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Seifert Manifolds as Circle Manifolds

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

Seifert manifolds play a crucial role in the classification of 3-manifolds, as they occupy six of the eight geometries proposed by Thurston. In this thesis we determine which Seifert manifolds can be given a smooth action from the circle group. We show how the stabilizer groups and orbit space of this circle manifold relate to the Seifert invariants and base space of the Seifert manifold. In particular, we give a smooth circle action to Brieskorn manifolds and spherical space forms and calculate their stabilizer groups. Additionally, we show that the orbit space is an orbifold.

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CHAPTER 1

Introduction

Seifert manifolds were introduced and classified by Herbert Seifert in the 1930s. Interest in these manifolds has had a resurgence due to the work of William Thurston and his celebrated geometrization conjecture. This was proposed by Thurston in 1992 and proved later by Grigorij Perelman in 2002. The geometrization conjecture essentially states that any closed, oriented 3-manifold can be canonically decomposed into something called *geometric* 3-manifolds. In other words the classification of closed, oriented 3-manifolds can be reduced to the classification of geometric 3-manifolds. Thurston further shows that there are eight classes of geometric 3-manifolds (or eight *geometries*), seven of which are completely classified. The only remaining case is the spaces with a hyperbolic geometry.

The Seifert manifolds are special in that they occupy precisely six of these Thurston geometries, and one can identify their geometry by calculating their *Seifert invariants*. We give a short introduction to the geometrization conjecture and its relevance in this first introductory chapter.

Seifert manifolds are a generalization of circle bundles where we allow certain "singularities". One major part of this thesis to show that due to this circle "bundle" we can consider most Seifert manifolds as special G-manifolds where G is the circle group S^1, with Lie group structure inherited from $U(1)$. A G-manifold is a manifold with a group action of the Lie group G. We introduce Seifert manifolds and G-manifolds in general in the second chapter, and show that most Seifert manifolds are S^1-manifolds with a certain finiteness condition.

As the Seifert manifolds cover as many as six of the eight geometries, they naturally include a lot of different examples. In the third chapter we introduce some classes of manifolds and show that they are Seifert manifolds by identifying them as certain S^1-manifolds as explained in chapter 2.

In the fourth and final chapter we introduce the concept of an orbifold, a generalization of manifolds. We prove that the orbit space of G-manifolds with a certain finiteness condition are indeed orbifolds. In particular, all Seifert manifolds that are S^1-manifolds have this property. Since all the information of the Seifert manifold is encoded into this orbifold and its projection map, we conjecture that you can calculate the Seifert invariants by studying these orbifolds further.

The main sources used for this thesis is [Sco83] and [Mar16], which summarize the geometrization conjecture and the classification of seven of the geometries. Further we have used a standard textbook about smooth manifolds ([Tu08]), lecture notes by professors at the University of Oslo, and some articles for specific results.

1.1 Terminology and notation

For the most part we will use terminology as in [Tu08] for talking about differential topology. In particular, we assume that topological manifolds are second countable and we will use the term manifold to mean smooth manifold without boundary. Occasionally we will make remarks regarding manifolds with a boundary.

The symbol $\mathbb{H}^n$ is used to represent the hyperbolic n-space, while $\mathbb{H}$ is used in chapter 3 to denote the quaternions. Other than this we try to avoid any ambiguity, and the symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote as usual the integers, rational numbers, real numbers and complex numbers, respectively.

A couple of definitions that can be found in [Lee12, p. 327, 337] and that do not belong anywhere will be presented here:

Definition 1.1.1. Let M be a smooth manifold. A *Riemannian metric on M* is a smooth symmetric covariant 2-tensor field on M that is positive definite at each point (i.e. a choice of inner product at each tangent space that vary smoothly across M.) A *Riemannian manifold* (M, r) is a smooth manifold M with a Riemannian metric r.

We often refer to a Riemannian manifold (M, r) simply as M for simplicity.

The Riemannian metric is the extra structure that makes smooth manifolds into geometric objects. It is possible to define familiar notions such as distance and angle from the Riemannian metric. Importantly, every smooth manifold has a Riemannian metric.

One construction that will be important to us is the normal bundle.

Definition 1.1.2. Let (M, r) be an n-dimensional Riemannian manifold and $S \subseteq M$ is a k-dimensional submanifold. The *normal space to S at x* is the subspace $N_xS \subseteq T_xM$ consisting of all vectors that are normal to S at x with respect to $\langle\cdot,\cdot\rangle_r$. The *normal bundle of S* is the subset $NS \subseteq TM$ consisting of the union of all the normal spaces N_xS for $x \in S$.

The normal bundle is the total space of the vector bundle $NS \to S$, defined as the restriction to NS of the tangent bundle $TM \to M$. We also denote the vector bundle $NS \to S$ itself the normal bundle of S.

1.2 Thurston's geometrization conjecture

Our story starts with the prime decomposition of compact oriented 3-manifolds. This statement is found and proved in [Mar16, p. 277].

Theorem 1.2.1. *Every compact oriented 3-manifold M decomposes into prime manifolds:*

$$M = M_1 \# \dots \# M_k$$

This list of prime factors is unique up to permutations and adding/removing copies of S^3.

The term *prime* here means not diffeomorphic to S^3 and with the property that every separating 2-sphere in the manifold bounds a 3-ball.

This canonical decomposition of 3-manifolds along spheres is then followed by another canonical decomposition along tori and Klein bottles. This next result is a precise statement of the geometrization conjecture due to [Mor14, p. 1].

Theorem 1.2.2. *Any closed, orientable, prime 3-manifold M contains a disjoint union of embedded incompressible 2-tori and Klein bottles such that each connected component of the complement admits a complete, locally homogenous Riemannian metric of finite volume.*

The term *incompressible* means that the fundamental group of the surface injects into the fundamental group of the 3-manifold. We shall call a manifold *geometrizable* if there exists a complete, locally homogenous Riemannian metric of finite volume.

There is also a relative version of this statement ([Mor14, p. 6]), replacing closed manifolds by compact manifolds, showing that this is also true for manifolds with boundary.

What these two results imply is that every compact oriented 3-manifold can be decomposed into geometrizable manifolds in a canonical way. This allows us to classify all oriented, compact 3-manifolds by classifying all geometrizable manifolds.

We say that a Riemannian 3-manifold N has a *geometric structure modelled on M* if every point $p \in N$ has a neighbourhood isometric to some open set on M. If M has a complete, homogenous Riemannian metric, then so will N. It turns out that every geometrizable manifold is modelled on one of the following complete, homogenous Riemannian manifolds ([Mar16, p. 363]):

$$S^3, \mathbb{R}^3, \mathbb{H}^3, S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \mathrm{Nil}, \mathrm{Sol}, \widetilde{SL_2}.$$

We call these *Thurston's eight geometries*, or simply *the eight geometries*. Furthermore, if N is modelled on one of these eight geometries, that geometry is unique ([Sco83, p. 476]). Seifert manifolds occupy precisely six of these geometries ([Mar16, p. 364]):

Theorem 1.2.3. *A closed orientable 3-manifold has a geometric structure modelled on one of the following six geometries:*

$$S^3, \mathbb{R}^3, S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \mathrm{Nil}, \widetilde{\mathrm{SL}_2}$$

if and only it is a Seifert manifold.

It is worth noting that the Seifert manifolds are classified and placed according to their geometries, as are the manifolds with the Sol geometry. Classifying the manifolds modelled on $\mathbb{H}^3$ i.e. the hyperbolic manifolds, is still an open problem ([Mor14, p. 3]).

CHAPTER 2

Seifert manifolds as G-manifolds

In this chapter we introduce a way to define Seifert manifolds, namely as Seifert fiber spaces. This point of view will allow us to show that Seifert manifolds under certain conditions are S^1-manifolds. We also introduce the basic theory of G-manifolds that will be required to show this.

2.1 Seifert manifolds

This section will vaguely follow [Sco83, p.428-430], unless otherwise stated. The aim is to introduce Seifert manifolds and their Thurston geometries.

Definition 2.1.1. A trivial fibered solid torus is the space $S^1 \times D^2$ where the circles $F_x = S^1 \times \{x\}$ for $x \in D^2$ are called fibers. A fibered solid torus is the space $S^1 \times D^2$ finitely covered by a trivial fibered solid torus. If π is the covering map, the fibers of a fibered solid torus are $F_{\pi(x)} := \pi(F_x)$.

A fibered solid torus can be constructed from a trivial fibered solid torus by cutting it open along $\{z\} \times D^2$ for any $z \in S^1$, rotating one of the resulting discs q/p times of a full turn and gluing the discs back together. Therefore we use the notation $\mathbf{T}_{(p,q)}$ for a solid torus, where p and q are coprime integers. The fibers of $\mathbf{T}_{(p,q)}$ can be parameterized as

$$F_x(t) = (e^{2\pi it}, e^{2\pi it\frac{q}{p}}x)$$

for $x \in D^2$.

We say that two fibered solid tori $\mathbf{T}_{(p,q)}$ and $\mathbf{T}_{(p',q')}$ are isomorphic if there exists a diffemorphism between them sending fibers to fibers. We must have $p = p'$ for two tori to be isomorphic, since they are p-fold and p'-fold covered by a trivial fibered solid torus, respectively. However it is sufficient to have $q = \pm q' \pmod p$ for them to be isomorphic. We can verify this by looking at how the diffeomorphisms

$$\begin{aligned} \phi : \mathbf{T}_{(p,q)} &\to \mathbf{T}_{(p',q')} \\ (z,x) &\mapsto (z, e^{2\pi int}x) \end{aligned}$$

for $n \in \mathbb{Z}$ and

$$\begin{aligned} \psi : \mathbf{T}_{(p,q)} &\to \mathbf{T}_{(p',q')} \\ (z,x) &\mapsto (z, -x) \end{aligned}$$

act on the fiber $F_x \subseteq \mathbf{T}_{(p,q)}$ for $x \in D^2$. We get

$$\begin{aligned}\phi(F_x(t)) &= \phi(e^{2\pi it}, e^{2\pi it\frac{q}{p}}x) \\ &= (e^{2\pi it}, e^{2\pi it}e^{2\pi it\frac{q}{p}}x) \\ &= (e^{2\pi it}, e^{2\pi it\frac{q+pn}{p}}x) \\ &= F_x \subseteq \mathbf{T}_{(p,q+pn)}\end{aligned}$$

and

$$\begin{aligned}\psi(F_x(t)) &= \psi(e^{2\pi it}, e^{2\pi it\frac{q}{p}}x) \\ &= (e^{2\pi it}, -e^{2\pi it\frac{q}{p}}x) \\ &= (e^{2\pi it}, e^{2\pi it\frac{-q}{p}}x) \\ &= F_x \subseteq \mathbf{T}_{(p,-q)}.\end{aligned}$$

This shows that we can choose the pair (p, q) such that $p \in \mathbb{Z} \setminus \{0\}$ and $0 \leq q < p$ where p and q are coprime. We call (p, q) under these restrictions the *orbit invariants* of $\mathbf{T}_{(p,q)}$. For $q = 0$ and any p, the fibered solid torus $\mathbf{T}_{(p,q)}$ is trivial, so we say that $(1, 0)$ are the orbit variants of the trivial fibered solid torus and denote it simply by $\mathbf{T}_0$.

We can use these fibered solid tori to define a Seifert fiber space.

Definition 2.1.2. A Seifert fiber space is a 3-manifold M with a decomposition of M into disjoint circles, called fibers, such that each circle has a neighbourhood in M which is a union of fibers and is isomorphic to a fibered solid torus.

We note that it is possible to extend the definition to include neighbourhoods which are diffeomorphic to fibered Klein bottles as well, but we will not need that.

Definition 2.1.3. Let M be a Seifert manifold and F_x be the fiber at the point $x \in M$. We say that F_x is singular if it is the central fiber (i.e. the fiber $F_0(t)$ in $\mathbf{T}_{(p,q)}$) of a non-trivial fibered solid torus, and regular otherwise.

We note that there are only finitely many singular fibers, and they are all separated.

The reason for the term fiber is that we can consider a Seifert fiber space as a kind of circle bundle. If E is a Seifert fiber space we define B to be the quotient of E obtained by identifying all fibers to a point. In particular, every circle bundle is a Seifert fiber space with no singular fibers. In general B is not a manifold, but it is something called an orbifold, as we will discuss in chapter 4. This means that $E \to B$ is not in general a circle bundle of manifolds, but it is possible to make sense of it as a circle bundle of orbifolds. We will not need this, but the space B is called the *base space* of E and makes sense topologically, and it will be relevant for classifying the geometries of Seifert manifolds.

It is possible to define Seifert manifolds in terms of Dehn surgery. In [Mar16, p. 310] it is shown that these are equivalent to Seifert fiber spaces. The Seifert fiber space definition is more useful for our purposes, but because of this equivalence we will refer to Seifert fiber spaces as Seifert manifolds from now on.

From [Mar16, p. 311] we know that all Seifert manifolds M are characterized by its base space and the orbit invariants of its singular fibers. So we write $M =$

$(S, (p_1, q_1), \ldots, (p_k, q_k))$ where S is the base space of M and $(p_1, q_1), \ldots, (p_k, q_k)$ are the orbit invariants of the singular fibers.

Much like compact 2-manifolds can be classified into geometries based on their Euler characteristic, there is a similar result for Seifert manifolds. We now state the definition of the relevant invariants ([Mar16, p. 37, p. 159, p. 312]).

Definition 2.1.4. The Euler characteristic of an n-complex X is given by the integer

$$\chi(X) = \sum_{i=1}^{n} (-1)^i C_i$$

where C_i is the number of i-cells in X.

Definition 2.1.5. The Euler characteristic of a Seifert manifold

$$M = (S, (p_1, q_1), \ldots, (p_k, q_k))$$

is given by

$$\chi(M) = \chi(S) - \sum_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

Definition 2.1.6. The Euler number e of a Seifert manifold

$$M = (S, (p_1, q_1), \ldots, (p_k, q_k))$$

is given by

$$e(M) = \sum_{i=1}^{k} \frac{q_i}{p_i}$$

This is only defined up to modulo $\mathbb{Z}$ if M has a boundary.

While the notation $M = (S, (p_1, q_1), \ldots, (p_k, q_k))$ is not unique, the Euler characteristic and Euler number are invariant under different notations. From [Mar16, p. 364] these invariants are sufficient to classify the geometry of Seifert manifolds, as shown in the following table:

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{SL_2}$

2.2 G-manifolds

In this section we introduce basic definitions of G-manifolds and the equivariant tubular neighbourhood theorem from [Jän68, p. 1-4].

Definition 2.2.1. Let G be a compact Lie group and M a smooth manifold. By an *action of* G on M we will mean a smooth map $G \times M \to M$, where the image of (g, x) is written gx, with the following properties:
(i) $g_1(g_2x) = (g_1g_2)x$ (Compatibility)
(ii) $1x = x$ (Identity)
A smooth manifold M together with a smooth action by G on M is called a *G-manifold*, simply denoted by M when the G-action is understood.

For each $g \in G$ we can define the (left-)action of g by

$$\begin{aligned} l_g : M &\to M \\ x &\mapsto gx \end{aligned}$$

By the previous definition this map is smooth and has a smooth inverse $l_{g^{-1}}$, hence being a diffeomorphism.

Some examples of G-manifolds include any Lie group acting on itself, and this also induces an H-manifold structure on G for any subgroup $H \subseteq G$.

We now define the concepts we need to know about G-manifolds.

Definition 2.2.2. Let M be a G-manifold and let $x \in M$. Then the set $Gx = \{gx \mid g \in G\}$ is the *orbit* of the point x, and $G_x = \{g \mid gx = x\}$ is the *stabilizer group* of the point x. Further we denote by M/G the *orbit space*, that is the set of all orbits $\{Gx \mid x \in M\}$, equipped with the quotient topology, that is: A set of orbits is open in M/G whenever their union is open in M.

Let M be a G-manifold and $x \in M$. The Lie group G is a G-manifold by definition, and hence also a G_x-manifold, since G_x is a subgroup of G. The orbit space G/G_x is by definition

$$G/G_x = \{(G_x)g \mid g \in G\} = \{gG_x \mid g \in G\}$$

which yields an isomorphism

$$\begin{aligned} G/G_x &\to Gx \\ gG_x &\mapsto gx \end{aligned}$$

showing the connection between the orbits and stabilizer groups in a different way.

Definition 2.2.3. Let M and N be two G-manifolds. A map $\phi : M \to N$ is said to be equivariant if for every $g \in G$ and $x \in M$ we have

$$\phi(gx) = g\phi(x)$$

Definition 2.2.4. Let M be a G-manifold. A vector bundle over M together with a G-action on its total space E is called a *G-vector bundle* over M, when the fiber E_x is mapped isomorphically to E_{gx} for $g \in G$.

The tangent bundle TM of a G-manifold M is a G-vector bundle in a natural way, the map from T_xM to $T_{gx}M$ being induced by the differential of $l_g : M \to M$.

We want to show that if S is an equivariantly embedded submanifold of a G-manifold M, then the normal bundle NS is a G-vector bundle over M.

First let G/H be a homogenous space, that is G is a Lie group and $H \subseteq G$ is a subgroup, and let $E \to G/H$ be a G-vector bundle. Denote the fiber at the point $1H \in G/H$ by V. Then V is an H-module, because every $h \in H$ fixes the point $1H \in G/H$, so it sends V to V.

Now let us consider the fiber bundle $G \times_H V \to G/H$, associated with the principal H-fiber bundle $G \to G/H$. Here, $G \times_H V$ is defined to be $(G \times V)/H$ where the equivalence classes are $[g, v] = [gh, h^{-1}v]$. Then $G \times_H V \to G/H$ is in particular a vector bundle, and when we for $g \in G$ declare that $g[g, v] = [gg, v]$, then $G \times_H V$ becomes a G-vector bundle over G/H. From the map $[g, v] \mapsto gv$ we obtain a map

$$G \times_H V \to E$$

which is indeed a G-vector bundle isomorphism. We conclude that any G-vector bundle over G/H is determined by its H-module at the point $1H$.

Now let M be a G-manifold and $x \in M$. We denote by $N_x = T_xM/T_xGx$ the normal space of the orbit Gx at the point x. The identification of Gx with G/G_x as we showed earlier and the identification of G-vector bundles over homogenous spaces shows that we can idnetify the normal bundle of Gx in M with the bundle $G \times_{G_x} N_x \to G/G_x$.

We can now formulate the equivariant tubular neighbourhood theorem:

Theorem 2.2.5. *Let M be a G-manifold. There exists an equivariant diffeomorphism between a G-invariant open neighbourhood of the zero section in $G \times_{G_x} N_x$ and a G-invariant open neighbourhood of Gx in M, which sends the zero section G/G_x to the orbit Gx in a canonical way.*

This is actually just a version of the tubular neighbourhood theorem found in [Mil74, p.115-117] for G-manifolds:

Theorem 2.2.6. *Let M be a Riemannian manifold and S be a smoothly embedded submanifold of M. There exists an open neighbourhood of S in M which is diffeomorphic to the total space of the normal bundle which maps each point $x \in S$ to the zero normal vector at x.*

We will not reproduce all the details of the proofs, but we will sketch the proofs.

For the tubular neighbourhood theorem one denotes the total space of the normal bundle as E and defines the set

$$E(\varepsilon) := \{(x, v) \in E \mid \|v\| < \varepsilon\}$$

where $x \in S$ and v is a normal vector at x. It is then shown that this set maps diffeomorphically to a neighbourhood of S in M by the exponential map

$$\begin{aligned} \mathrm{Exp} : E(\varepsilon) &\to M \\ (x, v) &\mapsto \gamma(1) \end{aligned}$$

where $\gamma : [0,1] \to M$ is a geodesic arc with $\gamma(0) = x$ and velocity vector equal to v at 0, which exists for sufficiently small ε. It is then shown that $E(\varepsilon)$ is indeed diffeomorphic to E, proving the result.

For the equivariant version it is sufficient to show that we can find an equivariant Riemannian metric on M for the tubular neighbourhood to be equivariant. Here it is shown that for any Riemannian metric p on M one can construct an equivariant Riemannian metric $\overline{p}$ on M. Note that the diffeomorphism between $E(\varepsilon)$ and E is not generally equivariant, so the result does not extend to all of E unless M is complete.

2.3 Circle manifolds

As mentioned in [Sco83, p. 430], a Seifert manifold M has the structure of a S^1-manifold (where S^1 has the Lie group structure of $U(1)$) if and only if we can coherently orient the fibers. In this section we make sense of what this means and fill out the details.

Firstly, not all S^1-manifolds are Seifert manifolds. We need to restrict ourselves to only look at those S^1-manifolds with the following finiteness property:

Definition 2.3.1. A G-manifold is said to be finitely stabilized if every stabilizer group is finite and only finitely many stabilizer groups are non-trivial.

As will be evident, the requirement that we can only have finitely many stabilizer groups corresponds to the fact that Seifert manifolds have only finitely many singular fibers. The requirement that all stabilizer groups have to be finite corresponds to the fact that $\mathbf{T}_{(p,q)}$ is finitely covered by $\mathbf{T}_0$, i.e. that q/p is rational.

Secondly, not every Seifert manifold is a S^1-manifold. We need to have a definition of what it means for fibers to be coherently oriented.

Definition 2.3.2. Let M be a Seifert manifold and let F_p be the fiber of p. A point-wise fiber orientation μ of M is a choice of orientation of the tangent space $T_p(F_p)$ for every $p \in M$. A point-wise fiber orientation is said to be continuous if for every $p \in M$ there exists a neighbourhood $U \ni p$ and a continuous vector field X on U such that X_p gives $T_p(F_p)$ the same orientation as μ (where $T_p(F_p)$ is identified with $T_p(M)$). A Seifert manifold M is said to be fiber orientable if there exists a continuous point-wise fiber orientation of M.

The special case where $\pi : M \to B$ is a circle bundle where M is orientable and B is not orientable, M will not be fiber orientable despite being a Seifert manifold. Assuming the opposite, if X is a continuous vector field on $U \subseteq M$ corresponding to the fiber orientation of M and $[X, Y, Z]$ is a continuous frame on U corresponding to the orientation on M, then $[Y|_B, Z|_B]$ is a continuous frame on $\pi(U) \subseteq B$. Since this is true for all U, it would imply that B is orientable, which is a contradiction.

We are now ready to prove the following:

Proposition 2.3.3. *Let M be a compact 3-dimensional S^1-manifold. If M is finitely stabilized, then the orbits the S^1-action give a Seifert structure on M.*

Proof. We start by noting that all orbits of M are circles. The orbit of S^1 is homeomorphic to S^1/S^1_x. Since the stabilizer S^1_x is finite by assumption, the orbit S^1x is homeomorphic to a finite quotient of S^1, which is again homeomorphic to S^1.

By the equivariant tubular neighbourhood theorem there exists a S^1-invariant neighbourhood U of S^1x that is diffeomorphic to $E(\varepsilon) \subseteq E$. Here E is the total space of the normal bundle of S^1x in M and $E(\varepsilon)$ is the set of normal vectors of length less than ε for some $\varepsilon > 0$. Since we only have finitely many non-trivial stabilizer groups, we can choose $\varepsilon > 0$ small enough such that U contains no orbits with non-trivial stabilizer group other than possibly S^1x itself.

The identification of $S^1 \times_{S^1_x} N_x$ with E from the equivariant tubular neighbourhood theorem shows that U is indeed a fibered solid torus. Since S^1 is

1-dimensional and M is 3-dimensional, N_x is 2-dimensional. Thus N_x is diffeomorphic to D^2, showing that E is a solid torus under the quotient by the finite stabilizer group S^1_x. The S^1-vector bundle structure on $E(\varepsilon)$ inherited from E reveals that every orbit in U winds $p = |S^1_x|$ times around S^1x, showing that U has the structure of $\mathbf{T}_{(p,q)}$ for some $0 \leq q < p$. Hence M is a Seifert manifold. ■

Proving the converse requires a few lemmas which we will now state and prove.

Lemma 2.3.4. *Let X and Y be π-related vector fields on the manifolds M and N, respectively, where $\pi : M \to N$ is a smooth map. If γ is an integral curve of X then $\pi \circ \gamma$ is an integral curve of Y.*

Proof. By the chain rule we have

$$(\pi \circ \gamma)_{*,t} = \pi_{*,\gamma(t)} \circ \gamma_{*,t}$$

and by the definition of an integral curve we have

$$\pi_{*,\gamma(t)} \circ \gamma_{*,t} = \pi_{*,\gamma(t)} \circ X_{\gamma(t)}.$$

Finally by π-relatedness we have

$$\pi_{*,\gamma(t)} \circ X_{\gamma(t)} = Y_{(\pi\circ\gamma)(t)}$$

showing that $\pi \circ \gamma$ is indeed an integral curve of Y. ■

Lemma 2.3.5. *There exists a smooth vector field X on the fibered solid torus $\boldsymbol{T}_{(p,q)}$ such that each integral curve is a fiber.*

Proof. We use the fact that $\mathbb{R}$ is a Lie group to define the left-invariant vector field Y generated by $\frac{d}{dt}\big|_0 \in T_e(\mathbb{R})$. This vector field is smooth by [Tu08, p.181]. We can use this to define a smooth vector field $\widetilde{X}$ on the infinite solid cylinder $\mathbb{R} \times D^2$ as the following composition:

$$\mathbb{R} \times D^2 \xrightarrow{\pi} \mathbb{R} \xrightarrow{Y} T\mathbb{R} \xhookrightarrow{i} T\mathbb{R} \oplus TD^2 \simeq T(\mathbb{R} \times D^2).$$

The projection π is smooth because projections are always smooth, the map Y is smooth because Y is a smooth vector field, and the final inclusion i is smooth because inclusions are always smooth. Hence the vector field $\widetilde{X} = i \circ Y \circ \pi$ on $\mathbb{R} \times D^2$ is smooth. Furthermore since Y is left-invariant, $\widetilde{X}$ is invariant under the $\mathbb{R}$-action

$$r(t, x) = (rt, x)$$

on $\mathbb{R} \times D^2$ for $r, t \in \mathbb{R}$ and $x \in D^2$. The integral curves of $\widetilde{X}$ are the fibers $\mathbb{R} \times \{x\}$.

The map

$$\phi : \mathbb{R} \times D^2 \to S^1 \times D^2 \simeq \mathbf{T}_{(p,q)}$$
$$(t, x) \mapsto (e^{2\pi it}, e^{2\pi it\frac{q}{p}} x)$$

is a covering map sending each fiber of the cylinder to a fiber of the torus. We will now show that the pushforward $X := \phi(\widetilde{X})$ is a well-defined smooth vector field on $\mathbf{T}_{(p,q)}$. It will then have the fibers of $\mathbf{T}_{(p,q)}$ as integral curves, since ϕ sends integral curves to integral curves by Lemma 2.3.4.

Since ϕ is a covering map and $\mathbb{R} \times D^2$ is simply connected, the infinite cylinder is the universal cover of $\mathbf{T}_{(p,q)}$. The deck transformation group is homomorphic to the fundamental group $\mathbb{Z}$ of $\mathbf{T}_{(p,q)}$. If $n \in \mathbb{Z}$ then this group acts on $\mathbb{R} \times D^2$ as $n \cdot (t, x) \mapsto (t + pn, x)$. Since $\widetilde{X}$ is invariant under the previously defined $\mathbb{R}$-action, it is also invariant under this $\mathbb{Z}$-action. Hence if V is any connected component of $\phi^{-1}(U)$ where $U \subseteq \mathbf{T}_{p,q}$ is evenly covered and $\psi : U \to V$ is the inverse of $\pi|_V$, then the composition $\widetilde{X} \circ \psi$ is independent on the choice of V. Thus the composition

$$\mathbf{T}_{(p,q)} \xrightarrow{\psi} \mathbb{R} \times D^2 \xrightarrow{\widetilde{X}} T(\mathbb{R} \times D^2) \xrightarrow{\pi_*} T(\mathbf{T}_{(p,q)})$$

is well-defined.

The map ψ can be expressed explicitly as $\psi(z, x) = (\frac{\log(z)}{2\pi i}, e^{-2\pi i t \frac{q}{p}} x)$, where the value of $\log(z)$ lies in V. This is clearly smooth as it is the composition of smooth elementary functions. The map $\widetilde{X}$ is smooth because it is a smooth vector field, and the differential π_* is smooth because all differentials are smooth. This means that $X = \pi_* \circ \widetilde{X} \circ \psi$ is a smooth vector field on $\mathbf{T}_{(p,q)}$.

■

Note that the vector field X constructed in the above proof makes $\mathbf{T}_{(p,q)}$ fiber oriented. This is because it is smooth, hence continuous, and restricts to a basis for each fiber on each fiber. Furthermore the vector field $-X$ has the opposite orientation of X, so we can always find a smooth vector field corresponding to either orientation.

Proposition 2.3.6. *Every Seifert manifold M with oriented fibers is a finitely stabilized $U(1)$-manifold.*

Proof. Let $\{U_j\}$ be a cover of the Seifert manifold M where each U_j is a fibered solid torus, and let $\{X_j\}$ be a collection of smooth vector fields on $\{U_j\}$ with fibers as integral curves, as constructed in 2.3.5. We choose our X_j's such that they all have the same fiber orientation. We can now find a partition of unity subordinate to U_i to construct a smooth vector field X on M where all the integral curves are fibers. We need to modify this vector field in such a way that the period of all the integral curves of non-singular fibers is the same.

For any fibered solid torus $U_j \simeq \mathbf{T}_{(p,q)}$ in our cover we can pull back the vector field $X|_{U_j}$ to a vector field $\widetilde{X}_j$ on $\mathbb{R} \times D^2$. Since we know that the images of the integral curves of this vector field are the fibers $\mathbb{R} \times \{x\}$, this vector field must be given by $(\widetilde{X}_j)_{(t,x)} = f \left.\frac{\partial}{\partial t}\right|_{(t,x)}$ for some smooth, strictly positive function f. Now let $\gamma_x : \mathbb{R} \to \mathbb{R}$ be the integral curve of the fiber $\mathbb{R} \times \{x\}$ (evaluated only in the first coordinate) such that $\gamma_x(0) = 0$. By the definition of integral curves we have $\gamma_x'(t) = (\widetilde{X}_j)_{\gamma_x(t)}$. In terms of calculus this leads to the differential equation

$$\dot{\gamma}_x(t) = f(\gamma_x(t))$$
$$\frac{\dot{\gamma}_x(t)}{f(\gamma_x(t))} = 1.$$

Now let F be any function such that $\dot{F} = \frac{1}{f}$. Since f is smooth, so is F. We then get

$$\dot{F}(\gamma_x(t)) = 1$$
$$F(\gamma_x(t)) = t + a$$
$$\gamma_x(t) = F^{-1}(t + a).$$

Note that we can choose F such that $a = 0$ to avoid the extra constant.

Let t_x be the value such that $\gamma_x(t_x) = p$. If U_j is a non-trivial fibred solid torus and the fiber of x is non-singular, the period of the projection of γ_x in U_j is the t_x. Otherwise the period of the projection of γ_x in U_j is t_x/p. We want all non-singular fibers to have period 1, so let us consider a function $c(x)$ that is constant on each fiber and the vector field $c\widetilde{X}_j$. This vector field will be smooth if $c(x)$ is smooth, and the images of the integral curves will be the same as for $\widetilde{X}_j$. If $\omega_x : \mathbb{R} \to \mathbb{R}$ is the integral curve of the fiber $\mathbb{R} \times \{x\}$ (evaluated only in the first coordinate) such that $\omega_x(0) = 0$, then we have $\omega'_x(t) = (c\widetilde{X}_j)_{\omega_x(t)}$. In terms of calculus this leads to the differential equation

$$\dot{\omega}_x(t) = c(x) f(\omega_x(t))$$
$$\frac{\dot{\omega}_x(t)}{f(\omega_x(t))} = c(x)$$
$$\dot{F}(\omega_x(t)) = c(x)$$
$$F(\omega_x(t)) = c(x)t + b$$
$$\omega_x(t) = F^{-1}(c(x)t + b)$$

Since F was chosen such that $a = 0$, we get $F(\gamma_x(t)) = t$ which implies $F(0) = 0$. Since we defined $\omega_x(0) = 0$ this shows that $b = 0$ as well. By substitution we now have $\omega_x(t) = \gamma_x(c(x)t)$. Thus if t_x is the period of γ_x, then $c(x)t_x$ is the period of ω_x. So to have a period equal to 1 we need to have $c(x) = 1/t_x$. This is smooth, because $t_x = \gamma_x^{-1}(p)$ where γ_x as a function of x is the smooth flow of X, and because its differential is never zero γ^{-1} is smooth.

Because the period of an integral curve is independent of the choice of U_j, $c(x)$ can be extended globally to $x \in M$. Hence cX is a smooth vector field on M.

To define an S^1-action on M we set

$$e^{2\pi i t} x := \gamma_x(t)$$

where $\gamma_x : \mathbb{R} \to M$ is the integral curve of cX at $x \in M$ such that $\gamma_x(0) = x$. This action is smooth because cX is smooth. It is compatible:

$$\begin{aligned} e^{2\pi i t}(e^{2\pi i s} x) &= e^{2\pi i t} \gamma_x(s) \\ &= \gamma_{\gamma_x(s)}(t) \\ &= \gamma_x(t + s) \end{aligned}$$

and the identity acts as the identity:

$$\begin{aligned} e^{2\pi i0}x &= \gamma_x(0) \\ &= x. \end{aligned}$$

Every non-singular fiber has trivial stabilizer group, because they have period 1. Every singular fiber has finite stabilizer group of order p, because they have period $1/p$ which means that

$$e^{2\pi it}x = \gamma_x(t) = \gamma_x(t + q/p) = e^{2\pi itq/p}x$$

for $0 \leq q < p$. This shows that M is a finitely stabilized S^1-manifold.

■

From 2.3.3 and 2.3.6 we see that a Seifert manifold admits a finitely stabilzied S^1-structure if and only if it is fiber orientable. From the proofs we see that we can calculate the invariants $p_1, \ldots, p_k$ of a fiber oriented Seifert manifold $M = (S, (p_1, q_1), \ldots, (p_k, q_k))$ by finding the orders of the non-trivial stabilizer groups. Moreover the base space of a Seifert manifold is the orbit space M/S^1, since the points of the orbit space are the fibers of M by definition. We will show that this is an orbifold in chapter 4.

CHAPTER 3

Examples of Seifert manifolds

3.1 Brieskorn manifolds

A 3-dimensional Brieskorn manifold $\Sigma(p,q,r)$ for $p,q,r \geq 2$ is the intersection between the hyperplane in $\mathbb{C}^3$ given by

$$z_1^p + z_2^q + z_3^r = 0$$

and the odd sphere

$$S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^2 + |z_3|^2 \; = 1\}.$$

These are known for being 3-dimensional homology spheres when p, q and r are relatively prime (see [Mil75, p.176]). This means that they have the same homology groups as the n-sphere despite not being diffeomorphic to S^n. We are interested in these manifolds because they turn out to be Seifert manifolds. We will prove this by giving them a finitely stabilized circle action.

Proposition 3.1.1. *A Brieskorn manifold $\Sigma(p,q,r)$ gets a Seifert structure from the circle action*

$$u(z_1, z_2, z_3) := (u^{\frac{m}{p}} z_1, u^{\frac{m}{q}} z_2, u^{\frac{m}{r}} z_3)$$

where $u \in S^1$ and m is the least common multiple of p, q and r. If $\Sigma(p,q,r)$ is a homology sphere it has exactly three singular fibres with cyclic stabilizer groups of order p, q and r.

Proof. We start by checking that the given circle action is indeed a group action by checking that it satisfies the definition. It is clearly smooth since it is a composition of smooth functions. Let $z = (z_1, z_2, z_2) \in \Sigma(p,q,r)$ for the remainder of the proof. The the other two axioms are also satisfied:

$$\begin{aligned}(uv)(z) &= ((uv)^{\frac{m}{p}} z_1, (uv)^{\frac{m}{q}} z_2, (uv)^{\frac{m}{r}} z_3) \\ &= ((u)^{\frac{m}{p}} (v)^{\frac{m}{p}} z_1, (u)^{\frac{m}{q}} (v)^{\frac{m}{q}} z_2, (u)^{\frac{m}{r}} (v)^{\frac{m}{r}} z_3) \\ &= u(v^{\frac{m}{p}} z_1, v^{\frac{m}{q}} z_2, v^{\frac{m}{r}} z_3) \\ &= u(vz)\end{aligned}$$

and

$$\begin{aligned}1z &= (1^{\frac{m}{p}} z_1, 1^{\frac{m}{q}} z_2, 1^{\frac{m}{r}} z_3) \\ &= (z_1, z_2, z_3) \\ &= z\end{aligned}$$

Next we need to check that the circle action is well-defined by checking that uz is contained in $\Sigma(p,q,r)$.

$$\begin{aligned}(u^{\frac{m}{p}} z_1)^p + (u^{\frac{m}{q}} z_2)^q + (u^{\frac{m}{r}} z_3)^r) &= u^m z_1^p + u^m z_1^q + u^m z_1^r \\ &= u^m (z_1^p + z_2^q + z_3^r) \\ &= 0\end{aligned}$$

because $u^m \neq 0$.

$$\begin{aligned}|u^{\frac{m}{p}} z_1|^2 + |u^{\frac{m}{q}} z_2|^2 + |u^{\frac{m}{r}} z_3|^2 &= |u|^{\frac{2m}{p}} |z_1|^2 + |u|^{\frac{2m}{q}} |z_2|^2 + |u|^{\frac{2m}{r}} |z_3|^2 \\ &= |z_1|^2 + |z_2|^2 + |z_3|^2 \\ &= 1\end{aligned}$$

because $|u| = 1$.

To calculate the stabilizer groups of the circle action we consider two cases. The first case we consider is when $z_1, z_2, z_3 \neq 0$. Then u is in the stabilizer group of z if we have

$$\begin{aligned}uz &= z \\ (u^{\frac{m}{p}} z_1, u^{\frac{m}{q}} z_2, u^{\frac{m}{r}} z_3) &= (z_1, z_2, z_3) \\ u^{\frac{m}{p}} = u^{\frac{m}{q}} = u^{\frac{m}{r}} &= 1\end{aligned}$$

The last line shows that the order of u must divide $\frac{m}{p}$, $\frac{m}{q}$ and $\frac{m}{r}$. From elementary number theory we can find k_1, k_2 and k_3 such that

$$k_1 \frac{m}{p} + k_2 \frac{m}{q} + k_3 \frac{m}{r} = \gcd\left(\frac{m}{p}, \frac{m}{q}, \frac{m}{r}\right).$$

Let $d = \gcd\left(\frac{m}{p}, \frac{m}{q}, \frac{m}{r}\right)$. Then $\frac{m}{dp}$, $\frac{m}{dq}$ and $\frac{m}{dr}$ are integers, hence p, q and r divide $\frac{m}{d}$. But since m is defined to be the least common multiple of p, q and r this means that $d = 1$. It now follows that

$$\begin{aligned}u^1 &= u^{k_1 \frac{m}{p} + k_2 \frac{m}{q} + k_3 \frac{m}{r}} \\ u &= \left(u^{\frac{m}{p}}\right)^{k_1} \left(u^{\frac{m}{q}}\right)^{k_2} \left(u^{\frac{m}{r}}\right)^{k_3} \\ u &= 1\end{aligned}$$

Hence the stabilizer group of z is trivial.

The remaining case is to check when either of z_1, z_2 or z_3 is equal to zero. Note that we must have at least two non-zero coefficients in order to satisfy the requirements to be a point on the Brieskorn manifold. We start by checking for $z_3 = 0$. Similar to before we get

$$\begin{aligned}uz &= z \\ (u^{\frac{m}{p}} z_1, u^{\frac{m}{q}} z_2, u^{\frac{m}{r}} 0) &= (z_1, z_2, 0) \\ u^{\frac{m}{p}} = u^{\frac{m}{q}} &= 1.\end{aligned}$$

In this case the order of u must divide $\frac{m}{p}$ and $\frac{m}{q}$. We can find k_1 and k_2 such that

$$k_1 \frac{m}{p} + k_2 \frac{m}{q} = \gcd\left(\frac{m}{p}, \frac{m}{q}\right).$$

To simplify notation we set $d = \gcd\left(\frac{m}{p}, \frac{m}{q}\right)$. This leads to

$$\begin{aligned} u^d &= u^{k_1 \frac{m}{p} + k_2 \frac{m}{q}} \\ &= \left(u^{\frac{m}{p}}\right)^{k_1} \left(u^{\frac{m}{q}}\right)^{k_2} \\ &= 1 \end{aligned}$$

Hence the order of u divides $\gcd\left(\frac{m}{p}, \frac{m}{q}\right)$, which means that we have

$$u = e^{\frac{2\pi i n}{d}}.$$

with $n \in \mathbb{Z}$. This shows that the stabilizer group of z is the cyclic group of order d. In the particular case where p, q and r are relatively prime we get that

$$\begin{aligned} d &= \gcd\left(\frac{m}{p}, \frac{m}{q}\right) \\ &= \gcd\left(\frac{pqr}{p}, \frac{pqr}{q}\right) \\ &= \gcd(qr, pr) \\ &= r. \end{aligned}$$

By symmetry we also find that the stabilizer group of z for $z_2 = 0$ is the cyclic group of order $\gcd\left(\frac{m}{p}, \frac{m}{r}\right)$ (which is equal to q when p, q and r are relatively prime,) and the stabilizer group of z when $z_1 = 0$ is the cyclic group of order $\gcd\left(\frac{m}{q}, \frac{m}{r}\right)$ (which is equal to p when p, q and r are relatively prime).

It now remains to show that we only have finitely many orbits with non-trivial stabilizer group. Since we just showed that the only points with non-trivial stabilizer group are the ones with either z_1, z_2 or z_3 equal to zero, all orbits with non-trivial stabilizer groups are contained in the sets

$$V_k = \{z \in \Sigma(p, q, r) \mid z_k = 0\}$$

for $k = 1, 2, 3$. We need to show that each set V_k splits into finitely many orbits.

Assume that $z_3 = 0$. We then get the following relationship:

$$\begin{aligned} z_1^p + z_2^q &= 0 \\ z_1^p &= -z_2^q \\ |z_1|^p &= |z_2|^q \\ |z_1| &= |z_2|^{\frac{q}{p}} \\ \sqrt{1 - |z_2|^2} &= |z_2|^{\frac{q}{p}} \\ 1 - |z_2|^2 - |z_2|^{\frac{2q}{p}} &= 0 \end{aligned}$$

The final equation has a strictly decreasing function of $|z_2|$ on the left-hand side. This means that the equation has a unique solution $|z_2|$, and since the function yields 1 for $|z_2| = 0$ and -1 for $|z_2| = 1$, the solution lies between $0 < |z_2| < 1$. This shows that the length of z_2 (and hence z_1) is only dependent on p and q.

Now every orbit of $(z_1, z_2, 0) \in V_3$ contains a point on the form $(|z_1|, z_2, 0)$, since we can find $u \in S^1$ such that $u^{\frac{m}{p}} z_1 = |z_1|$. We can now reduce our problem to looking for the orbits of the points $(|z_1|, z_2, 0)$. Let

$$W = \{(|z_1|, z_2, 0) \in V_3\}$$

be the set of such points. These points have the property that

$$\begin{aligned} |z_1|^p + z_2^q &= 0 \\ |z_1|^p + (|z_2|e^{\theta i})^q &= 0 \\ 1 + \frac{|z_2|^q}{|z_1|^p}(e^{\theta i})^q &= 0 \\ (e^{\theta i})^q &= -1 \end{aligned}$$

which has q solution for θ, hence $|W| = q$.

The number of orbits of V_3 under the S^1-action is the same as the the number of orbits in W under the action of the subset of S^1 that leaves W fixed in V_3. This subgroup contains the elements u such that $u^{\frac{m}{p}} = 1$, hence it is the cyclic subgroup $C_{\frac{m}{p}}$ of order $\frac{m}{p}$. The stabilizer group of the $C_{\frac{m}{p}}$-action on W contains the elements u such that

$$\begin{aligned} u(|z_1|, z_2, 0) &= (|z_1|, z_2, 0) \\ u^{\frac{m}{q}} z_2 &= z_2 \\ u^{\frac{m}{q}} &= 1 \end{aligned}$$

Since the order of u also divides $\frac{m}{p}$, this means that the stabilizer group is cyclic of order $d = \gcd(\frac{m}{p}, \frac{m}{q})$. Since we are working with a finite group, the order of the orbit of $(|z_1|, z_2, 0)$ is the order of $C_{\frac{m}{p}}$ divided by the order of the stabilizer of $(|z_1|, z_2, 0)$, that is

$$\frac{\frac{m}{p}}{d} = \frac{m}{pd}.$$

The number of orbits is therefore the order of W divided by the order of the orbit, that is

$$\begin{aligned} \frac{q}{\frac{m}{pd}} &= \frac{pqd}{m} \\ &= \frac{pq\gcd(\frac{m}{p}, \frac{m}{q})}{m} \\ &= \frac{\gcd(mp, mq)}{m} \\ &= \gcd(p, q) \end{aligned}$$

This means that V_3 splits into $\gcd(p, q)$ orbits.

By symmetry we also get that V_2 splits into $\gcd(p, r)$ orbits and that V_1 splits into $\gcd(q, r)$ orbits. This shows that are finitely many orbits with finite stabilizer group. In particular we have only three singular orbits when p, q and r are relatively prime, since each V_k represents a single orbit.

■

3.2 Spherical space forms

In general a spherical space form is the quotient of S^n with a finite subgroup of its isometry group. For $n = 3$ and $\Gamma \subseteq SU(2)$ finite, the spherical space forms S^3/Γ are Seifert manifolds. We will show this by giving them an appropriate circle action.

There are multiple ways to think of S^3, and in this case it turns out to be useful to think of S^3 as the unit quaternions which are also diffeomorphic to the Lie Group $SU(2)$

$$\begin{aligned} SU(2) &\simeq \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 + d^2 = 1\} \\ &= \{\alpha + \beta j \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1\} \\ &\subseteq \mathbb{H}. \end{aligned}$$

Now S^3 inherits the Lie group structure of $SU(2)$, and by letting it act on itself it is therefore an $SU(2)$-manifold. Taking this further we know that $U(1) \subseteq SU(2)$ and that S^1 can be identified with $U(1)$, so we also inherit an S^1-action on S^3.

From [Mar16, p. 365] we have a group homomorphism

$$\phi : SU(2) \to SO(3)$$

with kernel $\{1, -1\}$. The preimage under ϕ of a subgroup of $SO(3)$ is called a binary group, and Martelli shows that all finite subgroups $\Gamma \in SU(2)$ up to conjugation are either cyclic or binary. The finite subgroups of $SO(3)$ are the cyclic groups C_n, the dihedral groups D_{2m}, the tetrahedral group T_{12}, the octahedral group O_{24} and the icosahedral group I_{60}. Thus the finite subgroups of $SU(2)$ up to conjugation are known to be the cyclic groups C_n, the binary dihedral groups D^*_{4n}, the tetrahedral group T^*_{24}, the binary octahedral group O^*_{48} and the binary icosahedral group I^*_{120}. Thought of in terms of quaternions they can be expressed as follows:

$$C_m = \{e^{\frac{2\pi i n}{m}} \mid 0 \leq n < m\} \qquad \text{(Cyclic groups)}$$

$$D^*_{4m} = \{e^{\frac{\pi i n}{m}}, e^{\frac{\pi i n}{m}} j \mid 0 \leq n < 2m\} \qquad \text{(Binary dihedral groups)}$$

$$T^*_{24} = D_8 \cup \left\{ \frac{\pm 1 \pm i \pm j \pm k}{2} \right\} \qquad \text{(Binary tetrahedral group)}$$

$$O^*_{48} = T^*_{24} \cup \left\{ \frac{a+b}{\sqrt{2}} \mid a, b \in D_8, a \neq b \right\} \qquad \text{(Binary octahedral group)}$$

$$I^*_{120} = T^*_{24} \cup \left\{ \pm\frac{a}{2} \pm \frac{(\sqrt{5}-1)b}{4} \pm \frac{(\sqrt{5}+1)c}{4} \mid (a, b, c, d) = \sigma(1, i, j, k) \right\} \qquad \text{(Binary icosahedral group)}$$

for all even permutations σ and where $D_8 = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the set of Lipschitz units.

The key to define a circle action is to let $\Gamma \subseteq SU(2)$ act on S^3 from the right and let S^1 act on S^3 from the left to obtain a S^1-action on S^3/Γ. We do this by defining

$$g[x] := [gx]$$

for $g \in S^1$ and $[x] \in S^3/\Gamma$. This is well-defined because

$$g(xh) = (gx)h \in [gh].$$

This S^1-action does not always give rise to a Seifert structure on S^3/Γ, however. In the case where $-1 \in \Gamma$ (that is, Γ having even order) all the stabilizer groups would contain -1, so in that case we use the fact that the quotient group $S^1/\{\pm 1\}$ is diffeomorphic to S^1 by the map

$$\begin{aligned} \phi : S^1/\{\pm 1\} &\to S^1 \\ [e^{\theta i}] &\mapsto e^{2\theta i} \end{aligned}$$

and define a circle action on S^3/Γ by using the quotient group instead. We do this by setting

$$g[x] := [gx]$$

for $[g] \in S^1/\{\pm 1\}$. Note that this definition does not depend on the representative g, since we either have $g = e^{i\theta}$ or $g = -e^{i\theta}$, and $[gx] = [-gx]$ when the order of Γ is even. It is easy to see that if $S^1_{[x]}$ is the stabilizer group of $[x] \in S^3/\Gamma$ under the S^1-action, then $(S^1/\{\pm 1\})_{[x]} = (S^1_{[x]})/\{\pm 1\}$. Thus we can calculate the stabilizer groups of the $S^1/\{\pm 1\}$-action from the S^1-action. In particular if $S^1_{[x]}$ has finite order n, then $(S^1/\{\pm 1\})_{[x]}$ has order $n/2$.

For the remainder of this section let G denote S^1 for Γ of odd order and denote $S^1/\{\pm 1\}$ for Γ of even order. We will now prove that this G-action yields a Seifert structure on S^3/Γ. We start by proving the following proposition.

Proposition 3.2.1. *Let $\Gamma \in SU(2)$ be a finite subgroup. The number of orbits of the set*

$$\Gamma_0 = \left\{ \frac{Im(h)}{|Im(h)|} \mid h \in \Gamma, Im(h) \neq 0 \right\}$$

under the action of Γ-conjugation is the same as the number of orbits of S^3/Γ with non-trivial stabilizer group under the previously defined G-action.

Proof. The stabilizer group of $[x] \in S^3/\Gamma$ by the G-action is given by

$$\begin{aligned} G_{[x]} &= \{g \in G \mid [gx] = [x]\} \\ &= \{g \in G \mid \exists h \in \Gamma, gx = xh\} \\ &= \{g \in G \mid \exists h \in \Gamma, g = xh\overline{x}\}. \end{aligned}$$

In other words, g is in the stabilizer group of $[x]$ if and only if there exists $h \in \Gamma$ such that $g = xh\overline{x}$. To find the elements $[x]$ with non-trivial stabilizer group we need to solve the equation $g = xh\overline{x}$ for g with non-zero imaginary part, that is $g = \cos(\psi) + i\sin(\psi)$ for $\sin(\psi) \neq 0$. We note that any quaternion $h \in \mathbb{H}$ can be written as $h = \cos(\theta) + q\sin(\theta)$ where $q \in \mathbb{H}_0$ is the normalized imaginary part of h (see for instance [Frø19]). The set Γ_0 is the set of normalized imaginary parts of elements $h \in \Gamma$, so we can write $h = \cos(\theta) + q\sin(\theta)$ for $q \in \Gamma_0$. We then get that

$$\begin{aligned} g &= xh\overline{x} \\ \cos(\psi) + i\sin(\psi) &= x(\cos(\theta) + \sin(\theta)q)\overline{x} \\ \cos(\psi) + i\sin(\psi) &= x\cos(\theta)\overline{x} + x\sin(\theta)q\overline{x} \\ \cos(\psi) + i\sin(\psi) &= \cos(\theta) + \sin(\theta)xq\overline{x} \end{aligned}$$

$$\cos(\psi) = \cos(\theta) \qquad\qquad \sin(\psi)i = \sin(\theta)xq\overline{x}$$

The bottom two equations imply that $\sin(\theta) = \pm\sin(\psi)$ which yields

$$xq\overline{x} = \pm i.$$

Thus $[x] \in S^3/\Gamma$ has non-trivial stabilizer group if and only if there exists $q \in \Gamma_0$ such that $xq\overline{x} = \pm 1$. We can improve this by noting that if $q \in \Gamma_0$ is the normalized imaginary part of $h \in \Gamma$ then the normalized imaginary part of $\overline{h} \in \Gamma$ is $-q \in \Gamma_0$, so if $xq\overline{x} = -i$ then $x(-q)\overline{x} = i$. With this improvement we see that $[x] \in S^3/\Gamma$ has non-trivial stabilizer group if and only if there exists $q \in \Gamma_0$ such that $xq\overline{x} = i$.

Now we need to check that conjugation by Γ is a group action on Γ_0. Clearly conjugation is compatible because $\overline{h}_2\overline{h}_1 = \overline{h_1 h_2}$ for $h_1, h_2 \in \Gamma$, and $h = 1$ acts as the identity. So we just need to check that conjugation by Γ is a closed action. We know that conjugation by $h \in \Gamma$ (for Γ of even order) is the rotation of $\phi(h) \in SO(3)$, and the normalized imaginary part $q \in \Gamma_0$ of h is a point on the axis of rotation of $\phi(h)$. Since $SO(3)$ is a group of symmetries, it preserves the axes of rotation, hence conjugation by Γ leaves Γ_0 fixed. (In the case where the order of Γ is odd we have $\Gamma = C_n$ for n odd, which has the same properties as C_n for even n in terms of conjugation.) The orbit of $q \in \Gamma_0$ is the set of conjugations $hq\overline{h}$ for $h \in \Gamma$.

Let $x, xh \in S^3$ for $h \in \Gamma$ be two representatives for $[x] \in S^3/\Gamma$ and let $[x]$ be on a singular G-orbit. Then there exists $q, q' \in \Gamma_0$ such that

$$xq\overline{x} = (xh)q'\overline{xh} = i$$

which shows that $q = hq'\overline{h}$. Thus q and q' are Γ-conjugate i.e. on the same orbit in Γ_0. If $g[x] \in S^3/\Gamma$ for $g \in G$ is another point on the same orbit as $[x]$ then there exists $u \in G$ and $h \in \Gamma$ such that

$$\begin{aligned} g(ux) &= (ux)h \\ (\overline{u}gu)x &= xh \\ gx &= xh \end{aligned}$$

showing there exists $q \in \Gamma_0$ such that

$$xq\overline{x} = (gx)q\overline{(gx)} = i.$$

We have now shown that for every G-orbit in S^3/Γ there exists an orbit in Γ_0.

Conversely, let $q, q' \in \Gamma_0$ be Γ-conjugate, that is $q' = hq\overline{h}$ for $h \in \Gamma$. There exist $[x], [y] \in S^3/\Gamma$ on singular orbits such that

$$xq\overline{x} = yq'\overline{y} = i.$$

This leads to

$$xq\overline{x} = (yh)q\overline{(yh)} = i.$$

Due to the Lie group structure on $S^3 \simeq SU(2)$ we can find an element $g \in SU(2)$ such that $x = g(yh)$. We then get

$$\begin{aligned} (yh)q\overline{(yh)} &= i \\ g(xq\overline{x})\overline{g} &= i \\ gi\overline{g} &= i \end{aligned}$$

showing that conjugating by g leaves i fixed. The only non-identity elements that leave i fixed under conjugation are the elements that rotate around the axis through i, so we get

$$g = \cos(\theta) + i\sin(\theta) \in G.$$

Thus we have $x = gyh$ for $g \in G$ and $h \in \Gamma$, showing that $[x]$ and $[y]$ are on the same G-orbit in S^3/Γ. Therefore for every orbit in Γ_0 there exists a G-orbit in S^3/Γ. ■

From this proposition it readily follows that S^3/Γ is a Seifert manifold.

Corollary 3.2.2. *Let $\Gamma \subseteq SU(2)$ be finite. Then S^3/Γ is a Seifert manifold.*

Proof. From the previous proposition we know that the number of singular G-orbits of S^3/Γ is the same as the number of orbits of Γ_0 under Γ-conjugation. Since Γ_0 is finite, hence has finitely many orbits, there are only finitely many singular orbits of S^3/Γ.

Now recall that $g \in G$ is in the stabilizer group of $[x]$ if and only if there exists $h \in \Gamma$ such that $g = xh\overline{x}$. Since there are only finitely many $h \in \Gamma$, there are only finitely many $g \in G_{[x]}$, hence all stabilizer groups are finite.

Since the G-action on S^3/Γ is finitely stabilized, S^3/Γ is a Seifert manifold. ■

The next thing to do is to calculate the stabilizer groups of the spherical space forms. We denote by $\mathrm{Cl}(q)$ for $q \in \Gamma_0$ the orbit of q under conjugation by Γ. By $\mathrm{Cl}(q)^*$ for $q \in \Gamma_0$ we shall mean the set

$$\mathrm{Cl}(q)^* := \{h = \cos(\theta) + q'\sin(\theta) \in \Gamma \mid q' \in \mathrm{Cl}(q)\}.$$

Now if $[x] \in S^3/\Gamma$ is a point with non-trivial stabilizer group corresponding to $\mathrm{Cl}(q)$ and $g \in G$ has non-real imaginary part, all $h \in \Gamma$ that solve the equation

$$gx = xh$$

are precisely the $h \in \mathrm{Cl}(q)^*$. Due to the Lie group structure on S^3 we have exactly one solution $g \in S^1$ for each solution $h \in \Gamma$, so the number of non-real elements in the stabilizer group $S^1_{[x]}$ is $|\mathrm{Cl}(q)^*|$. Thus $G_{[x]}$ has order $|\mathrm{Cl}(q)^*| + 1$ if Γ has odd order, and order $|\mathrm{Cl}(q)^*|/2 + 1$ if Γ has even order (since we have to add the identity element, which is real). We now note that all finite subgroups of S^1 are cyclic, hence uniquely decided by its order.

We now have all we need to find the stabilizer groups of S^3/Γ for all cases of Γ.

Cyclic groups

Let $\Gamma = C_n$. Then $\Gamma_0 = \{\pm i\}$. Since Γ consists of rotations about the axis through i and $-i$, both i and $-i$ are left fixed, so the orbits are $\mathrm{Cl}(i)$ and $\mathrm{Cl}(-i)$.

Every non-real element of C_n has normalized imaginary part $\pm i$, hence all non-real elements are sent to both conjugacy classes. Therefore if n is odd we have $|\mathrm{Cl}(\pm i)^*| = n-1$, so both the orbit of $\mathrm{Cl}(i)$ and $\mathrm{Cl}(-i)$ have stabilizer group of order $n-1+1=n$. If instead n is even we have $|\mathrm{Cl}(\pm i)^*| = n-2$, so both the orbit of $\mathrm{Cl}(i)$ and $\mathrm{Cl}(-i)$ have stabilizer group of order $(n-2)/2+1 = n/2$.

In summary, S^3/C_n has two singular orbits when $n \geq 3$, with stabilizer groups C_n for n odd, and with stabilizer groups $C_{n/2}$ for n even.

Binary dihedral groups

Let $\Gamma = D^*_{4m}$. Then $\Gamma_0 = \{\pm i, e^{\frac{\pi i n}{m}} j \mid 0 \leq n < 2m\}$. Now Γ consists of $\frac{n\pi}{m}$-rotations about the axis through i and π-rotations about the axis through $e^{\frac{\pi i n}{m}} j$. The $\frac{n\pi}{m}$-rotations about the axis through i leave i and $-i$ fixed while taking any $e^{\frac{\pi i n_1}{m}} j$ to any other $e^{\frac{\pi i n_2}{m}} j$. The π-rotations about the axis through $e^{\frac{\pi i n}{m}} j$ take i to $-i$ and $e^{\frac{\pi i n}{m}} j$ to $-e^{\frac{\pi i n}{m}} j$. Hence we get the two orbits $\mathrm{Cl}(i)$ and $\mathrm{Cl}(j)$.

Every non-real element $e^{\frac{\pi i n}{m}}$ has normalized imaginary part $\pm i$ and every element $e^{\frac{\pi i n}{m}} j$ is itself normalized and imaginary. Therefore we get $|\mathrm{Cl}(i)^*| = 2m-2$ and $|\mathrm{Cl}(j)^*| = 2m$. So the orbit of $\mathrm{Cl}(i)$ has stabilizer group of order $(2m-2)/2+1 = m$, and the orbit of $\mathrm{Cl}(j)$ has stabilizer group of order $(2m)/2+1 = m+1$.

In summary, S^3/D^*_{4m} has two singular orbits when $m \geq 2$, with stabilizer groups C_m and C_{m+1}. For $m=1$ it has one singular orbit with stabilizer group C_2.

Binary groups of platonic solids

The final three cases are when Γ is the binary group of a platonic solid. We know that Γ_0 consists of points on the rotational axes of the platonic solid that Γ corresponds to. We know that all these axes pass through either a vertex, the midpoint of an edge or the centroid of a face on the given platonic solid. Hence the set Γ_0 actually consist of the normalized vertices, normalized midpoints of edges and normalized centroids of faces. Conjugating Γ_0 by Γ is the same as rotating the platonic solid corresponding to Γ such that every vertex is sent to another vertex, every edge is sent to another edge and every face is sent to another face. It is therefore clear that Γ_0 has precisely three orbits under Γ-conjugation, one consisting of normalized vertices, one consisting of normalized midpoints of edges and one consisting of normalized centroids of faces. Identifying which orbit the normalized part of $h \in \Gamma$ can simply be done by considering the real part of h. The real part of h represents the rotation when conjugating by h, and different types of rotational axes have different rotations (see for instance [CR17, p. 20]). For the tetrahedron we have

i. Rotating $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ about the axis through a vertex and the centroid of the opposing face.

ii. Rotating π about the axis through the midpoints of opposing edges.

For the cube we have

i. Rotating $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$ about the axis through the centroid of opposing faces.

ii. Rotating π about the axis through the midpoints of opposing edges.

iii. Rotating $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ about the axis through opposing vertices.

For the dodecahedron we have

i. Rotating $\frac{2\pi}{5}$, $\frac{4\pi}{5}$, $\frac{6\pi}{5}$ and $\frac{8\pi}{5}$ about the axis through the centroid of opposing faces.

ii. Rotating π about the axis through the midpoints of opposing edges.

iii. Rotating $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ about the axis through opposing vertices.

The only thing we have to be careful about is the vertex-face rotation of the tetrahedron, as in every other case we know whether $\mathrm{Cl}(q)$ represents the vertices, edges or faces from the real part of $h = \cos(\theta) + q\sin(\theta) \in \Gamma$. In the case of the tetrahedron, because every vertex is opposed by a face, we have that $\mathrm{Cl}(q)$ and $\mathrm{Cl}(-q)$ are different orbits unless q is a normalized edge. In this case every element $h \in \Gamma$ belongs to $\mathrm{Cl}(q)^*$ if and only if it also belongs to $\mathrm{Cl}(-q)^*$.

Now let $\Gamma = T^*_{24}$. From the above discussion we see that the three orbits are the normalized edges $\mathrm{Cl}(i)$, and the normalized vertices and faces $\mathrm{Cl}(\frac{i+j+k}{\sqrt{3}})$ and $\mathrm{Cl}(\frac{-i-j-k}{\sqrt{3}})$ for the tetrahedron. We further see that

$$\mathrm{Cl}(i)^* = \{\pm i, \pm j, \pm k\}$$

and

$$\mathrm{Cl}\left(\frac{i+j+k}{\sqrt{3}}\right)^* = \mathrm{Cl}\left(\frac{-i-j-k}{\sqrt{3}}\right)^* = \left\{\frac{\pm 1 \pm i \pm j \pm k}{2}\right\}.$$

Thus we have $|\mathrm{Cl}(i)^*| = 6$ and $|\mathrm{Cl}\left(\frac{i+j+k}{\sqrt{3}}\right)^*| = |\mathrm{Cl}\left(\frac{-i-j-k}{\sqrt{3}}\right)^*| = 16$. So the orbit of $\mathrm{Cl}(i)^*$ has stabilizer group of order $6/2+1 = 4$, and the orbits $\mathrm{Cl}(\frac{i+j+k}{\sqrt{3}})$ and $\mathrm{Cl}(\frac{-i-j-k}{\sqrt{3}})$ have stabilizer groups of order $16/2+1 = 9$.

Next let $\Gamma = O^*_{48}$. We now have the orbits of normalized faces $\mathrm{Cl}(i)$, normalized vertices $\mathrm{Cl}(\frac{i+j+k}{\sqrt{3}})$ and of normalized edges $\mathrm{Cl}(\frac{i+j}{\sqrt{2}})$ of the cube, with

$$|\mathrm{Cl}(i)^*| = 18$$
$$\left|\mathrm{Cl}\left(\frac{i+j+k}{\sqrt{3}}\right)^*\right| = 16$$
$$\left|\mathrm{Cl}\left(\frac{i+j}{\sqrt{2}}\right)^*\right| = 12$$

So the orbit of $\mathrm{Cl}(i)$ has stabilizer group of order $18/2+1 = 10$, the orbit of $\mathrm{Cl}(\frac{i+j+k}{\sqrt{3}})$ has stabilizer group of order $16/2+1 = 9$, and the orbit of $\mathrm{Cl}(\frac{i+j}{\sqrt{2}})$ has stabilizer group of order $12/2+1 = 7$.

Finally let $\Gamma = I^*_{120}$. We now have the orbits of normalized edges $\mathrm{Cl}(i)$, normalized vertices $\mathrm{Cl}(\frac{i+j+k}{\sqrt{3}})$ and of normalized faces $\mathrm{Cl}\left(\sqrt{\frac{5+\sqrt{5}}{10}}i + \sqrt{\frac{5-\sqrt{5}}{10}}j\right)$

of the dodecahedron, with

$$\begin{aligned} |\mathrm{Cl}(i)^*| &= 30 \\ \left|\mathrm{Cl}\left(\frac{i+j+k}{\sqrt{3}}\right)^*\right| &= 40 \\ \left|\mathrm{Cl}\left(\sqrt{\frac{5+\sqrt{5}}{10}}i + \sqrt{\frac{5-\sqrt{5}}{10}}j\right)^*\right| &= 48 \end{aligned}$$

So the orbit of $\mathrm{Cl}(i)$ has stabilizer group of order $30/2+1 = 16$, the orbit of $\mathrm{Cl}(\frac{i+j+k}{\sqrt{3}})$ has stabilizer group of order $40/2+1 = 21$, and the orbit of $\mathrm{Cl}\left(\sqrt{\frac{5+\sqrt{5}}{10}}i + \sqrt{\frac{5-\sqrt{5}}{10}}j\right)$ has stabilizer group of order $48/2+1 = 25$.

In summary, S^3/T^*_{24} has three singular orbits, one with stabilizer group C_4 and two with stabilizer group C_9. S^3/O^*_{48} has three singular orbits with stabilizer groups C_{10}, C_9 and C_7. S^3/I^*_{120} has three singular orbits with stabilizer groups C_{16}, C_{21} and C_{25}.

CHAPTER 4

Orbifolds

In this chapter we introduce the concept of an orbifold and show in particular that the base space of a fiber oriented Seifert manifold is an orbifold.

An orbifold is a generalization a manifold where we allow charts to be finite quotients of open sets in $\mathbb{R}^n$. The following definition is due to [Thu, ch.13, p.6]

Definition 4.0.1. An orbifold O is a Hausdorff space X_O with a covering of open sets $\{U_i\}$ closed under finite intersections. To each U_i there is an associated finite group Γ_i acting on an open subset $\widetilde{U}_i$ of $\mathbb{R}^n$ and a homeomorphism $\phi_i : U_i \simeq \widetilde{U}_i/\Gamma_i$. Additionaly, for every inclusion $U_i \subseteq U_j$ there should be an injective homomorphism $f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$ and a Γ_i-equivariant smooth embedding $\psi_{ij} : U_i \hookrightarrow U_j$ such that $\phi_j \circ \psi_{ij} = \phi_j$.

The triple (U_i, Γ_i, ϕ_i) is called an *orbifold chart.* We will add the additional restriction that $\Gamma_i \subseteq \mathrm{O}(n)$, like in [Mar16, p.101].

The following result gives us a way of constructing orbifolds as orbit spaces of finitely stabilized G-manifolds.

Proposition 4.0.2. *Let G be a compact Lie-group and let P be a finitely stabilized G-manifold. Then $M = P/G$ is an orbifold.*

Proof. Let P and G be n- and m-dimensional, respectively, and let NG_x be the normal bundle of $Gx \subset P$. Since $Gx \simeq G/G_x$ and G_x is finite, Gx must also be m-dimensional. Hence each fiber of NG_x is k-dimensional for $k = n - m$. Call all points with trivial stabilizer group *regular*, and points with non-trivial stabilizer group *singular*.

By the equivariant tubular neighbourhood theorem we can choose a G-invariant metric on P such that the subset $E(\varepsilon) \subseteq E$ of the total space E of NG_x is equivariantly diffeomorphic to some neighbourhood $Gx(\varepsilon)$ of Gx for sufficiently small $\varepsilon > 0$ with respect to this metric.

Now let $U(\varepsilon)$ be the image of the restriction $E(\varepsilon)|_{N_x}$ by the exponential map, where N_x is the normal space at x. This is a k-submanifold of P, since $E(\varepsilon)|_{N_x}$ is an open ε-ball of the normal space N_x which is a k-vector space, and it is embedded in P by the exponential map. Thus we have a chart $\phi : U(\varepsilon) \leftarrow U \subset \mathbb{R}^k$.

Choose $\varepsilon > 0$ small enough such that $Gx(\varepsilon)$ (and hence also $U(\varepsilon)$) contains only regular points, except possibly the points of Gx. This is possible since we only have finitely many non-trivial stabilizer groups.

Now let V be the image of $U(\varepsilon)$ under the projection π. From the projection

$$\pi : U(\varepsilon) \to V \subset M$$

we note that V is open, since $\pi^{-1}(V) = Gx(\varepsilon)$ is open.

Recall that $U(\varepsilon)$ is G_x-invariant, because if $h \in G_x$ and $v \in N_x$ then $hv \in N_{hx} = N_x$. Thus the quotient map $\theta : U(\varepsilon) \to U(\varepsilon)/G_x$ is well-defined.

Since the induced G-action from P (and hence the induced G_x-action) acts trivially on M (and hence on V) we can factor π through θ:

$$U(\varepsilon) \xrightarrow{\theta} U(\varepsilon)/G_x \xrightarrow{\psi} V, \qquad \pi = \psi \circ \theta$$

The map ψ sends each equivalent class $[y] = \{gy \mid y \in U(\varepsilon), g \in G_x\}$ to the orbit Gy. This is clearly injective, because if $Gy \neq Gz$ then $[y] \neq [z]$ since $[y] \subset Gy$ and $[z] \subset Gz$. ψ is also surjective because π is a projection. The continuity of ψ and its inverse is clear from the topologies of $U(\varepsilon)/G_x$ and V: A subset $\{[y] \in U(\varepsilon)/G_x \mid y \in W\}$ of $U(\varepsilon)/G_x$ is open if and only if W is open in $U(\varepsilon)$. A subset $\{Gy \in V \mid y \in W\}$ is open if and only if the set $\{gy \in P \mid g \in G, y \in W\}$ is open in P. However, W is open in $U(\varepsilon)$ if and only if it is the restriction of an open set in P, showing that a subset of M is open if and only if it is the image of a subset of $U(\varepsilon)/G_x$ under ψ. Hence ψ is a homeomorphism.

The triple $(V, G_x, \phi \circ \psi)$ an oribfold chart, because $U(\varepsilon)$ is homeomorphic to $U \subset \mathbb{R}^k$, which means that $U(\varepsilon)/G_x$ is homeomorphic to U/G_x. Since G_x is distance preserving it is orthogonal, and since it is also finite it is homeomorphic to a finite subgroup of $O(n)$.

By our construction all the transition maps between charts are trivially compatible, since none of the charts containing singular points intersect. We complete the orbifold atlas by adding every finite intersection of every chart.

Finally, from [Lee12, p.543] we have that M is Hausdorff and second countable since G is compact and acts smoothly on P.

∎

As mentioned earlier, the orbit space M/S^1 of a fiber oriented Seifert manifold M is the base space of M. We have now shown that this space has an orbifold structure.

Final remarks

We have shown that we can find the invariants $(p_1, \ldots, p_k)$ and base space S of a fiber oriented Seifert manifold M by considering it as a finitely stabilized S^1-manifold. This allows us to calculate the Euler characteristic χ of M. To calculate the Euler number e of M we need to also know the invariants $q_1, \ldots, q_k$. These invariants are decided by the particular way S^1 acts on M, which means that it should be able to recover them from the projection map to M/S^1 somehow.

By developing a theory of smooth orbifolds, similar to smooth manifolds, we might be able to calculate the Euler number e of M directly by looking at the projection $M \to M/S^1$. We leave others to investigate this perspective further.

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