ i_\epsilon = (i_\epsilon \times 1)^*(\mathscr{H}^\chi_\theta)$, where $\epsilon = 0,1$. It follows that $\operatorname{supp} \mathscr{H}^\chi_\epsilon = (i_\epsilon \times 1)^{-1}((H_\theta \times 1)(\Gamma_{p_X}))$, and this closed subset is determined by those points $(x,u) \in \mathscr{X}$ satisfying $h_\theta(x,u) = \epsilon$. In other words, $\operatorname{supp} \mathscr{H}^\chi_\epsilon = \mathscr{D}_\epsilon$. ■

Lemma I.9.7. *There is an invertible regular function λ on U such that*

$$\mathscr{H}^\chi_0 = \mathrm{can} \circ \langle \lambda \rangle + j \circ \Phi'_0$$

and

$$\mathscr{H}^\chi_1 = j \circ \Phi_1,$$

where $\Phi'_0, \Phi_1 \in \widetilde{\mathrm{Cor}}_k(U, X \setminus Z)$ and $\langle \lambda \rangle \in \mathbf{K}^{\mathrm{MW}}_0(U)$.

Proof. By Lemmas I.9.4 and I.9.6 we have $\operatorname{supp}\mathscr{H}_0^X = \Delta(U) \amalg \mathscr{D}_0'$, where $\mathscr{D}_0' \subseteq \mathscr{X} \setminus \mathscr{Z}$. By Lemma I.2.4 we may therefore write $\mathscr{H}_0^X = \alpha + \beta$ where $\alpha \in \widetilde{\mathrm{Cor}}_k(U, X)$ is supported on $\Delta(U)$ and $\beta \in \widetilde{\mathrm{Cor}}_k(U, X)$ is supported on $\mathscr{D}_0'$. Since $\operatorname{supp}\beta = \mathscr{D}_0' \subseteq \mathscr{X} \setminus \mathscr{Z}$, Corollary I.2.6 ensures that there exists a unique finite MW-correspondence $\Phi_0' \in \widetilde{\mathrm{Cor}}_k(U, X \setminus Z)$ such that $j \circ \Phi_0' = \beta$. Hence $\mathscr{H}_0^X$ is of the form $\mathscr{H}_0^X = \alpha + j \circ \Phi_0'$ for $\Phi_0' \in \widetilde{\mathrm{Cor}}_k(U, X \setminus Z)$. The same reasoning shows that, since $\operatorname{supp}\mathscr{H}_1^X = \mathscr{D}_1 \subseteq \mathscr{X} \setminus \mathscr{Z}$, there is a unique MW-correspondence $\Phi_1 \in \widetilde{\mathrm{Cor}}_k(U, X \setminus Z)$ such that $\mathscr{H}_1^X = j \circ \Phi_1$.

It therefore only remains to understand the finite MW-correspondence $\alpha \in \widetilde{\mathrm{CH}}^{d_X}_{\Delta(U)}(U \times X, \omega_X)$. Recall that, by definition,

$$\mathscr{H}_0^X = (i_0 \times 1)^*(H_\theta \times 1)_*(\Gamma_{p_X})_*(\langle 1\rangle).$$

Let $i_{\Delta(U)}$ and $i_{\mathscr{D}_0}$ denote the respective inclusions $i_{\Delta(U)}\colon \Delta(U) \subseteq \mathscr{X}$ and $i_{\mathscr{D}_0}\colon \mathscr{D}_0 \subseteq \mathscr{X}$. The base change formula [CF17, Proposition 3.2] applied to the pullback square

$$\begin{tikzcd}
(\Delta(U) \amalg \mathscr{D}_0') \times X \arrow[r, "i_{\mathscr{D}_0} \times 1"] \arrow[d, "H_\theta|_{\mathscr{D}_0} \times 1"'] & \mathscr{X} \times X \arrow[d, "H_\theta \times 1"] \\
U \times X \arrow[r, "i_0 \times 1"] & \mathbf{A}^1 \times U \times X
\end{tikzcd}$$

reveals that $\alpha = (H_\theta|_{\Delta(U)} \times 1)_*(i_{\Delta(U)} \times 1)^*(\Gamma_{p_X})_*(\langle 1\rangle)$. Using that $\Delta\colon U \to \mathscr{X}$ is an isomorphism onto its image and that $H_\theta|_{\Delta(U)}\colon \Delta(U) \to U$ is an isomorphism, we may write $\alpha = (\Delta \times 1)^*(\Gamma_{p_X})_*(\langle 1\rangle)$. Next, consider the pullback diagram

$$\begin{tikzcd}
U \arrow[r, "\Delta"] \arrow[d, "\Gamma_{\mathrm{can}}"'] & \mathscr{X} \arrow[d, "\Gamma_{p_X}"] \\
U \times X \arrow[r, "\Delta \times 1"] & \mathscr{X} \times X.
\end{tikzcd}$$

Using base change once more, we obtain $\alpha = (\Gamma_{\mathrm{can}})_*\Delta^*(\langle 1\rangle)$. Comparing this expression with the definition $\widetilde{\gamma}_{\mathrm{can}} := (\Gamma_{\mathrm{can}})_*(\langle 1\rangle)$ of $\widetilde{\gamma}_{\mathrm{can}}$, we see that two possibly different trivializations of the line bundle ω_U are involved. Letting $\lambda \in k[U]^\times$ be the fraction of these two trivializations, it follows that $\alpha = \widetilde{\gamma}_{\mathrm{can}} \circ \langle\lambda\rangle$. ■

Proof of Theorem I.9.1. In the notation of Lemma I.9.7, define $\mathscr{H}_\theta := \mathscr{H}_\theta^X \circ \langle\lambda^{-1}\rangle$ and $\Phi := (\Phi_1 - \Phi_0') \circ \langle\lambda^{-1}\rangle$. By Lemma I.9.7, $\mathscr{H}_\theta$ provides a homotopy $\mathrm{can} \sim_{\mathbf{A}^1} j \circ \Phi$. ■

I.10 Nisnevich excision

The setting of this section is as follows. Suppose that $X, X' \in \mathrm{Sm}_k$ are smooth affine k-schemes such that there is an elementary distinguished Nisnevich square

$$\begin{tikzcd} V' \arrow[r] \arrow[d] & X' \arrow[d, "\Pi"] \\ V \arrow[r] & X. \end{tikzcd} \tag{I.2}$$

Define the closed subschemes $S := (X \setminus V)_{\mathrm{red}} \subseteq X$ and $S' := (X' \setminus V')_{\mathrm{red}} \subseteq X'$. Let $x \in S$ and $x' \in S'$ be two points satisfying $\Pi(x') = x$. Moreover, we set $U := \mathrm{Spec}(\mathcal{O}_{X,x})$ and $U' := \mathrm{Spec}(\mathcal{O}_{X',x'})$. Let $\mathrm{can}\colon U \to X$ and $\mathrm{can}'\colon U' \to X'$ be the canonical inclusions and let $\pi := \Pi|_{U'}\colon U' \to U$. We can summarize the situation with the following diagram:

$$\begin{tikzcd} V' \arrow[r] \arrow[d] & X' \arrow[d, "\Pi"] & U' \arrow[l, "\mathrm{can}'"'] \arrow[d, "\pi"] \\ V \arrow[r] & X & U. \arrow[l, "\mathrm{can}"] \end{tikzcd} \tag{I.3}$$

The main result of this section is the following excision theorem for Nisnevich squares.

Theorem I.10.1 (Nisnevich excision)**.** *Let $\mathscr{F}$ be a homotopy invariant presheaf on $\widetilde{\mathrm{Cor}}_k$. Given any elementary distinguished Nisnevich square as* (I.2)*, the induced morphism*

$$\pi^*\colon \frac{\mathscr{F}(U \setminus S)}{\mathscr{F}(U)} \to \frac{\mathscr{F}(U' \setminus S')}{\mathscr{F}(U')}$$

is an isomorphism.

The proof of Theorem I.10.1 relies on the two following results, establishing respectively injectivity and surjectivity of π^*:

Theorem I.10.2 (Injectivity of Nisnevich excision)**.** *With the notations in* (I.3)*, there exist finite* MW*-correspondences*

$$\Phi \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus S), (X', X' \setminus S'))$$

and

$$\Theta \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus S), (X \setminus S, X \setminus S))$$

such that

$$\overline{\Pi} \circ \overline{\Phi} - \overline{j}_X \circ \overline{\Theta} = \overline{\mathrm{can}}$$

in $\mathrm{h}\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus S), (X, X \setminus S))$*. Here $j_X\colon (X \setminus S, X \setminus S) \hookrightarrow (X, X \setminus S)$ is the inclusion.*

Theorem I.10.3 (Surjectivity of Nisnevich excision)**.** *Keep the notations as in* (I.3)*. Then there exist finite* MW*-correspondences*

$$\Psi \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus S), (X', X' \setminus S'))$$

and

$$\Xi \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U', U' \setminus S'), (X' \setminus S', X' \setminus S'))$$

such that

$$\overline{\Psi} \circ \overline{\pi} - \overline{j}_{X'} \circ \overline{\Xi} = \overline{\mathrm{can}'}$$

in $\mathrm{h}\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U', U' \setminus S'), (X', X' \setminus S'))$*. Here* $j_{X'} \colon (X' \setminus S', X' \setminus S') \hookrightarrow (X', X' \setminus S')$ *is the inclusion.*

Assuming Theorems I.10.2 and I.10.3, Theorem I.10.1 now follows:

Proof of Theorem I.10.1. Let $\mathscr{F}$ be a homotopy invariant presheaf with MW-transfers. First, note that Theorem I.9.1 implies that the restriction maps $\mathscr{F}(U) \to \mathscr{F}(U \setminus S)$ and $\mathscr{F}(U') \to \mathscr{F}(U' \setminus S')$ are injective. Indeed, suppose that $s_x \in \mathscr{F}(U)$ maps to 0 in $\mathscr{F}(U \setminus S)$. We may assume that s_x is represented by a section $s \in \mathscr{F}(W)$ for some Zariski open neighborhood W of x, such that $s|_{W \setminus S} = 0$. But then $s_x = 0$ by Corollary I.9.2. Hence $\mathscr{F}(U) \to \mathscr{F}(U \setminus S)$ is injective. It follows similarly that $\mathscr{F}(U') \to \mathscr{F}(U' \setminus S')$ is injective.

Now, as the MW-correspondence Θ of Theorem I.10.2 maps to $(X \setminus S, X \setminus S)$, $j_X \circ \Theta$ induces the trivial map

$$(j_X \circ \Theta)^* = 0 \colon \frac{\mathscr{F}(X \setminus S)}{\mathscr{F}(X)} \to \frac{\mathscr{F}(U \setminus S)}{\mathscr{F}(U)}.$$

Hence $\Phi^* \circ \Pi^* = \mathrm{can}^*$. Similarly, $\Xi^* = 0$ and hence $\pi^* \circ \Psi^* = (\mathrm{can}')^*$. We use this to show that π^* is an isomorphism.

To show that π^* is injective, let us assume that $s_x \in \mathscr{F}(U \setminus S)/\mathscr{F}(U)$ is a germ such that $\pi^*(s_x) = 0$. As

$$\frac{\mathscr{F}(U \setminus S)}{\mathscr{F}(U)} = \varinjlim_{W \ni x} \frac{\mathscr{F}(W \setminus S)}{\mathscr{F}(W)},$$

we may assume that s_x is represented by a section $s \in \mathscr{F}(W \setminus S)/\mathscr{F}(W)$ for some affine k-smooth Zariski open neighborhood W of x. Thus s is a section satisfying $\mathrm{can}^*(s) = s_x$ and $\pi^*(s_x) = 0$. Now, since $\pi^*(s_x) = 0$ in $\mathscr{F}(U' \setminus S')/\mathscr{F}(U')$, there is some affine k-smooth Zariski open neighborhood W' of x' in $X' \times_X W$ such that $\Pi^*(s)|_{W'} = 0$. Replacing X by W and X' by W', we may then apply Theorem I.10.2 to obtain a finite MW-correspondence $\Phi \in \widetilde{\mathrm{Cor}}_k(U, X')$ such that $\Phi^* \circ \Pi^* = \mathrm{can}^*$. But then $s_x = \mathrm{can}^*(s) = \Phi^*(\Pi^*(s)) = 0$. Hence π^* is injective.

To show surjectivity, let $s'_{x'} \in \mathscr{F}(U' \setminus S')/\mathscr{F}(U')$. Similarly as above, we may assume that $s'_{x'}$ is represented by a section $s' \in \mathscr{F}(X' \setminus S')/\mathscr{F}(X')$, i.e., $(\mathrm{can}')^*(s') = s'_{x'}$. By Theorem I.10.3, there is a finite MW-correspondence $\Psi \in \widetilde{\mathrm{Cor}}_k(U, X')$ such that $\pi^* \circ \Psi^* = (\mathrm{can}')^*$. We then have

$$s'_{x'} = (\mathrm{can}')^*(s') = \pi^*(\Psi^*(s')),$$

and thus π^* is surjective. ■

We proceed to prove Theorems I.10.2 and I.10.3.

I.11 Injectivity of Nisnevich excision

In this section we aim to prove Theorem I.10.2. As preparation, we need to perform a shrinking process similar to that in Section I.9. By [GP18, Lemma 9.4], there is a Zariski open subscheme $X^\circ \subseteq X$ along with an almost elementary fibration $q\colon X^\circ \to B$ such that $\omega_{B/k} \cong \mathcal{O}_B$ and $\omega_{X^\circ/k} \cong \mathcal{O}_{X^\circ}$. By [GP18, §9] we may replace X by X° and X' by $\Pi^{-1}(X^\circ)$. We regard X' as a B-scheme via the map $q \circ \Pi$.

Let Δ denote the morphism $\Delta := (\mathrm{id}, \mathrm{can})\colon U \to U \times_B X$, and let p_X and $p_{\mathbf{A}^1\times U}$ denote the projections from $\mathbf{A}^1 \times U \times_B X$ onto X respectively $\mathbf{A}^1 \times U$.

Proposition I.11.1 ([GP18, Proposition 9.9])**.** *Let θ be the coordinate of $\mathbf{A}^1$. There exists a function $h_\theta \in k[\mathbf{A}^1 \times U \times_B X]$ such that the following properties hold for the functions h_θ, $h_0 := h_\theta|_{0\times U\times_B X}$ and $h_1 := h_\theta|_{1\times U\times_B X}$:*

(*a*) *The morphism $H_\theta := (p_{\mathbf{A}^1\times U}, h_\theta)\colon \mathbf{A}^1 \times U \times_B X \to \mathbf{A}^1 \times U \times \mathbf{A}^1$ is finite and surjective. Letting $Z_\theta := h_\theta^{-1}(0) \subseteq \mathbf{A}^1 \times U \times_B X$, it follows that Z_θ is finite, surjective and flat over $\mathbf{A}^1 \times U$.*

(*b*) *Let $Z_0 := h_0^{-1}(0) \subseteq U \times_B X$. Then there is the equality of schemes $Z_0 = \Delta(U) \amalg G$, where $G \subseteq U \times_B (X \setminus S)$.*

(*c*) *The closed subscheme $\mathcal{V}((\mathrm{id}_U \times \Pi)^*(h_1)) \subseteq U \times_B X'$ is a disjoint union of two closed subschemes $Z_1' \amalg Z_2'$. Moreover, the map $(\mathrm{id}_U \times \Pi)|_{Z_1'}$ identifies Z_1' with $Z_1 := h_1^{-1}(0)$.*

$$\begin{tikzcd} U \times_B X' \arrow[r, "1\times\Pi"] & U \times_B X \arrow[r, "h_1"] & \mathbf{A}^1 \\ Z_1' \arrow[u, hook] \arrow[r, "\cong"] & Z_1 = \mathcal{V}(h_1) \arrow[u, hook] & \end{tikzcd}$$

(*d*) *We have $Z_\theta \cap (\mathbf{A}^1 \times (U \setminus x) \times_B X) \subseteq \mathbf{A}^1 \times (U \setminus x) \times_B (X \setminus x)$.*

Corollary I.11.2 ([GP18, Remark 9.10])**.** *We have the following inclusions:*

(1) $Z_\theta \cap (\mathbf{A}^1 \times (U \setminus S) \times_B X) \subseteq \mathbf{A}^1 \times (U \setminus S) \times_B X \setminus S$.

(2) $Z_0 \cap ((U \setminus S) \times_B X) \subseteq (U \setminus S) \times_B (X \setminus S)$.

(3) $Z_1 \cap ((U \setminus S) \times_B X) \subseteq (U \setminus S) \times_B (X \setminus S)$.

(4) $Z_1' \cap ((U \setminus S) \times_B X') \subseteq (U \setminus S) \times_B (X' \setminus S')$.

Definition I.11.3. Choose a trivialization χ of $\omega_{H_\theta\times 1}$. We define the homotopy

$$\mathscr{H}_\theta^\chi \in \widetilde{\mathrm{Cor}}_k(\mathbf{A}^1\times U, X)$$

as the image of the projection $p_X \in \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathbf{A}^1\times U\times_B X\times X, \omega_X)$ under the composition

$$\begin{aligned}
&\widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathbf{A}^1\times U\times_B X\times X, \omega_X)\\
&\xrightarrow{\cong} \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathbf{A}^1\times U\times_B X\times X, \omega_{H_\theta\times 1}\otimes\omega_X)\\
&\xrightarrow{(H_\theta\times 1)_*} \widetilde{\mathrm{CH}}^{d_X}_{(H_\theta\times 1)(\Gamma_{p_X})}(\mathbf{A}^1\times U\times \mathbf{A}^1\times X, \omega_X)\\
&\xrightarrow{(1\times i_0\times 1)^*} \widetilde{\mathrm{CH}}^{d_X}_{T}(\mathbf{A}^1\times U\times X, \omega_X),
\end{aligned}$$

where $d_X := \dim X$, $T := (1\times i_0\times 1)^{-1}((H_\theta\times 1)(\Gamma_{p_X}))$, and where the first isomorphism is induced by χ.

Lemma I.11.4. *The finite* MW*-correspondence $\mathscr{H}_\theta^\chi$ is supported on Z_θ. Moreover, for $\epsilon = 0, 1$ we have* $\operatorname{supp}\mathscr{H}_\epsilon^\chi = Z_\epsilon$ *(where $\mathscr{H}_\epsilon^\chi := \mathscr{H}_\theta^\chi\circ i_\epsilon$).*

Proof. Let T denote the support of $\mathscr{H}_\theta^\chi$. As indicated in Definition I.11.3 we have $T = (1\times i_0\times 1)^{-1}((H_\theta\times 1)(\Gamma_{p_X}))$. By the same argument as in Lemma I.9.6, $H_\theta\times 1$ injects Γ_{p_X} onto its image, hence $(H_\theta\times 1)(\Gamma_{p_X})\cong \mathbf{A}^1\times U\times_B X$. Thus T consists of those points $(t,u,x)\in\mathbf{A}^1\times U\times_B X$ such that $h_\theta(t,u,x) = 0$, i.e., $T = Z_\theta$.

Turning to the support of $\mathscr{H}_\epsilon^\chi$, note that $\mathscr{H}_\epsilon^\chi$ is the image of p_X under the map

$$\widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathbf{A}^1\times U\times_B X\times X, \omega_X)\to \widetilde{\mathrm{CH}}^{d_X}_{\operatorname{supp}\mathscr{H}_\epsilon^\chi}(\epsilon\times U\times X, \omega_X)$$

given as the composition $(i_\epsilon\times 1)^*\circ(1\times i_0\times 1)^*\circ(H_\theta\times 1)_*$. By the same reasoning as above, pulling back along $i_\epsilon\times 1$ amounts to substituting $\theta=\epsilon$ in h_θ, which yields the desired result. ■

Lemma I.11.5. *There are finite* MW*-correspondences $\Theta\in\widetilde{\mathrm{Cor}}_k(U, X\setminus S)$ and $\Phi\in\widetilde{\mathrm{Cor}}_k(U, X')$ along with an invertible regular function λ on U such that $\mathscr{H}_0^\chi = \mathrm{can}\circ\langle\lambda\rangle + j_X\circ\Theta$ and $\mathscr{H}_1^\chi = \Pi\circ\Phi$.*

Proof. By Proposition I.11.1 (b), we can write $\mathscr{H}_0^\chi = \alpha + \Theta'$, where $\Theta'\in\widetilde{\mathrm{Cor}}_k(U,X)$ is supported on G and $\alpha\in\widetilde{\mathrm{Cor}}_k(U,X)$ is supported on $\Delta(U)$. Using Proposition I.11.1 (b), Lemma I.2.5 ensures that there is a unique finite MW-correspondence $\Theta\in\widetilde{\mathrm{Cor}}_k(U, X\setminus S)$ such that $\Theta' = j_X\circ\Theta$. We proceed similarly for $\mathscr{H}_1^\chi$: by Proposition I.11.1 (c), the pullback

$$(1\times\Pi)^*(\mathscr{H}_1^\chi)\in\widetilde{\mathrm{CH}}^{d_X}_{(1\times\Pi)^{-1}(Z_1)}(U\times X', \omega_{X'})$$

is supported on $Z_1' \amalg Z_2'$, and $(1 \times \Pi)|_{Z_1'}$ is an isomorphism from Z_1' onto Z_1. It follows that we have an isomorphism

$$(1 \times \Pi)_* \colon \widetilde{\mathrm{CH}}^{d_X}_{Z_1'}(U \times X', \omega_{X'}) \xrightarrow{\cong} \widetilde{\mathrm{CH}}^{d_X}_{Z_1}(U \times X, \omega_X).$$

Hence $\Phi := (1\times\Pi)_*^{-1}(\mathscr{H}_1^{\chi}) = (1\times\Pi)^*(\mathscr{H}_1^{\chi}) \in \widetilde{\mathrm{Cor}}_k(U, X')$ satisfies $\Pi\circ\Phi = \mathscr{H}_1^{\chi}$.

It remains to show that $\alpha = \mathrm{can} \circ \langle\lambda\rangle$, the proof of which being similar as in the proof of Lemma I.9.7. As

$$(1 \times i_0 \times 1) \circ (i_0 \times 1) = (i_0 \times 1 \times i_0 \times 1) \colon U \times X \to \mathbf{A}^1 \times U \times \mathbf{A}^1 \times X,$$

we can write $\mathscr{H}_0^{\chi} = (i_0 \times 1 \times i_0 \times 1)^*(H_\theta \times 1)_*(\Gamma_{p_X})_*(\langle 1\rangle)$. Using the base change formula twice as in Lemma I.9.7, we find that

$$\alpha = (H_\theta|_{\Delta(U)} \times 1)_*(i_{\Delta(U)} \times 1)^*(\Gamma_{p_X})_*(\langle 1\rangle) = (\Gamma_{\mathrm{can}})_*(\langle 1\rangle) \circ \langle\lambda\rangle = \widetilde{\gamma}_{\mathrm{can}} \circ \langle\lambda\rangle,$$

where $\lambda \in k[U]^\times$ is the fraction of two trivializations of ω_U, and $i_{\Delta(U)} \colon \Delta(U) \hookrightarrow U \times_B X$ is the inclusion. ■

Lemma I.11.6. *Let $j_U \colon U \setminus S \hookrightarrow U$, $j_X \colon X \setminus S \hookrightarrow X$ and $j_{X'} \colon X' \setminus S' \hookrightarrow X'$ denote the inclusions, and set:*

$$\begin{aligned}
\mathscr{H}_\theta^{\chi,S} &:= (1 \times j_U \times j_X)^*(\mathscr{H}_\theta^{\chi}) \in \widetilde{\mathrm{Cor}}_k(\mathbf{A}^1 \times (U \setminus S), X \setminus S).\\
\Phi^S &:= (j_U \times j_{X'})^*(\Phi) \in \widetilde{\mathrm{Cor}}_k(U \setminus S, X' \setminus S').\\
\Theta^S &:= (j_U \times 1)^*(\Theta) \in \widetilde{\mathrm{Cor}}_k(U \setminus S, X \setminus S).
\end{aligned}$$

Then we have:

$$\begin{aligned}
(\mathscr{H}_\theta^{\chi}, \mathscr{H}_\theta^{\chi,S}) &\in \widetilde{\mathrm{Cor}}^{\mathrm{pr}}_k(\mathbf{A}^1 \times (U, U \setminus S), (X, X \setminus S)).\\
(\Phi, \Phi^S) &\in \widetilde{\mathrm{Cor}}^{\mathrm{pr}}_k((U, U \setminus S), (X', X' \setminus S')).\\
(\Theta, \Theta^S) &\in \widetilde{\mathrm{Cor}}^{\mathrm{pr}}_k((U, U \setminus S), (X \setminus S, X \setminus S)).
\end{aligned}$$

Proof. In light of Corollary I.11.2, this follows from Lemma I.2.7. ■

Proof of Theorem I.10.2. Replacing $\mathscr{H}_\theta^{\chi}$, Θ and Φ by the respective precompositions with $\langle\lambda^{-1}\rangle$, it follows from Lemmas I.11.5 and I.11.6 that we have the identity

$$\overline{\Pi} \circ \overline{\Phi} - \overline{j}_X \circ \overline{\Theta} = \overline{\mathrm{can}}$$

in $\mathrm{h}\widetilde{\mathrm{Cor}}^{\mathrm{pr}}_k$. ■

I.12 Surjectivity of Nisnevich excision

We proceed to prove Theorem I.10.3. Performing a similar shrinking process as in Section I.11, we may assume that there is an almost elementary fibration

$q\colon X \to B$ such that $\omega_{B/k} \cong \mathcal{O}_B$ and $\omega_{X/k} \cong \mathcal{O}_X$. Since Π is étale, it follows that $\omega_{X'/k} \cong \mathcal{O}_{X'}$.

Let $\Delta' := (\mathrm{id}, \mathrm{can}')\colon U' \to U' \times_B X'$, and let $p_{X'}$ and $p_{\mathbf{A}^1 \times U'}$ denote the projections from $\mathbf{A}^1 \times U' \times_B X'$ to X' respectively $\mathbf{A}^1 \times U'$. First we recall the following fact from [GP18]:

Proposition I.12.1 ([GP18, Proposition 11.6]). *Let $\mathbf{A}^1$ have coordinate θ. There exist functions $F \in k[U \times X']$ and $h'_\theta \in k[\mathbf{A}^1 \times U' \times_B X']$ such that the following properties hold for the functions F, h'_θ, $h'_0 := h'_\theta|_{0 \times U' \times_B X'}$, and $h'_1 := h'_\theta|_{1 \times U' \times_B X'}$:*

(a) *The morphism $H'_\theta := (p_{\mathbf{A}^1 \times U'}, h'_\theta)\colon \mathbf{A}^1 \times U' \times_B X' \to \mathbf{A}^1 \times U' \times \mathbf{A}^1$ is finite and surjective. Letting $Z'_\theta := (h'_\theta)^{-1}(0) \subseteq \mathbf{A}^1 \times U' \times_B X'$, it follows that Z'_θ is finite, surjective and flat over $\mathbf{A}^1 \times U'$.*

(b) *Let $Z'_0 := (h'_0)^{-1}(0)$. Then there is the equality of schemes*

$$Z'_0 = \Delta'(U') \amalg G',$$

where $G' \subseteq U' \times_B (X' \setminus S')$.

(c) *$h'_1 = (\pi \times \mathrm{id}_{X'})^*(F)$. We write $Z'_1 := (h'_1)^{-1}(0)$.*

(d) *$Z'_\theta \cap (\mathbf{A}^1 \times (U' \setminus S') \times_B X') \subseteq \mathbf{A}^1 \times (U' \setminus S') \times_B (X' \setminus S')$.*

(e) *The morphism $(\mathrm{pr}_U, F)\colon U \times X' \to U \times \mathbf{A}^1$ is finite and surjective. Letting $Z_1 := F^{-1}(0)$, it follows that Z_1 is finite and surjective over U.*

(f) *$Z_1 \cap ((U \setminus S) \times X') \subseteq (U \setminus S) \times (X' \setminus S')$.*

Corollary I.12.2 ([GP18, Remark 11.7]). *We have the following inclusions:*

(1) $Z'_\theta \cap (\mathbf{A}^1 \times (U' \setminus S') \times_B X') \subseteq \mathbf{A}^1 \times (U' \setminus S') \times_B (X' \setminus S')$.

(2) $Z'_0 \cap ((U' \setminus S') \times_B X') \subseteq (U' \setminus S') \times_B (X' \setminus S')$.

(3) $Z'_1 \cap ((U' \setminus S') \times_B X') \subseteq (U' \setminus S') \times_B (X' \setminus S')$.

(4) $Z_1 \cap ((U \setminus S) \times X') \subseteq (U \setminus S) \times (X' \setminus S')$.

Definition I.12.3. Choose a trivialization χ of $\omega_{H'_\theta \times 1}$. We define the homotopy

$$\mathscr{H}^\chi_\theta \in \widetilde{\mathrm{Cor}}_k(\mathbf{A}^1 \times U', X')$$

as the image of the projection $p_{X'} \in \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_{X'}}}(\mathbf{A}^1 \times U' \times_B X' \times X', \omega_{X'})$ under the composition

$$\begin{aligned}
&\widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_{X'}}}(\mathbf{A}^1 \times U' \times_B X' \times X', \omega_{X'}) \\
&\xrightarrow{\cong} \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_{X'}}}(\mathbf{A}^1 \times U' \times_B X' \times X', \omega_{H'_\theta \times 1} \otimes \omega_{X'}) \\
&\xrightarrow{(H'_\theta \times 1)_*} \widetilde{\mathrm{CH}}^{d_X}_{(H'_\theta \times 1)(\Gamma_{p_{X'}})}(\mathbf{A}^1 \times U' \times \mathbf{A}^1 \times X', \omega_{X'}) \\
&\xrightarrow{(1 \times i_0 \times 1)^*} \widetilde{\mathrm{CH}}^{d_X}_{T'}(\mathbf{A}^1 \times U' \times X', \omega_{X'}),
\end{aligned}$$

where $T' := (1 \times i_0 \times 1)^{-1}((H'_\theta \times 1)(\Gamma_{p_{X'}}))$, and where the first isomorphism is induced by χ'.

The same argument as in Lemma I.11.4 readily yields:

Lemma I.12.4. *The finite* MW*-correspondence* $\mathscr{H}_\theta^\chi$ *is supported on* Z'_θ*. Moreover, for* $\epsilon = 0, 1$ *we have* $\operatorname{supp} \mathscr{H}_\epsilon^\chi = Z'_\epsilon$ *(where, as usual,* $\mathscr{H}_\epsilon^\chi := \mathscr{H}_\theta^\chi \circ i_\epsilon$*).*

Lemma I.12.5. *There are finite* MW*-correspondences* $\Psi' \in \widetilde{\mathrm{Cor}}_k(U, X')$ *and* $\Xi \in \widetilde{\mathrm{Cor}}_k(U', X' \setminus S')$ *along with an invertible regular function* λ' *on* U' *such that* $\mathscr{H}_0^\chi = \mathrm{can}' \circ \langle \lambda' \rangle + j_{X'} \circ \Xi$ *and* $\mathscr{H}_1^\chi = \Psi' \circ \pi$*.*

Proof. The claim about $\mathscr{H}_0^\chi$ follows from an identical argument as in the proof of Lemma I.11.5 by using Proposition I.12.1 (b), so let us turn our attention to $\mathscr{H}_1^\chi$. By Proposition I.12.1 (c), the morphism $\pi \times 1$ identifies Z'_1 with Z_1. By étale excision [CF17, Lemma 3.7], $\pi \times 1$ induces an isomorphism

$$(\pi \times 1)^* \colon \widetilde{\mathrm{CH}}^{d_X}_{Z_1}(U \times X', \omega_{X'}) \xrightarrow{\cong} \widetilde{\mathrm{CH}}^{d_X}_{Z'_1}(U' \times X', \omega_{X'}).$$

Arguing similarly to the proof of Lemma I.11.5, it follows that there exists a unique element $\Psi' \in \widetilde{\mathrm{CH}}^{d_X}_{Z'_1}(U' \times X', \omega_{X'}) \subseteq \widetilde{\mathrm{Cor}}_k(U', X')$ such that $\mathscr{H}_1^\chi = \Psi' \circ \pi$. ∎

Let us check also that the finite MW-correspondences constructed above are in fact morphisms of pairs:

Lemma I.12.6. *Let* $j_{U'} \colon U' \setminus S' \hookrightarrow U'$ *denote the inclusion, and define:*

$$\begin{aligned}
\mathscr{H}_\theta^{\chi, S'} &:= (1 \times j_{U'} \times j_{X'})^*(\mathscr{H}_\theta^\chi) \in \widetilde{\mathrm{Cor}}_k(\mathbf{A}^1 \times (U' \setminus S'), X' \setminus S'). \\
\Psi^{S'} &:= (j_U \times j_{X'})^*(\Psi) \in \widetilde{\mathrm{Cor}}_k(U \setminus S, X' \setminus S'). \\
\Xi^{S'} &:= (j_{U'} \times 1)^*(\Xi) \in \widetilde{\mathrm{Cor}}_k(U' \setminus S', X' \setminus S').
\end{aligned}$$

Then

$$\begin{aligned}
(\mathscr{H}_\theta^\chi, \mathscr{H}_\theta^{\chi, S'}) &\in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}(\mathbf{A}^1 \times (U', U' \setminus S'), (X', X' \setminus S')). \\
(\Psi, \Psi^{S'}) &\in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus S), (X', X' \setminus S')). \\
(\Xi, \Xi^{S'}) &\in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U', U' \setminus S'), (X' \setminus S', X' \setminus S')).
\end{aligned}$$

Proof. By Corollary I.12.2, the supports of the given MW-correspondences satisfy the hypothesis of Lemma I.2.7. ∎

We are almost in position to prove Theorem I.10.3. However, as opposed to the situation in Section I.11 we cannot immediately precompose the homotopy $\mathscr{H}_\theta^\chi$ of Definition I.12.3 with $\langle (\lambda')^{-1} \rangle$ and obtain a homotopy of the desired form. In order to remedy this, we need the following lemma (see also [Dru18, Proof of Proposition 6.7]):

Lemma I.12.7. *Let X, S, U and* can *be as in Theorem I.10.1, and suppose that $\lambda \in k[U]^\times$ is an invertible regular function satisfying $\lambda|_{U\cap S} = 1$. Then*

$$\mathrm{can} \circ \langle \lambda \rangle \sim_{\mathbf{A}^1} \mathrm{can} \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus S), (X, X \setminus S)).$$

Proof. The germ λ is represented by an invertible section μ on some smooth affine Zariski open neighborhood W of x in X. Moreover, by assumption there is some affine Zariski open neighborhood $W' \subseteq W$ of x such that $\mu|_{S\cap W'} = 1$. Replacing X by W', we may assume that μ is an invertible regular function on the smooth affine k-scheme X satisfying $\mu|_S = 1$.

Define an étale covering $\Pi\colon X' \to X$ by letting

$$X' := \mathrm{Spec}(k[X][t]/(t^2 - \mu)).$$

Consider the closed subscheme $S' := \mathrm{Spec}(k[S][t]/(t-1))$ of X'. As Π induces an isomorphism $S' \xrightarrow{\cong} S$, it follows that we have an elementary distinguished Nisnevich square

$$\begin{tikzcd} X' \setminus S' \arrow[r] \arrow[d] & X' \arrow[d, "\Pi"] \\ X \setminus S \arrow[r] & X. \end{tikzcd}$$

Thus Theorem I.10.2 provides us with a finite Milnor–Witt correspondence

$$\Phi \in \widetilde{\mathrm{Cor}}_k(U, X')$$

such that $\Pi \circ \Phi \sim_{\mathbf{A}^1} \mathrm{can}$ as correspondences of pairs. But then

$$\begin{aligned} \mathrm{can} \circ \langle \lambda \rangle &= \langle \mu \rangle \circ \mathrm{can} \\ &\sim_{\mathbf{A}^1} \langle \mu \rangle \circ \Pi \circ \Phi \\ &= \Pi \circ \langle \Pi^*(\mu) \rangle \circ \Phi \\ &= \Pi \circ \langle t^2 \rangle \circ \Phi \\ &= \Pi \circ \Phi \sim_{\mathbf{A}^1} \mathrm{can}, \end{aligned}$$

where we have used the left and right actions of $\mathbf{K}_0^{\mathrm{MW}}$ on finite Milnor–Witt correspondences [CF17, Example 4.14], along with the fact that $\langle a^2 \rangle = 1$ in $\mathrm{K}_0^{\mathrm{MW}}$. ■

Proof of Theorem I.10.3. Using that $k(x) \cong k(x')$, we can find an invertible regular function ν on U such that $\pi^*(\nu)(x') = \lambda'(x')^{-1}$. Then put

$$\mathscr{H}_\theta := \mathscr{H}_\theta^\chi \circ \langle \pi^*(\nu) \rangle$$

and

$$\Psi := \Psi' \circ \langle \nu \rangle.$$

By Lemmas I.12.5 and I.12.6, $\mathscr{H}_\theta$ provides a homotopy of correspondences of pairs

$$\mathrm{can}' \circ \langle \lambda' \cdot \pi^*(\nu) \rangle \sim_{\mathbf{A}^1} \Psi' \circ \pi \circ \langle \pi^*(\nu) \rangle = \Psi' \circ \langle \nu \rangle \circ \pi = \Psi \circ \pi.$$

We conclude by noting that $\mathrm{can}' \circ \langle \lambda' \cdot \pi^*(\nu) \rangle \sim_{\mathbf{A}^1} \mathrm{can}'$ by Lemma I.12.7. ■

I.13 Homotopy invariance

In this section we show, following [GP18, Proof of Theorem 2.1] and [Dru14], how homotopy invariance of the sheaves $\mathscr{F}_{\mathrm{Zar}}$ and $\mathscr{F}_{\mathrm{Nis}}$ follows from the excision theorems along with injectivity for local schemes. Throughout this section $\mathscr{F}$ will denote a homotopy invariant presheaf with MW-transfers, and $X \in \mathrm{Sm}_k$ will denote a smooth irreducible k-scheme with generic point $\eta\colon \mathrm{Spec}(k(X)) \to X$. Write $K := k(X)$ for the function field of X.

In this section, the field k is assumed to be of characteristic 0.

Homotopy invariance of $\mathscr{F}_{\mathbf{Zar}}$

Below we will use Zariski excision along with injectivity for local schemes to show homotopy invariance of the Zariski sheaf $\mathscr{F}_{\mathrm{Zar}}$ associated to $\mathscr{F}$. Let $x \in X$ be a closed point of X. We may write $\mathscr{F}(\mathrm{Spec}(\mathcal{O}_{X,x}))$ or $\mathscr{F}(\mathcal{O}_{X,x})$ for the stalk $\mathscr{F}_x$ of $\mathscr{F}$ at x in the Zariski topology.

Lemma I.13.1. *The natural map $\eta^*\colon \mathscr{F}(\mathcal{O}_{X,x}) \to \mathscr{F}(K)$ is injective.*

Proof. For $U := \mathrm{Spec}(\mathcal{O}_{X,x})$ we have $\mathscr{F}(U) = \varinjlim_{V \ni x} \mathscr{F}(V)$, and $\mathscr{F}(K) = \varinjlim_{W \neq \varnothing} \mathscr{F}(W)$. Let $s_x \in \mathscr{F}(\mathcal{O}_{X,x})$ be a germ mapping to 0 in $\mathscr{F}(K)$. This means that there is some nonempty open $W \subseteq X$ such that $s|_W = 0$. If $x \in W$ then $s_x = 0$ in $\mathscr{F}(\mathcal{O}_{X,x})$ and we are done. So suppose that $x \notin W$, and let Z denote the closed complement of W in X. Then $s|_{X \setminus Z} = 0$, and thus Corollary I.9.2 applies, yielding $s_x = 0$ in $\mathscr{F}(\mathcal{O}_{X,x})$. ■

Corollary I.13.2. *The map $\eta^*\colon \mathscr{F}_{\mathrm{Zar}}(X) \to \mathscr{F}_{\mathrm{Zar}}(K)$ is injective.*

Proof. Suppose that $s \in \mathscr{F}_{\mathrm{Zar}}(X)$ maps to 0 in $\mathscr{F}_{\mathrm{Zar}}(K)$. By Lemma I.13.1, the germs $s_x \in \mathscr{F}_x$ of s vanish at all closed points of X, which yields $s = 0$. ■

Corollary I.13.3. *For any nonempty open subscheme $i\colon V \hookrightarrow X$, the map $i^*\colon \mathscr{F}_{\mathrm{Zar}}(X) \to \mathscr{F}_{\mathrm{Zar}}(V)$ is injective.*

Proof. We know that $K = k(V)$, hence Corollary I.13.2 ensures that there are injections $\mathscr{F}_{\mathrm{Zar}}(X) \hookrightarrow \mathscr{F}(K)$ and $\mathscr{F}_{\mathrm{Zar}}(V) \hookrightarrow \mathscr{F}(K)$ induced by the generic point. Since $\mathscr{F}_{\mathrm{Zar}}(X) \hookrightarrow \mathscr{F}(K)$ factors through $\mathscr{F}_{\mathrm{Zar}}(V)$, the result follows. ■

For the next lemma we will need to pass to the presheaf $\mathscr{F}^X$ on $\widetilde{\mathrm{Cor}}^X_K$, defined in Section I.8.

Lemma I.13.4. *Let x be a closed point in $\mathbf{A}^1_K$, and write $U_x := \mathrm{Spec}(\mathcal{O}_{\mathbf{A}^1_K,x})$ for its local scheme. Then the restriction map*

$$\frac{\mathscr{F}^X(\mathbf{A}^1_K \setminus x)}{\mathscr{F}^X(\mathbf{A}^1_K)} \xrightarrow{\cong} \frac{\mathscr{F}^X(U_x \setminus x)}{\mathscr{F}^X(U_x)}$$

is an isomorphism.

Proof. We have

$$\frac{\mathscr{F}^X(U_x \setminus x)}{\mathscr{F}^X(U_x)} = \varinjlim_{W \ni x} \frac{\mathscr{F}^X(W \setminus x)}{\mathscr{F}^X(W)},$$

and so Zariski excision on $\mathbf{A}^1_K$ (Lemma I.8.10) applied to the pair $x \in W \subseteq \mathbf{A}^1_K$ yields an isomorphism

$$\frac{\mathscr{F}^X(\mathbf{A}^1_K \setminus x)}{\mathscr{F}^X(\mathbf{A}^1_K)} \xrightarrow{\cong} \frac{\mathscr{F}^X(W \setminus x)}{\mathscr{F}^X(W)}.$$

The isomorphism is given by the pullback along the inclusion, so it is compatible with the transition maps in the directed system. It follows that the natural map from $\mathscr{F}^X(\mathbf{A}^1_K \setminus x)/\mathscr{F}^X(\mathbf{A}^1_K)$ to the colimit $\mathscr{F}^X(U_x \setminus x)/\mathscr{F}^X(U_x)$ is an isomorphism. ∎

Lemma I.13.5. *The sheafification map $\psi\colon \mathscr{F}^X(\mathbf{A}^1_K) \to \mathscr{F}^X_{\mathrm{Zar}}(\mathbf{A}^1_K)$ is an isomorphism.*

Proof. Let ξ be the generic point of $\mathbf{A}^1_K$. Since stalks remain the same after sheafification, the commutative diagram

$$\begin{tikzcd}
\mathscr{F}^X(K) \arrow[r, "\cong", "p^*"'] \arrow[dr] \arrow[ddr, bend right, "\cong"'] & \mathscr{F}^X(\mathbf{A}^1_K) \arrow[d, "\psi"] \\
 & \mathscr{F}^X_{\mathrm{Zar}}(\mathbf{A}^1_K) \arrow[d, "i_0^*"] \\
 & \mathscr{F}^X_{\mathrm{Zar}}(K)
\end{tikzcd}$$

(in which p^* is an isomorphism by Remark I.8.6) shows that ψ is injective. It remains to show surjectivity.

Let $s \in \mathscr{F}^X_{\mathrm{Zar}}(\mathbf{A}^1_K)$ be a section, mapping to the germ $s_\xi \in \mathscr{F}^X_\xi$ at the generic point ξ of $\mathbf{A}^1_K$ under the morphism $\xi^*\colon \mathscr{F}^X_{\mathrm{Zar}}(\mathbf{A}^1_K) \to \mathscr{F}^X_\xi$. As

$$\mathscr{F}^X_\xi = \varinjlim_{V \subseteq \mathbf{A}^1_K} \mathscr{F}^X(V),$$

we can find a nonempty Zariski open $V \subseteq \mathbf{A}^1_K$ and a section $s' \in \mathscr{F}^X(V)$ such that $\psi(s') = s|_V \in \mathscr{F}^X_{\mathrm{Zar}}(V)$. Thus $s'_v = s_v$ for any $v \in V$. The idea from here is to extend the section $s' \in \mathscr{F}^X(V)$ to a global section $s'' \in \mathscr{F}^X(\mathbf{A}^1_K)$.

We may assume that $V = \mathbf{A}^1_K \setminus x$, where x is a closed point. Indeed, the general case follows by induction since V is then the complement of finitely many closed points. For $U_x := \mathrm{Spec}(\mathcal{O}_{\mathbf{A}^1_K,x})$, the commutative diagram

$$\begin{tikzcd}
U_x \setminus x \arrow[r, hook] \arrow[d, hook] & \mathbf{A}^1_K \setminus x \arrow[d, hook] \\
U_x \arrow[r, hook] & \mathbf{A}^1_K
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
\dfrac{\mathscr{F}^X(V)}{\mathscr{F}^X(\mathbf{A}^1_K)} \arrow[r, "\cong"] & \dfrac{\mathscr{F}^X(U_x \setminus x)}{\mathscr{F}^X(U_x)} \\
\mathscr{F}^X(V) \arrow[u] \arrow[r] & \mathscr{F}^X(U_x \setminus x) = \mathscr{F}^X_\xi \arrow[u] \\
\mathscr{F}^X(\mathbf{A}^1_K) \arrow[u] \arrow[r] & \mathscr{F}^X(U_x). \arrow[u]
\end{tikzcd}$$

Here the upper horizontal arrow is an isomorphism by Lemma I.13.4. Moreover, note that $\mathscr{F}^X(U_x \setminus x)$ and $\mathscr{F}^X(U_x)$ are both stalks, as $\mathscr{F}^X(U_x \setminus x) = \mathscr{F}^X_\xi$. Thus we have the isomorphism

$$\frac{\mathscr{F}^X(U_x \setminus x)}{\mathscr{F}^X(U_x)} \cong \frac{\mathscr{F}^X_{\mathrm{Zar}}(U_x \setminus x)}{\mathscr{F}^X_{\mathrm{Zar}}(U_x)}.$$

We want to lift $s' \in \mathscr{F}^X(V)$ to $\mathscr{F}^X(\mathbf{A}^1_K)$, which is possible if and only if s' maps to 0 in the cokernel of the map $\mathscr{F}^X(\mathbf{A}^1_K) \to \mathscr{F}^X(V)$. But s' maps to s_ξ under the map $\mathscr{F}^X(V) \to \mathscr{F}^X(U_x \setminus x)$ by the choice of s'. Moreover, $s_\xi \in \mathscr{F}^X_\xi$ is the image of the germ $s_x \in \mathscr{F}^X(U_x)$ of s at x. Hence s_ξ vanishes in $\mathscr{F}^X(U_x \setminus x)/\mathscr{F}^X(U_x)$. By the excision isomorphism we conclude that s' vanishes in $\mathscr{F}^X(V)/\mathscr{F}^X(\mathbf{A}^1_K)$, and hence there is a section $s'' \in \mathscr{F}^X(\mathbf{A}^1_K)$ such that $s''|_V = s'$.

Finally, we need to check that $s'' \in \mathscr{F}^X(\mathbf{A}^1_K)$ maps to s under the morphism $\psi\colon \mathscr{F}^X(\mathbf{A}^1_K) \to \mathscr{F}^X_{\mathrm{Zar}}(\mathbf{A}^1_K)$. It suffices to show that the germs of s'' and s coincide at every point of $\mathbf{A}^1_K$. For the points $v \in V$ we know that $s|_V = \psi(s') = \psi(s''|_V)$, so it remains to check that $s''_x = s_x$ in $\mathscr{F}^X(U_x)$. By Lemma I.13.1 we have an injection

$$\xi^*\colon \mathscr{F}^X(U_x) \hookrightarrow \mathscr{F}^X_\xi.$$

Since $\xi^*(s_x) = \xi^*(s''_x) = s_\xi$, we conclude that $s''_x = s_x$. ∎

Theorem I.13.6. *If $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$ is a homotopy invariant presheaf with* MW-*transfers, then $\mathscr{F}_{\mathrm{Zar}}$ is homotopy invariant.*

Proof. Let i_0 be the zero section $i_0\colon X \to X \times \mathbf{A}^1$, and write K for the function field $k(X)$ of X. We then have $p \circ i_0 = \mathrm{id}_X$, where $p\colon X \times \mathbf{A}^1 \to X$ is the projection. Hence the induced map $i_0^*\colon \mathscr{F}_{\mathrm{Zar}}(X \times \mathbf{A}^1) \to \mathscr{F}_{\mathrm{Zar}}(X)$ is split surjective, and it remains to show that i_0^* is injective. Consider the commutative diagram

$$\begin{tikzcd}
\mathscr{F}_{\mathrm{Zar}}(X \times \mathbf{A}^1) \arrow[r, "(\eta\times 1)^*"] \arrow[d, "i_0^*"'] & \mathscr{F}_{\mathrm{Zar}}(\mathbf{A}^1_K) \arrow[r, equal] \arrow[d, "i_0^*"] & \mathscr{F}^X_{\mathrm{Zar}}(\mathbf{A}^1_K) \arrow[d, "(i_0^X)^*"] \\
\mathscr{F}_{\mathrm{Zar}}(X) \arrow[r, "\eta^*"] & \mathscr{F}_{\mathrm{Zar}}(K) \arrow[r, equal] & \mathscr{F}^X_{\mathrm{Zar}}(K),
\end{tikzcd}$$

where the right hand vertical map is the map on $\mathscr{F}^X_{\mathrm{Zar}}$ induced by the zero section. The homomorphisms η^* and $(\eta\times 1)^*$ are injective by Corollary I.13.2 and Corollary I.13.3, respectively. Now, notice that $(i_0^X)^* = i_0^*$ by the definition of the presheaf $\mathscr{F}^X$. Hence the right hand square is commutative, and thus it suffices to show that the map $i_0^*\colon \mathscr{F}^X_{\mathrm{Zar}}(\mathbf{A}^1_K) \to \mathscr{F}^X_{\mathrm{Zar}}(K)$ is injective. Using Lemma I.13.5 along with homotopy invariance of the presheaf $\mathscr{F}^X$ (see Remark I.8.6) we find

$$\mathscr{F}^X_{\mathrm{Zar}}(\mathbf{A}^1_K) \cong \mathscr{F}^X(\mathbf{A}^1_K) \cong \mathscr{F}^X(K) = \mathscr{F}^X_{\mathrm{Zar}}(K).$$

Hence the right hand vertical map is an isomorphism. We conclude that $i_0^*\colon \mathscr{F}_{\mathrm{Zar}}(X \times \mathbf{A}^1) \to \mathscr{F}_{\mathrm{Zar}}(X)$ is injective. ■

Homotopy invariance of $\mathscr{F}_{\mathbf{Nis}}$

We proceed to prove homotopy invariance of the associated Nisnevich sheaf $\mathscr{F}_{\mathrm{Nis}}$, the proof being similar to the one for Zariski sheafification using Nisnevich excision. If A is a local ring, let A^h denote the henselization of A. We may write $\mathscr{F}(\mathrm{Spec}(\mathcal{O}^h_{X,x}))$ or $\mathscr{F}(\mathcal{O}^h_{X,x})$ for the stalk of $\mathscr{F}$ at x in the Nisnevich topology. Thus $\mathscr{F}(\mathcal{O}^h_{X,x}) = \varinjlim_V \mathscr{F}(V)$, where the colimit runs over the filtered system of étale neighborhoods of x in X, i.e., étale morphisms $p\colon V \to X$ such that $p^{-1}(x) \cong x$.

Lemma I.13.7. *For $U^h_x := \mathrm{Spec}(\mathcal{O}^h_{X,x})$, the natural map $\mathscr{F}(U^h_x) \to \mathscr{F}(k(U^h_x))$ is injective.*

Proof. Suppose that $s \in \mathscr{F}(U^h_x)$ maps to 0 in $\mathscr{F}(k(U^h_x))$. This means that there is some étale neighborhood $p\colon W \to X$ such that $s|_W = 0$. Replacing W by its open image, we may assume that $W \subseteq X$. Let Z be the closed complement of W in X. If $x \in W$ then $s = 0$ in $\mathscr{F}(U^h_x)$; if not then $x \in Z$, and thus Corollary I.9.2 shows that $s|_V = 0$ for some Zariski neighborhood V of x. Since V is also an étale neighboorhood, it follows that $s = 0$ in $\mathscr{F}(U^h_x)$. ■

The next two corollaries follow from Lemma I.13.7 similarly to the Zariski case.

Corollary I.13.8. *The map $\eta^*\colon \mathscr{F}_{\mathrm{Nis}}(X) \to \mathscr{F}_{\mathrm{Nis}}(K)$ is injective.*

Corollary I.13.9. *For any nonempty open subscheme $i\colon V \hookrightarrow X$, the map $i^*\colon \mathscr{F}_{\mathrm{Nis}}(X) \to \mathscr{F}_{\mathrm{Nis}}(V)$ is injective.*

Lemma I.13.10. *Let x be a closed point in $\mathbf{A}^1_K$. Write $U_x := \mathrm{Spec}(\mathcal{O}_{\mathbf{A}^1_K,x})$ and $U^h_x := \mathrm{Spec}(\mathcal{O}^h_{\mathbf{A}^1_K,x})$. Then there is a natural isomorphism*

$$\frac{\mathscr{F}^X(U_x \setminus x)}{\mathscr{F}^X(U_x)} \xrightarrow{\cong} \frac{\mathscr{F}^X(U^h_x \setminus x)}{\mathscr{F}^X(U^h_x)}.$$

Proof. We have

$$\begin{aligned}
\frac{\mathscr{F}^X(U_x^h \setminus x)}{\mathscr{F}^X(U_x^h)} &= \varinjlim_{W \to \mathbf{A}^1_K} \frac{\mathscr{F}^X(W \setminus x)}{\mathscr{F}^X(W)} \\
&= \varinjlim_{W \to \mathbf{A}^1_K} \varinjlim_{W' \subseteq W} \frac{\mathscr{F}^X(W' \setminus x)}{\mathscr{F}^X(W')} \\
&= \varinjlim_{W \to \mathbf{A}^1_K} \frac{\mathscr{F}^X(\mathrm{Spec}(\mathcal{O}_{W,x}) \setminus x)}{\mathscr{F}^X(\mathrm{Spec}(\mathcal{O}_{W,x}))} \\
&\xrightarrow{\cong} \varinjlim_{W \to \mathbf{A}^1_K} \frac{\mathscr{F}^X(U_x \setminus x)}{\mathscr{F}^X(U_x)} = \frac{\mathscr{F}^X(U_x \setminus x)}{\mathscr{F}^X(U_x)}.
\end{aligned}$$

Here W runs over all étale neighborhoods of x in $\mathbf{A}^1_K$; W' runs over all Zariski open neighborhoods of x in W; and the fourth isomorphism is given by Nisnevich excision. ∎

Lemma I.13.11. *The sheafification map $\psi\colon \mathscr{F}^X(\mathbf{A}^1_K) \to \mathscr{F}^X_{\mathrm{Nis}}(\mathbf{A}^1_K)$ is an isomorphism.*

Proof. Let ξ be the generic point of $\mathbf{A}^1_K$. By the same reasoning as in the proof of Lemma I.13.5, the map $\mathscr{F}^X(\mathbf{A}^1_K) \to \mathscr{F}^X_{\mathrm{Nis}}(\mathbf{A}^1_K)$ is injective, and it remains to show surjectivity. Let $s \in \mathscr{F}^X_{\mathrm{Nis}}(\mathbf{A}^1_K)$ be a section. Since the stalks $\mathscr{F}^X_\xi$ and $(\mathscr{F}^X_{\mathrm{Nis}})_\xi$ coincide, there exists a Zariski open subscheme $V \subseteq \mathbf{A}^1_K$ and a section $s' \in \mathscr{F}^X(V)$ such that $\psi(s') = s|_V$ in $\mathscr{F}^X_{\mathrm{Nis}}(V)$. We wish to extend $s' \in \mathscr{F}^X(V)$ to a global section $s'' \in \mathscr{F}^X(\mathbf{A}^1_K)$. Considering one point at a time, we may assume that $V = \mathbf{A}^1_K \setminus x$ for some closed point x. Let $U_x := \mathrm{Spec}(\mathcal{O}_{\mathbf{A}^1_K,x})$ and $U_x^h := \mathrm{Spec}(\mathcal{O}^h_{\mathbf{A}^1_K,x})$. A lift of s' to a section of $\mathscr{F}^X(\mathbf{A}^1_K)$ exists if and only if s' maps to 0 in the quotient $\mathscr{F}^X(V)/\mathscr{F}^X(\mathbf{A}^1_K)$. Consider the sequence

$$\frac{\mathscr{F}^X(V)}{\mathscr{F}^X(\mathbf{A}^1_K)} \xrightarrow{\cong} \frac{\mathscr{F}^X(U_x \setminus x)}{\mathscr{F}^X(U_x)} \xrightarrow{\cong} \frac{\mathscr{F}^X(U_x^h \setminus x)}{\mathscr{F}^X(U_x^h)} \xrightarrow{\cong} \frac{\mathscr{F}^X_{\mathrm{Nis}}(U_x^h \setminus x)}{\mathscr{F}^X_{\mathrm{Nis}}(U_x^h)}.$$

Here the left hand map is an isomorphism by Lemma I.13.4; the middle map is an isomorphism by Lemma I.13.10; and the right hand map is an isomorphism since both $\mathscr{F}^X(U_x^h \setminus x)$ and $\mathscr{F}^X(U_x^h)$ are stalks in the Nisnevich topology. Thus it is enough to show that $s' \in \mathscr{F}^X(V)$ maps to 0 in $\mathscr{F}^X_{\mathrm{Nis}}(U_x^h \setminus x)/\mathscr{F}^X(U_x^h)$. But this follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathscr{F}^X(V) \arrow[r] & \mathscr{F}^X(U_x^h \setminus x) \\
\mathscr{F}^X(\mathbf{A}^1_K) \arrow[u] \arrow[r] & \mathscr{F}^X(U_x^h). \arrow[u]
\end{tikzcd}$$

Hence we can lift s' to $s'' \in \mathscr{F}^X(\mathbf{A}^1_K)$. It remains to check that s'' maps to $s \in \mathscr{F}^X_{\mathrm{Nis}}(\mathbf{A}^1_K)$. Knowing that $\psi(s''|_V) = s|_V \in \mathscr{F}^X_{\mathrm{Nis}}(V)$, it remains to show that $s''_x = s_x \in \mathscr{F}^X_{\mathrm{Nis}}(U^h_x) = \mathscr{F}^X(U^h_x)$. As $\mathscr{F}^X(U^h_x)$ injects into $\mathscr{F}^X(U^h_x \setminus x) = \mathscr{F}^X(k(U^h_x))$ by Lemma I.13.7, it is sufficient to prove the equality in the latter stalk. This follows from the commutativity of the above diagram, using that both s and s'' map to $s|_V$ in $\mathscr{F}^X_{\mathrm{Nis}}(V)$. ■

Theorem I.13.12. *If $\mathscr{F}$ is a homotopy invariant presheaf on $\widetilde{\mathrm{Cor}}_k$, then $\mathscr{F}_{\mathrm{Nis}}$ is also homotopy invariant.*

Proof. We must show that the map $i_0^*\colon \mathscr{F}_{\mathrm{Nis}}(X \times \mathbf{A}^1) \to \mathscr{F}_{\mathrm{Nis}}(X)$ induced by the zero section is injective. As in the proof of Theorem I.13.6 we consider the commutative diagram

$$\begin{tikzcd} \mathscr{F}_{\mathrm{Nis}}(X \times \mathbf{A}^1) \arrow[r, "(\eta\times 1)^*"] \arrow[d, "i_0^*"'] & \mathscr{F}^X_{\mathrm{Nis}}(\mathbf{A}^1_K) \arrow[d, "i_0^*"] \\ \mathscr{F}_{\mathrm{Nis}}(X) \arrow[r, "\eta^*"] & \mathscr{F}^X_{\mathrm{Nis}}(K). \end{tikzcd}$$

The homomorphisms η^* and $(\eta \times 1)^*$ are injective by Corollary I.13.8 and Corollary I.13.9, respectively. Using Lemma I.13.11 along with homotopy invariance of the presheaf $\mathscr{F}^X$, we find that the right hand vertical map i_0^* is an isomorphism. Hence $i_0^*\colon \mathscr{F}_{\mathrm{Nis}}(X \times \mathbf{A}^1) \to \mathscr{F}_{\mathrm{Nis}}(X)$ is injective. ■

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Paper II

On modules over motivic ring spectra

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II

Abstract

We provide an axiomatic framework that characterizes the stable ∞-categories that are module categories over a motivic spectrum. This is done by invoking Lurie's ∞-categorical version of the Barr–Beck theorem. As an application, this gives an alternative approach to Röndigs and Østvær's theorem relating Voevodsky's motives with modules over motivic cohomology and to Garkusha's extension of Röndigs and Østvær's result to general correspondence categories, including the category of Milnor–Witt correspondences in the sense of Calmès and Fasel. We also extend these comparison results to regular Noetherian schemes over a field (after inverting the residue characteristic), following the methods of Cisinski and Déglise.

Contents

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Keywords and Phrases: motivic homotopy theory, generalized motivic cohomology, Milnor–Witt K-theory, Barr–Beck–Lurie theorem, ∞-categories.

II.1 Introduction

In [RØ06] and [RØ08], Röndigs and Østvær employed the technology of motivic functors developed in [DRØ03] to prove an important structural result regarding motivic cohomology, namely that there is an equivalence of model categories between motives and modules over motivic cohomology (at least over fields of characteristic zero). In particular, this implies that Voevodsky's triangulated category of motives, introduced in [Voe00], is equivalent to the homotopy category of modules over the motivic Eilenberg–Mac Lane spectrum. This result has been extended to bases which are regular schemes over a field in the work of Cisinski–Déglise on integral mixed motives in the equicharacteristic case [CD15]. More recently, Röndigs and Østvær's result was extended to general categories of correspondences over a field by Garkusha in [Gar19]. These theorems provide pleasant reinterpretations of Voevodsky's category of motives as modules over a highly structured ring spectrum. The analog in topology is the result that chain complexes over a ring R are equivalent (in an appropriate model categorical sense) to modules over the Eilenberg–Mac Lane spectrum $\mathrm{H}R$. This result was first obtained by Schwede and Shipley in [SS03] as part of the characterization of stable model categories in *loc. cit.*[1]

In the present paper, we aim to provide a general axiomatic approach to the above results. More precisely, by making use of Lurie's ∞-categorical version of the Barr–Beck theorem we derive a characterization of those stable ∞-categories that are equivalent to a module category over a motivic spectrum. These categories are instances of *motivic module categories* as defined in Definition II.3.1. Examples include $\mathbf{DM}(k)$ in the sense of Voevodsky [MVW06] and $\widetilde{\mathbf{DM}}(k)$ in the sense of Déglise–Fasel [DF17]. Our characterization then reads as follows:

Theorem II.1.1 (See Theorem II.5.2)**.** *Let k be a field of exponential characteristic e, and suppose that $\mathscr{M}(k)$ is a motivic module category on k. Then there is an equivalence of presentably symmetric monoidal stable ∞-categories*

$$\mathscr{M}(k)\left[\tfrac{1}{e}\right] \simeq \mathrm{Mod}_{R_{\mathscr{M}}\left[\frac{1}{e}\right]}(\mathbf{SH}(k)),$$

where $R_{\mathscr{M}}$ is a motivic $\mathcal{E}_\infty$-ring spectrum in $\mathbf{SH}(k)$ corresponding to the monoidal unit in $\mathscr{M}(k)$. In particular, the associated triangulated categories are equivalent.

[1] We remark that an ∞-categorical treatment of Schwede and Shipley's results can be found in [Lur17, Theorem 7.1.2.1].

In fact, we formulate a parametrized version of motivic module categories and, under further hypotheses, we show that Theorem II.1.1 extends to regular schemes over fields (see Theorem II.5.5). The proof of the latter follows the approach of Cisinski–Déglise [CD15], while the proof of Theorem II.1.1 breaks down into three steps:

(1) Invoke the Barr–Beck–Lurie theorem to prove that a motivic module category $\mathscr{M}(k)$ on k is equivalent to the category of modules over some monad on $\mathbf{SH}(k)$.

(2) Produce a functor from modules over the monad to modules over a corresponding motivic spectrum (Lemma II.3.6).

(3) Determine when this functor is an equivalence.

In practice, the proof of item (3) above breaks further down into two steps:

(3a) Show that the functor is an equivalence under the assumption of the *projection formula* (see Definition II.3.4). This is done in Theorem II.3.5.

(3b) Verify the projection formula in the relevant cases, which we do in Section II.5.

After proving Theorem II.1.1 we proceed to give a way to engineer several examples of motivic module categories via the notion of *correspondence categories*, to which one can apply the usual constructions of motivic homotopy theory.

II.1.1 Overview

Here is an outline of this paper:

- In Section II.2 we collect some background material on the Barr–Beck–Lurie theorem, on compact rigid generation in motivic homotopy theory, and on premotivic categories.
- In Section II.3 we provide an axiomatic framework characterizing the stable ∞-categories that are module categories over motivic spectra.
- In Section II.4 we move on to discuss examples of categories satisfying the axioms of Section II.3. The most prominent example are those arising from some sort of correspondences.
- Finally, in Section II.5 we prove that the axioms of Section II.3 are satisfied for the correspondence categories constructed in Section II.4 in various situations.

II.1.2 Conventions and notation

We will rely on the language of ∞-categories following Lurie's books [Lur09] and [Lur17]. By a *base scheme* we mean a Noetherian scheme S of finite dimension. We denote by Sch the category of Noetherian schemes, and by Sm_S the category of smooth schemes of finite type over S. The symbol T will denote the Thom space of the trivial vector bundle of rank 1 over the base S. Thus we have the standard motivic equivalences $\mathrm{T} \simeq \mathbf{A}^1/\mathbf{A}^1 \setminus 0 \simeq \mathbf{P}^1$. We set $S^{p,q} := (S^1)^{\otimes(p-q)} \otimes \mathbf{G}_m^{\otimes q}$ and $\Sigma^{p,q}M := S^{p,q} \otimes M$, suitably interpreted in the category of motivic spaces or spectra. We reserve the symbol $\mathbf{1}$ for the motivic sphere spectrum in $\mathbf{SH}(k)$ and write $\Sigma^{p,q}\mathbf{1}$ for the (p,q)-suspension of $\mathbf{1}$. If τ is a topology on Sm_S, we write $\mathbf{H}_\tau(S)$ (resp. $\mathbf{SH}_\tau(S)$) for the unstable (resp. the T-stable) motivic homotopy ∞-category. If $\tau = \mathrm{Nis}$ we may drop the decoration.

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II.2 Preliminaries

II.2.1 The Barr–Beck–Lurie Theorem

Let us start out by recalling the Barr–Beck–Lurie theorem characterizing modules over a monad, in the setting of ∞-categories. We use the terminology of [GR17, §3.7].

Let $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ be an adjunction. Then the endofunctor $GF : \mathscr{C} \to \mathscr{C}$ is a monad, and the functor $G : \mathscr{D} \to \mathscr{C}$ factors as

$$\mathscr{D} \xrightarrow{G^{\mathrm{enh}}} \mathrm{LMod}_{GF}(\mathscr{C}) \xrightarrow{u} \mathscr{C},$$

where u is the forgetful functor. Moreover, the functor $G^{\mathrm{enh}} : \mathscr{D} \to \mathrm{LMod}_{GF}(\mathscr{C})$ admits a left adjoint

$$F^{\mathrm{enh}} : \mathrm{LMod}_{GF}(\mathscr{C}) \to \mathscr{D}.$$

II.2.1.1

The net result is that the adjunction $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ factors as

$$
\begin{tikzcd}
\mathscr{C} \arrow[rr, bend left, "F"] \arrow[dr, bend right, "\mathrm{Free}_{GF}"'] & & \mathscr{D} \arrow[ll, "G"'] \arrow[dl, "G^{\mathrm{enh}}"] \\
& \mathrm{LMod}_{GF}(\mathscr{C}). \arrow[ul, "u"] \arrow[ur, bend right, "F^{\mathrm{enh}}"'] &
\end{tikzcd}
$$

Here the functor $\mathrm{Free}_{GF}\colon \mathscr{C} \to \mathrm{LMod}_{GF}(\mathscr{C})$ is simply the left adjoint to the functor u appearing in the factorization of G above, and thus deserves to be called the "free GF-module" functor.

II.2.1.2

The Barr–Beck–Lurie theorem provides necessary and sufficient conditions for the functor $G^{\mathrm{enh}}\colon \mathscr{D} \to \mathrm{LMod}_{GF}(\mathscr{C})$ to be an equivalence. Before stating the theorem, recall first that a simplicial object $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathscr{D}$ is *split* if it extends to a split augmented object; in other words it extends to a functor $U\colon \Delta^{\mathrm{op}}_{-\infty} \to \mathscr{D}$. Here $\Delta_{-\infty}$ is the category whose objects are integers ≥ -1, and where $\mathrm{Hom}_{\Delta_{-\infty}}(n,m)$ consists of nondecreasing maps $n \cup \{-\infty\} \to m \cup \{-\infty\}$. Every split augmented simplicial diagram is a colimit diagram so that the map $\operatorname{colim} X_\bullet \to X_{-1}$ is an equivalence. If $G\colon \mathscr{D} \to \mathscr{C}$ is a functor, we say that a simplicial object $X_\bullet$ in $\mathscr{D}$ is *G-split* if $G \circ X_\bullet$ is split.

Theorem II.2.1 (Barr–Beck–Lurie [Lur17, Theorem 4.7.3.5])**.** *Let $G\colon \mathscr{D} \to \mathscr{C}$ be a functor of ∞-categories admitting a left adjoint $F\colon \mathscr{C} \to \mathscr{D}$. Then the following are equivalent:*

1. *The functor G^{enh} and F^{enh} are mutually inverse equivalences.*

2. *The functor G^{enh} is conservative, and for any simplicial object $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathscr{D}$ which is G-split, $X_\bullet$ admits a colimit in $\mathscr{D}$. Furthermore, any extension $\overline{X_\bullet}\colon (\Delta^{\mathrm{op}})^{\triangleright} \to \mathscr{D}$ is a colimit diagram if and only if $G \circ \overline{X_\bullet}$ is.*

Any adjunction (F, G) satisfying the equivalent conditions above is called *monadic.*

II.2.2 Compact and rigid objects in motivic homotopy theory

We now recall some facts about compact-rigid generation in motivic stable ∞-categories.

II.2.2.1

For now we work over an arbitrary base S. Denote by:

1. $\mathbf{SH}^{\omega}(S)$ the full subcategory of $\mathbf{SH}(S)$ spanned by the compact objects, and

2. $\mathbf{SH}^{\mathrm{rig}}(S)$ the full subcategory of $\mathbf{SH}(S)$ spanned by the strongly dualizable objects.

The ∞-category $\mathbf{SH}(S)$ is generated under sifted colimits by $\Sigma^q_{\mathrm{T}}\Sigma^\infty_{\mathrm{T}}X_+$, where X is an affine smooth scheme over S and $q \in \mathbf{Z}$ [Kha16, Proposition 4.2.4]. Furthermore, each generator is a compact object in $\mathbf{SH}(S)$ since Nisnevich sheafification preserves filtered colimits (see, for example, [Hoy17b, Proposition 6.4] where we set the group of equivariance to be trivial). Hence the ∞-category $\mathbf{SH}^\omega(S)$ is generated under finite colimits and retracts by $\Sigma^q_{\mathrm{T}}\Sigma^\infty_{\mathrm{T}}X_+$, where $q \in \mathbf{Z}$ and X is affine. In particular the unit in $\mathbf{SH}(S)$ is compact and we have an inclusion

$$\mathbf{SH}^{\mathrm{rig}}(S) \subseteq \mathbf{SH}^\omega(S). \tag{II.1}$$

Over fields this inclusion is an equality—at least after an appropriate localization:

Lemma II.2.2. *Let k be a field and suppose that ℓ is a prime which is coprime to the exponential characteristic e of k. Let $L_{(\ell)}\colon \mathbf{SH}(k) \to \mathbf{SH}(k)$ be the localization endofunctor at ℓ. Then* (II.1) *induces equalities*

$$\mathbf{SH}^{\mathrm{rig}}(k)_{(\ell)} = \mathbf{SH}^\omega(k)_{(\ell)}$$

and

$$\mathbf{SH}^{\mathrm{rig}}(k)\left[\tfrac{1}{e}\right] = \mathbf{SH}^\omega(k)\left[\tfrac{1}{e}\right].$$

Proof. Since $\mathbf{SH}^\omega(k)$ is generated as a stable subcategory which is closed under retracts by $\Sigma^\infty_{\mathrm{T}}X_+$, where X is a smooth affine scheme, $\mathbf{SH}^\omega(k)_{(\ell)}$ is generated by the image of the same objects under $L_{(\ell)}$. Now, $\Sigma^\infty_{\mathrm{T}}X_+$ is dualizable whenever X is smooth and proper by [Rio05], hence it suffices to prove that $L_{(\ell)}(\Sigma^\infty_{\mathrm{T}}X_+)$ is a retract of some $L_{(\ell)}(\Sigma^\infty_{\mathrm{T}}Y_+)$, where Y is a smooth projective k-scheme. If k is perfect then this is [Lev+19, Corollary B.2]. We note that this result is extended to the case of arbitrary fields in [EK19, Theorem 3.2.1]. The result for e-inverted motivic spectra follows. ∎

Example II.2.3. If S is a positive dimensional base scheme, we should not expect (II.1) to be an equality in general even after localization; see [CD12, Corollary 3.2.7].

II.2.2.2

We adopt the following terminology:

Definition II.2.4. Let k be a field and suppose that $L\colon \mathbf{SH}(k) \to \mathbf{SH}(k)$ is a localization endofunctor. We say that $L(\mathbf{SH}(k))$, or simply L, has *compact-rigid generation* if (II.1) is an equality after applying L.

Hence Lemma II.2.2 tells us that $\mathbf{SH}(k)_{(\ell)}$ and $\mathbf{SH}(k)\left[\tfrac{1}{e}\right]$ have compact-rigid generation.

II.2.3 Premotivic categories and adjunctions

Lastly, we recall Cisinski and Déglise's notion of a *premotivic category* [CD19]. For a detailed treatment of this notion using the language of ∞-categories we refer the reader to the thesis of Khan [Kha16]. In particular, the results used in this paper can also be found in [Kha16, Chapter 2, Section 3.6].

Suppose that $\mathscr{S}$ is a full subcategory of the category Sch of Noetherian schemes, and let $\mathscr{P}$ denote a class of admissible morphisms [CD19, §1]. In fact, the only example we care about is when $\mathscr{P}$ is the class of smooth morphisms. As in [CD19, §1] (see also [CD16, Appendix A] for a more succinct discussion), a functor

$$\mathscr{M}\colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

is called a *$\mathscr{P}$-premotivic category over $\mathscr{S}$* if for each morphism $f\colon T \to S$ in $\mathscr{S}$, the induced functor $f^*\colon \mathscr{M}(S) \to \mathscr{M}(T)$ admits a right adjoint f_*, and if f is admissible, it admits a left adjoint $f_\#$. The left adjoints are furthermore required to satisfy the *$\mathscr{P}$-base change formula*, i.e., the exchange morphism $Ex^*_\#\colon q_\# g^* \to f^* p_\#$ is an equivalence whenever

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ S \arrow[r, "p"] & T \end{tikzcd}$$

is a Cartesian diagram in $\mathscr{S}$ such that p is a $\mathscr{P}$-morphism. See [CD19, §1.1.9] for details.

If the context is clear, we simply refer to $\mathscr{M}$ as a premotivic category. We may also speak of premotivic categories taking values in other (large) ∞-categories such as the ∞-category of symmetric monoidal ∞-categories $\mathrm{Cat}_\infty^\otimes$, or the ∞-category of stable ∞-categories $\mathrm{Cat}_{\infty,\mathrm{stab}}$. Another candidate that will often appear is the ∞-category Pr^L of locally small presentable ∞-categories [Lur09, Definition 5.5.0.1]. We recall that an ∞-category $\mathscr{C}$ is presentable if and only if it is an accessible localization of the ∞-category of presheaves on some small ∞-category [Lur09, Theorem 5.5.1.1].

II.2.3.1

We also have the appropriate notion of an adjunction between premotivic categories (see [CD19, Definition 1.4.6], [CD16, Definition A.1.7]). Indeed, if $\mathscr{M}$ and $\mathscr{M}'$ are premotivic categories, then a *premotivic adjunction* is a transformation $\gamma^*\colon \mathscr{M} \to \mathscr{M}'$ such that

1. for each $S \in \mathscr{S}$, the functor $\gamma^*_S\colon \mathscr{M}(S) \to \mathscr{M}'(S)$ admits a right adjoint γ_{S*}.
2. For each morphism $f\colon T \to S \in \mathscr{S}$, the canonical transformation $f_\#\gamma^*_S \to \gamma^*_T f_\#$ is an equivalence.

Furthermore, we say that a premotivic adjunction γ^* is a *localization of premotivic categories* (or, simply, a *localization*) if for each $S \in \mathscr{S}$ the functor γ_{S*} is fully faithful, i.e., a localization in the sense of [Lur09, Definition 5.2.7.2]. We say that a localization of premotivic categories is *smashing* if γ_{S*} preserves colimits. Suppose further that $\mathscr{M}$ takes values in $\mathrm{Cat}_\infty^\otimes$. In particular, the functors f^* are strongly symmetric monoidal. Then a localization L is *symmetric monoidal* if given any $S \in \mathscr{S}$ and any $E \in \mathscr{M}(S)$ that is L-local, then for any $E' \in \mathscr{M}(S)$, $E \otimes E'$ is L-local as well. This last condition implies that the symmetric monoidal structure on $\mathscr{M}(S)$ descends to one on the subcategory of L-local objects and that the localization functor is strongly symmetric monoidal [Lur17, Proposition 2.2.1.9].

II.2.3.2

We recall two conditions on $\mathscr{M}$ which will be relevant to us later. In order to formulate them, we will now assume that $\mathscr{M}$ takes values in stable ∞-categories. Let $S \in \mathscr{S}$ be a scheme. Suppose that $i \colon Z \to S$ is a closed subscheme, and let $j \colon U \to S$ be its open complement.

Definition II.2.5. Let $\mathscr{M} : \mathscr{S}^{\mathrm{op}} \to \mathrm{Cat}_{\infty,\mathrm{stab}}$ be a premotivic category, and let $Z \xrightarrow{i} S \xleftarrow{j} U$ be as above. We say that $\mathscr{M}$ satisfies (Loc_i) if

$$\mathscr{M}(Z) \xrightarrow{i_*} \mathscr{M}(S) \xrightarrow{j^*} \mathscr{M}(U)$$

is a cofiber sequence of stable ∞-categories. We say that $\mathscr{M}$ satisfies (Loc) if (Loc_i) is satisfied for any closed immersion i.

Now let $c = (c_\alpha)_{\alpha \in I}$ be a collection of Cartesian sections of $\mathscr{M}$ (the only case we consider is $\{\Sigma^{p,q}\mathbf{1}\}_{p,q \in \mathbf{Z}}$). We denote by $\mathscr{M}_c(S) \subseteq \mathscr{M}(S)$ the smallest thick subcategory of $\mathscr{M}(S)$ which contains $f_\# f^* c_{\alpha,X}$ for any smooth morphism $f \colon T \to S$. Following [CD15, Definition 2.3], we call objects in $\mathscr{M}_c(S)$ *c-constructible*. We say that $\mathscr{M}$ is *c-generated* if for all $X \in \mathscr{S}$ the stable ∞-category $\mathscr{M}(S)$ is generated by $\mathscr{M}_c(S)$ under all small colimits.

Definition II.2.6. Let $\mathscr{M} : \mathscr{S}^{\mathrm{op}} \to \mathrm{Cat}_{\infty,\mathrm{stab}}$ be a premotivic category. Suppose that $\mathscr{A} \subseteq \mathscr{S}^{\Delta^1}$ is a collection of morphisms in $\mathscr{S}$. We say that $\mathscr{M}$ is *continuous with respect to* $\mathscr{A}$ if the following holds. Suppose that $X \colon I \to \mathscr{S}$ is a cofiltered diagram in $\mathscr{S}$ whose transition maps belong to $\mathscr{A}$ and whose limit $X := \lim_{\alpha \in I} X_\alpha$ exists in $\mathscr{S}$. Then the canonical map

$$\mathscr{M}_c(X) \to \lim_{\alpha \in I} \mathscr{M}_c(X_\alpha).$$

is an equivalence.

II.3 Motivic module categories

In this section we formulate the notion of *motivic module categories* and relate it to categories of modules over a motivic $\mathcal{E}_\infty$-ring spectrum.

II.3.0.1

Let $\mathscr{S}$ be a full subcategory of Sch. By [Ayo07; CD19] we then have a premotivic category

$$\mathbf{SH}|_{\mathscr{S}} \colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Pr}^{L,\otimes}_{\mathrm{stab}}$$

whose value at $S \in \mathscr{S}$ is the motivic stable homotopy category $\mathbf{SH}(S)$ over S. Here $\mathrm{Pr}^{L,\otimes}_{\mathrm{stab}}$ denotes the ∞-category of presentably symmetric monoidal stable ∞-categories [Lur09, Definition 5.5.0.1], [Lur17, Definition 2.0.0.7].

Definition II.3.1. Let $\mathscr{S}$ be as above, and suppose that $L \colon \mathbf{SH}|_{\mathscr{S}} \to L(\mathbf{SH})|_{\mathscr{S}}$ is a localization which is symmetric monoidal in the sense of §II.2.3.1. We then define the following:

1. Let $S \in \mathscr{S}$. An *L-local motivic module category on S* is a presentably symmetric monoidal stable ∞-category $\mathscr{M}(S)$ equipped with an adjunction

$$\gamma_S^* : L(\mathbf{SH}(S)) \rightleftarrows \mathscr{M}(S) : \gamma_{S*}$$

   such that the left adjoint γ_S^* is symmetric monoidal, and the right adjoint γ_{S*} is conservative and preserves sifted colimits.

2. An *L-local motivic module category over $\mathscr{S}$* (or, simply, a *motivic module category* if the context is clear) is a premotivic category

$$\mathscr{M} \colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Pr}^{L,\otimes}_{\mathrm{stab}}$$

   valued in presentably symmetric monoidal stable ∞-categories, along with a premotivic adjunction

$$\gamma^* \colon L(\mathbf{SH})|_{\mathscr{S}} \to \mathscr{M}; \quad S \mapsto (\gamma_S^* \colon L(\mathbf{SH}(S)) \to \mathscr{M}(S)),$$

   which evaluates to an L-local motivic module category $\mathscr{M}(S)$ on S for each $S \in \mathscr{S}$.

If L is the identity functor, then we simply say that $\mathscr{M}$ is a *motivic module category*. When the localization L is clear, we may denote a motivic module category by a pair $(\mathbf{SH}|_{\mathscr{S}}, \mathscr{M})$. Moreover, if the scheme S is implicitly understood, we may drop the S from the notation $(\gamma_S^*, \gamma_{S*})$.

In §II.4 we will give a way to construct motivic module categories using very general inputs.

Lemma II.3.2. *Let $S \in \mathscr{S}$, and let $\mathbf{1}_S \in \mathbf{SH}(S)$ denote the motivic sphere spectrum over S. If $\mathscr{M}$ is an L-local motivic module category, then the spectrum $L\gamma_*\gamma^*(\mathbf{1}_S) \in \mathbf{SH}(S)$ is an $\mathcal{E}_\infty$-ring spectrum.*

Proof. As γ_* is lax symmetric monoidal, it follows that γ_* preserves $\mathcal{E}_\infty$-algebras. Since γ^* is symmetric monoidal, $\gamma^*(\mathbf{1}_S)$ is the unit object in $\mathscr{M}$ and is thus an $\mathcal{E}_\infty$-algebra. As L is symmetric monoidal, we conclude that $\gamma_*\gamma^*(\mathbf{1}_S)$ is an $\mathcal{E}_\infty$-ring spectrum. ∎

II.3.0.2

The Barr–Beck–Lurie theorem ensures that a motivic module category on S is always equivalent to modules over a monad, as the following lemma records. We will subsequently investigate when we can further enhance this equivalence to modules over the $\mathcal{E}_\infty$-ring spectrum $L\gamma_*\gamma^*(\mathbf{1}_S)$.

Lemma II.3.3. *If $\mathscr{M}(S)$ is a motivic module category on S, then the induced adjunction*

$$\gamma^{*,\mathrm{enh}} : \mathrm{LMod}_{\gamma_*\gamma^*}(L(\mathbf{SH}(S))) \rightleftarrows \mathscr{M}(S) : \gamma_*^{\mathrm{enh}}$$

is an equivalence of ∞-categories.

Proof. By assumption, the conditions of Theorem II.2.1 are satisfied. ■

II.3.1 Motivic module categories versus categories of modules

The following definition will be essential in relating a motivic module category to a category of modules over a motivic $\mathcal{E}_\infty$-ring spectrum.

Definition II.3.4. Suppose that $\mathscr{M}$ is an L-local motivic module category over $\mathscr{S}$ and let $S \in \mathscr{S}$. We say that the pair $(\mathbf{SH}|_{\mathscr{S}}, \mathscr{M})$ *admits the projection formula at S* if there is an equivalence

$$\gamma_*\gamma^*(\mathbf{1}_S) \otimes (-) \xrightarrow{\simeq} \gamma_*\gamma^*$$

of endofunctors on $L(\mathbf{SH}(S))$. If $(\mathbf{SH}|_{\mathscr{S}}, \mathscr{M})$ admits the projection formula at any $S \in \mathscr{S}$, we say that $(\mathbf{SH}|_{\mathscr{S}}, \mathscr{M})$ *admits the projection formula.*

Theorem II.3.5. *Let $\mathscr{M}$ be an L-local motivic module category over $\mathscr{S}$. Suppose that $S \in \mathscr{S}$ is a scheme such that $(\mathbf{SH}|_{\mathscr{S}}, \mathscr{M})$ admits the projection formula at S. Then there is an equivalence of presentably symmetric monoidal stable ∞-categories*

$$\mathscr{M}(S) \simeq \mathrm{Mod}_{L\gamma_*\gamma^*(\mathbf{1}_S)}(\mathbf{SH}(S)).$$

Consequently, if $(\mathbf{SH}|_{\mathscr{S}}, \mathscr{M})$ admits the projection formula, then we have an equivalence of premotivic categories

$$\mathscr{M} \simeq \mathrm{Mod}_{L\gamma_*\gamma^*(\mathbf{1})}(\mathbf{SH}(-)).$$

II.3.1.1

In light of Lemma II.3.3, we can prove Theorem II.3.5 by means of relating modules over the monad $\gamma_*\gamma^*$ with modules over the motivic spectrum $\gamma_*\gamma^*(\mathbf{1}_S)$. Thus, given $S \in \mathscr{S}$ our task is to formulate a relationship between the two ∞-categories

$$\mathrm{LMod}_{\gamma_*\gamma^*}(\mathbf{SH}(S)) \quad \text{and} \quad \mathrm{LMod}_{\gamma_*\gamma^*(\mathbf{1}_S)\otimes(-)}(\mathbf{SH}(S)).$$

To do so, it suffices produce a map of monads

$$c\colon \gamma_*\gamma^*(\mathbf{1}_S) \otimes (-) \to \gamma_*\gamma^*,$$

which will induce a functor

$$c^*\colon \mathrm{LMod}_{\gamma_*\gamma^*(\mathbf{1}_S)\otimes(-)}(\mathbf{SH}(S)) \to \mathrm{LMod}_{\gamma_*\gamma^*}(\mathbf{SH}(S)).$$

For this, we appeal to a general lemma.

Lemma II.3.6. *Let $\mathscr{C}, \mathscr{D}$ be symmetric monoidal ∞-categories and suppose that we have an adjunction $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ such that F is symmetric monoidal (so that G is lax symmetric monoidal). Then there is a map of monads*

$$c\colon GF(1) \otimes (-) \to GF, \tag{II.2}$$

which gives rise to a commutative diagram of adjunctions

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, bend left, "\mathrm{Free}_{GF}"] \arrow[dr, bend right, "GF(1)\otimes(-)"'] & & \mathrm{LMod}_{GF}(\mathscr{C}) \arrow[ll, "u"'] \arrow[dl, "c_*"] \\
& \mathrm{LMod}_{GF(1)\otimes(-)}(\mathscr{C}). \arrow[ul, "u"] \arrow[ur, bend right, "c^*"'] &
\end{tikzcd}$$

Proof. Since F is monoidal and G is lax monoidal, the functor GF is lax monoidal. Hence $GF(1)$ is an algebra object of $\mathscr{C}$, and thus $GF(1) \otimes (-)$ is indeed a monad. We construct the map of monads $c\colon GF(1) \otimes (-) \to GF(-)$ by letting c be the composite of the following maps of monads:

$$\begin{aligned}
GF(1) \otimes (-) &\simeq (GF(1) \otimes (-)) \circ \mathrm{id} \\
&\xrightarrow{\mathrm{id}\circ\epsilon} (GF(1) \otimes (-)) \circ GF(-) \\
&\xrightarrow{\mu} G(F(1) \otimes F(-)) \\
&\simeq GF.
\end{aligned}$$

Here ϵ is the unit of the adjunction (F, G). The transformation ϵ is a map of monads via the triangle identities, and the map $\mathrm{id} \circ \epsilon$ is a map of monads since we are $\circ$-tensoring two maps of monads. The map μ is given by the lax monoidal structure of G; more precisely, we note that the endofunctor $G(A \otimes F(-))$ is a monad for any algebra object A, and so $G(F(1) \otimes F(-))$ is in particular a monad. We have a canonical equivalence of monads

$$(GF(1) \otimes (-)) \circ GF(-) \simeq GF(1) \otimes GF(-).$$

The lax structure of G then provides a morphism of endofunctors

$$GF(1) \otimes GF(-) \to G(F(1) \otimes F(-)) \simeq GF(-),$$

and the lax structure also verifies that this is a map of monads. This gives rise to a functor $c_*\colon \mathrm{LMod}_{GF}(\mathscr{C}) \to \mathrm{LMod}_{GF(1)\otimes(-)}(\mathscr{C})$, which has a left adjoint by the adjoint functor theorem.

To obtain the desired factorizations, we note that we have the following commutative diagram of forgetful functors

$$\begin{tikzcd} \mathscr{C} & & \mathrm{LMod}_{GF}(\mathscr{C}) \arrow[ll, "u"'] \arrow[dl, "c_*"] \\ & \mathrm{LMod}_{GF(1)\otimes(-)}(\mathscr{C}). \arrow[ul, "u"] & \end{tikzcd}$$

Thus the left adjoints also commute. ■

II.3.1.2

We can now apply Lemma II.3.6 to prove Theorem II.3.5.

Proof of Theorem II.3.5. We claim that the adjunction of Lemma II.3.6,

$$c^* : \mathrm{LMod}_{\gamma_*\gamma^*(\mathbf{1}_S)}(\mathbf{SH}(S)) \rightleftarrows \mathrm{LMod}_{\gamma_*\gamma^*}(\mathbf{SH}(S)) : c_*,$$

is an equivalence. By the construction in the proof of Lemma II.3.6, the above adjunction arises from a map of monads given by $c\colon \gamma_*\gamma^*(\mathbf{1}_S)\otimes(-) \to \gamma_*\gamma^*$. Since $(\mathbf{SH}|_{\mathscr{S}}, \mathscr{M})$ satisfies the projection formula, we conclude that the adjunction (c^*, c_*) is an equivalence.

Now, note that Theorem II.2.1 and Lemma II.3.6 are phrased for $\mathcal{E}_1$-algebras and left modules. However, as $\gamma_*\gamma^*(\mathbf{1})$ is an $\mathcal{E}_\infty$-ring spectrum by Lemma II.3.2, the ∞-categories of left and right $\gamma_*\gamma^*(\mathbf{1})$-modules are equivalent. We thus conclude that there is an natural equivalence

$$\mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1}_S)}(\mathbf{SH}(S)) \simeq \mathscr{M}(S)$$

of ∞-categories, which carries $\gamma_*\gamma^*(\mathbf{1}_S)$ to the unit object $\gamma^*(\mathbf{1}_S)$ of $\mathscr{M}(S)$. Finally, if $\mathscr{M}$ satisfies the projection formula at any $S \in \mathscr{S}$, then the naturality of the above equivalence furnishes the equivalence of premotivic categories $\mathscr{M} \simeq \mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1})}(\mathbf{SH}(-))$. ■

Remark II.3.7. In fact, the above reduction can be achieved using a more refined version of Lurie's Barr–Beck theorem [Lur17, Proposition 4.8.5.8].

Remark II.3.8. We were also informed by Niko Naumann that the above result is a consequence of [MNN17, Proposition 5.29].

In the following Sections II.4 and II.5 we will provide examples for which the hypotheses of Theorem II.3.5 are satisfied.

II.4 Correspondence categories

The prime examples of motivic module categories are built from various notions of correspondences. In this section we will give an axiomatization of ∞-categories that behave like the category of framed correspondences as in [Elm+19a]; Suslin–Voevodsky's category of finite correspondences [VSF00], [MVW06, Chapters 1 and 2]; Calmès and Fasel's finite Milnor–Witt correspondences [CF17; DF17]; Grothendieck–Witt correspondences [Dru18]; and, more recently, the categories of correspondences studied in [DK20] and [Elm+20]. These examples will be discussed in §II.4.2. To begin with, consider the discrete category Sch_{S+}, whose objects are S-schemes of the form $X_+ := X \amalg S$ and morphisms which preserve the base point. We consider the subcategory $\mathrm{Sm}_{S+} \subseteq \mathrm{Sch}_{S+}$ spanned by smooth S-schemes of finite type. We will use heavily the *nonabelian derived* ∞*-category* $\mathrm{PSh}_\Sigma(\mathscr{C})$ associated to an ∞-category $\mathscr{C}$ with finite products; more detailed treatments of this construction can be found in [BH18, Chapter 1] and [Lur09, §5.5.8].

Definition II.4.1. A *correspondence category* (over a base scheme S) is a preadditive[2] ∞-category $\mathscr{C}$ equipped with a *graph functor*

$$\gamma_{\mathscr{C}} \colon \mathrm{Sm}_{S+} \to \mathscr{C} \tag{II.3}$$

satisfying the following conditions:

1. The functor $\gamma_{\mathscr{C}}$ is essentially surjective and preserves finite coproducts,[3] so that we get an induced functor

$$\gamma_{\mathscr{C}*} \colon \mathrm{PSh}_\Sigma(\mathscr{C}) \to \mathrm{PSh}(\mathrm{Sm}_S); \quad \mathscr{F} \mapsto \mathscr{F} \circ \gamma_{\mathscr{C}}.$$

2. The composite functor

$$\mathrm{Sm}_{S+} \to \mathscr{C} \to \mathrm{PSh}_\Sigma(\mathscr{C}) \xrightarrow{\gamma_{\mathscr{C}*}} \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+}) \tag{II.4}$$

has a right lax Sm_{S+}-linear structure. We abusively denote the composite (II.4) by $\gamma_{\mathscr{C}}(-)$; the context will always make it clear what is meant.

The ∞-category CorrCat of correspondence categories is defined as a full subcategory of the (large) ∞-category $\mathrm{PreAdd}_{\infty,\mathrm{Sm}_{S+}/}$ of small preadditive ∞-categories and functors that preserve finite coproducts equipped with a finite coproduct-preserving functor from Sm_{S+}.[4]

II.4.0.1

We begin with a couple of clarifying remarks and an example.

Remark II.4.2. Informally, the Sm_{S+}-linear structure on $\gamma_{\mathscr{C}}(-)$ encodes, for any $X, Y \in \mathrm{Sm}_S$, maps

$$X_+ \otimes \gamma_{\mathscr{C}}(Y_+) \to \gamma_{\mathscr{C}}(X_+ \otimes Y_+)$$

[2] Recall that a preadditive ∞-category is one that is pointed, has finite products and coproducts, and is such that the map $X \amalg Y \to X \times Y$ is an equivalence for all $X, Y \in \mathscr{C}$.

[3] By requiring the functor $\gamma_{\mathscr{C}}$ to preserve finite coproducts we include also the empty coproduct, ensuring that $\gamma_{\mathscr{C}}$ preserves the base point of Sm_{S+}.

[4] More succinctly, CorrCat is the pullback of ∞-categories $\mathrm{PreAdd} \times_{\mathrm{Cat}^{\amalg}_\infty} \{\mathrm{Sm}_{S+}\}$, where $\mathrm{Cat}^{\amalg}$ denotes ∞-categories with finite coproducts and finite coproduct-preserving functors.

in $\mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+}) \simeq \mathrm{PSh}_\Sigma(\mathrm{Sm}_S)_*$ which are subject to various compatibilites. For example, if $f\colon X_+ \to Z_+$ is a map in Sm_{S+} then we have a 2-cell witnessing the commutativity of

$$\begin{tikzcd} X_+ \otimes \gamma_{\mathscr{C}}(Y_+) \arrow[r] \arrow[d, "f\otimes \mathrm{id}"'] & \gamma_{\mathscr{C}}(X_+ \otimes Y_+) \arrow[d, "\gamma_{\mathscr{C}}(f\otimes \mathrm{id})"] \\ Z_+ \otimes \gamma_{\mathscr{C}}(Y_+) \arrow[r] & \gamma_{\mathscr{C}}(Z_+ \otimes Y_+). \end{tikzcd}$$

Similarly, if $g\colon Y_+ \to Z_+$ is a map in Sm_{S+} then we have a 2-cell witnessing the commutativity of

$$\begin{tikzcd} X_+ \otimes \gamma_{\mathscr{C}}(Y_+) \arrow[r] \arrow[d, "\mathrm{id}\otimes g"'] & \gamma_{\mathscr{C}}(X_+ \otimes Y_+) \arrow[d, "\gamma_{\mathscr{C}}(\mathrm{id}\otimes g)"] \\ X_+ \otimes \gamma_{\mathscr{C}}(Z_+) \arrow[r] & \gamma_{\mathscr{C}}(X_+ \otimes Z_+). \end{tikzcd}$$

These cells are required to satisfy an infinite list of coherences.

Remark II.4.3. The Sm_{S+}-linearity assumption will be satisfied if $\mathscr{C}$ has a symmetric monoidal structure and the functor $\gamma_{\mathscr{C}}$ is symmetric monoidal. In more detail, we denote by $\mathrm{CorrCat}^{\otimes}$ the ∞-category of preadditive ∞-categories with a symmetric monoidal structure such that the graph functor $\gamma_{\mathscr{C}}\colon \mathrm{Sm}_{S+} \to \mathscr{C}$ is symmetric monoidal, essentially surjective and preserves finite coproducts. There is a forgetful functor $\mathrm{CorrCat}^{\otimes} \to \mathrm{CorrCat}$; the second part of Definition II.4.1 is obtained from the strong symmetric monoidality of $\gamma_{\mathscr{C}}$. This is the case in the examples considered in this paper, but we include it as an axiom to clarify proofs of certain properties.

Example II.4.4. Let Cor_S denote the discrete category whose objects are smooth S-schemes and morphisms are spans $X \leftarrow Y \to Z$. This is a preadditive category by [BH18, Lemma C.3]. The graph functor witnesses Cor_S as a correspondence category.

II.4.0.2

We now provide some elementary properties of a correspondence category.

Proposition II.4.5. *Let $\mathscr{C}$ be a preadditive ∞-category equipped with an essential surjection*

$$\gamma_{\mathscr{C}}\colon \mathrm{Sm}_S \to \mathscr{C}$$

which preserves coproducts, and let $\gamma_{\mathscr{C}*}$ denote the induced functor

$$\gamma_{\mathscr{C}*}\colon \mathrm{PSh}_\Sigma(\mathscr{C}) \to \mathrm{PSh}_\Sigma(\mathrm{Sm}_S); \quad \mathscr{F} \mapsto \mathscr{F} \circ \gamma_{\mathscr{C}}.$$

Then the following properties hold:

1. *The ∞-category $\mathrm{PSh}_\Sigma(\mathscr{C})$ is presentable and preadditive.*

2. *The functor $\gamma_{\mathscr{C}*}$ preserves sifted colimits.*

3. *The functor $\gamma_{\mathscr{C}*}$ is conservative.*

Proof. Presentability of $\mathrm{PSh}_\Sigma(\mathscr{C})$ is [Lur09, Proposition 5.5.8.10 (1)], while PSh_Σ applied to a preadditive ∞-category is again preadditive by [GGN15, Corollary 2.4]. The functor $\gamma_{\mathscr{C}*}$ preserves sifted colimits since sifted colimits are computed pointwise (a direct consequence of parts (4) and (5) of [Lur09, Proposition 5.5.8.10]), while $\gamma_{\mathscr{C}*}$ is conservative since $\gamma_{\mathscr{C}}$ is essentially surjective. ∎

II.4.0.3

The composite of $\gamma_{\mathscr{C}}$ with the Yoneda functor $\mathrm{Sm}_{S+} \xrightarrow{\gamma_{\mathscr{C}}} \mathscr{C} \xrightarrow{y} \mathrm{PSh}_\Sigma(\mathscr{C})$ has a canonical sifted colimit-preserving extension $\gamma_{\mathscr{C}}^*\colon \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+}) \to \mathrm{PSh}_\Sigma(\mathscr{C})$. It is easy to check that $\gamma_{\mathscr{C}*}$ is the right adjoint to $\gamma_{\mathscr{C}}^*$ and thus $\gamma_{\mathscr{C}}^*$ preserves all small colimits. As a result, we have an adjunction

$$\gamma_{\mathscr{C}}^* : \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+}) \rightleftarrows \mathrm{PSh}_\Sigma(\mathscr{C}) : \gamma_{\mathscr{C}*}. \tag{II.5}$$

It is also easy promote the Sm_{S+}-linear structure given by the second axiom of a correspondence category to a $\mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$-linear structure so that the functor

$$\gamma_{\mathscr{C}*} \circ \gamma_{\mathscr{C}}^*\colon \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+}) \to \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$$

extends to a right lax $\mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$-linear functor.

II.4.0.4

Now we would like to do motivic homotopy theory on $\mathscr{C}$. Recall that if $X, Y \in \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$, then X is $\mathbf{A}^1$*-homotopy equivalent to* Y if there are maps $f\colon X \to Y$, $g\colon Y \to X$ and $\mathbf{A}^1$-homotopies $H\colon \mathbf{A}^1_+ \otimes X \to X$, $H'\colon \mathbf{A}^1_+ \otimes Y \to Y$ from gf and fg to the respective identity morphisms. We note that any $\mathbf{A}^1$-homotopy equivalence is an $\mathrm{L}_{\mathbf{A}^1}$-equivalence [MV99, §2 Lemma 3.6].

Lemma II.4.6. *The functor $\gamma_{\mathscr{C}}\colon \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+}) \to \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$ preserves $\mathbf{A}^1$-homotopy equivalences.*

Proof. Suppose that we have a homotopy $H\colon \mathbf{A}^1_+ \otimes X_+ \to Y$ between maps $f, g\colon X \to Y$. We obtain, using the right lax-structure, a homotopy

$$\mathbf{A}^1_+ \otimes \gamma_{\mathscr{C}}(X) \to \gamma_{\mathscr{C}}(\mathbf{A}^1 \times X) \to \gamma_{\mathscr{C}}(Y)$$

between $\gamma_{\mathscr{C}}(f)$ and $\gamma_{\mathscr{C}}(g)$. ∎

Lemma II.4.7. *The functor $\gamma_{\mathscr{C}}\colon \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+}) \to \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$ preserves $\mathrm{L}_{\mathbf{A}^1}$-equivalences.*

Proof. By definition the class of $\mathrm{L}_{\mathbf{A}^1}$-equivalences is the strong saturation, in the sense of [Lur09, Proposition 5.5.4.5], of the maps in $\mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$ by the (Yoneda image of) $\mathbf{A}^1$-projections $\pi_X\colon (\mathbf{A}^1 \times X)_+ \simeq \mathbf{A}^1_+ \otimes X_+ \to X_+$ for $X \in \mathrm{Sm}_S$. According to [BH18, Lemma 2.10] the class of $\mathrm{L}_{\mathbf{A}^1}$-equivalences is then generated under 2-out-of-3 and sifted colimits by maps of the form $\pi_X \amalg \mathrm{id}_{Y_+}$ where $Y \in \mathrm{Sm}_S$.

Since π_X is an $\mathbf{A}^1$-homotopy equivalence, it follows from Lemma II.4.6 that $\gamma_{\mathscr{C}}(\pi_X)$ is an $\mathbf{A}^1$-homotopy equivalence. Since $\gamma_{\mathscr{C}}$ preserves coproducts by assumption, the same is true for the morphism

$$\gamma_{\mathscr{C}}(\pi_X \amalg \mathrm{id}_{Y_+}) \simeq \gamma_{\mathscr{C}}(\pi_X) \amalg \gamma_{\mathscr{C}}(\mathrm{id}_{Y_+}).$$

The functor $\gamma_{\mathscr{C}}$ clearly preserves the 2-out-of-3-property. Lastly, the functor $\gamma_{\mathscr{C}}$ preserves sifted colimits by definition and sifted colimits are computed valuewise in $\mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})^{\Delta^1}$. Hence we conclude that $\gamma_{\mathscr{C}}$ preserves $\mathrm{L}_{\mathbf{A}^1}$-equivalences. ■

II.4.0.5

Now we take into account a topology that we might want to put on Sm_{S+}, namely, the topology of *coproduct decomposition.* This is a topology on Sm_{S+} defined by a cd-structure, denoted by $\amalg$, generated by squares

$$\begin{tikzcd} S \arrow[r] \arrow[d] & U_+ \arrow[d] \\ V_+ \arrow[r] & X_+ \end{tikzcd}$$

where U and V are clopen subschemes of X such that $U \amalg V = X$. Sheaves with respect to the topology generated by this cd-structure is precisely the nonabelian derived category on $\mathscr{C}$. In other words we have

$$\mathrm{Shv}_{\amalg}(\mathrm{Sm}_{S+}) \simeq \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$$

by [BH18, Lemma 2.4]. Hence all topologies τ considered in this paper satisfy $\mathrm{Shv}_\tau(\mathrm{Sm}_{S+}) \subseteq \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$.

Definition II.4.8. Let τ be a topology on Sm_S, and let $\mathscr{C}$ be a correspondence category with graph functor $\gamma_{\mathscr{C}}\colon \mathrm{Sm}_{S+} \to \mathscr{C}$. Then $\mathscr{C}$ *is compatible with* τ if for every τ-sieve $U \hookrightarrow X$ in Sm_S, the natural map

$$\gamma_{\mathscr{C}}(U_+) \to \gamma_{\mathscr{C}}(X_+)$$

is an L_τ-equivalence in $\mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$.

Lemma II.4.9. *Suppose that $\mathscr{C}$ is a correspondence category which is compatible with τ. Then the functor*

$$\gamma_{\mathscr{C}}\colon \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+}) \to \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$$

preserves L_τ-equivalences.

Proof. By definition, the class of L_τ-equivalences is the strong saturation, in the sense of [Lur09, Proposition 5.5.4.5], of the maps in $\mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$ by the (Yoneda image of the) maps $i_+ : U_+ \hookrightarrow X_+$ where $X \in \mathrm{Sm}_S$ and i is a τ-sieve. According to [BH18, Lemma 2.10], the class of L_τ-equivalences is then generated under 2-out-of-3 and sifted colimits by maps of the form $\pi_X \amalg \mathrm{id}_{Y_+}$ for $Y \in \mathrm{Sm}_S$. By the same reasoning as in Proposition II.4.7 we need only check that $\gamma_{\mathscr{C}}(U_+) \to \gamma_{\mathscr{C}}(X_+)$ is an L_τ-equivalence which is true by hypothesis. ■

From now on, whenever we consider a correspondence category $\mathscr{C}$ we make the following assumption on the topologies we discuss:

- The topology τ is at least as fine as the Nisnevich topology and is compatible in the sense of Definition II.4.8.

II.4.0.6

If $\mathscr{C}$ is a correspondence category, then we can construct its unstable motivic homotopy ∞-category in the usual way, as we now do. We consider two full subcategories of $\mathrm{PSh}_\Sigma(\mathscr{C})$ spanned by objects $\mathscr{F}$ satisfying the following two axioms on homotopy invariance and τ-descent:

(Htpy) The presheaf $\mathscr{F} \circ \gamma_{\mathscr{C}} : \mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{Spc}$ is $\mathbf{A}^1$-invariant. We denote the ∞-category spanned by such $\mathscr{F}$'s by $\mathrm{PSh}_{\mathbf{A}^1}(\mathscr{C})$.

(τ-Desc) The presheaf $\mathscr{F} \circ \gamma_{\mathscr{C}} : \mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{Spc}$ is a τ-sheaf. We denote the ∞-category spanned by such $\mathscr{F}$'s by $\mathrm{Shv}_\tau(\mathscr{C})$.

Since $\mathrm{PSh}_\Sigma(\mathscr{C})$ is preadditive by Proposition (II.4.5), we have a canonical equivalence $\mathrm{CMon}(\mathrm{PSh}_\Sigma(\mathscr{C})) \simeq \mathrm{PSh}_\Sigma(\mathscr{C})$. The ∞-category of *unstable $\mathscr{C}$-motives*, denoted by $\mathbf{H}_\tau(\mathscr{C})$, is then defined as $\mathrm{PSh}_{\mathbf{A}^1}(\mathscr{C}) \cap \mathrm{Shv}_\tau(\mathscr{C}) \subseteq \mathrm{PSh}_\Sigma(\mathscr{C})$. As usual we have localization functors $\mathrm{L}_\tau^{\mathscr{C}} : \mathrm{PSh}_\Sigma(\mathscr{C}) \to \mathrm{Shv}_\tau(\mathscr{C})$, $\mathrm{L}_{\mathbf{A}^1}^{\mathscr{C}} : \mathrm{PSh}_\Sigma(\mathscr{C}) \to \mathrm{PSh}_{\mathbf{A}^1}(\mathscr{C})$ and $\mathrm{L}_{\mathrm{mot},\tau}^{\mathscr{C}} : \mathrm{PSh}_\Sigma(\mathscr{C}) \to \mathbf{H}_\tau(\mathscr{C})$. From the construction of these localizations and the assumption on τ, the adjunction (II.5) descends to an adjunction

$$\gamma_{\mathscr{C}}^* : \mathbf{H}_\tau(\mathrm{Sm}_{S+}) \simeq \mathbf{H}_\tau(S)_* \rightleftarrows \mathbf{H}_\tau(\mathscr{C}) : \gamma_{\mathscr{C}*} \tag{II.6}$$

Lemma II.4.10. *The ∞-category $\mathbf{H}_\tau(\mathscr{C})$ is preadditive. Hence we have a canonical equivalence* $\mathrm{CMon}(\mathbf{H}_\tau(\mathscr{C})^\times) \simeq \mathbf{H}_\tau(\mathscr{C})$.

Proof. The ∞-category $\mathbf{H}_\tau(\mathscr{C})$ is closed under finite products by checking that the conditions (Htpy) and (τ-Desc) are preserved under taking products which are computed pointwise. The statement follows since $\mathrm{PSh}_\Sigma(\mathscr{C})$ is preadditive by Proposition II.4.5. ■

Definition II.4.11. The ∞-category of *effective $\mathscr{C}$-motives* $\mathbf{H}_\tau(\mathscr{C})^{\mathrm{gp}}$ is defined to be the full subcategory of $\mathbf{H}_\tau(\mathscr{C})$ spanned by the grouplike objects, in the sense of [GGN15, Definition 1.2].

II.4.0.7

The next proposition captures the main property of categories of correspondences from the point of view of motivic homotopy theory.

Proposition II.4.12. *Suppose that $\mathscr{C}$ is a correspondence category which is compatible with τ. Then the functor*

$$\gamma_{\mathscr{C}*} \colon \mathbf{H}_\tau(\mathscr{C}) \to \mathbf{H}_\tau(S)_*$$

preserves sifted colimits and is conservative. Furthermore, $\mathbf{H}_\tau(\mathscr{C})$ is canonically an $\mathbf{H}(S)_*$-module.

Proof. For the first claim it suffices, after Proposition II.4.5, to check that

$$\gamma_{\mathscr{C}*} \colon \mathrm{PSh}_\Sigma(\mathscr{C}) \to \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+}) \simeq \mathrm{PSh}_\Sigma(\mathrm{Sm}_S)_*$$

sends $\mathrm{L}^{\mathscr{C}}_{\mathrm{mot},\tau}$-equivalences to $\mathrm{L}_{\mathrm{mot},\tau}$-equivalences. This holds by Lemma II.4.7 and Lemma II.4.9. The assertion that $\mathbf{H}_\tau(\mathscr{C})$ is an $\mathbf{H}(S)_*$-module follows from the right lax structure of $\gamma_{\mathscr{C}*}$. ∎

Remark II.4.13. If τ is a topology finer than the Nisnevich topology, then the fully faithful functor $\mathbf{H}_\tau(S)_* \to \mathbf{H}(S)_*$ need not preserve colimits. Hence the composite $\mathbf{H}_\tau(\mathscr{C}) \to \mathbf{H}_\tau(S)$ need not preserve colimits.

II.4.0.8

From the above point of view, we see that $\gamma_{\mathscr{C}*}$ is very close to preserving all colimits—we need only show that it preserves finite coproducts. The universal way to enforce this is to take commutative monoid objects on both sides with respect to Cartesian monoidal structures. We can do this for $\mathbf{H}_\tau(S)_*$ since it has finite products, and $\mathrm{CMon}(\mathbf{H}_\tau(\mathscr{C})^\times) \simeq \mathbf{H}_\tau(\mathscr{C})$ since it is preadditive [GGN15, Proposition 2.3]. We remark that the symmetric monoidal structure on $\mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$ given by Day convolution is not Cartesian.[5]

Thus, to see that $\gamma_{\mathscr{C}*}$ preserves all colimits, consider its left adjoint

$$\gamma^*_{\mathscr{C}} \colon \mathbf{H}_\tau(S)_* \to \mathbf{H}_\tau(\mathscr{C}),$$

which preserves all small colimits. According to the universal property of CMon [GGN15, Corollary 4.9] we obtain an essentially unique functor

$$\gamma^*_{\mathscr{C}} \colon \mathrm{CMon}(\mathbf{H}_\tau(S)^\times_*) \to \mathbf{H}_\tau(\mathscr{C})$$

[5]On the other hand, the symmetric monoidal structure on $\mathrm{PSh}_\Sigma(\mathrm{Sm}_S)$ given by Day convolution is Cartesian, and the natural sifted-colimit preserving functor $\mathrm{PSh}_\Sigma(\mathrm{Sm}_S) \to \mathrm{PSh}_\Sigma(\mathrm{Sm}_{S+})$ is symmetric monoidal.

since $\mathbf{H}_\tau(\mathscr{C})$ is preadditive by Proposition II.4.5.1. This functor admits a right adjoint $\gamma_{\mathscr{C}*}\colon \mathbf{H}_\tau(\mathscr{C}) \to \mathrm{CMon}(\mathbf{H}_\tau(S)_*^\times)$ which fits into a commutative diagram

$$\begin{tikzcd} & \mathrm{CMon}(\mathbf{H}_\tau(S)_*^\times) \arrow[d] \\ \mathbf{H}_\tau(\mathscr{C}) \arrow[ur, "\gamma_{\mathscr{C}*}"] \arrow[r, "\gamma_{\mathscr{C}*}"] & \mathbf{H}_\tau(S)_*. \end{tikzcd} \tag{II.7}$$

In other words, the functor $\gamma_{\mathscr{C}*}$ factors through the forgetful functor

$$\mathrm{CMon}(\mathbf{H}_\tau(S)_*^\times) \to \mathbf{H}_\tau(S)_*.$$

We use this to conclude:

Proposition II.4.14. *Suppose that $\mathscr{C}$ is a correspondence category which is compatible with τ. Then the functor*

$$\gamma_{\mathscr{C}*}\colon \mathbf{H}_\tau(\mathscr{C}) \to \mathbf{H}_\tau(S)_*$$

preserves all small colimits and is conservative.

Proof. By the diagram (II.7), the diagonal functor $\gamma_{\mathscr{C}*}$ preserves sifted colimits because the horizontal arrow preserves sifted colimits by Proposition II.4.12 and the vertical arrow preserves sifted colimits as a special case of [GGN15, Proposition B.4]. Since it is a right adjoint it preserves finite products, but since its domain and codomain are preadditive it preserves finite coproducts as well and we are done by [BH18, Lemma 2.8]. The conservativity statement follows from Proposition II.4.12 and the fact that the forgetful functor from commutative monoid objects is conservative. ∎

II.4.0.9 T-stability

We now introduce the notion of T-stability along with the weaker notion of T-prestability. This is inspired by the treatment of [Lur18, Appendix C] on prestable ∞-categories.

Definition II.4.15. Let $\mathscr{C}$ be an $\mathbf{H}(S)_*$-module in Cat_∞. Then $\mathscr{C}$ is T-*prestable* if the endofunctor

$$\mathrm{T} \otimes (-)\colon \mathscr{C} \to \mathscr{C} \tag{II.8}$$

is fully faithful. The ∞-category $\mathscr{C}$ is T-*stable* if the endofunctor (II.8) is invertible.

Remark II.4.16. The notion of a T-stable ∞-category is a familiar one in motivic homotopy theory; indeed, the motivic stable homotopy category $\mathbf{SH}(S)$ is T-stable. In fact, T-prestability is a familiar concept as well: it is inspired by Voevodsky's *cancellation theorem* [Voe10] which asserts that $\mathbf{DM}^{\mathrm{eff}}(k, \mathbf{Z})$ is T-prestable for any perfect field k. The analogous statement holds for Milnor–Witt motives as proved in [FØ17]. For the ∞-category of framed motivic spaces, cancellation holds by [Elm+19a, Theorem 3.5.8], which in turn relies on the cancellation theorem of Ananyevskiy, Garkusha and Panin [AGP18]. Moreover, for any base scheme S, the subcategory $\mathbf{SH}(S)^{\mathrm{eff}} \subseteq \mathbf{SH}(S)$ of effective motivic spectra is T-prestable.

II.4.0.10

From now on we assume that the correspondence category $\mathscr{C}$ is such that $\mathbf{H}(\mathscr{C})_*$ is prestable. The thesis of Robalo [Rob15] provides a way to invert T and obtain a symmetric monoidal stable ∞-category—in fact one that is a module over $\mathbf{SH}(S)$. We define the stable ∞-category of *$\mathscr{C}$-motives* simply by

$$\mathbf{SH}_\tau(\mathscr{C}) := \mathbf{H}_\tau(\mathscr{C})[\mathrm{T}^{\otimes -1}],$$

with notation as in [Rob15, Definition 2.6]. We then have the basic adjunction

$$\Sigma^\infty_{\mathrm{T},\mathscr{C}} : \mathbf{H}_\tau(\mathscr{C}) \rightleftarrows \mathbf{SH}_\tau(\mathscr{C}) : \Omega^\infty_{\mathrm{T},\mathscr{C}}.$$

The following summarizes the basic properties of $\mathbf{SH}_\tau(\mathscr{C})$:

Proposition II.4.17. *If $\mathscr{C}$ is a correspondence category, then the following hold:*

1. *The ∞-category $\mathbf{SH}_\tau(\mathscr{C})$ is a presentably symmetric monoidal stable ∞-category, and*
2. *is generated under sifted colimits by objects of the form*
$$\{\mathrm{T}^{\otimes n} \otimes \Sigma^\infty_{\mathrm{T},\mathscr{C}} X\}_{n\in\mathbf{Z}, X\in\mathscr{C}}.$$
3. *The ∞-category $\mathbf{SH}_\tau(\mathscr{C})$ is computed as the colimit in $\mathrm{Mod}_{\mathbf{H}(\mathrm{Sm}_S)_*}(\mathrm{Pr}^L)$ of*
$$\mathbf{H}_\tau(\mathscr{C}) \xrightarrow{\mathrm{T}\otimes(-)} \mathbf{H}_\tau(\mathscr{C}) \xrightarrow{\mathrm{T}\otimes(-)} \mathbf{H}_\tau(\mathscr{C}) \xrightarrow{\mathrm{T}\otimes(-)} \cdots . \tag{II.9}$$
4. *The functor*
$$\gamma_{\mathscr{C}*} : \mathbf{SH}_\tau(\mathscr{C}) \to \mathbf{SH}_\tau(\mathrm{Sm}_S)$$
is conservative and preserves colimits.

Proof. Stability follows from the standard equivalence $\mathrm{T} \simeq S^1 \otimes \mathbf{G}_m$ in $\mathbf{SH}(S)$, which remains true for modules over $\mathbf{SH}(S)$. The second assertion follows from the third via [Lur09, Lemma 6.3.3.7] and the fact that $\mathbf{H}_\tau(\mathscr{C})$ is generated under sifted colimits by representables which are smooth affine by the argument of [Kha16, Proposition 2.2.9] (which works for any topology τ finer than Nis), while the third comes from [Rob15, Corollary 2.22]. The last assertion follows from Proposition II.4.14. ■

II.4.0.11

The last part of Proposition II.4.17 is the main point of our axiomatization: the adjunction $\mathbf{SH}_\tau(S) \rightleftarrows \mathbf{SH}_\tau(\mathscr{C})$ is monadic. In particular, if $\tau = \mathrm{Nis}$, then $\mathbf{SH}(S) \rightleftarrows \mathbf{SH}(\mathscr{C})$ is monadic.

II.4.1 From categories of correspondences to motivic module categories

Suppose that we have a functor

$$\mathscr{C}: \mathscr{S}^{\mathrm{op}} \to \mathrm{CorrCat}^{\otimes}$$

which carries a morphism of schemes $f: T \to S$ to $f^*: \mathscr{C}_S \to \mathscr{C}_T$. By naturality of the preceding constructions[6] we obtain a functor

$$\mathbf{SH}_\tau \circ \mathscr{C}: \mathscr{S}^{\mathrm{op}} \to \mathrm{Pr}^{L,\otimes}_{\mathrm{stab}}$$

equipped with a transformation $\mathbf{SH}|_{\mathscr{S}} \to \mathbf{SH}_\tau \circ \mathscr{C}$. We impose an additional assumption on $\mathscr{C}$, inspired by [CD19, Lemma 9.3.7]:

- For each $p: T \to S$, a smooth morphism in $\mathscr{S}$, the functor p^* admits a left adjoint $p_\#$ such that the transformation $p_\#\gamma_{\mathscr{C}_T} \to \gamma_{\mathscr{C}_S}p_\#$ is an equivalence.

In this case, we say that $\mathscr{C}$ is *adequate*.

II.4.1.1

We employ the following additional notation: if $L: \mathbf{SH}(S) \to \mathbf{SH}(S)$ is a localization, denote by $L(\mathbf{SH}_\tau(\mathscr{C}_S))$ the subcategory of $\mathbf{SH}_\tau(\mathscr{C}_S)$ spanned by objects X such that $\gamma_{\mathscr{C}*}(X)$ is L-local. Since $\gamma_{\mathscr{C}*}$ preserves limits, the inclusion $L(\mathbf{SH}_\tau(\mathscr{C}_S)) \hookrightarrow \mathbf{SH}_\tau(\mathscr{C}_S)$ is closed under limits and there is a localization functor (by the adjoint functor theorem)

$$L_{\mathscr{C}_S}: \mathbf{SH}_\tau(\mathscr{C}_S) \to L(\mathbf{SH}_\tau(\mathscr{C}_S))$$

rendering the following diagram commutative (since their right adjoints commute):

$$\begin{tikzcd}
\mathbf{SH}(S) \arrow[r, "\gamma^*_{\mathscr{C}_S}"] \arrow[d, "L"'] & \mathbf{SH}_\tau(\mathscr{C}_S) \arrow[d, "L_{\mathscr{C}_S}"] \\
L(\mathbf{SH}(S)) \arrow[r, "\gamma^*_{\mathscr{C}_S}"] & L(\mathbf{SH}_\tau(\mathscr{C}_S)).
\end{tikzcd}$$

Proposition II.4.18. *If $\mathscr{C}: \mathscr{S}^{\mathrm{op}} \to \mathrm{CorrCat}^{\otimes}$ is adequate, then the following hold:*

1. *We have premotivic adjunctions $\mathbf{SH}|_{\mathscr{S}} \rightleftarrows \mathbf{SH}_\tau \circ \mathscr{C}$.*

2. *If L is smashing and a symmetric monoidal localization of $\mathbf{SH}|_{\mathscr{S}}$, then we have a premotivic adjunction $L(\mathbf{SH})|_{\mathscr{S}} \leftrightarrows L(\mathbf{SH} \circ \mathscr{C})$.*

[6] The most nontrivial of which is the universal property of T-stabilization for which we can appeal to [BH18, Lemma 4.1].

3. *If τ is a topology such that for each $S \in \mathscr{S}$, the functor $L(\mathbf{SH}_\tau(S)) \to L(\mathbf{SH}(S))$ preserves sifted colimits, then the premotivic adjunction*

$$L(\mathbf{SH})|_{\mathscr{S}} \rightleftarrows L(\mathbf{SH}_\tau \circ \mathscr{C})$$

is a motivic module category (in particular, this holds when $\tau = \mathrm{Nis}$).

Proof. The proof of (1) follows as in the case of Grothendieck abelian categories [CD19, Corollary 10.3.11] and Voevodsky's $\mathscr{C} = \mathrm{Cor}$ (in the sense of [CD19, §9]); we give only the main points. Since $\mathscr{C}$ is adequate, we get that the equivalence $p_\# \gamma_{\mathscr{C}_T} \to \gamma_{\mathscr{C}_S} p_\#$ persists on the level of T-stabilizations. What we need to verify, just as in [CD19, Proposition 10.3.9], is that the transformation $L_\tau \gamma_{\mathscr{C}*} \simeq \gamma_{\mathscr{C}*} L_\tau$ is an equivalence on the unstable level, i.e., the "forgetful" functor $\mathbf{H}_\tau \circ \mathscr{C} \to \mathbf{H}|_{\mathscr{S}}$ preserves τ-local objects. This is given by Lemma II.4.9 under the standing assumption that $\mathscr{C}$ is compatible with τ. The next two statements are then immediate from the definition of motivic module categories and the last statement of Proposition II.4.17. ∎

II.4.2 Examples

We now discuss examples of the above constructions and results.

Example II.4.19. Let $\mathscr{S} = \mathrm{Sch}_S$ and suppose that E is an $\mathcal{E}_\infty$-ring spectrum in $\mathbf{SH}(S)$. Then $\mathrm{Mod}_E = (E \otimes (-)) \circ \mathbf{SH}$ furnishes the first examples of motivic module categories. We can also consider further localizations of the premotivic category Mod_E, such as in [Elm+19b] where $\mathscr{S} = \mathrm{Sch}_{\mathbf{Z}[\frac{1}{\ell}]}$ the localization functor is given by the composite of ℓ-completion and étale localization, and E is MGL; see *loc. cit.* for more details where results in this paper are used to describe the ∞-category of modules over étale cobordism.

Example II.4.20. Consider a localization $L\colon \mathbf{SH}|_{\mathscr{S}} \to L(\mathbf{SH}|_{\mathscr{S}})$. If L is smashing, then $L(\mathbf{SH}|_{\mathscr{S}})$ is a motivic module category. Examples of these smashing localizations are given by *periodization of elements*; we refer the reader to [Hoy17a, Section 3] for an extensive discussion in our context. For example, a theorem of Bachmann [Bac18] proves that periodizing the element ρ yields real étale localization. Consider $x\colon \Sigma^{p,q}\mathbf{1} \to \mathbf{1}$. Then the results of [Hoy17a, §3] (or apply [Bac18, Lemma 15]) tell us that $\mathbf{1}[x^{-1}]$ is an $\mathcal{E}_\infty$-ring and the projection formula holds, hence the category of x-periodic motivic spectra are modules over $\mathbf{1}[x^{-1}]$.

Example II.4.21. The basic example of a category of correspondences is Voevodsky's category of finite correspondences Cor_S in the sense of [MVW06, Appendix 1A], [CD19, §9], which is defined for any Noetherian scheme S [CD19, §9.1]. When S is essentially smooth over a perfect base field, the category of finite Milnor–Witt correspondences $\widetilde{\mathrm{Cor}}_S$ of Calmès and Fasel [CF17] is defined and is also a category of correspondences. Over a perfect field (where both categories are defined), these categories are generalized by Garkusha's axioms in [Gar19]. When defined, these categories are adequate in the sense of §II.4.1. All of these are examples of categories of correspondences, and thus give rise to motivic module categories.

Example II.4.22. Let k be a perfect field. Given any $S \in \mathrm{Sm}_k$ and any good cohomology theory A on Sm_S in the sense of [DK20, §2], then [DK20, §3] defines an adequate category of correspondences Cor_S^A on Sm_S.

Example II.4.23. The ∞-category of framed correspondences of [Elm+19a] is another example of a category of correspondences and is defined for any qcqs scheme S. The main theorem of [Hoy18] asserts that the corresponding motivic module category is equivalent to $\mathbf{SH}(S)$, relying on the "recognition principle" of [Elm+19a].

Example II.4.24. If $E \in \mathbf{SH}(S)$ is a homotopy associative ring spectrum, then [Elm+20] defines an hSpc-enriched category hCor_S^E of *finite E-correspondences*, which the authors expect to be the homotopy category of an ∞-category Cor_S^E whenever E is an $\mathcal{A}_\infty$-ring. Setting $\mathscr{C} = \mathrm{Cor}_S^E$, the ∞-category $\mathbf{SH}(\mathscr{C})$ in this paper corresponds to $\mathbf{DM}^E(S)$ in *loc. cit.* We will return to this example in the next section.

II.5 Module categories over regular schemes

In this section we show that the hypotheses of Theorem II.3.5 are satisfied for module categories over a field k, and more generally for module categories over regular k-schemes.

II.5.1 The case of fields

We start by verifying that the projection formula holds at a field k. In this case, we can use the following computation to reduce to the case of compact-rigid generation:

Lemma II.5.1. *Suppose that we have an adjunction of symmetric monoidal ∞-categories*

$$F : \mathscr{C} \rightleftarrows \mathscr{D} : G$$

such that F is strongly symmetric monoidal. Let $1 \in \mathscr{C}$ denote the unit object of $\mathscr{C}$. If $E \in \mathscr{C}$ is a strongly dualizable object, then the map $c\colon GF(1) \otimes E \to GF(E)$ is an equivalence.

Proof. This follows from a standard computation: let $E' \in \mathscr{C}$ be arbitrary, then we have a string of equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathscr{C}}(E', GF(1) \otimes E) &\simeq \mathrm{Map}_{\mathscr{C}}(E' \otimes E^\vee, GF(1)) \\
&\simeq \mathrm{Map}_{\mathscr{D}}(F(E' \otimes E^\vee), F(1)) \\
&\simeq \mathrm{Map}_{\mathscr{D}}(F(E') \otimes F(E)^\vee, F(1)) \\
&\simeq \mathrm{Map}_{\mathscr{D}}(F(E'), F(E)) \\
&\simeq \mathrm{Map}_{\mathscr{C}}(E', GF(E)),
\end{aligned}$$

which shows the claim. ■

II.5.1.1

Thus, if $\mathbf{SH}(S)$ is generated by strongly dualizable objects, it follows that the projection formula holds:

Theorem II.5.2. *Let k be a field. Suppose that ℓ is a prime which is coprime to the exponential characteristic e of k and let $\mathscr{M}$ be a motivic module category on k. Then we have the following equivalences of presentably symmetric monoidal stable ∞-categories:*

$$L_{(\ell)}(\mathscr{M}(k)) \simeq \mathrm{Mod}_{L_{(\ell)}\gamma_*\gamma^*(\mathbf{1}_S)}(\mathbf{SH}(k)),$$

and

$$\mathscr{M}(k)\left[\tfrac{1}{e}\right] \simeq \mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1})\left[\frac{1}{e}\right]}(\mathbf{SH}(k)).$$

Proof. In light of Theorem II.3.5 we need to verify the appropriate projection formulas. By assumption, the functor γ_* preserves sifted colimits and thus the functors $\gamma_*\gamma^*(\mathbf{1}_S)\otimes(-)$ and $\gamma_*\gamma^*(-)$ do as well. Now Lemma II.5.1 tells us that the projection formula holds for strongly dualizable objects in $\mathbf{SH}(k)_{(\ell)}$. Thus we will be done if we can prove that the inclusion of (II.1), $\mathbf{SH}^{\mathrm{rig}}(k)_{(\ell)} \subseteq \mathbf{SH}^{\omega}(k)_{(\ell)}$, is an equality. This amounts to showing that $\mathbf{SH}(k)_{(\ell)}$ is in fact generated under sifted colimits by strongly dualizable objects. But this follows by Lemma II.2.2, which also verifies the theorem for the e-inverted case. ■

II.5.1.2

We now obtain the following extension of [RØ08, Theorem 1], [HKØ17, Theorem 5.8], [Gar19, Theorem 5.3] and [BF18, Lemma 5.3]:

Corollary II.5.3. *Let k be a field of exponential characteristic e and let*

$$\gamma_{\mathscr{C}} \colon \mathrm{Sm}_k \to \mathscr{C}$$

be a correspondence category over k. Then there is an equivalence of presentably symmetric monoidal stable ∞-categories

$$\mathbf{SH}(\mathscr{C})\left[\tfrac{1}{e}\right] \simeq \mathrm{Mod}_{\gamma_{\mathscr{C}*}\gamma_{\mathscr{C}}^*(\mathbf{1})\left[\frac{1}{e}\right]}(\mathbf{SH}(k)).$$

II.5.2 The case $\mathscr{S} = \mathbf{Reg}_k$

Following [CD15] we can extend the previous result to the category Reg_k of finite dimensional Noetherian schemes that are regular over a field k, provided that we impose some additional assumptions on $\mathscr{M}$. For the rest of this section, we will therefore assume that $\mathscr{M}$ is a motivic module category which in addition satisfies the following property:

- The premotivic category $\mathscr{M}$ satisfies localization (Definition II.2.5) and continuity (Definition II.2.6).

Lemma II.5.4. *Suppose that $f\colon T \to S$ is a morphism in Reg_k. In the following cases, the transformation*

$$f^*\gamma_* \to \gamma_* f^*$$

is an equivalence:

1. *The scheme T is an inverse limit $\varprojlim_\alpha T_\alpha$ of S-schemes T_α such that the transition maps $T_\alpha \to T_\beta$ are dominant, affine and smooth.*

2. *The map f is a closed immersion and $S \cong \varprojlim_\alpha S_\alpha$, where each S_α is a smooth, separated k-scheme of finite type with flat affine transition maps.*

Proof. Under the continuity and localization assumption on $\mathscr{M}$, the proof in [CD15, Lemma 3.20] for the case of $\mathscr{M} = \mathbf{DM}(-, R)$ applies verbatim. ∎

II.5.2.1

We now have the following extension of Theorem II.5.2.

Theorem II.5.5. *Let k be a field of exponential characteristic e, and let $\mathscr{M}$ be a motivic module category on Reg_k. Then the functor $\gamma^*\colon \mathbf{SH} \to \mathscr{M}$ induces a canonical equivalence*

$$\mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1})\left[\frac{1}{e}\right]}(\mathbf{SH}(-)) \xrightarrow{\simeq} \mathscr{M}\left[\tfrac{1}{e}\right]$$

of premotivic categories on Reg_k.

Proof. After Theorem II.3.5, our goal is to verify that $(\mathbf{SH}|_{\mathrm{Reg}_k}, \mathscr{M})$ satisfies the projection formula. Suppose that $S \in \mathrm{Reg}_k$, and let $E \in \mathbf{SH}(S)$. We claim that the map

$$\gamma_*\gamma^*(\mathbf{1}_S) \otimes E \to \gamma_*\gamma^*(E) \tag{II.10}$$

is an equivalence. To show this, we follow closely the logic of [CD15, Theorem 3.1].

First, assume that S is an essentially smooth scheme over a field. For each $x \in S$, we write S_x for the localization of S at x. Then the family of functors

$$\{\mathbf{SH}(S) \to \mathbf{SH}(S_x)\}$$

is conservative by [CD19, Proposition 4.3.9]. Hence we are reduced to proving that the map (II.10) is an equivalence in the case S is furthermore *local*. In this case, let $i\colon x \hookrightarrow S_x$ be the closed point and write $j\colon U_x \to S_x$ for the open complement. By our assumption on S, U_x has dimension $<$ dim S. We consider the following commutative diagram, where the rows are cofiber sequences:

$$\begin{tikzcd}
j_!(j^*\gamma_*\gamma^*(\mathbf{1}_S) \otimes j^*E) \arrow[r] \arrow[d] & \gamma_*\gamma^*(\mathbf{1}_S) \otimes E \arrow[r] \arrow[d] & i_*(i^*\gamma_*\gamma^*(\mathbf{1}_S) \otimes i^*E) \arrow[d] \\
j_!j^*\gamma_*\gamma^*(E) \arrow[r] \arrow[d, "f_1"'] & \gamma_*\gamma^*(E) \arrow[r] \arrow[d, "="'] & i_*i^*\gamma_*\gamma^*(E) \arrow[d, "f_2"] \\
j_!\gamma_*\gamma^*j^*E \arrow[r] & \gamma_*\gamma^*E \arrow[r] & i_*\gamma_*\gamma^*i^*E.
\end{tikzcd} \tag{II.11}$$

Now,

- The left vertical composite is an equivalence because (1) j^* commutes with γ_* by definition of a morphism of premotivic categories, and (2) by the induction hypothesis.

- The right vertical composite is an equivalence using (1) Lemma II.5.4.2 and (2) the case of fields, Theorem II.5.2.

It therefore remains to show that f_1 and f_2 are equivalences.

- The map f_1 is an equivalence because j^* commutes with γ_*.

- That f_2 is an equivalence follows from Lemma II.5.4.2.

Now, following the "*General case*" of [CD15], we explain how the bootstrap to regular k-schemes work. By continuity (appealing to [CD19, Proposition 4.3.9] again), we may again assume that S is a *Henselian local* regular k-scheme. As explained in *loc. cit.*, there is a sequence of regular Noetherian k-schemes

$$T \xrightarrow{f} S' \xrightarrow{q} S$$

such that the following hold:

- The scheme S' has infinite residue field and the functor $q^* \colon \mathbf{SH}(S)\left[\frac{1}{e}\right] \to \mathbf{SH}(S')\left[\frac{1}{e}\right]$ is conservative.

- The scheme T is the ∞-*gonflement* of $\Gamma(S', \mathcal{O}_{S'})$ [CD15, Definition 3.21] and the functor $f^* \colon \mathbf{SH}(S')\left[\frac{1}{e}\right] \to \mathbf{SH}(T)\left[\frac{1}{e}\right]$ is conservative.

- Both f and q satisfy the hypotheses of Lemma II.5.4.1, and thus f^* and q^* commute with γ_*.

Hence, to check that the map (II.10) is an equivalence it suffices to check that it is an equivalence after applying $(qf)^*$. Since T is, by construction, the spectrum of a filtered union of its smooth subalgebras we invoke continuity of $\mathbf{SH}$ to conclude. ∎

II.5.2.2

Lastly, we provide the following class of examples of motivic module categories for which localization and continuity holds. We will make the following assumption:

- for a base scheme S and an $\mathcal{A}_\infty$-ring spectrum $E \in \mathbf{SH}(S)$, there exists an ∞-category Cor_S^E such that its homotopy category is the hSpc-enriched category hCor_S^E of [Elm+20].

With this assumption in play, any motivic $\mathcal{A}_\infty$-ring spectrum E gives rise to the motivic module category $\mathbf{DM}^E$ as explained in Example II.4.24 and [Elm+20]. While this makes the next results conditional, we will explain unconditional instances of these results in Example II.5.9.

Proposition II.5.6. *Let $\mathscr{S} \subseteq \mathrm{Sch}_S$. Then, for any $\mathcal{A}_\infty$-ring spectrum $E \in \mathbf{SH}(S)$, the premotivic category $\mathbf{DM}^E\colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ satisfies continuity for dominant affine morphisms.*

Proof. We first claim that the analog of [CD19, Proposition 9.3.9] holds for finite E-correspondences. Let $(X_i)_{i\in I}$ be a cofiltered diagram of separated S-schemes of finite type with affine dominant transition morphisms. Let $X = \varprojlim_i X_i$, which is assumed to exist in Sch_S and is assumed to be Noetherian. Then we claim that for any separated S-scheme Y of finite type, the map

$$\operatorname*{colim}_{i\in I^{\mathrm{op}}} \mathrm{Cor}_S^E(X_i, Y) \to \mathrm{Cor}_S^E(X, Y) \tag{II.12}$$

is an equivalence.

To do so, we use the dual of [Elm+19a, Lemma 4.1.26]. Denote by c_{X_i} (resp. c_X) the filtered poset of reduced subschemes of $X_i \times_S Y$ (resp. $X \times_S Y$) which are finite and universally open over X_i (resp. X). Furthermore, we denote by $\mathrm{Sub}(c_X)$ the poset of full sub-posets of c_X. We then have a functor $K\colon I \to \mathrm{Sub}(c_X)$ given by $i \mapsto K_i := c_{X_i}$, where c_{X_i} is regarded as a full sub-poset in the obvious way. By continuity of $\mathbf{SH}$, the functor $E^{\mathrm{BM}}(-/X)\colon c_X \to \mathrm{Spc}$ of Borel–Moore E-homology spaces [Elm+20, §2] restricts to a functor $E^{\mathrm{BM}}(-/X_i)\colon c_{X_i} \to \mathrm{Spc}$. Hence the map (II.12) is, by [Elm+20, Definition 4.1.1], equal to the map

$$\operatorname*{colim}_{I^{\mathrm{op}}} \operatorname*{colim}_{c_{X_i}} E^{\mathrm{BM}}(Z_i/X_i) \to \operatorname*{colim}_{Z\in c_X} E^{\mathrm{BM}}(Z/X),$$

which we claim is an equivalence. The hypotheses of [Elm+19a, Lemma 4.1.26] follow easily (under the hypotheses that the transition maps are affine and dominant) by [CD19, Propositions 8.3.6, 8.3.9]. Hence the desired claim follows. The rest of the proof follows as in the case of $\mathbf{DM}$ from [CD19, Theorem 11.1.24]. ∎

Proposition II.5.7. *Let k be a field and let $E \in \mathbf{SH}(k)$ be an $\mathcal{A}_\infty$-ring spectrum. Then the premotivic category $\mathbf{DM}^E\colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ satisfies (Loc_i) whenever i is a closed immersion of regular schemes.*

Proof. Since $\mathbf{DM}^E$ is constructed from Nisnevich local objects, it is Nisnevich separated. By [CD19, Proposition 6.3.14], it has the weak localization property, i.e., it has (Loc_i) for any closed immersion with smooth retractions. Arguing as in [CD19, Corollary 6.3.15], it has the localization property with respect to any closed immersion between smooth schemes. The rest of the argument then follows as in [CD15, Proposition 3.12], which uses the continuity results established in Proposition II.5.6 as above. ■

II.5.2.3

From this we conclude:

Corollary II.5.8. *Let k be a field of exponential characteristic e and let $E \in \mathbf{SH}(k)$ be an $\mathcal{A}_\infty$-ring spectrum. Then we have a canonical equivalence*

$$\mathbf{DM}^E\left[\tfrac{1}{e}\right] \simeq \mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1})\left[\frac{1}{e}\right]}(\mathbf{SH}(-))$$

of premotivic categories on Reg_k.

Example II.5.9. As explained in [Elm+20, §4.1.19], the hypothetical ∞-category Cor_S^E is equivalent to hCor_S^E whenever S is essentially smooth over a perfect field k and E is pulled back from the heart of the effective homotopy t-structure $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ over k. Hence Theorem II.5.8 holds unconditionally whenever E is pulled back from the prime subfield of k and lies in the heart of the effective homotopy t-structure there.

Examples of such spectra include the motivic cohomology spectrum $\mathrm{H}\mathbf{Z}$ and its Milnor–Witt counterpart $\mathrm{H}\widetilde{\mathbf{Z}}$. Furthermore, in [Elm+20, Proposition 4.3.6] (resp. [Elm+20, Proposition 4.3.19]) it is proved that $\mathbf{DM}^{\mathrm{H}\mathbf{Z}}(S) \simeq \mathbf{DM}(S)$ (resp. $\mathbf{DM}^{\mathrm{H}\widetilde{\mathbf{Z}}}(S) \simeq \widetilde{\mathbf{DM}}(S)$) whenever S is essentially smooth over a Dedekind domain (resp. essentially smooth over a perfect field) [Elm+20, Proposition 4.3.8] (resp. [Elm+20, Proposition 4.3.19]). By the continuity result of Proposition II.5.6 we can enhance the comparison results for $\mathbf{DM}$ to regular schemes over fields. While $\widetilde{\mathbf{DM}}(S)$ is not defined outside of smooth schemes over perfect fields, Corollary II.5.8 promotes the comparison results between $\widetilde{\mathbf{DM}}$ and modules over $\mathrm{H}\widetilde{\mathbf{Z}}$ of [Gar19] and [BF18] at least to smooth schemes over fields. We contend, however, that $\mathbf{DM}^{\mathrm{H}\widetilde{\mathbf{Z}}}(S)$ is a decent definition for $\widetilde{\mathbf{DM}}(S)$ in general.

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Paper III

Cohomological correspondence categories

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Abstract

We prove that homotopy invariance and cancellation properties are satisfied by any category of correspondences that is defined, via Calmès and Fasel's construction, by an underlying cohomology theory. In particular, this includes any category of correspondences arising from the cohomology theory defined by an MSL-algebra.

III

Contents

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III.1 Introduction

Originally envisioned by Grothendieck, the theory of motives was set in new light by Beilinson's conjecture on the existence of certain *motivic complexes*, from which it should be possible to derive a satisfactory motivic cohomology theory. This point of view ultimately led to Suslin and Voevodsky's construction of the derived category of motives $\mathbf{DM}(k)$ over any field k [Voe00b]. The basic ingredient of this construction is the category Cor_k of finite correspondences over k. Finite correspondences define an additive category, and presheaves on this category—baptized *presheaves with transfers*—are exceptionally well behaved. Indeed, presheaves with transfers carry a very rich theory, satisfying fundamental properties such as preservation of homotopy invariance under sheafification [Voe00a], and a cancellation property with respect to smashing with $\mathbf{G}_m$ [Voe10]. These results are crucial in order to obtain a good category of motivic complexes.

Shortly after Suslin and Voevodsky's introduction of motivic complexes, a "nonlinear" version of $\mathbf{DM}(k)$ was defined by Morel and Voevodsky [MV99] in the context of motivic homotopy theory. In this more general setting, the motivic stable homotopy category $\mathbf{SH}(k)$ was constructed, most notably via the $\mathbf{A}^1$-localization and the $\mathbf{P}^1$-stabilization process. The category $\mathbf{SH}(k)$ is equipped with an adjunction

$$\gamma^* : \mathbf{SH}(k) \rightleftarrows \mathbf{DM}(k) : \gamma_* \tag{III.1}$$

such that the image of the unit for the symmetric monoidal structure on $\mathbf{DM}(k)$ is mapped to the motivic Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbf{Z}$ in $\mathbf{SH}(k)$ under γ_*. In fact, this adjunction exhibits $\mathbf{DM}(k)$ as the category of modules over the ring spectrum $\mathrm{H}\mathbf{Z}$ (at least after inverting the exponential characteristic of k) [RØ08]. Furthermore, the restriction of γ_* to the heart of the homotopy t-structure on $\mathbf{DM}(k)$ is fully faithful. In fact, with rational coefficients, the category $\mathbf{SH}(k)_{\mathbf{Q}}$ splits into a plus part and a minus part, where the plus part is equivalent to $\mathbf{DM}(k, \mathbf{Q})$ [CD19]. Informally we can think of $\mathbf{DM}(k, \mathbf{Q})$ as consisting of the oriented part of $\mathbf{SH}(k)_{\mathbf{Q}}$.

Several alternative and refined versions of the category of correspondences have been introduced in the wake of Suslin and Voevodsky's pioneering work, many of which attempt to provide a better approximation to the motivic stable homotopy category than $\mathbf{DM}(k)$. In particular, it is desirable to construct correspondences that capture also the unoriented information contained in $\mathbf{SH}(k)$. Examples include

- the category $\mathbf{Z}\mathrm{F}_*(k)$ of linear framed correspondences, introduced by Voevodsky and further developed by Garkusha and Panin [GP18a];
- $\mathrm{K}_0^{\oplus}$- and K_0-correspondences, studied by Suslin and Walker in [Sus03; Wal96];
- the category $\widetilde{\mathrm{Cor}}_k$ of finite Milnor–Witt correspondences, introduced by Calmès–Déglise–Fasel [CF17; DF17a]; and
- the category GWCor_k of finite Grothendieck–Witt correspondences defined by the first author in [Dru18b].

To exemplify to what extent the above categories succeed in providing better approximations to $\mathbf{SH}(k)$, let us mention that framed correspondences classify infinite $\mathbf{P}^1$-loop spaces [Elm+19], and the heart of the category $\widetilde{\mathbf{DM}}(k)$ associated to $\widetilde{\mathrm{Cor}}_k$ is equivalent to the heart of $\mathbf{SH}(k)$ (with respect to the homotopy t-structure) [AN19].

Along with the introduction of each new category of correspondences follows the need to prove fundamental properties like strict homotopy invariance and cancellation in order to produce a satisfactory associated derived category of motives. For the above examples, this is achieved in [AGP18; DF17a; Dru18a; Dru18c; FØ17; GP18b; Sus03]. The aim of this note is to establish these properties simultaneously for a certain class of correspondence categories, namely those that are defined by an underlying cohomology theory (see Definition III.3.1 for the precise meaning). This includes Voevodsky's finite correspondences—which can be defined using the cohomology theory CH^* of Chow groups—as well as finite Milnor–Witt correspondences $\widetilde{\mathrm{Cor}}_k$, which are defined using Chow–Witt groups $\widetilde{\mathrm{CH}}^*$. More generally, any ring spectrum $E \in \mathbf{SH}(k)$ that is an algebra over Panin and Walter's algebraic cobordism spectrum MSL [PW18] gives rise to a cohomological correspondence category.

III.1.1 Outline

In Section III.2 we introduce the axioms for a cohomology theory A^* needed to build the associated category Cor_k^A of finite A-correspondences. The definition of the category Cor_k^A is given in Section III.3. In addition we give in Section III.3 a number of constructions in the category Cor_k^A. Most notably, Construction III.3.11 ensures that a regular function on a smooth relative curve along with a trivialization of the relative canonical class gives rise to a finite A-correspondence; this construction is used to define all the finite A-correspondences needed to prove strict homotopy invariance and cancellation.

Section III.4 is a brief comparison between our construction of finite A-correspondences and framed correspondences. This is done by constructing a functor from the category of framed correspondences $\mathrm{Fr}_*(k)$ to Cor_k^A.

Sections III.5, III.6, III.7 and III.8 are devoted to the proof of the strict homotopy invariance property of homotopy invariant presheaves on Cor_k^A. The proof breaks down into several excision results as well as a moving lemma, each of which is treated in its own section.

In Section III.9 we show the cancellation theorem for finite A-correspondences, following the technique in Voevodsky's original proof [Voe10].

Finally, in Section III.10 we use the previous results to establish a well behaved category of motivic complexes $\mathbf{DM}_A(k)$ associated to the category Cor_k^A, and we show several properties expected of this category. In particular, we define A-motivic cohomology in this category, and show that $\mathbf{DM}_A(k)$ comes equipped with an adjunction to $\mathbf{SH}(k)$ parallelling (III.1). Note that these constructions are for the most part standard. For this reason we keep it rather brief on certain formal aspects of the constructions, and refer the interested reader to, e.g., [MVW06; Voe00b] or [DF17a] for further details.

Section III.A is a collection of the geometric results used in the proofs of the excision theorems.

III.1.2 Relationship to other works

In the independent project [Elm+20], the construction of the category Cor_k^E of Section III.3.1.1 is generalized to arbitrary ring spectra in $\mathbf{SH}(S)$ over a base scheme S. Let us also mention that functors from the category of framed correspondences to other correspondence categories have been considered by several authors. The original construction of a functor $\mathrm{Fr}_*(k) \to \widetilde{\mathrm{Cor}}_k$ from framed correspondences to finite Milnor–Witt correspondences was given by Déglise and Fasel in [DF17a]. In [Elm+20, §4.2], the functor of Déglise and Fasel was refined to an hSpc-enriched functor $\Phi^E \colon \mathrm{h}\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sch}_S) \to \mathrm{h}\mathbf{Corr}^E(\mathrm{Sch}_S)$ from the homotopy category of the ∞-category of framed correspondences to finite E-correspondences.

III.1.3 Conventions and notation

Throughout, the symbol k will denote a field, and the symbol $\mathbf{G}_m := \mathrm{Spec}(k[t^{\pm 1}])$ will denote the multiplicative group scheme over k. In certain sections we will also need to put some restrictions on the field k; this will be stated in the beginning of the relevant section.

By a *base scheme* we mean a noetherian scheme of finite Krull dimension. If S is a base scheme, we let Sm_S denote the category of schemes that are smooth, separated and of finite type over S. By an *essentially smooth scheme* we mean a scheme that is a projective limit of open immersions of smooth ones. We denote the category of essentially smooth schemes by EssSm_S. If $f \colon X \to Y$ is a morphism in Sm_S (or EssSm_S), we let $\omega_f := \omega_{X/S} \otimes f^*\omega_{Y/S}^{-1}$ denote the relative canonical sheaf. Moreover, we may write simply ω_Y for $\omega_{X \times_S Y/X}$. In the case of smooth (or essentially smooth) schemes $X, Y \in \mathrm{Sm}_k$ (or EssSm_k) over a field k, we will often abbreviate $X \times_k Y$ to $X \times Y$; $\mathbf{A}_k^n$ to $\mathbf{A}^n$ and $\mathbf{P}_k^n$ to $\mathbf{P}^n$. Throughout, we will let i_0 and i_1 denote the zero- respectively the unit section $i_0, i_1\colon \operatorname{Spec} k \to \mathbf{A}^1$. If we for example need to emphasize that $\mathbf{A}^2$ has coordinates (x,y), we may for brevity denote this by $\overset{(x,y)}{\mathbf{A}^2}$. This notation will in particular be used in the tables in Sections III.5, III.6, III.7 and III.8.

If $\mathscr{L}$ is a line bundle on a scheme X and $s \in \Gamma(X, \mathscr{L})$ is a section of $\mathscr{L}$, we will denote by $Z(s) \subseteq X$ the vanishing locus of s. We say that a section $s \in \Gamma(X, \mathscr{L})$ is *invertible* if the homomorphism $\mathcal{O}_X \to \mathscr{L}$ defined by s is an isomorphism.

We denote by $\operatorname{Map}_{\mathscr{C}}(X,Y)$ the mapping spaces of an ∞-category $\mathscr{C}$, and write $[X,Y]_{\mathscr{C}} := \pi_0 \operatorname{Map}_{\mathscr{C}}(X,Y)$. If $\mathscr{C}$ is any category, we denote by $\operatorname{PSh}(\mathscr{C}) := \operatorname{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Spc})$ the ∞-category of presheaves on $\mathscr{C}$, and for a ring R we denote by $\operatorname{PSh}(\mathscr{C}; R)$ the ∞-category of presheaves of R modules on $\mathscr{C}$. Moreover, we let $\operatorname{PSh}_{\Sigma}(\mathscr{C})$ denote the full subcategory of $\operatorname{PSh}(\mathscr{C})$ spanned by presheaves that carry finite coproducts to finite products [Lur09, §5.5.8].

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III.2 Twisted cohomology theories with support

Let S be a base scheme. We denote by $\mathrm{SmOp}_S^{\mathrm{L}}$ the category of triples $(X, U, \mathscr{L})$, where $X \in \mathrm{Sm}_S$ is separated, smooth and of finite type over S, U is an open subscheme of X and $\mathscr{L}$ is a line bundle on X. A morphism $(X, U, \mathscr{L}) \to (Y, V, \mathscr{M})$ in $\mathrm{SmOp}_S^{\mathrm{L}}$ consists of a pair (f, α) of a morphism of S-schemes $f\colon X \to Y$ such that $f(U) \subseteq V$, and an isomorphism $\alpha\colon \mathscr{L} \xrightarrow{\cong} f^*\mathscr{M}$. Note that there is an embedding $\mathrm{Sm}_S \to \mathrm{SmOp}_S^{\mathrm{L}}$ given by $X \mapsto (X, \varnothing, \mathcal{O}_X)$. For any $(X, U, \mathscr{L}) \in \mathrm{SmOp}_S^{\mathrm{L}}$, we will write i_U for the inclusion $i_U\colon U \to X$ and j_U for the inclusion $j_U\colon (X, \varnothing, \mathscr{L}) \to (X, U, \mathscr{L})$. In the case when $U = \varnothing$, we will often denote the triple $(X, \varnothing, \mathscr{L}) \in \mathrm{SmOp}_S^{\mathrm{L}}$ simply by $(X, \mathscr{L})$.

Definition III.2.1. A *twisted pre-cohomology theory* is a graded functor

$$A^*\colon (\mathrm{SmOp}_S^{\mathrm{L}})^{\mathrm{op}} \to \mathrm{Ab}^{\mathbf{Z}}$$

which satisfies the following properties:

(a) (Localization) There is a natural transformation

$$\partial\colon A^*(X,U,\mathscr{L}) \to A^{*+1}(U, i_U^*\mathscr{L})$$

of degree +1 which fits into an exact sequence

$$A^*(X,\mathscr{L}) \xrightarrow{i_U^*} A^*(U, i_U^*\mathscr{L}) \xrightarrow{\partial} A^{*+1}(X,U,\mathscr{L}) \xrightarrow{j_U^*} A^{*+1}(X,\mathscr{L}).$$

(b) (Étale excision) Suppose that $f\colon X \to Y$ is an étale morphism of smooth S-schemes. Assume moreover that $Z \subseteq Y$ is a closed subset such that $f|_{f^{-1}(Z)}\colon f^{-1}(Z) \to Z$ is an isomorphism. Then the pullback homomorphism

$$f^*\colon A^n(Y, Y \setminus Z, \mathscr{L}) \to A^n(X, X \setminus f^{-1}(Z), f^*\mathscr{L})$$

is an isomorphism for any line bundle $\mathscr{L}$ on Y and any $n \in \mathbf{Z}$.

If $(X,U,\mathscr{L}) \in \mathrm{SmOp}_S^{\mathrm{L}}$, let $Z := X \setminus U$ be the closed complement of U. We then write $A_Z^*(X,\mathscr{L}) := A^*(X,U,\mathscr{L})$. The map $j_U^*\colon A_Z^*(X,\mathscr{L}) \to A^*(X,\mathscr{L})$ is called the *extension of support-homomorphism*.

Remark III.2.2. Definition III.2.1 is but a twisted version of Panin and Smirnov's definition of a cohomology theory considered for example in [Pan09], except that for our purposes we need not assume the axiom of homotopy invariance. In the case of oriented homotopy invariant theories, our definition coincides with Panin and Smirnov's definition.

Remark III.2.3. The axiom of étale excision in Definition III.2.1 implies that there is a canonical isomorphism $A^*_{Z_1 \amalg Z_2}(X,\mathscr{L}) \cong A^*_{Z_1}(X,\mathscr{L}) \oplus A^*_{Z_2}(X,\mathscr{L})$, i.e., that the cohomology theory A^* also satisfies Zariski excision. In fact, Zariski excision is enough to prove most of the results below. The only places where we need étale excision are in the construction of the functor from framed correspondences to A-correspondences in Section III.4, and in the proof that A-transfers are preserved under Nisnevich sheafification (Theorem III.10.1). Furthermore, the latter case only requires étale excision on local schemes. In Corollary III.8.5 we show that a *homotopy invariant* cohomology theory satisfying Zariski excision will automatically satisfy étale excision on local schemes.

Definition III.2.4. Let A^* be a twisted pre-cohomology theory. Suppose that we in addition are given the following data:

1. (Pushforward) For any morphism $f\colon X \to Y \in \mathrm{Sm}_S$ of smooth equidimensional S-schemes of constant relative dimension d, and any closed subset $Z \subseteq X$ such that $f|_Z$ is finite, we have a *pushforward* homomorphism

$$f_*\colon A_Z^n(X, \omega_f \otimes f^*\mathscr{L}) \to A^{n-d}_{f(Z)}(Y,\mathscr{L})$$

for any $n \geq 0$ and any line bundle $\mathscr{L}$ on Y.

2. (External product) The cohomology theory is a *ring cohomology theory*, i.e., there is an associative product structure

$$\times\colon A^n_{Z_1}(X,\mathscr{L})\otimes A^m_{Z_2}(Y,\mathscr{M})\to A^{n+m}_{Z_1\times_S Z_2}(X\times_S Y,\mathscr{L}\boxtimes\mathscr{M})$$

and a unit $1\in A^0(S)$.

We say that a pre-cohomology theory A^* equipped with the homomorphisms f_* and the product $\times$ as above forms a *good cohomology theory* if the following properties hold:

3. (Pushforward functoriality) The homomorphisms f_* are functorial in the sense that $\mathrm{id}_*=\mathrm{id}$, and if $(X_1,U_1,\mathscr{L}_1)\xrightarrow{f}(X_2,U_2,\mathscr{L}_2)\xrightarrow{g}(X_3,U_3,\mathscr{L}_3)$ are composable morphisms in $\mathrm{SmOp}^{\mathrm{L}}_S$ finite on the supports $Z_i:=X_i\setminus U_i$, then the diagram

$$\begin{tikzcd}
A^{n-d_f}_{f(Z_1)}(X_2,\omega_g\otimes g^*\mathscr{L}_3) \arrow[r, "g_*"] & A^{n-d_{gf}}_{gf(Z_1)}(X_3,\mathscr{L}_3) \\
A^n_{Z_1}(X_1,\omega_f\otimes f^*\mathscr{L}_2) \arrow[u, "f_*"] \arrow[ur, "(gf)_*"'] &
\end{tikzcd}$$

is commutative. Here d_f, d_g and d_{gf} are the respective relative dimensions of the morphisms.

4. (External product functoriality) The external product $\times$ commutes with pullbacks in the sense that if $f\colon(X,f^*\mathscr{L})\to(Y,\mathscr{L})$ and $g\colon(X',g^*\mathscr{L}')\to(Y',\mathscr{L}')$ are morphisms in $\mathrm{SmOp}^{\mathrm{L}}_S$, then the diagram

$$\begin{tikzcd}
A^n(Y,\mathscr{L})\otimes A^m(Y',\mathscr{L}') \arrow[r, "\times"] \arrow[d, "f^*\otimes g^*"'] & A^{n+m}(Y\times_S Y',\mathscr{L}\boxtimes\mathscr{L}') \arrow[d, "(f\times g)^*"] \\
A^n(X,f^*\mathscr{L})\otimes A^m(X',g^*\mathscr{L}') \arrow[r, "\times"] & A^{n+m}(X\times_S X',f^*\mathscr{L}\boxtimes g^*\mathscr{L}')
\end{tikzcd}$$

is commutative.

5. (Base change) For any strongly transversal square (defined in Definition III.2.6) that is equipped with a set of compatible line bundles (defined in Definition III.2.7) the diagram

$$\begin{tikzcd}
A^n_{\phi_Y^{-1}(Z)}(Y',\mathscr{M}') \arrow[r, "i'_*"] & A^{n-d'}_{i'(\phi_Y^{-1}(Z))}(X',\mathscr{L}') \\
A^n_Z(Y,\mathscr{M}) \arrow[u, "\phi_Y^*"] \arrow[r, "i_*"] & A^{n-d}_{i(Z)}(X,\mathscr{L}), \arrow[u, "\phi_X^*"']
\end{tikzcd}$$

is commutative.

6. (Projection formula) Suppose that $f\colon X \to Y$ and $Z \subseteq X$ satisfy the hypotheses of (1), and let $W \subseteq Y$ be a closed subset. Let moreover $\mathscr{L}$ and $\mathscr{M}$ be two line bundles on Y. Given any two cohomology classes $\alpha \in A^n_Z(X, \omega_f \otimes f^*\mathscr{L})$ and $\beta \in A^m_W(Y, \mathscr{M})$, we then have
$$f_*(\alpha) \smile \beta = f_*(\alpha \smile f^*\beta).$$
7. (Graded commutativity) For any $\alpha \in A^n_Z(X, \mathscr{L})$ and $\beta \in A^m_Z(X, \mathscr{L})$, we have
$$\alpha \smile \beta = \langle -1 \rangle^{nm}(\beta \smile \alpha).$$
Here $\langle -1 \rangle \in A^0(S)$ is given as the pushforward $\langle -1 \rangle := (\mathrm{id}_S, -1)_*(1)$; see Definition III.3.16. Hence the ring $A^*(S)$ is $\langle -1 \rangle$-graded commutative.

Remark III.2.5. The existence of an external product $\times$ as in Definition III.2.4 (2) is equivalent to the existence of a cup product
$$\smile\colon A^n_{Z_1}(X, \mathscr{L}) \otimes A^m_{Z_2}(X, \mathscr{M}) \to A^{n+m}_{Z_1 \cap Z_2}(X, \mathscr{L} \otimes \mathscr{M});$$
see [Pan09, Definition 1.5] for further details on this.

Definition III.2.6. Let
$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "\phi_Y"'] & X' \arrow[d, "\phi_X"] \\ Y \arrow[r, "i"] & X \end{tikzcd} \tag{III.2}$$
be a Cartesian square of smooth S-schemes. The square (III.2) is called *transversal* if the corresponding sequence
$$0 \to g^*(\Omega_X) \to \phi_Y^*(\Omega_Y) \oplus {i'}^*(\Omega_{X'}) \to \Omega_{Y'} \to 0$$
is exact, where $g := \phi_X \circ i' = i \circ \phi_Y$. Note that for any transversal square, the isomorphism $d\phi_Y$ induces an isomorphism $d\phi_Y : \phi_Y^*\omega_i \xrightarrow{\cong} \omega_{i'}$.

A transversal square (III.2) is called *strongly transversal* if one of the following two conditions are satisfied:

- The morphisms i and i' are closed embeddings.
- The morphisms ϕ_X and ϕ_Y are smooth and surjective.

Definition III.2.7. Suppose that the square (III.2) is strongly transversal. Then a *compatible set of line bundles* on the square (III.2) consists of the following data:

- Line bundles $\mathscr{L}, \mathscr{L}', \mathscr{M}, \mathscr{M}'$ on respectively X, X', Y and Y'.
- Isomorphisms of line bundles
$$\alpha\colon \phi_X^*\mathscr{L} \xrightarrow{\cong} \mathscr{L}'; \quad \gamma\colon i^*\mathscr{L} \otimes \omega_i \xrightarrow{\cong} \mathscr{M};$$
$$\beta\colon \phi_Y^*\mathscr{M} \xrightarrow{\cong} \mathscr{M}'; \quad \delta\colon (i')^*\mathscr{L}' \otimes \omega_{i'} \xrightarrow{\cong} \mathscr{M}'.$$

We furthermore require that $\beta \circ \phi_Y^*(\gamma)$ corresponds to $\delta \circ ((i')^*(\alpha) \otimes \mathrm{id}_{\omega_{i'}})$ under the isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{Y'}}(\phi_Y^* i^* \mathscr{L} \otimes \phi_Y^* \omega_i, \mathscr{M}') \cong \mathrm{Hom}_{\mathcal{O}_{Y'}}((i')^* \phi_X^* \mathscr{L} \otimes \omega_{i'}, \mathscr{M}')$$

induced by the canonical isomorphism $\phi_Y^* \omega_i \cong \omega_{i'}$ for the transversal square.

III.3 Cohomological correspondences

We are now ready to extend Calmès and Fasel's definition of finite Milnor–Witt correspondences [CF17] to our setting:

Definition III.3.1. Let S be a connected base scheme, and suppose that A^* is a good cohomology theory on $\mathrm{SmOp}_S^{\mathrm{L}}$. Assume further that $p\colon X \to S$ is a smooth map of constant relative dimension d. Denote by $\mathcal{A}_0(X/S)$ the set of *admissible subsets*[1] *of X relative to S*—that is, closed subsets T of X such that each irreducible component of T_{red} is finite and surjective over S via the morphism p. The set $\mathcal{A}_0(X/S)$ is partially ordered by inclusions. As the empty set has no irreducible components, it is admissible. If X is connected, we define the group of *finite relative A-cycles on X* as

$$\mathrm{C}_0^A(X/S) := \varinjlim_{T \in \mathcal{A}_0(X/S)} A_T^d(X, \omega_{X/S}).$$

If X is not connected, we may write $X = \coprod_j X_j$ where the X_j's are the connected components of X. We then set $\mathrm{C}_0^A(X/S) := \prod_j \mathrm{C}_0^A(X_j/S)$.

Now let k be a field, and suppose further that $S \in \mathrm{Sm}_k$. Let Cor_S^A denote the category whose objects are the same as the objects of Sm_S, i.e., smooth separated schemes of finite type over S, and morphisms defined as follows. Let $X, Y \in \mathrm{Sm}_S$, and suppose first that X and Y are connected. We define the group of *finite relative A-correspondences from X to Y as*

$$\mathrm{Cor}_S^A(X, Y) := \mathrm{C}_0^A(X \times_S Y/X).$$

Note in particular that $\mathrm{Cor}_S^A(X, S) = A^0(X)$ for any $X \in \mathrm{Sm}_S$. If X or Y is not connected, let $X = \coprod_i X_i$ and $Y = \coprod_j Y_j$ denote the connected components of X and Y. Then we put $\mathrm{Cor}_S^A(X, Y) := \prod_{i,j} \mathrm{Cor}_S^A(X_i, Y_j)$. If $S = \operatorname{Spec} k$, we refer to $\mathrm{Cor}_k^A(X, Y)$ simply as the group of *finite A-correspondences from X to Y*.

Composition of finite relative A-correspondences is defined in an identical manner as [CF17, §4.2]. Indeed, if $\alpha \in \mathrm{Cor}_S^A(X, Y)$ and $\beta \in \mathrm{Cor}_S^A(Y, Z)$, we put

$$\beta \circ \alpha := (p_{XZ})_*(p_{XY}^* \alpha \smile p_{YZ}^* \beta). \tag{III.3}$$

[1]Note that for any $X, Y \in \mathrm{Sm}_k$ we have $\mathcal{A}_0(X \times Y/X) = \mathcal{A}(X, Y)$, where $\mathcal{A}(X, Y)$ is the set of admissible subsets of $X \times Y$ in the sense of [CF17, Definition 4.1].

Here we write p_{XY} for the projection $p_{XY}\colon X\times_S Y\times_S Z\to X\times_S Y$, and similarly for the other two maps. An identical proof as that of [CF17, Lemma 4.13] then shows that the groups $\mathrm{Cor}^A_S(X,Y)$ form the mapping sets of a (discrete) category Cor^A_S whose objects are the same as those of Sm_S. We refer to Cor^A_S as the category of *finite relative A-correspondences*. In the case when $S=\operatorname{Spec} k$, we refer to Cor^A_k simply as the category of *finite A-correspondences*.

Finally, we define the *homotopy category* $\overline{\mathrm{Cor}}^A_S$ of Cor^A_S as follows. The objects of $\overline{\mathrm{Cor}}^A_S$ are the same as those of Cor^A_S, and the morphisms are given by

$$\begin{aligned}\overline{\mathrm{Cor}}^A_S(X,Y) &:= \mathrm{Cor}^A_S(X,Y)/\sim_{\mathbf{A}^1}\\ &= \operatorname{coker}\left(\mathrm{Cor}^A_S(\mathbf{A}^1_S\times_S X,Y)\xrightarrow{i_0^*-i_1^*}\mathrm{Cor}^A_S(X,Y)\right).\end{aligned}$$

We write $[\alpha]$ for the class in $\overline{\mathrm{Cor}}^A_S$ of a finite relative A-correspondence α from X to Y.

III.3.0.1 Graph functors

We define a graph functor $\gamma_{A,S}\colon \mathrm{Sm}_S\to\mathrm{Cor}^A_S$ similarly as [CF17, §4.3]: the functor $\gamma_{A,S}$ is the identity on objects, and if $f\colon X\to Y$ is a morphism in Sm_S, we let $\gamma_{A,S}(f):=i_*(1)$. Here $i\colon\Gamma_f\to X\times_S Y$ is the embedding of the graph of f, and $i_*\colon A^0(\Gamma_f,\mathcal{O}_{\Gamma_f})\to A^{\dim Y}_{\Gamma_f}(X\times_S Y,\omega_Y)$ is the induced pushforward. If $S=\operatorname{Spec} k$, we will write γ_A for the graph functor. We will often abuse notation and write simply f instead of $\gamma_{A,S}(f)$.

III.3.0.2 Symmetric monoidal structure

Defining $X\oplus Y:=X\amalg Y$ turns Cor^A_S into an additive category with zero-object the empty scheme. Moreover, Cor^A_S is symmetric monoidal, with tensor product $\otimes$ defined by $X\otimes Y:=X\times_S Y$ on objects, and given by the external product on morphisms.

Lemma III.3.2. *The category Cor^A_k is a (discrete) correspondence category in the sense of [EK20, Definition 4.1] (see also [Gar19, §2]).*

Proof. This follows from [EK20, Proposition 4.5]. ∎

III.3.0.3

For S a smooth k-scheme there is a functor $\mathrm{ext}_S\colon\mathrm{Cor}^A_k\to\mathrm{Cor}^A_S$ defined as follows. For any $X\in\mathrm{Sm}_k$, let $X_S:=X\times_k S$. Let $X,Y\in\mathrm{Sm}_k$; by working with one connected component at a time, we may assume that X and Y are connected. By the universal property of fiber products we have a morphism $f\colon X_S\times_S Y_S\to X\times Y$, which induces a pullback morphism

$$f^*\colon A^{\dim Y}_T(X\times Y,\omega_Y)\to A^{\dim Y}_{f^{-1}(T)}(X_S\times_S Y_S,f^*\omega_Y)$$

for any $T \in \mathcal{A}_0(X \times Y/X)$. As finiteness and surjectivity are preserved under base change we have $f^{-1}(T) \in \mathcal{A}_0(X_S \times_S Y_S/X_S)$. Moreover, the canonical sheaf $\omega_{X/k}$ pulls back over X_S to $\omega_{X_S/S}$, and similarly for $\omega_{Y/k}$. Hence $f^*\omega_{X\times Y/X} \cong \omega_{X_S\times_S Y_S/X_S}$. Since pullbacks commute with extension of support, we get an induced map on the colimit

$$\mathrm{ext}_S\colon \mathrm{Cor}_k^A(X,Y) \to \mathrm{C}_0^A(X_S \times_S Y_S/X_S) = \mathrm{Cor}_S^A(X_S, Y_S).$$

It follows from the base change axiom applied to the diagram

$$\begin{tikzcd}
X_S \times_S Y_S \times_S Z_S \arrow[r, "p_{X_S Y_S}"] \arrow[d, "f_{XYZ}"'] & X_S \times_S Y_S \arrow[d, "f_{XY}"] \\
X \times Y \times Z \arrow[r, "p_{XY}"] & X \times Y
\end{tikzcd}$$

that the map ext_S preserves composition of finite A-correspondences. Thus we obtain a functor $\mathrm{ext}_S\colon \mathrm{Cor}_k^A \to \mathrm{Cor}_S^A$.

III.3.0.4

In the opposite direction there is a "forgetful" functor $\mathrm{res}_S\colon \mathrm{Cor}_S^A \to \mathrm{Cor}_k^A$ induced by pushforwards. Indeed, let $X, Y \in \mathrm{Sm}_S$. Then there is a Cartesian diagram

$$\begin{tikzcd}
X \times_S Y \arrow[r, "i_{XY}"] \arrow[d] & X \times Y \arrow[d] \\
\Delta_S \arrow[r, "i"] & S \times S,
\end{tikzcd}$$

where $\Delta_S \subseteq S \times S$ denotes diagonal. Moreover, we have isomorphisms

$$\omega_{X\times_S Y} \otimes i_{XY}^*\omega_{X\times Y}^{-1} \cong \omega_{i_{XY}} \cong \omega_i \cong \omega_S^{-1}.$$

Thus there is, for any $T \in \mathcal{A}_0(X \times_S Y/X)$, a pushforward homomorphism

$$(i_{XY})_*\colon A_T^{\dim_S Y}(X \times_S Y, \omega_{Y/S}) \to A_{i_{XY}(T)}^{\dim Y}(X \times Y, \omega_Y).$$

Passing to the colimit, we obtain a map $\mathrm{res}_S\colon \mathrm{Cor}_S^A(X,Y) \to \mathrm{Cor}_k^A(X,Y)$. To show that this homomorphism preserves composition in the category Cor_S, first note that the commutative diagram

$$\begin{tikzcd}
X \times_S Y \times_S Z \arrow[r, "i_{XYZ}"] \arrow[d, "p_{X\times_S Z}"'] & X \times Y \times Z \arrow[d, "p_{XZ}"] \\
X \times_S Z \arrow[r, "i_{XZ}"] & X \times Z
\end{tikzcd}$$

yields $(i_{XZ})_*(p_{X\times_S Z})_* = (p_{XY})_*(i_{XYZ})_*$. By decomposing the morphism i_{XYZ} as

$$i_{XYZ}\colon X \times_S Y \times_S Z \xrightarrow{i_X} X \times Y \times_S Z \xrightarrow{i_Y} X \times Y \times Z$$

and applying the projection formula twice, we obtain the claim. Hence the maps res_S above define a functor $\mathrm{res}_S\colon \mathrm{Cor}_S^A \to \mathrm{Cor}_k^A$.

III.3.0.5

For any $X \in \mathrm{Sm}_S$, $Y \in \mathrm{Sm}_k$ and any admissible subset T of $X \times Y$ we have a natural isomorphism

$$A_T^{\dim Y}(X \times Y, \omega_Y) \cong A_T^{\dim_S Y_S}(X \times_S Y_S, \omega_{X \times_S Y_S/X}).$$

These isomorphisms define a natural isomorphism $\mathrm{Cor}_k^A(X, Y) \cong \mathrm{Cor}_S^A(X, Y_S)$. Similarly as in [CF17, §6.2] we deduce from this that the functors res_S and ext_S form an adjunction $\mathrm{res}_S : \mathrm{Cor}_S^A \rightleftarrows \mathrm{Cor}_k^A : \mathrm{ext}_S$.

III.3.1 Examples of cohomological correspondence categories

Different choices for the cohomology theory A^* recover various known correspondence categories, as well as new ones. For example, if $A^* = \mathrm{CH}^*$ is the theory of Chow groups, then the definition of Cor_k^A gives back Voevodsky's category Cor_k of finite correspondences. If the ground field k is perfect and of characteristic not 2, then we can let A^* be Chow–Witt theory, i.e., $A^* = \widetilde{\mathrm{CH}}^*$. In this case we obtain Calmès–Déglise–Fasel's category $\widetilde{\mathrm{Cor}}_k$ of finite Milnor–Witt correspondences. On the other hand, we can also define a good cohomology theory A^* by letting $A_T^n(X, \mathscr{L}) := \mathrm{H}_T^n(X, \mathbf{I}^n, \mathscr{L})$, where $\mathbf{I}^n$ is the Nisnevich sheaf of powers of the fundamental ideal. Then Cor_k^A is the category WCor_k of finite Witt-correspondences considered in [CF17, Remark 5.16]. Note that WCor_k thus defined differs from the category of Witt correspondences defined in [Dru16]; however, arguing similarly as in [BF18] one can show that the associated derived categories of motives are equivalent after inverting the exponential characteristic of the ground field.

III.3.1.1 Algebras over MSL

More generally, we claim that any ring spectrum $E \in \mathbf{SH}(k)$ that is an algebra over MSL defines a cohomological correspondence category. Here $\mathrm{MSL} \in \mathbf{SH}(k)$ denotes the ring spectrum constructed by Panin and Walter in [PW18].

In order to show this, let us first recollect a few notions from the formalism of six functors. Let $X \in \mathrm{Sm}_k$, and suppose that $i\colon Z \subseteq X$ is a closed subscheme. Let moreover $p\colon X \to \operatorname{Spec} k$ be the structure map. We then have adjunctions

$$p^* : \mathbf{SH}(k) \rightleftarrows \mathbf{SH}(X) : p_*$$

and

$$i_! : \mathbf{SH}(Z) \rightleftarrows \mathbf{SH}(X) : i^!.$$

If $q\colon \mathcal{E} \to X$ is a vector bundle on X, let $s\colon X \to \mathcal{E}$ denote the zero section. Recall from [Hoy17, §5.2] that this defines *Thom transformations*

$$\Sigma^{\mathcal{E}} := q_\# s_* : \mathbf{SH}(X) \rightleftarrows \mathbf{SH}(X) : s^! q^* =: \Sigma^{-\mathcal{E}}.$$

In fact, these functors are defined for any $\xi \in \mathrm{K}(X)$ [BH18, §16.2].

Definition III.3.3 ([DF17b; Elm+20])**.** Let $E \in \mathbf{SH}(k)$ be a spectrum and let X, Z be as above. Let furthermore $\xi \in \mathrm{K}(Z)$. The *ξ-twisted cohomology of X with support on Z and coefficients in E* is the space

$$E_Z(X, \xi) := \mathrm{Map}_{\mathbf{SH}(k)}(\mathbf{1}_k, p_* i_! \Sigma^\xi i^! p^* E),$$

where $\mathbf{1}_k \in \mathbf{SH}(k)$ denotes the motivic sphere spectrum. The associated bigraded *twisted cohomology groups with support* are then given as

$$E_Z^{p,q}(X, \xi) := [\mathbf{1}_k, \Sigma^{p,q} p_* i_! \Sigma^\xi i^! p^* E]_{\mathbf{SH}(k)}.$$

Proposition III.3.4. *Suppose that $E \in \mathbf{SH}(k)$ is an* MSL*-algebra. Let $X \in \mathrm{Sm}_k$, and suppose that $i\colon Z \subseteq X$ is a closed subscheme. For any line bundle $\mathscr{L}$ on X, set*

$$A_Z^n(X, \mathscr{L}) := E_Z^{2n,n}(X, i^*\mathscr{L})$$

Then $A_Z^*(X, \mathscr{L})$ defines a good cohomology theory and hence a cohomological correspondence category Cor_k^E.

Proof. The proposition follows from the six operations on $\mathbf{SH}(k)$, as explained in [DF17b; DJK18] or [Elm+20]. Indeed, for the contravariant functoriality we refer to [DF17b, §2.2], and for the definition of the cup product, see [DF17b, §2.3.1]. The pushforward is given by the Gysin map

$$f_!\colon E_Z(X, f^*\xi + \mathbf{L}_f) \to E_{f(Z)}(Y, \xi),$$

where $\mathbf{L}_f \in \mathrm{K}(X)$ is the cotangent complex of f; see [DJK18; Elm+20]. In particular, for MSL we have the Thom isomorphism

$$\Sigma^\xi \mathrm{MSL} \simeq \Sigma^{2\,\mathrm{rk}\,\xi, \mathrm{rk}\,\xi} \Sigma^{\det \xi - \mathcal{O}} \mathrm{MSL};$$

see [BH18, Example 16.29]. When ξ is a line bundle $\mathscr{L}$, this gives the pushforward

$$f_*\colon A_Z^n(X, \omega_f \otimes f^*\mathscr{L}) \to A_{f(Z)}^{n-d}(Y, \mathscr{L}).$$

For the base change and projection formulas, see [DF17b, Proposition 2.2.5] and [DF17b, Remark 2.3.2]. ■

III.3.2 Presheaves on Cor_k^A

Our basic object of study is the ∞-category $\mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A; \mathbf{Z})$ of presheaves of abelian groups on Cor_k^A that take finite coproducts to finite products. More generally we may of course also consider, for any coefficient ring R, the ∞-category $\mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A; R)$ of presheaves of R-modules. For notational simplicity we will however mostly work with $R = \mathbf{Z}$.

Definition III.3.5. The objects of $\mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A; \mathbf{Z})$ will be referred to as *presheaves with A-transfers*.

A presheaf with A-transfers $\mathscr{F} \in \mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A; \mathbf{Z})$ is *homotopy invariant* if for any $X \in \mathrm{Sm}_k$, the map $\mathrm{pr}^*\colon \mathscr{F}(X) \xrightarrow{\cong} \mathscr{F}(X \times \mathbf{A}^1)$ induced by the projection $\mathrm{pr}\colon X \times \mathbf{A}^1 \to X$ is an isomorphism.

III.3.2.1

The ∞-category $\mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A;\mathbf{Z})$ inherits a symmetric monoidal structure from that on Cor_k^A via Day convolution. Moreover, the graph functor $\gamma_A\colon \mathrm{Sm}_k \to \mathrm{Cor}_k^A$ defines a "forgetful" functor $\gamma_*^A\colon \mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A;\mathbf{Z}) \to \mathrm{PSh}_\Sigma(\mathrm{Sm}_k)$ given by $\gamma_*^A(\mathscr{F}) := \mathscr{F} \circ \gamma_A$. Similarly as in [DF17a, §1.2], we deduce that the functor γ_*^A admits a left adjoint γ_A^* which is symmetric monoidal.

III.3.2.2 Sheaves on $\mathbf{Cor}_{k}^{A}$

For any Grothendieck topology τ, we define the ∞-category $\mathrm{Shv}_\tau(\mathrm{Cor}_k^A;\mathbf{Z})$ consisting of those presheaves $\mathscr{F} \in \mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A;\mathbf{Z})$ such that $\gamma_*^A(\mathscr{F})$ is a τ-sheaf on Sm_k. The adjunction (γ_A^*, γ_*^A) above then defines an adjunction

$$\gamma_A^* : \mathrm{Shv}_\tau(\mathrm{Sm}_k) \rightleftarrows \mathrm{Shv}_\tau(\mathrm{Cor}_k^A;\mathbf{Z}) : \gamma_*^A,$$

and the symmetric monoidal structure on $\mathrm{PSh}(\mathrm{Cor}_k^A;\mathbf{Z})$ restricts to a symmetric monoidal structure on $\mathrm{Shv}_\tau(\mathrm{Cor}_k^A;\mathbf{Z})$.

III.3.2.3

In this text, we will almost exclusively work with the case when $\tau = \mathrm{Nis}$ is the Nisnevich topology. We show below (see Theorem III.10.1) that the full inclusion $i\colon \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Cor}_k^A;\mathbf{Z}) \to \mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A;\mathbf{Z})$ admits a left adjoint

$$a_{\mathrm{Nis}}\colon \mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A;\mathbf{Z}) \to \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Cor}_k^A;\mathbf{Z}).$$

In particular, the Nisnevich sheafification of a presheaf on Cor_k^A comes equipped with A-transfers in a canonical way. Hence we can make the following definition:

Definition III.3.6. Let $X \in \mathrm{Sm}_k$ be a smooth k-scheme. Following the notation of [CF17], we let $\mathrm{c}_A(X) \in \mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A;\mathbf{Z})$ denote the representable presheaf on Cor_k^A given by $U \mapsto \mathrm{Cor}_k^A(U,X)$. Moreover, we let

$$\mathbf{Z}_A(X) := a_{\mathrm{Nis}}(\mathrm{c}_A(X)) \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Cor}_k^A;\mathbf{Z})$$

denote the Nisnevich sheaf associated to the presheaf $\mathrm{c}_A(X)$.

III.3.3 Correspondences of pairs

In the excision theorems of Sections III.6 and III.8 we are always in the setting of a pair of schemes $j\colon U \subseteq X$, and we are led to consider the associated quotient $\mathrm{coker}(j^*\colon \mathscr{F}(X) \to \mathscr{F}(U))$ for a given presheaf with A-transfers. In particular, if $U = X$ and j is the identity, then the associated quotient is zero. The notion of a correspondence of pairs provides a natural setting to study these objects.

Definition III.3.7. Let $\mathrm{Cor}_S^{A,\mathrm{pr}}$ denote the category whose objects are those of SmOp_S and whose morphisms are defined as follows. For $(X,U),(Y,V) \in$

SmOp_S, with open immersions $j_X\colon U \to X$ and $j_Y\colon V \to Y$, consider the complex

$$\mathrm{Cor}_S^A(X,V) \xrightarrow{d_0} \mathrm{Cor}_S^A(X,Y) \oplus \mathrm{Cor}_S^A(U,V) \xrightarrow{d_1} \mathrm{Cor}_S^A(U,Y)$$

in which $d_0 := ((j_Y)_*, j_X^*)$ and $d_1 := j_X^* - (j_Y)_*$. We define the group $\mathrm{Cor}_S^{A,\mathrm{pr}}((X,U),(Y,V))$ of *finite relative A-correspondences of pairs* as the homology of this complex, i.e.,

$$\mathrm{Cor}_S^{A,\mathrm{pr}}((X,U),(Y,V)) := \ker d_1 / \operatorname{im} d_0.$$

In particular, if $U = X$, then $\mathrm{Cor}_S^{A,\mathrm{pr}}((X,X),(Y,V)) = 0$. We denote the elements of $\mathrm{Cor}_S^{A,\mathrm{pr}}((X,U),(Y,V))$ by (α,β), where $\alpha \in \mathrm{Cor}_S^A(X,Y)$ and $\beta \in \mathrm{Cor}_S^A(U,V)$. If β is implicitly understood, we may write simply α instead of (α,β). The composition in $\mathrm{Cor}_S^{A,\mathrm{pr}}$ is defined by $(\alpha,\beta)\circ(\gamma,\delta) := (\alpha\circ\gamma, \beta\circ\delta)$.

Finally, we define the homotopy category $\overline{\mathrm{Cor}}_S^{A,\mathrm{pr}}$ of $\mathrm{Cor}_S^{A,\mathrm{pr}}$ as follows. The objects of $\overline{\mathrm{Cor}}_S^{A,\mathrm{pr}}$ are the same as those of $\mathrm{Cor}_S^{A,\mathrm{pr}}$, and the morphisms are given by

$$\begin{aligned}\overline{\mathrm{Cor}}_S^{A,\mathrm{pr}}((X,U),(Y,V)) &:= \mathrm{Cor}_S^{A,\mathrm{pr}}((X,U),(Y,V))/\sim_{\mathbf{A}^1}\\ &= \operatorname{coker}\left(\mathrm{Cor}_S^{A,\mathrm{pr}}(\mathbf{A}_S^1 \times_S (X,U),(Y,V)) \xrightarrow{i_0^*-i_1^*} \mathrm{Cor}_S^{A,\mathrm{pr}}((X,U),(Y,V))\right).\end{aligned}$$

Here $\mathbf{A}_S^1 \times_S (X,U)$ is shorthand for $(\mathbf{A}_S^1 \times_S X, \mathbf{A}_S^1 \times_S U)$. If

$$(\alpha,\beta) \in \mathrm{Cor}_S^{A,\mathrm{pr}}((X,U),(Y,V))$$

is a finite relative A-correspondence of pairs, we write $[(\alpha,\beta)]$, or simply $[\alpha]$, for the image of (α,β) in $\overline{\mathrm{Cor}}_S^{A,\mathrm{pr}}((X,U),(Y,V))$.

III.3.4 Correspondences between essentially smooth schemes

We will frequently encounter local- and henselian local schemes, and we need to consider correspondences also between such objects. The definitions and results below take care of this. We remind the reader that the definition of an étale neighborhood can be found in Definition III.A.4 in the appendix.

Definition III.3.8. Let $X = \varprojlim X_\alpha \in \mathrm{EssSm}_S$ be an essentially smooth S-scheme. Consider a closed subscheme $T = \varprojlim T_\alpha$ of X, where T_α is a closed subscheme of X_α for each α. Define

$$A_T^n(U \times_S X, \omega_X) := \varinjlim_\alpha A_{T_\alpha}^n(U \times_S X_\alpha, \omega_{X_\alpha}).$$

Furthermore, for any $U = \varprojlim_\alpha U_\alpha \in \mathrm{EssSm}_S$, and for any $X \in \mathrm{Sm}_S$, we define

$$\mathrm{Cor}_S^A(U,X) := \varinjlim_\alpha \mathrm{Cor}_S^A(U_\alpha, X).$$

Finally, for any $X \in \mathrm{Sm}_S$, any point $x \in X$, and any $U \in \mathrm{EssSm}_S$, we put

$$\mathrm{Cor}_S^A(U, X_x^h) := \varprojlim_v \mathrm{Cor}_S^A(U, X').$$

Here the limit ranges over all étale neighborhoods $v\colon (X', x) \to (X, x)$ of x in X.

Lemma III.3.9. *For any $X \in \mathrm{Sm}_S$ of relative dimension d over S, and for any henselian local scheme $U \in \mathrm{EssSm}_S$, we have*

$$A_T^d(U \times_S X, \omega_X) = \bigoplus_{x \in X} A_{T_x}^d(U \times_S X_x, \omega_X) = \bigoplus_{x \in X} A_{T_x^h}^d(U \times_S X_x^h, \omega_{X_x^h})$$

for any $T \in \mathcal{A}_0(U \times_S X/U)$. Here x ranges over the set of all (not necessarily closed) points of X, and $T_x := T \times_X X_x$; $T_x^h := T \times_X X_x^h$.

Proof. Since U is henselian local and $T \in \mathcal{A}_0(U \times_S X/U)$ is finite over U, it follows that T is a semi-local henselian scheme. In fact, $T = \coprod_{z \in T_{(0)}} T_z^h$, where z ranges over the set of closed points in T. Hence $T = \coprod_{x \in X} T_x$ and $T = \coprod_{x \in X} T_x^h$, where x ranges over the set of all points of X. In particular we have $T_x = T_x^h$. We note that T_x^h is semi-local henselian, but not necessarily local. By Zariski excision, we obtain

$$A_T^d(U \times_S X, \omega_X) = \bigoplus_{x \in X} A_{T_x}^d(U \times_S X, \omega_X),$$

and

$$A_{T_x}^d(U \times_S X', \omega_X) = A_{T_x}^d(U \times_S X, \omega_X)$$

for any open $X' \subseteq X$ containing x. This implies the first claim.

For the second equality, note that since the scheme T_x^h is semi-local henselian for any $x \in X$, it follows that T_x^h is isomorphic to its preimage under any étale neighborhood $v\colon (X', x) \to (X, x)$. Hence it follows from étale excision that $A_{T_x^h}^d(U \times_S X, \omega_X) = A_{T_x^h}^d(U \times_S X', \omega_X)$, and consequently $A_{T_x^h}^d(U \times_S X, \omega_X) = A_{T_x^h}^d(U \times_S X_X^h, \omega_{X_X^h})$. So the second equality follows. ■

Lemma III.3.10. *Let $X \in \mathrm{Sm}_S$ be as in Lemma III.3.9. Then, for any point $x \in X$ and for any henselian local scheme $U \in \mathrm{EssSm}_S$ we have*

$$\mathrm{Cor}_S^A(U, X_x) = \varinjlim_{T \in \mathcal{A}_0(U \times_S X_x/U)} A_T^d(U \times_S X_x, \omega_{X_x}),$$

$$\mathrm{Cor}_S^A(U, X_x^h) = \varinjlim_{T \in \mathcal{A}_0(U \times_S X_x^h/U)} A_T^d(U \times_S X_x^h, \omega_{X_x^h}).$$

Proof. The first claim follows from the first equality of Lemma III.3.9, by the following computation:

$$\begin{aligned}
\mathrm{Cor}^A_S(U, X_x) &= \varprojlim_v \varinjlim_{T \in \mathcal{A}_0(U \times_S X'/U)} A^d_T(U \times_S X', \omega_{X'}) \\
&= \varprojlim_v \varinjlim_{T \in \mathcal{A}_0(U \times_S X'/U)} \bigoplus_{x' \in X'} A^d_{T_{x'}}(U \times_S X'_{x'}, \omega_{X'_{x'}}) \\
&= \varprojlim_v \bigoplus_{x' \in X'} \varinjlim_{T \in \mathcal{A}_0(U \times_S X'_{x'}/U)} A^d_T(U \times_S X'_{x'}, \omega_{X'_{x'}}) \\
&= \varprojlim_v \varinjlim_{T \in \mathcal{A}_0(U \times_S X'_x/U)} A^d_T(U \times_S X'_x, \omega_{X'_x}) = A^d_T(U \times_S X_x, \omega_{X_x}).
\end{aligned}$$

Here $v \colon (X', x) \hookrightarrow (X, x)$ ranges over the set of Zariski neighborhoods of x in X. The second equality of the claim follows in a similar manner from the second equality of Lemma III.3.9 with X_x replaced by X^h_x, and with v ranging over the set of étale neighborhoods of x in X. ■

III.3.5 Constructing correspondences from functions and trivializations

From now on we will assume that the base scheme S is the spectrum of a field k. Later on we will also have to put more restrictions on k (e.g., infinite or perfect); the appropriate assumptions will be stated in the beginning of each section where they are needed.

III.3.5.1

We will now describe how to construct a finite A-correspondence from the data of a regular function on a relative curve together with a trivialization of the relative canonical class. This construction can be thought of as an analogous statement to the defining axiom of a pretheory in the sense of Voevodsky [Voe00a], and will be used throughout.

Construction III.3.11. Suppose that there is a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, "g"] & \mathbf{A}^1 \\
U & X
\end{tikzcd} \tag{III.4}$$

in Sm_k satisfying the following properties:

1. $p \colon \mathcal{C} \to U$ is a smooth relative curve, and $g \colon \mathcal{C} \to X$ is any morphism.

2. $Z(f) = Z \amalg Z'$, with Z finite over U.

3. There is an isomorphism $\mu \colon \mathcal{O}_{\mathcal{C}} \xrightarrow{\cong} \omega_{\mathcal{C}/U}$.

We can then define finite A-correspondences

$$\begin{aligned} &\mathrm{div}_U^A(f)_Z^\mu \in \mathrm{Cor}_U^A(U,\mathcal{C}); \quad \mathrm{div}^A(f)_Z^\mu \in \mathrm{Cor}_k^A(U,\mathcal{C}); \\ &\mathrm{div}_U^A(f)_Z^{\mu,g} \in \mathrm{Cor}_U^A(U,X); \quad \mathrm{div}^A(f)_Z^{\mu,g} \in \mathrm{Cor}_k^A(U,X) \end{aligned}$$

as follows:

Let Γ_f denote the graph of the morphism f, with embedding $i_f\colon \Gamma_f \hookrightarrow \mathcal{C}\times\mathbf{A}^1$. Consider the pushforward homomorphism

$$(i_f)_*\colon A^0(\Gamma_f,\mathcal{O}_{\Gamma_f}\otimes\omega_{i_f}) \to A^1_{\Gamma_f}(\mathcal{C}\times\mathbf{A}^1,\mathcal{O}_{\mathcal{C}\times\mathbf{A}^1}),$$

and let $dT\colon \mathcal{O}_{\mathbf{A}^1}\cong\omega_{\mathbf{A}^1}$ be the trivialization defined by the coordinate function T on $\mathbf{A}^1$. Using the trivializations $-dT$ and μ we then obtain a homomorphism

$$i_*\colon A^0(\Gamma_f,\mathcal{O}_{\Gamma_f}) \to A^1_{\Gamma_f}(\mathcal{C}\times\mathbf{A}^1,\omega_{\mathcal{C}\times\mathbf{A}^1/U\times\mathbf{A}^1}).$$

Consider the image

$$i_*(1)\in A^1_{\Gamma_f}(\mathcal{C}\times\mathbf{A}^1,\omega_{\mathcal{C}\times\mathbf{A}^1/U\times\mathbf{A}^1})$$

of $1\in A^0(\Gamma_f,\mathcal{O}_{\Gamma_f})$ under the map i_*.

Next we may pull back along the zero section,

$$i_0^*\colon A^1_{\Gamma_f}(\mathcal{C}\times\mathbf{A}^1,\omega_{\mathcal{C}\times\mathbf{A}^1/U\times\mathbf{A}^1}) \to A^1_{Z(f)}(\mathcal{C},\omega_{\mathcal{C}/U}).$$

Since $Z(f)=Z\amalg Z'$ we have $A^1_{Z(f)}(\mathcal{C},\omega_{\mathcal{C}/U}) = A^1_Z(\mathcal{C},\omega_{\mathcal{C}/U})\oplus A^1_{Z'}(\mathcal{C},\omega_{\mathcal{C}/U})$ by Remark III.2.3. We define the finite relative A-correspondence

$$\mathrm{div}_U^A(f)_Z^\mu \in \mathrm{Cor}_U^A(U,\mathcal{C})$$

as the image of $i_*(1)\in A^1_{\Gamma_f}(\mathcal{C}\times\mathbf{A}^1,\omega_{\mathcal{C}\times\mathbf{A}^1/U})$ under the composite homomorphism

$$A^1_{\Gamma_f}(\mathcal{C}\times\mathbf{A}^1,\omega_{\mathcal{C}/U}) \xrightarrow{i_0^*} A^1_{Z(f)}(\mathcal{C},\omega_{\mathcal{C}/U}) \to A^1_Z(\mathcal{C},\omega_{\mathcal{C}/U}) \to \mathrm{Cor}_U^A(U,\mathcal{C}).$$

Here the second map is the projection to the first coordinate, and the last map is the canonical homomorphism to the colimit. By composing with the morphism g we obtain the finite relative A-correspondence

$$\mathrm{div}_U^A(f)_Z^{\mu,g} := g\circ\mathrm{div}_U^A(f)_Z^\mu \in \mathrm{Cor}_U^A(U,X).$$

We readily obtain a nonrelative A-correspondence by applying the functor res_U. More precisely, we define

$$\mathrm{div}^A(f)_Z^{\mu,g} := g\circ\mathrm{res}_U(\mathrm{div}_U^A(f)_Z^\mu)\in\mathrm{Cor}_k^A(U,X).$$

If it is clear from the context, we might drop the trivialization μ or the map g from the notation. Moreover, if $Z=Z(f)$ and Z is finite over U, we may also abbreviate $\mathrm{div}^A(f)_{Z(f)}$ to $\mathrm{div}^A(f)$. We think of $\mathrm{div}^A(f)_Z^\mu$ as a divisor supported on Z whose multiplicity at each component of Z is given by an A-cohomology class.

Lemma III.3.12. *Let $\mathcal{C}$, Z, p, f and g be as in Construction III.3.11. Then* $\operatorname{div}^A(\lambda f)_Z^{\lambda\mu,g} = \operatorname{div}^A(f)_Z^{\mu,g}$ *for any* $\lambda \in \Gamma(U, \mathcal{O}_U^\times)$.

Proof. For any smooth U-scheme X, any closed subscheme $Z \subseteq X$, and any line bundle $\mathscr{L}$ on X, define the automorphism $\Lambda_X\colon A_Z^*(X, \mathscr{L}) \to A_Z^*(X, \mathscr{L})$ as the map induced by the automorphism $\mathscr{L} \to \mathscr{L}$ given by multiplication by λ.

Consider the homomorphisms

$$\begin{aligned} i_*\colon A^0(\Gamma_f, \mathcal{O}_{\Gamma_f}) &\to A^1_{\Gamma_f}(\mathcal{C} \times \mathbf{A}^1, \omega_{\mathcal{C}\times\mathbf{A}^1/U}),\\ i_*^{\lambda f,\lambda\mu}\colon A^0(\Gamma_{\lambda f}, \mathcal{O}_{\Gamma_{\lambda f}}) &\to A^1_{\Gamma_{\lambda f}}(\mathcal{C} \times \mathbf{A}^1, \omega_{\mathcal{C}\times\mathbf{A}^1/U}) \end{aligned}$$

in the constructions of $\operatorname{div}^A(f)_Z^{\mu,g}$ and $\operatorname{div}^A(\lambda f)_Z^{\lambda\mu,g}$. Let moreover $i_*^{\lambda\mu}$ denote the homomorphism

$$i_*^{\lambda\mu}\colon A^0(\Gamma_f, \mathcal{O}_{\Gamma_f}) \to A^1_{\Gamma_f}(\mathcal{C} \times \mathbf{A}^1, \omega_{\mathcal{C}\times\mathbf{A}^1/U})$$

given by the trivialization $dT \otimes \lambda\mu$. Define automorphisms

$$\begin{aligned} H^\lambda\colon \mathbf{A}^1 \times \mathcal{C} &\to \mathbf{A}^1 \times \mathcal{C}, \quad (T, x) \mapsto (\lambda T, x),\\ H^{\lambda^{-1}}\colon \mathbf{A}^1 \times \mathcal{C} &\to \mathbf{A}^1 \times \mathcal{C}, \quad (T, x) \mapsto (\lambda^{-1} T, x). \end{aligned}$$

Then $H^{\lambda^{-1}}(\Gamma_{\lambda f}) = \Gamma_f$, and $H_*^{\lambda^{-1}}(dT) = \lambda^{-1} dT$. Hence

$$H_*^{\lambda^{-1}} \circ i_*^{\lambda f,\lambda\mu} = (\Lambda_{\mathcal{C}\times\mathbf{A}^1})^{-1} \circ i_*^{\lambda\mu} = i_*,$$

and the claim follows. ■

Lemma III.3.13. *Let $\mathcal{C}$, Z, p and f be as in* (III.4) *and suppose that $Z = Z_1 \amalg Z_2$ with both Z_1 and Z_2 finite over U. Then*

$$\operatorname{div}^A(f)_Z^{\mu,g} = \operatorname{div}^A(f)_{Z_1}^{\mu,g} + \operatorname{div}^A(f)_{Z_2}^{\mu,g}.$$

Proof. The claim follows from the definition and Remark III.2.3. ■

Definition III.3.14. Let $\mathcal{C}$, U, μ, Z, X, p, f and g be as above and suppose that $U' \subseteq U$ and $X' \subseteq X$ are open subschemes such that $Z \times_U U' \subseteq g^{-1}(X')$. Write $f' := f\big|_{\mathcal{C}\times_U U'}$ and $g' := g\big|_{\mathcal{C}\times_U U'}$. This data defines a correspondence of pairs

$$\Big(\operatorname{div}^A(f)_Z^{\mu,g}, \operatorname{div}^A(f')_{Z\times_U U'}^{\mu,g'}\Big) \in \operatorname{Cor}_k^{A,\mathrm{pr}}((U, U'), (X, X')).$$

Suppose furthermore that $\pi\colon (\mathcal{C}', Z') \to (\mathcal{C}, Z)$ is an étale neighborhood (see Definition III.A.4) satisfying $Z' \times_U U' \subseteq v^{-1}(X')$, where $v := g \circ \pi$. Then this data defines a finite A-correspondence of pairs

$$\operatorname{div}^A(\tilde{f})_Z^{\tilde{\mu},v} \in \operatorname{Cor}_k^{A,\mathrm{pr}}((U, U'), (X, X')),$$

where $\tilde{f} := \pi^*(f)$ and $\tilde{\mu} := \pi^*(\mu)$. If the morphism π is implicitly understood from the context, we may sometimes abuse notation and write simply $\operatorname{div}^A(f)_Z^{\mu,g}$ for this A-correspondence.

Lemma III.3.15. *Let $\mathcal{C}$, U, μ, Z, X, p, f and g be as above and suppose that $U' \subseteq U$ and $X' \subseteq X$ are open subschemes. If $Z \cap g^{-1}(X \setminus X') = \varnothing$, then* $\operatorname{div}^A(f)_Z^{\mu,g} = 0 \in \operatorname{Cor}_k^{A,\mathrm{pr}}((U,U'),(X,X'))$.

Proof. The correspondence $\operatorname{div}^A(f)_Z^{\mu,g} \in \operatorname{Cor}_k^A(U,X')$ defines the diagonal in the diagram

$$\begin{tikzcd} X' \arrow[r] & X \\ U' \arrow[u] \arrow[r] & U. \arrow[u] \arrow[ul] \end{tikzcd}$$

Moreover, the vertical arrows in the above diagram define the correspondence of pairs $\operatorname{div}^A(f)_Z^{\mu,g} \in \operatorname{Cor}_k^{A,\mathrm{pr}}((U,U'),(X,X'))$; it follows that $\operatorname{div}^A(f)_Z^{\mu,g}$ factors through (X',X') and is therefore zero. ■

Definition III.3.16. Let $U \in \mathrm{Sm}_k$ and suppose that λ is an invertible regular function on U. We can then consider the morphism

$$(\mathrm{id},\lambda)\colon (U \times U, \omega_U) \to (U \times U, \omega_U)$$

in $\mathrm{SmOp}_k^{\mathrm{L}}$. We denote by

$$\langle\lambda\rangle \in \operatorname{Cor}_k^A(U,U)$$

the image of $\mathrm{id}_U \in \operatorname{Cor}_k^A(U,U)$ under the corresponding pushforward map $(\mathrm{id},\lambda)_*$. In particular, if $\lambda = -1$, we will write ϵ for the finite A-correspondence $\epsilon := -\langle -1\rangle \in \operatorname{Cor}_k^A(U,U)$.

Example III.3.17. Suppose that $A^* = \widetilde{\mathrm{CH}}^*$, so that Cor_k^A is the category of finite Milnor–Witt correspondences. Then $\langle\lambda\rangle \in \operatorname{Cor}_k^A(U,U)$ is the Milnor–Witt correspondence $\langle\lambda\rangle \cdot \mathrm{id}_U \in \widetilde{\operatorname{Cor}}_k(U,U)$ given by multiplication with the quadratic form $\langle\lambda\rangle \in \mathbf{K}_0^{\mathrm{MW}}(U)$. In particular, the finite A-correspondence ϵ coincides with the usual ϵ defined in Milnor–Witt K-theory.

Lemma III.3.18. *Let U, $\mathcal{C}$, p, f and g be as in* (III.4). *Suppose also that p induces an isomorphism $Z(f) \cong U$, so that $Z(f)$ defines a section $s\colon U \to \mathcal{C}$ of p. Then the following hold:*

(a) *There is an invertible regular function λ on U such that*

$$\operatorname{div}^A(f)_{Z(f)}^{\mu,g} = g \circ s \circ \langle\lambda\rangle$$

in $\operatorname{Cor}_k^A(U,X)$.

(b) *If moreover $\mu|_{Z(f)} = df$, where df denotes the trivialization of the normal bundle $N_{Z(f)/\mathcal{C}}$ defined by f, then* $\operatorname{div}^A(f)_{Z(f)}^{\mu,g} = g \circ s$.

Proof. (a) Let $j\colon Z(f) \to \Gamma_f$, $j_f\colon Z(f) \to \mathcal{C}$ and $i_f\colon \Gamma_f \to \mathcal{C} \times \mathbf{A}^1$ denote the closed embeddings. Consider the following diagram consisting of two squares of varieties equipped with compatible sets of line bundles (in which we have also included the relevant line bundles in the notation):

$$\begin{tikzcd}
(Z(f), \mathcal{O}_{Z(f)}) \arrow[r, "(\mathrm{id},\mu)"] \arrow[d, "j"] & (Z(f), \omega_{Z(f)/U}) \arrow[r, "j_f"] \arrow[d, "j"] & (\mathcal{C}, \omega_{\mathcal{C}/U}) \arrow[d, "i_0"] \\
(\Gamma_f, \mathcal{O}_{\Gamma_f}) \arrow[r, "(\mathrm{id},\mu)"] & (\Gamma_f, \omega_{\Gamma_f/U}) \arrow[r, "i_f"] & (\mathcal{C} \times \mathbf{A}^1, \omega_{\mathcal{C}\times\mathbf{A}^1/U}).
\end{tikzcd}$$

The first square is evidently transversal (and strongly transversal). To prove that the second one is (strongly) transversal, it is enough to note that the homomorphism $k[\mathcal{C}][T] = k[\mathcal{C} \times \mathbf{A}^1] \to k[\mathcal{C}]$ given by $T \mapsto 0$ takes the function $f - T$ to f and induces an isomorphism

$$N_{\Gamma_f/\mathcal{C}\times\mathbf{A}^1} \otimes k[\mathcal{C} \times 0] = (f-T)/(f-T)^2 \otimes k[\mathcal{C}][T]/(T) \cong (f)/(f)^2 = N_{Z(f)/\mathcal{C}}.$$

Hence the base change axiom gives us the following commutative diagram:

$$\begin{tikzcd}
A^0(\Gamma_f, \mathcal{O}_{\Gamma_f}) \arrow[r, "\mu"] \arrow[d, "j^*"'] & A^0(\Gamma_f, \omega_{\Gamma_f/U}) \arrow[r, "(i_f)_*"] \arrow[d, "j^*"'] & A^1_{\Gamma_f}(\mathcal{C} \times \mathbf{A}^1, \omega_{\mathcal{C}\times\mathbf{A}^1/U}) \arrow[d, "i_0^*"'] \\
A^0(Z(f), \mathcal{O}_{Z(f)}) \arrow[r, "\mu_{Z(f)}"] \arrow[rrd, "(j_f,\nu)_*"'] & A^0(Z(f), \omega_{Z(f)/U} \otimes \omega_j) \arrow[r, "(j_f)_*"] & A^1_{Z(f)}(\mathcal{C}, \omega_{\mathcal{C}/U} \otimes \omega_{i_0}) \arrow[d, "-dT"] \\
& & A^1_{Z(f)}(\mathcal{C}, \omega_{\mathcal{C}/U}).
\end{tikzcd}$$

Here j^* and i_0^* are defined via the canonical isomorphisms $j^*(\omega_{\Gamma_f/U}) \cong \omega_{Z(f)/U} \otimes \omega_j$ and $i_0^*(\omega_{\mathcal{C}\times\mathbf{A}^1/U}) \cong \omega_{\mathcal{C}/U} \otimes \omega_{i_0}$ induced by the short exact sequences of vector bundles

$$0 \to T_{Z(f)} \to j^*(T_{\Gamma_f}) \to N_{Z(f)/\Gamma_f} \to 0$$

and

$$0 \to T_{\mathcal{C}\times 0} \to i_0^*(T_{\mathcal{C}\times\mathbf{A}^1}) \to N_{\mathcal{C}\times 0/\mathcal{C}\times\mathbf{A}^1} \to 0.$$

Moreover, the homomorphism $\mu_{Z(f)}$ is given as the composition of $\mu|_{Z(f)}$ and the isomorphism $j^*\omega_{\Gamma_f/U} \cong \omega_{Z(f)/U} \otimes \omega_j$; the homomorphism $(j_f)_*$ is defined via the isomorphism $j_f^*(\omega_{i_0}) \cong \omega_j$ induced by the canonical isomorphism $\Gamma_f \cong \mathcal{C}$; and the diagonal homomorphism $(j_f, \nu)_*$ is induced by some trivialization $\nu\colon \mathcal{O}_{Z(f)} \cong \omega_{Z(f)/U}$.

It follows from the construction that $\mathrm{div}^A(f)^{\mu}_{Z(f)} = -dT(i_0^*(i_f)_*\mu(1))$. Since the diagram is commutative we thus obtain $\mathrm{div}^A(f)^{\mu}_{Z(f)} = (j_f, \nu)_* j^*(1) = s \circ \langle\lambda\rangle$, where λ is given as the fraction of ν and the canonical isomorphism $\omega_{Z(f)/U} \cong \mathcal{O}_U$ induced by the isomorphism $p\colon Z(f) \overset{\cong}{\to} U$.

(b) A straightforward computation with isomorphisms of line bundles shows that $(j_f)_*$ is given as the product of the canonical isomorphism $\mathcal{O}_{Z(f)} \cong \omega_{Z(f)/U}$ with the invertible function $\mu\big|_{Z(f)} \otimes df^{-1}$, where $df\colon \mathcal{O}_{Z(f)} \cong \omega_{Z(f)/U}$ denotes the trivialization induced by the choice of the generator $-f$ of the ideal $(f) = I(Z(f))$. So $\lambda = 1$, and the claim follows. ∎

III.3.6 Some homotopies

We now give a computation with A-correspondences that will come in handy later on, especially in the proof of Lemma III.8.3.

Lemma III.3.19. *Suppose that the base field k is infinite. Let U be an essentially smooth local scheme over k and let $\lambda \in \Gamma(U, \mathcal{O}_U^\times)$. Suppose that $\lambda = w^2$ for some invertible section w on U. Then*

$$\langle \lambda \rangle \sim_{\mathbf{A}^1} \mathrm{id}_U \in \mathrm{Cor}_k^A(U, U).$$

Similarly

$$\langle \lambda \rangle \sim_{\mathbf{A}^1} \mathrm{id}_{(U,V)} \in \mathrm{Cor}_k^{A,\mathrm{pr}}((U,V),(U,V))$$

for any open subscheme $V \subseteq U$.

Proof. Assume first that $V = \varnothing$ and $\lambda(x) \neq 1$, where $x \in U$ is the closed point. Let $\alpha := (\lambda - 1)^{-1}$, and define the regular function

$$h := (1-\nu)\alpha(t-\lambda)(t-1) + \nu\alpha(t-w)^2 \in \Gamma(\overset{t}{\mathbf{G}_m} \times U \times \overset{\nu}{\mathbf{A}^1}, \mathcal{O}).$$

Keeping the notation of (III.4) in mind, consider the following diagram:

$$\begin{tikzcd} \mathbf{G}_m \times U \times \mathbf{A}^1 \arrow[r, "h"] \arrow[d, "p"'] \arrow[dr, "\mathrm{pr}"] & \mathbf{A}^1 \\ U \times \mathbf{A}^1 & U. \end{tikzcd}$$

Here the morphisms p and pr are the projections. We aim to apply Construction III.3.11 to this diagram. To this end, notice that h is a polynomial in t with leading term α, which is invertible on U. Moreover, the substitution $t \mapsto 0$ takes h to $(1-\nu)\alpha\lambda + \nu\alpha w^2 = \alpha\lambda$, which is invertible too. Hence $Z(h) \subseteq \mathbf{G}_m \times U \times \mathbf{A}^1$ is finite over $U \times \mathbf{A}^1$. Using the trivialization tdt of the canonical class of $\mathbf{G}_m$, we get from Construction III.3.11 a finite relative A-correspondence

$$\Theta := \mathrm{div}_U^A(h)^{tdt,\mathrm{pr}} \in \mathrm{Cor}_U^A(U \times \mathbf{A}^1, U).$$

Let $i_0, i_1 \colon U \to U \times \mathbf{A}^1$ denote the zero- and unit sections. We then have

$$\begin{aligned} &\Theta \circ i_0 \\ &= \mathrm{div}_U^A(\alpha(t-\lambda)(t-1))^{tdt,\mathrm{pr}} \\ &= \mathrm{div}_U^A((\lambda-1)^{-1}(t-\lambda)(t-1))^{tdt,\mathrm{pr}}_{Z(t-\lambda)} + \mathrm{div}_U^A(-(1-\lambda)^{-1}(t-\lambda)(t-1))^{tdt,\mathrm{pr}}_{Z(t-1)} \\ &= \langle \lambda \rangle + \langle -1 \rangle. \end{aligned}$$

On the other hand,

$$\Theta \circ i_1 = \mathrm{div}_U^A(\alpha(t-w)^2)^{tdt,\mathrm{pr}} = \mathrm{div}_U^A(\alpha w^{-1}(t-w)^2)^{dt,\mathrm{pr}},$$

where the second equality follows from Lemma III.3.12. Thus we see that

$$\langle\lambda\rangle + \langle -1\rangle \sim_{\mathbf{A}^1} \operatorname{div}_U^A(\alpha w^{-1}(t-w)^2)^{dt,\mathrm{pr}} \in \mathrm{Cor}_U^A(U,U).$$

We now construct yet another homotopy similar to the one in the proof of [GP18b, Lemma 13.15], which is in turn inspired by [Nes18, Lemma 7.3]. Put $\alpha' := \alpha w^{-1}$. Consider the regular function

$$h' = (1-\nu)\alpha'(t-w)^2 + \nu\alpha'(t-\alpha'^{-1})t \in \Gamma(\overset{t}{\mathbf{A}^1} \times U \times \overset{\nu}{\mathbf{A}^1}, \mathcal{O}),$$

along with the diagram

$$\begin{tikzcd}
\mathbf{A}^1 \times U \times \mathbf{A}^1 \arrow[r, "h'"] \arrow[d, "p'"'] \arrow[dr, "\mathrm{pr}'"] & \mathbf{A}^1 \\
U \times \mathbf{A}^1 & U
\end{tikzcd}$$

in which p' and pr' are the projections. As h' is a polynomial in t with leading term α', which is invertible on U, it follows that $Z(h') \subseteq \mathbf{A}^1 \times U \times \mathbf{A}^1$ is finite over $U \times \mathbf{A}^1$. Using the trivialization dt of the canonical class of $\mathbf{A}^1$, we then get from Construction III.3.11 a finite A-correspondence

$$\Theta' := \operatorname{div}_U^A(h')^{dt,\mathrm{pr}'} \in \mathrm{Cor}_U^A(U \times \mathbf{A}^1, U).$$

By definition of h', the A-correspondence Θ' satisfies

$$\begin{aligned}
\Theta' \circ i_0 &= \operatorname{div}_U^A(\alpha'(t-w)^2)^{dt,\mathrm{pr}'},\\
\Theta' \circ i_1 &= \operatorname{div}_U^A(\alpha'(t-\alpha'^{-1})t)^{dt,\mathrm{pr}'}\\
&= \operatorname{div}_U^A(\alpha'(t-\alpha'^{-1})t)^{dt,\mathrm{pr}'}_{Z(t-\alpha'^{-1})} + \operatorname{div}_U^A(\alpha'(t-\alpha'^{-1})t)^{dt,\mathrm{pr}'}_{Z(t)}\\
&= \langle 1\rangle + \langle -1\rangle.
\end{aligned}$$

Thus we see that

$$\operatorname{div}_U^A(\alpha'(t-w)^2)^{dt,\mathrm{pr}'} \sim_{\mathbf{A}^1} \langle 1\rangle + \langle -1\rangle \in \mathrm{Cor}_U^A(U,U).$$

Now, since $\operatorname{div}_U^A(\alpha'(t-w)^2)^{dt,\mathrm{pr}'} = \operatorname{div}_U^A(\alpha'(t-w)^2)^{dt,\mathrm{pr}}$, we get

$$\langle\lambda\rangle + \langle -1\rangle \sim_{\mathbf{A}^1} \langle 1\rangle + \langle -1\rangle \in \mathrm{Cor}_U^A(U,U).$$

Thus the claim follows from the fact that $\langle 1\rangle = \mathrm{id}_U$.

We have now proved the claim in the case $\lambda(x) \neq 1$. In the general case when $\lambda \in \Gamma(U, \mathcal{O}_U^\times)$, consider a function $u \in \Gamma(U, \mathcal{O}_U^\times)$ such that $u(x) \neq w(x)^{-1}$ and $u(x) \neq 1$. Such a function exists, since the base field is infinite by assumption. Then we have by the above that $\langle\lambda u^2\rangle \sim_{\mathbf{A}^1} \mathrm{id}_U$ and $\langle u^2\rangle \sim_{\mathbf{A}^1} \mathrm{id}_U$ in $\mathrm{Cor}_U^A(U,U)$. Thus, since $\langle\lambda u^2\rangle = \langle\lambda\rangle \circ \langle u^2\rangle$, the claim follows.

So the claim of the lemma is done for $V = \varnothing$. The case of a pair (U,V) with $V \neq \varnothing$ follows, since all the constructed homotopies are relative homotopies over U, i.e., they are elements of $\mathrm{Cor}_U^A(U \times \mathbf{A}^1, U)$. Consequently all the homotopies defined are elements in $\mathrm{Cor}_U^{A,\mathrm{pr}}((U,V) \times \mathbf{A}^1, (U,V))$ as well. ■

III.4 Connection to framed correspondences

Using similar techniques as in Construction III.3.11 we can define a functor $\Upsilon\colon \mathrm{Fr}_*(k) \to \mathrm{Cor}_k^A$ from the category of framed correspondences [GP18a] to the category Cor_k^A. See also [Elm+20] for an alternative approach using Thom classes [Elm+20, Lemma 4.3.24].

Construction III.4.1. Let $\Phi = (Z, \mathcal{V}, \phi; g) \in \mathrm{Fr}_n(X, Y)$ be an explicit framed correspondence. Thus Z is a closed subset in $\mathbf{A}^n \times X = \mathbf{A}^n_X$; $(\mathcal{V}, Z) \to (\mathbf{A}^n_X, Z)$ is an étale neighborhood of Z in $\mathbf{A}^n_X$; $\phi = (\phi_i)$, where the ϕ_i's are regular functions on $\mathcal{V}$ such that $Z = Z(\phi)$; and g is a morphism $g\colon \mathcal{V} \to Y$. For any unit $\lambda \in k^\times$ we define a finite A-correspondence $\Upsilon_\lambda(\Phi) \in \mathrm{Cor}_k^A(X, Y)$ in the following way.

Let $dt\colon \omega_{\mathbf{A}^1} \cong \mathcal{O}_{\mathbf{A}^1}$ denote the standard trivialization of the canonical class, and consider further two trivializations $\mu_1, \mu_2\colon \omega_{\mathbf{A}^n} \cong \mathcal{O}_{\mathbf{A}^n}$ given by $\mu_1 = (dt)^{\wedge n}$ and $\mu_2 = \lambda^n \mu_1$. Let Γ denote the graph $\Gamma \subseteq \mathbf{A}^n_X \times_X \mathcal{V} = \mathbf{A}^n \times \mathcal{V}$ of the relative morphism $\mathcal{V} \to \mathbf{A}^n_X$ over X. Then there is a canonical projection $\Gamma \to \mathbf{A}^n_X$. Denote by $i_X\colon X \to \mathbf{A}^n_X$ and $i_{\mathcal{V}}\colon \mathcal{V} \to \mathbf{A}^n \times \mathcal{V}$ the embeddings given by the zero sections. Let furthermore $g'\colon \mathcal{V} \to X \times Y$ denote the product of g and the projection to X. The following diagram summarizes the situation:

$$\begin{tikzcd} & Y & \\ \mathbf{A}^n \times \mathcal{V} \arrow[d] & \mathcal{V} \arrow[l, shift left, "\Gamma"'] \arrow[l, shift right, "i_{\mathcal{V}}"] \arrow[u, "g"] \arrow[r] \arrow[d] & \mathbf{A}^n_X \\ \mathbf{A}^n \times X & X. \arrow[l, "i_X"'] & \end{tikzcd} \tag{III.5}$$

We then define $\Upsilon_\lambda(\Phi) := g'_*(i_{\mathcal{V}}^*(\Gamma_*(1)))$, where we use the trivialization μ_1 of the canonical class $\omega_{\mathbf{A}^n}$, and the trivialization of $\omega_{\mathcal{V}/X}$ defined by the pullback of μ_2 along the étale morphism $\mathcal{V} \to \mathbf{A}^n \times X$.

In other words, the finite A-correspondence $\Upsilon_\lambda(\Phi)$ is obtained as the image of $i_{\mathcal{V}}^*(\Gamma_*(1)) \in A^n_Z(\mathcal{V}, \omega_{\mathcal{V}/X})$ under the composition

$$A^n_Z(\mathcal{V}, \omega_{\mathcal{V}/X}) \to \mathrm{Cor}_X^A(X, \mathcal{V}) \xrightarrow{\mathrm{res}_X} \mathrm{Cor}_k^A(X, \mathcal{V}) \xrightarrow{g_*} \mathrm{Cor}_k^A(X, Y)$$

in which the last map is given by composition with g.

Theorem III.4.2. *For each unit $\lambda \in k^\times$, Construction III.4.1 defines a functor $\Upsilon_\lambda\colon \mathrm{Fr}_*(k) \to \mathrm{Cor}_k^A$ that carries the framed correspondence*

$$\sigma = (0, \mathbf{A}^1, t, \mathrm{pr}\colon \mathbf{A}^1 \to \mathrm{pt}) \in \mathrm{Fr}_1(\mathrm{pt}, \mathrm{pt})$$

to $\langle \lambda \rangle \in \mathrm{Cor}_k^A(\mathrm{pt}, \mathrm{pt})$. Moreover, Υ_λ factors through the category $\mathbf{Z}\mathrm{F}_*(k)$ of linear framed correspondences.

Proof. Construction III.4.1 gives rise to a map Υ_λ depending on the fraction $\lambda \in k^\times$ of the two trivializations of the canonical classes. To show that Υ_λ is in fact a functor, we need to check the following:

(1) Equivalent explicit framed correspondences give rise to the same finite A-correspondence.

(2) Let $\mathrm{id}_X \in \mathrm{Fr}_0(X, X)$ be the identity morphism in the graded category $\mathrm{Fr}_*(k)$. Then $\Upsilon_\lambda(\mathrm{id}_X)$ is equal to the identity morphism in the category Cor_k^A.

(3) For any $\Phi_1 \in \mathrm{Fr}_{n_1}(X_1, X_2)$ and $\Phi_2 \in \mathrm{Fr}_{n_2}(X_2, X_3)$, we have $\Upsilon_\lambda(\Phi_2 \circ \Phi_1) = \Upsilon_\lambda(\Phi_2) \circ \Upsilon_\lambda(\Phi_1)$.

(4) For any $\Phi = (Z, \mathcal{V}, \phi; g) \in \mathrm{Fr}_n(X_1, X_2)$ such that $Z = Z_1 \amalg Z_2$, we have $\Upsilon_\lambda(\Phi) = \Upsilon_\lambda(Z_1, \mathcal{V}, \phi; g) + \Upsilon_\lambda(Z_2, \mathcal{V}, \phi; g)$.

All points are straightforward from the properties of the cohomology theory A^*. ∎

Remark III.4.3. Note that Theorem III.10.3 on strict homotopy invariance of presheaves on Cor_k^A follows from the existence of a functor from framed correspondences to Cor_k^A along with the fact that this theorem holds for framed correspondences by work of Garkusha–Panin [GP18b]. Below we will however give an explicit proof not relying on framed correspondences.

III.5 Injectivity on the relative affine line

In this section we prove the following theorem, which is the first in a series of ingredients necessary to establish strict homotopy invariance (Theorem III.10.3):

Theorem III.5.1. *Let U be an affine smooth k-scheme, and suppose that*

$$V_1 \subseteq V_2 \subseteq \mathbf{A}^1_U$$

are two open subschemes such that $\mathbf{A}^1_U \setminus V_2$ and $V_2 \setminus V_1$ are finite over U. Let $i\colon V_1 \subseteq V_2$ denote the inclusion. Then, for any homotopy invariant presheaf with A-transfers $\mathscr{F} \in \mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A; \mathbf{Z})$, the restriction homomorphism

$$i^*\colon \mathscr{F}(V_2) \to \mathscr{F}(V_1)$$

is injective.

III.5.0.1

We deduce Theorem III.5.1 from the following result, which ensures the existence of a left inverse to i^*:

Lemma III.5.2. *Suppose that $V_1 \subseteq V_2 \subseteq \mathbf{A}^1_U$ are open subschemes as in Theorem III.5.1. Then there is a finite A-correspondence $\Phi \in \mathrm{Cor}_k^A(V_2, V_1)$ such that $[i \circ \Phi] = [\mathrm{id}_{V_2}] \in \overline{\mathrm{Cor}}_k^A(V_2, V_2)$.*

Proof. To prove the claim we must construct a finite A-correspondence $\Phi \in \mathrm{Cor}_k^A(V_2, V_1)$ along with a homotopy $\Theta \in \mathrm{Cor}_k^A(\mathbf{A}^1 \times V_2, V_2)$ satisfying $\Theta \circ i_0 = i \circ \Phi$ and $\Theta \circ i_1 = \mathrm{id}_{V_2}$. To do this, we will make use of the following functions:

$f \in k[\overset{y}{\mathbf{A}^1} \times \overset{(x,u)}{V_2}]$ $f = y^n + a_1 y^{n-1} + \cdots + a_n$	$h \in k[\overset{y}{\mathbf{A}^1} \times \overset{(x,u)}{V_2} \times \overset{\lambda}{\mathbf{A}^1}]$ $h = y^n + b_1 y^{n-1} + \cdots + b_n$	$g \in k[\overset{y}{\mathbf{A}^1} \times \overset{(x,u)}{V_2}]$ $g = y^{n-1} + c_1 y^{n-2} + \cdots + c_{n-1}$
$f\big\rvert_{(\mathbf{A}^1_U \setminus V_1) \times_U V_2} = 1$	$h\big\rvert_{\mathbf{A}^1 \times V_2 \times 0} = f$ $h\big\rvert_{(\mathbf{A}^1_U \setminus V_2) \times_U V_2} = 1$	$h\big\rvert_{\mathbf{A}^1 \times V_2 \times 1} = (y-x)g$ $g\big\rvert_{(\mathbf{A}^1_U \setminus V_2) \times_U V_2} = (y-x)^{-1}$ $g\big\rvert_{(V_2 \setminus V_1) \times_U V_2} = 1$ $g\big\rvert_{Z(y-x)} = 1$

The functions f and g can be constructed for any n big enough by using the Chinese remainder theorem III.A.3. Having f and g we then put

$$h := (1-\lambda)f + \lambda(y-x)g.$$

We now aim to apply Construction III.3.11 to the regular functions f and h. Keeping the notations as in (III.4), consider the following diagrams:

$$\begin{tikzcd} \overset{y}{V_1} \times_U \overset{x}{V_2} \arrow[r, "f"] \arrow[d] \arrow[dr, "\mathrm{pr}_1^{12}"] & \mathbf{A}^1 \\ \overset{x}{V_2} & \overset{y}{V_1}, \end{tikzcd} \qquad \begin{tikzcd} \overset{y}{V_2} \times_U \overset{x}{V_2} \arrow[r, "(y-x)g"] \arrow[d] \arrow[dr, "\mathrm{pr}_2^{22}"] & \mathbf{A}^1 \\ \overset{x}{V_2} & \overset{y}{V_2}, \end{tikzcd}$$

$$\begin{tikzcd} \overset{y}{V_2} \times_U \overset{x}{V_2} \times \mathbf{A}^1 \arrow[r, "h"] \arrow[d] \arrow[dr, "\mathrm{pr}_2"] & \mathbf{A}^1 \\ \overset{x}{V_2} \times \mathbf{A}^1 & \overset{y}{V_2}\,. \end{tikzcd}$$

Here $\mathrm{pr}_1^{12}, \mathrm{pr}_2^{22}$ and pr_2 are projections. Since f, $(y-x)g$ and h are monic polynomials in the variable y, it follows that $Z(f)$, $Z((y-x)g)$ and $Z(h)$ are finite over V_2 and $\mathbf{A}^1 \times V_2$, respectively. Hence Construction III.3.11 yields finite A-correspondences

$$\Phi' := \mathrm{div}^A(f)_{Z(f)}^{dy,\mathrm{pr}_1^{12}} \in \mathrm{Cor}_k^A(V_2, V_1),$$
$$\Theta' := \mathrm{div}^A(h)_{Z(h)}^{dy,\mathrm{pr}_2} \in \mathrm{Cor}_k^A(\mathbf{A}^1 \times V_2, V_2).$$

The properties of f and h above imply that

$$\Theta' \circ i_0 = i \circ \Phi',$$
$$\Theta' \circ i_1 = \mathrm{div}^A((y-x)g)_{Z(y-x)}^{dy,\mathrm{pr}_2^{22}} + \mathrm{div}^A((y-x)g)_{Z(g)}^{dy,\mathrm{pr}_2^{22}}\,.$$

Now, according to Lemma III.3.18, the first summand in the last equality is equal to $\langle \nu \rangle \in \mathrm{Cor}_k^A(V_2, V_2)$ for some invertible function ν. Therefore, if we let $\Phi := \Phi^+ - \Phi^-$, where

$$\Phi^+ := \Phi' \circ \langle \nu^{-1} \rangle, \quad \Phi^- := \mathrm{div}^A((y-x)g)_{Z(g)}^{dy,\mathrm{pr}_2^{22}} \circ \langle \nu^{-1} \rangle,$$

it follows that

$$[\mathrm{id}_{V_2}] = \mathrm{div}^A((y-x)g)_{Z(y-x)}^{dy,\mathrm{pr}_2^{22}} \circ \langle \nu^{-1} \rangle = [i \circ \Phi] \in \overline{\mathrm{Cor}}_k^A(V_2, V_2),$$

as desired. ■

III.5.0.2

We will need the following two particular cases of Theorem III.5.1:

Corollary III.5.3. *Suppose that $\mathscr{F}$ is a homotopy invariant presheaf with A-transfers over a field k. Then, for any pair of open subschemes $V_1 \subseteq V_2 \subseteq \mathbf{A}_k^1$, the restriction homomorphism $\mathscr{F}(V_2) \to \mathscr{F}(V_1)$ is injective.*

Corollary III.5.4. *Let $\mathscr{F}$ be a homotopy invariant presheaf with A-transfers over a field k, and let U be an open subscheme of $\mathbf{G}_m \times \mathbf{G}_m$ such that the complement $(\mathbf{G}_m \times \mathbf{G}_m) \setminus U$ is finite over the first copy of $\mathbf{G}_m$. Then the restriction homomorphism $\mathscr{F}(\mathbf{G}_m \times \mathbf{G}_m) \to \mathscr{F}(U)$ is injective.*

III.6 Excision on the relative affine line

The aim of this section is the prove the following excision result for open subsets of a relative affine line:

Theorem III.6.1. *Suppose that $U \in \mathrm{Sm}_k$ is an affine scheme, and let*

$$V_1 \subseteq V_2 \subseteq \mathbf{A}_U^1$$

be a pair of open subschemes such that $0_U \in V_1$. Let $i \colon V_1 \subseteq V_2$ denote the inclusion. Then, for any homotopy invariant presheaf with A-transfers $\mathscr{F} \in \mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A; \mathbf{Z})$, the restriction homomorphism i^* induces an isomorphism

$$i^* \colon \mathscr{F}(V_2 \setminus 0_U)/\mathscr{F}(V_2) \xrightarrow{\cong} \mathscr{F}(V_1 \setminus 0_U)/\mathscr{F}(V_1).$$

Remark III.6.2. By Theorem III.5.1, the restriction maps $\mathscr{F}(V_i) \to \mathscr{F}(V_i \setminus 0)$ are injective for $i = 1, 2$, which justifies the notation $\mathscr{F}(V_i \setminus 0_U)/\mathscr{F}(V_i)$.

III.6.0.1

To prove the above theorem, we will show that i^* is injective and surjective, which amounts to constructing appropriate correspondences of pairs up to homotopy. Let us first show that i^* is injective:

Lemma III.6.3. *Suppose that* $i\colon V \subseteq \mathbf{A}^1_U$ *is an open subscheme with* $0_U \in V$. *Then there is a finite* A*-correspondence of pairs*

$$\Phi \in \mathrm{Cor}_k^{A,\mathrm{pr}}((\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U), (V, V \setminus 0_U))$$

such that

$$[i \circ \Phi] = [\mathrm{id}_{(\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U)}] \in \overline{\mathrm{Cor}}_k^{A,\mathrm{pr}}((\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U), (\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U)).$$

Proof. We need to construct a finite A-correspondence of pairs

$$\Phi \in \mathrm{Cor}_k^{A,\mathrm{pr}}((\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U), (V, V \setminus 0_U))$$

along with a homotopy

$$\Theta \in \mathrm{Cor}_k^{A,\mathrm{pr}}(\mathbf{A}^1 \times (\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U), (\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U))$$

such that $\Theta \circ i_0 = i \circ \Phi$ and $\Theta \circ i_1 = \mathrm{id}_{(\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U)}$. To do this, we will make use of the following sections:

$s \in \Gamma\left(\overset{[t_0:\,t_\infty]}{\mathbf{P}^1}_{U \times \overset{x}{\mathbf{A}^1}}, \mathcal{O}(n)\right)$	$\tilde{s} \in \Gamma\left(\overset{[t_0:\,t_\infty]}{\mathbf{P}^1}_{U \times \overset{x}{\mathbf{A}^1} \times \overset{\lambda}{\mathbf{A}^1}}, \mathcal{O}(n)\right)$	$s' \in \Gamma\left(\overset{[t_0:\,t_\infty]}{\mathbf{P}^1}_{U \times \overset{x}{\mathbf{A}^1}}, \mathcal{O}(n-1)\right)$
$s\big\vert_{((\mathbf{P}^1 \times U)\setminus V)\times \mathbf{A}^1} = t_0^n$ $s\big\vert_{0 \times U \times \mathbf{A}^1} = t_\infty^{n-1}(t_0 - xt_\infty)$	$\tilde{s}\big\vert_{\mathbf{P}^1 \times U \times \mathbf{A}^1 \times 0} = s$ $\tilde{s}\big\vert_{\infty \times U \times \mathbf{A}^1 \times \mathbf{A}^1} = t_0^n$ $\tilde{s}\big\vert_{0 \times U \times \mathbf{A}^1 \times \mathbf{A}^1} = t_\infty^{n-1}(t_0 - xt_\infty)$	$\tilde{s}\big\vert_{\mathbf{P}^1 \times U \times \mathbf{A}^1 \times 1} = (t_0 - xt_\infty)s'$ $s'\big\vert_{\infty \times U \times \mathbf{A}^1} = t_0^{n-1}$ $s'\big\vert_{0 \times U \times \mathbf{A}^1} = t_\infty^{n-1}$ $s'\big\vert_{Z(t_0 - xt_\infty)\times U} = t_\infty^{n-1}$

Since U is affine, it follows that $\mathcal{O}(1)$ is ample on $\mathbf{P}^1 \times U \times \mathbf{A}^1$ and $\mathbf{P}^1 \times U \times \mathbf{A}^1 \times \mathbf{A}^1$. Hence, for n big enough, Serre's theorem III.A.2 ensures the existence of the sections s and s' as above. Having s and s', we then put $\tilde{s} := (1-\lambda)s + \lambda(t_0 - xt_\infty)s'$.

It follows by Lemma III.A.7 that $Z(s)$ and $Z(\tilde{s})$ are finite over $U \times \mathbf{A}^1$ and $U \times \mathbf{A}^1 \times \mathbf{A}^1$ respectively. Let $y := t_0/t_\infty$ be the coordinate on the affine line $\mathbf{A}^1 \subseteq \mathbf{P}^1$, and consider the trivialization dy of the canonical class of $\mathbf{A}^1$. Let moreover $p\colon \mathbf{A}^1 \times V \to \mathbf{A}^1_U$ denote the composition of the projection onto V followed by the inclusion $V \subseteq \mathbf{A}^1_U$, and let $p'\colon \mathbf{A}^1 \times \mathbf{A}^1 \times U \times \mathbf{A}^1 \to \mathbf{A}^1_U \times \mathbf{A}^1$ be the projection onto the last two coordinates. Applying Construction III.3.11 to the diagrams

$$\begin{tikzcd}
\overset{y}{V} \times \overset{x}{\mathbf{A}^1} \arrow[r, "s/t_\infty^n"] \arrow[d, "p"'] \arrow[dr, "\mathrm{pr}"] & \mathbf{A}^1 \\
\overset{x}{\mathbf{A}^1_U} & \overset{x}{V},
\end{tikzcd}
\qquad
\begin{tikzcd}
\overset{y}{\mathbf{A}^1} \times U \times \overset{x}{\mathbf{A}^1} \times \overset{\lambda}{\mathbf{A}^1} \arrow[r, "\widetilde{s}/t_\infty^n"] \arrow[d, "p'"'] \arrow[dr, "\mathrm{pr}_2"] & \mathbf{A}^1 \\
\overset{x}{\mathbf{A}^1_U} \times \overset{\lambda}{\mathbf{A}^1} & \overset{x}{\mathbf{A}^1_U},
\end{tikzcd}$$

we thus obtain finite A-correspondences

$$\Phi' := \mathrm{div}^A(s/t_\infty^n)^{dy,\mathrm{pr}} \in \mathrm{Cor}_k^{A,\mathrm{pr}}((\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U), (V, V \setminus 0_U)),$$
$$\Theta' := \mathrm{div}^A(\tilde{s}/t_\infty^n)^{dy,\mathrm{pr}_2} \in \mathrm{Cor}_k^{A,\mathrm{pr}}(\mathbf{A}^1 \times (\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U), (\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U)).$$

It then follows from the properties of s and $\tilde{s}$ above that

$$\begin{aligned} \Theta' \circ i_0 &= i \circ \Phi', \\ \Theta' \circ i_1 &= \operatorname{div}^A((y-x)g)_{Z(y-x)} + \operatorname{div}^A((y-x)g)_{Z(g)}, \end{aligned}$$

where $g := s'/t_\infty^{n-1} \in k[\mathbf{A}^1 \times \mathbf{A}^1 \times U]$. By Lemma III.3.18 the first summand in the last equality is equal to $\langle \nu \rangle$ for some $\nu \in k[\mathbf{A}^1_U]^\times$. The second summand, $\operatorname{div}^A((y-x)g)_{Z(g)}$, is zero by Lemma III.3.15 since $Z(g) \cap (0 \times \mathbf{A}^1 \times U) = \varnothing$. Now we define $\Phi := \Phi' \circ \langle \nu^{-1} \rangle$ and $\Theta := \Theta' \circ (\langle \nu^{-1} \rangle \times \mathrm{id}_{\mathbf{A}^1})$. Then $\Theta' \circ i_1 = \mathrm{id}_{(\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U)}$, and the claim follows. ∎

III.6.0.2

The next step is to show surjectivity of i^*:

Lemma III.6.4. *Suppose that $i\colon V \subseteq \mathbf{A}^1_U$ is an open subscheme with $0_U \in V$. Then there is a finite A-correspondence of pairs*

$$\Psi \in \mathrm{Cor}_k^{A,\mathrm{pr}}((\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U), (V, V \setminus 0_U))$$

such that

$$[\Psi \circ i] = [\mathrm{id}_{(V, V\setminus 0_U)}] \in \overline{\mathrm{Cor}}_k^{A,\mathrm{pr}}((V, V \setminus 0_U), (V, V \setminus 0_U)).$$

Proof. To prove the claim we need to construct a finite A-correspondence of pairs

$$\Psi \in \mathrm{Cor}_k^{A,\mathrm{pr}}((\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U), (V, V \setminus 0_U))$$

along with a homotopy

$$\Theta \in \mathrm{Cor}_k^{A,\mathrm{pr}}(\mathbf{A}^1 \times (V, V \setminus 0_U), (V, V \setminus 0_U))$$

such that $\Theta \circ i_0 = \Psi \circ i$ and $\Theta \circ i_1 = \mathrm{id}_{(V, V \setminus 0_U)}$. We do this via the following sections:

$s \in \Gamma\left(\overset{[t_0:\, t_\infty]}{\mathbf{P}^1}_{U \times \overset{x}{\mathbf{A}^1}}, \mathcal{O}(n)\right)$	$\tilde{s} \in \Gamma\left(\overset{[t_0:\, t_\infty]}{\mathbf{P}^1}_{\overset{x}{V} \times \overset{\lambda}{\mathbf{A}^1}}, \mathcal{O}(n)\right)$	$s' \in \Gamma\left(\overset{[t_0:\, t_\infty]}{\mathbf{P}^1}_{\overset{x}{V}}, \mathcal{O}(n-1)\right)$
$s\big\vert_{D \times \mathbf{A}^1} = t_0^n$ $s\big\vert_{0 \times U \times \mathbf{A}^1} = t_0 - xt_\infty$	$\tilde{s}\big\vert_{\mathbf{P}^1 \times V \times 0} = s$ $\tilde{s}\big\vert_{D \times V \times \mathbf{A}^1} = t_0^n$ $\tilde{s}\big\vert_{0 \times V \times \mathbf{A}^1} = t_0 - xt_\infty$	$\tilde{s}\big\vert_{\mathbf{P}^1 \times V \times 1} = (t_0 - xt_\infty)s'$ $g\big\vert_{D \times \mathbf{A}^1} = t_0^n (t_0 - xt_\infty)^{-1}$ $s'\big\vert_{0 \times V} = t_\infty^n$ $s'\big\vert_{Z(t_0 - xt_\infty)} = t_\infty^{n-1}$

Here $D := (\mathbf{P}^1 \times U) \setminus V$ denotes the reduced closed complement, $g := s'/t_\infty^{n-1} \in k[\mathbf{A}^1 \times V]$, and $Z(t_0 - xt_\infty) \subseteq \mathbf{P}^1 \times V$ denotes vanishing locus of the section

$$t_0 - xt_\infty \in \Gamma(\mathbf{P}^1 \times V, \mathcal{O}(1)),$$

with $[t_0\colon t_\infty]$ being coordinates on $\mathbf{P}^1$, and x the one on V. Since U is affine, it follows that $\mathcal{O}(1)$ is ample on $\mathbf{P}^1 \times \mathbf{A}^1 \times U$ and $\mathbf{P}^1 \times \mathbf{A}^1 \times U \times \mathbf{A}^1$. Hence Serre's theorem III.A.2 ensures the existence of the sections s and s' as above, provided n is big enough. Having s and s', we then put $\tilde{s} := (1-\lambda)s + \lambda(t_0 - xt_\infty)s'$.

Next, it follows by Lemma III.A.7 that $Z(s)$ and $Z(\widetilde{s})$ are finite over $U \times \mathbf{A}^1$ and $V \times \mathbf{A}^1$, respectively. Let $y := t_0/t_\infty$ be the coordinate on the affine line $\mathbf{A}^1 \subseteq \mathbf{P}^1$, and let us use the trivialization dy of the canonical class of $\mathbf{A}^1$. Consider the diagrams

$$\begin{tikzcd} \overset{y}{V} \times \overset{x}{\mathbf{A}^1} \arrow[r, "s/t_\infty^n"] \arrow[d] \arrow[dr, "\mathrm{pr}"] & \mathbf{A}^1 \\ \overset{x}{\mathbf{A}^1_U} & \overset{y}{V}, \end{tikzcd} \qquad \begin{tikzcd} \overset{y}{V} \times_U \overset{x}{V} \times \overset{\lambda}{\mathbf{A}^1} \arrow[r, "\widetilde{s}/t_\infty^n"] \arrow[d] \arrow[dr, "\mathrm{pr}'"] & \mathbf{A}^1 \\ \overset{x}{V} \times \overset{\lambda}{\mathbf{A}^1} & \overset{y}{V}\,. \end{tikzcd}$$

Here the map $\mathrm{pr}\colon \mathbf{A}^1 \times V \to V$ is the projection, while the map $\mathrm{pr}'\colon \mathbf{A}^1 \times V \times \mathbf{A}^1 \to \mathbf{A}^1_U$ is the composition of the projection onto V followed by the inclusion $V \subseteq \mathbf{A}^1_U$. Applying Construction III.3.11 to these diagrams we get finite A-correspondences of pairs

$$\begin{aligned} \Psi' &:= \mathrm{div}^A(s/t_\infty^n)^{dy,\mathrm{pr}} \in \mathrm{Cor}_k^{A,\mathrm{pr}}((\mathbf{A}^1_U, \mathbf{A}^1_U \setminus 0_U), (V, V \setminus 0_U)), \\ \Theta' &:= \mathrm{div}^A(\tilde{s}/t_\infty^n)^{dy,\mathrm{pr}'} \in \mathrm{Cor}_k^{A,\mathrm{pr}}(\mathbf{A}^1 \times (V, V \setminus 0_U), (V, V \setminus 0_U)), \end{aligned}$$

The properties of s and s' above imply that

$$\begin{aligned} \Theta' \circ i_0 &= \Psi' \circ i, \\ \Theta' \circ i_1 &= \mathrm{div}^A((y-x)g)_{Z(y-x)} + \mathrm{div}^A((y-x)g)_{Z(g)}. \end{aligned}$$

By Lemma III.3.18, the first summand in the last equality is equal to $\langle \nu \rangle \in \mathrm{Cor}_k^{A,\mathrm{pr}}((V, V \setminus 0_U), (V, V \setminus 0_U))$ for some $\nu \in k[V]^\times$. The second summand is zero by Lemma III.3.15, since $Z(g) \cap (0 \times V) = \varnothing$. Hence the A-correspondences $\Psi := \langle \nu^{-1} \rangle \circ \Psi'$ and $\Theta := \langle \nu^{-1} \rangle \circ \Theta'$ have the desired properties. ∎

Proof of Theorem III.6.1. Lemma III.6.3 and Lemma III.6.4 immediately imply the claim for the case of $V_2 = \mathbf{A}^1_U$. In general, it follows that we have natural isomorphisms

$$\mathscr{F}(V_2 \setminus 0_U)/\mathscr{F}(V_2) \cong \mathscr{F}(\mathbf{A}^1_U \setminus 0_U)/\mathscr{F}(\mathbf{A}^1_U) \cong \mathscr{F}(V_1 \setminus 0_U)/\mathscr{F}(V_1),$$

which shows the claim. ∎

III.6.0.3

Arguing similarly as in the proof of Theorem III.6.1, we obtain also an excision result for a nonrelative affine line:

Theorem III.6.5. *Consider the function field $K := k(U)$ of some integral scheme $U \in \mathrm{Sm}_k$. Let z be a closed point in $\mathbf{A}^1_K$, and let $i\colon V_1 \subseteq V_2$ be an inclusion of two open subschemes of $\mathbf{A}^1_K$ such that $z \in V_1$. Then, for any homotopy invariant* *presheaf with A-transfers $\mathscr{F} \in \mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A; \mathbf{Z})$, the restriction homomorphism i^* induces an isomorphism*

$$i^* : \mathscr{F}(V_2 \setminus z)/\mathscr{F}(V_2) \xrightarrow{\cong} \mathscr{F}(V_1 \setminus z)/\mathscr{F}(V_1).$$

Proof. The proof is parallel to the proof of Theorem III.6.1. All we need to do is to replace the line bundle $\mathcal{O}(1)$ by $\mathcal{O}(d)$, where $d := \deg_K k(z)$; the section $t_0 \in \Gamma(\mathbf{P}^1_{\mathbf{A}^1_K}, \mathcal{O}(1))$ by a section $\nu \in \Gamma(\mathbf{P}^1_{\mathbf{A}^1_K}, \mathcal{O}(d))$ such that $Z(\nu) = z \times \mathbf{A}^1_K$; and the section t_∞ by t_∞^d. ■

III.7 Injectivity for semilocal schemes

In this section we will assume that the base field k is infinite.

Theorem III.7.1. *Let X be a smooth k-scheme and let $x_1, \ldots, x_r \in X$ be finitely many closed points. Let $U := \operatorname{Spec} \mathcal{O}_{X,x_1,\ldots,x_r}$ and write $j : U \to X$ for the canonical inclusion. Let $Z \hookrightarrow X$ be a closed subscheme with $x_1, \ldots, x_r \in Z$, and let $i : U \setminus Z \to U$ be the immersion of the open complement to the semilocalization of Z at the points $x_1, \ldots, x_r$. Then, for any homotopy invariant presheaf with A-transfers $\mathscr{F} \in \mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A; \mathbf{Z})$, the homomorphism $i^* : \mathscr{F}(U) \to \mathscr{F}(U \setminus Z)$ is injective.*

III.7.0.1

Theorem III.7.1 is an immediate consequence of the following moving lemma:

Lemma III.7.2. *Assume the hypotheses of Theorem III.7.1. Then there exists a finite A-correspondence $\Phi \in \mathrm{Cor}_k^A(U, X \setminus Z)$ such that the diagram*

$$\begin{tikzcd} & X \setminus Z \arrow[d, "i"] \\ U \arrow[ur, "\Phi"] \arrow[r, "j"] & X \end{tikzcd}$$

commutes up to homotopy.

III.7.0.2

We prove Lemma III.7.2 by constructing an appropriate relative curve $\mathcal{C}$ over U along with a good compactification $\overline{\mathcal{C}}$ of $\mathcal{C}$. The desired finite A-correspondence will then be defined by using certain sections on $\overline{\mathcal{C}}$.

Lemma III.7.3. *Assume the hypotheses of Theorem III.7.1. Then there exists a diagram*

$$X \xleftarrow{v} \mathcal{C} \xrightarrow{j} \overline{\mathcal{C}} \xrightarrow{p} U$$

in EssSm_k, satisfying the following properties:

(1) *$p\colon \overline{\mathcal{C}} \to U$ is a relative projective curve, $j\colon \mathcal{C} \to \overline{\mathcal{C}}$ is an open immersion, and the composition $p \circ j$ is smooth.*

(2) *The map $p \circ j$ admits a section $\Delta\colon U \to \mathcal{C}$. By abuse of notation, we write Δ also for the image of the morphism Δ.*

(3) *Let $\mathcal{Z} := v^{-1}(Z) \subseteq \mathcal{C}$. Then $\mathcal{Z}$ is finite over U.*

(4) *$D := \overline{\mathcal{C}} \setminus \mathcal{C}$ is finite over U.*

(5) *The relative curve $\overline{\mathcal{C}}$ has an ample line bundle $\mathcal{O}_{\overline{\mathcal{C}}}(1)$.*

(6) *There is a trivialization $\mu\colon \mathcal{O}_{\mathcal{C}} \xrightarrow{\cong} \omega_{\mathcal{C}/U}$.*

Proof. We apply Lemma III.A.5 with $\pi = \mathrm{id}\colon X \to X$. ■

Proof of Lemma III.7.2. First of all we apply Lemma III.7.3. Then it follows from Serre's theorem III.A.2 that there is an integer $l \gg 0$ and a section $d \in \Gamma(\overline{\mathcal{C}}, \mathcal{O}(l))$ such that $D \subseteq Z(d)$, $Z(d) \cap \mathcal{Z} = \varnothing$ and $Z(d)$ is finite over U. For notational simplicity, let us redenote $\mathcal{O}(l)$ by $\mathcal{O}(1)$, and redenote $D := Z(d)$. Now our aim is to construct the following sections:

$s \in \Gamma(\overline{\mathcal{C}}, \mathcal{O}(n))$	$\widetilde{s} \in \Gamma(\overline{\mathcal{C}} \times \overset{\lambda}{\mathbf{A}^1}, \mathcal{O}(n))$	$s' \in \Gamma(\overline{\mathcal{C}}, \mathcal{O}(n) \otimes \mathscr{L}(\Delta)^{-1})$	$\delta \in \Gamma(\overline{\mathcal{C}}, \mathscr{L}(\Delta))$
$Z(s\vert_{\mathcal{Z} \amalg D}) = \varnothing$	$\widetilde{s}\vert_{\overline{\mathcal{C}} \times 0} = s$ $\widetilde{s}\vert_{\overline{\mathcal{C}} \times 1} = s' \otimes \delta$ $\widetilde{s}\vert_{D \times \mathbf{A}^1} = s$	$Z(s'\vert_{\mathcal{Z} \amalg D \amalg \Delta}) = \varnothing$	$Z(\delta) = \Delta$

To do this, let δ be a section of $\mathscr{L}(\Delta)$ with $Z(\delta) = \Delta$, and choose, using Lemma III.A.2, an integer $n \gg 0$ such that the restriction maps

$$\Gamma(\overline{\mathcal{C}}, \mathcal{O}(n) \otimes \mathscr{L}(\Delta)^{-1}) \to \Gamma(\mathcal{Z} \amalg D \amalg \Delta, \mathcal{O}(n) \otimes \mathscr{L}(\Delta)^{-1}),$$
$$\Gamma(\overline{\mathcal{C}}, \mathcal{O}(n)) \to \Gamma(\mathcal{Z} \amalg D, \mathcal{O}(n))$$

are surjective. We can then find a global section s' of $\mathcal{O}(n) \otimes \mathscr{L}(\Delta)^{-1}$ such that $s'|_{\mathcal{Z} \amalg D \amalg \Delta}$ is invertible. Let s be a lift of $s'\delta|_{\mathcal{Z} \amalg D} \in \Gamma(\mathcal{Z} \amalg D, \mathcal{O}(n))$, and define $\widetilde{s} := (1 - \lambda)s + \lambda s' \otimes \delta$. We now aim to apply Construction III.3.11 to the diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "s'\otimes\delta/d^n"] \arrow[r, shift right, "s/d^n"'] \arrow[d, "p\circ j"'] \arrow[drr, "v"] & \mathbf{A}^1 & \\ U & & X, \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \times \mathbf{A}^1 \arrow[r, "\widetilde{s}/d^n"] \arrow[d, "(p\circ j)\times\mathbf{A}^1"'] \arrow[dr, "v\circ\mathrm{pr}"] & \mathbf{A}^1 \\ U \times \mathbf{A}^1 & X. \end{tikzcd}$$

Here $\mathrm{pr}\colon \mathcal{C} \times \mathbf{A}^1 \to \mathcal{C}$ is the projection. By Lemma III.A.7, the vanishing loci $Z(s)$ and $Z(\widetilde{s})$ are finite over U and $U \times \mathbf{A}^1$, respectively. Hence we obtain finite

A-correspondences

$$\begin{aligned}\Phi' &:= \mathrm{div}^A(s/d^n)^{\mu,v}_{Z(s)} - \mathrm{div}^A(s'\otimes\delta/d^n)^{\mu,v}_{Z(s')}\\ &\in \mathrm{Cor}^A_k(U, X\setminus Z),\\ \Theta' &:= \mathrm{div}^A(\tilde{s}/d^n)^{\mu,v\circ\mathrm{pr}^{\mathcal{C}\times\mathbf{A}^1}_{\mathcal{C}}}_{Z(\widetilde{s})} - \mathrm{div}^A(s'\otimes\delta/d^n)^{\mu,v}_{Z(s')}\circ\mathrm{pr}^{U\times\mathbf{A}^1}_U\\ &\in \mathrm{Cor}^A_k(U\times\mathbf{A}^1, X).\end{aligned}$$

Then the properties of the sections above imply that $\Theta'\circ i_0 = i\circ\Phi'$, and Lemma III.3.18 implies that $\Theta'\circ i_1 = j\circ\langle\nu\rangle$ for some $\nu\in k[U]^\times$. Now let $\Phi := \Phi'\circ\langle\nu^{-1}\rangle$. Then $\Theta := \Theta'\circ\langle\nu^{-1}\rangle$ gives the required homotopy, satisfying $\Theta\circ i_0 = i\circ\Phi$ and $j = \Theta\circ i_1$. ■

III.8 Nisnevich excision

In this section we assume that the base field is infinite. The main result of the section is the following Nisnevich excision result for homotopy invariant presheaves with A-transfers:

Theorem III.8.1. *Let $X\in\mathrm{Sm}_k$ and suppose that $\pi\colon(X',Z')\to(X,Z)$ is an étale neighborhood of Z in X. Assume also that $z\in Z$ and $z'\in Z'$ are two closed points such that $\pi(z')=z$. Write $U := X_z = \mathrm{Spec}\,\mathcal{O}_{X,z}$ for the corresponding local scheme, and similarly $U' := X'_{z'}$. Then, for any homotopy invariant presheaf with A-transfers $\mathscr{F}\in\mathrm{PSh}_\Sigma(\mathrm{Cor}^A_k;\mathbf{Z})$, the map π^* induces an isomorphism*

$$\pi^*\colon \mathscr{F}(X_z\setminus Z_z)/\mathscr{F}(X_z)\xrightarrow{\cong}\mathscr{F}(X'_{z'}\setminus Z'_{z'})/\mathscr{F}(X'_{z'}).$$

III.8.0.1

The proof of Theorem III.8.1 relies on some geometric input. Our main tool for this is Lemma III.A.5; we refer the reader to the appendix for details around this construction.

Having Lemma III.A.5 at hand, we start out by showing that the map π^* is injective:

Lemma III.8.2. *Under the assumptions of Theorem III.8.1 there is a finite A-correspondence $\Phi\in\mathrm{Cor}^A_k(U,X')$ satisfying $\pi\circ\Phi\sim_{\mathbf{A}^1} i$, where $i\colon U\to X$ denotes canonical embedding.*

Proof. Applying Lemma III.A.5 we obtain a morphism of relative curves $\varpi\colon\mathcal{C}'\to\mathcal{C}$ over U, with compactification $\overline{\varpi}\colon\overline{\mathcal{C}'}\to\overline{\mathcal{C}}$, and subschemes $D,\Delta,\mathcal{Z}\subseteq\overline{\mathcal{C}}$, $D',\Delta'_Z,\mathcal{Z}'\subseteq\overline{\mathcal{C}'}$ as in Lemma III.A.5. Let $\delta\in\Gamma(\overline{\mathcal{C}},\mathscr{L}(\Delta))$ be a section such that $Z(\delta)=\Delta$. Our first aim is to prove that there is an integer N such that for all $n\geq N$, there exist sections satisfying the following conditions:

$s \in \Gamma(\overline{\mathcal{C}}, \mathcal{O}(n))$	$\tilde{s} \in \Gamma(\overline{\mathcal{C}} \times \overset{\lambda}{\mathbf{A}^1}, \mathcal{O}(n))$	$s' \in \Gamma(\overline{\mathcal{C}}, \mathcal{O}(n) \otimes \mathscr{L}(\Delta)^{-1})$
	$\tilde{s}\big\|_{\overline{\mathcal{C}} \times 0} = s$	$\tilde{s}\big\|_{\overline{\mathcal{C}} \times 1} = \delta \otimes s'$
$Z(s\big\|_{D}) = \varnothing$	$Z(\tilde{s}\big\|_{D \times \mathbf{A}^1}) = \mathrm{pr}^*(s)$	
$s\big\|_{\mathcal{Z}} = \delta \otimes s'$	$\tilde{s}\big\|_{\mathcal{Z} \times \mathbf{A}^1} = \delta \otimes s'$	$Z(s'\big\|_{\mathcal{Z}}) = \varnothing$
	$Z(\tilde{s}) \cap Z(d) = \varnothing$	

In addition, we will require that $Z(s) = Z_0 \amalg Z_0'$ and that there exists a regular map $l \colon Z_0 \to \mathcal{C}'$ satisfying $\varpi \circ l = \mathrm{id}_{Z_0}$. Here $\mathrm{pr} \colon \overline{\mathcal{C}} \times \mathbf{A}^1 \to \overline{\mathcal{C}}$ is the canonical projection.

To do this we start the following preparations. Let $\mathcal{O}_{\mathcal{C}'}(1) := \overline{\varpi}^*(\mathcal{O}(1))$. Then, since $\overline{\varpi}$ is finite, $\mathcal{O}_{\mathcal{C}'}(1)$ is an ample bundle on $\overline{\mathcal{C}'}$. Since ϖ induces isomorphisms $\mathcal{Z}' \cong \mathcal{Z}$ and $\Delta_Z' \cong \Delta \times_{\mathcal{C}} \mathcal{Z}$, there is a section $\delta' \in \Gamma(\mathcal{Z}', \mathscr{L}')$ such that $Z(\delta') = \Delta_Z'$ for some line bundle $\mathscr{L}'$ on $\mathcal{Z}'$. Since $\mathcal{Z}'$ is a finite scheme over a local scheme U, $\mathcal{Z}'$ is semilocal and any line bundle on $\mathcal{Z}'$ is trivial. Hence there is an isomorphism $\mathscr{L}' \cong \mathcal{O}_{\mathcal{C}'}(m)\big|_{\mathcal{Z}'}$ for any $m \in \mathbf{Z}$. Similarly, since the subscheme $D' \subseteq \overline{\mathcal{C}'}$ is finite over U, for any $m \in \mathbf{Z}$, the line bundle $\mathcal{O}_{\mathcal{C}'}(m)\big|_{D'}$ is trivial. Now, applying Lemma III.A.6 to the morphism $\overline{\varpi} \colon \overline{\mathcal{C}'} \to \overline{\mathcal{C}}$ and the subschemes D' and $\mathcal{Z}$ we construct, for some $m \in \mathbf{Z}$, a section $\xi \in \Gamma(\mathcal{C}', \mathcal{O}_{\mathcal{C}'}(m))$ such that there is a closed embedding $Z(\xi) \to \mathcal{C}$, and such that $Z(\xi\big|_{\overline{\varpi}^{-1}(\mathcal{Z})}) = \Delta_Z'$. Define $Z_0 := \overline{\varpi}(Z(\xi)) \subseteq \mathcal{C} \subseteq \overline{\mathcal{C}}$ and put $\mathscr{L} := \mathscr{L}(Z)$. Let $\zeta \in \Gamma(\overline{\mathcal{C}}, \mathscr{L})$ be a section with $Z(\zeta) = Z$. Then $Z(\zeta\big|_{\mathcal{Z}}) = \Delta_Z$.

Using Serre's theorem III.A.2 we can choose an integer $N \in \mathbf{Z}$ such that for all $n \geq N$, the restriction homomorphisms

$$\Gamma(\overline{\mathcal{C}}, \mathcal{O}(n) \otimes \mathscr{L}^{-1}) \to \Gamma(\mathcal{Z} \amalg D, \mathcal{O}(n) \otimes \mathscr{L}^{-1}),$$
$$\Gamma(\overline{\mathcal{C}}, \mathcal{O}(n)) \to \Gamma((\mathcal{Z} \cup \Delta) \amalg D, \mathcal{O}(n))$$

are surjective. Then, since $\mathcal{Z} \amalg D$ is semilocal, there is a section

$$\zeta' \in \Gamma(\overline{\mathcal{C}}, \mathcal{O}(n) \otimes \mathscr{L}^{-1})$$

such that $\zeta'\big|_{\mathcal{Z} \amalg D}$ is invertible. Define $s := \zeta \otimes \zeta' \in \Gamma(\overline{\mathcal{C}}, \mathcal{O}(n))$.

Now choose a section $s_1 \in \Gamma(\overline{\mathcal{C}}, \mathcal{O}(n))$ such that $s_1\big|_{\Delta} = 0$ and $s_1\big|_{\mathcal{Z}} = s$. We then put $\tilde{s} := (1 - \lambda)s + \lambda s_1$. Since $s_1\big|_{\Delta} = 0$, there is a section

$$s' \in \Gamma(\overline{\mathcal{C}}, \mathcal{O}(n) \otimes \mathscr{L}(\Delta)^{-1})$$

such that $s_1 = \delta \otimes s'$, where $\delta \in \Gamma(\overline{\mathcal{C}}, \mathscr{L}(\Delta))$ satisfies $Z(\delta) = \Delta$. Moreover, since by construction $Z(s_1\big|_{\mathcal{Z}}) = \Delta_Z = Z(\delta\big|_{\mathcal{Z}})$, it follows that $s'\big|_{\mathcal{Z}}$ is invertible and so $Z(s'\big|_{\mathcal{Z}}) = \varnothing$. Hence the desired sections s, $\tilde{s}$, and s' are constructed. It now follows by Lemma III.A.7 that $Z(s)$ and $Z(\tilde{s})$ are finite over U and $U \times \mathbf{A}^1$ respectively.

By construction, the morphism ϖ induces an isomorphism between the closed subschemes $l(Z_0) \subseteq \mathcal{C}'$ and Z_0. Since ϖ is étale, it follows that $\varpi^{-1}(Z_0) = l(Z_0) \amalg$

$\widehat{Z}_0$. Hence we can define an étale neighborhood $\varpi^+ : (\mathcal{C}' \setminus \widehat{Z}_0, l(Z_0)) \to (\mathcal{C}, Z_0)$ such that $\varpi^+(Z_0) = l(Z_0)$. Consider the diagrams

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\varpi^*(s/d^n)"] \arrow[d, "p\circ j\circ\varpi"'] \arrow[dr, "v'"] & \mathbf{A}^1 \\ U & X', \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \times \mathbf{A}^1 \arrow[r, "\widetilde{s/d^n}"] \arrow[d, "(p\circ j)\times\mathbf{A}^1"'] \arrow[dr, "v\circ\mathrm{pr}"] & \mathbf{A}^1 \\ U \times \mathbf{A}^1 & X, \end{tikzcd}$$

where $\mathrm{pr} : \mathcal{C} \times \mathbf{A}^1 \to \mathcal{C}$ is the projection. Applying Construction III.3.11 to these diagrams we obtain finite A-correspondences

$$\begin{aligned} \Phi' &:= \mathrm{div}^A(\varpi^*(s/d^n))_{Z_0}^{\varpi^*(\mu),v'} \in \mathrm{Cor}_k^{A,\mathrm{pr}}((U, U \setminus Z \times_X U), (X', X' \setminus Z')), \\ \Theta' &:= \mathrm{div}^A(\tilde{s}/d^n)^{v\circ\mathrm{pr}} \in \mathrm{Cor}_k^{A,\mathrm{pr}}(\mathbf{A}^1 \times (U, U \setminus Z \times_X U), (X, X \setminus Z)). \end{aligned}$$

It follows from the list of properties above, Lemma III.3.18, and Lemma III.3.15 that $\Theta' \circ i_1 = i \circ \langle \nu \rangle$ for some invertible function $\nu \in k[U]^\times$. If we let $\Phi := \Phi' \circ \langle \nu^{-1} \rangle$ and $\Theta := \Theta' \circ \langle \nu^{-1} \rangle$, it follows that $\Theta \circ i_1 = i$. So to prove the lemma it is enough to show that $\Theta \circ i_0 \sim_{\mathbf{A}^1} \pi \circ \Phi$.

Since $\overline{\varpi}$ is finite, it is affine. Hence for some Zariski neighborhood V' of $l(Z_0)$ in $\mathcal{C}' \setminus \widehat{Z}_0$, the restriction $\varpi\big|_V$ is affine. Then, for some Zariski neighborhood V of Z_0 in $\mathcal{C}$, there is a closed embedding $c : V'' \subseteq \mathbf{A}^r \times V$, where $V'' := V' \cap \varpi^{-1}(V)$, which is such that $c(l(Z_0)) = 0 \times Z_0$. Let $f_1, \ldots, f_r \in k[\mathbf{A}^r \times V]$ be functions satisfying $f_i\big|_{c(V'')} = 0$ and $f_i\big|_{\mathbf{A}^r \times Z_0} = x_i$, where the x_i's denote the coordinate functions on $\mathbf{A}^r$. For $i = 1, \ldots, r$, let $\widetilde{f}_i := (1 - \lambda) f_i + \lambda x_i$ and consider the closed subscheme $Z(\widetilde{f}_1, \ldots, \widetilde{f}_r) \subseteq \mathbf{A}^r \times V \times \mathbf{A}^1$. Then the projection $\mathrm{pr} : Z(\tilde{f}_1, \ldots, \tilde{f}_r) \to V \times \mathbf{A}^1$ is étale over $Z_0 \times \mathbf{A}^1$. Let $W \subseteq Z(\tilde{f}_1, \ldots, \tilde{f}_r)$ be a Zariski neighborhood of $0 \times Z_0 \times \mathbf{A}^1$ such that the restriction of the projection $\mathrm{pr}_W : W \to V \times \mathbf{A}^1$ is étale. Furthermore, let t be the pullback of s/d^n from V to W, and let $i_V : V \to \mathcal{C}$ denote the open embedding. Applying Construction III.3.11 to the diagram

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "v\times\mathbf{A}^1"'] \arrow[dr, "v\circ i_V\circ\mathrm{pr}_W"] & \mathbf{A}^1 \\ U \times \mathbf{A}^1 & X, \end{tikzcd}$$

we obtain a homotopy

$$\mathrm{div}^A(t)_{0\times Z_0\times\mathbf{A}^1}^{\mathrm{pr}_W^*(\omega), v\circ i_V\circ\mathrm{pr}_W} \in \mathrm{Cor}_k^{A,\mathrm{pr}}(\mathbf{A}^1 \times (U, U \setminus Z \times_X U), (X, X \setminus Z))$$

connecting $\pi \circ \Phi' = \mathrm{div}^A(\varpi^*(s/d^n))^{\varpi^*(\omega), v\circ\varpi}$ and $\Theta \circ i_0 = \mathrm{div}^A(s/d^n)^{\omega, v}$. ■

III.8.0.2

Before we move on to the surjective part of Nisnevich excision, we need the following lemma:

Lemma III.8.3. *Suppose that* $\operatorname{char} k \neq 2$*, and let* $X \in \mathrm{Sm}_k$*. Let* $Z \subseteq X$ *be a closed subscheme and* $z \in X$ *a closed point. Write* U *for the essentially smooth local scheme* $U := X^h_z = \operatorname{Spec} \mathcal{O}^h_{X,z}$*, and let* $\lambda \in k[U]^\times$ *be an invertible regular function satisfying* $\lambda\big|_{Z\times_X U} = 1$*. Then*

$$i \circ \langle\lambda\rangle \sim_{\mathbf{A}^1} i \in \mathrm{Cor}^{A,\mathrm{pr}}_k((U, U \setminus Z \times_X U), (X, X \setminus Z)),$$

where i *denotes the canonical morphism* $i\colon U \to X$*.*

Proof. Lift λ to an invertible section on some affine Zariski neighborhood $V \subseteq X$ of the point $z \in X$. Then $\lambda\big|_{Z\times_X V'} = 1$ for some other Zariski neighborhood $V' \subseteq V$ of z; shrinking X to V' we may assume that $\lambda \in k[X]^\times$ with $\lambda\big|_Z = 1$.

Consider the étale covering $\pi\colon X' \to X$, where $X' = \operatorname{Spec} k[X][w]/(w^2 - \lambda)$. Let Z' be the closed subscheme of X' given by $Z' := \operatorname{Spec} k[Z][w]/(w-1)$, so that $Z' \cong Z$. Then $(X', Z') \to (X, Z)$ is an étale neighborhood. By Lemma III.8.2 there exists a finite A-correspondence of pairs

$$\Phi \in \mathrm{Cor}^{A,\mathrm{pr}}_k((U, U \setminus Z \times_X U), (X', X' \setminus Z'))$$

such that $\pi \circ \Phi \sim_{\mathbf{A}^1} i$ in $\mathrm{Cor}^{A,\mathrm{pr}}_k((U, U \setminus Z \times_X U), (X, X \setminus Z))$. On other hand, Lemma III.3.19 implies that

$$\langle\lambda\rangle \circ \pi = \pi \circ \langle\pi^*(\lambda)\rangle = \pi \circ \langle w^2\rangle \sim_{\mathbf{A}^1} \pi \in \mathrm{Cor}^{A,\mathrm{pr}}_k((X, X \setminus Z), (X, X \setminus Z)).$$

Hence $i \circ \langle i^*(\lambda)\rangle = \langle\lambda\rangle \circ i \sim_{\mathbf{A}^1} \langle\lambda\rangle \circ \pi \circ \Phi \sim_{\mathbf{A}^1} \pi \circ \Phi \sim_{\mathbf{A}^1} i$. ∎

Lemma III.8.4. *Let* $i'\colon U' = X'_{z'} \to X'$ *denote the canonical embedding. Then under the assumptions of Theorem III.8.1, there exists* $\Phi \in \mathrm{Cor}^A_k(U, X')$ *such that* $\Phi \circ \pi \sim_{\mathbf{A}^1} i'$*.*

Proof. Using Lemma III.A.5 we construct relative projective curves $p'\colon \overline{\mathcal{C}}' \to U$, $p''\colon \overline{\mathcal{C}}'' \to U'$, along with the other data related to the first two rows of the diagram (III.23).

Since U' is essentially smooth, we have $\Delta'' \cong U'$. Moreover, since $p''\colon \mathcal{C}'' \to U''$ is a smooth morphism with fibers of dimension one, it follows that Δ'' is a smooth divisor on $\mathcal{C}''$. Hence it is a smooth divisor on $\overline{\mathcal{C}}''$ as well and there is an invertible bundle $\mathscr{L}(\Delta'')$ on $\overline{\mathcal{C}}''$ and a section $\delta \in \Gamma(\overline{\mathcal{C}}'', \mathscr{L}(\Delta''))$ such that $Z(\delta) = \Delta''$.

Since $\mathcal{Z}'$ is finite over the local scheme U, $\mathcal{Z}'$ is semilocal. Let $\delta' \in k[\mathcal{Z}']$ be a regular function such that $\delta'\big|_{\Delta'_Z} = 0$, and such that δ' is invertible on the closed points of $\mathcal{Z}'$ outside Δ'_Z. Then the closed fibers of $Z(\delta')$ and Δ'_Z coincide. Now $Z(\delta')$ is finite over U since it is a closed subset in $\mathcal{Z}'$. Moreover, Δ_Z is finite over U since Δ_Z is isomorphic to the closed subscheme $U \times_X Z$ in U. Hence $Z(\delta') = \Delta_Z$ by Nakayama's lemma.

Using the notations of Lemma III.A.5, define $\mathcal{O}_{\overline{\mathcal{C}}'}(1) := \overline{\varpi}'^*(\mathcal{O}(1))$ and $\mathcal{O}_{\overline{\mathcal{C}}''}(1) := \overline{\varpi}^*\overline{\varpi}'^*(\mathcal{O}(1))$. Then, since $\mathcal{O}(1)$ is ample and $\overline{\varpi}, \overline{\varpi}'$ are finite, it follows that $\mathcal{O}_{\overline{\mathcal{C}'}}(1)$ and $\mathcal{O}_{\overline{\mathcal{C}''}}(1)$ are ample. Serre's theorem III.A.2 then tells us that there is an integer $n \in \mathbf{Z}$ such that the restriction homomorphisms

$$\Gamma(\overline{\mathcal{C}}', \mathcal{O}(n)) \to \Gamma(\mathcal{Z}'' \amalg D'', \mathcal{O}(n) \otimes \mathscr{L}(\Delta'')), \tag{III.6}$$

$$\Gamma(\overline{\mathcal{C}}'', \mathcal{O}(n) \otimes \mathscr{L}(\Delta'')) \to \Gamma(\mathcal{Z}'' \amalg D'', \mathcal{O}(n) \otimes \mathscr{L}(\Delta'')) \tag{III.7}$$

are surjective. As mentioned above, $\mathcal{Z}$ and D are finite over U, so it follows that $\mathcal{Z}'$ and D' are semilocal, and moreover that there are trivializations $\xi_Z \colon \mathcal{O}_{\mathcal{Z}'} \overset{\cong}{\to} \mathcal{O}_{\overline{\mathcal{C}'}}(1)\big|_{\mathcal{Z}'}$ and $\xi_D \colon \mathcal{O}_{D'} \overset{\cong}{\to} \mathcal{O}_{\overline{\mathcal{C}'}}(1)\big|_{D'}$. Now using surjectivity of the map (III.6) we find a section

$$s \in \Gamma(\overline{\mathcal{C}'}, \mathcal{O}(n)), \quad s\big|_{\mathcal{Z}'} = \delta \otimes \xi_Z^{\otimes n}, \quad s\big|_{D'} = \xi_D^{\otimes n}.$$

By the same reason as above there is some trivialization

$$\xi'_Z \colon \mathcal{O}_{\mathcal{Z}''} \overset{\cong}{\to} \mathscr{L}(\Delta'')\big|_{\mathcal{Z}''}.$$

Then $b_1 = \varpi^*(\delta')$ and $b_2 = \delta \otimes {\xi'_Z}^{-1}$ are two regular functions on $\mathcal{Z}''$ such that $Z(b_1) = Z(b_2) = \Delta''_Z$. Hence there is an invertible function $\nu \in k[\mathcal{Z}'']^\times$ such that $\varpi^*(\delta')\nu = \delta \otimes {\xi'_Z}^{-1}$. Indeed, ν is uniquely defined by the equality $b_1\nu = b_2$ on the closed subscheme $Z(I) \subseteq \mathcal{Z}''$. Here $I := \ker(m^{b_1})$, where $m^{b_1} \in \mathrm{End}(k[\mathcal{Z}''])$ is defined as multiplication by b_1. Moreover, the equality $b_1\nu = b_2$ implies that ν is invertible on $Z(I)$, and any lift of ν to a regular function on $\mathcal{Z}''$ satisfies the equality $b_1\nu = b_2$ as well. So it is enough to choose a lift such that ν is nonzero at the closed points of $\mathcal{Z}'' \setminus Z(I)$.

Using surjectivity of the second map (III.7), we find a section

$$s' \in \Gamma(\overline{\mathcal{C}''}, \mathcal{O}(n) \otimes \mathscr{L}(\Delta'')^{-1}), \quad s'\big|_{\mathcal{Z}''} = \overline{\varpi}'^*(\xi_Z)^{\otimes n}\nu, \quad s'\big|_{D''} = \overline{\varpi}^*(\xi_D^{\otimes n}) \otimes \delta\big|_{D''}^{-1}.$$

Note that the section $\delta\big|_{D''}^{-1}$ is well defined since $\Delta'' \cap D'' = \varnothing$. Now define $\tilde{s} := (1-\lambda)s + \lambda s'$. Then we have:

$s \in \Gamma(\overline{\mathcal{C}'}, \mathcal{O}(n))$	$\tilde{s} \in \Gamma(\overline{\mathcal{C}''} \overset{\lambda}{\times} \mathbf{A}^1, \mathcal{O}(n))$	$s' \in \Gamma(\overline{\mathcal{C}''}, \mathcal{O}(n) \otimes \mathscr{L}(\Delta')^{-1})$
	$\tilde{s}\big|_{\overline{\mathcal{C}''}\times 0} = \varpi'^*(s)$	$\tilde{s}\big|_{\overline{\mathcal{C}''}\times 1} = \delta \otimes s'$
$Z(s\big|_{D'}) = \varnothing$	$Z(\tilde{s}\big|_{D''}) = \mathrm{pr}^*(\varpi'^*(s))$	
$s\big|_{\mathcal{Z}'\times_U Z} = \delta' \otimes s'$	$\tilde{s}\big|_{\mathcal{Z}''\times \mathbf{A}^1} = \delta \otimes s'$	$Z(s'\big|_{\mathcal{Z}''}) = \varnothing$

We now aim to apply Construction III.3.11 to the diagrams

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "s/d^n"] \arrow[d, "p'\circ j'"'] \arrow[dr, "v'"] & \mathbf{A}^1 \\ U & X', \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}'' \times \mathbf{A}^1 \arrow[r, "\widetilde{s/d^n}"] \arrow[d, "(p''\circ j'')\times \mathbf{A}^1"'] \arrow[dr, "v''\circ \mathrm{pr}"] & \mathbf{A}^1 \\ U' \times \mathbf{A}^1 & X'. \end{tikzcd}$$

Here $\mathrm{pr}\colon \mathcal{C}'' \times \mathbf{A}^1 \to \mathcal{C}''$ is the projection. By Lemma III.A.7, $Z(s)$ and $Z(\tilde{s})$ are finite over U and $U' \times \mathbf{A}^1$, respectively. Hence Construction III.3.11 yields finite A-correspondences

$$\Phi' := \mathrm{div}^A(s/d^n)^{\mu',v'} \in \mathrm{Cor}_k^A(U, X'),$$
$$\Theta' := \mathrm{div}^A(\tilde{s}/d^n)^{\varpi^*(\mu'),v''\circ\mathrm{pr}} \in \mathrm{Cor}_k^A(U' \times \mathbf{A}^1, X').$$

Then, by construction,

$$\Theta' \circ i_0 = \Phi' \circ \pi,$$
$$\Theta' \circ i_1 = \mathrm{div}^A(\delta \otimes s'/d^n)_{\Delta''}^{\varpi^*(\mu'),v''} + \mathrm{div}^A(\delta \otimes s'/d^n)_{Z(s')}^{\varpi^*(\mu'),v''}.$$

By Lemma III.3.15 we have

$$\mathrm{div}^A(\delta \otimes s'/d^n)_{Z(s')}^{\varpi^*(\mu'),v''} = 0 \in \mathrm{Cor}_k^{A,\mathrm{pr}}((U', U' \setminus Z' \times_{X'} U'), (X', X' \setminus Z')).$$

Furthermore, Lemma III.3.18 tells us that $\mathrm{div}^A(\delta \otimes s'/d^n)_{\Delta''}^{\varpi^*(\mu'),v''} = i' \circ \langle \lambda' \rangle$ for some $\lambda' \in k[U']^\times$. Let $\omega \in k[U]^\times$ be an invertible function on U satisfying $\pi^*(\omega)(z) = \lambda'(z)^{-1}$. Define $\Phi := \Phi' \circ \langle \omega \rangle$ and $\Theta := \Theta' \circ \langle \pi^*(\omega) \rangle$. Then $\Theta \circ i_1 = i' \circ \langle \lambda' \cdot \pi^*(\omega) \rangle$ and so Lemma III.8.3 yields the claim. ■

Proof of Theorem III.8.1. Lemmas III.8.2 and III.8.4 establish respectively injectivity and surjectivity of the map π^*. ■

III.8.0.3

We finish this section with a result on the interplay between Zariski excision, Nisnevich excision, étale excision and homotopy invariance for the cohomology theory A^*.

Corollary III.8.5. *Suppose that A^* is a graded presheaf of abelian groups that satisfies all properties of a good cohomology theory except the étale excision axiom. Instead, assume that A^* satisfies Zariski excision and homotopy invariance. In other words, for any $X \in \mathrm{Sm}_k$, any line bundle $\mathscr{L}$ on X, any open subscheme $j\colon U \subseteq X$ and any closed subscheme $Z \subseteq X$ such that $Z \subseteq U$, the maps*

$$\mathrm{pr}^*\colon A^n(X, \mathscr{L}) \xrightarrow{\cong} A^n(X \times \mathbf{A}^1, \mathrm{pr}^* \mathscr{L}),$$
$$j^*\colon A^n(X, X \setminus Z, \mathscr{L}) \xrightarrow{\cong} A^n(U, U \setminus Z, j^* \mathscr{L})$$

are isomorphisms.

Then A^* satisfies the étale excision axiom on local schemes. In other words, for any $X \in \mathrm{Sm}_k$, $Z \subseteq X$, $\pi\colon (X', Z') \to (X, Z)$, $z \in Z$ and $z' \in Z'$ as in Theorem III.8.1, the morphism π induces an isomorphism

$$\pi^*\colon A^*(X_z, X_z \setminus Z_z) \xrightarrow{\cong} A^*(X'_{z'}, X'_{z'} \setminus Z'_{z'}).$$

Proof. Consider the category Cor_k^A of correspondences built from A^* in the sense of Definition III.3.1. First of all we see that the proofs of Lemmas III.8.2 and III.8.4 (as well as Construction III.3.11) do not use the étale excision axiom for A^*. Thus we have morphisms $\Phi_l, \Phi_r \in \mathrm{Cor}_k^A(U, X')$ such that $\pi \circ \Phi_r = i$, and $\Phi_r \circ \pi = i'$. Then Φ_l induces a right inverse $A^*(X'_{z'}, X'_{z'} \setminus Z'_{z'}) \to A^*(X_z, X_z \setminus Z_z)$ to π^*, and Φ_r induces a left inverse. ∎

III.9 The cancellation theorem

In this section we show the cancellation theorem for A-correspondences by suitably adapting Voevodsky's proof for the case of Cor_k [Voe10]; see Theorem III.9.8. For the sake of brevity we will omit the steps that are identical to Voevodsky's original proof, and rather focus on the details that are specific to our situation. We refer the interested reader to [Voe10] for the remaining formal aspects of the proof.

Definition III.9.1. The *Karoubi envelope* of Cor_k^A is the preadditive category whose objects are pairs (X, p) with $X \in \mathrm{Sm}_k$ and $p \in \mathrm{Cor}_k^A(X, X)$ an idempotent. The morphisms are given by

$$\mathrm{Cor}_k^A((X,p),(X',p')) = \mathrm{im}\left(\mathrm{Cor}_k^A(X,X') \xrightarrow{p' \circ (-) \circ p} \mathrm{Cor}_k^A(X,X')\right).$$

Any object $X \in \mathrm{Sm}_k$ can be considered as an object of the Karoubi envelope of Cor_k^A by $X \mapsto (X, \mathrm{id}_X)$. By abuse of notation, we will write Cor_k^A also for the Karoubi envelope of Cor_k^A.

Definition III.9.2. Define $X \wedge \mathbf{G}_m^{\wedge 1} := \ker(\mathrm{pr}_1 \colon X \times \mathbf{G}_m \to X)$ as an object of the Karoubi envelope of Cor_k^A. Let $\mathrm{pr}^\wedge \colon \mathbf{G}_m^{\times 2} \to \mathbf{G}_m^{\wedge 2}$ denote the canonical projection, and let $\iota^\wedge \colon \mathbf{G}_m^{\wedge 2} \to \mathbf{G}_m^{\times 2}$ denote the canonical injection. Note that $\mathrm{pr}^\wedge \circ \iota^\wedge = \mathrm{id}_{\mathbf{G}_m^{\wedge 2}}$. The external product on A-correspondences defines a functor $(-) \wedge \mathbf{G}_m^{\wedge 1} \colon \mathrm{Cor}_k^A \to \mathrm{Cor}_k^A$ given by $X \to X \wedge \mathbf{G}_m^{\wedge 1}$, $\alpha \mapsto \alpha \times \mathrm{id}_{\mathbf{G}_m^{\wedge 1}}$. Furthermore, for any $X \in \mathrm{Sm}_k$ we let $\mathrm{c}_A(X) \wedge \mathbf{G}_m^{\wedge 1}$ denote the presheaf

$$U \mapsto \mathrm{Cor}_k^A(U \wedge \mathbf{G}_m, X \wedge \mathbf{G}_m).$$

Lemma III.9.3. *Let $\tau^\times \colon \mathbf{G}_m^{\times 2} \to \mathbf{G}_m^{\times 2}$ denote the twist automorphism given by $\tau(x_1, x_2) := (x_2, x_1)$, and let*

$$\tau^\wedge := \mathrm{pr}^\wedge \circ \tau^\times \circ \iota^\wedge \colon \mathbf{G}_m^{\wedge 2} \to \mathbf{G}_m^{\wedge 2}.$$

Then $\tau^\wedge$ is $\mathbf{A}^1$-homotopic to $\epsilon = -\langle -1 \rangle \in \mathrm{Cor}_k^A(\mathbf{G}_m^{\wedge 2}, \mathbf{G}_m^{\wedge 2})$.

Proof. Let (x_1, x_2) denote the coordinates on $\mathbf{G}_m^{\times 2}$. Denote by $\Delta \subseteq \mathbf{G}_m^{\times 2}$ the diagonal, and by $\widehat{\Delta} \subseteq \mathbf{G}_m^{\times 2}$ the anti-diagonal, i.e.,

$$\Delta := Z(x_1 x_2^{-1} - 1), \quad \widehat{\Delta} := Z(x_1 x_2 - 1) \subseteq \mathbf{G}_m^{\times 2}.$$

Let us first show that $\mathrm{pr}^\wedge \circ \tau^\times \circ j \sim_{\mathbf{A}^1} \epsilon \circ j$, where $j\colon \mathbf{G}_m^{\times 2} \setminus (\Delta \cup \widehat{\Delta}) \to \mathbf{G}_m^{\times 2}$ denotes the inclusion and $\epsilon = -\langle -1 \rangle \in \mathrm{Cor}_k^A(\mathbf{G}_m^{\times 2}, \mathbf{G}_m^{\times 2})$. To do this, consider the diagram

$$\begin{tikzcd}
\overset{t}{\mathbf{G}_m} \times ((\overset{x_1}{\mathbf{G}_m} \times \overset{x_2}{\mathbf{G}_m}) \setminus (\Delta \cup \widehat{\Delta})) \arrow[rr, "f"] \arrow[d, "p"] \arrow[drr, "g"] & & \mathbf{A}^1 \\
(\overset{x_1}{\mathbf{G}_m} \times \overset{x_2}{\mathbf{G}_m}) \setminus (\Delta \cup \widehat{\Delta}) & & \mathbf{G}_m \times \mathbf{G}_m
\end{tikzcd}$$

in which $g(t, x_1, x_2) := (t, x_1 x_2 t^{-1})$. Then p is a smooth relative curve whose relative canonical class is trivialized by dt. Applying Construction III.3.11 to this diagram we obtain a finite A-correspondence

$$\mathrm{div}^A(f)_Z^{dt,g} \in \mathrm{Cor}_k^A(\mathbf{G}_m^{\times 2} \setminus (\Delta \cup \widehat{\Delta}), \mathbf{G}_m^{\times 2})$$

for any regular function f whose vanishing locus Z is finite over $\mathbf{G}_m^{\times 2} \setminus (\Delta \cup \widehat{\Delta})$. For simplicity, let us skip dt and g in the notation. Then the required $\mathbf{A}^1$-homotopy is given as follows:

$$\begin{aligned}
&(\tau^\times + \langle -1 \rangle) \circ j \\
&= \big(\mathrm{div}^A((t - x_1)(t - x_2))_{Z(t-x_2)} + \mathrm{div}^A((t - x_1)(t - x_2))_{Z(t-x_1)}\big) \circ j \circ \langle (x_2 - x_1)^{-1} \rangle \\
&= \mathrm{div}^A((t - x_1)(t - x_2)) \circ j \circ \langle (x_2 - x_1)^{-1} \rangle \\
&\sim_{\mathbf{A}^1} \mathrm{div}^A((t - x_1 x_2)(t - 1)) \circ j \circ \langle (x_2 - x_1)^{-1} \rangle \\
&= \big(\mathrm{div}^A((t - x_1 x_2)(t - 1))_{Z(t-1)} \\
&\qquad + \mathrm{div}^A((t - x_1 x_2)(t - 1))_{Z(t-x_1 x_2)}\big) \circ j \circ \langle (x_2 - x_1)^{-1} \rangle \\
&= (\nu_1 + \nu_2) \circ i \circ \langle (1 - x_1 x_2)(x_2 - x_1)^{-1} \rangle \in \mathrm{Cor}_k^A(\mathbf{G}_m^{\times 2} \setminus (\Delta \cup \widehat{\Delta}), \mathbf{G}_m^{\times 2}).
\end{aligned}$$

Here $\nu_1\colon \mathbf{G}_m^{\times 2} \to \mathbf{G}_m^{\times 2}$ is the morphism $(x_1, x_2) \mapsto (x_1 x_2, 1)$, while the morphism $\nu_2\colon \mathbf{G}_m^{\times 2} \to \mathbf{G}_m^{\times 2}$ is defined by $(x_1, x_2) \mapsto (1, x_1 x_2)$. Since $\mathrm{pr}^\wedge \circ \nu_1 = 0$ and $\mathrm{pr}^\wedge \circ \nu_2 = 0$ in $\mathrm{Cor}_k^A(\mathbf{G}_m^{\times 2} \setminus (\Delta \cup \widehat{\Delta}), \mathbf{G}_m^{\wedge 2})$, it follows that

$$\mathrm{pr}^\wedge \circ (\tau^\times + \langle -1 \rangle) \circ j = 0 \in \overline{\mathrm{Cor}}_k^A(\mathbf{G}_m^{\times 2} \setminus (\Delta \cup \widehat{\Delta}), \mathbf{G}_m^{\wedge 2}).$$

Now Corollary III.5.4 yields that

$$\mathrm{pr}^\wedge \circ (\tau^\times + \langle -1 \rangle) = 0 \in \overline{\mathrm{Cor}}_k^A(\mathbf{G}_m^{\times 2}, \mathbf{G}_m^{\wedge 2}), \tag{III.8}$$

since $\overline{\mathrm{Cor}}_k^A(-, \mathbf{G}_m^{\wedge 2})$ is a homotopy invariant presheaf with A-transfers. Finally, since

$$\epsilon = -\mathrm{pr}^\wedge \circ \langle -1 \rangle \circ \iota^\wedge \in \mathrm{Cor}_k^A(\mathbf{G}_m^{\wedge 2}, \mathbf{G}_m^{\wedge 2}),$$

we get the claim upon composing (III.8) with $\iota^\wedge$. ∎

Definition III.9.4. Let $\mathbf{G}_m \times \mathbf{G}_m$ have coordinates (t_1, t_2). For any $n \geq 1$, define the functions $g_n^+, g_n^- \in k[\mathbf{G}_m \times \mathbf{G}_m]$ by

$$g_n^+ := t_1^n + 1, \quad g_n^- := t_1^n + t_2.$$

Moreover, let $Z_n^{\pm}$ denote the support of the principal divisor $Z(g_n^{\pm})$ on $\mathbf{G}_m \times \mathbf{G}_m$ defined by $g_n^{\pm}$.

Remark III.9.5. The functions g_n^+/g_n^- differ by a sign from Voevodsky's functions g_n defined in [Voe10, §4]. However, the same proof as that of [Voe10, Lemma 4.1] goes through to show that for any closed subset T of $\mathbf{G}_m \times X \times \mathbf{G}_m \times Y$ finite and surjective over $\mathbf{G}_m \times X$, there is an integer N such that for all $n \geq N$, the divisor of g_n^+/g_n^- intersects T properly over X, and the associated cycle is finite over X. The only reason for our choice of functions is to make the finite A-correspondence in Lemma III.9.7 homotopic to $\langle 1 \rangle$, and not $\langle -1 \rangle$. Of course, in the situation of [Voe10] this choice does not matter, as Voevodsky's correspondences are oriented.

Definition III.9.6. Let $Y \in \mathrm{Sm}_k$, and recall from Definition III.9.2 the definition of the presheaf $\mathrm{c}_A(Y) \wedge \mathbf{G}_m^{\wedge 1}$. Given any integer $n \geq 1$, we will construct maps of presheaves

$$\mathrm{c}_A(Y) \underset{\rho_n}{\overset{\theta}{\rightleftarrows}} \mathrm{c}_A(Y) \wedge \mathbf{G}_m^{\wedge 1}$$

as follows.

Let $X \in \mathrm{Sm}_k$, and let T be any admissible subset of $X \times Y$. Then the homomorphism

$$\theta\colon A_T^{\dim Y}(X \times Y, \omega_Y) \to A_{T \times \Delta(\mathbf{G}_m)}^{\dim Y + 1}(X \times \mathbf{G}_m \times Y \times \mathbf{G}_m, \omega_{Y \times \mathbf{G}_m})$$

is defined by

$$\theta := (-) \times \mathrm{id}_{\mathbf{G}_m} = (-) \times \Delta_*(1),$$

where $\Delta\colon \mathbf{G}_m \to \mathbf{G}_m \times \mathbf{G}_m$ is the diagonal. Since for any admissible T in $X \times Y$ the subset $T \times \Delta(\mathbf{G}_m)$ is admissible in $X \times \mathbf{G}_m \times Y \times \mathbf{G}_m$, the map θ is well defined. It follows that θ induces a map of presheaves $\theta\colon \mathrm{c}_A(Y) \to \mathrm{c}_A(Y) \wedge \mathbf{G}_m^{\wedge 1}$. On the other hand, the map

$$\rho_n\colon A_T^{\dim Y + 1}(X \times \mathbf{G}_m \times Y \times \mathbf{G}_m, \omega_{Y \times \mathbf{G}_m}) \to A_{T \cap (Z_n^+ \cup Z_n^-)}^{\dim Y}(X \times Y, \omega_Y)$$

is defined in the following way. By applying Construction III.3.11 to the diagram

$$\begin{tikzcd}
\overset{t_1}{\mathbf{G}_m} \times \overset{t_2}{\mathbf{G}_m} \arrow[r, "g_n^{\pm}"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, "\mathrm{pr}_1"] & \mathbf{A}^1 \\
\overset{t_2}{\mathbf{G}_m} & \overset{t_1}{\mathbf{G}_m}
\end{tikzcd}$$

we obtain finite A-correspondences $\mathrm{div}^A(g_n^{\pm}) \in \mathrm{Cor}_k^A(\mathbf{G}_m, \mathbf{G}_m)$. We then define ρ_n by the formula

$$\rho_n := p_*\Big((-) \smile q^*\Big(\mathrm{div}^A(g_n^+) - \mathrm{div}^A(g_n^-)\Big)\Big),$$

where p and q are the projections

$$p\colon X \times \mathbf{G}_m \times Y \times \mathbf{G}_m \to X \times Y, \quad q\colon X \times \mathbf{G}_m \times Y \times \mathbf{G}_m \to \mathbf{G}_m \times \mathbf{G}_m.$$

Thus ρ_n is defined whenever the subset $T \cap (Z_n^+ \cup Z_n^-)$ is admissible in $X \times Y$. Now, note that for any $f\colon X' \to X$ and $\Phi \in A_T^{\dim Y+1}(X \times \mathbf{G}_m \times Y \times \mathbf{G}_m, \omega_{Y\times\mathbf{G}_m})$, the element $\rho_n(f^*(\Phi))$ is defined whenever $\rho_n(\Phi)$ is defined, and $\rho_n(f^*(\Phi)) = f^*(\rho_n(\Phi))$. Secondly, for any $\Phi, \Psi \in A_T^{\dim Y+1}(X \times \mathbf{G}_m \times Y \times \mathbf{G}_m, \omega_{Y\times\mathbf{G}_m})$ the element $\rho_n(\Phi + \Psi)$ is defined whenever $\rho_n(\Phi)$ and $\rho_n(\Psi)$ are defined and $\rho_n(\Phi + \Psi) = \rho_n(\Phi) + \rho_n(\Psi)$. In this regard we refer to ρ_n a *partially defined map of presheaves.*

III.9.0.1

The maps ρ_n form an exhausting sequence of partially defined homomorphisms in the sense that for any finite subset $F \subseteq \mathrm{Cor}_k^A(X \wedge \mathbf{G}_m, Y \wedge \mathbf{G}_m)$, there is an integer $N(F)$ such that for all $n \geq N(F)$, $\rho_n(\alpha)$ is defined for all $\alpha \in F$. Indeed, this condition is satisfied by Remark III.9.5.

Lemma III.9.7. *Let $q'\colon \mathbf{G}_m \times \mathbf{G}_m \to \operatorname{Spec} k$ denote the projection, and let*

$$\Delta\colon \mathbf{G}_m \to \mathbf{G}_m \times \mathbf{G}_m$$

be the diagonal. Then there is an $\mathbf{A}^1$-homotopy

$$q'_*\Big(\Delta_*\Big(\mathrm{div}^A(\Delta^*(g_n^+)) - \mathrm{div}^A(\Delta^*(g_n^-))\Big)\Big) \sim_{\mathbf{A}^1} \langle 1 \rangle \in A^0(\operatorname{Spec} k, \mathcal{O}_{\operatorname{Spec} k}).$$

Proof. We deduce the claim from the following computation:

$$\begin{aligned}
&q'_*\Big(\Delta_*\Big(\mathrm{div}^A(\Delta^*(g_n^+)) - \mathrm{div}^A(\Delta^*(g_n^-))\Big)\Big) && \text{(III.9)}\\
&= \mathrm{div}^A(\Delta^*(g_n^+))^{\mathrm{pr}_{\mathrm{pt}}^{\mathbf{G}_m}} - \mathrm{div}^A(\Delta^*(g_n^-))^{\mathrm{pr}_{\mathrm{pt}}^{\mathbf{G}_m}} && \text{(III.10)}\\
&= \mathrm{div}^A(\Delta^*(g_n^+))^{\mathrm{pr}_{\mathrm{pt}}^{\mathbf{A}^1}} - \mathrm{div}^A(\Delta^*(g_n^-))_{Z(g_n^-|_{\mathbf{G}_m})}^{\mathrm{pr}_{\mathrm{pt}}^{\mathbf{A}^1}} && \text{(III.11)}\\
&= \mathrm{div}^A(t^n+1)^{\mathrm{pr}_{\mathrm{pt}}^{\mathbf{A}^1}} - \mathrm{div}^A(t^n+t)_{Z(t^{n-1}+1)}^{\mathrm{pr}_{\mathrm{pt}}^{\mathbf{A}^1}} && \text{(III.12)}\\
&\sim_{\mathbf{A}^1} \mathrm{div}^A(t^n+t)_{Z(t^n+t)}^{\mathrm{pr}_{\mathrm{pt}}^{\mathbf{A}^1}} - \mathrm{div}^A(t^n+t)_{Z(t^{n-1}+1)}^{\mathrm{pr}_{\mathrm{pt}}^{\mathbf{A}^1}} && \text{(III.13)}\\
&= \mathrm{div}^A(t^n+t)_{Z(t)}^{\mathrm{pr}_{\mathrm{pt}}^{\mathbf{A}^1}} = \langle 1 \rangle. && \text{(III.14)}
\end{aligned}$$

Here the homotopy (III.13) is given by $t^n + \lambda t + (1-\lambda) \in k[\mathbf{A}^1 \times \mathbf{A}^1]$. ■

III.9.0.2

We are now ready to prove the cancellation theorem for A-correspondences.

Theorem III.9.8. *For any $X, Y \in \mathrm{Sm}_k$, the map $\theta = (-) \wedge \mathbf{G}_m^{\wedge 1}$ induces a quasi-isomorphism of complexes of presheaves with A-transfers*

$$\mathrm{C}_*(\theta)\colon\ \mathrm{Cor}_k^A(\Delta^\bullet \times X, Y) \simeq \mathrm{Cor}_k^A((\Delta^\bullet \times X) \wedge \mathbf{G}_m^{\wedge 1}, Y \wedge \mathbf{G}_m^{\wedge 1}).$$

Here $\Delta^\bullet$ denotes the standard cosimplicial scheme over k, whose n-simplices Δ^n are given by $\operatorname{Spec} k[x_0, \dots, x_n]/(\sum_i x_i - 1)$.

Proof. The proof follows the same approach as Voevodsky's cancellation theorem for the category Cor_k [Voe10]. Thus many aspects of the proof will be the same as those of Voevodsky's proof, and we will therefore focus on the details that are specific to our context.

To prove that $\mathrm{C}_*(\theta)$ is a quasi-isomorphism it is enough to show that the maps ρ_n and θ are inverse to each other up to natural $\mathbf{A}^1$-homotopy. To this end, first note that the functions g_n^+ and g_n^- enjoy the following properties:

(1) $g_n^+\big|_\Delta = t^n + a_1 t^{n-1} + \cdots + a_{n-1}t + 1$, and $g_n^-\big|_\Delta = t^n + b_1 t^{n-1} + \cdots + b_{n-2}t^2 + t$ (in fact, $g_n^+\big|_\Delta = t^n + 1$ and $g_n^-\big|_\Delta = t^n + t$);

(2) $g_n^+\big|_{\mathbf{G}_m \times 1} = g_n^-\big|_{\mathbf{G}_m \times 1} \neq 0$.

Let p and q be the projections

$$p\colon X \times \mathbf{G}_m \times Y \times \mathbf{G}_m \to X \times Y, \quad q\colon X \times \mathbf{G}_m \times Y \times \mathbf{G}_m \to \mathbf{G}_m \times \mathbf{G}_m.$$

Moreover, denote by $p'\colon X \times Y \to \operatorname{Spec} k$ and $q'\colon \mathbf{G}_m \times \mathbf{G}_m \to \operatorname{Spec} k$ the structure maps. Thus we have a pullback square

$$\begin{tikzcd} X \times \mathbf{G}_m \times Y \times \mathbf{G}_m \arrow[r, "q"] \arrow[d, "p"'] & \mathbf{G}_m \times \mathbf{G}_m \arrow[d, "q'"] \\ X \times Y \arrow[r, "p'"] & \operatorname{Spec} k. \end{tikzcd}$$

Property (1) along with Lemma III.9.7 then implies that the composition $\rho_n \circ \theta$ is $\mathbf{A}^1$-homotopic to the identity, by the following computation:

$$p_*\Big((\alpha \times \Delta_*(1)) \smile q^*\Big(\mathrm{div}^A(g_n^+) - \mathrm{div}^A(g_n^-)\Big)\Big) \tag{III.15}$$
$$= p_*\Big(p^*(\alpha) \smile q^*\Big(\Delta_*(1) \smile \Big(\mathrm{div}^A(g_n^+) - \mathrm{div}^A(g_n^-)\Big)\Big)\Big) \tag{III.16}$$
$$= \alpha \smile p_*\Big(q^*\Big(\Delta_*(1) \smile \Big(\mathrm{div}^A(g_n^+) - \mathrm{div}^A(g_n^-)\Big)\Big)\Big) \tag{III.17}$$
$$= \alpha \smile (p')^*\Big(q'_*\Big(\Delta_*(1) \smile \Big(\mathrm{div}^A(g_n^+) - \mathrm{div}^A(g_n^-)\Big)\Big)\Big) \tag{III.18}$$
$$= \alpha \smile (p')^*\Big(q'_*\Big(\Delta_*\Big(\mathrm{div}^A(\Delta^*(g_n^+)) - \mathrm{div}^A(\Delta^*(g_n^-))\Big)\Big)\Big) \tag{III.19}$$
$$\sim_{\mathbf{A}^1} \alpha \smile (p')^*(\langle 1 \rangle) \tag{III.20}$$
$$= \alpha. \tag{III.21}$$

Here the equality (III.17) follows from the projection formula, (III.18) follows from base change applied to the diagram above, and the homotopy (III.20) is given by Lemma III.9.7.

Similarly, property (2) implies that for any $\alpha \in \mathrm{Cor}^A_k(X,Y)$, the classes $\rho_n((\alpha \times \mathrm{id}_{\mathbf{G}_m}) \circ i_X)$, $\rho_n(i_Y \circ (\alpha \times \mathrm{id}_{\mathbf{G}_m}) \circ i_X)$ and $\rho_n(i_Y \circ (\alpha \times \mathrm{id}_{\mathbf{G}_m}))$ are equal to 0 up to natural homotopy, where $i_X \colon X \to X \times \mathbf{G}_m$ and $i_Y \colon Y \to Y \times \mathbf{G}_m$ denote the morphisms given by the rational point $1\colon \operatorname{Spec} k \to \mathbf{G}_m$. Thus we see that $\rho_n \circ \theta \sim_{\mathbf{A}^1} \mathrm{id}_{\mathrm{c}_A(Y)}$.

Finally, Lemma III.9.3 implies that ρ_n is also right inverse up to $\mathbf{A}^1$-homotopy by the same argument as [Voe10, Theorem 4.6] (see also [AGP18, Lemma 7.5]). ∎

III.10 The category of A-motives

In this section we assume that the base field k is infinite, perfect and of characteristic different from 2.

III.10.1 Nisnevich localization

Theorem III.10.1. *The category of Nisnevich sheaves with A-transfers is abelian. The Nisnevich sheafification $\mathscr{F}_{\mathrm{Nis}}$ of any presheaf with A-transfers $\mathscr{F}$ is equipped with A-transfers in a unique and natural way, and there is a natural isomorphism*

$$\mathrm{Ext}^i_{\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Cor}^A_k;\mathbf{Z})}(\mathbf{Z}_A(X), \mathscr{F}_{\mathrm{Nis}}) \cong \mathrm{H}^i_{\mathrm{Nis}}(X, \mathscr{F}_{\mathrm{Nis}}).$$

Proof. By [Dru18b, Theorem 3.1] it is enough to show that

$$\mathrm{Cor}^A_k(U,X) \cong \bigoplus_{x \in X} \mathrm{Cor}^A_k(U, X^h_x),$$

where $x \in X$ ranges over the set of all (not necessary closed) points. Let d_X denote the dimension of X. Then we have

$$\begin{aligned}
\mathrm{Cor}^A_k(U,X) &= \varinjlim_{T \in \mathcal{A}_0(U \times X/U)} A^{d_X}_T(U \times X, \omega_X) \\
&\cong \varinjlim_{T \in \mathcal{A}_0(U \times X/U)} \bigoplus_{x \in X} A^{d_X}_{T^h_x}(U \times X^h_x, \omega_{X^h_x}) \\
&= \bigoplus_{x \in X} \varinjlim_{T \in \mathcal{A}_0(U \times X^h_x/U)} A^{d_X}_T(U \times X^h_x, \omega_{X^h_x}) \cong \bigoplus_{x \in X} \mathrm{Cor}^A_k(U, X^h_x),
\end{aligned}$$

where the isomorphism in the second row is given by Lemma III.3.9, and the isomorphism in the last row follows from Lemma III.3.10. ∎

Remark III.10.2. The category of finite A-correspondences Cor^A_k is a strict V-category of correspondences in the sense of [Gar19, Definition 2.3], and a V-ringoid in the sense of [GP14, Definition 2.4]. So, alternatively, Theorem III.10.1

can be proved by using the technique of [GP14]. Note also that the proof of Theorem III.10.1 could be obtained by following the original approach of Suslin and Voevodsky [Voe00a], that is, showing that the cone of the morphism $\mathrm{c}_A(\mathcal{U}^\bullet) \to \mathrm{c}_A(U)$ is acyclic. Here $\mathrm{c}_A(\mathcal{U}^\bullet)$ is the Čech complex associated to a Nisnevich covering $\mathcal{U} \to U$ of a smooth k-scheme U.

III.10.2 Strict homotopy invariance

Theorem III.10.3. *Let $\mathscr{F} \in \mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A; \mathbf{Z})$ be a homotopy invariant presheaf with A-transfers. Then the associated Nisnevich sheaf $\mathscr{F}_{\mathrm{Nis}}$ is strictly homotopy invariant, i.e., the projection $p\colon X \times \mathbf{A}^1 \to X$ induces an isomorphism*

$$p^*\colon \mathrm{H}^n_{\mathrm{Nis}}(X, \mathscr{F}_{\mathrm{Nis}}) \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{Nis}}(X \times \mathbf{A}^1, \mathscr{F}_{\mathrm{Nis}})$$

for all $X \in \mathrm{Sm}_k$ and all $n \geq 0$.

Proof. The theorem is a consequence of the injectivity and excision theorems proved in Sections III.5, III.6, III.7 and III.8. The deduction of strict homotopy invariance from these results is formal; see for example [GP18b] or [Dru18c]. ■

III.10.3 Effective A-motives

Definition III.10.4. The ∞-category $\mathbf{DM}^{\mathrm{eff}}_A(k)$ of *effective A-motives* is the localization of the derived ∞-category $\mathbf{D}^-(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Cor}_k^A; \mathbf{Z}))$ with respect to the morphisms of the form $X \times \mathbf{A}^1 \to X$. Let

$$\mathrm{M}^{\mathrm{eff}}_A\colon \mathrm{Sm}_k \to \mathbf{DM}^{\mathrm{eff}}_A(k)$$

be the functor defined as the composition of the localization

$$\mathbf{D}^-(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Cor}_k^A; \mathbf{Z})) \to \mathbf{DM}^{\mathrm{eff}}_A(k)$$

with the functor

$$\mathrm{Sm}_k \to \mathbf{D}^-(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Cor}_k^A; \mathbf{Z}))$$

given by $X \mapsto \mathbf{Z}_A(X)[0]$. For any $X \in \mathrm{Sm}_k$, we refer to $\mathrm{M}^{\mathrm{eff}}_A(X)$ as the *effective A-motive* of X. If $X = \mathrm{Spec}\, k$, we abbreviate $\mathrm{M}^{\mathrm{eff}}_A(\mathrm{Spec}\, k)$ to $\mathbf{Z}_A$. Finally, we define the Tate object $\mathbf{Z}_A(1)$ as

$$\mathbf{Z}_A(1) := \mathrm{cofib}(\mathbf{Z}_A \to \mathrm{M}^{\mathrm{eff}}_A(\mathbf{G}_m))[-1],$$

where $\mathbf{Z}_A \to \mathrm{M}^{\mathrm{eff}}_A(\mathbf{G}_m)$ is the map induced by the rational point $1\colon \mathrm{Spec}\, k \to \mathbf{G}_m$.

Remark III.10.5. Equivalently, $\mathbf{DM}^{\mathrm{eff}}_A(k)$ can be defined by starting with the ∞-category of presheaves of spaces $\mathrm{PSh}_\Sigma(\mathrm{Cor}_k^A)$ and then performing a motivic localization as in [BH18, §14]. Indeed, since Cor_k^A is additive, presheaves of spaces and presheaves of abelian groups agree.

III.10.3.1

Note that there is a symmetric monoidal structure on $\mathbf{DM}^{\mathrm{eff}}_A(k)$ inherited from that on $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Cor}^A_k;\mathbf{Z})$, satisfying $\mathrm{M}^{\mathrm{eff}}_A(X)\otimes\mathrm{M}^{\mathrm{eff}}_A(Y)\simeq\mathrm{M}^{\mathrm{eff}}_A(X\times Y)$. The motive of a point, $\mathbf{Z}_A$, is then the unit for this monoidal structure. For any $n\geq 1$, we can use the monoidal structure to define $\mathbf{Z}_A(n):=\mathbf{Z}_A(1)^{\otimes n}$.

Theorem III.10.6 (cf. [MVW06, Theorem 14.11])**.** *The ∞-category $\mathbf{DM}^{\mathrm{eff}}_A(k)$ of effective A-motives is equivalent to the full subcategory of $\mathbf{D}^-(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Cor}^A_k;\mathbf{Z}))$ spanned by* motivic complexes*, i.e., complexes whose cohomology sheaves are strictly homotopy invariant.*

Theorem III.10.7 (cf. [MVW06, Proposition 14.16])**.** *Let $X\in\mathrm{Sm}_k$, and let $\mathscr{F}^\bullet$ be a motivic complex. Then there is a natural isomorphism*

$$[\mathrm{M}^{\mathrm{eff}}_A(X),\mathscr{F}^\bullet[i]]_{\mathbf{D}^-(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Cor}^A_k;\mathbf{Z}))}\cong\mathbf{H}^i_{\mathrm{Nis}}(X,\mathscr{F}^\bullet)$$

for each $i\geq 0$.

III.10.4 The category of A-motives

As in the classical case, we obtain the category $\mathbf{DM}_A(k)$ of A-motives via a stabilization process with respect to tensoring with the Tate object.

Definition III.10.8. The ∞-category $\mathbf{DM}_A(k)$ of *A-motives* is obtained from $\mathbf{DM}^{\mathrm{eff}}_A(k)$ by $\otimes$-inverting $\mathbf{Z}_A(1)$. There is then a canonical functor

$$\Sigma^\infty\colon\mathbf{DM}^{\mathrm{eff}}_A(k)\to\mathbf{DM}_A(k),$$

and we define the functor $\mathrm{M}_A\colon\mathrm{Sm}_k\to\mathbf{DM}_A(k)$ as the composition of $\mathrm{M}^{\mathrm{eff}}_A$ and Σ^∞.

III.10.4.1

It follows similarly as in [DF17a] that $\mathbf{DM}_A(k)$ is a presentably symmetric monoidal stable ∞-category equipped with an adjunction

$$\Sigma^\infty:\mathbf{DM}^{\mathrm{eff}}_A(k)\rightleftarrows\mathbf{DM}_A(k):\Omega^\infty.$$

III.10.4.2

The following result is a consequence of the cancellation theorem for finite A-correspondences:

Theorem III.10.9. *The canonical functor $\Sigma^\infty\colon\mathbf{DM}^{\mathrm{eff}}_A(k)\to\mathbf{DM}_A(k)$ is fully faithful, and for any $X\in\mathrm{Sm}_k$ and any motivic complex*

$$\mathscr{F}^\bullet\in\mathbf{D}^-(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Cor}^A_k;\mathbf{Z})),$$

there is a natural isomorphism

$$[\mathrm{M}_A(X),\Sigma^\infty\mathscr{F}^\bullet]_{\mathbf{DM}_A(k)}\cong\mathbf{H}^i_{\mathrm{Nis}}(X,\mathscr{F}^\bullet).$$

Definition III.10.10. Let $X \in \mathrm{Sm}_k$. For any pair of integers $p, q \in \mathbf{Z}$, we define the *A-motivic cohomology of X in bidegree* (p, q) as

$$\mathrm{H}^{p,q}_A(X, \mathbf{Z}) := [\mathrm{M}_A(X), \mathbf{Z}_A(q)[p]]_{\mathbf{DM}_A(k)}.$$

III.10.4.3

The adjunction $\gamma_A^* : \mathrm{PSh}_\Sigma(\mathrm{Sm}_k) \rightleftarrows \mathrm{PSh}_\Sigma(\mathrm{Cor}^A_k; \mathbf{Z}) : \gamma^A_*$ descends to an adjunction

$$\gamma_A^* : \mathbf{SH}(k) \rightleftarrows \mathbf{DM}_A(k) : \gamma^A_* \tag{III.22}$$

of stable ∞-categories, which allows us to compare $\mathbf{DM}_A(k)$ with the motivic stable homotopy category $\mathbf{SH}(k)$.

Definition III.10.11. Denote by $\mathbf{1} \in \mathbf{SH}(k)$ the motivic sphere spectrum. In the adjunction (III.22) above, let $\mathrm{H}\mathbf{Z}_A \in \mathbf{SH}(k)$ denote the Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbf{Z}_A := \gamma^A_* \gamma^*_A(\mathbf{1})$.

Lemma III.10.12. *The spectrum* $\mathrm{H}\mathbf{Z}_A$ *is an* $\mathcal{E}_\infty$*-ring spectrum in* $\mathbf{SH}(k)$*.*

Proof. As the right adjoint γ^A_* is lax symmetric monoidal, it follows that it preserves $\mathcal{E}_\infty$-algebras. Now the left adjoint γ^*_A is symmetric monoidal, so $\gamma^*_A(\mathbf{1})$ is the unit in $\mathbf{DM}_A(k)$ and hence an $\mathcal{E}_\infty$-algebra. We conclude that $\mathrm{H}\mathbf{Z}_A = \gamma^A_*\gamma^*_A(\mathbf{1})$ is an $\mathcal{E}_\infty$-ring spectrum. ■

III.10.4.4

The cancellation theorem for A-correspondences implies that $\mathrm{H}\mathbf{Z}_A$ is an Ω_{T}-spectrum in $\mathbf{SH}(k)$ which represents A-motivic cohomology. More precisely, for any $X \in \mathrm{Sm}_k$ and any pair of integers p, q, there is a natural isomorphism $[\Sigma^\infty_{\mathrm{T}} X_+, \Sigma^{p,q}\mathrm{H}\mathbf{Z}_A]_{\mathbf{SH}(k)} \cong \mathrm{H}^{p,q}_A(X, \mathbf{Z})$.

III.10.4.5

The combination of Lemma III.3.2 and [EK20, Theorem 5.2] shows moreover that in the above adjunction (III.22), the right adjoint is monadic:

Theorem III.10.13. *Let e denote the exponential characteristic of k. Then there is an equivalence of presentably symmetric monoidal stable ∞-categories*

$$\mathrm{Mod}_{\mathrm{H}\mathbf{Z}_A[1/e]}(\mathbf{SH}(k)) \simeq \mathbf{DM}_A(k, \mathbf{Z}[1/e]),$$

where $\mathrm{Mod}_{\mathrm{H}\mathbf{Z}_A[1/e]}(\mathbf{SH}(k))$ *denotes motivic spectra equipped with an action from* $\mathrm{H}\mathbf{Z}_A[1/e]$*.*

Remark III.10.14. Recall that the ∞-category $\mathbf{SH}^{\mathrm{eff}}(k)$ of *effective spectra* is the stable subcategory of $\mathbf{SH}(k)$ generated under colimits by $\mathbf{P}^1$-suspension spectra of smooth k-schemes. We note that Bachmann and Fasel's effectivity criterion [BF18, Theorem 4.4] applies in our setting, showing that the spectrum $\mathrm{H}\mathbf{Z}_A \in \mathbf{SH}(k)$ is effective. G. Garkusha and I. Panin communicated to us orally that they proved this result independently using the category $\mathbf{Z}\mathrm{F}_*(k)$ of linear framed correspondences.

Appendix III.A Geometric ingredients

In this section we summarize the geometric facts and constructions used in the text. In particular, we formulate a version of Serre's theorem on the existence of sections satisfying relevant properties, which is used in the proofs in Sections III.6, III.7 and III.8. We then provide the construction of the relative curves used in Sections III.7 and III.8. Finally, we formulate a few lemmas that imply the finiteness conditions on the vanishing loci of the functions constructed in Sections III.6, III.7 and III.8.

All schemes considered in this appendix are assumed to be noetherian and separated.

Proposition III.A.1. *For any étale morphism $e\colon U \to Y$ there is a decomposition $U \xrightarrow{u} X \xrightarrow{p} Y$ with $p \circ u = e$, in which u is a dense open immersion and p is finite.*

Proof. This follows Zariski's Main Theorem [Har77, III Corollary 11.4]. ■

III.A.0.1 Serre's theorem

The following lemma is a consequence of [Har77, III Theorem 5.2], and is used in Sections III.6, III.7 and III.8. In the text we refer to this result simply as Serre's theorem.

Lemma III.A.2 (Serre)**.** *Let $\mathcal{O}(1)$ be an ample invertible sheaf on a scheme X, and $\mathscr{L}$ be an invertible sheaf on X. Then there is, for any closed subscheme $Z \subseteq X$, an integer $N \in \mathbf{Z}$ such that the restriction homomorphism*

$$\Gamma(X, \mathscr{L}(l)) \to \Gamma(Z, \mathscr{L}(l))$$

is surjective for all $l \geq N$. Here $\mathscr{L}(l) := \mathscr{L} \otimes \mathcal{O}(l)$.

Example III.A.3 (Chinese remainder theorem)**.** Let U be an affine scheme. Suppose that $Z \subseteq \mathbf{A}^1_U$ is a closed subscheme, and that $v \in \mathcal{O}_Z$ is a regular function on Z. Then, for all large enough n there is a monic polynomial $f \in \mathcal{O}_U[t] = \mathcal{O}_{\mathbf{A}^1_U}$ of degree n such that $f\big|_Z = v$.

III.A.0.2 Construction of relative curves

We now formulate the construction of relative curves used in the proofs of the Nisnevich excision theorems. For the proof we refer to [Dru18c, Lemma 3.7]. Before stating the result, let us first recall the notion of an étale neighborhood:

Definition III.A.4. Let X be a scheme and suppose that $Z \subseteq X$ is a closed subscheme. If $\pi\colon X' \to X$ is an étale morphism and $Z' \subseteq X'$ is a closed subscheme such that π induces an isomorphism $Z' \xrightarrow{\cong} Z$, then we say that $\pi\colon (X', Z') \to (X, Z)$ is an *étale neighborhood of Z in X*.

Lemma III.A.5 ([Dru18c, Lemma 3.7]). *Let k be a field and let X be a smooth k-scheme. Suppose we are given a closed subscheme $Z \subseteq X$ along with an étale neighborhood $\pi\colon (X', Z') \to (X, Z)$ of Z in X. Let moreover $z \in Z$ and $z' \in Z'$ be closed points such that $\pi(z') = z$, and write $U := X_z$ and $U' := X'_{z'}$ for the corresponding local schemes. Then there is a commutative diagram*

$$
\begin{tikzcd}
U' \arrow[d] & \overline{\mathcal{C}''} \arrow[l, "p''"'] \arrow[d, "\overline{\varpi}'"] & \mathcal{C}'' \arrow[l, "j''"'] \arrow[r, "v''"] \arrow[d, "\varpi'"] & X' \arrow[d, equal] \\
U \arrow[d, equal] & \overline{\mathcal{C}'} \arrow[l, "p'"'] \arrow[d, "\overline{\varpi}"] & \mathcal{C}' \arrow[l, "j'"'] \arrow[r, "v'"] \arrow[d, "\varpi"] & X' \arrow[d, "\pi"] \\
U & \overline{\mathcal{C}} \arrow[l, "p"'] & \mathcal{C} \arrow[l, "j"'] \arrow[r, "v"] & X
\end{tikzcd}
\tag{III.23}
$$

in Sm_k, such that the following properties hold:

(1) *p, p', p'' are relative projective curves; j, j', j'' are open immersions; ϖ, ϖ' are étale; $\overline{\varpi}$, $\overline{\varpi}'$ are finite; and $p \circ j$, $p' \circ j'$, $p'' \circ j''$ are smooth. Moreover, $\mathcal{C}'' = \mathcal{C}' \times_U U'$; $\overline{\mathcal{C}}'' = \overline{\mathcal{C}}' \times_U U'$; and there are trivializations of the relative canonical classes $\mu\colon \mathcal{O}_{\mathcal{C}} \cong \omega_{\mathcal{C}/U}$ and $\mu'\colon \mathcal{O}_{\mathcal{C}'} \cong \omega_{\mathcal{C}'/U}$.*

(2) *The schemes $\mathcal{Z} := v^{-1}(Z)$, $\mathcal{Z}' := v'^{-1}(Z')$ and $\mathcal{Z}'' := v''^{-1}(Z')$ are finite over U and U', respectively.*

(3) *There are closed subschemes $\Delta_Z \subseteq \mathcal{Z}$, $\Delta'_Z \subseteq \mathcal{Z}'$ and $\Delta''_Z \subseteq \mathcal{Z}''$ such that p, p' and p'' induce isomorphisms $w\colon \Delta_Z \cong Z' \times_X U$, $w'\colon \Delta'_Z \cong Z \times_X U$ and $w''\colon \Delta''_Z \cong Z' \times_{X'} U'$. Moreover, $v\big|_{\mathcal{Z}} \circ w^{-1} = \mathrm{pr}_Z^{Z \times_X U}$, $v'\big|_{\mathcal{Z}'} \circ w'^{-1} = \pi\big|_{Z'} \circ \mathrm{pr}_{Z'}^{Z' \times_X U}$, and $v''\big|_{\mathcal{Z}''} \circ w''^{-1} = \mathrm{pr}_{Z'}^{Z' \times_{X'} U'}$.*

(4) *There are closed subschemes $\Delta \subseteq \mathcal{C}$ and $\Delta' \subseteq \mathcal{C}''$ such that $\Delta \times_U Z = \Delta_Z$, $\Delta' \times_{U'} Z' = \Delta''_Z$ and such that p and p'' induce isomorphisms $p\big|_{\Delta}\colon \Delta \cong U$ and $p''\big|_{\Delta'}\colon \Delta' \cong U'$. Moreover, the compositions $v \circ p\big|_{\Delta}^{-1}$ and $v \circ p''\big|_{\Delta'}^{-1}$ are equal to the canonical morphisms $U \to X$ and $U' \to X'$, respectively.*

(5) *The schemes $D := \overline{\mathcal{C}} \setminus \mathcal{C}$, $D' := \overline{\mathcal{C}}' \setminus \mathcal{C}'$ and $D'' := \overline{\mathcal{C}''} \setminus \mathcal{C}''$ are finite over U and U' respectively. Furthermore, $D'' \cong \overline{\varpi}'^{-1}(D')$, and $D' \supseteq \overline{\varpi}^{-1}(D)$.*

(6) *There is an ample line bundle $\mathcal{O}(1)$ on $\overline{\mathcal{C}}$ and a section $d \in \Gamma(\overline{\mathcal{C}}, \mathcal{O}(1))$ such that $Z(d) = D$.*

III.A.0.3 Finiteness of vanishing loci

The following lemmas are used to prove that the zero loci of the functions constructed in Sections III.6, III.7 and III.8 are finite over the relevant schemes.

Lemma III.A.6 ([Dru18c, Lemma 4.1]). *Let U be a local scheme, and let $x \in U$ denote the closed point. Suppose that the residue field $k := k(x)$ is infinite. Let*

$$\begin{tikzcd} D' \arrow[r, hook, "i"] \arrow[dr] & \overline{\mathcal{C}'} \arrow[r, "\pi"] \arrow[d, "p'"] & \overline{\mathcal{C}} \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

be a commutative diagram such that

- *p' and p are projective morphisms of relative dimension one;*
- *i is a closed immersion, and*
- *π and $p' \circ i$ are finite.*

Suppose furthermore that we are given the following data:

- *an ample line bundle $\mathcal{O}(1)$ on $\overline{\mathcal{C}'}$;*
- *a section $d \in \Gamma(\overline{\mathcal{C}'}, \mathcal{O}(1))$ such that $Z(d) \subseteq D'$;*
- *an invertible section $s_\infty \in \Gamma(D', \mathcal{O}(1))$;*
- *a closed subscheme $\mathcal{Z} \subseteq \overline{\mathcal{C}}$ satisfying $\mathcal{Z}' \cap D' = \varnothing$, where $\mathcal{Z}' := \pi^{-1}(\mathcal{Z}) \subseteq \overline{\mathcal{C}'}$;*
- *a section $s_{\mathcal{Z}'} \in \Gamma(\mathcal{Z}', \mathcal{O}(1))$ such that π induces an isomorphism $Z(s_{\mathcal{Z}'}) \cong \pi(Z(s_{\mathcal{Z}'}))$.*

Then there is an integer $L \in \mathbf{Z}$ such that for all $l \geq L$, there is a section $s \in \Gamma(\overline{\mathcal{C}'}, \mathcal{O}(l))$ satisfying

(1) $s\big|_{D'} = s_\infty^l$, $s\big|_{\mathcal{Z}'} = s_{\mathcal{Z}'} d^{l-1}$;

(2) *π induces an isomorphism $Z(s) \cong \pi(Z(s))$.*

Lemma III.A.7. *Let U be a scheme and suppose that $\overline{\mathcal{C}} \to U$ is a projective morphism of pure dimension one. Let $\mathscr{L}$ be an ample line bundle on $\overline{\mathcal{C}}$. Then, for any pair of sections $d, e \in \Gamma(\overline{\mathcal{C}}, \mathscr{L})$ such that $Z(d) \cap Z(e) = \varnothing$, the vanishing loci $Z(e)$ and $Z(d)$ are finite over U.*

Proof. We prove that $Z(e)$ is finite over U; the case of $Z(d)$ follows by symmetry. Since $\overline{\mathcal{C}}$ is projective over U, the same holds also for the closed subscheme $Z(e)$. As $\overline{\mathcal{C}}$ is of pure dimension one, it follows that $Z(e)$ is finite over U unless $Z(e)$ contains at least one irreducible component C of the fiber $\overline{\mathcal{C}} \times_U x$ for some point $x \in U$. But since $\mathscr{L}$ is ample, $\mathscr{L}\big|_C$ is nontrivial and hence $Z(d\big|_C) \neq \varnothing$. So $Z(e)$ cannot contain an irreducible component of the fiber $\overline{\mathcal{C}} \times_U x$. ■

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