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The Birch Tate conjecture and Somekawa K-theory

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Contents

CHAPTER 1

Introduction

The Birch-Tate conjecture states that the following formula:

$$\zeta_K(-1) = \frac{|K_2(\mathcal{O}_K)|}{|H^0(K, \mathbb{Q}/\mathbb{Z}(2))|}$$

holds up to a power of 2. Here K is a totally real numberfield, ζ_K is the Dedekind zeta function and $K_2(\mathcal{O}_K)$ is the algebraic K_2-group. The formula was conjectured after Tate's proof of the same formula for function fields, a proof that heavily relied upon ideas of Weil's solution to the Riemann hypothesis for curves. The main idea of Weil was that the essential part of the zeta-function is the determinant of the inverse Forbenius acting on the Jacobian of the curve. Hence began the search for an analogue for the Jacobian in the number field setting initiated by Iwasawa. This was the birth of what we today call Iwasawa-theory, whose main conjecture would imply the Birch-Tate conjecture. The main conjecture of Iwasawa-theory for totally real fields was proved by Andrew Wiles in 1990 for odd primes p, and later in [**rognes**] for $p = 2$.

In a paper in 2008, Takao Yamazaki extended Somekawa K-theory to rings of integers and proved that the more general formula:

$$L(X(T), -1) = \frac{|K^T(\mathcal{O}_K)|}{|H^0(K, X(T) \otimes \mathbb{Q}/\mathbb{Z}(2))|},$$

holds up to a power of 2. Here:

1. L/K is a finite extension of totally real fields.
2. T is a torus over K split by L and is assumed to admit *motivic interpretation*.
3. $X(T)$ is the cocharacter group.
4. $L(X(T), s)$ is the Artin L-function attached to the Artin representation $X(T) \otimes C$.

This thesis is for the most part an exposition of ideas comming together in this formula. We focus especially on the function field case, because it is the easiest one. Our main focus has been has been on the big picture, and relate concepts coming form different areas in algebraic number theory. Our main theorem is that we prove Yamazaki's formula at the prime $p = 2$, i.e. extend Yamazaki's result.

Acknowledgements

I want to thank my supervisor Paul Arne Østvær for giving me such challenging and fun topic for the thesis. It was his idea that Yamazaki's theorem could possibly be extended at the prime 2 and gave me the task to figure it out. I also want to thank Martin Helsø for help with Latex.

CHAPTER 2

Symbols

2.1 The general reciprocity law.

In the beginning of 20th-century, David Hilbert published 23 problems which in his point of view was the biggest unsolved mystries in mathematics. Some of the are now solved, some of them are partly solved and some of then are not solved. In Hilbert's opinion, the 8th problem which is the Riemann hypothesis, was the most important one and it is considered far away from being solved. On the other hand, the 9th problem was concering the quest for finding "the most general reciprocity law", he considered to be "more special". Anyhow, the 9th problem is one of the solved problems and it is the topic of this section.

Quadratic reciprocity

A very interesting activity to do early in ones mathematical career is to think about squares mod p. One need only elementary knowledge to comprehend the stiking phenomena the quadratic reciprocity relation among prime numbers yields. We start by considering two examples., i.e., Consider

$$(\mathbb{Z}/11\mathbb{Z})^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

We then square every element and take mod 11. We get:

$$(\mathbb{Z}/11\mathbb{Z}^*)^2 = \{1, 4, 9, 5, 3, 3, 9, 4, 1\}.$$

We also want to consider:

$$(\mathbb{Z}/13\mathbb{Z})^*)^2 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\},$$

whose square is:

$$((\mathbb{Z}/13\mathbb{Z})^*)^2 = \{1, 4, 6, 3, 12, 10, 10, 12, 3, 9, 4, 1\}.$$

A pattern one spots immediately is the symmetry about the middle. That has a trivial but nice explaination. $\mathbb{Z}/11\mathbb{Z}$ can be written like:

$$\mathbb{Z}/11\mathbb{Z} = \{1, 2, 3, 4, 5, -5, -4, -3, -2, -1\},$$

so when we square all the numbers the minus sign does not matter.

Another pattern that is easy to spot is that the squares appear symmetrically in $\mathbb{Z}/13\mathbb{Z}$, but not in $\mathbb{Z}/11\mathbb{Z}$. This is the case since $13 \equiv 1 \mathrm{mod} 4$, and $11 \equiv 3 \mathrm{mod} 4$.

We will later see that this corresponds to even and odd characters respectivly, a crucial concept in Iwasawa-theory. We will now return to the general case and look at how the patterns in squares mod p is captured formally.

The Frobenius map

The Frobenius map is one of the most important maps in all of number theory. Although it is very classical, it plays a central role in modern mathematics. For instance, "It governs class field theory like a king" [Neu99] and is essential in the Weil conjectures and in Scholze's perfectoid spaces, just to metion a few topics where it occours.

For our purposes now, the Frobenius map is a field automorphism:

$$\begin{aligned} \mathbb{Z}/p\mathbb{Z} &\xrightarrow{F} \mathbb{Z}/p\mathbb{Z} \\ a &\longmapsto a^p \end{aligned}$$

This map is obiously multiplicative:

$$F(ab) = F(a)F(b),$$

but it turns out that by the binomial formula combined that we take mod p, makes this map additive also:

$$F(a+b) = F(a) + F(b).$$

In our case the Frobenius map is the identity by Fermat's little theorem.

Suppose we now ask the question whether a number a is a square mod p. To formulate the question terms of algebra, we make the ring:

$$\frac{\mathbb{Z}/p\mathbb{Z}[x]}{x^2 - a},$$

and ask if this ring splits. We pick a random element $u + \sqrt{a}$ and apply the Frobenius:

$$F(u + \sqrt{a}v) = (u + \sqrt{a}v)^p = u^p + v^p(\sqrt{a})^p = u + vd^{p-1/2}\sqrt{a}.$$

But what is $a^{p-1/2}$? We apply the Frobenius yet again:

$$0 = a^p - a = a(a^{p-1/2} - 1)(a^{p-1/2} + 1).$$

One of the three terms must be zero, and since $(a,p) = 1$ it must be one of the last two. We put $\left(\frac{a}{p}\right) = 1$ if $a^{p-1/2} = 1$, and put $\left(\frac{a}{p}\right) = 1$ if $a^{p-1/2} = -1$.

The conclusion from the above is that the Frobenius map acts as the identity incase a is a square mod p, and it acts as complex conjugation of it not.

Given a prime number p, we can associate to it the so called Legendre symbol:

$$\begin{aligned} \mathbb{Z}/p\mathbb{Z} &\xrightarrow{\left(\frac{\cdot}{p}\right)} \{-1, 1\} \\ a \bmod p &\longmapsto \left(\frac{a}{p}\right) \end{aligned}$$

$\left(\frac{a}{p}\right) = 1$ if a is a square mod p, and $\left(\frac{a}{p}\right) = -1$ if a is not a square mod p. At first this might look strange, but already the next result shows the usefulness of this tool:

Proposition 2.1.1. *$\left(\frac{\cdot}{p}\right)$ is the unique character of order 2, i.e. it is multiplicative.*

Proof. There are only four cases to consider. (1) Let both a and b be squares mod p. Then ab is a square mod p. Thus $1 \cdot 1 = 1$. (2) Let a be square and b is not square mod p. Then ab is not a square mod p. Thus $1 \cdot -1 = -1$. (3) Symmetric of (2). Same argument yields $-1 \cdot 1 =$. (4) Suppose neither a nor b are squares mod p. If we combine the facts that half of the integers mod p are squares and multiplication with an element prime to p yields an automorphism of $\mathbb{Z}/p\mathbb{Z}$, we see that ab must be a square mod p. Thus $-1 \cdot -1 = 1$, and we are done. ■

Remark 2.1.2. To obtain the results above we remark that the assuption that the field is finite is crucial. It yields symmetry and lets us make use of powerfull couting arguments we otherwise would not have.

As we have seen above, the Legendre symbol $\left(\frac{a}{p}\right)$ is easy to describe for varying a. The dual problem, namely describing how it bahaves for varying p is the subject of quadratic reciprocity.

Since the Legendre symbol is multiplicative and we have the fundamental theorem of arithmetic, we have that:

$$\left(\frac{a}{p}\right) = \left(\frac{\pm 1}{p}\right)\left(\frac{2}{p}\right)\left(\frac{q_1}{p}\right)\cdots\left(\frac{q_r}{p}\right).$$

Hence we see that the problem reduces to three special cases: $-1, 2$ and for an arbitrary prime p. We then state the quadratic reciprocity law, which was conjectured by Euler and proven by the 21 year old Gauss.

Theorem 2.1.1. *The Legendre symbol satisfies the following relations:*

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}\left(\frac{q}{p}\right)$$

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}, \quad \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}.$$

A more understandable way to state the first relation would be: Given that p is a square mod q, then q is a square mod p unless both are congruent to 3 mod 4. That means that if we know what p is mod $4a$, then we know what $\left(\frac{a}{p}\right)$ is. There are many proofs of these formulas. They are quite formal and technical, and in my opinion not very conceptual. In addition, it will not be that relevant for the thesis. However, showing the generalization of this theorem using Hilbert symbols is both conceptual and very relevant as we will see later.

Definition 2.1.3. Let p be a prime number. We associate to it the Hilbert symbol $\left(\frac{a,b}{p}\right)$. It is equal to 1 if:

$$ax^2 + by^2 = 1, \text{ has a solution in } \mathbb{Q}_p.$$

Otherwise it is -1.

Remark 2.1.4. It is subtle matter that the Hilbert symbols vanish on the complex places and perhaps it is due to $\mathbb{C}^*$ being connected. The vanishing seems to equivalent to the fact that there are $r_2 + 1$ cyclotomic extensions of a number field or that the rank of the Iwasawa module X is r_2. Anyhow, on $\mathbb{Q}_p$ the Hilbert symbol is a quadratic form, and over $\mathbb{C}$ all quadratic forms of the same dimension are equivalent.

Theorem 2.1.2 (Hilbert)**.**

1. $\left(\frac{a,b}{p}\right)$ *is symmetric and bimultiplicative.*

2. *If* p *is odd and* $(a,p) = 1$, *then* $\left(\frac{a,b}{p}\right) = \left(\frac{a}{p}\right)^{v_p(b)}$.

3. *The product formula:* $\prod_p \left(\frac{a,b}{p}\right) = 1$

2.2 The K_2 functor

A symbol on a field F is a bilinear map:

$$F^* \times F^* \xrightarrow{(\ ,\ )} C$$

satisfying the relation $(a, 1-a)$. A symbol on F factors through $K_2(F)$ by the universal property. We will take this as the definition in favour of the Steinberg group one.

2.3 The norm residue symbol

Section 3.1 Let μ_m be the m-th roots of unity in F_s^* and suppose $(char(F), m) = 1$. Then we have short exact sequence:

$$0 \longrightarrow \mu_m \longrightarrow F_s^* \xrightarrow{m} F_s^* \longrightarrow 0$$

Passing to Galois cohomology yields:

$$F^* \xrightarrow{m} F^* \longrightarrow H^1(F_s, \mu_m) \longrightarrow 0$$

Hence we get an isomorphism:

$$F^*/(F^*)^m \xrightarrow{\simeq} H^1(F_s, \mu_m)$$

Using the cup product Section 3.1 we construct a symbol on F:

$$\begin{tikzcd}
F^* \times F^* \arrow[r, "(\ ,\ )"] \arrow[dr] \arrow[rrr, bend left, "\tilde{h}_1"] & F^*/(F^*)^m \times F^*/(F^*)^m \arrow[r, "\delta_m \times \delta_m"] & H^1(F,\mu_m) \times H^1(F,\mu_m) \arrow[r, "\cup"] & H^2(F,\mu_m^{\otimes 2}) \\
& K_2(F) \arrow[urr, "h_1"] & &
\end{tikzcd}$$

If $\mu_m \subset F$, then:

$$H^2(F,\mu_m^{\otimes 2}) \xrightarrow{=} H^2(F,\mu_m) \otimes \mu_m \xrightarrow{=} Br_m(F) \otimes \mu_m$$

Thus, choosing a $z \in \mu_m$, we have a symbol on F:

$$\begin{aligned} F^* \times F^* &\xrightarrow{\tilde{h}_1} Br_m(F) \otimes \mu_m \\ (a,b) &\longmapsto (a,b)_m \otimes z \end{aligned}$$

Thus we have obained a symbol on F with values in the the Brauer group $Br(F)_m \otimes \mu_m$.

The Hilbert symbol

The above construction is a cohomological reformulation of the classical Hilbert symbol on a number field which we define locally for each place:

$$F_v^*/(F_v^*)^n \times F_v^*/(F_v^*)^n \xrightarrow{\left(\frac{\ ,\ }{v}\right)} \mu_v$$

which is the generalization of the Hilbert symbol defined above on $\mathbb{Q}_p$. By local class field theory and Kummer theory we obtain $F_v^*/(F_v^*)^n \simeq G(K(\sqrt[n]{K^*})/K)$ and $F_v^*/(F_v^*)^n \simeq Hom(G(K(\sqrt[n]{K^*})/K),\mu_v)$ respectively. The pairing is canonical:

$$\begin{aligned} Hom(G(K(\sqrt[n]{K^*})/K),\mu_v) \times G(K(\sqrt[n]{K^*})/K) &\xrightarrow{\left(\frac{\ ,\ }{v}\right)} \mu_v \\ (\chi,\sigma) &\longmapsto \chi(\sigma) \end{aligned}$$

Putting all the local Hilbert symbols together, we get a global one:

$$K_2(F) \xrightarrow{\lambda} \coprod_{v\ n.c.} \mu_v$$

Definition 2.3.1. A finite F-algebra A is called simple if the only bilateral ideals of A are the whole algebra A or the trivial ideal $\{0\}$. An F-algebra is called simple if F is the center of A.

Every central simple F-algebra is isomorphic to the matrix algebra $M_n(D)$ where D is a skew field over F, where n and D is determined by A up to isomorphism.

Definition 2.3.2. We say that two central simple algebras A and A' are isomorphic if their corresponding skew fields, D and D' are the same.

Definition 2.3.3 (The Brauer group)**.** The Brauer group Br(F) is the group consisting of isomorphism classes of central simple algebras and where the tensor product is the group operation.

Remark 2.3.4. Using norm and restriction arguments , on shows that the Brauer group is a torsion group.

After having defined the Brauer group, we now return to our symbol. By the universal property of K_2 we have the norm residue homomorphism:

$$\begin{aligned} K_2(F) &\xrightarrow{h_1} Br_m(F) \otimes \mu_m \\ \{a, b\} &\longmapsto (a, b)_m \otimes z \end{aligned}$$

$(a, b)_m$ is the element in $Br(F)$ represented by the central simple algebra:

$$\frac{F[\alpha, \beta]}{(\alpha^m - a), (\beta^m - b), (\beta\alpha - z\alpha\beta)}$$

Proposition 2.3.5. *Let a, b be elements in F^*. Then the following statements are equivalent:*

$\{a, b\} \in mK_2(F)$.

$h_1(\{a, b\}) = 0$.

$(a, b)_m = 0$.

$\exists \gamma \in F(a^{\frac{1}{m}})$ *such that* $N_{F(a^{\frac{1}{m}})/F}\gamma = b$.

Proof. (1) $\implies$ (2): Let $\{a, b\} \in mK_2(F)$. Then $\{a, b\} = m\{r, s\}$ for some $r, s \in F^*$. Then we have:

$$h_1(\{a, b\} = h_1(m\{r, s\}) = mh_1(\{r, s\} = m(z \otimes (a, b)_m)) = 0.$$

The last equality follows since both groups μ_m and $Br_m(F)$ has order m.
(2) $\implies$ (3): By assuption we have:

$$0 = h_1(\{a, b\}) = z \otimes (a, b)_m.$$

Since z is a fixed root of unity, we must have: $(a, b)_m = 0$. (3) $\implies$ (4): This is a classical result of Milnor [Mil72].
(4) $\implies$ (1): Suppose $\exists \gamma \in F(a^{\frac{1}{m}})$ such that $N_{F(a^{\frac{1}{m}})/F}\gamma = b$. then we once again make use of the transfer maps **??**:

$$\{a, b\} = \{a, N_{F(a^{\frac{1}{m}})/F}(\gamma)\} = N_{F(a^{\frac{1}{m}})/F}\{a, \gamma\} = mN_{F(a^{\frac{1}{m}})/F}\{a^{\frac{1}{m}}, \gamma\} \in mK_2(F)$$

■

CHAPTER 3

Galois cohomology and K_2

3.1 Galois cohomology

Definition of group cohomology

In this section we are following [Neu]. Group cohomology arises from a canonical diagram:

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; G\times G\times G\times G \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; G\times G\times G \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; G\times G \;\substack{\longrightarrow\\\longrightarrow}\; G$$

The arrows are canonical projections:

$$\begin{aligned} G^{n+1} &\xrightarrow{d_i} G^n \\ (\sigma_0,\dots,\sigma_n) &\longmapsto (\sigma_0,\dots,\hat{\sigma_i},\dots,\sigma_n) \end{aligned}$$

Let A be a G-module, in particular an abelian group that comes equipped with a G-action. Let

$$X^n = X^n(G,A) = Hom_{cont}(G^{n+1},A).$$

If $x \in X^n$, then

$$\begin{aligned} G^{n+1} &\xrightarrow{x} A \\ (\sigma_0,\dots,\sigma_n) &\longmapsto x(\sigma_0,\dots,\sigma_n) \end{aligned}$$

X^n gets an induced G-module stucture, i.e. a G-action:

$$\begin{aligned} G^{n+1} &\xrightarrow{\sigma x} A \\ (\sigma_0,\dots,\sigma_n) &\longmapsto \sigma x(\sigma^{-1}\sigma_0,\dots,\sigma^{-1}\sigma_n) \end{aligned}$$

Which we observe is a conjugation of the elements of X^n by G:

$$\begin{tikzcd} G^{n+1} \arrow[r, "\sigma^{-1}"] \arrow[rrr, bend left, "\sigma x"] & G^{n+1} \arrow[r, "x"] & A \arrow[r, "\sigma"] & A \end{tikzcd}$$

The following diagram:

$$\begin{tikzcd}
\dots \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[r, shift right=4] & G \times G \times G \times G \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[d] & G \times G \times G \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & G \times G \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d] & G \arrow[d] \\
\dots \arrow[r, "id"] & A \arrow[r, "id"] & A \arrow[r, "id"] & A \arrow[r, "id"] & A
\end{tikzcd}$$

shows how we obtain the diagram:

$$\begin{tikzcd}
A \arrow[r] & X \arrow[r, shift left=1] \arrow[r, shift right=1] & X^2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X^3 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X^4 \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[r, shift right=4] & \dots
\end{tikzcd}$$

To get a complex we form the differentials as alternating sums:

$$X^{n-1} \xrightarrow{\partial^n} X^n$$
$$x \longmapsto \sum_0^n (-1)^i d_i^* x$$

Proposition 3.1.1. *The sequence*

$$0 \longrightarrow A \xrightarrow{\partial^0} X \xrightarrow{\partial^1} X^2 \xrightarrow{\partial^2} X^3 \xrightarrow{\partial^3} \dots$$

is exact.

Proof. By construction we have $\partial^{n+1} \circ \partial^n = 0$, because the alternating sums cancel. Hence we have $\mathrm{Im}(\partial^n) \subset \mathrm{Ker}(\partial^{n+1}$
To prove exactness, we define the following maps:

$$X^{n+1} \xrightarrow{D^n} X^n$$
$$x(\sigma_0, \dots, \sigma_n) \longmapsto x(1, \sigma_0, \dots, \sigma_{n-1})$$

We combine this map with the differential to two get maps:

$$G^{n+1} \xrightarrow{\partial^n D^{n-1}(x)} A$$
$$x(\sigma_0, \dots, \sigma_n) \longmapsto \sum_{i=0}^n (-1)^i x(1, \sigma_0, \dots, \hat{\sigma}_i, \dots, \sigma_n)$$

and

$$G^{n+1} \xrightarrow{D^n \partial^{n+1}(x)} A$$
$$x(\sigma_0, \dots, \sigma_n) \longmapsto x(\sigma_0, \dots, \sigma_n) + \sum_{i=1}^{n+1} (-1)^i x(1, \sigma_0, \dots, \hat{\sigma}_i, \dots, \sigma_n).$$

Hence

$$\partial^n \circ D^{n-1} + D^n \circ \partial^{n+1} = id.$$

If $x \in \mathrm{Ker}(\partial^{n+1})$, then $x = \mathrm{id}(x) = \partial^n \circ D^{n-1}(y)$ for some $y \in X^n$. Thus exactness is proved. ■

The sequence in the proposition is an injective resolution of A, and is known as the *standard resolution.* The family $(D^n)_n$ is called a *contracting homotopy* of it.
With an injective resolution of the G-module A, we are now in shape to define the associated cohomology groups. Analogously to sheaf cohomology where we take the global sections functor on the injective resloution, we instead take G-invariance. We let:

$$C^n(G, A) := X^n(G, A)^G.$$

We easily see that $\partial \circ \partial = 0$, since the alternating sums still cancel. Hence we get a complex, called the homogeneous cochain complex:

$$C^0(G, A) \xrightarrow{\partial^1} C^1(G, A) \xrightarrow{\partial^2} C^2(G, A) \xrightarrow{\partial^3} C^3(G, A) \xrightarrow{\partial^4} \dots$$

The boundary maps are now

Definition 3.1.2. We define n-cocycles to be:

$$Z^n(G, A) = \mathrm{Ker}(C^n(G, A) \to C^{n+1}(G, A),$$

and n-coboundaries to be:

$$B^n(G, A) = \mathrm{Im}(C^{n-1}(G, A) \to C^n(G, A)).$$

Finally we define the n-th cohomology group to be:

$$H^n(G, A) = Z^n(G, A)/B^n(G, A).$$

For computational purposes one replaces the homogeneous cochain complex above by the inhomogeneous cochain complex which we explain now. We construct isomorphisms:

$$\begin{aligned} C^0(G, A) &\xrightarrow{\simeq} \mathscr{C}^0(G, A) = A \\ x(\sigma) &\longmapsto x(1) \\ C^n(G, A) &\xrightarrow{\simeq} \mathscr{C}^n(G, A) \\ x(\sigma_0, \dots, \sigma_n) &\longmapsto x(1, \sigma_1, \sigma_1\sigma_2, \dots, \sigma_1 \cdots \sigma_n) \end{aligned}$$

We obtain the inhomogeneous cochain complex:

$$\mathscr{C}^0(G, A) \xrightarrow{\partial^1} \mathscr{C}^1(G, A) \xrightarrow{\partial^2} \mathscr{C}^2(G, A) \xrightarrow{\partial^3} \dots$$

The boundary maps are:

$$\begin{aligned} \mathscr{C}^0(G, A) &\xrightarrow{\partial^1} \mathscr{C}^1(G, A) \\ \partial^1 a(\sigma) &\longmapsto \sigma a - a \\ \mathscr{C}^1(G, A) &\xrightarrow{\partial^2} \mathscr{C}^2(G, A) \\ \partial^2 y(\sigma, \tau) &\longmapsto \sigma y(\tau) - y(\sigma\tau) + y(\sigma) \\ \mathscr{C}^2(G, A) &\xrightarrow{\partial^3} \mathscr{C}^3(G, A) \\ \partial^3 y(\sigma, \tau, \lambda) &\longmapsto \sigma y(\tau, \lambda) - y(\sigma\tau, \lambda) + y(\sigma, \tau\lambda) - y(\sigma, \tau) \end{aligned}$$

and so on.

Profinite groups

Given a field K, we can associate a group to it, namely the absolute Galois group $G = Gal(\overline{K}/K)$.

Some cohomological tools

Suppose we have a short exact sequence of topological G-modules:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

This induce a short exact sequence among the inhomogeneous chain complexes just defined:

$$0 \longrightarrow \mathscr{C}^\bullet(G,A) \longrightarrow \mathscr{C}^\bullet(G,B) \longrightarrow \mathscr{C}^\bullet(G,C) \longrightarrow 0$$

By the sanke lemma we obtain a long exact sequence in cohomology:

$$\dots \longrightarrow H^i(G,B) \longrightarrow H^i(G,C) \longrightarrow H^{i+1}(G,A) \longrightarrow H^{i+1}(G,B) \longrightarrow \dots$$

Norm and restriction

We have proved an analoguos result.

Lemma 3.1.3. *Let E/F be a Galois extension of degree n with $G = Gal(E/F)$. Then for any integer i and topological G-module M, we have isomorphisms:*

$$K_2(F) \simeq (K_2(L))^G, \text{ and } H^i(F,M) \simeq (H^i(L,M))^G.$$

Proof.

$$\begin{tikzcd}
K_2(L) \arrow[d, shift right, "N_{L/F}"'] & (K_2(L))^G \arrow[l, hook'] \arrow[dl, dashed, bend left] & H^i(L,M) \arrow[d, shift right, "N_{L/F}"'] & (H^i(L,M))^G \arrow[l, hook'] \arrow[dl, dashed, bend left] \\
K_2(F) \arrow[u, shift right, "R_{L/F}"'] \arrow[ur, dashed, "f"] & & H^i(F,M) \arrow[u, shift right, "R_{L/F}"'] \arrow[ur, dashed, "f"] &
\end{tikzcd}$$

We have that $N_{L/F}(R_{L/F}(x)) = nx$, and $R_{L/F}(N_{L/F}(y)) = \sum_{\sigma \in G} \sigma y$. Thus we see that both the kernel and cokernel of f are killed by n, and since $(n,l) = 1$ we have proved the statements. ■

3.2 Results in group cohomology

Proposition 3.2.1. *Let G be a profinite group and U is an open subgroup. Then for every G-module A such that $\hat{H}^n(U,A) = 0$, we have:*

$$(G:U)\hat{H}^n(U,A) = 0$$

Let H be an arbitrary closed subgroup of G. For every H-module A, we consider the G-module:

$$Ind_G^H(A) = \{x : G \to A |\, x(\tau\sigma) = \tau x(\sigma)\}.$$

If $[G : H]$ is finite, then

$$Ind_G^H(A) = \oplus_{i=1}^n \sigma_i A, \sigma_i \in G/H.$$

When A is a G-module, then we have a canonical isomorphism:

$$\begin{aligned} Ind_G^H(A) &\xrightarrow{\simeq} Map(G/K, A) \\ x(\sigma) &\longmapsto y(\sigma H) = \sigma x(\sigma^{-1}) \end{aligned}$$

Proposition 3.2.2.

$$H^n(G, Ind_G A) = 0, \ for\ all\ n > 0.$$

Proposition 3.2.3 (Shapiro's lemma)**.** *Let H be a closed subgroup of G and A an H-module. Then we have canonical isomorphisms:*

$$H^n(G, Ind_G^H(A)) \simeq H^n(H, A), \ for\ all\ n \geq 0.$$

3.3 Tate theorems

Let $lim_{\leftarrow} \mu_{l^i} = \mathbb{Z}_l(1)$ be the Tate module. We can tensor this module with itself with the following rule:

$$\mathbb{Z}_l(n) \otimes \mathbb{Z}_l(m) = \mathbb{Z}_l(n+m).$$

It is a topological G-module and as a topological group we want it to be finitely generated as a $\mathbb{Z}_l$-module with a linear G-action. Let T be a topological G-module. As a topological group it is finitely generated free $\mathbb{Z}_l$-module with a linear G-action.The next proposition is very useful, and its proof is elementary reasoning in homological algebra.

Proposition 3.3.1. *The group $H^n(G, \mathbb{Z}_l(n))$ contains no non-zero subgroup which is l-divisible.*

$$0 \longrightarrow \mathbb{Z}_l \longrightarrow \mathbb{Q}_l \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow 0$$

Tensoring this sequence by the Tate module $lim_{\leftarrow} \mu_{l^i} = \mathbb{Z}_l(1)$, we get:

$$0 \longrightarrow \mathbb{Z}_l(n) \longrightarrow \mathbb{Q}_l(n) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l(n) \longrightarrow 0$$

We see that $\mathbb{Z}_l(n)$ is an open compact subgroup of the finite dimensional vector space $\mathbb{Q}_l(n)$. Since $\mathbb{Q}_l/\mathbb{Z}_l(1) = lim_{\rightarrow} \mu_{l^i}$ we see that it is a discrete divisible $l-$primary torsion group.

Proposition 3.3.2. *We have an exact sequence:*

$$0 \longrightarrow ker(\delta) \longrightarrow H^{n-1}(G, \mathbb{Q}_l/\mathbb{Z}_l(n)) \xrightarrow{\delta} H^n(G, \mathbb{Z}_l(n)) \longrightarrow coker(\delta) \longrightarrow 0 \ ,$$

in which:

3.3.2.1. *$ker(\delta)$ is the maximal divisible subgroup of $H^{n-1}(G, \mathbb{Q}_l/\mathbb{Z}_l(n))$.*

3.3.2.2. *$im(\delta)$ is the torsion subgroup of $H^n(G, T)$.*

Proof. Since $H^{n-1}(G, \mathbb{Q}_l(n))$ is divisible, its image $= ker(\delta)$ is divisible too. By the previous proposition $H^n(G, \mathbb{Z}_l(n))$ has no nonzero divisible subgroup, so all all divisible subgroups in $H^n(G, \mathbb{Q}_l/\mathbb{Z}_l(n))$ must be in $ker(\delta)$. This settles (1).

We pick an element $x \in H^n(G, \mathbb{Q}_l/\mathbb{Z}_l(2))$ and represent it by a cocycle:

$$G^{n-1} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}(n)$$

Since $\mathbb{Q}_l/\mathbb{Z}_l(n)$ is discrete, we know that the inverse images of the points will be an open cover of G^{n-1}. But G is profinite, hence compact, and therefore we only need the inverse images of finitely many points to cover all of G^{n-1}. This means that f takes only finitely many values. Combined with the fact that $\mathbb{Q}_l/\mathbb{Z}_l(n)$ is a torsion group, it follows that $H^n(G, \mathbb{Z}_l(n))$ is a torsion group. Additionally, $im(\delta)$ is the kernel of a map into a torsion free divisible group $H^n(G, \mathbb{Q}(n))$. Thus all the torsion in $H^n(G, \mathbb{Z}_l(n))$ is in $im(\delta)$. ∎

The universal continuous symbol

When we constructed the norm residue homomorphism, we took Galois cohomology of the exact sequence:

$$0 \longrightarrow \mu_l \longrightarrow F_s^* \xrightarrow{l} F_s^* \longrightarrow 0 \ ,$$

and combined with the cup-product we made a symbol on the form:

$$\begin{tikzcd} F^*/(F^*)^l \times F^*/(F^*)^l \arrow[r] \arrow[d] & H^2(F, \mu_l^{\otimes 2}) \\ K_2(F)/lK_2(F) \arrow[ur, dashed, "h_l"] & \end{tikzcd}$$

We will now modify the situation above to construct a *continuous* symbol on F. We start by making the following machine:

$$\begin{tikzcd} 0 \arrow[r] & \mu_l \arrow[r] & F_s^* \arrow[r, "l"] & F_s^* \arrow[r] & 0 \\ 0 \arrow[r] & \mu_{l^2} \arrow[r] \arrow[u, "l"] & F_s^* \arrow[r, "l^2"] \arrow[u, "l"] & F_s^* \arrow[r] \arrow[u, "\mathbb{1}"] & 0 \\ \vdots & \vdots \arrow[u, "l"] & \vdots \arrow[u, "l"] & \vdots \arrow[u, "\mathbb{1}"] & \vdots \\ 0 \arrow[r] & \mu_{l^i} \arrow[r] \arrow[u, "l"] & F_s^* \arrow[r, "l^i"] \arrow[u, "l"] & F_s^* \arrow[r] \arrow[u, "\mathbb{1}"] & 0 \\ 0 \arrow[r] & \mu_{l^{i+1}} \arrow[r] \arrow[u, "l"] & F_s^* \arrow[r, "l^{i+1}"] \arrow[u, "l"] & F_s^* \arrow[r] \arrow[u, "\mathbb{1}"] & 0 \\ & \vdots \arrow[u, "l"] & \vdots \arrow[u, "l"] & \vdots \arrow[u, "\mathbb{1}"] & \end{tikzcd}$$

Taking the limit of this diagram, we obtain the exact sequence:

$$0 \longrightarrow \mathbb{Z}_l(1) \longrightarrow lim_{\leftarrow} F_s^* \longrightarrow F_s^* \longrightarrow 0$$

We take Galois cohomology and obtain the boundary map:

$$H^0(F, F_s^*) = F^* \xrightarrow{\delta} H^1(F, \mathbb{Z}_l(1))$$

Using the cup-product we make the map:

$$\begin{aligned} F^* \times F^* &\xrightarrow{\delta \times \delta} H^1(F, \mathbb{Z}_l(1)) \times H^1(F, \mathbb{Z}_l(1)) \xrightarrow{\cup} H^2(F, \mathbb{Z}_l(2)) \\ (a, b) &\longmapsto (\delta a, \delta b) \longmapsto \delta a \cup \delta b \end{aligned}$$

We get:

$$F^* \times F^* \xrightarrow{\tilde{h}} H^2(F, \mathbb{Z}_l(2))$$

We want to show that this map factors through $K_2(F)$, i.e. we must show that $\tilde{h}$ is a symbol. The fact that this map is bilinear follows from the definition of the cup-product and that δ is a group homomorphism. The Steinberg relation is more subtle, but turns out to hold:

Theorem 3.3.1. *$\tilde{h}$ induces a unique group homomorphism:*

$$\begin{aligned} K_2(F) &\xrightarrow{h} H^2(F, \mathbb{Z}_l(2)) \\ \{a, b\} &\longmapsto \delta a \cup \delta b \end{aligned}$$

Proof. By Proposition 3.3.1 it is enough to show that all elements on the form $N_{E/F}(\delta a \cup \delta(1-a))$ for $a \in E^*$, where E/F is a finite extension, are l-divisible. We make the polynomial $f(x) = x^l - a$ which splits into irreducibles $\prod_i f_i(x)$ which again linear factors $(x - a_i)$, where a_i's live in the finite extensions $E(a_i)/E$. By the definition of the cohomological transfer we get $N_{E_i/E}(1 - a_i) = f_i(x)$. Evaluating $f(x)$ at 1 yields:

$$1 - a = \prod_i f_i(x) = \prod_i N_{E_i/E}(1 - a_i).$$

We now combine the equation above with nice properties of δ and the cup-product:

$$\delta a \cup \delta(1 - a) = \delta a \cup \delta(\prod_i N_{E_i/E}(1 - a_i)) = \sum_i \delta a \cup \delta(N_{E_i/E}(1 - a_i)).$$

Because of the commutativity of the diagram for any finite extension E/F:

$$\begin{array}{ccc} E^* & \xrightarrow{\delta} & H^1(E, \mathbb{Z}_l(1)) \\ {\scriptstyle N_{E/F}}\downarrow & & \downarrow{\scriptstyle N_{E/F}} \\ F^* & \xrightarrow{\delta} & H^1(F, \mathbb{Z}_l(1)) \end{array},$$

combined with the projection formula for the transfer, we have:

$$\sum_i \delta a \cup N_{E_i/E}\delta((1-a_i)) = \sum_i N_{E_i/E}(\delta R_{E_i/E} a \cup \delta(1-a_i),$$

This becomes:

$$\sum_i N_{E_i/E}(a_i^l \cup \delta(1-a_i) = \sum_i l N_{E_i/E}(\delta a_i \cup \delta(1-a_i).$$

We conclude that any element in $N_{E/F}(\delta a \cup \delta(1-a))$ is l-divisible, hence 0. ■

A fundamental diagram

Suppose $\mu_l \subset F$ and pick a $z \in \mu_l$. We have:

$$\begin{aligned} \mu_l \otimes F^* &\xrightarrow{\gamma} K_2(F) \\ z \otimes a &\longmapsto \{z, a\} \\ \mu_l \otimes F^* &\xrightarrow{i\,\simeq} \mu_l \otimes H^1(F, \mu_l) = H^1(F, \mu_l^{\otimes 2}) \\ z \otimes a &\longmapsto z \cup \delta a \end{aligned}$$

Again i is an isomorphism because of Hilbert 90.

Suppose $\mu_l \not\subseteq F$. We then have to modify the maps above $E = F(\mu_l)$, and make the diagram:

$$\begin{tikzcd} H^1(F, \mu_l^{\otimes 2}) \arrow[d, "R_{E/F}"] & (\mu_l \otimes E^*)^G \arrow[l, "i\,\simeq"'] \arrow[r, "\gamma"] \arrow[d, hook] & K_2(F)[l] \arrow[d, "R_{E/F}"] \\ H^1(E, \mu_l^{\otimes 2}) & \mu_l \otimes E^* \arrow[l, "i_E\,\simeq"] \arrow[r, "\gamma_E"] & K_2(E)[l] \end{tikzcd}$$

We know that i is an isomophism.

A fundamental diagram

All the main players introduced so far fit together in a commutative diagram:

$$\begin{tikzcd} (\mu_l \otimes E^*)^G \arrow[r, "\gamma"] \arrow[d, "i\simeq"] & K_2(F) \arrow[r, "l"] \arrow[d, "h"] & K_2(F) \arrow[r] \arrow[d, "h"] & K_2(F)/lK_2(F) \arrow[d, "h_1\simeq"] \\ H^1(F, \mu_l^{\otimes 2}) \arrow[r, "\delta"] & H^2(F, \mathbb{Z}_l(2)) \arrow[r, "l"] & H^2(F, \mathbb{Z}_l(2)) \arrow[r] & H^2(F, \mu_l^{\otimes 2}) \end{tikzcd} \tag{3.1}$$

The next results follow as a diagram chase of diagram (3.3).

Theorem 3.3.2. *$ker(h) = K_2(F)_{l-div}$ and $coker(h)$ is l-torsion free.*

Corollary 3.3.3. $K_2(F)[l^\infty] \xrightarrow{\simeq} H^2(F, \mathbb{Z}_l(2))_{Tors} \xrightarrow{\simeq} H^1(F, \mathbb{Q}_l/\mathbb{Z}_l(2))/div$

It is a fact that

$$K_2(F) = \oplus_{(l, char(F))=1} K_2(F)[l^\infty],$$

so we have a cohomological description of $K_2(F)$.

Theorem 3.3.3. *Both rows in the diagram are exact.*

Definition of $K_2(\mathcal{O}_{F,S})$

Let S be a finite set of places of F, including the archemedian ones. Let $\mathcal{O}_{F,S}$ be the ring of S-integers in F. We identify it with the kernel of a map induced by tame symbols.

$$0 \longrightarrow K_2(\mathcal{O}_{F,S}) \longrightarrow K_2(F) \xrightarrow{d^S} \coprod_{v \notin S} k(v)^* \longrightarrow 0 \tag{3.2}$$

Remark 3.3.4 (Important). There is a well known theorem of Garland using Riemannian geometry and harmonic forms, showing that $K_2(\mathcal{O}_{F,S})$ is finite. It is therefore a trival remark that $K_2(F)$ is an extension of finite cyclic groups by a finite group. In the light of Corollary 3.3.3 and Proposition 3.3.1 it is a torsion group with no nonzero divisible subgroup.

Results involving $K_2(\mathcal{O}_{F,S})$

Let S be a finite set of places of F, including the archemedian ones and the ones above l in the number field case. Let S_c denote the complex places, and there are r_2 of them.

Theorem 3.3.4. *Let $\mu_l \subset F$. Then there is an exact sequence:*

$$0 \longrightarrow \mu_l \otimes Pic(\mathcal{O}_{F,S}) \longrightarrow K_2(\mathcal{O}_{F,S})/lK_2(\mathcal{O}_{F,S}) \longrightarrow \left(\coprod_{v \in (S-S_c)} \mu_l\right)_0 \longrightarrow 0$$

Proof. For $v \notin S$ there are isomorphisms:

$$k(v)^*/(k(v)^*)^l \simeq (k(v)^*)_l \simeq \mu_l$$

There is a diagram with exact top row:

$$\begin{array}{ccccccccc}
(K_2(F))_l & \xrightarrow{d^S} & \coprod_{v \notin S} \mu_l & \longrightarrow & K_2(\mathcal{O}_{F,S})/lK_2(\mathcal{O}_{F,S}) & \longrightarrow & K_2(F)/lK_2(F) & \longrightarrow & \coprod_{v \notin S} \mu_l \\
\gamma \uparrow & & = \uparrow & & & & h_1 \simeq \downarrow & & pr \uparrow \\
\mu_l \otimes F^* & \longrightarrow & \mu_l \otimes I_S & & & & \mu_l \otimes Br_l(F) & \xrightarrow{\simeq} & \left(\coprod_{v \notin S_c} \mu_l\right)_0
\end{array}$$

This induces the exact sequence:

$$0 \longrightarrow \mu_l \otimes I_S/(\mu_l \otimes F^*) \longrightarrow K_2(\mathcal{O}_{F,S}))/lK_2(\mathcal{O}_{F,S})) \longrightarrow (\coprod_{v \notin S_c} \mu_l)_0 \longrightarrow \prod_{v \notin S} \mu_l \longrightarrow 0$$

After identification on the left hand side with the Picard group and shortening of the sequence on the right we get:

$$0 \longrightarrow \mu_l \otimes Pic(\mathcal{O}_{F,S}) \longrightarrow K_2(\mathcal{O}_{F,S})/lK_2(\mathcal{O}_{F,S}) \longrightarrow \left(\coprod_{v \in (S-S_c)} \mu_l\right)_0 \longrightarrow 0$$

■

Theorem 3.3.5. *The kernel of γ in the diagram in Section 3.3 has order l^{r_2+1}*

Proof. We set S just large enough such that $Pic(\mathcal{O}_{F,S}) = 0$. We have:

$$\begin{tikzcd}
0 \arrow[r] & ker(\gamma_{\mathcal{O}_{F,S}}) \arrow[r] \arrow[d] & ker(\gamma) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mu_l \otimes \mathcal{O}_{F,S}^* \arrow[r] \arrow[d, "\gamma_{\mathcal{O}_{F,S}}"] & \mu_l \otimes F^* \arrow[r] \arrow[d, "\gamma"] & \mu_l \otimes I_S \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & (K_2(\mathcal{O}_{F,S}))_l \arrow[r] \arrow[d] & (K_2(F))_l \arrow[r] \arrow[d] & \coprod_{v \notin S} \mu_l \arrow[r] \arrow[d] & 0 \\
& coker(\gamma_{\mathcal{O}_{F,S}}) \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}$$

From the diagram we see that $ker(\gamma_{\mathcal{O}_{F,S}}) = ker(\gamma)$ and that $ker(ker(\gamma_{\mathcal{O}_{F,S}})$ is surjective. Thus we get:

$$0 \longrightarrow ker(\gamma) \longrightarrow \mu_l \otimes \mathcal{O}_{F,S}^* \longrightarrow (K_2(\mathcal{O}_{F,S}))_l \longrightarrow 0$$

■

Theorem 3.3.6.

$$H^1(F, \mathbb{Z}_l(2)) \simeq \mathbb{Z}_l^{r_2} \times H^0(F, \mathbb{Q}_l/\mathbb{Z}_l(2)) \simeq \mathbb{Z}_l^{r_2} \times \mathbb{Z}/l^m\mathbb{Z}.$$

Proof. The proof of this is also a diagram chase in diagram (3.3). We have isomorphisms:

$$H^1(F, \mathbb{Z}_l(2))/lH^1(F, \mathbb{Z}_l(2)) \simeq ker\delta \simeq ker(\gamma),$$

where the first isomorphism is trivial and the other follows since h is injective by Corollary 3.3.3 and diagram (3.2). The last group has order l^{r_2+1} by the theorem just proved. That takes care of the free part. The torsion part comes from $H^0(F, \mathbb{Q}_l/\mathbb{Z}_l(2))$ by Proposition 3.3.2. ■

Theorem 3.3.7. *There is a commuting diagram:*

$$\begin{tikzcd}
0 \arrow[r] & (\mathbb{Q}/\mathbb{Z})^{r_2} \arrow[r] \arrow[d] & H^1(F, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r] \arrow[d] & K_2(F) \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & (\mathbb{Q}_l/\mathbb{Z}_l)^{r_2} \arrow[r] & H^1(F, \mathbb{Q}_l/\mathbb{Z}_l(2)) \arrow[r, "\delta"] & H^2(F, \mathbb{Z}_l(2)) &
\end{tikzcd}$$

Proof. We can sumarize most results so far in a diagram:

$$\begin{tikzcd}
0 \arrow[r] & ker(g) \arrow[r] \arrow[d] & H^1(F, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r] \arrow[d] & K_2(F)[l^\infty] \arrow[r] \arrow[d, "h \simeq"] & 0 \\
0 \arrow[r] & (\mathbb{Q}_l/\mathbb{Z}_l)^{r_2} \arrow[r] & H^1(F, \mathbb{Q}_l/\mathbb{Z}_l(2)) \arrow[r, "\delta"] & H^2(F, \mathbb{Z}_l(2))_{Tors} \arrow[r] & 0
\end{tikzcd}$$

Because of the fact that the functor $H^i(F, \cdot)$ commutes with direct sums, we have the claim. ■

We arrive at an important result for us.

Corollary 3.3.5. *If F is a function field or a totally real number field, we have an isomorphism:*

$$H^1(F, \mathbb{Q}/\mathbb{Z}_l(2)) \xrightarrow{\simeq} K_2(F)$$

CHAPTER 4

Birch-Tate conjecture for function fields

4.1 Curves over finite fields

A very old question i mathematics is the following: Given a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ in $\mathbb{Z}[x]$, does it exist an integer b such that $f(b) = 0$? A slight generalizion of the problem arise if we consider the polynomial ring

$$\frac{\mathbb{Z}[x_1, \ldots, x_n]}{(f_1 \cdots f_r)}$$

and we ask if there exits a n-tuple of integers $(b_1, \ldots, b_n)$ such that all the f_i's vanish? The questions stated above are several tousands of years old and has turned out to be notoriously difficult to answer. However, introducing ideas form algebraic geometry makes the second question somewhat attackable.
We first assume that one of the f_i's is a prime number p. This simplifies matters considerably since the integers $\mathbb{Z}$ are of course far more mysterious that the field $\mathbb{Z}/p\mathbb{Z}$. We also let another f_j be of the form $x^q - x$, where $q = p^s$. The ring in question now looks like:

$$\frac{\mathbb{F}_q[x_1, \ldots, x_n]}{(f_1 \cdots f_r)},$$

where $\mathbb{F}_q$ is the unique finite field with q elements. We then glue such rings together to form a scheme X, and put assumptions on it to make it more manageble:

Definition 4.1.1 (Assumptions RH)**.**

1. $dim(X) = 1$, i.e. we force the residue fields to be finite. This is because we at this point want to study curves.

2. X is smooth, i.e. $\Omega_{X|Spec(\mathbb{F}_q)}$ is a locally free sheaf.

3. We have an embedding $X \hookrightarrow \mathbb{P}^m$ for some $m >> 0$, i.e. X is projective.

4. X should be geometrically connected. That means that $X(\mathbb{C})$ is connected as a topological space.

Divisors and linbundles

Definition 4.1.2. We define $Pic^0(F)$ to be the kernel of the canonical map:

$$\begin{aligned} Pic(X) &\xrightarrow{div} \mathbb{Z} \\ \mathcal{O}_X(D) &\longmapsto deg(D) \end{aligned}$$

It is a well known fact in algebraic geometry that there for smooth schemes is an equivalense between these for categories [Ott20], p.229:

$$\{Weil\ Divisors\} \longleftrightarrow \{Cartier\ divisors\} \longleftrightarrow \{Invertible\ sheaves\} \longleftrightarrow \{Line\ bundles\}$$

Geometry of the curve

In this section we add the assumption that X is hyperelliptic. This is just a temporarily assumption to make our discussion more explicit. The ring of integers is on the form

$$\frac{\mathbb{Z}/p\mathbb{Z}[x,y]}{f(x,y)}$$

on the affine charts. Picture of a hyperelliptic curve **??**.

$$X(\mathbb{F}_q), X(\mathbb{R}), X(\mathbb{C}).$$

Rational points of the variety

$$\begin{tikzcd} X \arrow[d] & \mathrm{Spec}(\mathbb{F}_q) \arrow[l] \arrow[dl] \\ \mathrm{Spec}(\mathbb{F}_q) & \end{tikzcd}$$

Remark 4.1.3. We remark that determining the existence of a real solution is very easy. One just finds two n-tuples α and β such that $f(\alpha) > 0$ and $f(\beta) < 0$ and then argue by the mean value theorem from Calculus that there must be atleast one zero in between.

The Weil-zeta function

Definition 4.1.4. Let X be a scheme over $\mathrm{Spec}(\mathbb{Z})$. Let X_0 be the set of closed points such that the corresponding residue fields $k(x)$ are finite. We define the zeta-function to be:

$$\zeta_X(s) = \prod_{x \in X_0} \frac{1}{1 - |k(x)|^{-s}}$$

To make calculations simpler it is normal to work with "the other" zeta-function:

$$Z(X,t) = \prod_{x \in X_0} \frac{1}{1 - t^{deg(x)}}$$

Remark 4.1.5. The two functions are equal up to a variable change, $Z(X, q^{-s}) = \zeta_X(s)$.

The next theorem was one of the big motivations for Andre Weil when he introduced zeta functions for schemes of finite type over finite fields and the statement in the theorem is quite spectacular. It is basically saying that the zeta function is determined by the sequence $(X(\mathbb{F}_q), X(\mathbb{F}_{q^2}), X(\mathbb{F}_{q^3}), \ldots)$ and vica versa.

Theorem 4.1.1.

$$Z(X,t) = \exp \sum_{m \geq 0} \frac{|X(\mathbb{F}_{q^m})|}{m} t^m$$

Proof.

$$\sum_{m \geq 1} \frac{|X(\mathbb{F}_{q^m})|}{m} t^n = \sum_{m \geq 1} \frac{\sum_{d|m} a_d d}{m} t^m = \sum_{d \geq 1} a_d \sum_{e \geq 1} \frac{t^{de}}{e} = \sum_{d \geq 1} a_d \log \frac{1}{1-t^d}$$
$$= \log \prod_{d \geq 1} \left(\frac{1}{1-t^d} \right)^{a_d} = \log \prod_{x \in X_0} \frac{1}{1-t^{deg(x)}}.$$

Here, a_d are the number of points with degree d. ■

Remark 4.1.6. Even though the proof is a short explicit computation, we observe that each step is quite nontivial. The proof relies on powerful theorems in Calculus regarding the log function and summing geometric series, but also the Frobenius map really shines. We see that the assumption that X lives over a finite field is essential.

The zeta function of simple schemes

In this section we study the zeta function for some easy schemes. The sequence $(X(\mathbb{F}_q), X(\mathbb{F}_{q^2}), X(\mathbb{F}_{q^3}), \ldots)$ can be determine if the scheme is simple enough, and hence by Theorem 4.1.1 we can determine the zeta function.

$X = \mathbf{Spec}(\mathbb{F}_{q^a})$

We must first determine $|X(\mathbb{F}_{q^r})|$. An element in $X(\mathbb{F}_{q^r})$ is by the Section 4.1 a map $Spec(\mathbb{F}_{q^r}) \to X$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d] & Spec(\mathbb{F}_{q^r}) \arrow[l] \arrow[dl] \\ Spec(\mathbb{F}_q) & \end{tikzcd}$$

By contravariance among rings and affine schemes in algebraic geometry, we have:

$$|X(\mathbb{F}_{q^r})| = \mathrm{Hom}_{\mathrm{Spec}(\mathbb{F}_q)}(\mathrm{Spec}(\mathbb{F}_{q^a}), \mathrm{Spec}(\mathbb{F}_{q^r})) = \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^a}, \mathbb{F}_{q^r}) = a.$$

$$Z(X,t) = \exp\left(\sum_{m \geq 1} \frac{|X(\mathbb{F}_{q^m})|}{m} t^m \right) = \exp\left(\sum_{i \geq 1} \frac{a}{ai} t^{ai} \right) = \exp\left(\sum_{i \geq 1} \frac{(t^a)^i}{i} \right)$$
$$= \exp\left(\log\left(\frac{1}{1-t^a} \right) \right) = \frac{1}{1-t^a}.$$

$\boldsymbol{X = \mathbb{A}^1_{\mathbb{F}_q}}$

Determining $|X(\mathbb{F}_{q^r})|$ is equivalent to determine $|Hom_{\mathbb{F}_q}(\mathbb{F}_q[t], \mathbb{F}_{q^r})|$. The last group has order q^r, because thats the number of different substitutions for the variable t. Thus:

$$\begin{aligned} Z(X,t) &= \exp\left(\sum_{m\geq 1} \frac{|X(\mathbb{F}_{q^m})|}{m} t^m\right) = \exp\left(\sum_{m\geq 1} \frac{q^m}{m} t^m\right) \\ &= \exp\left(\log\left(\frac{1}{1-qt}\right)\right) = \frac{1}{1-qt}. \end{aligned}$$

in the same way we calculate:

$$Z(\mathbb{A}^n_{\mathbb{F}_q}, t) = \frac{1}{1-q^n t}$$

Remark 4.1.7 (Motivic behaviour of the zeta function). For the next calculation we need to exploit a property of the zeta function which follows from Theorem 4.1.1. Since the zeta function is determined by the number of rational points it has over the finite fields $\mathbb{F}_{q^r}$, it is an obivous remark that it must satisfy a Mayer Vietoris property. This means that if X is on the form $X = U \bigcup V$, i.e. decomposable into two disjoint subvarieties U and V, then we have that:

$$Z(X,t) = Z(U,t)Z(Y,t).$$

$\boldsymbol{X = \mathbb{P}^1_{\mathbb{F}_q}}$

We know that $\mathbb{P}^1_{\mathbb{F}_p} = \mathbb{A}^1_{\mathbb{F}_p} \cup \{\infty\}$. Thus:

$$Z(\mathbb{P}^1_{\mathbb{F}_q}, t) = Z(\mathbb{A}^1_{\mathbb{F}_q})Z(\mathrm{Spec}(\mathbb{F}_q), t) = \frac{1}{(1-qt)(1-t)}.$$

Additionally, since $\mathbb{P}^n_{\mathbb{F}_p} = \mathbb{A}^n_{\mathbb{F}_p} \cup \mathbb{P}^{n-1}_{\mathbb{F}_q}$, It follows that:

$$\begin{aligned} Z(\mathbb{P}^n_{\mathbb{F}_q}, t) &= Z(\mathbb{A}^n_{\mathbb{F}_q})Z(\mathbb{P}^{n-1}_{\mathbb{F}_q}, t) \\ &= Z(\mathbb{A}^n_{\mathbb{F}_q})Z(\mathbb{A}^{n-1}_{\mathbb{F}_q}, t)Z(\mathbb{P}^{n-2}_{\mathbb{F}_q}, t) \\ &= \frac{1}{(1-q^n t)(1-q^{n-1}t)\cdots(1-t)}. \end{aligned}$$

We have the next theorem by using the Riemann-Roch theorem.

Theorem 4.1.2 (Rationality of the zeta function)**.** *Let X be a curve satisfying the conditions in Definition 4.1.1. Then the zeta function is rational, i.e.*

$$Z(X,t) \in \mathbb{Q}(t)$$

In particular it is on the form:

$$Z(X,t) = \frac{f(t)}{(1-qt)(1-t)}$$

where f(t) is a polynomial of degree $2g$.

Proof. The proof presented here is based upon [**Sam**].

$$\begin{aligned}
Z(X,t) &= \exp\left(\sum_{m\geq 1}\frac{|X(\mathbb{F}_{q^m})|}{m}t^n\right) = \prod_{x\in X}\frac{1}{1-t^{deg(x)}} = \prod_{x\in X}\sum_{n\geq 0}t^{deg(x)n} \\
&= (1+t^{\deg(x_1)}+t^{2\deg(x_1)}+\cdots)(1+t^{\deg(x_2)}+t^{2\deg(x_2)}+\cdots)\cdots \\
&= \sum_{D\in Div(X)}t^{deg(D)} = \sum_{\mathscr{L}\in Pic(X)}|\mathbb{P}(\Gamma(X,\mathscr{L}))|t^{\deg(\mathscr{L})} \\
&= \sum_{\mathscr{L}\in Pic(X)}\frac{q^{h^0(\mathscr{L})}-1}{q-1}t^{\deg(\mathscr{L})} \\
&= \sum_{0\leq\deg(\mathscr{L})\leq 2g-2}\frac{q^{h^0(\mathscr{L})}-1}{q-1}t^{\deg(\mathscr{L})} + \sum_{2g-1\leq\deg(\mathscr{L})}\frac{q^{h^0(\mathscr{L})}-1}{q-1}t^{\deg(\mathscr{L})}.
\end{aligned}$$

We observe that the first summand is going to be on the satifactory form. We continue to work with the second summand:

$$\begin{aligned}
\sum_{2g-1\leq\deg(\mathscr{L})}\frac{q^{h^0(\mathscr{L})}-1}{q-1}t^{\deg(\mathscr{L})} &= \sum_{2g-1\leq\deg(\mathscr{L})}\frac{q^{\deg\mathscr{L}+1-g}-1}{q-1}t^{\deg(\mathscr{L})} \\
&= |Pic^0(X)|\sum_{2g-1\leq n}\frac{q^{n+1-g}-1}{q-1}t^n \\
&= \frac{|Pic^0(X)|}{q-1}\left(q^{1-g}\sum_{2g-1\leq n}(qt)^n - \sum_{2g-1\leq n}t^n\right) \\
&= \frac{|Pic^0(X)|}{q-1}\left(q^{1-g}\frac{(qt)^{2g-1}}{1-qt} - \frac{t^{2g-1}}{1-t}\right) \\
&= \frac{f(t)}{(1-qt)(1-t)}.
\end{aligned}$$

We see that $f(t)$ has degree $2g$ and we are done ∎

An observation to make in the proof is that the zeta function is essentially a power series where we are summing over the degree of line bundles. There is a symmetry among line bundles over a X within the degree intervall 0 to $2g-2$ which is expressed in famous theorem of Serre:

Theorem 4.1.3 (Serre duality)**.** *Let X be a smooth projective variety of dimension n and let D be a Cartier divisor on X. Suppose ω is the canonical divisor on X. Then:*

$$\dim(H^i(X,\mathcal{O}_X(D))) \cong \dim(H^{n-i}(X,\mathcal{O}_X(\omega-D))).$$

Serre duality enable us to express $Z(X,\frac{1}{qt})$ in terms of $Z(X,t)$ giving rise to a functional equation for the zeta function:

Theorem 4.1.4 (The functional equation of the zeta function)**.** *Given the assuptions of* **??***, we have:*

$$Z(X,\frac{1}{qt}) = q^{1-g}t^{2-2g}Z(X,t)$$

Proof. See **??** for the details. ■

Remark 4.1.8. We make a pause just to gather the information in the two last theorems and make a non-trivial remark. We see that the zeta function is on the following form:

$$Z(X,t) = \frac{f(t)}{(1-t)(1-qt)} = \frac{1}{(1-t)(1-qt)} \prod_{0 \le i \le 2g} (1-\omega_i t),$$

where $f(t)$ is a polynomial of degree $2g$ and the ω_i's are its complex zero's. By the functional equation we have:

$$\frac{f(\frac{1}{qt})}{(1-\frac{1}{qt})(1-\frac{1}{t})} = \frac{t^{2-2g}q^{1-g}f(t)}{(1-t)(1-qt)}.$$

Thus $t^{2g}q^g f(\frac{1}{qt}) = f(t)$, or more explicitly:

$$\prod_{1 \le i \le 2g} (t-\omega_i) = q^{-g} \prod_{1 \le i \le 2g} (1-\omega_i t).$$

Remark 4.1.9. There are two canonical permutations on the set of roots of the zeta function. The first one arises because of the following. Both sides of the equation above have obiously the same vanishing points: The left hand side and the right hand side have $(\frac{\omega_i}{q})_i$ and $(\frac{1}{\omega_j})_j$ as roots respectivly. Thus the map $\omega_j \mapsto q/\omega_j = \omega_i$ will be a permutation of the roots. We therefore have:

$$\omega_i \omega_{\sigma(i)} = q,$$

for some $\sigma \in S_{2g}$. The other permutation is complex conjugation. Since the zeta function is a rational polynomial, the roots comes in complex conjugate pairs. A natural question to ask is if these two permutations are the same. Since the norm is invariant under a galois action, the question is whether or not the second permutation changes the norm or not. The answer is that that the two permutations coincide because of the following deep fact:

Theorem 4.1.5 (The Riemann hypothesis for curves)**.** *Given a curve X satisfying* **??**, *then*

$$|\omega_i| = \frac{1}{2}$$

To set up the machinery to prove this theorem will take us to far astray into intersection theory that will not be relevant for the rest of the thesis. We therefore once again refer to **??** for a proof.

4.2 Using etale cohomology

So far we have discussed three properties of the zeta function for a curve over a finite field: Rationality, functional equation and the Riemann hypthesis. These properties where conjectured by Emil Artin in his PhD-thesis, where he was studiying the relationship between function fields and number fields. Helmut Hasse proved them in the case of elliptic curves, and later Andre Weil proved them for curves of any genus [Wei49]. A very beautiful idea of Andre Weil was to count the rational points of the curve by letting the Frobenius automorphism act on the Jacobian variety. Before we get into these ideas we review quicly etale cohomology and some of its features.

Etale cohomology

In the end of the paper "Numbers of solutions of equations in finite fields" [Wei49], some conjectures where presented stating that rationality, functional equation and Riemann hypothesis where true for a variety of any dimension over a finite field. These became known as "the Weil conjectures". Many mathematicians tried to extend Weil's proof by letting the Frobenius act on some cohomology group. It then became apperent that the current algebraic geometry and its cohomology theories where not satisfactory developed. The search began for a "Weil cohomology theory", i.e. a cohomology theory that would be suited for proving the Weil conjectures. Grothendieck extracted axioms that had to be fullfilled by such a theory, like Künneth-formula, Poincare duality and Lefschetz trace formula. The latter is the essense in the idea of Weil of counting points and we are going to use it later.

Lefschetz trace formula

Theorem 4.2.1 (Lefschetz fixed-point formula)**.** *Let X be a smooth projective variety of dimesion n over an algebraically closed field. Let $X \xrightarrow{\phi} X$ be a regular map. Then we have the following formula for the intersection product:*

$$\Gamma_\phi \cdot \Delta = \sum_{i=0}^{2n} (-1)^i Tr(\phi | H^i_{et}(\overline{X}, \mathbb{Q}_l).$$

Proof. [**milne**] ∎

Let now X be a smooth projective variety of dimension n. A consequence of the Lefschets trace formula is that to count the number of rational points of X over $\mathbb{F}_q$, we take some sort of Euler characteristic of X. The reason for this is that when we count points on a variety, we do it the following way:
Suppose for simplicity that $X = \bigcup_{i=1}^m U_i$ is a finite cover of X by affine charts in the Zariski topology. Now, we first count all the points in this union. But then we have counted the intersections twice, so we must subract this. Then we need to add the tripple intersections and so on. To summarize:

$$\begin{aligned} |X(\mathbb{F}_q)| = \bigcup_{i=1}^{m} |U_i(\mathbb{F}_q)| - \bigcup_{i=1} |U_i \cap U_j(\mathbb{F}_q)| + \bigcup_{i=1} |U_i \cap U_j \cap U_k(\mathbb{F}_q)| - \ldots \\ + (-1)^{n-1} |U_1 \cap \ldots \cap U_n(\mathbb{F}_q) \end{aligned}$$

Since the U_i's are open in the Zariski topology, the dimension drops by one from one summand to the other. The relation to the Euler characteristic is clear.

Etale cohomology

The cohomology groups appearing in Theorem 4.2.1 are the *etale cohomology groups*, and here we give a very brief discussion of them. They are constructed analogous to the ordinary sheaf cohomology groups in algebraic geometry, using cech cohomology. The big difference is that instead of using the Zariski topology, one uses the *etale topology.* This topology is a *Grothendieck topology*, which means that the topology is defined from what the open covers $\bigcup_i U_i$ are of a space X. In the Zariski topology, all the U_i's are open immesions of X, but in the etale topology we say that there should be an *etale map*:

$$U_i \xrightarrow{etale} X \ ,$$

which means unramified and smooth.
The basic idea is that there are many more open sets in the etale topology. In the Zariski topology, all open sets $U \hookrightarrow X$ are dense. In the etale topology on the other hand, the opens $U \xrightarrow{etale} X$ can have dimension less than X, and intuition from the Euclidean topology is useful. The consequence of more open sets is that the cohomology theory captures much more information regarding algebraic cycles on X, and hence makes it suitable for the study of algebraic varieties.
The by far most used cohomology theory in this thesis is Galois cohomology. Therefore, what will be of most importance with respect to the etale cohomology, is its relation to Galois cohomology.

Relation to Galois cohomology

In this section we let F be any global field and we will state some results regarding relations between the etale and Galois cohomology with respect to F and its ring of integers $\mathcal{O}_F$. We start by making the identification [Kol02]:

$$H^1(F, \mu_{l^m}^{\otimes n}) \xrightarrow{=} H^1_{et}(F, \mu_{l^m}^{\otimes n})$$

A very useful tool in etale cohomology available is the Soule's exact localization sequence:

Proposition 4.2.1. *[Sou79]*

$$0 \longrightarrow H^1_{et}(\mathcal{O}_F[\tfrac{1}{l}], \mu_{l^m}^{\otimes n}) \longrightarrow H^1_{et}(F, \mu_{l^m}^{\otimes n}) \longrightarrow \oplus_v H^0_{et}(k(v), \mu_{l^m}^{\otimes n-1}) \longrightarrow H^2_{et}(\mathcal{O}_F[\tfrac{1}{l}], \mu_{l^m}^{\otimes n}) \longrightarrow$$

where v runs over the finite places of v.

Corollary 4.2.2. *If $n \neq 1$, then we have an isomorphism:*

$$H^1(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(n)) \simeq H^1(F, \mathbb{Z}_l(n))$$

Proof. The isomorphism follows after passing to the projective limit as above and noting the vanishing of $H^0_{et}(k(v), \mathbb{Z}_l(n-1))$. ■

To explain the leftmost group in the sequence, we pick as usual an l prime to $char(F)$ and let Ω^l_F denote the maximal algebraic extension of F, that is unramified outside l and infinite primes. We let $G^l_F = Gal(\Omega^l_F/F)$.

$$H^*(G^l_F, \mu_{l^m}^{\otimes n}) \xrightarrow{\simeq} H^*_{et}(\mathcal{O}_F[\tfrac{1}{l}], \mu_{l^m}^{\otimes n})$$

The group on the right hand side is etale cohomology of the scheme $spec(\mathcal{O}_F[\frac{1}{l}])$ with values in the etale sheaf $\mu_{l^m}^{\otimes n}$. This group is roughly explained above. Another rough description with closer analogy to Galois cohomology **??** is the following: let

$$\mu_{l^m}^{\otimes n} \longrightarrow \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \mathcal{I}^2 \xrightarrow{d^2} \mathcal{I}^3 \xrightarrow{d^3} \dots$$

be an injective resolution of the sheaf $\mu_{l^m}^{\otimes n}$, which always exists [Voe05]. Pick an open $U \subset spec(\mathcal{O}_F[\frac{1}{l}])$ and evaluate the exact sequence of sheaves on it. We obtain a complex:

$$\mathcal{I}^0(U) \xrightarrow{d^0} \mathcal{I}^1(U) \xrightarrow{d^1} \mathcal{I}^2(U) \xrightarrow{d^2} \mathcal{I}^3(U) \xrightarrow{d^3} \dots$$

analogous to the complex in **??**. Taking $ker(d^n)/im(d^{n-1})$ yields the n-th cohomology group $H^n_{et}(U, \mu_{l^m}^{\otimes n})$.

We then denote:

$$H^*_{et}(\mathcal{O}[\frac{1}{l}], \mathbb{Z}_l(n)) = lim_{\leftarrow} H^*_{et}(\mathcal{O}[\frac{1}{l}], \mu_{l^m}^{\otimes n}),$$

and

$$H^*_{et}(\mathcal{O}[\frac{1}{l}], \mathbb{Q}_l/\mathbb{Z}_l(n)) = lim_{\rightarrow} H^*_{et}(\mathcal{O}[\frac{1}{l}], \mu_{l^m}^{\otimes n}).$$

We then know from **??** that:

$$H^1_{et}(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(n))_{Tors} \simeq H^1_{et}(\mathcal{O}_F[\frac{1}{l}], \mathbb{Q}/\mathbb{Z}_l(n))/div,$$

where mod div means taking modulo the maximal divisble subgroup.
For the sheaf $\mathbb{G}_m$, the first cohomology groups are known [Mil]:

$$H^0_{et}(\mathcal{O}[\frac{1}{l}], \mathbb{G}_m) \simeq \mathbb{G}_m(\mathcal{O}_F[\frac{1}{l}]) = \mathcal{O}_F[\frac{1}{l}]^*$$

$$H^1_{et}(\mathcal{O}[\frac{1}{l}], \mathbb{G}_m) \simeq Cl(\mathcal{O}_F[\frac{1}{l}])$$

$$H^2_{et}(\mathcal{O}[\frac{1}{l}], \mathbb{G}_m) \simeq Br(\mathcal{O}_F[\frac{1}{l}]).$$

Finally, we need to discuss the relation between cohomology of the curve X and the localized ring of intergers $\mathcal{O}_F[\frac{1}{l}]$ of the function field F above. We have a diagram:

$$\begin{tikzcd} \mathcal{O}_F[\frac{1}{l}] \arrow[r, hook] & F \\ (\mathbb{F}[t])_l \arrow[r, hook] \arrow[u, hook] & \mathbb{F}(t) \arrow[u, hook] \end{tikzcd}$$

Here, $\mathcal{O}_F[\frac{1}{l}]$ sits as the integral closure of $\mathbb{F}[t]$ in F. The presence of infinite primes in $\mathcal{O}_F[\frac{1}{l}]$ are not shared by X. Fortunately, the localization sequence enables us to calculate the difference:

$$0 \longrightarrow H^1_{et}(X, \mathbb{Q}_l/\mathbb{Z}(n)_l) \longrightarrow H^1(\mathcal{O}_F[\tfrac{1}{l}], \mathbb{Q}_l/\mathbb{Z}_l(n))$$

$$\longrightarrow \oplus_{v|\infty} H^0(F_v, \mathbb{Q}_l/\mathbb{Z}_l(n-1)) \longrightarrow H^2_{et}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \longrightarrow 0$$

From [Kol02] p.203 we have

$$H^0(F_v, \mathbb{Q}_l/\mathbb{Z}_l(n-1)) = N(v)^{n-1} - 1.$$

Since the cohomology groups we deal with here are finite, we have isomorphisms Section 4.2

$$H^i_{et}(\mathcal{O}_F[\frac{1}{l}], \mathbb{Q}_l/\mathbb{Z}_l(n)) \simeq H^{i+1}_{et}(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(n)). \tag{4.1}$$

We use this isomorphism to prove:

$$H^2_{et}(\mathcal{O}_F[\frac{1}{l}], \mathbb{Q}_l/\mathbb{Z}_l(n)) \simeq H^3(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(n)) = 0, \tag{4.2}$$

where the last equality follows since X is a curve and by the localization sequence this cohomology group is an extension of the cohomology group of X by torsion.

Etale cohomology agrees with ordinary cohomology

[Mil] Let X be a smooth complete curve of genus g, then

$$H^1(X(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}^{2g},$$

but since $\pi(X)$ is profinite, its discrete quotients are finite. Hence :

$$H^1_{et}(X, \mathbb{Z}) = Hom_{crossed}(\pi(X), \mathbb{Z}) = 0.$$

However, with finite coefficients the story is exactly what we where looking for:

$$H^1(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g} \simeq H^1_{et}(X, \mathbb{Z}/n\mathbb{Z}).$$

Weil's theorem in cohomological language

We need an elemtrary result from linear algebra:

Lemma 4.2.3. *Suppose $\varphi : V \to V$ is an endomorhism of a finite dimensional vector space. The characteristic polynomial of φ is by definition:*

$$P_\varphi = det(\mathbb{1} - t\varphi|V).$$

Then we have the equality:

$$\log\left(\frac{1}{P_\varphi(t)}\right) = \sum_{m=1}^{\infty} Tr(\varphi^m|V)\frac{t^m}{m}$$

We know return to the zeta function for function fields and attack it with the machinery of etale cohomology just introduced:

Theorem 4.2.2. *Let X be a smooth (nonsingular **??**) projective variety of dimension n. Then the zeta function has the following form:*

$$Z(X,t) = \frac{P_1(t)\cdots P_{2n-1}}{P_0(t)\cdots P_{2n}(t)},$$

where the P_i's are on the form:

$$P_i(t) = det(\mathbb{1} - tF|H^i_{et}(\overline{X}, \mathbb{Q}_l))$$

Proof.

$$
\begin{aligned}
Z(X,t) &= \exp\left(\sum_{m\geq 1}\frac{|X(\mathbb{F}_{q^m})|}{m}t^n\right)\\
&= \exp\left(\sum_{m\geq 1}\frac{\sum_{i=0}^{i=2n}(-1)^i Tr(F|H^i_{et}(\overline{X},\mathbb{Q}_l))}{m}t^n\right)\\
&= \prod_{i=0}^{2n}\left(\exp\left(\sum_{m\geq 1}^{2n}\frac{Tr(F^m|H^i_{et}(\overline{X},\mathbb{Q}_l)t^m}{m}\right)\right)^{(-1)^i}\\
&= \prod_{i=0}^{2n}\left(\frac{det(\mathbb{1}-tF|H^1_{et}(\overline{X},\mathbb{Q}_l))\cdots det(\mathbb{1}-tF|H^{2n-1}_{et}(\overline{X},\mathbb{Q}_l))}{det(\mathbb{1}-tF|H^0_{et}(\overline{X},\mathbb{Q}_l))\cdots det(\mathbb{1}-tF|H^{2n}_{et}(\overline{X},\mathbb{Q}_l))}\right)\\
&= Z(X,t) = \frac{P_1(t)\cdots P_{2n-1}}{P_0(t)\cdots P_{2n}(t)}.
\end{aligned}
$$

■

Remark 4.2.4. The cohomology groups are the one's desribed above

The case of curves

Let X be are usual curve over $\mathbb{F}_q$ satisfying **??**. The theorem above specializes to :

$$
\begin{aligned}
\zeta_X(s) &= \frac{det(1-q^{-s}F|H^1_{et}(\overline{X},\mathbb{Q}_l))}{det(1-q^{-s}F|H^0_{et}(\overline{X},\mathbb{Q}_l))det(1-q^{-s}F|H^2_{et}(\overline{X},\mathbb{Q}_l))}\\
\zeta_X(1-n) &= \frac{det(1-q^{n-1}F|H^1_{et}(\overline{X},\mathbb{Q}_l))}{det(1-q^{n-1}F|H^0_{et}(\overline{X},\mathbb{Q}_l))det(1-q^{n-1}F|H^2_{et}(\overline{X},\mathbb{Q}_l))}
\end{aligned}
$$

Thus in our case:

$$\zeta_X(-1) = \frac{det(1-qF|H^1_{et}(\overline{X},\mathbb{Q}_l))}{det(1-qF|H^0_{et}(\overline{X},\mathbb{Q}_l))det(1-qF|H^2_{et}(\overline{X},\mathbb{Q}_l))}$$

One prime at the time

We have so far obtained two expressions (Remark 4.1.8) for the zeta function attached to X. We therefore equate the two:

$$\frac{det(1-q^{n-1}F|H^1_{et}(\overline{X},\mathbb{Q}_l))}{det(1-q^{n-1}F|H^0_{et}(\overline{X},\mathbb{Q}_l))det(1-q^{n-1}F|H^2_{et}(\overline{X},\mathbb{Q}_l))} = \frac{\prod_{0\leq i\leq 2g}(1-\omega_i q^{n-1})}{(1-q^{n-1})(1-q^n)}$$

We "secretly" know that the cohomology group in the numerator on the left hand side is the Jacobian variety for $\overline{X}$, and we want to show it. We therefore do the trick one always does with the Jacobian: we look at its l-primary part. We now state some results that are basically consequences of the relation between Galois and etale cohomology. First we extend the result **??**:

$$H^1_{et}(X,\mathbb{Q}_l/\mathbb{Z}_l(n)) \simeq H^1_{et}(\overline{X},\mathbb{Q}_l/\mathbb{Z}_l(n))^\Gamma = ker(\mathbb{1}-q^{n-1}F|H^1_{et}(\overline{X},\mathbb{Q}_l/\mathbb{Z}_l(n)).$$

There is therefore a reformulation of Section 4.2 where we only look at the prime l:

$$\zeta_X(1-n) \sim_l \frac{|H^1_{et}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))|}{|H^0_{et}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))||H^2_{et}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))|}. \tag{4.3}$$

The Birch-Tate conjecture for function fields

We want to show the Birch-Tate conjecture for the function field F, so the first thing we must do is to define the zeta function attached to it.

Definition 4.2.5.

$$\zeta_F(s) = \prod_{\mathfrak{p}\ finite} \frac{1}{(1 - N(\mathfrak{p})^{-s})}.$$

If we add the contribution of the infinite primes of F, we get the equality:

$$\zeta_X(1-n) = \zeta_F(1-n) \prod_{\mathfrak{p}|\infty} \frac{1}{(1 - N(\mathfrak{p})^{n-1})}.$$

This is analagous to how the completion of Riemann- and Dedekind zeta function concern adding the contribution from the infinite places. We have the theorem:

Theorem 4.2.3.

$$\zeta_F(1-n) \sim_l \frac{|H^2_{et}(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(n))|}{|H^0(F, \mathbb{Q}_l/\mathbb{Z}_l(n))}$$

Proof. The proof is a straight forward computation using several results gathered so far. By Equation (4.3):

$$\begin{aligned}
\zeta_F(1-n) = \zeta_X(1-n) \prod_{\mathfrak{p}|\infty}(1 - N(\mathfrak{p})^{n-1}) &\sim_l \frac{|H^1_{et}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))|}{|H^0_{et}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))||H^2_{et}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))|}(1 - N(\mathfrak{p})^{n-1}) \\
&= \frac{(1 - N(\mathfrak{p})^{n-1})^{-1}|H^1_{et}(\mathcal{O}_F[\frac{1}{l}], \mathbb{Q}_l/\mathbb{Z}_l(n))|}{|H^0_{et}(\mathcal{O}_F[\frac{1}{l}], \mathbb{Q}_l/\mathbb{Z}_l(n))||H^2_{et}(\mathcal{O}_F[\frac{1}{l}], \mathbb{Q}_l/\mathbb{Z}_l(n))|}(1 - N(\mathfrak{p})^{n-1}) \ \ Proposition\ 4.2.1 \\
&= \frac{|H^1_{et}(\mathcal{O}_F[\frac{1}{l}], \mathbb{Q}_l/\mathbb{Z}_l(n))|}{|H^0_{et}(F, \mathbb{Q}_l/\mathbb{Z}_l(n))|} = \frac{|H^2_{et}(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}_l(n))|}{|H^0(F, \mathbb{Q}_l/\mathbb{Z}_l(n))} \ \ Equation\ (4.1), Corollary\ 4.2.2, Equation
\end{aligned}$$

■

Relation to K-theory

In this section we are going to relate the result to K-theory. This could have been done by making use of the Quillen-Lichtenbaum conjecture:

Theorem 4.2.4. *The etale Chern characters induces an isomorphism for $i = 1, 2$:*

$$K_{2n-i}(\mathcal{O}_F) \otimes \mathbb{Z}_l \xrightarrow{\simeq} H^i_{et}(\mathcal{O}_F[\tfrac{1}{s}], \mathbb{Z}_l(n))$$

This isomorphism is a consequence of the famous Bloch-Kato conjecture. However, we do not need to make use of such extremely powerful results that are far beyond the scope of this thesis, because we are primarily interested in K_2. We will now show that the Quillen-Lichtenbaum conjecture is true for K_2, using tools already introduced.
We first make a definition.

Definition 4.2.6.

$$H^1_{et}(\mathcal{O}_F, \mathbb{Q}/\mathbb{Z}(2)) = \prod_l H^1_{et}(\mathcal{O}_F, \mathbb{Q}_l/\mathbb{Z}_l(2)),$$

$$H^2_{et}(\mathcal{O}_F, \mathbb{Z}(2)) = \prod_l H^1_{et}(\mathcal{O}_F, \mathbb{Z}_l(2)),$$

where l runs though all prime numbers. Let S_l be the set of infinite places of F and all places above l. We set:

$$K_2(\mathcal{O}_F) = \prod_l K_2(\mathcal{O}_{F,S_l}).$$

From the last theorem the next result is plain:

Corollary 4.2.7.

$$\zeta_F(1-n) = \frac{|H^2_{et}(\mathcal{O}_F, \mathbb{Z}(n))|}{|H^0(F, \mathbb{Q}/\mathbb{Z}(n))}$$

Theorem 4.2.5 (The Birch-Tate conjecture/theorem (Function fields))**.**

$$\zeta_F(-1) = \frac{|K_2(\mathcal{O}_F)|}{|H^0(F, \mathbb{Q}/\mathbb{Z}(2))}|$$

Proof. Let S as above and recall the exact sequence from Equation (3.2):

$$0 \longrightarrow K_2(\mathcal{O}_{F,S}) = K_2(\mathcal{O}_F) \longrightarrow K_2(F) \xrightarrow{d^S} \coprod_{v \notin S} k(v)^* \longrightarrow 0$$

On the other hand we have the localization sequence:

$$0 \longrightarrow H^1_{et}(\mathcal{O}_F, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow H^1_{et}(F, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \oplus_v H^0_{et}(k(v), \mathbb{Q}/\mathbb{Z}(1)) \longrightarrow 0$$

Making use of the identification of etale and Galois cohomology and a theorem of Tate Corollary 3.3.5, we obtain a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_2(\mathcal{O}_F) \arrow[r] \arrow[d] & K_2(F) \arrow[r] \arrow[d, "\simeq"] & \coprod_{v \notin S} k(v)^* \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & H^1_{et}(\mathcal{O}_F, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r] & H^1_{et}(F, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r] & \oplus_v H^0_{et}(k(v), \mathbb{Q}/\mathbb{Z}(1)) \arrow[r] & 0
\end{tikzcd}$$

Hence the leftmost map is an isomorphism which gives us:

$$K_2(\mathcal{O}_F) \xrightarrow{\simeq} H^2(\mathcal{O}_F, \mathbb{Z}(2))$$

By Corollary 4.2.7 we are done. ■

the l-primary part of the Jacobian

To state the next result we need set up some elementrary Iwasawa theory. Let F be the function field to X and suppose it contains μ_p. We let $F_\infty = F(\mu_{p^\infty})$ and let $X_\infty = Gal(M/F_\infty)$, where M is the maximal abelian extension. Since H^1 in Galois cohomology can be interpreted as crossed homomorphism and that $\Gamma = Gal(F_\infty/F) = \mathbb{Z}_p$ acts trivially on X_∞ gives us the isomorphism:

$$H^1_{et}(\overline{X}, \mathbb{Q}_l/\mathbb{Z}_l(n)) \simeq Hom(X_\infty, \mathbb{Q}_l/\mathbb{Z}_l(n)).$$

We will introduce Iwaswa theory later and use the fact that F is a function field to prove the following:

$$Hom(X_\infty, \mathbb{Q}_l/\mathbb{Z}_l(n)) \simeq J_l(n-1).$$

Since we have:

$$|(J_l(n-1))^\Gamma| = |ker(\mathbb{1} - q^{n-1}F|J_l)|,$$

we draw the conclusion that:

Corollary 4.2.8.

$$\zeta_X(1-n) \sim_l \frac{|ker(\mathbb{1} - q^{n-1}F|J_l)|}{|H^0_{et}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))||H^2_{et}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))|}$$

We do as above Theorem 4.2.3 and pass to the number field analog setting. Taking the product over all l, gives us the correct value of the zeta function. What we will need from all this is:

Corollary 4.2.9.

$$\zeta_F(-1) = \frac{(J(1))^G}{|H^0_{et}(F, \mathbb{Q}_l/\mathbb{Z}_l(2))|}$$

Remark on $H^0(F, \mathbb{Q}/\mathbb{Z}(2))$

$W_2(F) := H^0(F, \mathbb{Q}/\mathbb{Z}(2))$.

The largest integer m such that $Gal(\overline{F}/F)$ acts trivially on $\mu_m \times \mu_m$.

$$H^0(F, \mathbb{Q}/\mathbb{Z}(2)) = (\mathbb{Q}/\mathbb{Z}(2))^G.$$

$\mathbb{Q}_l/\mathbb{Z}_l = \mu_l \to \mu_{l^2} \to \mu_{l^3} \to \ldots = lim_{\to}\mu_{l^i}$

$$H^0(F, \mathbb{Q}_l/\mathbb{Z}_l(2)) = (\mathbb{Q}_l/\mathbb{Z}_l(2))^G = (lim_{\to}\mu_{l^i} \otimes lim_{\to}\mu_{l^i})^G.$$

$|H^0(F, \mathbb{Q}_l/\mathbb{Z}_l(2))|$ is the largest integer m such that $F(\mu_m)$ is contained in a composite of quadratic extensions of F.

$$W_2(F) = \mu(F)^2 \prod_{[E:F]=2} \left(\frac{\mu(E)}{\mu(F)}\right).$$

4.3 Yet another point of view

Let us recall the situation for the convinience of the reader. Let X a smooth projective geometrically connected curve over $\mathbb{F}_q$. We associate to $\overline{X} = \overline{\mathbb{F}}_q \otimes X$ its Jacobian variety, a g-dimesional variety that has the structure of an algebraic variety as well as an abelian group. The l-primary part of the Jacobian J_l has a simple group theoretical structure:

$$J_l = (\mathbb{Q}_l/\mathbb{Z}_l)^{2g}.$$

We recall the geometric Frobenius, whose action on an affine chart of X in our hyperelliptic case looks like:

$$\frac{\mathbb{F}_q[x,y]}{f(x,y)} \xrightarrow{F_g} \frac{\mathbb{F}_q[x,y]}{f(x,y)}$$
$$(x,y) \longmapsto (x^q, y^q)$$

The arithmetic Frobenius F is the one relavant for us and it is the inverse of the geometric one. The graph of this map Γ_F defines an algebraic correspondence on X because the Frobeinus automorphism is finite of degree q. This can be summarized in a diagram:

$$\begin{tikzcd} & \Gamma_F \subset X \times X \arrow[dl, "p_F"'] \arrow[dr, "p_{id}"] & \\ X & & X \arrow[ll, dashed, "p_F \circ p_{id}^{-1} \in Cor_{\mathbb{F}_q}(X,X)"] \end{tikzcd}$$

Finite correspondences between varieties are the arrows in Voveodsky's category DM. An algebraic correspondence of a curve induces an automorphism of its Jacobian, thus also on its l-primary part:

$$(\mathbb{Q}_l/\mathbb{Z}_l)^{2g} \xrightarrow{M(F)} (\mathbb{Q}_l/\mathbb{Z}_l)^{2g} \ ,$$

where $M(F)$ is the associated matrix acting on the $\mathbb{Q}_l/\mathbb{Z}_l$-vector space. By the discussion above Corollary 4.2.8, we have that the characteristic polynomial $char(M(F))$ can be identified with the numerator in Corollary 4.2.8. Before we finish the function field case we combine the discussion of $M(F)$, Corollary 4.2.8 and Section 4.1 to state:

Corollary 4.3.1.

$$char(M(F))(1-n) = \zeta_X(1-n) \cdot \zeta_{\mathbb{P}^1}$$

4.4 The Birch-Tate conjecture: Tate's proof

Unramified means surjective transfer

Theorem 4.4.1. *Given a tower of fields* $K \hookrightarrow L \hookrightarrow M$ *. Then one has:*

$$\Delta_{M/K} = \Delta_{L/K}^{[M:L]} N_{L/K} \Delta_{M/L}.$$

Corollary 4.4.1. *If the extension* L/K *is unramified, then the norm map* $N_{L/K}$ *is surjective.*

Proof. Let $a \in K$ be any element. The corresponding ideal (a) corresponds to finitely genereated $\mathcal{O}_K$-module which admit a $\mathbb{Z}$-basis:

$$(a) = \mathbb{Z}\alpha_1 + \ldots + \mathbb{Z}\alpha_n.$$

To such a basis we may associate a discriminant

$$d(\alpha_1, \ldots, \alpha_n) = det((\sigma_i \alpha_j))^2.$$

We can make a tower like the above such that

$$\Delta_{M/K} = d(\alpha_1, \ldots, \alpha_n).$$

Since the extension L/K is unramified, $\Delta_{L/K} = 0$. From the theorem above we have:

$$\Delta_{M/K} = N_{L/K} \Delta_{M/L}.$$

Hence we see that the norm map $N_{L/K}$ is surjective. ∎

4.5 Birch-Tate: Function field case

We are now going to attack this theorem yet again, but without the heavy machinery of etale cohomology. The proof we here present is based upon the one in [Tat]: The first ever published proof of the theorem. This proof is entirely different in nature, because we do all primes at once. Not just one at the time as we did in Section 4.2. The set up for this proof is basically the same as it was when we proved the Weil conjectures for curves over finite field, but we repeat it here for the convinience of the reader.
Suppose X is a curve over a finite field $\mathbb{F}_q$ and let F be its function field. For simplicity we put some mild restriction on the genus such that the function field is on the following simple form on affine charts:

$$\frac{\mathbb{F}_q(x, y)}{f(x, y)}.$$

We want to pass to the algebraic closure $\overline{X}$ of X, which we obtain by a cartesian diagram:

$$\begin{array}{ccc} \overline{F} & \longrightarrow & F \\ \downarrow & & \downarrow \\ Spec(\overline{\mathbb{F}}_q) & \longrightarrow & Spec(\mathbb{F}_q) \end{array}$$

We see that the Galois group $G(\overline{F}/F)$ is just $G(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. The latter group is easily calculated to be $\hat{\mathbb{Z}}$. We then see that we can interpret $\overline{\mathbb{F}}_q^*$ as the group of roots of unity μ of $\overline{\mathbb{F}}_q$. We then recall the class group sequence for a number field K introduced:

$$0 \longrightarrow \mathcal{O}^* \longrightarrow K^* \longrightarrow J_K \longrightarrow Cl_K \longrightarrow 0 \ .$$

This sequence has a direct translation to the function field case for our field $\overline{F}$:

$$0 \longrightarrow \mu \longrightarrow \overline{F} \longrightarrow D \longrightarrow Cl_{\overline{F}} \longrightarrow 0 \ ,$$

where we us the letter D for the divisor group since we want to use the J for something else.
The next step is crucial and it is yet another brilliant idea of Tate. We tensor the sequence above with the roots of unity μ, to obtain the exact sequence:

$$0 \longrightarrow Tor(\mu, Cl_{\overline{F}}) \longrightarrow \mu \otimes \overline{F} \longrightarrow \mu \otimes D_{\overline{F}} \longrightarrow \mu \longrightarrow 0 \ .$$

The exactness above is justified by the following lemma:

Lemma 4.5.1. *$J_{\overline{F}}$ is divisible.*

Proof. Let $[\mathfrak{a}]$ be a degree 0 class in the class group. Since $\overline{F}$ contains all roots of unity, we can move the ideal in the tower of field extensions degree n upwards for any n. Because $(n,p) = 1$, the extension will be unramified and thus the norm map is surjective Corollary 4.4.1. This means that there is a class $[\mathfrak{b}]$ of degree 0 such that $[\mathfrak{a}] = N(R([\mathfrak{b}]))$, where N and R are the norm and restriction respectivly. As we know that composing the two in this order is equivalent as to multiply by n **??**, we are done. ■

Corollary 4.5.2. $\mu \otimes Cl_{\overline{F}} = \mu$

Proof. We have by **??** an exact sequence:

$$0 \longrightarrow J_{\overline{F}} \longrightarrow Cl_{\overline{F}} \longrightarrow \mathbb{Z} \longrightarrow 0 \ .$$

Tensoring by μ and using the previous lemma, we get the isomorphism:

$$\mu \otimes Cl_{\overline{F}} \xrightarrow{\simeq} \mu$$

■

Now is a good time to recall the exact sequence for $K_2(\overline{F})$ for :

$$0 \longrightarrow K_2(\mathcal{O}_{\overline{F}}) \longrightarrow K_2(\overline{F}) \xrightarrow{\lambda} \coprod_{v \in Pl_{n.c.}(\overline{F})} \mu_v \longrightarrow coker(\lambda_{\overline{F}}) \longrightarrow 0$$

Here is a nice fact:

Theorem 4.5.1. *There is a commuting diagram:*

$$\begin{tikzcd}
0 \arrow[r] & Tor(\mu, Cl_{\overline{F}}) \arrow[r] \arrow[d, "\simeq"] & \mu \otimes \overline{F} \arrow[r] \arrow[d, "\simeq"] & \mu \otimes D_{\overline{F}} \arrow[r] \arrow[d, "\simeq"] & \mu \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & K_2(\mathcal{O}_{\overline{F}}) \arrow[r] & K_2(\overline{F}) \arrow[r, "\lambda"] & \coprod_{v \in Pl_{n.c.}(\overline{F})} \mu_v \arrow[r] & coker(\lambda_{\overline{F}}) \arrow[r] & 0
\end{tikzcd}$$

in which the vertical arrows are isomorphisms.

Theorem 4.5.2. *The following diagram commutes and the vertical arrows are isomorphisms:*

$$\begin{tikzcd}
0 \arrow[r] & (Tor(\mu, Cl_{\overline{F}}))^G \arrow[r] \arrow[d, "\simeq"] & (\mu \otimes \overline{F})^G \arrow[r] \arrow[d, "\simeq"] & (\mu \otimes D_{\overline{F}})^G \arrow[r] \arrow[d, "\simeq"] & (\mu)^G \arrow[d, "\simeq"] \\
0 \arrow[r] & (K_2(\mathcal{O}_{\overline{F}}))^G \arrow[r] \arrow[d, "\simeq"] & (K_2(\overline{F}))^G \arrow[r, "\lambda"] \arrow[d, "\simeq"] & (\coprod_{v \in Pl_{n.c.}(\overline{F})} \mu_v)^G \arrow[r] \arrow[d, "\simeq"] & (coker(\lambda_{\overline{F}}))^G \arrow[d, "\simeq"] \\
0 \arrow[r] & K_2(\mathcal{O}_F) \arrow[r] & K_2(F) \arrow[r] & \oplus_{v \in Pl_{n.c}(F)} \mu_v \arrow[r] & coker(\lambda)
\end{tikzcd}$$

Thus we have:

$$K_2(\mathcal{O}_F) \simeq Tor(\mu, Cl_{\overline{F}})^G \simeq Tor(\mu, J_{\overline{F}})^G = J(1)^G$$

. Combined with Corollary 4.2.9:

$$\zeta_X(-1) = \frac{(J(1))^G}{|H^0_{et}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))||H^2_{et}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))|}.$$

Thus:

Corollary 4.5.3.

$$\zeta_F(-1) = \frac{|K_2(\mathcal{O}_F)|}{|H^0(F, \mathbb{Q}_l/\mathbb{Z}_l(2))|}$$

4.6 A short note on Artin L-functions

Before we present Somekawa K-theory, we give a short note on Artin L-functions. They capture the essence of class field theory and were begining of the Langlands program. The theory behind the Artin L-function is vast, and we only need a small fraction of it to be able to proove our main theorem. Therefore we will be brief and informal.

Dirichlet L-functions

One of the most important functions in all of mathematics is the Riemann zeta function:

$$\zeta(s) = \sum_n \frac{1}{n^s}$$

which has been an object of study by leading mathematicians in all generations ever since Euler introduced it in the 18th century. A generalization of this function is the Dirichlet L-function:

$$L(\chi, s) = \sum_n \frac{\chi(n)}{n^s},$$

where χ is a Dirichlet character:

$$(\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\chi} \mathbb{C}^* \ ,$$

Example 4.6.1. If $\chi = 1$, then:

$$L(\chi, s) = \zeta(s),$$

so we see that it is indeed a generalization of the Riemann zeta function. Another example is the Legandre symbol $\left(\frac{-}{p}\right)$ we discussed. Since they are multiplicative, they have the right to be called Dirichlet characters.

The main attribute about these functions is their *motivic behaviour*, by which we mean the following: Given a character group G, the L-function (the "molecule") splits into its "atoms":

$$L(G, s) = \prod_\chi L(\chi, s).$$

Suppose we have a cyclotomic extension $\mathbb{Q}(\mu_p)/\mathbb{Q}$. As a topological space, this is just p points on the unit circle in the comlpex plane equipped with a structure map down to $spec(\mathbb{Q})$. All these points exept for the point 1 will be non-trivial Dirichlet characters and will have Dirichlet L-funtions attached to them. The character 1 corresponds to the Riemann zeta function as in the example above.Hence informally, we have a character group. We have a splitting:

$$L(\mathbb{Q}(\mu_p)) = \zeta(s) \prod_\chi L(\chi, s).$$

As a corollary of this very splitting one proves the famous theorem of Dirichlet on primes in arithmetic progressions.

Dedekind and Hecke

One could write the Riemann zeta function in a visually nicer way for an algebraic number theorist:

$$\zeta_{\mathbb{Q}}(s) = \sum_{\mathfrak{a} \in J_{\mathbb{Q}}} \frac{1}{n^s},$$

where we are summing ideals. It is then clear that to any number field one can associate its Dedekind zeta function:

$$\zeta_K = \sum_{\mathfrak{a} \in J_K} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})}.$$

We can now do the same as above and attach characters to obtain the Hecke L-function:

$$L(K, \chi, s) = \sum_{\mathfrak{a} \in J_K} \frac{\chi(\mathfrak{a})}{N_{K/\mathbb{Q}}(\mathfrak{a})},$$

where the characters are the Hecke characters:

$$J^{\mathfrak{f}}/P^{\mathfrak{f}} \xrightarrow{\chi} \mathbb{C}^*$$

These characters do more or less exactly the same for a number field K as the Dirchlet characters do for $\mathbb{Q}$: They associate to each element in the ideal group a value on the unit circle in the complex plane. But the a big difference is that a Hecke character breaks up into a finite and an infinite part, as a number field consists of finite and infinite places. The letter $\mathfrak{f}$ is the conductor of the character, $J^{\mathfrak{f}}$ means that we look at ideals prime to $\mathfrak{f}$. The anology to the Dirichlet character is that we look at all places in $\mathbb{Z}$ prime to p:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[rr, bend right] & (\mathbb{Z}/p\mathbb{Z})^* \arrow[r, "\chi"] & \mathbb{C}^* \end{tikzcd}$$

It is well known that the zeta- and L-functions introduce so far admit a meromorphic continuation to the complex plane, and have a functional equation. For Hecke L-series this has been proven analytically by Hecke himself, and Tate reproved it in his thesis using ideas of his supervisor Emil Artin.

Artin L-series

Attached to a Galois extension we have the Artin L-series:

$$L(L/K, \chi, s) = \prod_{\mathfrak{p} \in J_K} \frac{1}{det(1 - \chi(\varphi_{\mathfrak{P}})N(\mathfrak{p})^{(-s)}|V^{I_{\mathfrak{P}}}}$$

4.7 Milnor K-theory

Recall the definition of the Milnor ring of a field.

Definition 4.7.1. Let F be any field. Consider the tensor algebra $T_n(F) = F \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} F$. We make a relation among the symbols:

$$\{a_1, \ldots, a_n | a_i + a_j = 1\}$$

called the Steinberg relation. The Milnor n-th Milnor K-group is defined to be the tensor algebra above modded out by the Steinberg relation, i.e.:

$$K_n^M(F) = \frac{F \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} F}{(\{a_1, \ldots, a_n\} | a_i + a_j = 1)}$$

Some fundamental results

Lemma 4.7.2. *Let F be a field. The following relations are immediate in the Milnor ring:*

4.7.2.1. $\{a, -a\} = 0.$

4.7.2.2. $\{a, b\} + \{b, a\} = 0.$

4.7.2.3. $\{a, a\} = \{-1, a\}.$

Proof. Let $a \in F^*$. The assuption that F is a field yields the relation $-a = \frac{1-a}{1-a^{-1}}$. Hence:

$$\{a, -a\} = \{a, \frac{1-a}{1-a^{-1}}\} = \{a, 1-a\} - \{a, 1-a^{-1}\} = \{a^{-1}, 1-a^{-1}\} = 0.$$

This proves (1), and by using it we show (2)and (3):

$$0 = \{ab, -ab\} = \{a, -a\} + \{a, b\} + \{b, a\} + \{b, -b\} = \{a, b\} + \{b, a\}.$$

$$\{a, a\} = \{-1, a\} + \{-a, a\} = \{-1, a\}.$$

■

Proposition 4.7.3. *Let $F^* = (F^*)^m$. Then $K_n(F)$ is m-divisible*

Proof. Let $\{a_1, \ldots, a_n\}$ be a symbol in $K_n^M(F)$. Pick one of the a_i's. By assumption there exists an α_i such that $a_i = \alpha_i^m$. The artihmetic in the Milnor ring yields:

$$\{a_1, \ldots, a_i, \ldots, a_n\} = \{a_1, \ldots, \alpha_i^m, \ldots, a_n\} = m\{a_1, \ldots, \alpha_i, \ldots, a_n\} \in K_n^M(F).$$

■

Corollary 4.7.4. *If F is algbraically closed, then $K_n(F)$ is uniquely divisible.*

Proposition 4.7.5. *Let F be a finite field. Then $K_n(F) = 0$ for all $n \geq 2$.*

Proof. It suffices to prove the statement for $K_2^M(F)$. Let $\{a, b\}$ be a symbol and let θ be a generator for F. Then $\{a, b\} = \{\theta^l, \theta^k\} = lk\{\theta, \theta\}$, hence it suffices to show that $\{\theta, \theta\} = 0$. It follows from 4.7.2 (2) that $2\{\theta, \theta\} = 0$. If char(F)=2, then F^* has order $2^m - 1$. Thus

$$0 = (2^m - 1)\{\theta, \theta\} = -\{\theta, \theta\} = \{\theta, \theta\}.$$

If $char(F) \neq 2$, then there are exactly $(p^m - 1)/2$ squares and $(p^m - 1)/2$ non-squares. The map $\alpha \to 1 - \alpha$ cannot take all non-squares into squares because 1 is not in its image. Hence there must exist som odd k and l such that $\theta^k = 1 - \theta^l$. Hence for som $r \in \mathbb{N}$ we have:

$$0 = \{\theta^k, \theta^l\} = lk\{\theta, \theta\} = (2r + 1)\{\theta, \theta\} = \{\theta, \theta\}.$$

This finishes the proof. ■

Fundamental maps between the Milnor K-groups

For any field extension L/F, the canonical base change gives us a homomorphism:

$$K_n^M(F) \xrightarrow{j_{L/F}} K_n^M(L)$$

We now define "the boundary" homomorphism, also known as the local symbol. It is very technical but it plays a fundamental role in theory. We fix some notation. - F; discrete valuation field with v being its valuation.
- $\mathcal{O}_v$; the ring of integers.
- U_v; the group of units.
- $\bar{F}_v$; the residue field.
- Let $\bar{\alpha} \in F_v$ be the image of $\alpha \in \mathcal{O}_v$.

Skal skrive ut detaljer her

$$\begin{tikzcd} F^* \times \ldots \times F^* \arrow[r, "\partial"] \arrow[d] & K_{n-1}^M(\bar{F}_v) \\ K_n^M(F) \arrow[ur, dashed] & \end{tikzcd}$$

In degree 2, it is the tame symbol:

$$F^* \times F^* \to K_1^M(\bar{F}_v) \simeq \mu_n$$

The two maps just defined have nice properties when appering thogether. We have the fundamental

Theorem 4.7.1 (Bass-Tate)**.** *There is a split exact sequence:*

$$0 \longrightarrow K_n^M(F) \xrightarrow{j_{F(X)/F}} K_n^M(F(X)) \xrightarrow{\oplus\partial} \oplus_{v \neq v_\infty} K_{n-1}^M(\overline{F(X)}_v) \longrightarrow 0$$

where v runs over all non-arcimedean places of $F(X)$.

Transfer maps, so called "other way" maps, show up in many parts of mathematics and play a prominent role for example in the theory of motives. Among the Milnor K-groups they are called norm maps, beacuse in degree 1 they coincide with the classical norm maps defined in algebraic number theory.

Recall that thm 1.8 above states that a certain sequence is exact when we let v range over all finite places. Let now v range over all places.

$$0 \longrightarrow K_n(F) \xrightarrow{j_{E/F}} K_n(E) \xrightarrow{\oplus\partial_v} \oplus_v K_{n-1}(F(v)) \longrightarrow 0$$

Note that this sequence is exact everywhere except at the last term. We have that $\partial_{v_\infty} j_{E/F} K_n(F) = 0$.
For every place v we introduce the norm map:

$$K_n(F(v)) \xrightarrow{N_v} K_n(F)$$

, where N_{v_∞} is the identity. The norm maps should extend the sequence above and make it exact:

$$0 \longrightarrow K_n(F) \xrightarrow{j_{E/F}} K_n(E) \xrightarrow{\oplus\partial_v} \oplus_v K_{n-1}(F(v)) \xrightarrow{\oplus N_v} K_{n-1}(F) \longrightarrow 0$$

Another way to introduce the norm map is to state the following theorem:

Theorem 4.7.2. *There exists a unique family of natural homomorphisms*

$$K_n^M(k') \xrightarrow{N_{k'/k}} K_n^M(k)$$

associated to with finite field extensions k'/k such that $N_{k/k} = id$ and such that the reciprocity formula holds: Let k(t) be the field of rational functions in one variable over a field k. Then, for all $x \in K_*^M(k(t)$, the Weil reciprocity formula holds:

$$\sum_v N_{k(v)/k}(\partial_v(x)) = 0$$

, where v rages over all discrete valuations of $k(t)$ that are trivial on k.

In degree 0, the norm map $N_{k'/k}$ is multiplication by the degree $[k' : k]$. The formula in the theorem states that for every $f \in k(t)^*$:

$$\sum_v [k(v) : k] v(f) = 0.$$

Since $k[t]$ is a unique factorization domain, we have

$$f = lead(f) \prod_{v \neq v_\infty} \pi_v^{v(f)}.$$

Thus: the formula is statisfied:

$$\sum_v [k(v) : k] v(f) = \sum_{v \neq v_\infty} deg(\pi_v) v(f) + [k(v_\infty) : k] v_\infty(f) = deg(f) - deg(f) = 0.$$

In degree 1, we have the classical norm map from algebraic number theory. Let $x \in k'$. Multiplication by x gives us the diagram:

$$\begin{tikzcd} k' \arrow[r, "x"] & k' \\ k \arrow[u, hook] \arrow[r, "x|k"] & k \arrow[u, hook] \end{tikzcd}$$

Since k'/k is finite, we can observe what multiplication by x does to the basis of k' as a k-vector space. Hence we can associate a matrix to the top horisontal map in the diagram. The norm is by definition the determinant of this matrix. In this situation we have seen that ∂_v is the tame symbol. Thus the statement in the teorem becomes equivalent to the Weil reciprocity formula:

$$\prod_v N_{k(v)/k}((f,g)_v) = 1.$$

Proposition 4.7.6. *Composition*
$$\begin{tikzcd} K_n^M(F) \arrow[r, "j"] \arrow[rr, bend right, "{|[\overline{F(X)}_v : F]|}"'] & K_n^M(\overline{F(X)}_v) \arrow[r, "N_v"] & K_n^M(F) \end{tikzcd}$$

Theorem 4.7.3 (Bass-Tate-Kato)**.** *Let L/F be a finite extension. Then there exists a unique norm map:*

$$K_*^M(L) \xrightarrow{N_{L/F}} K_*^M(F)$$

with the following properties:

1. $N_{L/K} = N_{\alpha_1,\dots,\alpha_n}$*, for* $L = F(\alpha_1,\dots,\alpha_n)$.
2. *It is functorial. Given M/F in L/F we have:*
$$N_{L/F} = N_{M/F} \circ N_{L/M}.$$
3. *In degree 0 it is multiplication by the degree, and in degree 1 it is the classical norm map from algebraic number theory.*
4. $N_{L/F} \circ j_{L/F} = |[L:F]|$.
5. *Let L_1/F be a Galois extension containing L/F. Then*
$$j_{L_1/F} \circ N_{L/F} = m \sum \sigma_i$$
where $m = insep[L:F]$ and
$$K_*^M(L) \xrightarrow{\sigma_i} K_*^M(L_1)$$
are induced by the embeddings of L into L_1.
6. *If σ is an automorphism of L over F, then $N_{L/F} \circ \sigma = N_{L/F}$, where $K_*^M(L) \xrightarrow{\sigma} K_*^M(L)$ is induced by σ.*

[Mer] Let n be the number of roots of unity in k. Recall the Hilbert symbol of degree n which defines a surjective homomorphism:

$$K_2(k) \xrightarrow{\omega_n} \mu_n$$

We collect some results form [Tate1].

Theorem 4.7.4.

1. $ker(\omega_n) = nK_2(k)$.
2. $nK_2(F)$ *is a divisible group.*
3. $nK_2(F)$ *has no* l*-torsion for any prime integer* l *different from the residue characteristic.*

Corollary 4.7.7. ω_n *is a split homomorphism and*

$$K_2(F) = \{cyclic\, group\, of\, order\, n\} \oplus nK_2(F)$$

Tate conjectured in [Tate1] that $nK_2(F)$ has no torsion. This is now a theorem by Merkurjev:

Theorem 4.7.5. $nK_2(F)$ *is uniquely divisible.*

Corollary 4.7.8. *Let* L/F *be a finite abelian extension of degree* d*. Then the norm map:*

$$K_2^M(L) \xrightarrow{N_{L/F}} K_2^M(F)$$

is surjective.

Proof. We have

$$K_2(F) = N_{L/F}(j_{L/F}K_2^M(F)) = dK_2^M(F)$$

and

$$dK_2^M(F) \simeq mK_2^M(F) \oplus \mu_n^d \simeq mK_2^M(F) \oplus \mu_n \simeq K_2^M(F)$$

∎

Proposition 4.7.9 (Hilbert 90 for K_2)**.** *Let* L/F *be a cyclic extension of degree* $l \neq char(F)$*,* σ *be a generator for* $Gal(L/F)$ *and* $K_2^M(L) \to K_2^M(L)$ *be the map induced by* σ*. Then the following sequence is exact:*

$$K_2^M(L) \xrightarrow{1-\sigma} K_2^M(L) \xrightarrow{N_{L/F}} K_2^M(F)$$

4.8 Generalized Jacobians

In this chapter, we let X be a projective, irreducible, and non-singular algebraic curve and let G be an abelian algebraic group. Suppose $X \xrightarrow{f} G$ is a rational map, and let S be the finite set where f is not regular. If D is a divisor prime to S, we have:

$$D = \sum n_i p_i \mapsto \sum n_i f(p_i)$$

Theorem 4.8.1. *If* G *is an abelian variety, then:*

1. $S = 0$*, i.e a rational map into an abelian variety is regular.*
2. *If* $D = (\phi)$ *is a principal divisor, then* $f(D)$*=0.*
3. $f(D)$ *depends only on the class of* D *in the class group.*

If G is just an abelian algebraic group, we need to modify the notion of class.

Definition 4.8.1. A modulus $\mathfrak{m}$ with support on S is an effective divisor:

$$\mathfrak{m} = \sum_{p_i \in S} n_i p_i$$

We say that a rational function ϕ is congruent to 1 mod ϕ, if $v_{p_i}(1-\phi) \geq 0$ for all $p_i \in S$.

Theorem 4.8.2. *For every rational map* $X \xrightarrow{f} G$ *regular away from* S*, there exists a modulus* $\mathfrak{m}$ *with support* S *such that* $f(D) = 0$ *for every divisor* $D = (\phi)$ *with* $\phi \equiv 1 \, mod \, \mathfrak{m}$*.*

A converse result holds.

Theorem 4.8.3. *For every modulus* $\mathfrak{m}$*, there exists a commutative algebraic group* $J_{\mathfrak{m}}$ *and a rational map* $X \xrightarrow{f_{\mathfrak{m}}} J_{\mathfrak{m}}$ *, with the following universal property: Given* $X \xrightarrow{f} G$ *satisfying the assumptions in the previous theorem with respect to* $\mathfrak{m}$*, we have the commuting diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f_{\mathfrak{m}}"'] & G \\ & J_{\mathfrak{m}} \arrow[u, dashed, "\exists ! \theta"'] \end{tikzcd}$$

Theorem 4.8.4. *There is a short exact sequence:*

$$0 \longrightarrow R_{\mathfrak{m}}/\mathbb{G}_m \longrightarrow J_{\mathfrak{m}} \longrightarrow J \longrightarrow 0$$

, i.e. $J_{\mathfrak{m}}$ *is an extension of* J *by* $R_{\mathfrak{m}}/\mathbb{G}_m$*.*

4.9 Local symbol

Recall the set up of a rational map $X - S \xrightarrow{f} G$ with modulus supported on S, and $g \in k(X)^*$ is any rational function.

Definition 4.9.1. We say that $\mathfrak{m}$ is a modulus for the map f, if:

$$f((g)) = \sum_{p_i \notin S} v_{p_i}(g) f(p_i) = 0$$

Definition 4.9.2. A *local symbol on* X is a pairing of f and g assoiated to a point p:

$$G(k(X)_p) \times k(X)_p^* \xrightarrow{(\,,\,)_p} k \ ,$$

that satisfies the four conditions:

1. $(f, gg')_p = (f, g)_p + (f, g')_p$.
2. $(f, g)_p = 0$ if $p \in S$ and $g \equiv 1$ mod $\mathfrak{m}$ at p.

3. $(f,g)_p = v_p(g)f(p)$ if $p \in X - S$.
4. $\sum_{p\in X}(f,g)_p = 0$.

Proposition 4.9.3. *If $\mathfrak{m}$ is a modulus for f, then there exists a unique local symbol on X.*

4.10 A multiplicative local symbol: the tame symbol

As before, let X be a projective, irreducible, and non-singular algebraic curve. Suppose we are given a rational map $X - S \xrightarrow{f} \mathbb{G}_m$ and S is the set of zeros and poles for f. Let $g \in k(X)^*$. The tame symbol is the pairing: $k(X)^* \times k(X)^* \xrightarrow{(\,,\,)} k^*$ given locally as:

$$(f,g)_p = (-1)^{v_p(g)v_p(f)}\frac{f(p)^{v_p(g)}}{g(p)^{v_p(f)}}$$

Proposition 4.10.1. *The tame symbol is a local symbol.*

Proof. We must show that the four properties in Definition 4.9.2 are satisfied.

(1): We must show that $(f,gg')_p = (f,g)_p(f,g')_p$. But this is trivial since:

$$(f,gg')_p = (-1)^{v_p(gg')}\frac{f(p)^{v_p(gg')}}{(gg')^{v_p(f)}} = (-1)^{v_p(g)+v_p(g')}\frac{f(p)^{v_p(g)+v_p(g')}}{g(p)^{v_p(f)}g'(p)^{v_p(f)}} = (f,g)_p(f,g')_p.$$

(2): We must show that $(f,g)_p = 1$ if $p \in S$ and $g \equiv 1 \bmod \mathfrak{m}$ at p. We calculate:

$$(f,g)_p = (-1)^{v_p(g)v_p(f)}\frac{f(p)^{v_p(g)}}{g(p)^{v_p(f)}} = (-1)^0\frac{f(p)^0}{g(p)^{v_p(f)}} = \frac{1}{g(p)^{v_p(f)}} = 1.$$

(3): We must show $(f,g)_p = v_p(g)f(p)$ if $p \in X - S$. Since p is not in S, $v_p(f) = 0$. Then:

$$(f,g)_p = (-1)^{v_p(g)v_p(f)}\frac{f(p)^{v_p(g)}}{g(p)^{v_p(f)}} = f(p)^{v(p)}.$$

(4): The last property is:

$$\prod_{p\in X}(f,g)_p = 1.$$

The proof is more involved, so we skip it for now. ■

Corollary 4.10.2. *Suppose $f,g \in k(X)^*$ are relativly prime. Then:*

$$f((g)) = g((f)).$$

Proof. $f((g)) = f((\prod_p p^{v_p(g)})) = f(p)^{v_p(g)}$. Likewise for $g((f))$. The result follows now fram the product formula (4). ■

4.11 Algebraic varieties over a finite field

In this section we are following [**Serre**] p.111-112. Let V be an algebraic variety over $\mathbb{F}_q$. Suppose V is defined by charts U_i, where we have an isomophism:

$$U_i \xrightarrow{\simeq \ As \ \ k-varieties} W_i$$

If $x = (x_1, \ldots, x_r)$ is a point in an affine space, we write Fx or x^q for the point $(x_1, \ldots, x_n)$. The map $x \mapsto Fx$ commutes with polynomial maps with coefficients in k. Because of:

$$U_i \xrightarrow{\ \ F \ \simeq \ \ } U_i$$

"gluing" operates on V.

$$\begin{aligned} V &\xrightarrow{\ \ F\ \ } V \\ x &\longmapsto x^q \end{aligned}$$

Remark 4.11.1. Over $\overline{\mathbb{F}}$ Supose for a second that V is over $\mathbb{C}$. Then Frobenius F is bijective, bicontinuous and identifies the regular functions on V^p with the p-powers of the regular functions of V. One can say that F is the "maximal height 1 purely inseperable covering" of V^p.

$$\begin{aligned} V &\xrightarrow{\ \ F\ \ } V^p \\ x &\longmapsto x^q \end{aligned}$$

Proposition 4.11.2. *The $\mathbb{F}_q$-variety structure of V is unambiguously defined by its structure of variety over $\overline{\mathbb{F}_q}$ and by the Frobenius map F.*

Proposition 4.11.3. *Let X and X' be two varieties over $\mathbb{F}_q$ and if:*

$$X \xrightarrow{\varphi} X'$$

is a rational map with grapf Γ_φ, then there is one and only one rational map denoted φ^F with graph $F\Gamma_\varphi$. There is a formula:

$$\varphi^F \circ F = F \circ \varphi.$$

We have:

$$\varphi \circ F = F \circ \varphi \ \leftrightarrow \ \varphi^F = \varphi \ \leftrightarrow \ \varphi \, is \, defined \, over \, \mathbb{F}_q.$$

Extension and decent of the base field

There is a map corresponding to $\mathbb{F}_{q^m}$, namely F^m. Conversely, given a variety X over $\mathbb{F}_{q^m}$ and a map:

$$V \xrightarrow{F} V$$

we might ask under what conditions can we decend the base field of X to $\mathbb{F}_q$ such that F corresponds to:

$$V \xrightarrow{\ \ F\ \ } V$$

$$x \longmapsto x^q$$

The answer is here:

Theorem 4.11.1. *{It is possible to decend the base field}*

$$\leftrightarrow$$

$f = \varphi \circ \theta$, where:

$$X \xrightarrow{\theta} X^q \ ,$$

where θ is the canonical map of V to V^q and where φ is a biregular isomorphism:

$$X^q \xrightarrow{\simeq\ \varphi} X$$

Tori over a finite field

Let T be a torus of dimension r, i.e. $T \simeq (\mathbb{G}_m)^r$ and $\mathbb{F}_q^m$ is its splitting field. We ask: can we decend the base field to $\mathbb{F}_q$? Let say we can, how should we do it?

Decend base field

We have:

$$(\mathbb{G}_m)^r \xrightarrow{F} (\mathbb{G}_m)^r$$

. F corresponds to a square matrix M_F of degree r, with coefficients in $\mathbb{Z}$.

Lemma 4.11.4.

$$\{Identify\ F\ and\ M_F.\}$$

$$\leftrightarrow$$

$$F = q\Phi,$$

where $\phi \in Gl(r, \mathbb{Z})$, and $\Phi^m = 1$

Lemma 4.11.5. *If we have:*

$$T \xrightarrow{\simeq\ over\ \overline{\mathbb{F}}_q} T \ ,$$

$$\leftrightarrow$$

$$\{M_T \simeq M_{T'}\}$$

Proposition 4.11.6.

$$\{Classes\ of\ matrix\ groups\ up\ to\ conjugacy\} = G_M$$

$$\leftrightarrow$$

{Classes of representations of degree r with integer coefficients of a finite cyclic group}= G_R

Theorem 4.11.2. *If $G_M = G_R = l$, where l is a prime, then representations of G_R can be completely determined using $Cl_{\mathbb{Q}(\mu_l)}$.*

Tate, more intrinsic definition

One considers $X(G)$, the set of rational characters of G over $\overline{\mathbb{F}_q}$, and lets the Galois group $G(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ act on $X(G)$, i.e. we consider the group action:

$$X(G) \times G(\overline{\mathbb{F}_q}/\mathbb{F}_q) \longrightarrow X(G)$$

Relation to the zeta function and the Artin L-function.

Given G, we have an associated Φ.

Question: How many rational points does G have over $\mathbb{F}_{q^n}$? The answer is:

$$|G(\mathbb{F}_{q^n})| = det(q^n - \Phi^n) = \sum_{h=0}^{r}(-1)^i q^{n(r-h)} \sum_{i_1 \leq \ldots \leq i_h} (\lambda_{i_1} \cdots \lambda_{i_h}),$$

where the λ_i's are the Frobenius eigenvalues, i.e. the eigenvalues of Φ. We have:

Theorem 4.11.3.

$$\zeta_G(s;k) = \prod_{h=0}^{h=r} \prod_{i_1 \leq \ldots \leq i_h} (1 - \lambda_{i-1} \cdots \lambda_{i_h} q^{r-h} t)^{(-1)^{h+1}}.$$

Let denote the h-th exterior power of the matrix Φ by Φ_h. The we can rewrite the above such that:

$$\zeta_G(S;k) = \prod_{h=0}^{h=r} det(1 - q^{r-h} t \Phi_h)^{(-1)^{h+1}}$$

The factors in the last expression of the zeta function are the one appearing in Artin's L-function. Let us explain this:

Let L/K be a finite Galois extension with Galois group G. Let us consider a representation of G:

$$G \xrightarrow{\rho} \mathbb{C}^* \ .$$

Using the representation ρ we decend the base field of $(\mathbb{G}_m)^r$ from L to K, thus we obtain a torus T over K. For every prime ideal $\mathfrak{p} \in K$, we have the reduction $T_\mathfrak{p}$. Thus we have a splitting as above:

$$\zeta_T(s) = \prod_{\mathfrak{p}} \zeta_{T_\mathfrak{p}}(s).$$

Hence we arrive at:

$$\zeta_T = \prod_{h=0}^{h=r} L_h(s - r + h)^{(-1)^h}.$$

Stated in the language of this thesis, it means that:

$$\zeta_L(s) = L(X(T), s) = \prod_{\chi} L(\chi, s).$$

We now arrive at the theorem we are going to use in the proof of the generalization of the Birch Tate conjecture: depends only on the representation ρ from the rational point of view, i.e:

Theorem 4.11.4. *The zeta function and the Artin L-function depends only on the representation ρ from the rational point of view, i.e they are stable under isogeny.*

CHAPTER 5

Somekawa K-theory

In [Som90], Somekawa introduced what is today called Somekawa K-theory. This definition was probably inspired by ideas of his supervisor Kazuya Kato, a leading mathmatichan in algebraic number theory and cofounder higher dimensional class field theory. Somekawa K-theory is a generalization of Milnor K-theory.

5.1 Somekawa K-groups

In order to be able to define the Somekawa K-groups, we must first introduce a generalization of the local symbol considered in Definition 4.9.2. Let us first quickly recapitulate the situation. Let K be a finitely generated extension of transcendence degree 1 over a field k, i.e. we are considering a curve. Recall that for a place v, the symbol in Definition 4.9.2 is a pairing:

$$G(K_v) \otimes K_v^* \xrightarrow{(\,,\,)} G(k)$$

If the algebraic group G is a semi-abelian variety, i.e. an extension of an abelian variety by a torus, then this symbol can be canonically extended to a *Somekawa symbol.*

The Somekawa symbol

Suppose we are given a semi-abelian variety G. Such a variety sits in an exact sequence:

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0 \ ,$$

where T is a torus and A is an abelian variety. Lets consider these group schemes over one place a the time. Since the T is defined over K, it is also defined over K_v. There exists a finite unramified Galois extension L/K_v that the torus is splits over the residue field F of L, i.e. $T \times_k F \simeq \mathbb{G}_m^{\oplus n}$. We get a diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & T(\mathcal{O}_L) & \longrightarrow & G(\mathcal{O}_L) & \longrightarrow & A(\mathcal{O}_L) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \simeq & & \\
0 & \longrightarrow & T(L) & \longrightarrow & G(L) & \longrightarrow & A(L) & \longrightarrow & 0 \\
 & & ord_l \downarrow & & \downarrow r_l & & & & \\
 & & \mathbb{Z}^{\oplus n} & \xrightarrow{=} & \mathbb{Z}^{\oplus n} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

In the case $T = \mathbb{G}_m$, we can think of an element living in $T(L)$ as a nonzero polynomial $p(t)$ in one variable over k. It makes therefore sense to define the ord_l map, taking the order. Note that if $p \in \mathcal{O}_L^*$, then it is constant. The map r_l is the one defined last, and it is defined to be the one making the diagram commute.
In the general case where T is n-dimensional as above, for $h \in K_v^*$ and $g \in G(K_v)$ we set:

$$\begin{aligned}
K_v^* &\longrightarrow T(L) \\
h_i &\longmapsto (1, \dots, h, \dots, 1) \\
G(K_v) \times K_v^* &\xrightarrow{\epsilon} G(\mathcal{O}_L) \\
(g, h) &\longmapsto ((-1)^{ord_l(h) r_L^1(g)}, \dots, (-1)^{ord_L(h) r_L^n(g)})
\end{aligned}$$

We use this to obtain:

$$\begin{aligned}
G(K_v) \times K_v^* &\xrightarrow{\tilde{\partial}_v} G(\mathcal{O}_L) \\
(g, h) &\longmapsto \epsilon(g, h) g^{ord_L(h)} \prod_i^n h_i^{r_L^i}
\end{aligned}$$

We remark that $\tilde{\partial}_v$ looks very similar to the *tame symbol* Section 4.10.

$$\begin{tikzcd}
G(K_v) \otimes K_v^* \arrow[r, "\tilde{\partial}_v"] \arrow[rrr, bend left, "\partial_v"] & G(\mathcal{O}_L) \arrow[r] & G(F) \arrow[r] & G(k(v))
\end{tikzcd}$$

The last map is induced by the identity since $\tilde{\partial}_v$ is invariant by the Galois group $Gal(F/k(v))$.

Definition 5.1.1. We define the Somekawa symbol to be the composition ∂_v.

The Somekawa K-groups

Say we are given semi-abelian varieties $G_1, \dots, G_n$ and a field k. We define the Somekawa K-group $K(k; G_1, \dots, G_n)$ informally by writing:

$$K(k; G_1, \dots, G_n) = \oplus_{E/k} \frac{G_1(E) \otimes \dots \otimes G_n(E)}{Pr\&Wr}.$$

The above mean that the Somekawa K-group is generated by symbols on the form $\{g_1, \dots, g_n\}_{E/k}$. The arithmetic is just the same as for the Milnor K-group:

$$\{g_1, \dots, g_i + g_j, \dots, g_n\}_{E/k} = \{g_1, \dots, g_i, \dots, g_n\} + \{g_1, \dots, g_j, \dots, g_n\}.$$

We impose two different relations on the symbols:

$\boldsymbol{Pr}$: the projection formula

The projection formula is a standard formula that occours when there are restrictions, transfers and cup/tensor-products. Given a finite field extension:

$$E_1 \hookrightarrow E_2$$

and a semi-abelian variety, we have canonical norm and restiction maps:

$$\begin{tikzcd} G(E_1) \arrow[r, bend right, "R_{E_2/E_1}"'] & G(E_2) \arrow[l, bend right, "N_{E_2/E_1}"'] \end{tikzcd}$$

This yields the following relation among two elements in the Somekawa K-group:

$$\{R_{E_2/E_1}(g_1), \dots, g_i, \dots, R_{E_2/E_1}(g_n)\} = \{g_1, \dots, N_{E_2/E_1}(g_i), \dots, g_n\}$$

$\boldsymbol{Wr}$: Weil reciprocity

It was to decribe this relation we needed to define the Somekawa symbol.
Let K/k be a finitely generated extension of fields of transcendence degree 1. Let $g_i \in G_i(K)$ and $h \in K^*$. We assume that for any place v, there is an integer $0 \leq i(v) \leq n$ such that $g_i \in G(\mathcal{O}_v)$ for all $i \neq i(v)$.Then:

$$\sum_v \{g_1(v), \dots, \partial_v(g_{i(v)}, h), \dots, g_n(v)\}_{k(v)/k} = 0.$$

We have already seen several analogs of this. For instance the product formula for the local symbols and Hilbert symbols.

The first result on Somekawa K-theory

We are going to need the next lemma from [SY07].

Lemma 5.1.2. *Suppose we have $G_1, \dots, G_n$ semi-abelian varieties and for each G_i we have a exact sequence of commutative algebraic group:*

$$G_i' \longrightarrow G_i \longrightarrow G_i'' \longrightarrow 0$$

Then the following sequence is exact:

$$K(F; G_1', \dots, G_n') \longrightarrow K(F; G_1, \dots, G_n) \longrightarrow K(F; G_1'', \dots, G_n'') \longrightarrow 0$$

The first nice theorem regarding the Somekawa K-groups is that they agree with the Milnor K-groups when all the G_i's are the trivial tori:

Theorem 5.1.1.

$$K(k; \mathbb{G}_m, \ldots, \mathbb{G}_m) \simeq K_n^M(F)$$

The proof of this fact is quite formal and not so conceptual. We refer to [Som90] for the proof. We rather give a heuristic reason in the next paragraph. There are essentially two key facts that yields the isomorphism:

(1): We can imagine a web of finite field extensions over k and consider the object $G_1 \otimes \cdots \otimes G_n$, where the G_i's are semi abelian varieties. We can attach this object to each field in the web and when it is attached to a field E, we call the associated object $G_1(E) \otimes \cdots \otimes G_n(E)$ a Somekawa pregroup. We will then have one group per field and the groups will have nothing to do with each other. We then impose the projection formula relation and get $\frac{G_1(E) \otimes \cdots \otimes G_n(E)}{Pr}$. Now, the arithmetic in each group is very rigid since it is more or less determined by the arithmetic in the nearby fields in the web. The first part of the theorem says that all arithmetic is determined by what happens in the base $k \otimes \cdots \otimes k$.

(2): The second part says basically that when we have projected all the way down to the base, the Weil reciprocity relation becomes the Steinberg relation. The fact that the Steiberg relation implies Weil reciprocity in the base can be easyly shown using the Somekawa symbol. Suppose we have the following data:

$$K = k(t),\ h = t^{-1},\ g_i = 1 - a_i t^{-1},\ g_j = 1 - t \text{ and } g_l = a_l.$$

Suppose that $a_i + a_j = 1$. For the place $v = t - a_i$, we have:

$$\{g_1(v), \ldots, \partial_v(g_{i(v)}, h), \ldots, g_n(v)\} = \{a_1, \ldots, a_n\}$$

and for all other places it vanshies. By Section 5.1, we are done.

The next result connects the Somekawa K-groups to etale cohomology.

Proposition 5.1.3. *We have a commuting diagram:*

$$\begin{tikzcd}
G_1(k) \otimes \cdots \otimes G_n(k) \arrow[r, "\beta"] \arrow[d, "\alpha"'] & H^1_{et}(k, G_1[m]) \otimes \cdots \otimes H^1_{et}(k, G_n[m]) \arrow[d, "\cup"] \\
K(k; G_1, \ldots, G_m) \arrow[r, "c"] & H^m_{et}(k, G_1[m] \otimes \cdots \otimes G_m)
\end{tikzcd}$$

5.2 Yamazaki's version

Let us now turn our attention to the Somekawa K-groups in Yamazki's generalization of the Birch-Tate-conjecture. Let T be a torus. We have:

$$K^T(F) = K(F; T, \mathbb{G}_m) = \oplus_{E/F} \frac{T(E) \otimes E^*}{Pr\&Wr},$$

where the notation above is already explained above. Since these groups occupy a central part of this thesis we will give these groups a name, namely *Yamazki K-groups*. We begin by reviewing the set up for the main theorem of the thesis.

the generalization of the Birch-Tate-conjecture

As we have already proved in the function field case, we have the following formula:

$$\zeta_K(-1) = \frac{|K_2(\mathcal{O}_K)|}{|H^0(K, \mathbb{Q}/\mathbb{Z}(2))}.$$

In the number field case it is called the Birch-Tate-conjecture.

The weight two motivic complex $\mathbb{Z}(2)$

In order to be able to make certain constructions, we are dependent on having the complexes

Tori

The simplest torus one can think of is the trivial torus $\mathbb{G}_m$ and the next simplest are powers of it, i.e. $T = (\mathbb{G}_m)^{\oplus r}$. The reason for calling these tori is that in motivic homotopy theory, $\mathbb{G}_m$ is along with S^1 a motivic sphere. For example $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ which is homotopy equivalent to S^1, and $\mathbb{G}_(\mathbb{R}) = \mathbb{R}^*$ is homotopy eqivalent to S^0. If a *torus* is not a power of $\mathbb{G}_m$, then in comes equipped with a splitting field L. The extension L/K can be assumed to be finite Galois and when we base change we get: $T \times_K L = (\mathbb{G}_m)^r$. To the the torus T we associate the character group $X(T)^* = Hom(T, \mathbb{G}_m)$ and the cocharacter group $X(T) = Hom(\mathbb{G}_m, T)$. We will be mostly concerned with the cocharacter group, as it will be the coefficient module when take Galois cohomology. Note that for the trivial torus $Hom(\mathbb{G}_m, T) = Hom(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$. We make the following three definitions:

Definition 5.2.1.

1. A torus P over F is called quasi trivial if it is on the form:

   $$P = \oplus_{E_i/F} R_{E_i/F}\mathbb{G}_m,$$

   where the E_i's are finite field extensions.

2. A torus I over F is called invertible if there exist another torus I' over F such that $I \oplus I'$ is quasi trivial.

3. A torus Q is called flasque if it satifies the Hilbert 90 property for all finite extensions E/F, i.e:

   $$H^2(E, X(Q) \otimes \mathbb{Z}(1)) = 0.$$

Lemma 5.2.2. *We have the following implications for a torus T:*

$$\{T\, is\, quasi\, trivial\} \implies \{T\, is\, invertible\} \implies \{T\, is\, flasque\}$$

Proof. Let T be quasi trivial, i.e. $T = \oplus_{E_i/F} R_{E_i/F}\mathbb{G}_m$. Then by definition it is invertible. To see flasqueness, we first concider the trivial torus $T = \mathbb{G}_m$ an observe that flasqueness follows from Hilbert 90:

$$H^2(F, X \otimes \mathbb{Z}(1)) = H^1(F, \mathbb{G}_m = 0.$$

Let $T = \oplus_i R_{E_i/F}\mathbb{G}_m$. Then

$$H^2(F, X \otimes \mathbb{Z}(1)) = \oplus_i H^2(E_i, \mathbb{Z}(1)) = \oplus_i H^1(E_i, \mathbb{G}_m) = 0.$$

If the torus T is invertible, then $T \oplus T'$ is quasi trivial for some T', hence flasque. Thus T is flasque. ■

Here, we state a very important theorem in the classification of tori. We will treat this result purely formally, so we will omit the proof and rather refer to [Ono90] p. 114.

Theorem 5.2.1. *Let T be a torus over K. Then there are quasi trivial tori P and Q such that:*

$$T^{\oplus m} \oplus P \xrightarrow{\sim} Q$$

5.3 results

The universal symbol

Recall that we define the *universal symbol* on a field in Theorem 3.3.1. This symbol can also be defined for the Yamazaki's groups in the following way: Let E/F be a finite field extension. We have as in Theorem 3.3.1:

$$T(E) \otimes E^* \xrightarrow{\simeq} H^1(E, X \otimes \mathbb{Z}(1)) \otimes H^1(E, \mathbb{Z}(1)) \xrightarrow{\cup} H^2(E, X \otimes \mathbb{Z}(2)) \xrightarrow{N_{E/F}} H^2(F, X \otimes \mathbb{Z}(2))$$

Definition 5.3.1. A torus T has *motivic interpretation* if the associated continuous symbol is an isomorphism;

$$K^T(F) \xrightarrow{\simeq} H^2(F, X \otimes \mathbb{Z}(2))$$

There are some cases where it is easy to prove that a torus has motivic interpretation.

Example 5.3.2. (1) If $T = \mathbb{G}_m$, then the above isomorphism becomes:

$$K_2(F) \xrightarrow{\simeq} H^2(F, \mathbb{Z}(2)) \ .$$

A nice explicit and conceptual proof of this fact can be found in **??**.

(2) Let $T = \oplus_i R_{E_i/F}\mathbb{G}_m$. Then we have diagram which makes the statement obvious:

$$\begin{tikzcd} \oplus_i K_2(E_i) \arrow[r, "\simeq"] \arrow[d, "="'] & \oplus_i H^2(E_i, \mathbb{Z}(2)) \arrow[d, "="] \\ K^T(F) \arrow[r] & H^2(F, X \otimes \mathbb{Z}(2)) \end{tikzcd}$$

(3) Let T be invertible, then there is a T' such that $T \oplus T'$ is quasi trivial. Then the projection maps induces the isomorphisms we need:

$$\begin{tikzcd}
K^{T'}(F) \arrow[r] & H^2(F, X'(2)) \\
K^{T\oplus T'}(F) \arrow[r, "\simeq"] \arrow[u] \arrow[d] & H^2(F, X(2) \oplus x'(2)) \arrow[u] \arrow[d] \\
K^{T}(F) \arrow[r] & H^2(F, X(2))
\end{tikzcd}$$

Flasque resolutions

Let T be a torus over F split by L. Then there exists a flasque torus Q and a quasi trivial P, such that there is an exact sequence of group schemes:

$$0 \longrightarrow Q \longrightarrow P \longrightarrow T \longrightarrow 0$$

The above is called a flasque resolution.

Proposition 5.3.3. *Given a flasque resolution where Q is invertible, then T must have motivic interpretation.*

Proof. By Lemma 5.1.2, we have an exact sequence:

$$K^Q(F) \longrightarrow K^P(F) \longrightarrow K^T(F) \longrightarrow 0$$

We can also take Galois cohomology thus getting a long exact sequence. It is well known that $H^3(F, \mathbb{Z}(2)) = 0$ [Lic87]. Since Q is invertible we can find a Q' such that $Q \oplus Q' = \oplus_i R_{E_i/F}\mathbb{G}_m$. Thus:

$$H^3(F, X(Q \oplus Q') \otimes \mathbb{Z}(2)) = \oplus_i H^3(E_i, \mathbb{Z}(2)) = 0,$$

on the other hand:

$$\begin{tikzcd}
K^Q(F) \arrow[r] \arrow[d, "\simeq"] & K^P(F) \arrow[r] \arrow[d, "\simeq"] & K^T(F) \arrow[r] \arrow[d] & 0 \\
H^2(F, X(Q) \otimes \mathbb{Z}(2)) \arrow[r] & H^2(F, X(P) \otimes \mathbb{Z}(2)) \arrow[r] & H^2(F, X(T) \otimes \mathbb{Z}(2)) \arrow[r] & 0
\end{tikzcd}$$

■

Norm residue homomorphism and more

Recall the map from the Somekawa K-group into etale cohomology Proposition 5.1.3 which of course applies to the Yamazaki K-groups. We identify etale and Galois cohomology and construct the map:

$$K^T(F)/n \xrightarrow{h_1^T} H^2(F, T[n] \otimes \mu_n)$$

Construction

First we consider the short exact sequence:

$$0 \longrightarrow \mu_n \longrightarrow E_s^* \xrightarrow{n} E_s^* \longrightarrow 0 \ ,$$

which is the sequence from Section 2.3. Because of Hilbert 90 we have the isomorphism:

$$E^*/(E^*)^n \xrightarrow{\simeq} H^1(E_s, \mu_n)$$

Recall the norm residue homomorphism constructed in the classical situation in Section 2.3. Just as for the continuous symbol, the norm residue homomorphism has an extension to Yamazaki's K-groups. We consider the exact sequence in Section 2.3 with some modern modification. We say that if n is an integer invertible in F, we have a triangle:

$$\mathbb{Z}(2) \longrightarrow \mathbb{Z}(2) \longrightarrow \mu_n^{\otimes 2} \longrightarrow \mathbb{Z}(2)[1]$$

We tensor the sequence with X and obtain:

$$X(2) \longrightarrow X(2) \longrightarrow T[n] \otimes \mu_n \longrightarrow X(2)[1]$$

We take pass to cohomology and obtain the following results:

Proposition 5.3.4. *Assume that T admits motivic interpretation. Then we have an isomorphism:*

$$H^0(F, T[n] \otimes \mu_n) \simeq H^1(F, X \otimes \mathbb{Z}(2))[n],$$

and two short exact sequences:

$$0 \longrightarrow H^1(F, X \otimes \mathbb{Z}(2))/n \longrightarrow H^1(F, T[n] \otimes \mu_n) \longrightarrow K^T(F)[n] \longrightarrow 0 \ ,$$

$$0 \longrightarrow K^T(F)/n \xrightarrow{h_1^T} H^2(F, T[n] \otimes \mu_n) \longrightarrow H^3(F, X \otimes \mathbb{Z}(2))[n] \longrightarrow 0$$

Proof. The first isomorphism follows from the well known vanishing of $H^0(F, \mathbb{Z}(2))$ proved by Merkurjev. The other statements are tautologies. ■

Remark 5.3.5. We see that if $T = \mathbb{G}_m$, then:

$$K^T(F)/n \xrightarrow{h_1 \simeq} H^2(F, \mu_n \otimes \mu_n) \ ,$$

again because of the vanishing of $H^3(F, \mathbb{Z}(2))$. We remark that there might be a mistake here, because the above isomorphism is the famous Merkurjev-Suslin theorem, and we do not expect there to be a two sentence proof of that.

Tate's diagram for K_2

Recall the fundamental diagram discussed in Section 3.3, a diagram which is used in the proofs of almost all major result in that paper [Tat76]. These results have for example vast applications in Iwasawa theory, and it motivated Tate's student, Stephen Lichtenbaum to state the so called Quillen-Lichtenbaum conjecture. This conjecture is an extension of Tate's theorem for K_2 into all degrees, and it is now proven as it is a consequence of the Bloch Kato conjecture. Anyhow, it seems like a good idea to extend this diagram to our Yamazaki K-groups. Lets first modify the diagram a little so it fits nicely into our situation. Let us consider the diagram:

$$\begin{tikzcd}
\mathbb{G}_m(F)\otimes\mu_p \arrow[r, "\gamma"] \arrow[d, "\simeq"] & K^{\mathbb{G}_m}(F) \arrow[r, "p"] \arrow[d, "h"] & K^{\mathbb{G}_m}(F) \arrow[r] \arrow[d, "h"] & K^{\mathbb{G}_m}(F)/p \arrow[d, "h_1"] \\
H^1(F,\mu_p^{\otimes 2}) \arrow[r, "\delta"] & H^2(F,\mathbb{Z}_p(2)) \arrow[r, "p"] & H^2(F,\mathbb{Z}_p(2)) \arrow[r] & H^2(F,\mu_p^{\otimes 2})
\end{tikzcd}$$

This is how the diagram would look like if $T=\mathbb{G}_m$. Now, let T be any torus over F. There is a finite Galois extension L/K such that L is the splitting field for T, i.e. $T\simeq(\mathbb{G}_m)^r$. We set $\hat{X}_p(2)=X\otimes\mathbb{Z}_p(2)$ as the analogue of the twistet Tate module and get a diagram:

$$\begin{tikzcd}
\oplus_{i=1}^r\mathbb{G}_m(L)\otimes\mu_p \arrow[r, "\gamma"] \arrow[d, "\simeq"] & \oplus_{i=1}^r K^{\mathbb{G}_m}(L) \arrow[r, "p"] \arrow[d, "h"] & \oplus_{i=1}^r K^{\mathbb{G}_m}(L) \arrow[r] \arrow[d, "h"] & \oplus_{i=1}^r K^{\mathbb{G}_m}(L)/p \arrow[d, "h_1\simeq"] \\
\oplus_{i=1}^r H^1(L,\mu_p^{\otimes 2}) \arrow[r, "\delta"] & \oplus_{i=1}^r H^2(L,\mathbb{Z}_p(2)) \arrow[r, "p"] & \oplus_{i=1}^r H^2(L,\mathbb{Z}_p(2)) \arrow[r] & \oplus_{i=1}^r H^2(L,\mu_p^{\otimes 2}) \\
T(F)\otimes\mu_p \arrow[r, "\gamma^T"] \arrow[d] \arrow[uu, bend left] & K^T(F) \arrow[r, "p"] \arrow[d, "h^T"] \arrow[uu, bend left] & K^T(F) \arrow[r] \arrow[d, "h^T"] \arrow[uu, bend left] & K^T(F)/p \arrow[d, "h_1^T"] \arrow[uu, bend left] \\
H^1(F,T[p]\otimes\mu_p) \arrow[r, "\delta^T"] \arrow[uu, bend left] & H^2(F,\hat{X}_p(2)) \arrow[r, "p"] \arrow[uu, bend left] & H^2(F,\hat{X}_p(2)) \arrow[r] \arrow[uu, bend left] & H^2(F,T[n]\otimes\mu_p) \arrow[uu, bend left]
\end{tikzcd}$$

The above diagram basically sums up all we have done so far regarding Yamazaki's K-groups and also ordinary K_2 for that matter.

Remark 5.3.6. It is a very interesting question to ask what is the difference between the groups upstaris and downstairs. In Serre's note on tori over finite fields **??**, we saw that since T and $(\mathbb{G}_m)^r$ are equivalent over $\overline{\mathbb{F}}_q$, they must at least be isogeneous. If T is invertible, they are isomorphic by earlier discussions.

We now extend a major application of the "Main conjecture" of [Tat] which is proven in [Tat76]. The proof presented here is our own and differ quite a lot for the on by Yamazaki [**Yamazaki**]. The proof rests upon our lenghty treatment of Tate's results for ordinary K_2. We also implement a key idea in Yamazaki's proof of the generalization of the Birch Tate conjecture that makes everything fit thogether.

Theorem 5.3.1. *Assume T' admit motivic interpretation. Then we have isomorphisms:*

$$K^T(F)[p^\infty]\simeq H^2(F,X'\otimes\mathbb{Z}_p(2))_{Tors}\simeq H^1(F,X'\otimes\mathbb{Q}_p/\mathbb{Z}_p(2))/div.$$

Proof. By [Ono90] we can swap the torus T' by a quasi trivial torus T and they will be isogeneous. We can therefor assume that T is on the form $T = \oplus_{i=1}^r \mathbb{R}_{E_i/F} mathbbG_m$. We get a diagram:

$$\begin{tikzcd}
\oplus_{i=1}^r K^{\mathbb{G}_m}(E_i)[p^\infty] \arrow[r, "\simeq"] & \oplus_{i=1}^r H^2(E_i, \mathbb{Z}_p(2))_{Tors} & \oplus_{i=1}^r H^1(E_i, \mathbb{Q}_p/\mathbb{Z}_p(2))/div \arrow[l, "\simeq"'] \\
K^T(F)[p^\infty] \arrow[r, "\simeq"] \arrow[u, "="] & H^2(F, X \otimes \mathbb{Z}_p(2))_{Tors} \arrow[u, "="] & H^1(F, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))/div \arrow[l, "\simeq"'] \arrow[u, "="] \\
K^{T'}(F)[p^\infty] \arrow[r, "h"] \arrow[u, "\sim"] & H^2(F, X' \otimes \mathbb{Z}_p(2))_{Tors} \arrow[u, "\sim"] & H^1(F, X' \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))/div \arrow[l, "\delta"'] \arrow[u, "\sim"]
\end{tikzcd}$$

Here, $\sim$ means isogeneous. The $=$ signs going up are isomorphism induced by Shapiro's lemma **??**, which we have seen many times before. The top horisontal isomorphisms are due to the theorems of Tate discussed in **??**. The middle horisontal arrow become isomorphisms as a consequence. What is left is to show that h and δ are isomorphisms, but that is not that hard: (1) h and δ is surjective by a diagram chase. (2) h is injective by Proposition 3.3.1, i.e. $H^2(F, X' \otimes \mathbb{Z}_p(2))_{Tors}$ contains no p-divisible subgroup. (3) δ is injective because of **??** ∎

Remark 5.3.7. The technique in the proof might be applicable to many situations. The reason is that it really exposes the rigid nature of moving a torus defined over one field E_i to a smaller one T. This is very analogous to the discussion **??**.

$$\begin{tikzcd}
\oplus_{i=1}^r K^{\mathbb{G}_m}(E_i)[p^\infty] \arrow[r, "\simeq"] & \oplus_{i=1}^r H^2(E_i, \mathbb{Z}_p(2))_{Tors} & \oplus_{i=1}^r H^1(E_i, \mathbb{Q}_p/\mathbb{Z}_p(2))/div \arrow[l, "\simeq"'] \\
K^T(F)[p^\infty] \arrow[r, "\simeq"] \arrow[u, "="] & H^2(F, X \otimes \mathbb{Z}_p(2))_{Tors} \arrow[u, "="] & H^1(F, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))/div \arrow[l, "\simeq"'] \arrow[u, "="] \\
K^{T'}(F)[p^\infty] \arrow[r, "h"] \arrow[u, "\sim"] & H^2(F, X' \otimes \mathbb{Z}_p(2))_{Tors} \arrow[u, "\sim"] & H^1(F, X' \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))/div \arrow[l, "\delta"'] \arrow[u, "\sim"]
\end{tikzcd}$$

The Somekawa K-groups for local fields

Lemma 5.3.8. *If $p = char(k) > 0$, then $K^T(k) = 0$.*

Proof. Let k'/k be a finite Galois extension that splits T. We know by the theorem abover [Mer] that $K_2(F)$ is a direct sum of a finite group and a uniquely divisible group. By the norm argument $K^T(k)$ is a direct sum of a uniquely divisible group and a torsion group of finite exponent. If $char(k) = p > 0$, then we know from **??** that $K^T(k)[p^\infty]$ is p-divisible. Hence it is finite and divisible, thus it is trivial. ∎

Proposition 5.3.9. *Let k'/k be a finite extension. Then the norm map:*

$$K^T(k') \xrightarrow{N_k^{k'}} K^T(k)$$

is surjective.

Proof. We assume that k'/k is a Galois extension of degree l. We take an arbitrary element in the Somekawa K-group. It is represented by $(a,b)_{k_1/k}$ for some finite field extension k_1/k, where $a \in T(k_1)$ and $b \in k_1^*$. We will show that this element is in the image of the norm map. If $k' \subset k_1$, then $(a,b)_{k_1/k} = N_k^{k'}((a,b)_{k_1/k'})$. Hence we assume $k_1' = k_1k'/k_1$.

First we consider the case when $T = \mathbb{G}_m$. This case has already been proven above because Somekawa K-groups coincide with Milnor K-groups for $T = \mathbb{G}_m$. However, here is a very different proof that invokes fundamental ideas in local class field theory.

We take another extension k_2/k_1 of degree l. We have a map

$$k_1^*/N_{k_1}^{k_1'}k_1'^* \to k_2'/N_{k_2}^{k_2'}k_2'^*,$$

which is an isomorphism by the following argument. The existence theorem of local class field theory asserts the existence of a corresponding map between groups:

$$Gal(k_2'/k_2) \to Gal(k_1'/k_1),$$

which is by construction an isomorphism of cyclic groups of degree l. Hence we have

$$(a,b)_{k_1/k} = N_k^{k'}((a_1, R_{k_1}^{k_2'}(b))_{k_1'/k'} + (R_{k_2}^{k_2'}(a_2), b')_{k_2'/k'})$$

■

The result that the norm maps are surjective is a strong result. Yamazaki derives some consequences:

Proposition 5.3.10. *Let $m \in k$ be a natural number. Then the Galois symbol:*

$$K^T(k)/m \longrightarrow H^2(k, T[m] \otimes \mu_m)$$

is bijective.

Corollary 5.3.11. *Let $\hat{X}(r) = \lim_{\leftarrow} X(T) \otimes \mu_n^{\otimes r}$ and $\mathbb{Q}/\mathbb{Z}(r)` = lim_{\rightarrow}\mu_n^r$, where n runs through natural nubers prime to ther characteristic of k. Then we have isomorphisms:*

$$K^T(k)_{Tor} \xrightarrow{\simeq} K^T(k)/div \xrightarrow{\simeq} H^2(k, \hat{X}(2)) \xrightarrow{\simeq} H^1(k, X \otimes \mathbb{Q}/\mathbb{Z}(2)`)/div \xrightarrow{\simeq} \hat{X}(1)_{G_k}$$

Proposition 5.3.12. $K^T(F) = \{Finite\ group\} \oplus \{Uniquely\ divisible\}$

Yamazaki has conjectured that all tori admit motivic interpretation. In the proof of the generalization of the Birch Tate conjecture this will be one of the assumptions, hence we do not need the next result. We therefore state it without a proof:

Theorem 5.3.2. *Let T be a torus over a non-arcimedean local field k. Then T admits motivic interpretation.*

5.4 Non-archimedean local field

Since $K_2(\mathbb{C})$ is uniquely divisible, we get that $K^T(\mathbb{C})$ is uniquely divisible by applying the usual arguments related to the transport of structure by the norm and restriction maps:

$$\begin{tikzcd} K^T(\mathbb{C}) \arrow[r, bend left, "R"] & \oplus_{i=1}^{r} K^{\mathbb{G}_m}(\mathbb{C}) \arrow[l, bend left, "N"] \end{tikzcd} .$$

Yamazaki call this kind of reasoning the *norm argument*. We recall an elementary lemma from [**Milnorq**].

Lemma 5.4.1. $K_2(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} \oplus \{divisible\}$

Consider now the exact sequence:

$$0 \longrightarrow ker\left(Res_{\mathbb{R}}^{\mathbb{C}}\mathbb{G}_m \to \mathbb{G}_m\right) \longrightarrow Res_{\mathbb{R}}^{\mathbb{C}}\mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 0 \ .$$

Using Proposition 5.3.4 we obtain an exact sequence:

$$0 \longrightarrow K^T(\mathbb{R})/n \longrightarrow H^2(\mathbb{R}, T[n] \otimes \mu_n) \longrightarrow H^3(\mathbb{R}, X(T) \otimes \mathbb{Z}(2)) \longrightarrow 0 \ .$$

Example 5.4.2. If $T = \mathbb{G}_m$, then $K^T(\mathbb{R})_{Tor} = \mathbb{Z}/2\mathbb{Z}$.
If $T = Res_{\mathbb{R}}^{\mathbb{C}}\mathbb{G}_m$ or $T = \mathrm{Ker}(Res_{\mathbb{R}}^{\mathbb{C}}\mathbb{G}_m \to \mathbb{G}_m)$, then $K^T(\mathbb{R}) = 0$.
For all n, we have an exact sequence:

$$0 \longrightarrow K^T(\mathbb{R})/n \longrightarrow H^2(\mathbb{R}, T[n] \otimes \mu_n) \longrightarrow H^3(\mathbb{R}, X(T) \otimes \mathbb{Z}(2))$$

This sequence is known in the following special cases:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \qquad T = \mathbb{G}_m$$

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \qquad T = Res_{\mathbb{R}}^{\mathbb{C}}T$$

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \qquad T = Ker(Res_{\mathbb{R}}^{\mathbb{C}}T \to T)$$

5.5 Hasse principle

Theorem 5.5.1. *For all $i \geq 3$, we have isomorphisms:*

$$H^i(K, X(2)) \simeq \oplus_{v|\infty} H^i(K_v, X(2)).$$

Proof. We recall that we have the distinguished triangle whenever n is invertible:

$$X(2) \xrightarrow{n} X(2) \longrightarrow T[n] \otimes \mu_n \longrightarrow X(2)[1]$$

This triangle along with the inclusion $K \hookrightarrow K_v$ gives us a commutative diagram where the rows are the associated long exact sequences in cohomology.

Let n_i be the exponent of $H^i(K, X(2))$ for $i \geq 3$. Choosing n to be the prime to $char(K)$-part of $n_i n_{i+1}$ we get the diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^i(K,X(2)) \arrow[r] \arrow[d] & H^i(K,T[n]\otimes\mu_n) \arrow[r] \arrow[d, "\simeq"] & H^i(K,X(2)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_{v|\infty}H^i(K_v,X(2)) \arrow[r] & \oplus_{v|\infty}H^i(K_v,T[m]\otimes\mu_n) \arrow[r] & \oplus_{v|\infty}H^{i+1}(K_v,X(2)) \arrow[r] & 0
\end{tikzcd}$$

The middle vertical arrow is an isomophism by the Poitou-Tate theorem. We see immediately that all the f_i's are inejctive. Then since f_4 is injective, we see by a diagram chase that f_3 is surjective to. Hence all the f_i's are isomorphisms. ■

Theorem 5.5.2. *Assume T admits motivic interpretation. Let L/K be a finite separabel extension. There is an isomorphism of finite groups:*

$$K^T(K)/N_{L/K} \simeq \oplus_{v|\infty} K^T(K_v)/N_{L_{w(v)}/K}(L_{w(v)}).$$

Proof. We let S be the kernel of the canonical map $Res_K^L T \longrightarrow T$, and denote the associated cocharacter group $Y = X(S) = Hom(\mathbb{G}_m, S)$. We get a distinguished triangle: $Y(2) \longrightarrow Res_K^L X(2) \longrightarrow X(2) \longrightarrow Y(2)[1]$. We get two long exact sequences sitting thogether in a commuting diagram:

$$\begin{tikzcd}
\dots \arrow[r] & H^i(K,X(2)) \arrow[r] \arrow[d] & H^{i+1}(K,Y(2)) \arrow[r] \arrow[d] & H^{i+1}(L,X(2)) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \oplus_{v|\infty}H^i(K_v,X(2)) \arrow[r] & \oplus_{v|\infty}H^{i+1}(K_v,Y(2)) \arrow[r] & \oplus_{v|\infty}H^{i+1}(L_{w(v)},X(2)) \arrow[r] & \dots
\end{tikzcd}$$

We observe that the triangle is induced by the Norm map of tori. Using the assumption that T admits motivic interpretation we obtain the diagram:

$$\begin{tikzcd}
0 \arrow[r] & K^T(K)/N_K^L K^T(L) \arrow[r] \arrow[d] & H^3(K,Y(2)) \arrow[r] \arrow[d] & H^3(L,X(2)) \arrow[d] \\
0 \arrow[r] & \oplus_{v|\infty}K^T(K)/N_{K_v}^{L_{w(v)}} K^T(L_{w(v)}) \arrow[r] & \oplus_{v|\infty}H^3(K_v,Y(2)) \arrow[r] & \oplus_{v|\infty}H^3(L_{w(v)},X(2))
\end{tikzcd}$$

We remark that we only need to take the infinte places into account in the sum $\oplus_{v|\infty} K^T/N_K^L(L_{w(v)})$, since by theorem(?) the norm map is surjective for non-archimedean places. We also use that the norm map is independent of the place w we choose above v. By the last theorem the to vertical arrows on the right are isomorphisms, hence by the five-lemma we conclude that the left vertical map is an isomorphism too. ■

Let $T = \mathbb{G}_m$ and consider the "norm" map $Res_{\mathbb{R}}^{\mathbb{C}}\mathbb{G}_m \to \mathbb{G}_m$ with kernel S. We get the triangle:

$$S \longrightarrow Res_{\mathbb{R}}^{\mathbb{C}}\mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow S[1]$$

Recall that we are eventually going to consider the case when T is a torus over K split L, where K and L are totally real. That means that the completions above will just be the reals, i.e. $K_v = \mathbb{R}$ and $L_w = \mathbb{R}$. The diagram above becomes:

$$\begin{tikzcd}
0 \arrow[r] & K^T(K)/N_K^L K^T(L) \arrow[r] \arrow[d, "\simeq"] & H^3(K, Y(2)) \arrow[r] \arrow[d, "\simeq"] & H^3(L, X(2)) \arrow[d, "\simeq"] \\
0 \arrow[r] & \oplus_{v|\infty} K^T(K_v)/N_{K_v}^{L_{w(v)}} K^T(L_{w(v)}) \arrow[r] & \oplus_{v|\infty} H^3(\mathbb{R}, Y(2)) \arrow[r] & \oplus_{v|\infty} H^3(\mathbb{R}, X(2))
\end{tikzcd}$$

Corollary 5.5.1. *sds*

Proof. s ∎

Definition 5.5.2. Let $p \neq char(K)$, then we define:

$$K^T(C) = \mathrm{Ker}\left(K^T(K)[p^\infty] \to \prod_{v \nmid p, \infty} H^1(K_v^{nr}, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}_v}} \right)$$

The v-component is given by the composition:

$$K^T(K)[p^\infty] \longrightarrow K^T(K_v)[p^\infty] \xrightarrow{\simeq} H^1(K_v, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^1(K_v^{nr}, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}_v}}$$

Proposition 5.5.3. *If T admits motivic interpretation, then $K^T(C)$ is finite.*

Proof.

$$\begin{tikzcd}
 & K^T(C_L)[p^\infty]_G \arrow[r] \arrow[d] & K^T(L)[p^\infty]_G \arrow[r] \arrow[d] & \oplus_{v \nmid p,\infty} \left(\oplus_{w|v} H^1(L_w^{nr}, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}_v}} \right)_G \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & K^T(C)[p^\infty] \arrow[r] & K^T(K)[p^\infty] \arrow[r] & \oplus_{v \nmid p,\infty} H^1(K_v^{nr}, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}_v}} \arrow[r] & {}
\end{tikzcd}$$

∎

We fix now $\mathbb{Q}/\mathbb{Z}(2)' = \lim_{rightarrow} \mu_n$, where n runs over all natural numbers prime to the characteristic. The next result is a formula that is a key tool in the proof the main theorem. Yamazaki's statement in the number field case is that the formula hold up to a power of 2. We have extended the result such that no extra power of 2 is needed, by verifying Yamazaki's proof in the case $p = 2$.

Proposition 5.5.4. *Suppose $T_1 \xrightarrow{f} T_2$ is an isogeny among tori that are split by a totally real field in the number field case. Then the formula holds:*

$$\frac{|K^{T_1}(\mathcal{O}_K)|}{|H^0(K, X(T_1) \otimes \mathbb{Q}/\mathbb{Z}(2))|} = \frac{|K^{T_2}(\mathcal{O}_K)|}{|H^0(K, X(T_2) \otimes \mathbb{Q}/\mathbb{Z}(2))|}$$

Proof. Let S be the finite set of places where T_i have bad reduction and also the infinite places of K. We pick any prime p ($p = 2$ is allowed), and recall the localization sequences in etale cohomology Proposition 4.2.1 for $K^{T_i}(\mathcal{O}_K)$ and $H^1(\mathcal{O}_K[\frac{1}{pS}], X(T_i) \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))$. In order to have enough space for the diagram, let $M_i = X(T_i) \otimes \mathbb{Q}_p/\mathbb{Z}_p(2)$:

$$\begin{tikzcd}
0 \arrow[r] & K^T(\mathcal{O}_K)[p^\infty] \arrow[r] \arrow[d] & H^1(K, M_i) \arrow[r] \arrow[d, "="] & \oplus_{v \nmid p} H^1(K_v^{nr}, M_i)^{G_{\mathbb{F}_v}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(\mathcal{O}_K[\frac{1}{pS}], M_i) \arrow[r] & H^1(K, M_i)[p^\infty] \arrow[r] & \oplus_{v \nmid pS} T_{i,v}(\mathbb{F}_v)[p^\infty] \arrow[r] & 0
\end{tikzcd}$$

This implies that we have an exact sequence:

$$0 \longrightarrow K^T(\mathcal{O}_K)[p^\infty] \longrightarrow H^1(\mathcal{O}_K[\tfrac{1}{pS}], M_i) \longrightarrow \oplus_{v \nmid p, v \in S} H^1(K_v^{nr}, M_i)^{G_{\mathbb{F}_v}} \longrightarrow 0$$

We recall our result in etale cohomology, that

$$H^2(\mathcal{O}_K[\frac{1}{pS}], M_i) = H^2(K, M_i) = 0$$

vanishes by the localization sequence. If we are able to show that:

$$\frac{|H^1(\mathcal{O}_K[\frac{1}{pS}], M_1)|}{|H^0(\mathcal{O}_K[\frac{1}{pS}], M_1)|} = \frac{|H^1(\mathcal{O}_K[\frac{1}{pS}], M_2)|}{|H^0(\mathcal{O}_K[\frac{1}{pS}], M_2)|} \tag{5.1}$$

and that

$$| \oplus_{v \nmid p, v \in S} H^1(K_v^{nr}, M_i)^{G_{\mathbb{F}_v}} | = | \oplus_{v \nmid p, v \in S} H^1(K_v^{nr}, M_i)^{G_{\mathbb{F}_v}} | \tag{5.2}$$

we will be done by the localization sequence above. The isogeny $T_1 \xrightarrow{f} T_2$ gives us the exact sequence:

$$0 \longrightarrow C \otimes \mu_n \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0 \tag{5.3}$$

where $C = ker(f)$ and n is the p-power part of the order of C. There is an exact sequence:

$$0 \longrightarrow ker(a) \longrightarrow H^1(K_v^{nr}, M_1) \xrightarrow{a} H^1(K_v^{nr}, M_2) \longrightarrow 0$$

Taking $G_{\mathbb{F}_v}$ invariance we get:

$$0 \longrightarrow ker(a)^{G_{\mathbb{F}_v}} \longrightarrow H^1(K_v^{nr}, M_1)^{G_{\mathbb{F}_v}} \longrightarrow H^1(K_v^{nr}, M_2)^{G_{\mathbb{F}_v}} \longrightarrow H^1(\mathbb{F}_v, ker(a)) \longrightarrow 0\ ,$$

since

$$H^1(\mathbb{F}_v, H^1(K_v^{nr}, M_i)) = H^2(K_v, M_i) = K^T(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0,$$

because $K^T(F) = \{finite\} \oplus \{uniquely\, divisible\}$. Equation (5.2) follows now from the fact that $ker(a)^{G_{\mathbb{F}_v}}$ is a finite $G_{\mathbb{F}_v}$-module.

In [**neucoh**] p.427 we have the *global Euler-Poincare characteristic formula*, which in our case gives us the equation:

$$\frac{|H^0(\mathcal{O}_K[\frac{1}{pS}], C\otimes\mu_n)||H^2(\mathcal{O}_K[\frac{1}{pS}], C\otimes\mu_n)|}{|H^1(\mathcal{O}_K[\frac{1}{pS}], C\otimes\mu_n)|} = 1$$

Hence by the vanishing og H^2 and the exactness of the sequence (5.3), we are done.

∎

5.6 The generalization of the Birch Tate conjecture for Somekawa K-groups

We have now finally arrived at the main theorem. Here, we extend Yamazaki's theorem to also be correct at the prime $p = 2$.

Theorem 5.6.1 (Main theorem)**.** *Let T be a torus over a global field K and suppose T admits motivic interpretation.*

In the number field case we must additionally assume that L/K be an extension of totally real fields, where L is a splitting field for T. Then we have the formula:

$$L_K(X(T), -1) = \frac{|K^T(\mathcal{O}_K)|}{|H^0(K, X\otimes\mathbb{Q}/\mathbb{Z}(2))|}$$

Proof. We start by recalling Ono's classification theorem of tori Theorem 5.2.1: We can find quasi trival tori P and Q such that we have an isogeny:

$$T^{\oplus m}\oplus P \xrightarrow{\sim} Q$$

We have by Proposition 5.5.4 and Theorem 4.11.4 that both sides of the equation in the formula are stable under isogeny. Since both sides splits over direct sums:

$$L_K(X(T\oplus T'), -1) = L_K(X(T), -1)L_K(X(T'), -1)$$

and

$$\frac{|K^{T\oplus T'}(\mathcal{O}_K)|}{|H^0(K, X(T\oplus T')\otimes\mathbb{Q}/\mathbb{Z}(2))|} = \frac{|K^T(\mathcal{O}_K)||K^{T'}(\mathcal{O}_K)|}{|H^0(K, X(T)\otimes\mathbb{Q}/\mathbb{Z}(2))||H^0(K, X(T')\otimes\mathbb{Q}/\mathbb{Z}(2))|},$$

combined with the fact that $L_K(X(T), s)$ is real analytic, we have reduce the formula to the case of a 1-dimensional quasi trivial torus, i.e. we may assume that $T = R^M_K(\mathbb{G}_m)$ for some field M sitting in the tower: $K\subset M\subset L$. Note that M must be totally real. Now, we know that the Artin L-function depends only on the representation, so under induced represesentations we have:

$$L_K(X(T), s) = L(Ind^M_K X(\mathbb{G}_m), s) = \zeta_M(s).$$

Our Somekawa K-groups becomes equal to ordinary K_2, i.e:

$$K^T(\mathcal{O}_K) = K_2(\mathcal{O}_M).$$

Thus we are reduced to the question wether the following formula holds:

$$\zeta_M(-1) = \frac{|K^T(\mathcal{O}_M)|}{|H^0(M, X \otimes \mathbb{Q}/\mathbb{Z}(2))|},$$

but this is in the number field case a consquence of the main conjecture of Iwasawa theory for totally real fields proven by Wiles [**Wiles**] for p odd and in [**Rognes**] for $p = 2$. In the function field case it follows from Theorem 4.2.5. ■

d

CHAPTER 6

Somekawaformula

6.1 Main theorem

Theorem 6.1.1. *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be homotopy invariant Nisnevich sheaves with transfers. Then there is a an isomorphism:*

$$K(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \simeq Hom_{DM^{eff,-}_{Nis}}(\mathbb{Z}, \mathcal{F}_1[0], \dots, \mathcal{F}_n[0])$$

Recall the definition of the Somekawa K-group for a torus:

Definition 6.1.1.

$$K^T(F) = \oplus_{E/F} \frac{T(E) \otimes \mathbb{G}_m(E)}{PR\&WR}$$

.

Corollary 6.1.2.

$$K^T(K) = Hom_{DM^{eff,-}_{Nis}}(M(Spec(K)), T[0] \otimes \mathbb{G}_m[0])$$

6.2

We have a commuting diagram of symmetric monoidal categories, where the all functors are adjoints to canonical inclusions:

$$\begin{array}{ccc} PST & \xrightarrow{a_{Nis}} & PST \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_0^{Nis}} \\ HI & \xrightarrow{a_{Nis}} & HI_{Nis} \end{array}$$

Lemma 6.2.1. *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be homotopy invariant Nisnevich sheaves with transfers. Then there is a canonical isomorphism:*

$$(\mathcal{F}_1 \otimes_{HI} \dots \otimes_{HI} \mathcal{F}_n)(k) \xrightarrow{\simeq} Hom_{DM^{eff,-}_{Nis}}(\mathbb{Z}, \mathcal{F}_1[0], \dots, \mathcal{F}_n[0])$$

6.3 Local symbols

Let $\mathcal{G}$ be a presheaf with transfers. We define:

$$\mathcal{G}_{-1}(U) = Coker(U \times \mathbb{A}^1) \to \mathcal{G}(U \times (\mathbb{A}^1))).$$

If $\mathcal{G}$ is homotopy invariant, then $\mathcal{G}(U \times (\mathbb{A}^1 - \{0\}) = \mathcal{G}(U) \oplus \mathcal{G}_{-1}(U)$. Hence we get an exact endofunctor:

$$\begin{aligned} HI_{Nis} &\to HI_{Nis} \\ \mathcal{G} &\mapsto \mathcal{G}_{-1} \end{aligned}$$

Lemma 6.3.1. *Let $\mathcal{G}$ be homotopy invariant. For $i = 1$ we have:*

$$H^i(Spec(k(x)) \times (\mathbb{A}^1 - \{0\}), \mathcal{G})/H^i(Spec(k(x), \mathcal{G}) = \mathcal{G}_{-1}(k(x)).$$

For $i \neq 1$, the quotient is trivial.

We recall the definition of cohomology with support on a point:

$$H^i_x(X, U) = lim_{\to} H^i_{\{\bar{x}\} \cap U}(U, \mathcal{G}),$$

where the limit is taken over the partially ordered set of open neighbourhoods of x.

Proposition 6.3.2. *Let $\mathcal{G}$ be homotopy invariant. Then the for $i = 1$ we have:*

$$H^i_x(X, \mathcal{G}) = \mathcal{G}_{-1}(k(x))$$

The cohomology groups are trivial for $i \neq 1$.

Proof. We have a long exact sequence for cohomology with supports:

$$\ldots \longrightarrow H^0_{\{0\}}(\mathbb{A}^1, \mathcal{G}) \longrightarrow H^0(\mathbb{A}^1, \mathcal{G}) \longrightarrow H^0(\mathbb{A}^1 - \{0\}, \mathcal{G}) \longrightarrow H^1_{\{0\}}(\mathbb{A}^1, \mathcal{G}) \longrightarrow \ldots$$

By the lemma above we get the result ■

We use this result to define a canonical map:

$$\mathcal{G}(K) \xrightarrow{\partial_x} \mathcal{G}_{-1}(k(x))$$

Lemma 6.3.3 (Naturality)**.** *Let $Y \xrightarrow{f} X$ be a dominant morphism of smooth schemes. Suppose $y \in Y$ is a point of codimension 1 lying above x. Then we have diagrams reflecting the nice behaviour our symbol has with respect to pullback.*

$$\begin{array}{ccc} \mathcal{G}(L) & \xrightarrow{(\partial_y)} & \mathcal{G}_{-1}(k(y)) \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow \\ \mathcal{G}(K) & \xrightarrow{\partial_x} & \mathcal{G}_{-1}(k(x)) \end{array}$$

If f is finite and surjective, we have the following interplay with the pushforward:

$$\begin{array}{ccc} \mathcal{G}(L) & \xrightarrow{(\partial_y)} & \oplus_{y \in f^{-1}(x)} \mathcal{G}_{-1}(k(y)) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathcal{G}(K) & \xrightarrow{\partial_x} & \mathcal{G}_{-1}(k(x)) \end{array}$$

The analogy to the restriction, the norm and the boundary map in Milnor K-theory is clear.

Proposition 6.3.4. *Let $\mathcal{G}$ be homotopy invariant. Then we have a canonical isomorphism:*

$$\mathcal{G}_{-1} \to Hom(\mathbb{G}_m, \mathcal{G})$$

Proposition 6.3.5. *Let C be a smooth, proper, connected curve over K with function field K. For any $\mathcal{G} \in HI_{Nis}$ we have a canonical homomorphism:*

$$H^1_{Zar}(C, \mathcal{G}) \xrightarrow{Tr_{C/k}} \mathcal{G}_{-1}(k) \ ,$$

such that for any $x \in C$, the composition:

$$\begin{tikzcd} \mathcal{G}_{-1}(k(x)) \arrow[r, "\simeq"] \arrow[rrr, bend right, "Tr_{k(x)/k}"'] & H^1_x(C, \mathcal{G}) \arrow[r] & H^1_{Zar}(C, \mathcal{G}) \arrow[r, "Tr_{C/k}"] & \mathcal{G}_{-1}(k) \end{tikzcd}$$

equals the canonical transfer map:

$$Spec(k(x)) \xrightarrow{Tr_{k(x)/k}} Spec(k)$$

Proposition 6.3.6 (Reciprocity)**.** *Let C be a smooth, proper, connected curve over k, with function field K. Then we have a complex:*

$$\mathcal{G}(K) \xrightarrow{(\partial_x)} \oplus_{x \in C} \mathcal{G}_{-1}(k(x)) \xrightarrow{\sum_x Tr_{k(x)/k}} \mathcal{G}_{-1}(k)$$

6.4 The K-groups

$$K(k; G_1, \ldots, G_n) = \oplus_{E/k} \frac{G_1(E) \otimes \ldots \otimes G_n(E)}{PR\&WR}$$

Let $\mathcal{F}_1, \ldots, \mathcal{F}_n \in HI_{Nis}$. Our K-groups will be defined as qoutients of:

$$(\mathcal{F}_1 \otimes_{PST} \ldots \otimes_{PST} \mathcal{F}_n)(k)$$

We observe that the projection formula is already taken care of, since we are working with tensor products of Nisnevich sheaves with transfers. Hence we only need to introduce a Weil reciprocity relation to get our K-groups.

K-groups of Somekawa type

Let $\mathcal{F}_1, \ldots, \mathcal{F}_n \in HI_{Nis}$, and consider the following data:

1. a smooth proper connected curve C over k.
2. $h \in k(C)^*$.
3. $g_i \in \mathcal{F}_i(k(C))$ for each $i \in \{1, \ldots, n\}$ with the following condition: For any $c \in C$, there is an $i(c)$ such that $c \in R_i$ for all $i \neq i(c)$, where $R_i = \{c \in C | g_i \in Im(\mathcal{F}_i(\mathcal{O}_{C,c}) \to \mathcal{F}_i(k(C)))$.

Definition 6.4.1. Given a tripple $(C, h, (g_i))$ as above we define the Weil reciprocity of Somekawa type WR_S to be generated by elements of the form:

$$WR_S := \left\langle \sum_{c \in C} Tr_{k(c)/k}(g_1(c) \otimes \ldots \otimes \partial_c(g_{i(c)}, h) \otimes \ldots \otimes g_n(c)) | (C, h, (g_i)) \right\rangle$$

We then define the K-group of Somekawa type to be the quotient:

$$K(k; \mathcal{F}_1, \ldots, \mathcal{F}_n) := \frac{(\mathcal{F}_1 \otimes_{PST} \ldots \otimes_{PST} \mathcal{F}_n)(k)}{WR_S}$$

Theorem 6.4.1. *There is a surjective homomorphism:*

$$K(k; \mathcal{F}_1, \ldots, \mathcal{F}_n) \longrightarrow\!\!\!\!\!\rightarrow Hom_{DM^{eff,-}_{Nis}}(\mathbb{Z}, \mathcal{F}_1[0] \otimes \mathcal{F}_n[0])$$

K-groups of geometric type

Let now $\mathcal{F}_1, \ldots, \mathcal{F}_n \in PST$, and consider the folloing data:

1. C is a smooth projective connected curve C over k.
2. a surjective morphism $C \xrightarrow{f} \mathbb{P}^1$.
3. $g_i \in \mathcal{F}_i(C')$ for each $i \in \{1, \ldots, n\}$, where $C' = f^{-1}((\mathbb{P}^1 - \{1\})$.

Definition 6.4.2. We define the Weil reciprocity of geometric type to be genereted by elements of the form:

$$WR_G := \sum_{c \in C'} v_c(f) Tr_{k(c)/k}(g_1(c) \otimes \ldots \otimes g_n(c)).$$

We then define the K-group of geometric type to be:

$$K(k; \mathcal{F}_1, \ldots, \mathcal{F}_n) := \frac{(\mathcal{F}_1 \otimes_{PST} \ldots \otimes_{PST} \mathcal{F}_n)(k)}{WR_G}$$

Let now $\mathcal{F} \in PST$ and consider the following set of data:

1. A smooth projective curve C over k.
2. A surjective morphism $C \xrightarrow{f} \mathbb{P}^1$.
3. A section: $\mathbb{Z}_{tr}(C') \xrightarrow{\alpha} \mathcal{F}$.

To such a tripple (C, f, α) we associate an element:

$$\alpha(div(f)) = \sum_{c \in C'} v_c(f) Tr_{k(c)/k} \alpha(c)$$

Elements of this type generate the rational sections:

$$\mathcal{F}(k)_{rat} = \langle \alpha(div(f)) \rangle$$

Proposition 6.4.3. *Let $\mathcal{F} \in PST$. Then we have:*

$$h_0(\mathcal{F})(k) = \mathcal{F}(k)/\mathcal{F}(k)_{rat}.$$

Proof.

$$\begin{array}{ccc} \mathbb{Z}_{tr}(C')(\triangle) & \xrightarrow{\alpha} & \mathcal{F}(\triangle) \\ {\scriptstyle i_o^*-i_\infty^*}\downarrow & & \downarrow{\scriptstyle i_o^*-i_\infty^*} \\ \mathbb{Z}_{tr}(C')(k) & \xrightarrow[\alpha]{} & \mathcal{F}(k) \end{array}$$

■

Lemma 6.4.4. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_n \in PST$ and let $\mathcal{F} = \mathcal{F}_1 \otimes_{PST} \ldots \otimes_{PST} \mathcal{F}_n$. Let (C, f, α) be a rational tripple. Then α can be written as a finite sum:*

$$\alpha = \sum Tr_h(g_1 \otimes \ldots \otimes g_n),$$

Where we have sujective and finite morphisms of smooth projective curves, $D \xrightarrow{h} C \xrightarrow{f} \mathbb{P}^1$, $g_i \in \mathcal{F}_i(h^{-1}(C'))$ for $i \in \{1, \ldots, n\}$ and $\mathcal{F}(h^{-1}(C')) \xrightarrow{Tr_h} \mathcal{F}(C')$ is the canonical transfer.

Theorem 6.4.2. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_n \in HI_{Nis}$. Then we have an isomorphism:*

$$K'(k; \mathcal{F}_1, \ldots, \mathcal{F}_n) \xrightarrow{\simeq} Hom_{DM^{eff,-}_{Nis}}(\mathbb{Z}, \mathcal{F}_1[0], \ldots, \mathcal{F}_n[0])$$

Proof. ■

K-groups of Milnor type

Let $\mathcal{F}_1, \ldots \mathcal{F}_n \in HI_{Nis}$. Recall the cocharacter group $\mathcal{F}_{-1} = Hom_{HI_{Nis}}(\mathbb{G}_m, \mathcal{F})$, where the elements are cocharacters $\mathbb{G}_m \xrightarrow{\chi} \mathcal{F}$.

Definition 6.4.5. We define the Weil reciprocity of Milnor type to be generated by:

$$WR_M = \langle a_1 \otimes \ldots \otimes \chi_i(a) \otimes \ldots \otimes \chi_j(1-a) \otimes \ldots \otimes a_n \rangle$$

, where $a_r \in \mathcal{F}_r$ and $a \in (k^* - \{1\})$. We then define the K-theory of Milnor type to be:

$$\tilde{K}(k; \mathcal{F}_1, \ldots, \mathcal{F}_n) := \frac{(\mathcal{F}_1 \otimes_{PST} \ldots \otimes_{PST} \mathcal{F}_n)(k)}{WR_M}$$

Lemma 6.4.6. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_n \in HI_{Nis}$. Then we have a surjective homomorphism:*

$$\tilde{K}(k; \mathcal{F}_1, \ldots, \mathcal{F}_n) \twoheadrightarrow K(k; \mathcal{F}_1, \ldots, \mathcal{F}_n)$$

6.5 Main theorem

Proposition 6.5.1. *The following to statments are equivalent:*

6.5.1.1. *The homomorphism* $K(k; \mathcal{F}_1, \ldots, \mathcal{F}_n) \longrightarrow K'(k; \mathcal{F}_1, \ldots, \mathcal{F}_n)$ *is a bijection for* $\mathcal{F}_1, \ldots, \mathcal{F}_n \in HI_{Nis}$.

6.5.1.2. *We are given a tripple* $(C, f, (\tilde{g}))$ *giving rise to Weil reciprocity of geometric type, where* $\tilde{g}$ *is the section:*

$$\mathbb{Z}_{tr}(C') \xrightarrow{\tilde{g}} h_0^{Nis}(C')$$

Suppose $\mathcal{F}_1 = \ldots = \mathcal{F}_n = h_0^{Nis}(C')$, *and consider the modified tripple* $(C, f, (\tilde{g})$, *which give rise to Weil reciprocity of geometric type:*

$$WR_G = \sum_{c \in C'} v_c(f) Tr_{k(c)/k}(\tilde{g}(c) \otimes \ldots \otimes \tilde{g}(c))$$

Then WR_G *vanish in* $K(k; h_0^{Nis}(C'), \ldots, h_0^{Nis}(C'))$.

Definition 6.5.2. We say that a sheaf $\mathcal{F}$ is proper, if for any smooth curve C over a field k and any closed point $c \in C$, the canonical map:

$$\mathcal{F}(\mathcal{O}_{C,c}) \longrightarrow \mathcal{F}(k(C))$$

Definition 6.5.3. Let $\mathcal{F} \in HI_{Nis}$. We say that the sheaf $\mathcal{F}$ is *curve-like* if it sits in en exact sequence of the form:

$$0 \longrightarrow T \longrightarrow \mathcal{F} \longrightarrow \bar{\mathcal{F}} \longrightarrow 0$$

where $\bar{\mathcal{F}}$ is proper and T is a quasi-trivial torus.

Lemma 6.5.4. $h_0^{Nis}(C)$ *is curve-like. In fact, it is the Nisnevich sheaf with transfers associated to the presheaf of relative Picard groups:*

$$U \mapsto Pic(\overline{C} \times U, D \times U),$$

where $\overline{C}$ *is the projective completion and* D *is the normal crossing divisor* $\overline{C} - C$.

Proposition 6.5.5. *Let* C *be a smooth proper curve and let* $\mathcal{F}$ *be curve-like. Let* $Z \hookrightarrow C$ *be smooth and proper and* $K = k(C)$. *We have commutative diagram:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T(\mathcal{O}_{C,Z}) \arrow[r] \arrow[d] & \mathcal{F}(\mathcal{O}_{C,Z}) \arrow[r] \arrow[d] & \bar{\mathcal{F}}(\mathcal{O}_{C,Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T(K) \arrow[r] \arrow[d] & \mathcal{F}(K) \arrow[r] \arrow[d] & \bar{\mathcal{F}}(K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_i T_{-1}(k(c_i)) \arrow[r] \arrow[d] & \oplus_i \mathcal{F}_{-1}(k(c_i)) \arrow[r] \arrow[d] & \oplus_i \bar{\mathcal{F}}_{-1}(k(c_i)) = 0 \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Lemma 6.5.6. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be curve-like sheaves. Let $D \twoheadrightarrow C$ be a surjection of curves over k where C is smooth and proper. Let $f_i \in \mathcal{F}_i(k(D)$ and $v \in D$. Let $x = \{f_1, \ldots, f_n, f_{n+1}\} \in K(k; \mathcal{F}_1, \ldots, \mathcal{F}_n)$. Then:*

6.5.6.1. *If $v(f_{n+1} - 1) > 0$, then $\partial_v(x) = 0$.*

6.5.6.2. $\partial_v(x) = \{f_1(v), \ldots, \partial_v(f_i, f_{n+1}), \ldots, f_n)\}_{k(v)/k}$.

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